Advanced Data Structure and Algorithm

Randomized algorithms and QuickSort

WHAT IS A RANDOMIZED ALGORITHM?

- An algorithm that incorporates randomness as part of its operation.
- Basically, we'll make random choices during the algorithm:
    - Sometimes, we'll just hope that it works!
    - Other times, we'll just hope that our algorithm is fast!
- Let's formalize this...

LAS VEGAS vs. MONTE CARLO

LAS VEGAS ALGORITHMS

Guarantees correctness!

But the runtime is a random variable.

(i.e. there's a chance the runtime could take awhile)

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Correctness is a random variable.
(i.e. there's a chance the output is wrong)

But the runtime is guaranteed!

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LAS VEGAS ALGORITHMS

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We'll focus on these algorithms today
(BogoSort, QuickSort, QuickSelect)

MONTE CARLO ALGORITHMS

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But the runtime is guaranteed!

We'll see some examples of these later in the semester!

How do we measure the runtime of a randomized algorithm?

Scenario 1

1. You publish your algorithm.
2. Bad guy picks the input.
3. You run your randomized algorithm.

Scenario 2

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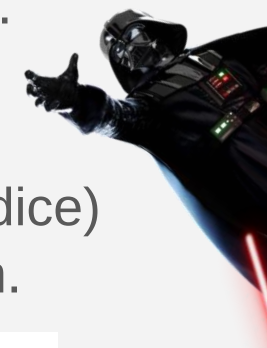
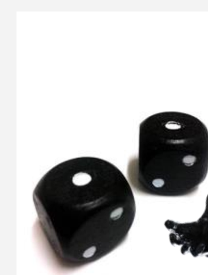
2. Bad guy picks the input.
3. Bad guy chooses the randomness (fixes the dice) and runs your algorithm.





- In Scenario 1, the running time is a **random variable.**
  - It makes sense to talk about **expected running time**.
- In Scenario 2, the running time is **not random**.
  - We call this the **worst-case running time** of the randomized algorithm.

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in both cases, we are still thinking about the *WORST-CASE INPUT*

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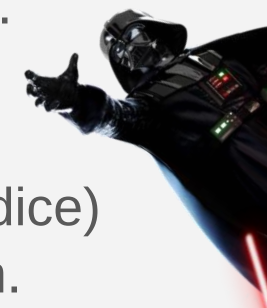

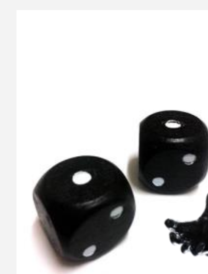

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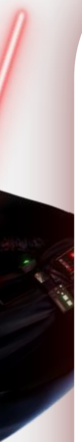

Don’t get confused!!!

Even with randomized algorithms, we are still considering the *WORST CASE INPUT*, regardless of whether we’re computing expected or worst-case runtime.

Expected runtime ***IS NOT*** runtime when given an expected input! We are taking the expectation over the random choices that our algorithm would make, ***NOT*** an expectation over the distribution of possible inputs.

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QUICK PROBABILITY EXERCISE

X is a Bernoulli/indicator random variable which is **1** with probability 1/100 and **0** with prob. 99/100.

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b. Suppose you draw *n* independent random variables $\mathbf{X_1}$, $\mathbf{X_2}$, …, $\mathbf{X_n}$, distributed like X. What is the expected value $\mathbb{E}[\sum_{i=1}^{n} X_i]$?

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c. Suppose I draw independent random variables $\mathbf{X_1}$, $\mathbf{X_2}$, …, $\mathbf{X_n}$, and I stop when I see the first "**1**". Let N be the last index that we draw. What is the expected value of N?

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N is a *geometric random variable*.
We can use the formula:

$$\mathbb{E}[N] = \frac{1}{p} = \frac{1}{1/100} = 100$$

GEOMETRIC RANDOM VARIABLE

If **N** represents "number of trials/attempts",
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$$\begin{aligned}
\mathbb{E}[N] &= 1(p) + (1 + \mathbb{E}[N])(1-p) \\
&= p + (1-p) + (1-p)\mathbb{E}[N] \\
&= 1 + (1-p)\mathbb{E}[N]
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[N](1-(1-p)) &= 1 \\
\mathbb{E}[N](p) &= 1 \\
\mathbb{E}[N] &= \frac{1}{p}
\end{aligned}$$

BOGOSORT

A bit silly, but a great pedagogical tool!

BOGOSORT

```
BOGOSORT(A):
    while True:
        A.shuffle()
        sorted = True
        for i in [0,...,n-2]:
            if A[i] > A[i+1]:
                  sorted = False
        if sorted:
            return A
```

This randomly permutes A (assume it takes O(n) time)

BOGOSORT: EXPECTED RUNTIME

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Probability that $\mathbf{X_i = 1}$ (A is sorted) = **1/n!**

since there are n! possible orderings of A and only one is sorted (assume A has distinct elements) ⇒ $E[X_i] = 1/n!$

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E[# of iterations/trials] = 1/(prob. of success on each trial)

= 1/(1/n!) = **n!**

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E[runtime on a list of length n]

= **E**[(# of iterations) * (time per iteration)]

= (time per iteration) * **E**[# of iterations]

= O(n) * **E**[# of iterations]

= O(n) * (n!)

= O(n * n!)

= ***REALLY REALLY BIG***

BOGOSORT: WORST-CASE RUNTIME?

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Worst-case runtime =

∞

This is as if the “bad guy” chooses all the randomness in the algorithm, so each shuffle could be unlucky… forever...

WHAT HAVE WE LEARNED?

EXPECTED RUNNING TIME

1. You publish your randomized algorithm
2. Bad guy picks an input
3. You get to roll the dice (leave it up to randomness)

WORST-CASE RUNNING TIME

1. You publish your randomized algorithm
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Don’t use BogoSort.

QUICKSORT

A much better randomized algorithm

QUICKSORT OVERVIEW

EXPECTED RUNNING TIME

$O(n \log n)$

WORST-CASE RUNNING TIME

$O(n^2)$

QUICKSORT OVERVIEW

EXPECTED RUNNING TIME

$O\ (n \log n)$

WORST-CASE RUNNING TIME

$O\ (n^2)$

In practice, it works great! It's competitive with MergeSort (& often better in some contexts!), and it runs *in place* (no need for lots of additional memory)

QUICKSORT: THE IDEA

Let's use DIVIDE-and-CONQUER again!

Select a pivot *at random*

Partition around it

Recursively sort L and R!

QUICKSORT: THE IDEA

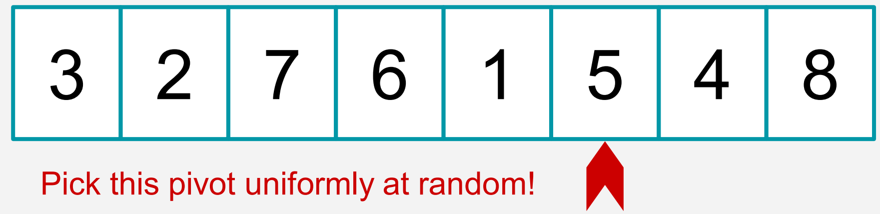
Select a pivot

3	2	7	6	1	5	4	8

Pick this pivot uniformly at random!

QUICKSORT: THE IDEA

Select a pivot



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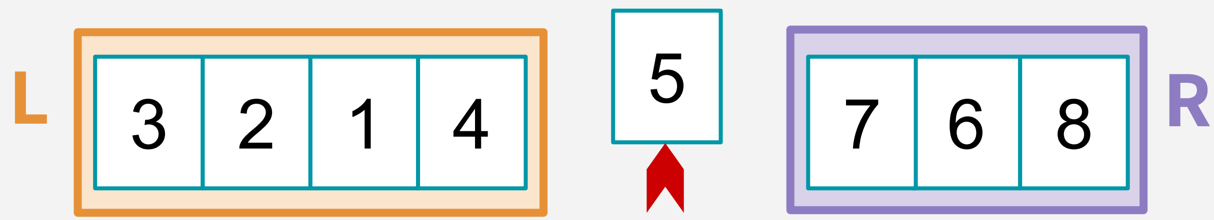

Partition around pivot: **L** has elements less than pivot, and **R** has elements greater than pivot.

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Partition around pivot: **L** has elements less than pivot, and **R** has elements greater than pivot.

Recurse!

Recursi ve magic

Recursively sort each side!

Recursi ve magic

L

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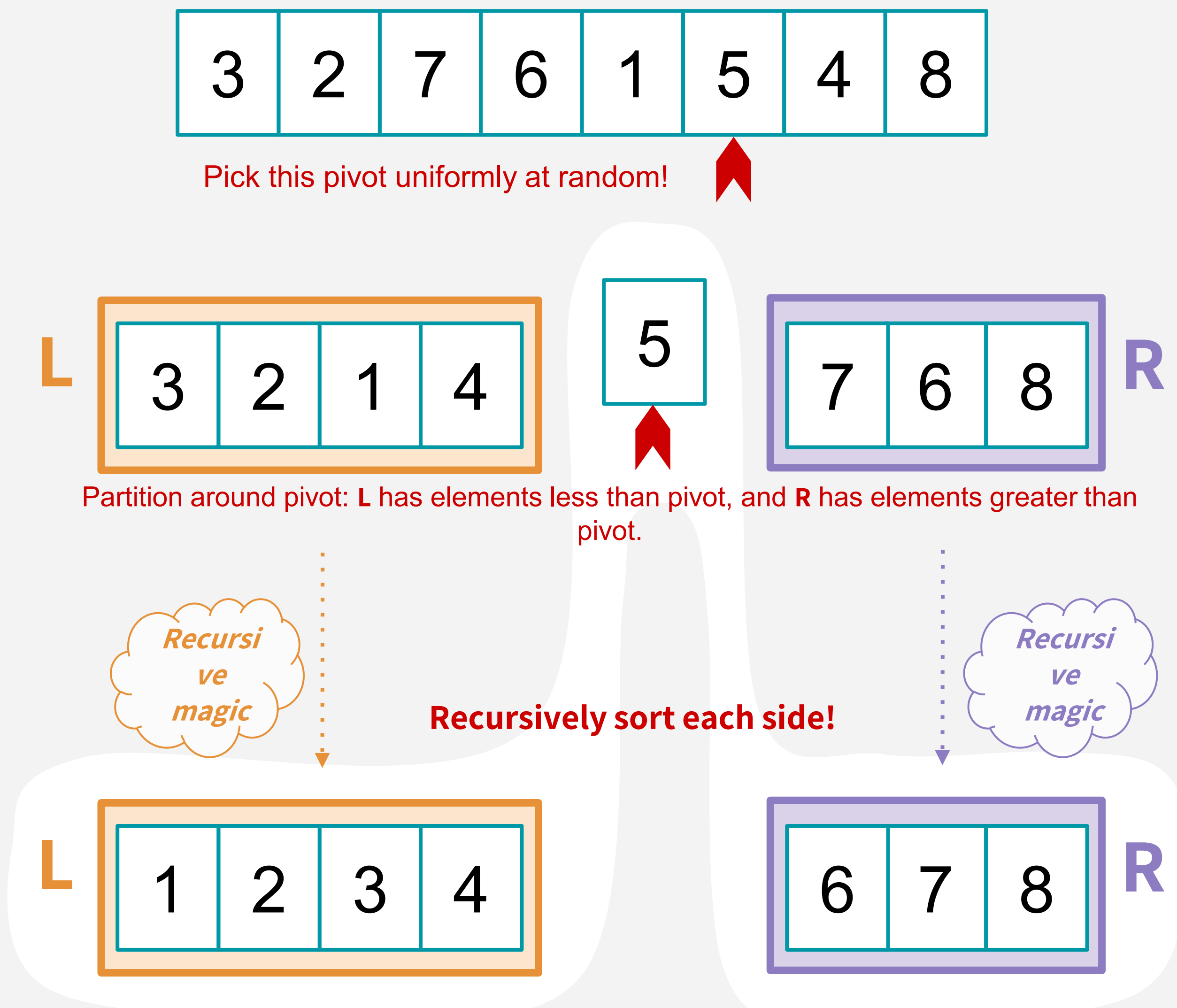
QUICKSORT: THE IDEA

Select a pivot

Partition around it

Recurse!



QUICKSORT: PSEUDO-PSEUDOCODE

Here's the high level outline:

(I've posted an IPython Notebook on the course website with actual code for QuickSort)

```
QUICKSORT(A):
    if len(A) <= 1:
        return
    pivot = random.choice(A)
    PARTITION A into:
        L (less than pivot) and
        R (greater than pivot)
    Replace A with [L, pivot, R]
    QUICKSORT(L)
    QUICKSORT(R)
```

RECURRENCE RELATION

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Recurrence Relation for QUICKSORT

$$T(n) = T(|L|) + T(|R|) + O(n)$$
$$T(0) = T(1) = O(1)$$

IDEAL RUNTIME?

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Ideal Runtime?

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Recurrence Relation for QUICKSORT

$$T(n) = T(|L|) + T(|R|) + O(n)$$
$$T(0) = T(1) = O(1)$$

In an ideal world, the pivot would split the array exactly in half, and we'd get:

$$T(n) = T(n/2) + T(n/2) + O(n)$$

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```

In an ideal world:

$$T(n) = 2 \cdot T(n/2) + O(n)$$

$$T(n) = \mathbf{O(n \log n)}$$

Recurrence Relation for QUICKSORT

$|) + T(|R|) + O(n)$

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Worst-Case Runtime?

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Recurrence Relation for QUICKSORT

$$T(n) = T(|L|) + T(|R|) + O(n)$$
$$T(0) = T(1) = O(1)$$

With the unluckiest randomness, the pivot would be either min(A) or max(A):

$$T(n) = T(0) + T(n-1) + O(n)$$

WORST-CASE RUNTIME

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With the worst “randomness”

$$T(n) = T(n-1) + O(n)$$

$$T(n) = O(n^2)$$

(recursion tree/table or substitution method!)

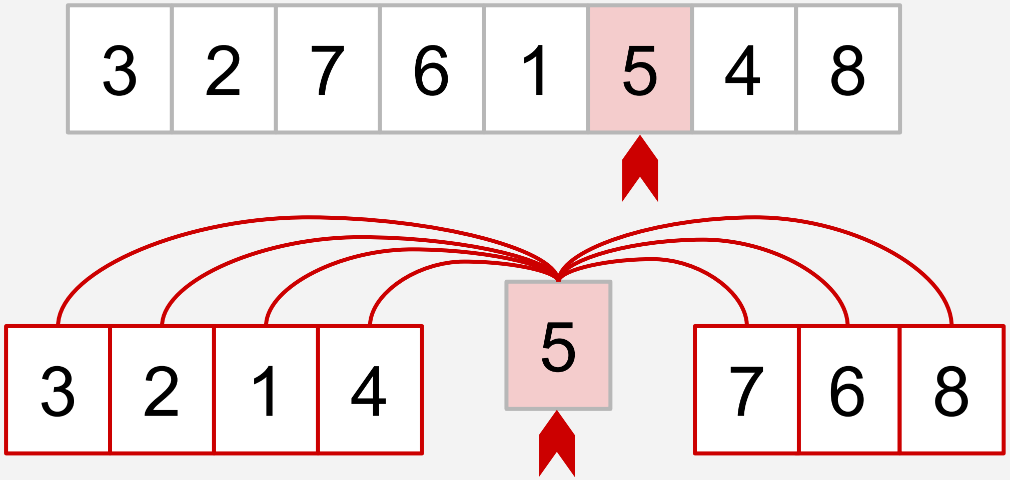
QUICKSORT O(n log n) EXPECTED RUNTIME

In order to prove this expected runtime:
Lets compute
How many times are any two items compared, in expectation?

HOW MANY COMPARISONS?

3	2	7	6	1	5	4	8

HOW MANY COMPARISONS?



Everything is compared to 5 once in this first step… and then never again with **5**.

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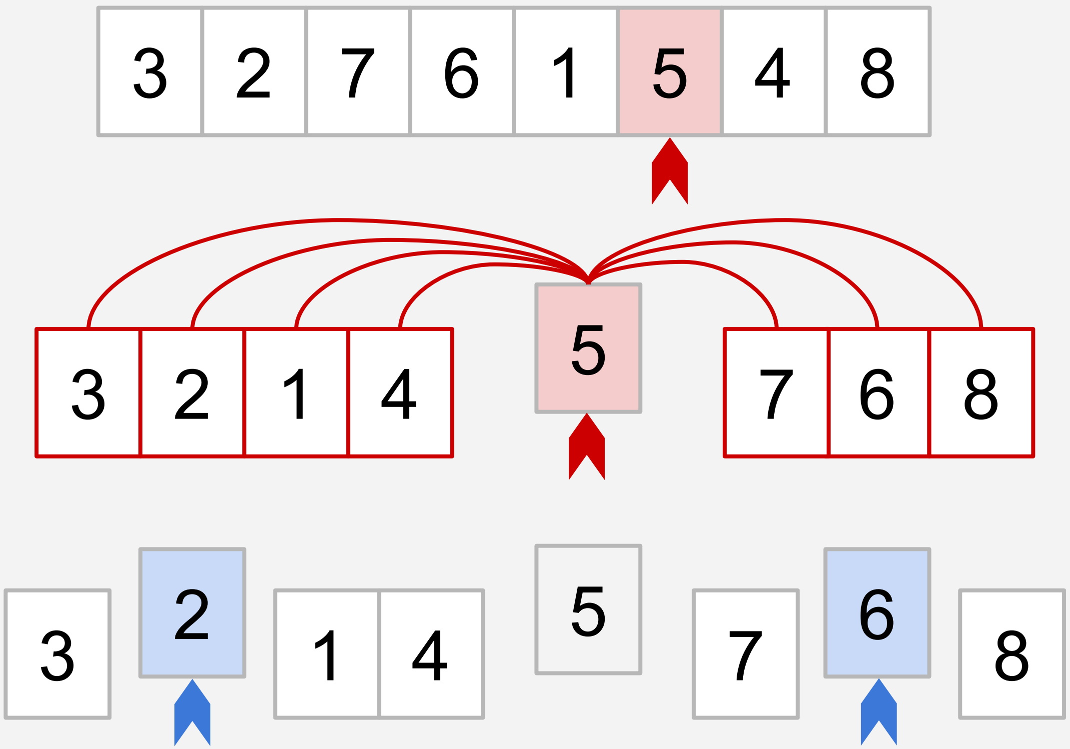

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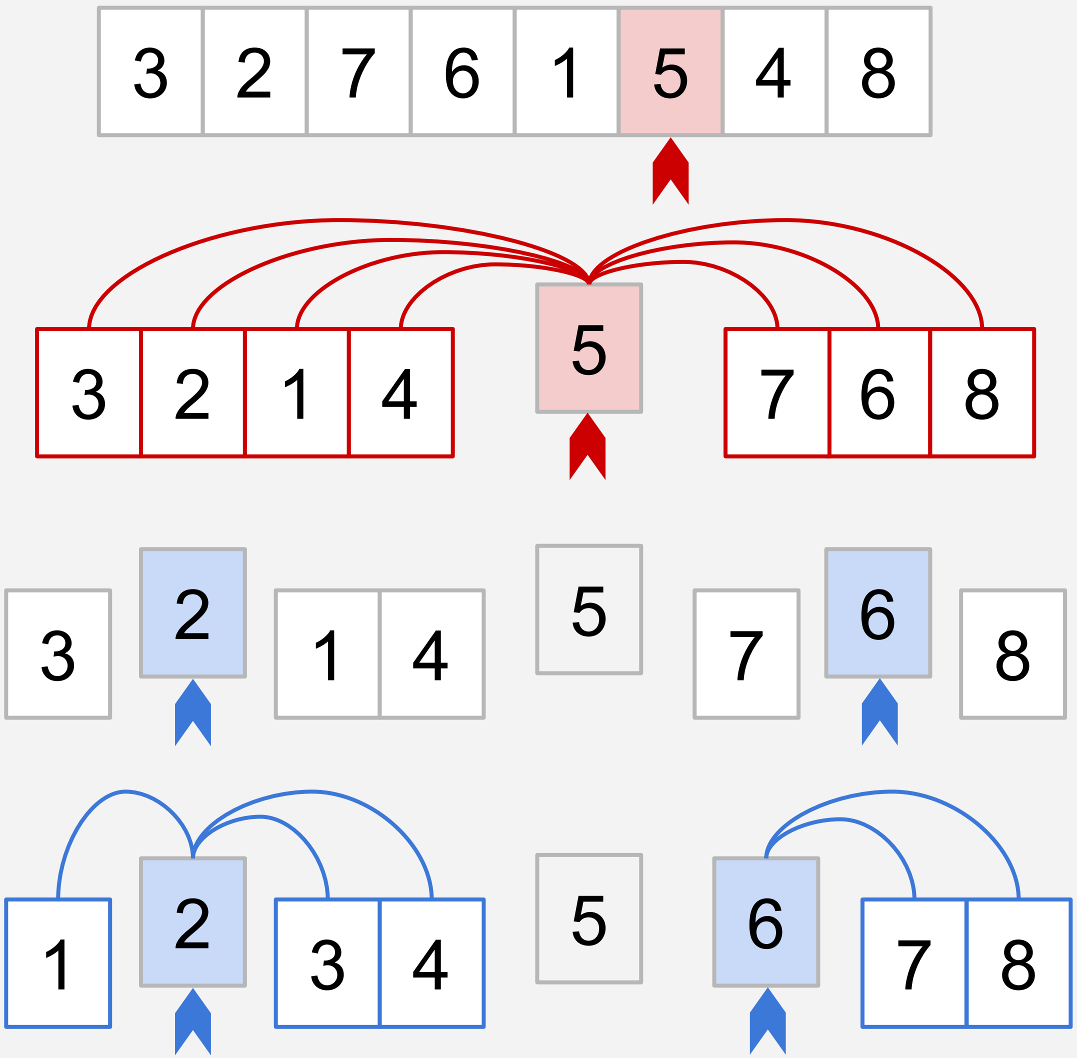

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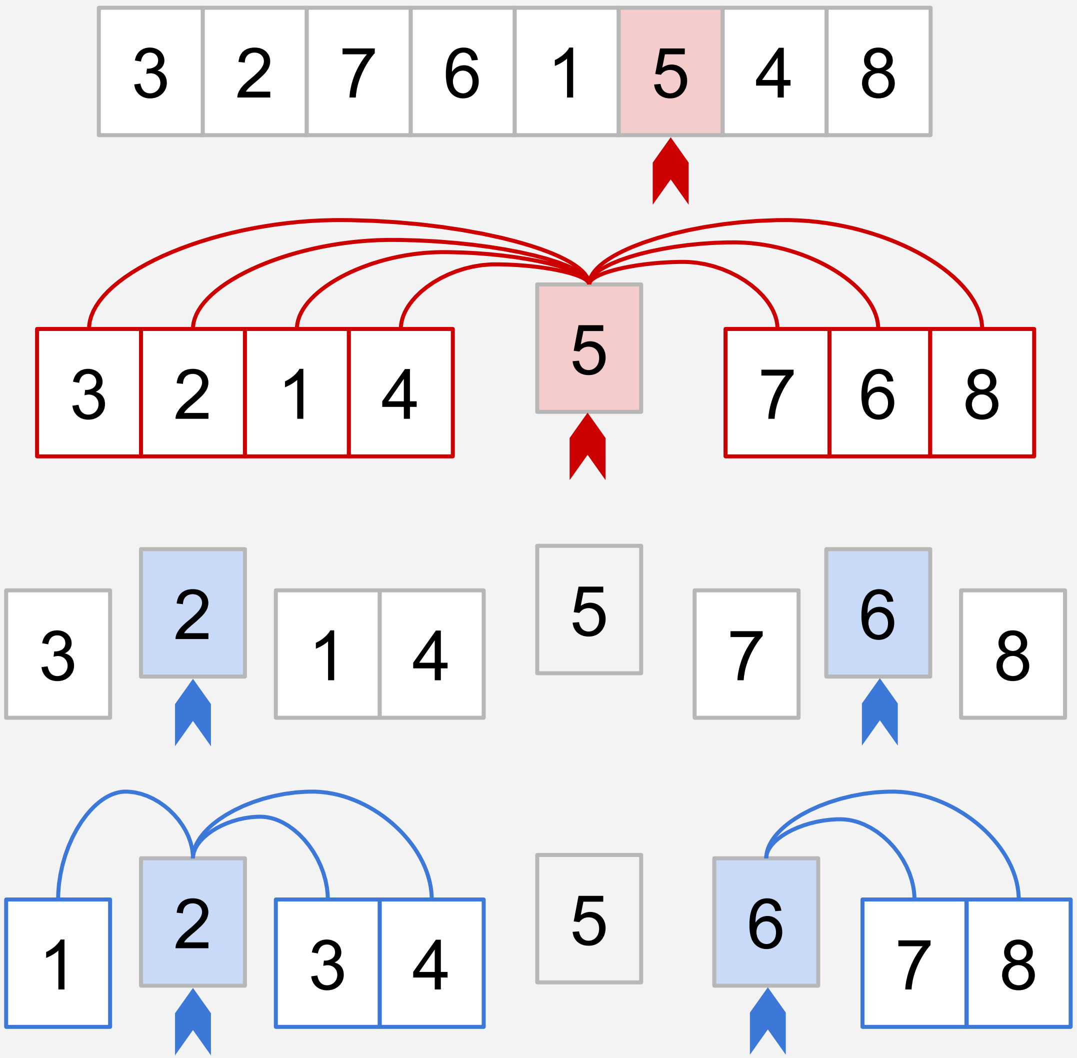


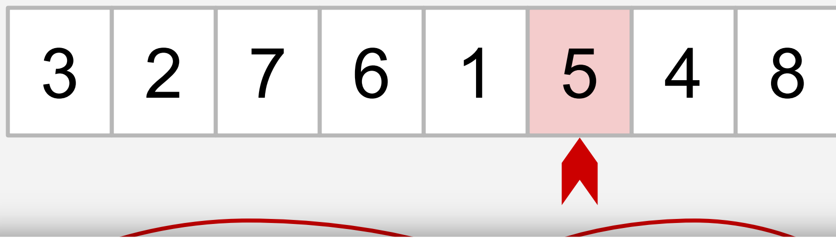
Everything is compared to 5 once in this first step… and then never again with **5**.

Only 1, 3, & 4 are compared to **2**.

And only 7 & 8 are compared with **6**.

No comparisons ever happen between two numbers on opposite sides of 5.

HOW MANY COMPARISONS?



Seems like whether or not two elements are compared has something to do with pivots...

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HOW MANY COMPARISONS?

Each pair of elements is compared either **0** or **1** times.

Let $\mathbf{X_{a,b}}$ be a Bernoulli/indicator random variable such that:

$\mathbf{X_{a,b} = 1}$ if **a** and **b** are compared

$\mathbf{X_{a,b} = 0}$ otherwise

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In our example, $\mathbf{X_{2,5}}$ took on the value **1** since **2** and **5** were compared. On the other hand, $\mathbf{X_{3,7}}$ took on the value **0** since **3** and **7** are *not* compared.

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Total number of comparisons =

$$\mathbb{E}\left[\sum_{a=0}^{n-2}\sum_{b=a+1}^{n-1} X_{a,b}\right]$$

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by linearity of expectation!

HOW MANY COMPARISONS?

Each pair of elements is compared either **0** or **1** times.

Let $\mathbf{X_{a,b}}$ be a Bernoulli/indicator random variable such that:

$\mathbf{X_{a,b} = 1}$ if **a** and **b** are compared

$\mathbf{X_{a,b} = 0}$ otherwise

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We need to figure out this value!

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1	2	**3**	4	5		**7**	8

If 4, 5, or 6 get picked as a pivot first, then 3 and 7 would be separated and never see each other again.

HOW MANY COMPARISONS?

So, what's $\mathbf{E}[X_{a,b}]$?

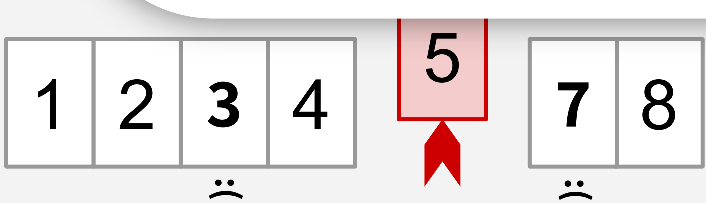
$\mathbf{P(X_{a,b} = 1)}$ *aka probability that **a** & **b** are compared*

=

probability that either **a** or **b** are selected as a pivot before elements between **a** and **b**.

=

$$\frac{2}{\text{(\# elements from } \mathbf{a} \text{ to } \mathbf{b}\text{, inclusive)}}$$



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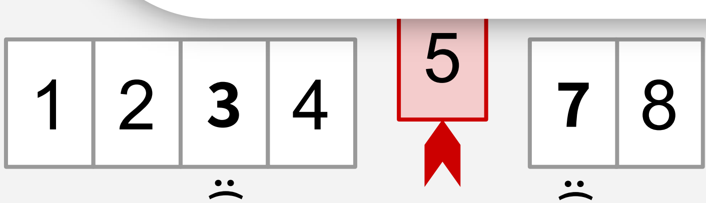
$$\frac{2}{\mathbf{b} - \mathbf{a} + 1}$$



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If E[# comparisons] = O(n log n), does this mean E[running time] is also O(n log n)?

YES! Intuitively, the runtime is dominated by comparisons.

QUICKSORT

```
QUICKSORT(A):
    if len(A) <= 1:
        return
    pivot = random.choice(A)
    PARTITION A into:
        L (less than pivot) and
        R (greater than pivot)
    Replace A with [L, pivot, R]
    QUICKSORT(L)
    QUICKSORT(R)
```

Worst case runtime: **$O(n^2)$**

Expected runtime: **$O(n \log n)$**

QUICKSORT IN PRACTICE

How is it implemented? Do people use it?

IMPLEMENTING QUICKSORT

In practice, a more clever approach is used to implement PARTITION, so that the entire QuickSort algorithm can be implemented "in-place"

(i.e. via swaps, rather than constructing separate L or R subarrays)

AN EXAMPLE IN-PLACE PARTITION

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7	2	3	6	1	5	4	8

Choose pivot & swap with last element so pivot is at the end.

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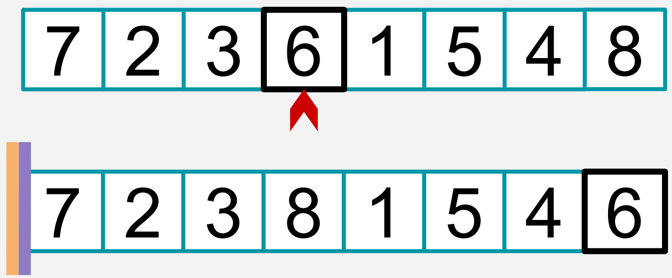

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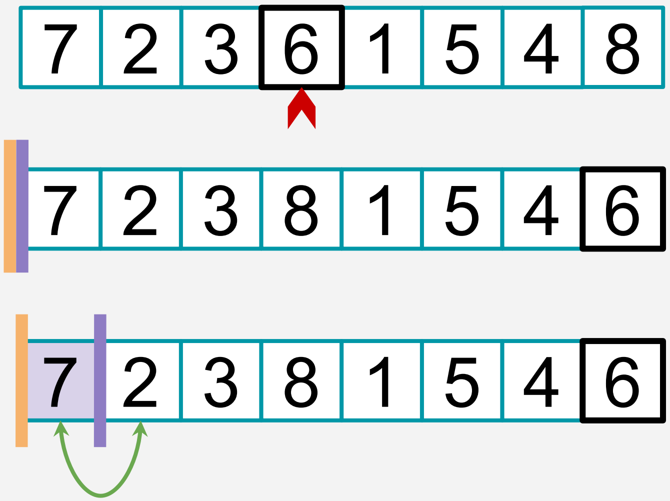

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Increment until it sees something smaller than pivot, **swap** the things ahead of the bars & increment both bars

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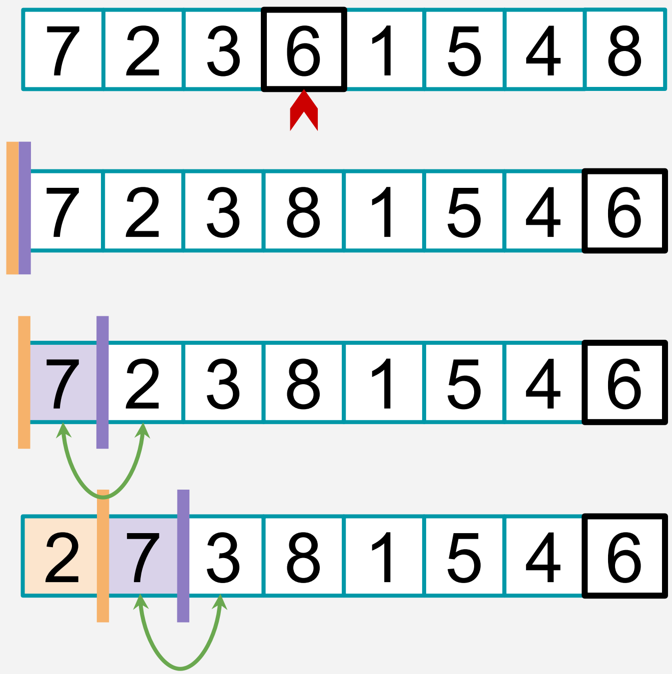

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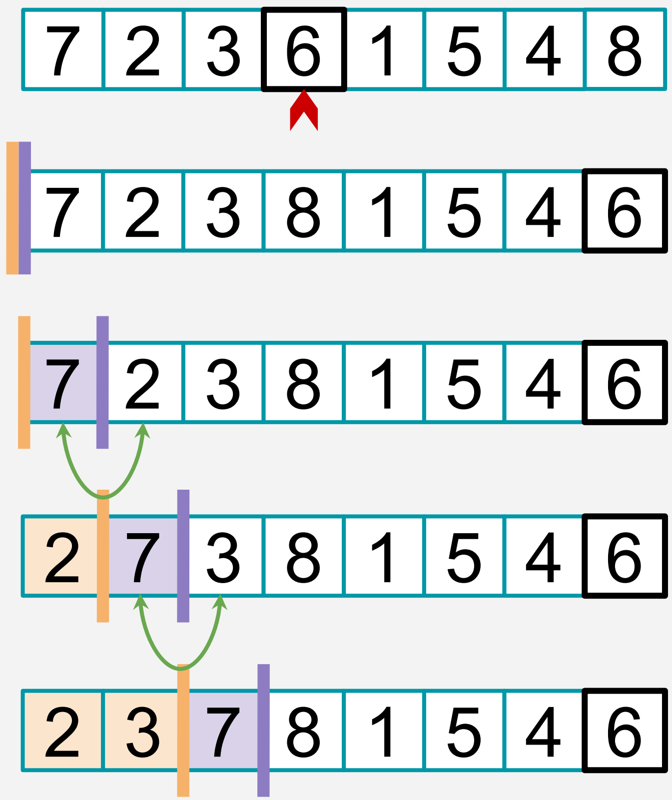

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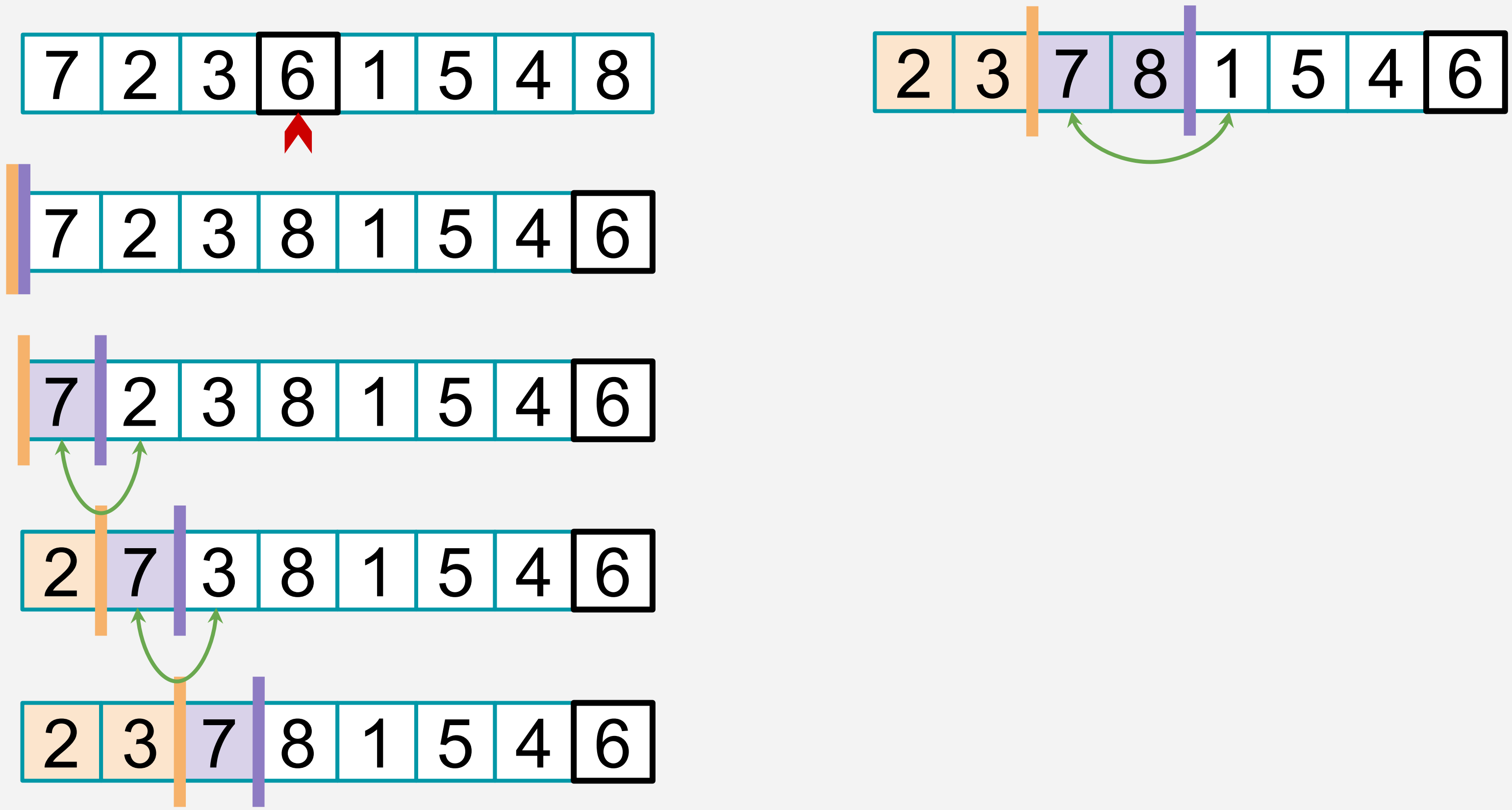

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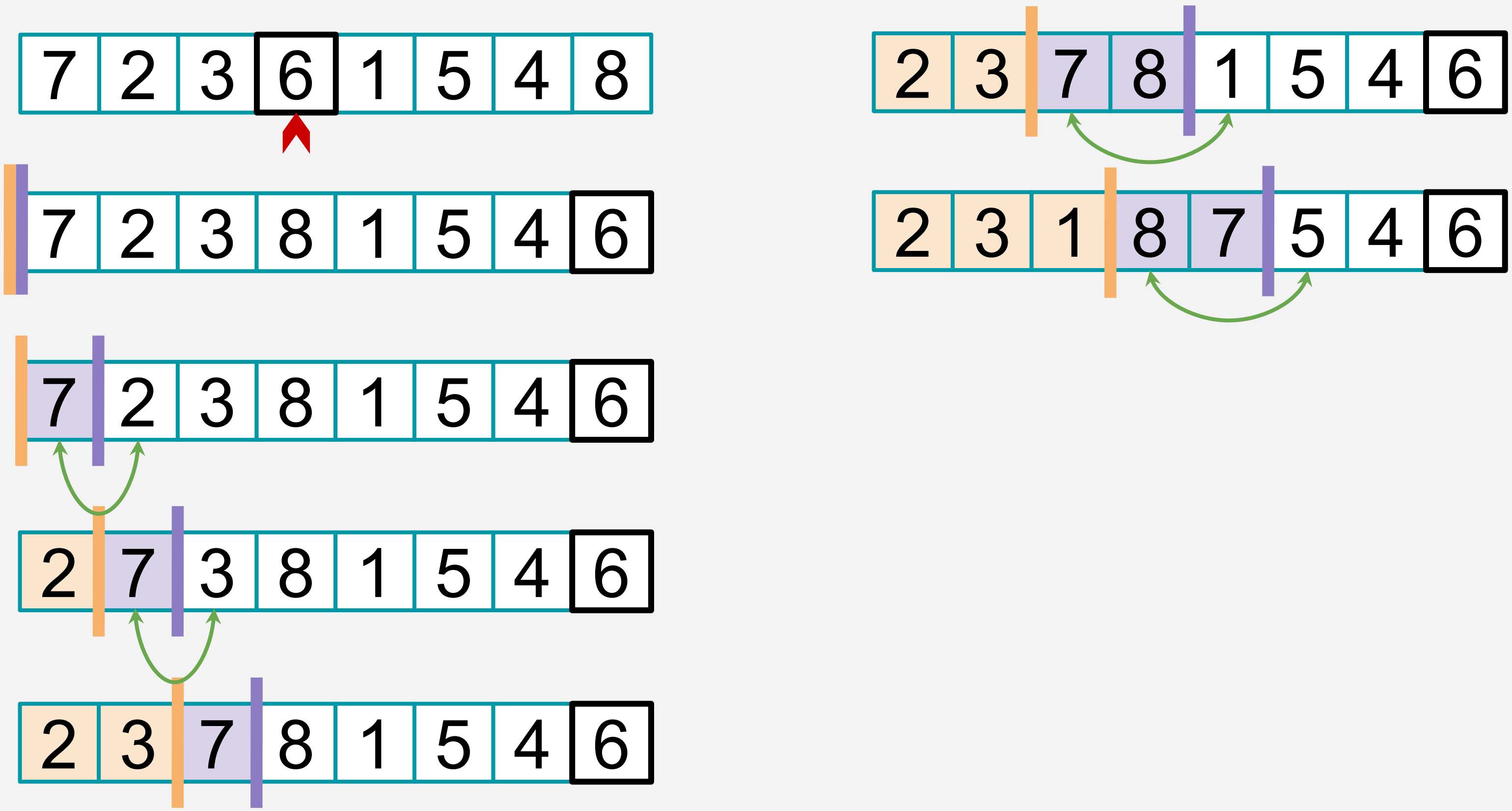

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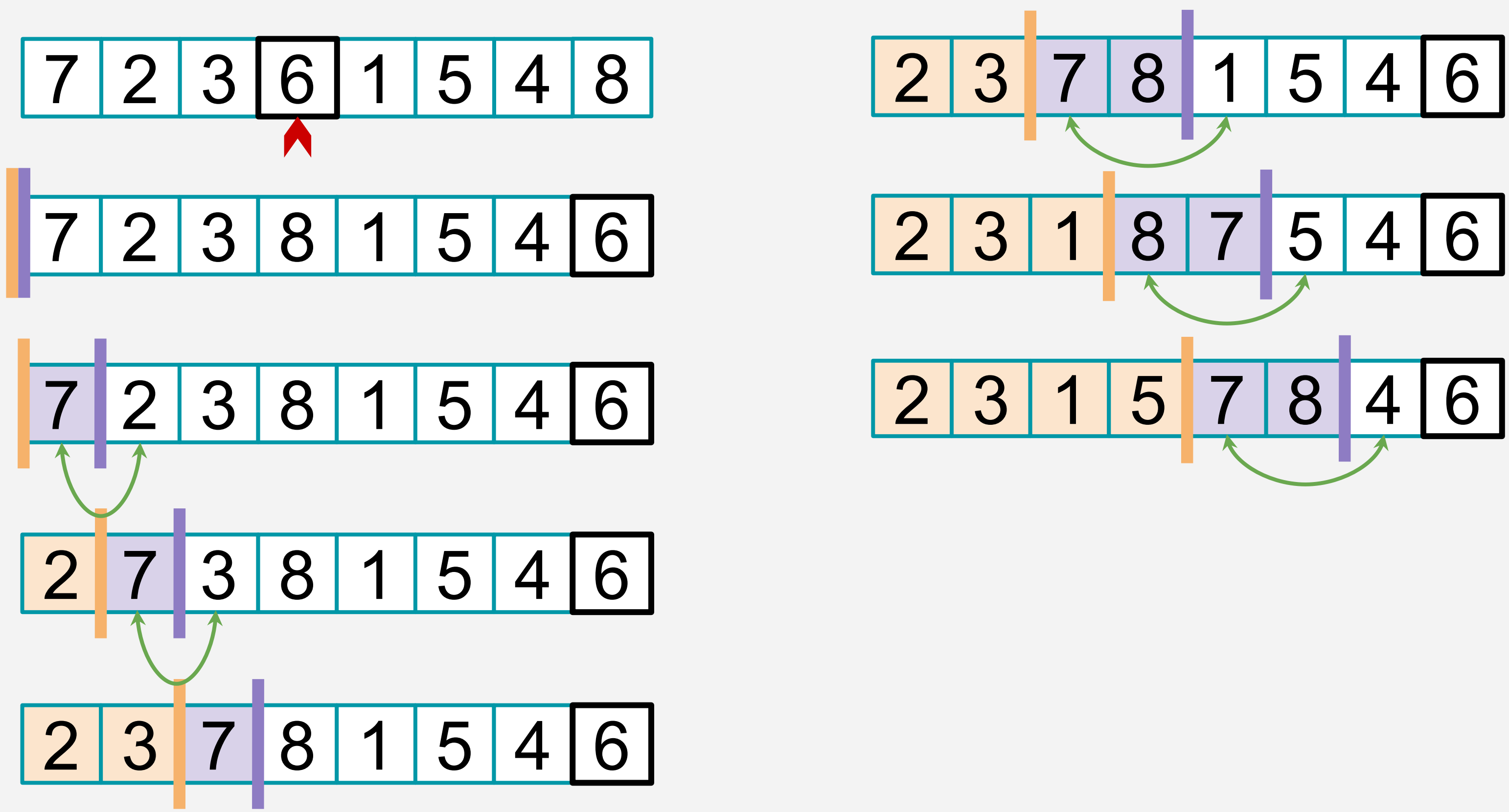

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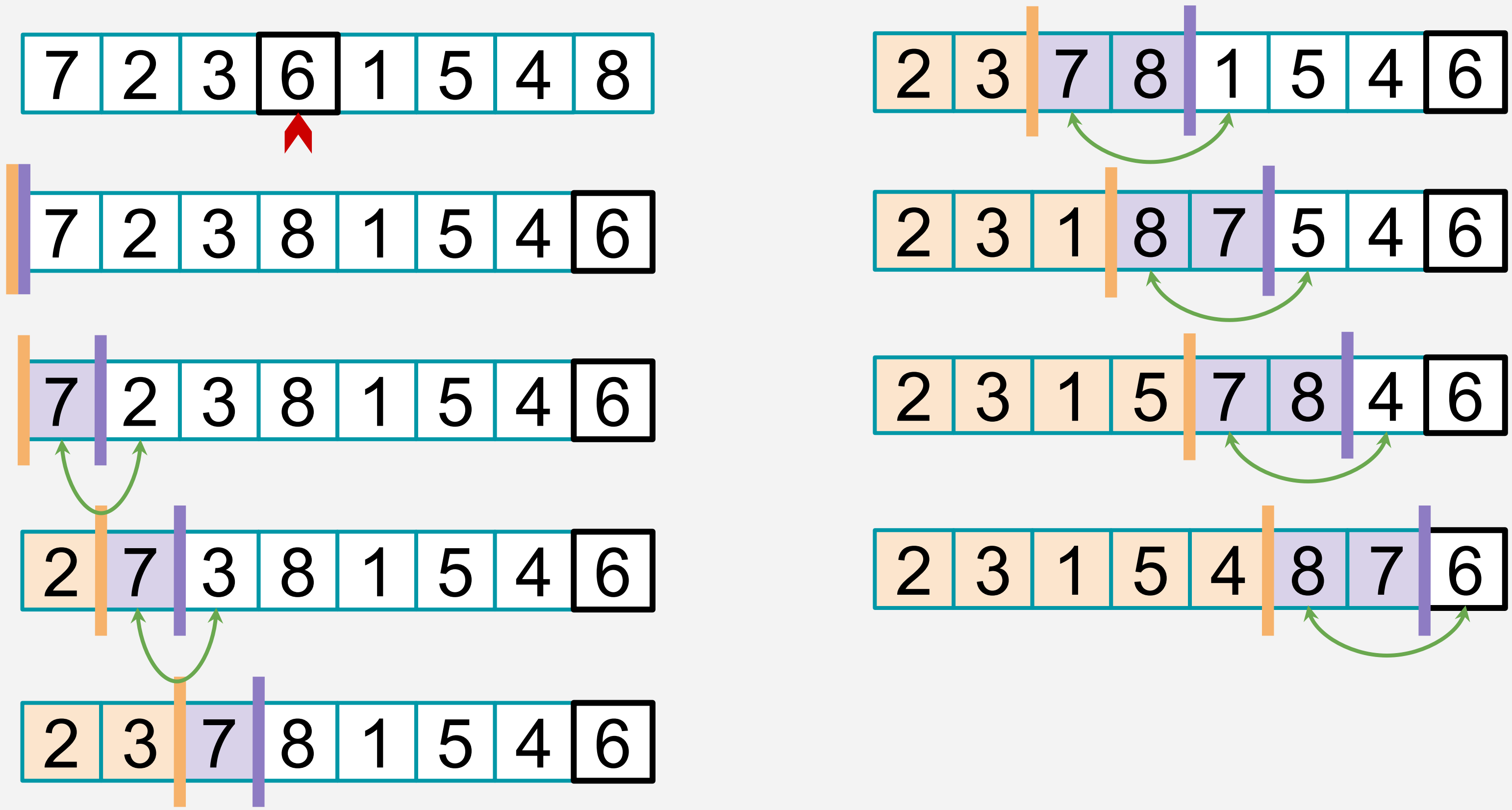

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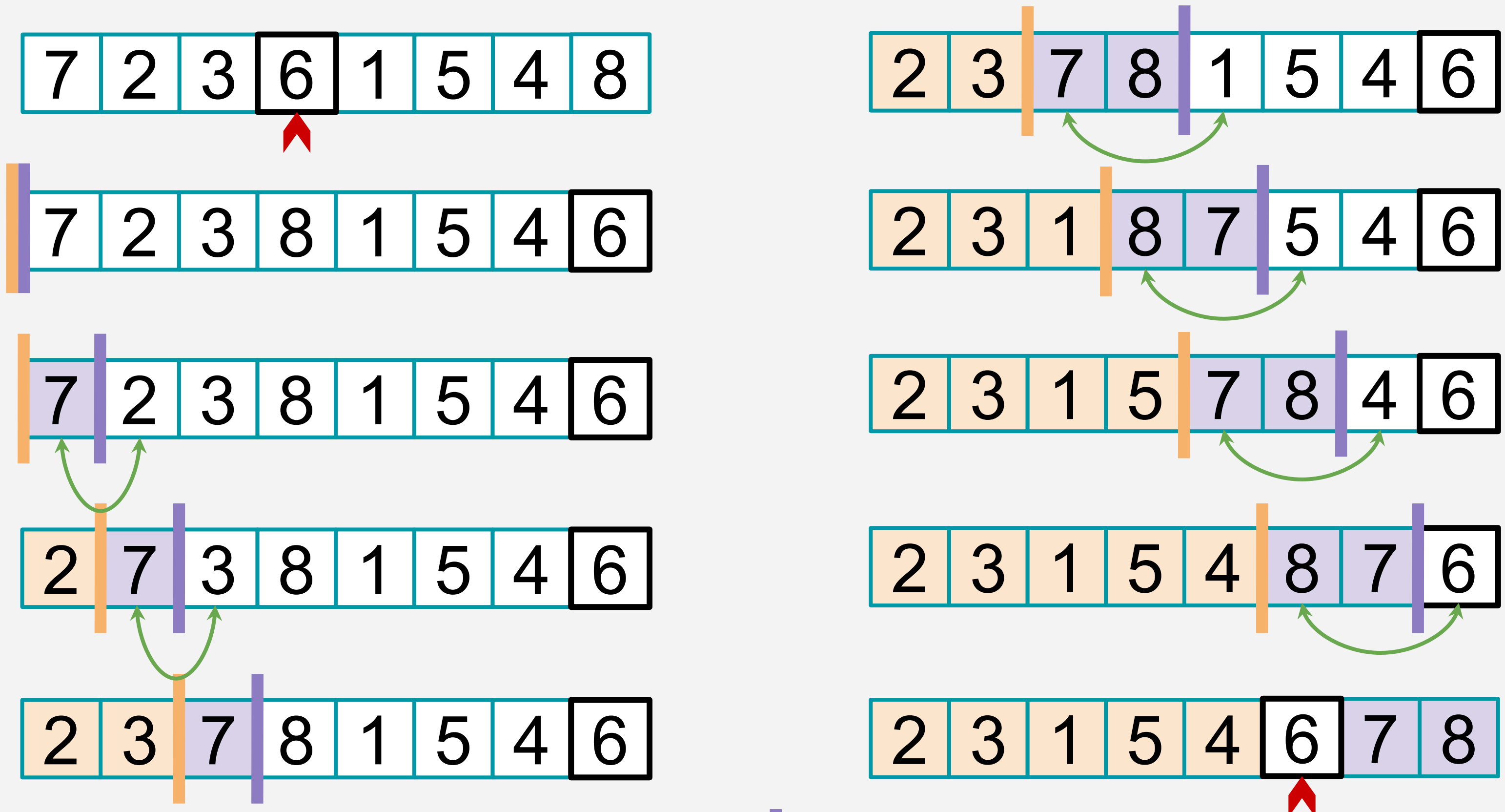

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IMPLEMENTING QUICKSORT

There's another in-place partition algorithm called Hoare Partition that's even more efficient as it performs less swaps.

(you're not responsible for knowing it in this class)

Check out these Hungarian Folk Dancers showing you how it's done!

QUICKSORT vs. MERGESORT

You do not need to understand any of this stuff

	QuickSort (random pivot)	MergeSort (deterministic)
Runtime	**Worst-case: $O(n^2)$ Expected: $O(n \log n)$**	**Worst-case: $O(n \log n)$**
Used by	Java (primitive types), C (qsort), Unix, gcc…	Java for objects, perl
In-place? (i.e. with $O(\log n)$ extra memory)	Yes, pretty easily!	Easy if you sacrifice runtime (O(nlogn) MERGE runtime). Not so easy if you want to keep runtime & stability.
Stable?	No	Yes

RECAP

- Runtimes of **randomized algorithms** can be measured in two main ways:
    - Expected runtime (you roll the dice)
    - Worst-case runtime (the bad guy gets to fix the dice)
- **QUICKSORT!**
    - Another *DIVIDE and CONQUER* sorting algorithm that employs randomness
    - Elegant, structurally simple, and actually used in practice!

NEXT TIME

- Can we sort faster than $\Theta(n \log n)$???

Acknowledgement

- Stanford University