# Planar Graphs II

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# Outline

# Section

# Planar Graph

- **Planar Graph:** A graph which can be drawn on a plane ($R^2$) without any of its edges crossing.
- **Plane Graph:** A graph drawn on a plane without any of its edges crossing.

# Examples

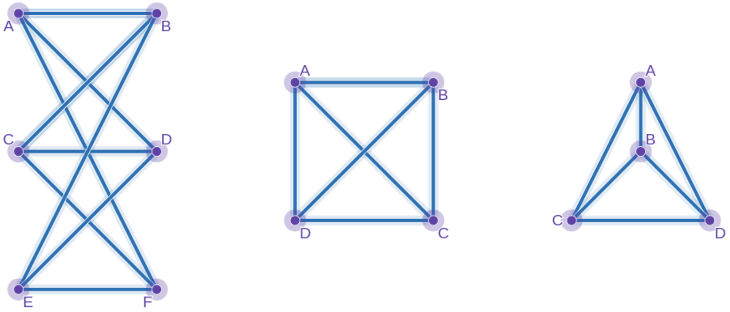


Figure 1: **a.** Non-Planar Graph **b.** Planar Graph **c.** Plane Graph

# Planar Graph Terminology

- **Regions or Faces**: When we draw a plane graph drawing, the plane gets divided into regions or faces. There are two types of faces:
  - **Bounded Faces**
  - **Unbounded Faces**
- **Boundary:** For each face of a plane graph, the set of vertices and edges that outline it form the boundary of the face.
- Every Planar graph has exactly one unbounded face, which is called the **exterior face**.

# Example

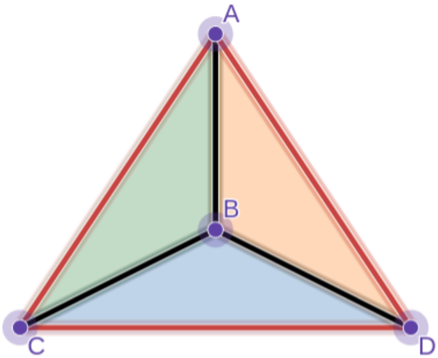


Figure 2: Plane Graph - Regions and Boundaries

# Section

# Convex Combination

Suppose we have two points in $R^2$, and we denote them as $A$ and $B$.

- The unique line passing through $A$ and $B$ can be denoted by $x = (1-t)A + tB, \;\; t \in R$
- The line segment joining the points $A$ and $B$ are denoted by $x = (1-t)A + tB, \;\; t \in [0,1]$

This can be extended for $n$ points, where we define the convex combination as:
For points $[a_0, a_1, ...., a_{n-1}], \;\; x = \sum_{i=0}^{n} t_i a_i$, where $\sum_{i=0}^{n} t_i = 1$ and $t_i \geq 0 \; \forall i \in [0, ...n-1]$

# Convex Hull

The set of all convex combinations of a point set is the *convex hull* for the point set.

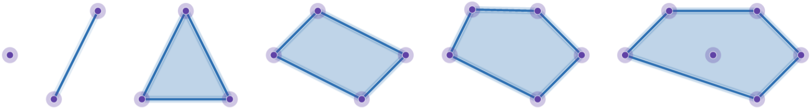

Figure 3: Examples of Convex Hull for different Point Sets

# Section

# Edge Skeleton of a Triangulation of a Disk

Let $G = (V, E)$ be such a graph that arises from the edge skeleton of a triangulation of a disk, i.e., suppose K is a triangulation of the disk, then the edge-skeleton consisting of the vertices and edges of K is a planar graph such that each interior face in bounded by three edges and their union is homeomorphic to $B^2$.

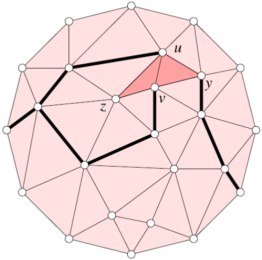


Figure 4: Edge Skeleton of a Triangulation of a Disk

# Strictly Convex Combination Mapping

$f : V \longrightarrow R^2$ is a *strictly convex combination* if for every interior vertex $u \in V$ there are real numbers $t_{uv}\ 0 with \sum_v t_{uv} = 1$ and $f(u) = \sum_v t_{uv} f(v)$, where both sums are over all neighbords v of u.

**Observation:** Since we are considering straight-line embedding, a *strictly convex combination mapping* suffices in defining the edge-skeleton for a planar graph.

# Inner Product

An inner product on a real vector space V is an operation which assigns to each pair of vectors u and v, a unique real number, denoted by <u, v>, which satisfies the following axioms $\forall\ u, v\ and\ w \in V\ and\ \forall\ k \in R$.

- $< u, v > = < v, u >$ $[Commutative\ Law]$
- $< u + v, w > = < u, w > + < v, w >$ $[Distributive\ Law]$
- $< ku, v > = k < u, v >$
- $< u, v > \geq 0,\ and\ < u, u > = 0 \iff u = 0.$

# Linear Function

We define a *linear function* $h : R^2 \longrightarrow R$ defined by $h(x) =< x, p > +c,$ *where* $p \in R^2$ is a non-zero vector, and $c \in R$.

**Lemma**

$$h(f(u)) = \sum_{v} t_{uv} h(f(v))$$

**Proof.**

$$h(f(u)) =< \sum_{v} t_{uv} f(v), p > + \sum_{v} t_{uv} c = \sum_{v} t_{uv} h(f(u))$$

## Boundary Edge, Interior Edge and Separating Edge

- **Boundary Edge:** An edge which is the part of the outer face of the planar graph is a *boundary edge*.
- **Interior Edge:** An edge which is not the part of the outer face of the planar graph is an *interior edge*.
- **Separating Edge:** An interior edge which connects to boundary vertices is a *separating edge*.

# Example

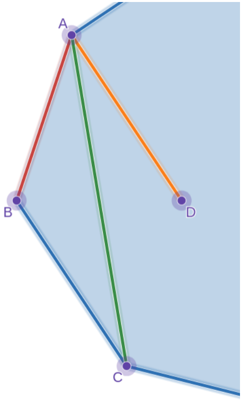


Figure 5: Types of Edges

# Connectedness

## Lemma

*Under the assumption of no separating edge, every interior vertex u can be connected to every boundary vertex by an interior path.*

## Proof.

- Begin expanding outwards from u, one edge at a time (BFS). Stop when we reach the first boundary vertex. Assume that this vertex is $w$ and its neighboring boundary vertices are $w_0$ *and* $w_1$.
- Since none of the edges separate, the neighbors of $w$ form a unique interior path connecting $w_0$ *to* $w_1$. Hence, there is an interior path from u to $w_0$

# Maximum Principle

## Lemma

*Let $K$ be a triangulation of $B^2$ without separating edges and $f : V \longrightarrow R$ be a convex combination function. If $f(u) \geq f(w)$ for an interior vertex $u$ and every boundary vertex $w$ then $(u) = f(v)$ for every vertex $v \in V$.*

## Proof.

- Suppose $u_0$ is an interior vertex which maximizes f (i.e., $f(u) \geq f(v) \; \forall \; v \in V$.
- Since f is a convex combination function, all the neighbors of $u_0$ have the same function value.

# Section

# Tutte's Theorem

**Theorem**

*If $f : v \longrightarrow R^2$ is a convex combination mapping that maps the boundary vertices to a strictly convex polygon then drawing the straight edges between the images of the vertices is a straight-line embedding of the edge-skeleton of $K$.*

# Interior Vertices Lemma

**Lemma**

*All interior vertices $u$ of $V$ map to the interior of the strictly convex polygon whose corners are the images of the boundary vertices.*

**Proof.**

- We choose our linear function such that it contains one of the boundary edge, and $h(f(w)) > 0$ for all boundary vertices other than the endpoints of that edge.
- Now, $h(f(u)) > 0$, otherwise the minimum principle would imply that value for all the boundary vertices is also 0.

# Implications of Interior Vertices Lemma

- All interior vertices lie in the interior of the polygon.
- Each triangle incident to a boundary edge is non-degenerate (i.e., the three vertices of that triangle are non-collinear).

# Positionality of Interior Vertices

## Lemma

*Let yuv and zuv be the two triangles sharing the interior edge uv in K, then the points f(y) and f(z) lie on the opposite side of* $f^{-1}(0)$ *which passes through f(u) and f(v).*

## Proof.

- Assume $h(f(y)) > 0$, and find a strictly rising path connecting y to the boundary.
- Find two strictly falling paths connecting u and v to the boundary, respectively.
- Now, we can get a piece of triangulation as defined by u, v, and the paths from them to the boundary.

# Example

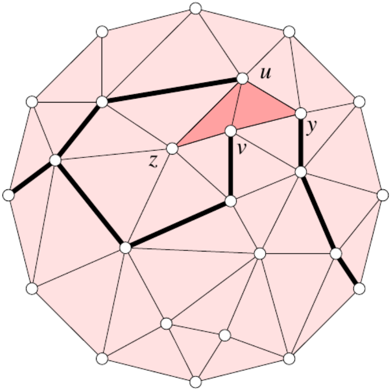


Figure 6: Positionality of Interior Vertices

# Triangles have Disjoint Interiors

**Lemma**

*The images of any two triangles in $K$ have disjoint interiors.*

**Proof.**

- Assume we have a point $x$ belonging to the interiors of two images, $T_1$ and $T_2$.
- Suppose a half-line from x, which avoids the image of all vertices. Such a half-line shall define a sequence of triangles from $T_1$ and $T_2$ to some boundary edge.
- Suppose the triangle intersected (which shares an edge with the boundary) is v. From here we trace our way backwards.

# Constructing Straight Line Embedding

**Input:** Edge skeleton of a triangulation of a disk, where no edge separates.
**Output:** Straight line embedding for the input planar graph.

- **Step 1** Re-index the vertices of the graph such that $u_1$ to $u_k$ are ordered along the boundary of the outer face, and $u_{k+1}$ to $u_n$ are the interior vertices of the graph.
- **Step 2** $f(u_i) = (cos(2i\pi/k), sin(2i\pi/k)), \ \forall \ i \in [1, .., k]$
- Hence, the vertices $u_1$ to $u_k$ form the boundary of a strictly convex polygon, as required.

# Continued

- **Step 3** We express the image of each interior vertex as a strictly convex combination of its neighbors as, $f(u_j) = (d_j)^{-1} \sum f(v)$, where $d_j$ is the degree of the vertex $u_j$, and $j \in [k+1, ...n]$.
- Hence, we get a system of (n-k) linear equations in (n-k) variables, which can be solved to obtain the embedding.