

# Planar Graphs II

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# Outline

- ① Introduction
- ② Convex Combinations Preliminaries
- ③ Formulations
- ④ Tutte's Theorem

# Section

- ① Introduction
- ② Convex Combinations Preliminaries
- ③ Formulations
- ④ Tutte's Theorem

# Planar Graph

- **Planar Graph:** A graph which can be drawn on a plane ( $\mathbb{R}^2$ ) without any of its edges crossing.
- **Plane Graph:** A graph drawn on a plane without any of its edges crossing.

# Examples

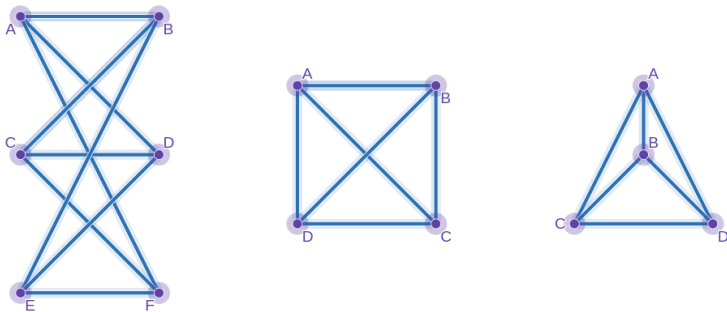


Figure 1: **a.** Non-Planar Graph **b.** Planar Graph **c.** Plane Graph

# Planar Graph Terminology

- **Regions or Faces:** When we draw a plane graph drawing, the plane gets divided into regions or faces. There are two types of faces:
  - **Bounded Faces**
  - **Unbounded Faces**
- **Boundary:** For each face of a plane graph, the set of vertices and edges that outline it form the boundary of the face.
- Every Planar graph has exactly one unbounded face, which is called the **exterior face**.

# Example

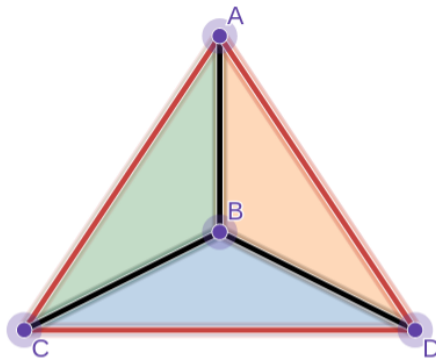


Figure 2: Plane Graph - Regions and Boundaries

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# Convex Combination

Suppose we have two points in  $R^2$ , and we denote them as  $A$  and  $B$ .

- The unique line passing through  $A$  and  $B$  can be denoted by  $x = (1 - t)A + tB$ ,  $t \in R$
- The line segment joining the points  $A$  and  $B$  are denoted by  $x = (1 - t)A + tB$ ,  $t \in [0, 1]$

This can be extended for  $n$  points, where we define the convex combination as:

For points  $[a_0, a_1, \dots, a_{n-1}]$ ,  $x = \sum_{i=0}^n t_i a_i$ , where  $\sum_{i=0}^n t_i = 1$  and  $t_i \geq 0 \forall i \in [0, \dots, n - 1]$

# Convex Hull

The set of all convex combinations of a point set is the *convex hull* for the point set.

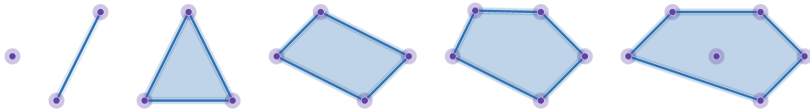


Figure 3: Examples of Convex Hull for different Point Sets

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# Edge Skeleton of a Triangulation of a Disk

Let  $G = (V, E)$  be such a graph that arises from the edge skeleton of a triangulation of a disk, i.e., suppose  $K$  is a triangulation of the disk, then the edge-skeleton consisting of the vertices and edges of  $K$  is a planar graph such that each interior face is bounded by three edges and their union is homeomorphic to  $B^2$ .

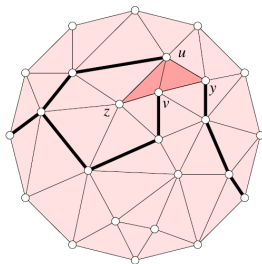


Figure 4: Edge Skeleton of a Triangulation of a Disk

# Strictly Convex Combination Mapping

$f : V \longrightarrow \mathbb{R}^2$  is a *strictly convex combination* if for every interior vertex  $u \in V$  there are real numbers  $t_{uv} > 0$  with  $\sum_v t_{uv} = 1$  and  $f(u) = \sum_v t_{uv} f(v)$ , where both sums are over all neighbors  $v$  of  $u$ .

**Observation:** Since we are considering straight-line embedding, a *strictly convex combination mapping* suffices in defining the edge-skeleton for a planar graph.

# Inner Product

An inner product on a real vector space  $V$  is an operation which assigns to each pair of vectors  $u$  and  $v$ , a unique real number, denoted by  $\langle u, v \rangle$ , which satisfies the following axioms

$\forall u, v \text{ and } w \in V \text{ and } \forall k \in \mathbb{R}$ .

- $\langle u, v \rangle = \langle v, u \rangle$  [*Commutative Law*]
- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$  [*Distributive Law*]
- $\langle ku, v \rangle = k \langle u, v \rangle$
- $\langle u, v \rangle \geq 0$ , and  $\langle u, u \rangle = 0 \iff u = 0$ .

# Linear Function

We define a *linear function*  $h : R^2 \rightarrow R$  defined by  $h(x) = \langle x, p \rangle + c$ , where  $p \in R^2$  is a non-zero vector, and  $c \in R$ .

## Lemma

$$h(f(u)) = \sum_v t_{uv} h(f(v))$$

## Proof.

$$h(f(u)) = \langle \sum_v t_{uv} f(v), p \rangle + \sum_v t_{uv} c = \sum_v t_{uv} h(f(v))$$

# Boundary Edge, Interior Edge and Separating Edge

- **Boundary Edge:** An edge which is the part of the outer face of the planar graph is a *boundary edge*.
- **Interior Edge:** An edge which is not the part of the outer face of the planar graph is an *interior edge*.
- **Separating Edge:** An interior edge which connects to boundary vertices is a *separating edge*.



# Example

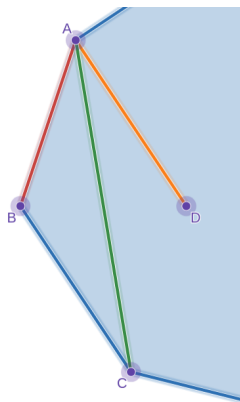


Figure 5: Types of Edges

# Connectedness

## Lemma

*Under the assumption of no separating edge, every interior vertex  $u$  can be connected to every boundary vertex  $v$  by an interior path.*

## Proof.

- Begin expanding outwards from  $u$ , one edge at a time (BFS). Stop when we reach the first boundary vertex. Assume that this vertex is  $w$  and its neighboring boundary vertices are  $w_0$  and  $w_1$ .
- Since none of the edges separate, the neighbors of  $w$  form a unique interior path connecting  $w_0$  to  $w_1$ . Hence, there is an interior path from  $u$  to  $w_0$

# Maximum Principle

## Lemma

*Let  $K$  be a triangulation of  $B^2$  without separating edges and  $f : V \rightarrow R$  be a convex combination function. If  $f(u) \geq f(w)$  for an interior vertex  $u$  and every boundary vertex  $w$  then  $f(u) = f(v)$  for every vertex  $v \in V$ .*

## Proof.

- Suppose  $u$  is an interior vertex which maximizes  $f$  (i.e.,  $f(u) \geq f(v) \forall v \in V$ ).
- Since  $f$  is a convex combination function, all the neighbors of  $u$  have the same function value.

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# Tutte's Theorem

## Theorem

*If  $f : V \rightarrow \mathbb{R}^2$  is a convex combination mapping that maps the boundary vertices to a strictly convex polygon then drawing the straight edges between the images of the vertices is a straight-line embedding of the edge-skeleton of  $K$ .*

# Interior Vertices Lemma

## Lemma

*All interior vertices  $u$  of  $V$  map to the interior of the strictly convex polygon whose corners are the images of the boundary vertices.*

## Proof.

- We choose our linear function such that it contains one of the boundary edge, and  $h(f(w)) > 0$  for all boundary vertices other than the endpoints of that edge.
- Now,  $h(f(u)) > 0$ , otherwise the minimum principle would imply that value for all the boundary vertices is also 0.

# Implications of Interior Vertices Lemma

- All interior vertices lie in the interior of the polygon.
- Each triangle incident to a boundary edge is non-degenerate (i.e., the three vertices of that triangle are non-collinear).

# Positionality of Interior Vertices

## Lemma

*Let  $yuv$  and  $zuv$  be the two triangles sharing the interior edge  $uv$  in  $K$ , then the points  $f(y)$  and  $f(z)$  lie on the opposite side of  $h^{-1}(0)$  which passes through  $f(u)$  and  $f(v)$ .*

## Proof.

- Assume  $h(f(y)) > 0$ , and find a strictly rising path connecting  $y$  to the boundary.
- Find two strictly falling paths connecting  $u$  and  $v$  to the boundary, respectively.
- Now, we can get a piece of triangulation as defined by  $u$ ,  $v$ , and the paths from them to the boundary.



# Example

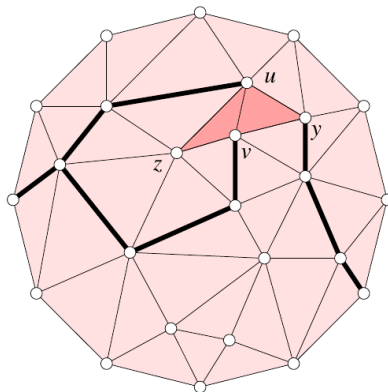


Figure 6: Positionality of Interior Vertices

# Triangles have Disjoint Interiors

## Lemma

*No two edges intersect except for at a single point (vertex).*

# Proof for non-intersecting edges

## Proof.

- Assume  $x$  is a point lying in the common interior of two edges ( $uv$  and  $u'v'$ ).
- Choose a half-line which emanates from  $x$  and avoids the images of all the vertices. This half-line shall then intersect exactly one of  $yu$ ,  $yv$ ,  $zu$  or  $zv$ .
- By construction we get a sequence half-lines starting from  $uv$  and  $u'v'$ , and ending at a boundary edge.
- While tracing back in the sequence, we pass from one edge to an unambiguously defined preceding edge, i.e.,  $uv = u'v'$ .

# Constructing Straight Line Embedding

**Input:** Edge skeleton of a triangulation of a disk, where no edge separates.

**Output:** Straight line embedding for the input planar graph.

- **Step 1** Re-index the vertices of the graph such that  $u_1$  to  $u_k$  are ordered along the boundary of the outer face, and  $u_{k+1}$  to  $u_n$  are the interior vertices of the graph.
- **Step 2**  $f(u_i) = (\cos(2i\pi/k), \sin(2i\pi/k))$ ,  $\forall i \in [1, \dots, k]$
- Hence, the vertices  $u_1$  to  $u_k$  form the boundary of a strictly convex polygon, as required.

# Continued

- **Step 3** We express the image of each interior vertex as a strictly convex combination of its neighbors as,  
$$f(u_j) = (d_j)^{-1} \sum f(v),$$
 where  $d_j$  is the degree of the vertex  $u_j$ , and  $j \in [k + 1, \dots, n]$ .
- Hence, we get a system of  $(n-k)$  linear equations in  $(n-k)$  variables, which can be solved to obtain the embedding.

# Example - Input

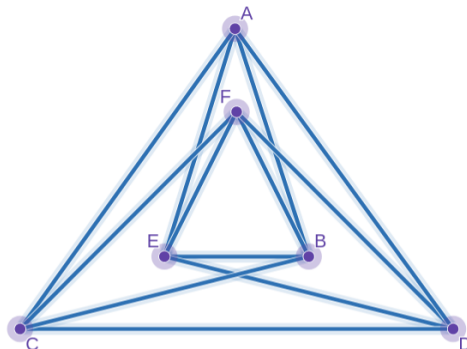


Figure 7: Planar Graph (Input)

# Example - Output

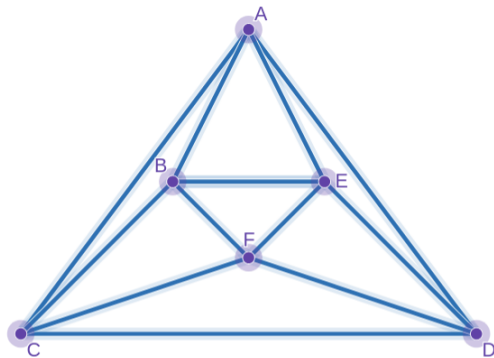


Figure 8: Straight Line Embedding for the Planar Graph(Output)

# References

[1] Herbert Edelsbrunner and John Harer.  
Computational Topology: An Introduction.

[2] Sarada Herke  
Graph Theory: 57. Planar Graphs and Graph Theory: 58. Euler's  
Formula for Plane Graphs.  
(<https://www.youtube.com/watch?v=wnYtITkWAYA>,  
<https://www.youtube.com/watch?v=5ywif1Zpeo4t=4s>)

[3] Gilbert Strang.  
Introduction to Linear Algebra.