Planar Graphs II

Amogh Johri
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Outline

Section

Planar Graph

- **Planar Graph:** A graph which can be drawn on a plane (R^2) without any of its edges crossing.
- **Plane Graph:** A graph drawn on a plane without any of its edges crossing.

Examples

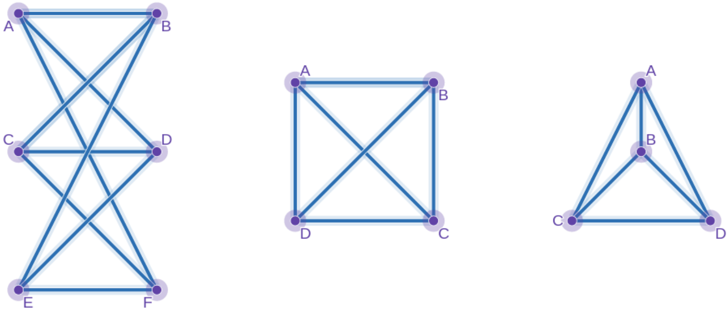

Figure 1: **a.** Non-Planar Graph **b.** Planar Graph **c.** Plane Graph

Planar Graph Terminology

- **Regions or Faces**: When we draw a plane graph drawing, the plane gets divided into regions or faces. There are two types of faces:
  - **Bounded Faces**
  - **Unbounded Faces**
- **Boundary:** For each face of a plane graph, the set of vertices and edges that outline it form the boundary of the face.
- Every Planar graph has exactly one unbounded face, which is called the **exterior face**.

Example

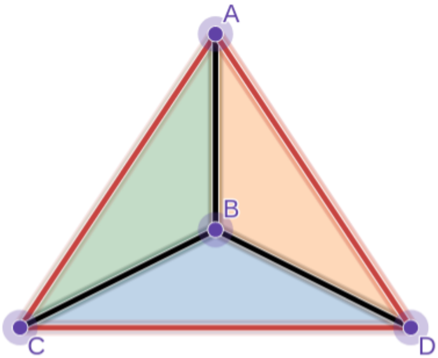

Figure 2: Plane Graph - Regions and Boundaries

Section

Convex Combination

Suppose we have two points in R^2, and we denote them as A and B.

- The unique line passing through A and B can be denoted by $x = (1-t)A + tB,\ \ t \in R$
- The line segment joining the points A and B are denoted by $x = (1-t)A + tB,\ \ t \in [0,1]$

This can be extended for n points, where we define the convex combination as:
For points $[a_0, a_1, ..., a_{n-1}],\ \ x = \sum_{i=0}^{n} t_i a_i$, where $\sum_{i=0}^{n} t_i = 1$ and $t_i \geq 0\ \forall i \in [0, ...n-1]$

Convex Hull

The set of all convex combinations of a point set is the *convex hull* for the point set.

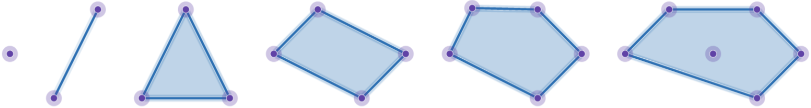

Figure 3: Examples of Convex Hull for different Point Sets

Section

Edge Skeleton of a Triangulation of a Disk

Let $G = (V, E)$ be such a graph that arises from the edge skeleton of a triangulation of a disk, i.e., suppose K is a triangulation of the disk, then the edge-skeleton consisting of the vertices and edges of K is a planar graph such that each interior face in bounded by three edges and their union is homeomorphic to B^2.

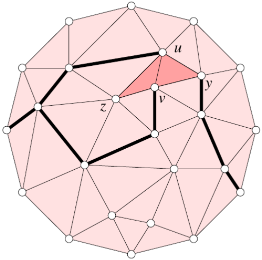

Figure 4: Edge Skeleton of a Triangulation of a Disk

Strictly Convex Combination Mapping

$f : V \longrightarrow R^2$ is a *strictly convex combination* if for every interior vertex $u \in V$ there are real numbers $t_{uv} > 0 with \sum_v t_{uv} = 1$ and $f(u) = \sum_v t_{uv} f(v)$, where both sums are over all neighbords v of u.

Observation: Since we are considering straight-line embedding, a *strictly convex combination mapping* suffices in defining the edge-skeleton for a planar graph.

Inner Product

An inner product on a real vector space V is an operation which assigns to each pair of vectors u and v, a unique real number, denoted by <u, v>, which satisfies the following axioms $\forall\ u, v\ and\ w \in V\ and\ \forall\ k \in R$.

- $< u, v > = < v, u >$ $[Commutative\ Law]$
- $< u + v, w > = < u, w > + < v, w >$ $[Distributive\ Law]$
- $< ku, v > = \ k < u, v >$
- $< u, v > \geq 0,\ and\ < u, u > = \ 0 \iff u = 0.$

Linear Function

We define a *linear function* $h : R^2 \longrightarrow R$ defined by $h(x) =< x, p > +c, \ where\ p \in R^2$ is a non-zero vector, and $c \in R$.

Lemma

$$h(f(u)) = \sum_{v} t_{uv} h(f(v))$$

Proof.

$$h(f(u)) =< \sum_{v} t_{uv} f(v), p > + \sum_{v} t_{uv} c = \sum_{v} t_{uv} h(f(u))$$

Boundary Edge, Interior Edge and Separating Edge

- **Boundary Edge:** An edge which is the part of the outer face of the planar graph is a *boundary edge*.
- **Interior Edge:** An edge which is not the part of the outer face of the planar graph is an *interior edge*.
- **Separating Edge:** An interior edge which connects to boundary vertices is a *separating edge*.

Example

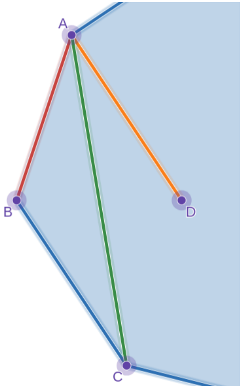

Figure 5: Types of Edges

Connectedness

Lemma

Under the assumption of no separating edge, every interior vertex u can be connected to every boundary vertex by an interior path.

Proof.

- Begin expanding outwards from u, one edge at a time (BFS). Stop when we reach the first boundary vertex. Assume that this vertex is w and its neighboring boundary vertices are w_0 *and* w_1.
- Since none of the edges separate, the neighbors of w form a unique interior path connecting w_0 *to* w_1. Hence, there is an interior path from u to w_0

Maximum Principle

Lemma

Let K be a triangulation of B^2 without separating edges and $f : V \longrightarrow R$ be a convex combination function. If $f(u) \geq f(w)$ for an interior vertex u and every boundary vertex w then $f(u) = f(v)$ for every vertex $v \in V$.

Proof.

- Suppose u_0 is an interior vertex which maximizes f (i.e., $f(u) \geq f(v) \; \forall \; v \in V$.
- Since f is a convex combination function, all the neighbors of u_0 have the same function value.

Section

Tutte's Theorem

Theorem

If $f : V \longrightarrow R^2$ is a convex combination mapping that maps the boundary vertices to a strictly convex polygon then drawing the straight edges between the images of the vertices is a straight-line embedding of the edge-skeleton of K.

Interior Vertices Lemma

Lemma

All interior vertices u of V map to the interior of the strictly convex polygon whose corners are the images of the boundary vertices.

Proof.

- We choose our linear function such that it contains one of the boundary edge, and $h(f(w)) > 0$ for all boundary vertices other than the endpoints of that edge.
- Now, $h(f(u)) > 0$, otherwise the minimum principle would imply that value for all the boundary vertices is also 0.

Implications of Interior Vertices Lemma

- All interior vertices lie in the interior of the polygon.
- Each triangle incident to a boundary edge is non-degenerate (i.e., the three vertices of that triangle are non-collinear).

Positionality of Interior Vertices

Lemma

Let yuv and zuv be the two triangles sharing the interior edge uv in K, then the points f(y) and f(z) lie on the opposite side of $h^{-1}(0)$ *which passes through f(u) and f(v).*

Proof.

- Assume $h(f(y)) > 0$, and find a strictly rising path connecting y to the boundary.
- Find two strictly falling paths connecting u and v to the boundary, respectively.
- Now, we can get a piece of triangulation as defined by u, v, and the paths from them to the boundary.

Example

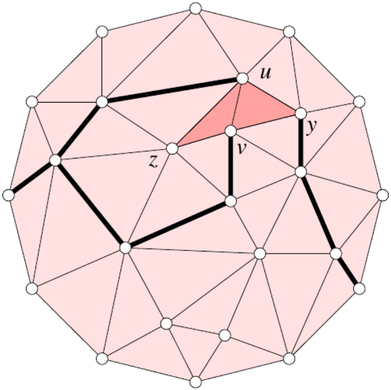

Figure 6: Positionality of Interior Vertices

Triangles have Disjoint Interiors

Lemma

No two edges intersect except for at a single point (vertex).

Proof for non-intersecting edges

Proof.

- Assume x is a point lying in the common interior of two edges (uv and u'v').
- Choose a half-line which emanates from x and avoids the images of all the vertices. This half-line shall then intersect exactly one of yu, yv, zu or zv.
- By construction we get a sequence half-lines starting from uv and u'v', and ending at a boundary edge.
- While tracing back in the sequence, we pass from one edge to an unambiguously defined preceding edge, i.e., uv = u'v'.

Constructing Straight Line Embedding

Input: Edge skeleton of a triangulation of a disk, where no edge separates.
Output: Straight line embedding for the input planar graph.

- **Step 1** Re-index the vertices of the graph such that u_1 to u_k are ordered along the boundary of the outer face, and u_{k+1} to u_n are the interior vertices of the graph.
- **Step 2** $f(u_i) = (cos(2i\pi/k), sin(2i\pi/k)), \ \forall\ i \in [1, .., k]$
- Hence, the vertices u_1 to u_k form the boundary of a strictly convex polygon, as required.

Continued

- **Step 3** We express the image of each interior vertex as a strictly convex combination of its neighbors as, $f(u_j) = (d_j)^{-1} \sum f(v)$, where d_j is the degree of the vertex u_j, and $j \in [k+1, ...n]$.
- Hence, we get a system of (n-k) linear equations in (n-k) variables, which can be solved to obtain the embedding.

Example - Input

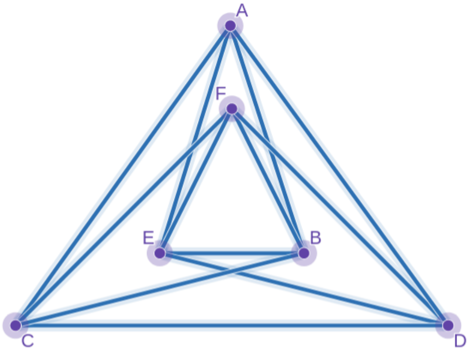

Figure 7: Planar Graph (Input)

Example - Output

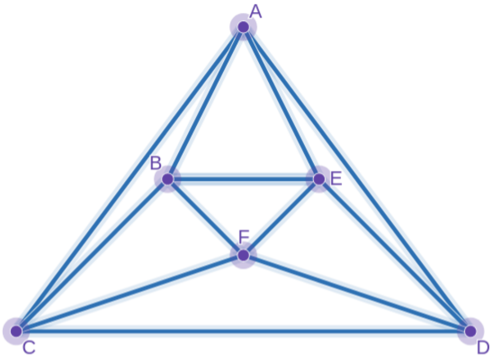

Figure 8: Straight Line Embedding for the Planar Graph(Output)

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[3] Gilbert Strang.
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