Asteroidal Triple Free Graphs

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1 Introduction

Definition 1.0.1. *In a graph G, an **asteroidal triple** is an independent set of three vertices, such that for any pair of the three vertices, there is a path between the two which avoids all the neighbors of the third vertex.*

Example: Let us look at probably the most simple example of an asteroidal triple The

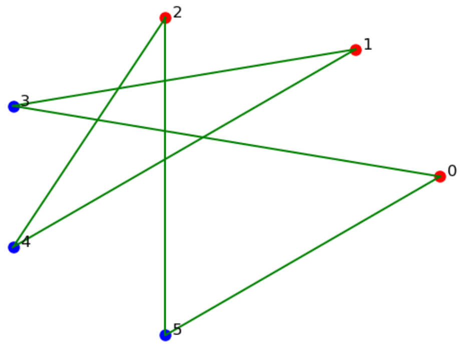


Figure 1: Asteroidal Triple

nodes colored in red, denote the asteroidal triple. Clearly, $\{v_0, v_1, v_2\}$ form an independent set, as they are pairwise independent. There is a path from v_0 to v_1 which can be given as $\{v_0, v_3, v_1\}$, where none of the three vertices belong in the neighborhood of v_2. Similarly, we can see there is a path from v_0 to v_2 which can be given as $\{v_0, v_5, v_2\}$, where v_5 is not in the neighborhood of v_1, and the path from v_1 to v_2 given by $\{v_1, v_4, v_2\}$, where v_4 is not in the neighborhood of v_0. Hence, we have a valid asteroidal triple.

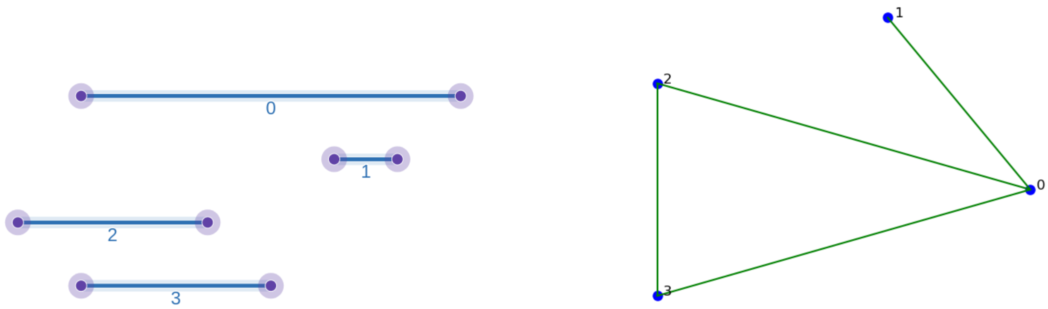
Definition 1.0.2. *A graph $G(V, E)$ is called an **asteroidal triple free graph (AT-Free Graph)** if for there exists no such triple, $u, v, w \in V$, such that u, v, w form an asteroidal triple.*

Example: There are a number of trivial examples of AT-Free graphs, such as any graph with less than 3 nodes, any graph $G(V, E), E = \phi$.

A major reason to motivate an extensive analysis of AT-free graphs is that, it forms a common generalization for various other graph classes, for example: *interval graphs*, *permutation graphs*, *trapezoid graphs* and *cocomparability graphs*.

Definition 1.0.3. *An* ***interval graph*** *is a graph $G(V, E)$, where each $v_i \in V$ corresponds to an interval $[i_l, i_r]$ on the real-line. There is an edge $(v_i, v_j) \in E$, if the intervals $[i_l, i_r]$ and $[j_l, j_r]$ overlap.*

Example: It is easy to see how the interval graph corresponds to the intervals. The



(a) Intervals (b) Interval Graph

Figure 2: Intervals and its corresponding Interval Graph

intervals however, are all supposed to lie on a 1-directional line (real-line), however, for a better representation, they have been placed at different y-coordinates.

Property: An interval graph is chordal and asteroidal triple-free, and every chordal and asteroidal-triple free graph is an interval graph.

Definition 1.0.4. *A graph $G(V, E)$ is called a* ***permutation graph*** *if V corresponds to elements of a permutation, and E represents elements that are reversed by the permutation.*

It is easier to understand them geometrically. Geometrically, a permutation graph can be created as follows:

- Select the number of nodes in the graph (n)
- Draw two parallel lines (x = 0, x = 1)
- Number the nodes from 0 to n - 1, and place each point on the parallel lines with the number corresponding to their x-coordinate
- Obtain a permutation of n numbers
- Place these n numbers from left to right on the other parallel line
- Now, join the points corresponding to the same number, lying on different parallel lines

- Every line corresponds to a node in the permutation graph
- If two lines intersect, we add an edge between their corresponding nodes in the permutation graph
- A more formal algorithm (in python) has been provided below

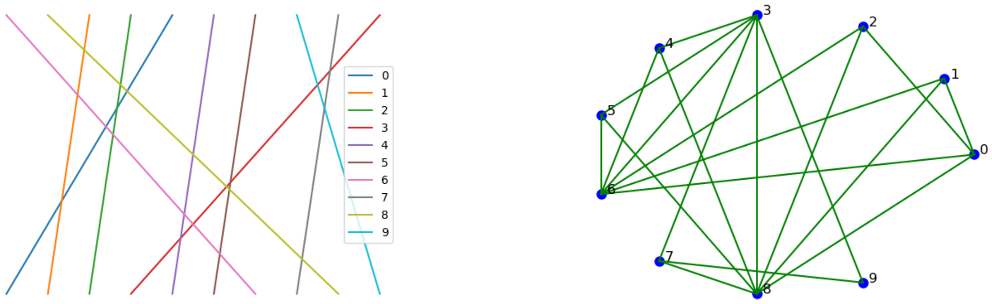
Algorithm 1: generateRandomPermutationGraph(n)

```
Result: A Permutation Graph with n Nodes
initialization;
parallelLines1 = [i for i in range(n)]
parallelLines2 = [i for i in range(n)]
random.shuffle(parallelLines2)
lines = []
i = 0
while i < len(parallelLines1) do
    lines.append(Line(Point(parallelLines1[i], 0), Point(parallelLines2[i], 1))
    i += 1
end
G = nx.Graph()
G.add_nodes_from(range(0, n))
i = 0
while i < len(lines)-1 do
    j = i
    while j < len(lines) do
        if Line.isIntersecting(lines[i], lines[j]) then
            G.add_edge(i, j)
        end
        j += 1
    end
end
```

Example: *Figure 3* represents an example of a permutation graph with 10 nodes. We represent the permutation lines on the left, and each line has been given a label, corresponding to which they form a node in the permutation graph. Note that for every line l_i in the permutation, its corresponding node in the graph is adjacent to all the lines that it intersects (for instance, here the line marked as 9 intersects the two lines marked as 3 and 7, hence, the node corresponding to 9 has two neighbors in the graph, which are precisely 3 and 7.)

Definition 1.0.5. *A **comparability graph** is an undirected graph $G(V, E)$, where there is a possible orientation of edges, such that the resultant graph has the following properties:*

- *Anti-symmetry: If we have an edge (u, v) in the resultant directed graph, then we cannot have the edge (v, u) in the same.*



(a) Permutation of Lines (b) Corresponding Permutation Graph

Figure 3: Permutation of lines, and its corresponding permutation graph

- *Transitivity: If we have two edges* (u, v) *and* (v, w) *in the resultant directed graph, then we also have the edge* (u, w) *in the same.*

It is easy to see that these graphs define something similar to a *partial-ordering.*
Example: *Figure 4* depicts a comparability graph. We can easily define an orientation on its edges (precisely, $\{(4, 1), (8, 6), (8, 0), (8, 9), (8, 2), (9, 6), (9, 0), (9, 2), (6, 2), (6, 0), (0, 2)\}$, and see that it follows both the rules of *anti-symmetry* and *transitivity.*

Definition 1.0.6. *A* ***co-comparability graph*** *is a graph who's complement is a comparability graph.*

Example: *Figure 4* depicts a co-comparability graph. While it might be hard to observe as is, but it is merely the complement of the comparability graph in *Figure 3.*

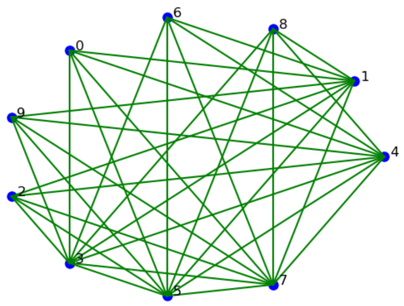


Figure 4: Co-comparability Graph

At this point, we shall discuss our first lemma.

Lemma 1.1. *Complement of an interval graph is a comparability graph.*

Proof. Suppose we have an interval graph, and we shall denote it as $G(V, E)$
Now, we consider its complement, $\bar{G}(V, E)$
For the graph $\bar{G}$, we have that $(u, v) \in \bar{G}(E) \iff$ the intervals corresponding to u and v

respectively, do not overlap.
Suppose we orient the graph in the following manner: we have an edge between two vertices who's corresponding intervals do not overlap. Hence, trivially one of the intervals lies completely to the left of the other. For any two such vertices, we include a directed edge corresponding to the interval on the left, to the interval on the right. Since clearly, if interval A is on the left of B, B cannot be on the left of A, we have that the resultant directed graph is anti-symmetric.
Now, we look at transitivity.
If $(u, v), (v, w) \in \bar{G}(E)$, then we have that the interval corresponding to u lies completely to the left of the interval corresponding to v, which in turn lies completely to the left of the interval corresponding to w. Hence, the interval corresponding to u lies completely to the left of the interval corresponding to w, and hence, we also have the edge (u, w) in the resultant directed graph. Hence, the transitivity property is satisfied.
Hence, we have come up with an orientation for the complement of the interval graph, such that it satisfies all the property for it to be an comparability graph. Hence, proved. □

An extremely trivial corollary of the last lemma is:
Corollary: An interval graph is a co-comparability graph.

2 Linear Structure in AT-free Graphs

2.1 Implication of Linear Structure

Many important graph classes like interval graphs, permutation graphs, comparability graphs, etc., show some kind of linear structure. Although, the notion of linearity usually differs for each class, it does result in emergence of generalizable structural properties. These structural properties also have algorithmic implications, i.e., many problems which are NP-hard for general graphs, admit polynomial time algorithms for these graph classes. Moreover, even for problems that do admit polynomial time algorithms in general graphs, the linear structure of such graph classes can be exploited to obtain an even more efficient algorithm for these.
Before proceeding further, we shall look at a concrete example:
Suppose the maximum weighted independent set problem. The problem can be defined as follows: Given a graph $G(V, E)$, with each vertex being assigned some weight $w, w \in R$, the *Maximum Weighted Independent Set* problem is to find a subset of vertices ($\subset G(V)$) whose weights sum to the maximum possible value without any two vertices being adjacent to another.
It is well known that this problem is NP-hard for general graphs. However, now suppose we have to solve this problem for the class of interval graphs. Suppose $G(V, E)$ is an interval graph. Hence, first we can find the corresponding interval model (**interval model** of an interval graph is a set of intervals that lead to the formation of an interval graph) in linear time. This problem is known as the *interval representation problem*, and has been solved by

[Booth and Lueker (1976)], [Hsu and Ma (1970)] and [Corneil, Olariu, and Stewart (2009)] in linear time. An extended discussion over these algorithms is outside the scope of the present study. However, once we have the corresponding interval model, we can use the following dynamic programming algorithm to find a solution in linear time.

$$Opt(j) = max(w_j + Opt(p(j)), Opt(j-1))$$

where

- $Opt(j)$ denotes the optimum solution when we have till the j^{th} interval (having sorted all intervals from left to right, in ascending order according to their left end-points)
- $p(j)$ refers to the last interval to the left that intersects with the j^{th} interval.
- w_j corresponds to the weight associated with the vertex associated with the j^{th} interval.

Using this, our dynamic programming algorithm can be defined as:

Algorithm 2: computeOpt(j)

Result: Weight of Maximum Weighted Independent Set
if $j == 0$ **then**
  return 0
else
  return $max(w_j + computeOpt(p(j)), computeOpt(j-1))$
end

Now, using the fact that j^{th} interval belongs to the optimal iff

$$opt(p(j)) + w_j \geq opt(j-1)$$

we can recover the solution from the values of the dynamic table.
A majority of the explicit details have been left out for the previous algorithm since it is a well-known algorithm. The points however, is that using the linear structure of the interval graphs, an NP-hard problem has been converted into a polynomial time solvable problem. The crux of the algorithm is sorting and searching the intervals from left to right, something which was only possible due to the underlying linear structure of interval graphs.

2.2 Intuition of Linearity

As mentioned earlier, the notions of linearity are different for each graph classes. For example, for interval graphs the notion of linearity is directly tied to the geometric position of the corresponding interval model. However, intuitively we can think of linearity to imply that at least for a subset of vertices in the graph $G(V, E)$, we can determine which vertices appear 'before', and which vertices appear 'after'.

Various properties and studies on AT-free graphs have suggested the existence of an underlying linear structure.

- Asteroidal triple-free graphs: showed that every AT-free graph has a dominating pair. This can be intuitively thought to impose a linear structure on the underlying graph, where the vertices of the dominating pair correspond to 'end vertices.'
- Möhring: showed that every minimal triangulation of an AT-free graph is an interval graph. Hence, no matter how chords are inserted in the induced cycles in an AT-free graph, the resultant graph has an inherent linear structure that can be seen similarly as for interval graphs. Later, it was also shown that this result gives a characterization of AT-free graphs.

To better understand the linear structure, and the properties that emerge through it, we first turn our attention towards a combinatorial structure, the *knotting graph*, which is capable of capturing the properties corresponding to the linear 'ordering' of the vertices in a rather elegant way.

2.3 Knotting Graph

Definition 2.3.1. *a graph $G(V, E)$, the **knotting graph** corresponding to G is given by $K_G(V_K, E_K)$. The vertex-set and edge-set are defined as follows:*

- *$K_G(V_K)$: $\forall v \in G(V), \exists\{v_1, v_2, ..., v_{i_v}\} \in V_K$, where i_v is the number of connected components of $N\bar{(}v)$ (and $N\bar{(}v)$ is the complement of the graph induced by $N(v)$).*
- *$K_G(E_K)$: $\forall e \in G(E), e = (u, v), u, v \in G(V), (u_i, v_j) \in E_K$ where v belongs to the i^{th} connected component of $N\bar{(}u)$, and u belongs to the j^{th} connected component of $N\bar{(}v)$*

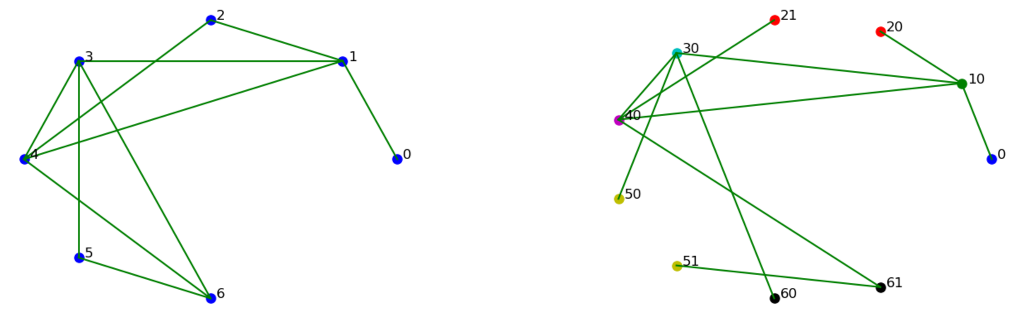
An algorithm to build a knotting-graph can be given as follows:

Algorithm 3: getKnottingGraph(G)

Result: Returns Knotting Graph: K_G corresponding to G

K_G = nx.Graph()
nodes = list(G.nodes)
offset = 10**(len(str(len(nodes))))
$connected\_components$ = {}
i = 0
while $i < len(nodes)$ **do**
  $connected\_components[node] = list(nx.connected\_components(nx.complement(G.subgraph(G.neighbors(nodes[i])))))$
  $K_G.add\_nodes\_from(range(node * offset, node * offset + len((connected\_components[node]))))$
end
edges = list(G.edges)
k = 0
while $k < len(list(G.edges))$ **do**
  $u, v = edges[k]$
  i = 0
  while $i < len((connected\_components[u]))$ **do**
    if $vin(connected\_components[u])[i]$ **then**
      break
    end
    i += 1
  end
  j = 0
  while $j < len((connected\_components[v]))$ **do**
    if $uin(connected\_components[v])[j]$ **then**
      break
    end
    j += 1
  end
  $K_G.add\_edge(u * offset + i, v * offset + j)$
end
return K_G

Example: Below we have shown an example of a random graph (G) and its corresponding knotting-graph. Going into the exact detail of the latter emerges from the prior shall become unnecessarily involved, and hence, we shall skip that. Do note that in the knotting graph, the copy of nodes from emerge from the single node in the original graph have been depicted in the same color. Also, the most significant bit in the labeling corresponds to the node that they are a copy of, with respect to the original graph.



(a) G (b) K_G

Figure 5: A graph and its corresponding Knotting graph

2.4 Results on Knotting Graph

We shall begin our analysis with a few important theorems on knotting graphs.

Theorem 2.1. *A graph $G(V, E)$ is transitively orientable if and only if its corresponding knotting-graph is bipartite.*

The proof for this theorem is rather involved, and we shall skip over it in this report. However, it can be found in the original work by Gallai (1967).

Theorem 2.2. *Let $G(V, E)$ be a graph, then $an(G) = \omega(K_G)$*

Proof. First we try to prove that if if $an(G) = k$, then we have a clique of size k in the knotting-graph of $G(\bar{V}, E)$ Suppose $an(G) = k$ and let $A = \{a_1, ..., a_k\}$ be an asteroidal set of $G(V, E)$. Then, we know that the vertices of A are pairwise independent (by the definition of asteroidal set.) Hence, when we consider $G(\bar{V}, E)$, the vertices of A form a K_k (complement of independent set is a clique.) Now, $\forall a_i \in A, A - \{a_i\} \in N_{\bar{G}}(a_i)$ (property of clique.)
Since A is an asteroidal set, we also have that $\forall a_j, a_k \in A - \{a_i\}, (j \neq k)$ there exists an a_j, a_k-path in $G(V, E)$ that only contains vertices from the set of non-neighbors of a_i (property of asteroidal set.) Hence, a_j, a_k belong to the same connected component of $G - N_G(a_i)$.
Now, considering the knotting-graph $K_{\bar{G}}$, we have that the edges corresponding to a_i, a_j and a_i, a_k, correspond to two edges which are incident to the same copy of a_i in $K_{\bar{G}}$ (in other words, we can say that these edges are *knotted* at the same point.) Hence, expanding this to other vertices, we have that for each pair of edges in the clique in $G(V, E)$, the edges are incident to the same copy of the vertex in the knotting-graph. Hence, we also have a K_k in the knotting-graph.
Now, we try to prove that if we have a clique of size k in the knotting-graph of $G(\bar{V}, E)$, then we have size k asteroidal set in the graph $G(V, E)$. We begin with the K_k clique. We know that there is a one-one correspondence between the edges of $G(\bar{V}, E)$ and between the edges of $K_{\bar{G}}$ (by the definition of knotting-graph.) We know that there is $K_k \in K_{\bar{G}}$,

which we shall denote by $A = \{a_1, ..., a_k\}$. By the definition of the knotting graph, for each vertex $a_i \in A$, the vertices of $A - \{a_i\}$ are contained in the same connected component of $G - N_G(a_i)$. Hence, A is an asteroidal set of $G(V, E)$.. Hence, we have our proof. □

Corollary: A graph is coAT-free if its knotting graph is triangle proof.

Proof. Suppose we have a graph $G(V, E)$ and its corresponding knotting-graph that we denote as K_G. Now, K_G is triangle free, hence, $\omega(K_G) \leq 2$
Hence, $an(\bar{G}) \leq 2$, thus we cannot have an asteroidal-triple in $G(\bar{V}, E)$. Hence $G(\bar{V}, E)$ is AT-free, and $G(V, E)$ is AT-free.
Hence, proved.
We shall prove a few more lemmas to get a good grip on knotting-graphs, and its properties. □

Lemma 2.3. *The pair of vertices a, b is a dominating pair in $G(\bar{V}, E)$ if and only if each common neighbor x of a, b in $G(V, E)$ has different vertex copies in K_G adjacent to the copy of a and the copy of b (i.e., xa and xb are not knotted at x.)*

Proof. First we prove that if a, b is a dominating pair in $G(\bar{V}, E)$, then each common neighbor x of a and b in $G(V, E)$ has different vertex copies in K_G adjacent to the copy of a and to the copy of b.
Suppose a, b is a dominating pair $G(\bar{V}, E)$.
Then, every a, b-path is a dominating set in $G(\bar{V}, E)$.
Suppose for contradiction. Suppose there is a common neighbor x of a and b, and that xa, xb are knotted at x in K_G.
Then, we have $(x_i, a_j), (x_i, b_k) \in K_G(E)$, where x_i is a copy of x, a_j is a copy of a and b_k is a copy of b, in the knotting graph respectively.
Then,
$(x, a) \in G(E)$, $a \in N(\bar{x})_i$
$(x, b) \in G(E)$, $b \in N(\bar{x})_i$
hence, there is an $ab-$path in the complement of the graph induced by $N(x)$
But now, no vertex in this path is adjacent to x (as it only contains vertices in $N(x)$, and all the edges from $N(x)$ to x are lost in the complement graph.)
Hence, this ab-path is not a dominating set in $G(\bar{V}, E)$, as the vertex x does not get dominated.
Hence, we have a contradiction.
Now we try and prove the other direction. Suppose each common neighbor x of a, v in $G(V, E)$ has different vertex in copies in K_G adjacent to the copy of a and the copy of b, then a, v is a dominating pair of $G(\bar{V}, E)$.
Suppose for contradiction. Suppose have an $ab-$path in $G(\bar{V}, E)$ such that there is a vertex y which is not dominated by this path.
Now, we pick any arbitrary ab-path. If this contains any vertex $v \in G(V) - N(y)$, then we have $(y, v) \in E(G$, and this will be a contradiction. Hence, this path contains only vertices

from $N(y)$.
If we can manage to find even one such path which satisfies the above assumption, we can prove the contradiction.
Now, suppose we have such a path in $G(\bar{V}, E)$. Then $a \in N(\bar{y})_i, b \in N(\bar{y})_i$ and they are knotted at y in K_G, and hence, we have a contradiction.
Hence, proved. $\square$

2.5 Intervals in Graphs

As discussed earlier, intuitively a linear structure on a graph implies that there are vertices that can be said to appear 'before' others (and similarly, some can be said to appear 'after' others.)
This would mean that we can also intuitively have a notion of *betweeness*, i.e., if there are three vertices a, b, c where a appears after b and a appears before c, then we can say that vertex a is in *between* vertex b and c. In other words, we can also say this as, vertex a appears in the *interval* defined by the vertex b and c (the terminology of interval makes it intuitive to understand what this would mean for interval graphs, and their corresponding interval models.) To give it a more formal notion, we shall use the following definition:

Definition 2.5.1. *Let $G(V, E)$ be a connected graph. A vertex $s \in V - \{x, y\}$ is said to be between x, y if x and s are in the same component of $G - N(y)$ and y, s are in the same component of $G - N(x)$. The **interval** $I(x, y)$ of $G(V, E)$ is defined to be the set of all vertices of G that are between x and y.*

Lemma 2.4. *Suppose we have a connected graph $G(V, E)$ and we denote the knotting-graph of its complement as $K_{\bar{G}}$. Then, the interval $I(x, y)$, for $x, y \in G(V)$ corresponds to the set of all vertices u such that the edge from u to x is incident to the same copy of $x \in K_{\bar{G}}(V)$ as the edge from y to x, and the edge from u to y is incident to the same copy of $y \in K_{\bar{G}}(V)$ as the edge from x to y.*

Proof. All vertices $s \in G(V) - \{x, y\}$ where x, s are in the same connected component of $G - N(y)$, and s, y are int he same component of $G - N(x)$, lie in the interval defined by $I(x, y)$ (by the definition of interval.)
Now, for each component in $G - N(x)$ we have a copy of $x \in K_{\bar{G}}(V)$.
Also, for two vertices $u, v \in G(V)$ that are non-adjacent to x in $G(V, E)$, these two vertices are adjacent to x in $G(\bar{V}, E)$, and are also connected in $G - N(x)$, hence, they are adjacent to the same copy of x in $K_{\bar{G}}$.
Hence, proved. $\square$

Lemma 2.5. *Suppose we have an AT-free graph $G(V, E)$, and $s, x, y \in G(V)$. Then, if $s \in I(x, y)$, x, y are in different components of $G - N(s)$.*

Proof. Suppose for contradiction.
Hence, x, y are in the same component of $G - N(s)$.
Now, now $s \in I(x, y)$, hence, by the definition of an asteroidal triple, if the vertices s, x, y are pairwise disjoint, then they form as asteroidal triple.

But this is not possible, hence, at least we have one of the three edges from $(s, x), (x, y), (s, y)$. Without loss of generality, suppose we have the edge (x, y) But now, $y \notin G - N(x)$, as $y \in N(x)$, hence, y does not belong to any connected component of $G - N(x)$, and we have a contradiction.

Hence, s, x, y form an independent set, and by the above definitions, they also form an asteroidal triple. Hence, this graph cannot be AT-free. Hence, our assumption was wrong, and x, y need to be in different components of $G - N(s)$. $\square$

Lemma 2.6. *Suppose we have an AT-free graph $G(V, E)$ and $s, x, y \in G(V)$. Then, $I(x, s) \cap I(x, y) = \phi$.*

Proof. We can use the last proposition to easily prove this one. For each edge in $G(\bar{V}, E)$, we only have one copy of it in $K_{\bar{G}}$ (by the property of knotting-graph.) Hence, if $I(x, s) \cap I(s, y) \neq \phi$, we have a contradiction as both x, y belong to separate components in the knotting-graph. $\square$

Lemma 2.7. *Let $G(V, E)$ be a comparability graph with $x, y \in G(V)$ and $(x, y) \in E$, and let $I(x, y)$ be the interval of x, y in $G(\bar{V}, E)$. Then, in any transitive orientation F of G, the vertices of $I(x, y)$ are between x, y i.e., for each $z \in I(x, y)$ either x is a predecessor of z and z is a predecessor of y or the other way (i.e., $x \leq z \leq y$ or $y \leq z \leq x$)*

Proof. Without loss of generality, suppose $x \leq y$ in $G(V, E)$.
Suppose there is some $z \in G(V), z \leq x$
Now, if $z \in I(x, y)$ in $G(\bar{V}, E)$
$y, z \in G(\bar{V}, E) - N_{\bar{G}}(x)$
$x, z \in G(\bar{V}, E) - N_{\bar{G}}(y)$
hence, there exists a $yz-$path in $G(\bar{V}, E) - N_{\bar{G}}(x)$
Obviously, this path cannot have any vertex adjacent to x in $G(\bar{V}, E)$.
Suppose we denote this path as $P = \{v_1, v_2, ..., v_k\}, v_1 = z, v_k = y$.
Now, v_2 is non-adjacent to z in $G(V, E)$
We have $x \leq y$ and $z \leq x$
Now, if $v_2 \geq x$, we ave vertices from z to y, where all the vertices are adjacent to x, which is a contradiction.
Hence, $v_2 \leq x$
By transitivity, v_2 cannot be adjacent to y in $G(\bar{V}, E)$, and hence, $k > 3$.
By similar arguments as above, v_{k-1} needs to be a successor of x in F.
Hence, v_2, v_{k-1} are not neighbors (by transitivity.)
Construction P this way, we can partition it into S_P, S_S, such that $\forall v_i \in S_P,\ \forall v_j \in S_S, i < j, i, j \in [k]$.
Now, each vertex from S_P has an edge to each vertex in S_S (by transitivity).
Hence, we cannot have a connected path in $G(\bar{V}, E)$, as S_P and S_S cannot be connected.
Hence, proved. $\square$

Lemma 2.8. *Suppose we have a AT-free graph $G(V, E)$, and $s, x, y, \in G(V)$. Then, $\forall s \in I(x, y)$, we have $I(x, s) \subset I(x, y)$ and $I(s, y) \subset I(x, y)$*

Proof. We have $s \in I(x, y)$. Hence, we have:
an $sx-$path in $G - N(y)$
an $sy-$path in $G - N(x)$
Now, suppose an arbitrary $z \in G(V), z \in I(x, s)$
then, this would imply:
a $zx-$path in $G - N(x)$
a $zs-$path in $G - N(x)$
To show $z \in I(x, y)$, we need to show that:
there exists $zx-$path in $G - N(y)$
and a $zy-$path in $G - N(x)$
Now, the $zs - path$ in $G - N(x)$ (from $z \in I(x, s)$) and the $sy-$path in $G - N(x)$ (from $s \in I(x, y)$) can be combined to get a $zy-$path in $G - N(x)$.
We also know that x, y are in different components of $G - N(s)$ (from previous lemma.)
Hence, we now show that the zx-path in $G - N(s)$ gives us a $zx-$path in $G - N(y)$
Suppose for contradiction, suppose we have a neighbor of y in the $zx - path$ in $G - N(s)$, then x, y are in the connected component $G - N(s)$, which is a contradiction.
Hence, we have proved that for any arbitrary $z \in I(x, s), z \in I(x, y)$. Hence, $I(x, s) \subseteq I(x, y)$.
Similarly, $I(s, y) \subseteq I(x, y)$
But now, $s \in I(x, y)$, but $s \notin I(s, y)$ and $s \notin I(x, s)$ (by definition of intervals), and hence, we have that $I(x, s) \subset I(x, y)$ and $I(s, y) \subset I(x, y)$.
Hence, proved. □

Theorem 2.9. *A graph $G(V, E)$ is AT-free if and only if for each interval $I(x, y)$ of $G(V, E)$ and each $z \in I(x, y)$, we have $I(x, z) \subseteq I(x, y)$ and $I(z, y) \subseteq I(x, y)$*

Proof. We have already proved this for one of the directions, for the previous lemma. Hence, all we need to prove is that if for each interval $I(x, y)$ of $G(V, E)$ and each $z \in I(x, y)$, we have $I(x, z) \subseteq I(x, y)$ and $I(z, y) \subseteq I(x, y)$, then the graph is AT-free.
Suppose for contradiction. Suppose that the graph $G(V, E)$ satisfying the property above, is not AT-free. Hence, suppose we have an asteroidal triple $(a, b, c) \in G(V)$.
Now, this gives us the following (from the definition of asteroidal triple):
an $ab-$path in $G - N(c)$
a $bc-$path in $G - N(a)$
an $ac-$path in $G - N(b)$
Now, $a \in I(b, c)$ and $b \in I(a, c)$ and $c \in I(a, b)$
But we know that if $a \in I(b, c)$, b, c lie in separate components of $G - N(a)$, however, we also have that there exists a $bc-$path in $G - N(a)$.
Clearly, this is a contradiction, and our assumption is wrong. Hence, we cannot have any triple of vertices $\in G(V)$ that form as asteroidal triple.
Hence, $G(V, E)$ is AT-free. □

Theorem 2.10. *Let $G(V, E)$ be an AT-free graph, let $I(x, y)$ be an interval of $G(V, E)$, and let*

$s \in I(x,y)$. Then, there exists components $C_1^s, C_2^s, ..., C_t^s$ of $(G - N(s))$ such that

$$I(x,y) - N(s) = I(x,s) \cup I(s,y) \cup \bigcup_{i=1}^{t} C_i^s$$

We do not need to formally prove this as it follows from the lemmas and propositions appearing before this. From this theorem we can further obtain the following:

Lemma 2.11. *Let $G(V,E)$ be an AT-free graph. Let $I(x,y)$ be an interval of $G(V,E)$, and $s \in I(x,y)$. Let C be a component of both $G-N(x)$ and $G-N(y)$. Then C is a component of $G-N(s)$.*

Proof. Since C is a component of both $G - N(x)$ and $G - N(y)$, we have
$\forall v \in N(C) \longrightarrow v \in N(x), v \in N(y)$
Suppose for contradiction. Suppose we have a vertex $u \in N(C)$ where $u \notin N(s)$
$u \in N(x), u \in N(y)$, hence, it is adjacent to both x, y.
Hence, we have an $xy-$path from u, which avoids s.
Now, as $s \in I(x,y)$, we already have $sx-$path in $G - N(y)$ and $sy-$path in $G - N(x)$
Hence, by the definition of asteroidal triple, s, x, y is an asteroidal triple.
Hence, we have a contradiction since $G(V,E)$ is AT-free.
Hence, our assumption is wrong, and $u \notin N(C)$. □

This notion of intervals has several algorithmic implications. For instance, we can use the properties discussed above to come up with a polynomial time algorithm to solve the independence number problem for AT-free graphs. We utilize the structure visible in the knotting-graph, and the most efficient algorithm for the same is $O(n\bar{m})$ or $O(n^3)$. Hence, linear structure of the graph is useful to understand and uncover various underlying structural properties for different graph classes, and can also be useful while solving optimization problems.

3 Characterization of AT-free graphs based on dominating pairs

3.1 Preliminaries

- A pair of vertices (x, y) is a dominating pair in a graph G if all x, y-paths in G are dominating sets.
- A set S of vertices of a graph G is said to be *dominating* if every vertex outside S is adjacent to some vertex in S.
- For vertices u, v, $D(u, v)$ denotes the set of vertices that intercept all the (u, v) paths.
- For vertices u, v, x of graph G, if $u \notin D(v, x)$ and $v \notin D(u, x)$ then we say that u, v are *unrelated* with respect to x.

3.2 Some useful claims and lemmas

The goal of this section is to provide a characterization of AT-free graphs based on dominating pairs. For that purpose, an adjacency condition is imposed on G with the dominating pair (x, y), where the connected component $G \quad x$ containing y has a dominating pair (x', y) with x' adjacent to x. As shown in figure 6, the graph C_6 fails this criterion.

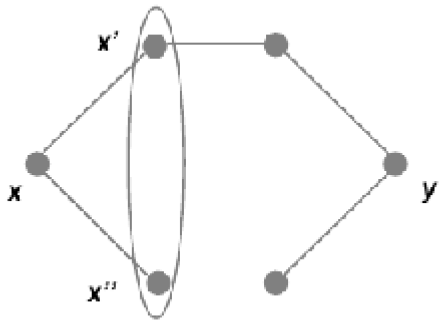


Figure 6: C_6

Claim 3.2.1. *Let u, v, y be vertices in a connected asteroidal triple-free graph, such that $v \notin D(u, y)$. If $D(u, y)D(v, y)$ then for some vertex w in $D(u, y)$, v and w are unrelated with respect to y.*

Proof. Let w be an arbitrary vertex in the set $D(u, y) \setminus D(v, y)$.
Since w does not belong in the set $D(v, y)$, it is clear that w misses some $v - y$ path.
Whereas w intercepts π and contains a $w - y$ path.
This proves that v and w are unrelated with respect to y. □

Claim 3.2.2. *No pair of paths among $\pi(x, y), \pi(x, z)$ and $\pi(y, z)$ has a cross point.*

Proof. Let G be a be graph containing an AT.
Let H be an induced subgraph of G such that there exists some AT in H.

The paths $\pi(x, y), \pi(x, z), \pi(y, z)$ demonstrate the AT in H.
Assume for contradiction that the paths $\pi(x, y)$ and $\pi(x, z)$ have a cross-point w such that $w = u_i = v_j$.
By definition of cross point and minimality of H, it is guaranteed that $3 \leq i$ and $3 \leq j$.
But in $H' = H \setminus v_{j-1}$, y misses the $x - z$ path and x misses the $y - z$ path.
This contradicts the minimality of H.

□

Claim 3.2.3. *Let i be the largest subscript for which there exists a subscript j such that $u_i = v_j$ and $u_{i+1} \neq v_{j+1}$. Then $i = j$ and $u_t = v_t \; \forall \; 1 \leq t \leq i$.*

Proof. By the claim 3.2.2, u_i cannot be a cross point, we must have $u_{i-1} = v_{j-1}$.
If we consider t as the minimum value for which $u_{i-t} \neq v_{j-t}$, then such t has to exist.
Therefore, $t \leq min\{i-2, j-2\}$.
Hence v_t can be removed from H while it still having an AT.
This contradiction completes the proof for the claim. □

Lemma 3.1. 1. *The unique path between x and x' is a prefix of both $\pi(x, y)$ and $\pi(x, z)$.*

2. *The unique path between y and y' is a prefix of both $\pi(y, x)$ and $\pi(y, z)$.*

3. *The unique path between z and z' is a prefix of both $\pi(z, x)$ and $\pi(z, y)$.*

There exist unique vertices x', y', z' in H such that –

Proof. Claim 3.2.3 shows that a vertex x can be associated with a x' to the largest subscript such that $u_i = v_i$.
In a similar way, y' and z' can be defined. □

Claim 3.2.4. *The vertices x', y', z' are all distinct or they coincide*

Proof. Assume for contradiction that only two of them coincide.
We can write $x' = u_i$ and $y' = w_{t-k+i}$.
From claim 3.2.4, we can say that subpaths $\pi(x, z)$ and $\pi(z, y)$ between z and x' coincide which is a contradiction. □

3.3 THE SPINE THEOREM

A graph G is asteroidal triple-free if and only if every connected induced subgraph H of G satisfied the *spine* property.
Spine property – For a connected graph with a dominating pair, if for every non-adjacent dominating pair (α, β), there exists a neighborhood α' such that (α', β) is a dominating pair of the connected component of $H \setminus \{\alpha\}$ containing β.

3.3.1 Proof of the "only if" part

Let H be any induced subgraph for a graph G.
We know H has a dominating pair (α, β).
Let C_β denote the connected component that contains β.
There will be a vertex $\bar{\alpha}$ such that $D(\bar{\alpha}, \beta) \subset D(t, \beta)$.
We see that $(\bar{\alpha}, \beta$ is a dominating pair in C_β.
From the above statement, we find w in $D(\bar{\alpha}, \beta)$ such that t and w are unrelated with respect to β.
So, we can say that w and t are in the same component of $N'(\beta)$ and are unrelated with respect to β.
This is contradiction which completes the "only-if" part.

3.3.2 Proof of the "if" part

Claim 3.3.1. *Vertices a, b are distinct from x, y, z, x', y', z'.*

Proof. Suppose $a = x$.
(a, b) is a dominating pair, which makes b belong to $\pi(y, z)$.
The $x - b$ path is missed by z unless b and z are adjacent.
Since H satisfies spine property and a and b are non-adjacent, we should be able to find a b' such that (a, b') is a dominating pair in $H \setminus b$. □

Claim 3.3.2. *Each pair of vertices x and x', y and y', and z and z' must coincide.*

Proof. First of all x', y', z' are distinct.
Using the path properties and previous claims we can show that –
vertices b and x are adjacent and
(b, y) is a dominating pair.
But if (b, y) is a dominating pair, we reach a contradiction since (b, y) cannot be a dominating pair according to claim 2.
So to prove that (b, y) is a dominating pair, consider a vertex c that misses the path $b - y$.
c must belong to $\pi(y, z)$.
But this makes c, x, y an AT.
This completes the proof of the spine theorem. □

4 Coloring Algorithm for 3-colorable AT-free Graphs

4.1 Preliminary Definitions

Definition 4.1.1. *Neighborhoods: $N(v)$ and $N[v]$*
In a graph G, the *neighborhood* of a vertex v, written as $N(v)$, denotes the set of all vertices adjacent to v in G. The set $N(v) \cup \{v\}$ is written as $N[v]$.

Definition 4.1.2. *Induced Subgraphs*

For a given graph G and a set $X \subseteq V(G)$, $G[X]$ denotes the subgraph induced by the vertices of X in G. $G - X$ denotes the subgraph induced by the set $V(G) \setminus X$

Definition 4.1.3. *Stable Set*
An independent vertex set in a graph is referred to as a *stable* set, i.e., a set $X \subseteq V(G)$ is stable if $G[X]$ contains no edges.

Definition 4.1.4. *Minimal Separator*
We say that a set $S \subseteq V(G)$ is a *minimal separator* of G if there exist vertices a and b such that S disconnects a and b but no proper subset of S disconnects them.

Definition 4.1.5. *Diamond*
A *diamond* refers to a graph on 4 vertices with 5 edges.

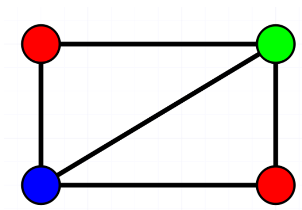


Figure 7: Diamond graph

Definition 4.1.6. *Externally Connected*
We say that a set $S \subseteq V(G)$ is *externally connected* in G if $\forall x \in V(G)$ s.t. $N[x] \cap S = \phi$, the set S is contained in a (single) connected component of $G - N[x]$

Definition 4.1.7. *Vertex Contraction of a Set of Vertices*
For a set $S \subseteq V(G)$, we write $G/_S$ for the graph we obtain from G by contracting all the vertices in S into a single vertex. That is,

$$V(G/_S) = V(G) \setminus S \cup \{s\}, \text{ where } s \notin V(G)$$
$$E(G/_S) = \{xy \in E(G) | x, y \notin S\} \cup \{sy | xy \in E(G) \wedge x \in S \wedge y \notin S\}$$

Definition 4.1.8. *Triangular Strip*
A *triangular strip* of order k is the cartesian product of an induced path on k vertices and a triangle. This the graph formed by taking three disjoint paths $P^1 = v_1^1, v_2^1, ..., v_k^1$, $P^2 = v_1^2, v_2^2, ..., v_k^2$ and $P^3 = v_1^3, v_2^3, ..., v_k^3$, and adding a triangle on v_i^1, v_i^2, v_i^3 for each $i = 1...k$.

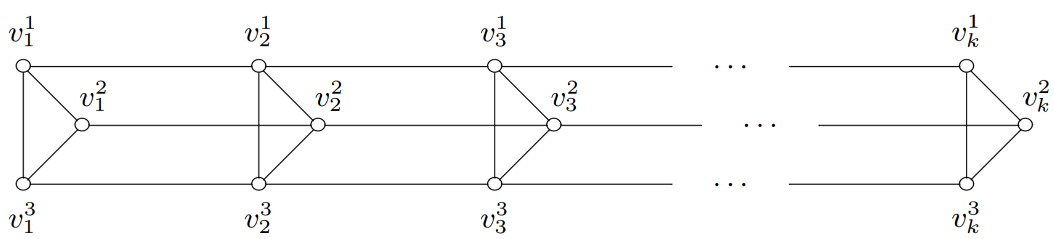


Figure 8: Triangular strip of order k

4.2 Algorithm

The algorithm for 3-coloring fundamentally operates on the fact that if an AT-free graph has no K_4 subset and no induced diamond, then it is always 3-colorable (we will prove this later). The algorithm works by recursively contracting appropriate sets of vertices of the given AT-free graph until the graph achieved is isomorphic to a triangular strip. The contractions in each step are done in such a way that the set of vertices contracted can be colored with the same color in the final coloring of the original graph.

We will now give a justification to the correctness of every step of the algorithm.

Lemma 4.1. *Let G is an AT-free graph, and $S \subseteq V(G)$ is an externally connected set in G. Then $G/_S$ is AT-free.*

Proof. Given that $S \subseteq V(G)$ is externally connected. By Definition 4.1.6, this implies that $\forall x \in V(G)$ s.t. $S \cap N[x] = \phi$, S lies in a connected component of $G \setminus N[x]$.
Assume for contradiction that vertices x, y, z form an AT in $G/_S$. We say that a path is *missed* by a vertex v, in no vertex on the path is adjacent to v.
Let P be a path from y to z in $G/_S$ that is *missed* by x. Let s be the vertex in $G/_S$ created from the contraction of the vertices in S.

- **Case 1:** s does not lie on P and $s \neq x$
  This means that all vertices and edges in path P also exist in G. Hence the triple $\{x, y, z\}$ is also an AT in G, which is a contradiction.

- **Case 2:** s does not lie on P, and $s = x$
  Since s misses P in $G/_S$, every vertex of S misses P in G. This means vertices of S along with y and z form ATs in G, which is a contradiction.

- **Case 3:** s lies on P, and s is not an end point of P, i.e., $s \neq y, s \neq z$
  Let u, v are the two neighbors of s in P. Then, since s is created by contraction of S, $\exists a, b \in S$ s.t. $ua, vb \in E(G)$.
  Also, since x misses P in $G/_S$, $xs \notin E(G/_S)$

  $$\begin{aligned}
  &\implies N[x] \cap S = \phi \\
  &\implies a, b \in S \text{ are in the same connected component of } G - N[x] \\
  &\implies u, v \text{ are in the same connected component of } G - N[x] \\
  &\implies \exists \text{ a path from } u \text{ to } v \text{ in } G - N[x] \text{ that misses } x \\
  &\implies \exists \text{ a path from } y \text{ to } z \text{ in } G - N[x] \text{ that misses } x
  \end{aligned}$$

  Hence, $\{x, y, z\}$ form an AT in G, which is a contradiction.

- **Case 4:** s is one of the end points of P
  W.l.o.g., let $y = s$. Then, by the same logic as in case 3, S lies in a connected component of $G - N[x]$. Therefore, for every vertex $v \in S$, there exists a path from v to z that is missed by x. Hence the set $\{v, x, z\}$ form an AT in G $\forall v \in S$. This is a contradiction.

Algorithm 4: 3-coloring algorithm for AT-free graphs

Input: An AT-free graph G
Output: A three coloring of G, or "G is not 3-colorable"

Function `3-color` (G)**:**
  if *G has K_4 as subgraph* **then**
    return "G is not 3-colorable"
  end
  `/* Now` G `doesn't contain any` K_4 `subgraph */`
  else if *G contains adjacent vertices u, v, with $|N(u) \cap N(v) \geq 2|$* **then**
    $S \leftarrow N(u) \cap N(v)$
    $G' \leftarrow G/_S$
    $G' \leftarrow$ `3-color`(G')
    return `color-map`(G', G, S)
  end
  `/* Now G has no induced diamond */`
  else if *G contains a cut vertex or is disconnected* **then**
    for *Each 2-connected component G' of G* **do**
      `3-color`(G')
      return G
    end
  end
  `/* Now` G `is at least 2-connected */`
  else if *G contains a minimum stable separator S with $|S| \geq 2$* **then**
    $G' \leftarrow G/_S$
    $G' \leftarrow$ `3-color`(G')
    return `color-map`(G', G, S)
  end
  `/* Now` G `is isomorphic to triangular strips of order` k`.`
  `Let the vertices of the` j^{th} `strip be denoted as` v_j^1, v_j^2, v_j^3
  `*/`
  for *All $v_j^i \in G$* **do**
    v_j^i.color $\leftarrow$ (($i + j$) mod 3) +1
  end
  return G

Function `color-map` (G', G, S)**:**
  for *each vertex v in G* **do**
    if $v \notin S$ **then**
      color v the same as it's color in G'
    end
  end
  for *vertices in S* **do**
    Color them all the same as the contraction vertex s in G'
  end

□

Using Lemma 4.1, we prove the following theorem.

Theorem 4.2. *If u and v are adjacent vertices of an AT-free graph G, and S is a maximal stable set in $N(u) \cup N(v)$, then $G/_S$ is AT-free. Additionally, G is 3-colorable if and only if $G/_S$ is 3-colorable.*

Proof. For the first part, by Lemma 4.1, it suffices to prove that S is externally connected in G.
Consider $x \in V(G)$ s.t. $N[x] \cap S = \phi$. This means that x is non-adjacent to every vertex in S. Since S is stable, this means that $S \cup \{x\}$ will also be stable. However, S is also the maximal stable set in $N(u) \cup N(v)$. This implies that x has to be non-adjacent to at least one of u and v. W.l.o.g., suppose that $xu \notin E(G)$. Then, since every vertex of S is in $N(u)$, $G[S \cup \{u\}]$ is a connected component. Also, since $xu \notin E(G)$, the connected component $G[S \cup \{u\}]$ lies completely in $G - N[x]$. By Definition 4.1.6, S is externally connected in G.

For the second part, let s be the vertex in $G/_S$ to which we contracted S. Since S is stable, in a 3-coloring of $G/_S$, whatever color is assigned to s can be given to all vertices of S in G, and the coloring will remain valid.
For the sufficiency, let there is a 3-coloring of G. Then since vertices u and v are adjacent, they must have different colors in this coloring. Also, since $\forall x \in S, ux, vx \in E(G)$, all vertices in S will have a third color. Hence this color can be used to color s in $G/_S$ to obtain a valid 3-coloring. □

Theorem 4.2 demonstrated the correctness of the block in line 5 of Algorithm 4. Now in the algorithm, at the end of line 16, G is 2-connected, has no K_4 and no induced diamond. The remaining algorithm claims that the graph either has a minimum stable separator of size at least 2, or is isomorphic to triangular strips. We will prove this claim as follows.

Lemma 4.3. *Let G be a 2-connected AT-free graph with no induced diamond and no K_4. Then every vertex of G is in exactly one triangle.*

- **Part 1:** *Every vertex of G is in at most 1 triangle*

  Proof. Assume for contradiction that there exists a vertex x in G that is a member of 2 different triangles. Let these triangles are labelled $\{x, u, v\}$ and $\{x, a, b\}$.
  Now, the vertices u, v and the vertices a, b are different, as if suppose $u = a$, then the set $\{u, v, x, b\}$ induces a diamond in G, which is a contradiction. For the same reason, the edges ua, ub, va, vb cannot be in $E((G)$.
  Since G is 2-connected, the graph $G - \{x\}$ is connected. This implies that there exists a path between the vertices u, v and a, b in $G - \{x\}$. W.l.o.g., let the path P from u to a is the shortest such path. Let y denote the vertex following u in P.
  Now, $yv \notin E(G)$ and also $xy \notin E(G)$, as if either of these edges exist, then the vertex set $\{y, v, u, x\}$ induces a diamond or a K_4 in G. Also, the edge $yb \notin E(G)$, as if edge

yb exists in G, then edge $ya \notin E(G)$. If both ya and yb are in G, then the vertex set $\{y, a, b, x\}$ induces a diamond in G. Therefore, if edge $yb \in E(G)$, then the shortest path from $\{u, v\}$ to $\{a, b\}$ will be $u \to y \to b$. This is a contradiction to the minimality of path P.

We have shown that edges $vy, xy, by \notin E(G)$. Hence, $v \to x \to b$ is a path from v to b that misses y. Similarly, edges $vb, ub, yb \notin E(G)$, hence the path $v \to u \to y$ is missed by the vertex b. By definition, the set $\{y, v, b\}$ is an asteroidal triple in G, which is a contradiction. □

- **Part 2:** *Every vertex of G is in at least 1 triangle*

  Proof. Assume for contradiction that there exists a vertex v which is not a part of any triangle. Then, either $V(G) = N[v]$, or $N[v]$ forms a stable cutset in G. If $V(G) = N[v]$, then v is a cut vertex in G, as G is assumed to have at least 3 vertices. This is a contradiction to the 2-connectedness of G. □

The above lemma implies that G contains a triangular strip of at least order 1. We can show that on finding the maximal triangular strip in G, we either get the whole graph G, or we find a stable cutset in G. More specifically:-

Lemma 4.4. *Let G be an AT-free graph with no induced diamond and no K_4. Let H is a subgraph of G isomorphic to a triangular strip. Then:*

1. *H is induced in G*
2. *No vertex of H has a neighbor in $G - V(H)$ except for the vertices of the triangles in H that are non-adjacent to any other triangles (also knows as the end-triangles of the triangular strip).*

The proof of Lemma 4.4 is very involved, and has hence been skipped for the purpose of this study. The original proof can be found in the original work by Stacho (2010)

In the algorithm, line 17 on-wards assumes that the graph either has a stable separator, or is isomorphic to a triangular strip. This can be proved using the above listed theorems and lemmas as follows.

Theorem 4.5. *Let G is an AT-free graph with at least three vertices and no induced diamond or K_4. Then either G is a triangular strip, or G contains a stable cutset*

Proof. From Lemma 4.4, we can assume that G is 2-connected and contains a triangular strip.

Let H be the largest triangular strip induced in G. We assume that $V(H) \neq V(G)$, as otherwise the theorem is trivially true. Hence, there exists a vertex $v \in V(G) \setminus V(H)$ adjacent to a vertex of H. Using Lemma 4.5, we can conclude that v will be adjacent to vertex c of an end-triangle of H. Let this end-triangle of H is $\{a, b, c\}$. Now, edges $va, vb \notin E(G)$, as otherwise, the set $\{v, a, b, c\}$ induces a diamond in G.

Note that $N(b) \setminus \{a\}$, and $N(a) \setminus \{b\}$ are stable sets. If this isn't true, then it would mean that either a or b is a member of two triangles, which is a violation of Lemma 4.4. Additionally, the sets $N(a) \cap N(v)$ and $N(b) \cap N(v)$ are both stable sets in G, as otherwise there would exist an induced diamond in G.

Let $\exists u \in N(a) \cap N(v), v \in N(b) \cap N(v)$ s.t. the vertices u and v are adjacent. We observe that if $u \in V(H)$, then u must belong to a triangle in $V(H)$. We also know that u belongs to the triangle $\{u, v, w\}$. However, this triangle is not in $V(H)$, since $v \notin V(H)$. This can only mean that u is a part of two different triangles, which is a contradiction to Lemma 4.4. Hence $u \notin V(H)$. By the same logic, $w \notin V(H)$. This implies that $G[V(H) \cup \{u, v, w\}]$ is a triangular strip in G strictly larger that H. This is a violation of our assumption that H is the maximal triangular strip in G. Therefore, an edge between vertices u and v cannot exist.

Suppose that $S = \big(N(b) \setminus \{a\}\big) \bigcup \big(N(a) \cap N(v)\big)$ is a stable set, i.e., there are no edges between $N(b) \setminus \{a\}$ and $N(a) \cap N(v)$. Let P be the shortest path in $G - S$ from a to v. Let z is the vertex following a on P. Since $N(a) \cap N(v) \subseteq S$, we conclude that the edge $zv \notin E(G)$. Also, the edges $zb, zc \notin E(G)$ as otherwise the set $\{a, b, c, z\}$ induces a diamond in G. Therefore, the path $v \to c \to b$ is missed by z, the path $z \to a \to b$ is missed by v, and the path $P \setminus \{a\}$ is a path from z to v missed by b, since $N(b) \setminus \{a\} \in S$. Hence the set $\{z, v, b\}$ forms an asteroidal triple in G, which is a contradiction. Therefore, path P cannot exist, and hence S must be a stable cutset of G separating a from v.

By symmetry, we can also show that if $S = \big(N(a) \setminus \{b\}\big) \bigcup \big(N(b) \cap N(v)\big)$ is a stable set, i.e., there are no edges between $N(a) \setminus \{b\}$ and $N(b) \cap N(v)$, then G contains a stable cutset. If not, we can show G to contain an asteroidal triple as above, which leads to a contradiction.

Hence, we have shown that if G is not a triangular strip, it must contain a stable cutset.

□

The above listed theorems and lemmas can hence be cascaded to prove the following theorem.

Theorem 4.6. *Every AT-free graph G with no induced diamond and no K_4 is 3-colorable. Moreover, if G contains a minimal stable separator S, then there is a 3-coloring of G in which all vertices of S have the same color.*

The proof of this theorem has been skipped in this study due to being too involved. However, it can be found in the original work by Stacho (2010)

Hence, Theorem 4.6 is the justification of the correctness of the `color-map` function used in Algorithm 4.

With this, we conclude the study on the coloring of AT-free graphs.

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