Advanced Data Structure and Algorithm

Hashing!

LAST TIME

- We learned about **Binary Search Trees**!
- Sometimes valid BSTs aren't balanced, which can lead to slow (O(n)) operations… so we discussed **self-balancing BSTs**!
    - They apply BST *rotations* to achieve balance.
    - **AVL Trees** are an example of a self-balancing BST!
    - **2-4 Trees** are an example of a self-balancing tree!
    - **Red-Black Trees** are also an example of a self-balancing BST! It maintains these slightly weird but very elegant properties as a proxy for balance.
        - RB Trees have complicated INSERT & DELETE routines in order to maintain those properties even when modifying the tree.

WHAT WE'LL COVER TODAY

- Hashing!
  - What operations are we trying to support?
  - Hash Functions
  - Dealing with collisions
  - What makes a good hash function?
  - *Universal* hash families are what we're looking for!

HASH TABLES OVERVIEW

What operations does it support?

THE TASK

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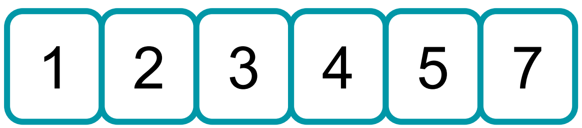
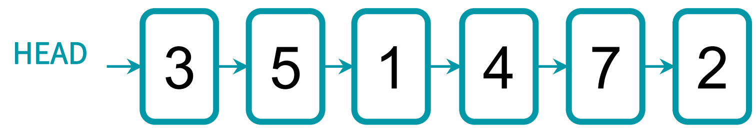
Sorted Arrays



O(n) INSERT/DELETE: first, find the relevant element (via SEARCH) and move a bunch of elements in the array

O(log n) SEARCH: use binary search to see if an element is in A

Linked Lists



O(1) INSERT: just insert the element at the head of the linked list

O(n) SEARCH/DELETE: since the list is not necessarily sorted, you need to scan the list (delete by manipulating pointers)

HASH TABLE MOTIVATION

OPERATION	SORTED ARRAY	UNSORTED LINKED LIST	HASH TABLES (HOPEFULLY)
SEARCH	O(log(n))	O(n)	O(1)
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What is a *naive* way to achieve these runtimes?

ATTEMPT 1: DIRECT ADDRESSING

Suppose you're storing numbers from 1 - 1000:

2	4	5	998	999

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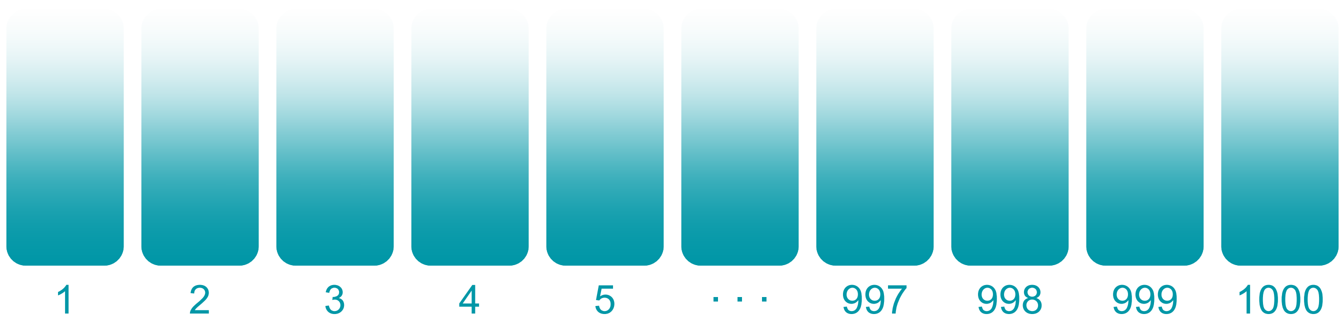

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2	4	5	998	999

Not bad!

But what’s the issue with this approach?

O(1) INSERT/DELETE/SEARCH: Just index into the bucket!

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2	3	1000	1002	10^{10}

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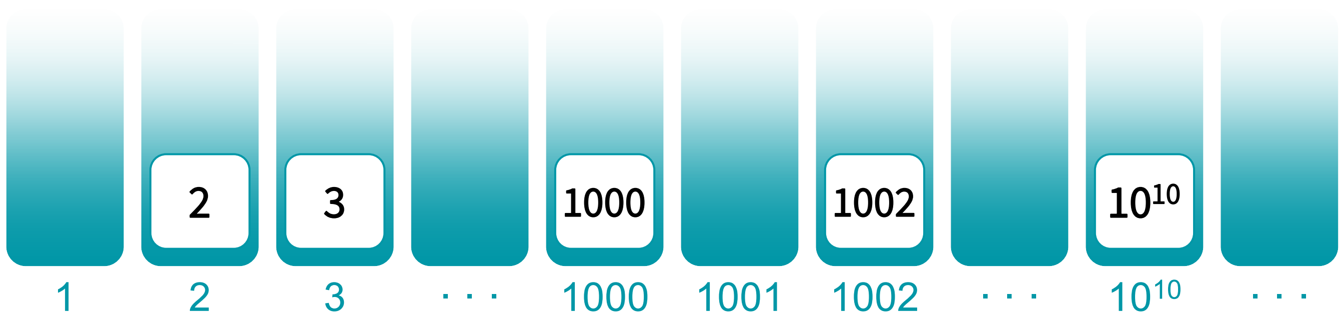

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But the space requirement is HUGE...

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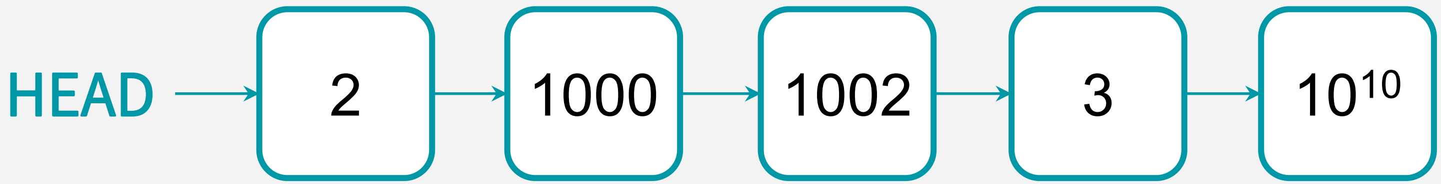

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HEAD → 2 → 1000 → 1002 → 3 → 10^{10}

Good news: Space is now proportional to the number of objects you deal with

Bad news: Searching for an object is now going to scale with the number of inputs you deal with… not close to our desired O(1)!

The direct-addressing approach still has merit because of it's fast object search/access

HOW DO WE IMPROVE THIS?

We like the **functionality of a direct-addressable** array for constant time access (super fast INSERT/DELETE/SEARCH)

But reserving an bucket/array slot for each possible key leads to unreasonable space requirements… (kind of like CountingSort)

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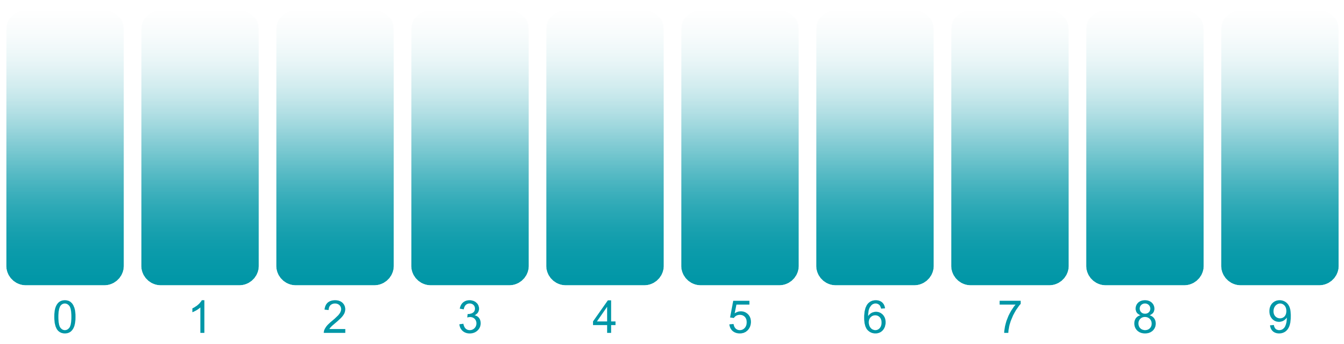
Let's try bucketing **by the least-significant digit**…

BUCKETING ATTEMPT 1:

Suppose you’re storing numbers from 1 - 10^{10}:

2 | 3 | 1000 | 1002 | 10^{10}

Bucket by last digit?



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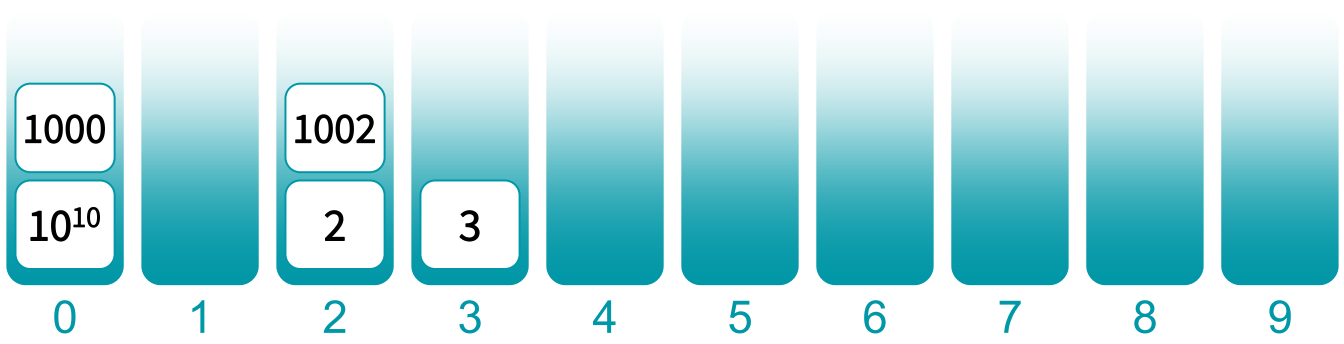

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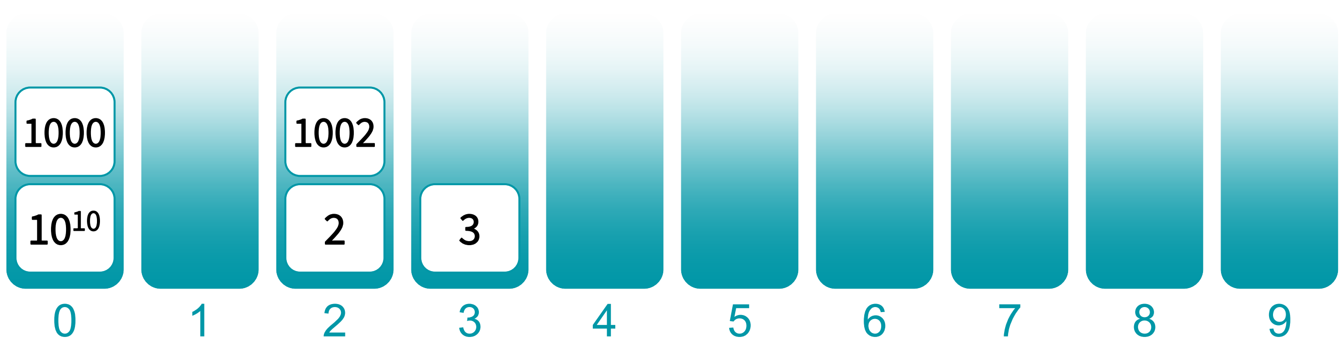

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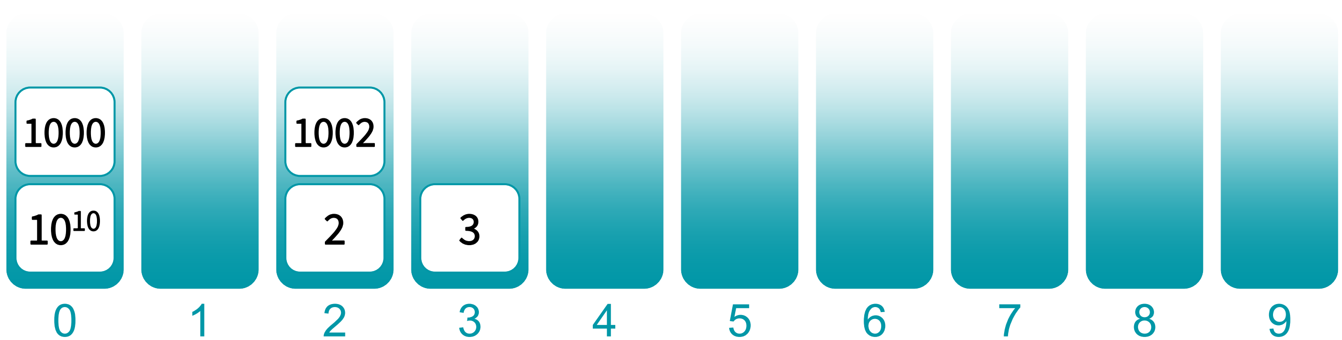

O(??????) SEARCH/DELETE:

Go visit bucket & search through until you find it…

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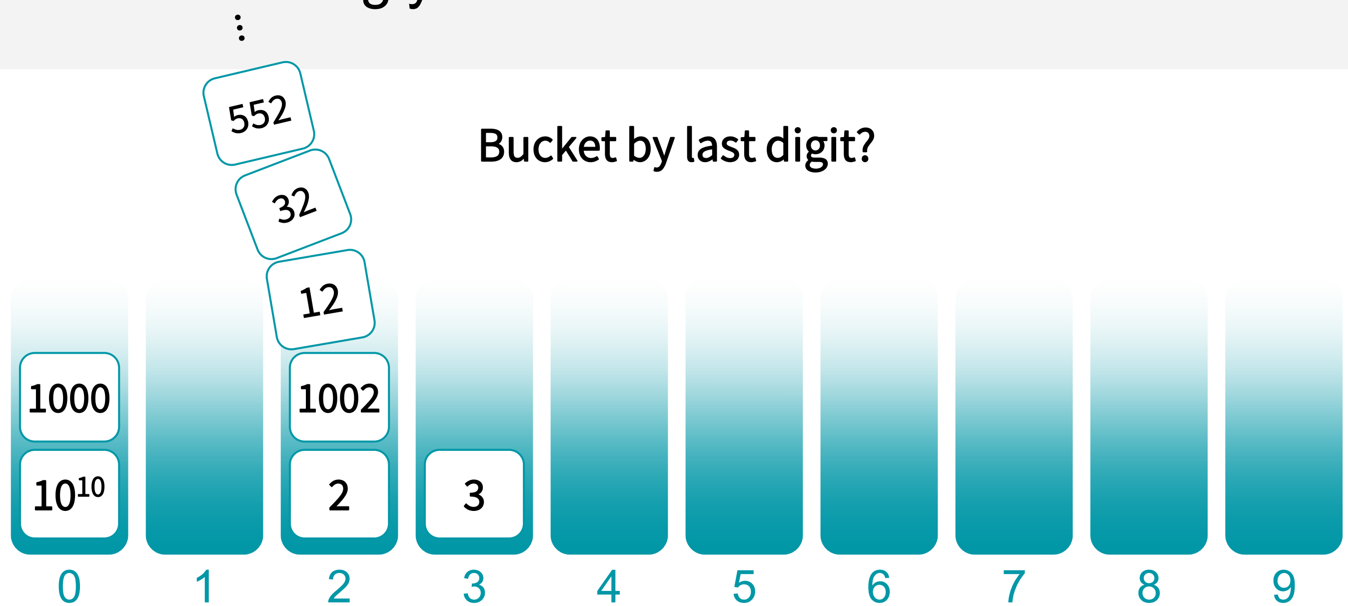

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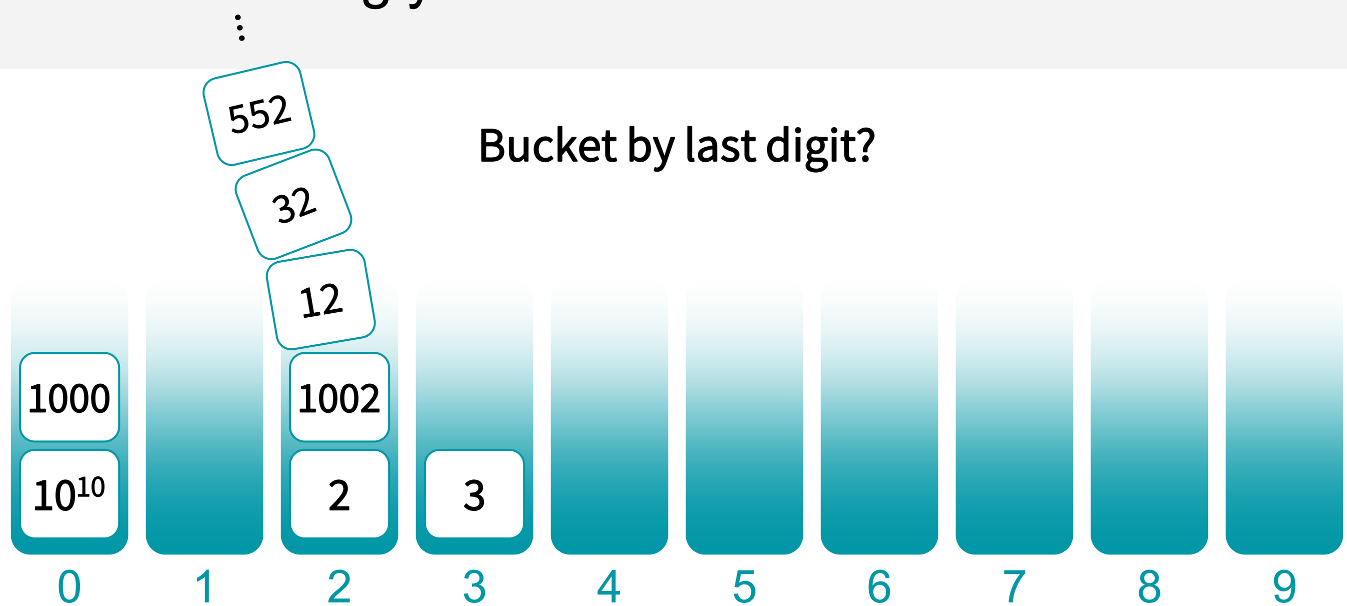

Maybe another bucketing scheme?

find it...

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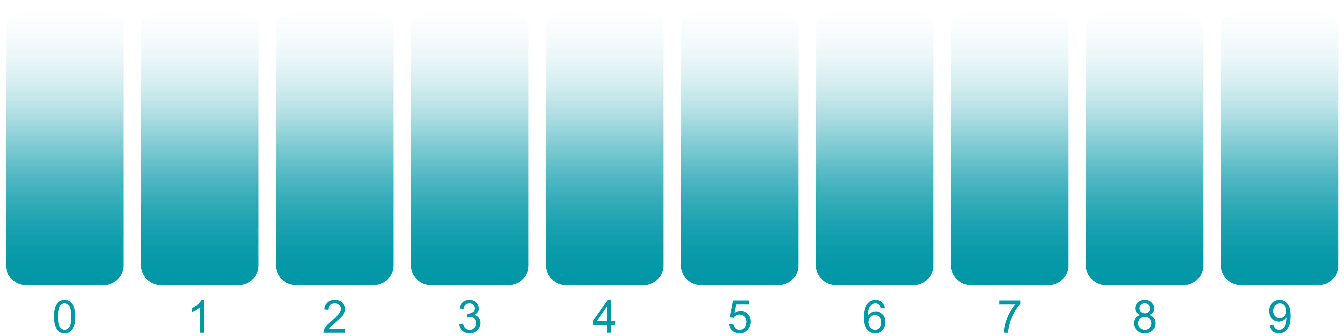
Bucket by last digit of (number * 7) mod 3



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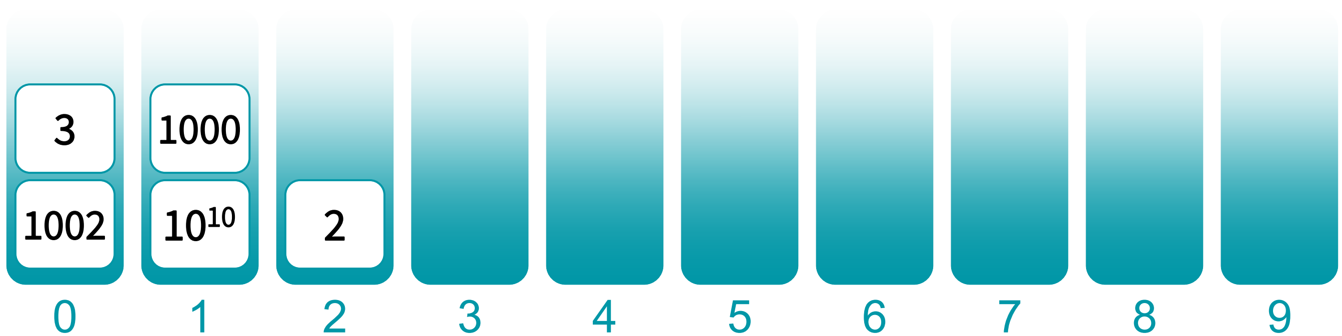

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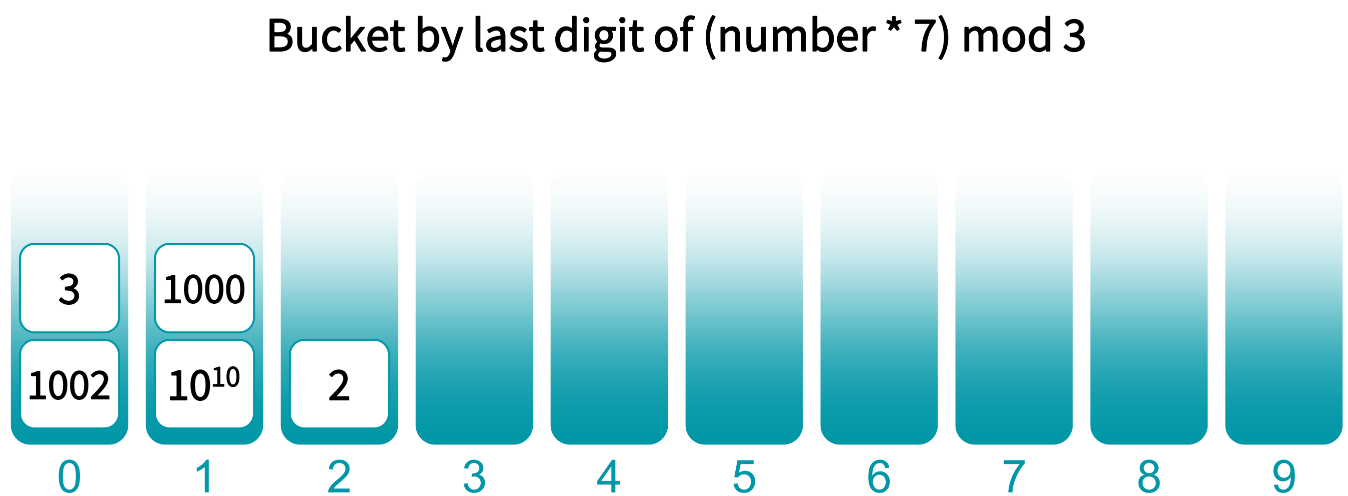

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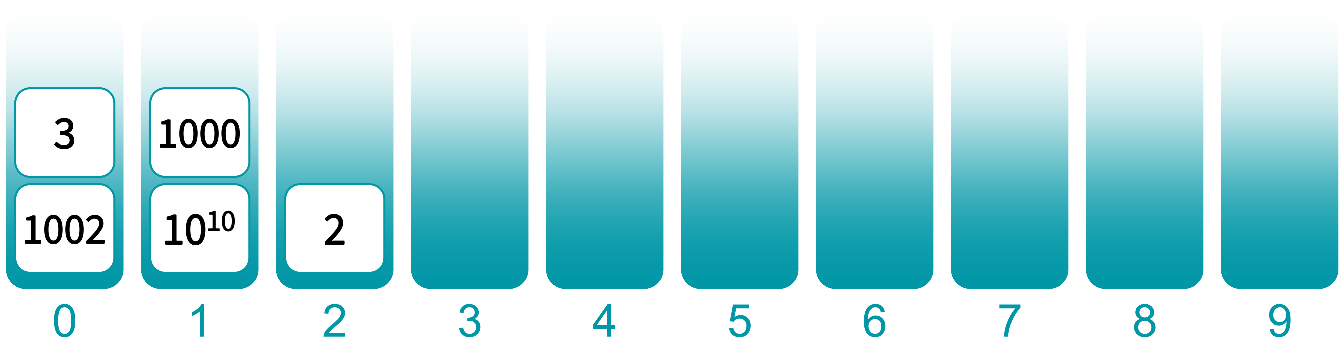

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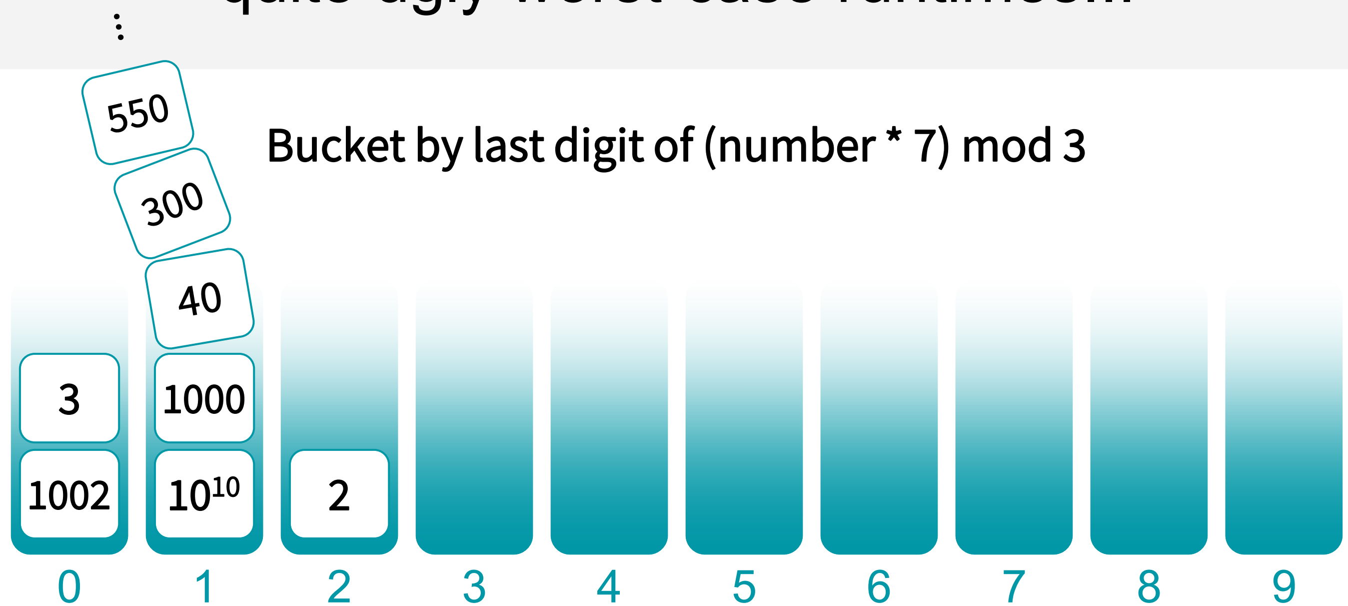

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⋮

550

Bucket by last digit of (number * 7) mod 3

Seems like a bad guy could still thwart us. There are other bucketing schemes we could use, so to reason about them more formally, let's talk about **HASH FUNCTIONS.**

find it...

HASH FUNCTIONS

What are “good” hash functions?

SOME TERMINOLOGY

There exists a universe U of keys, with size M.

Generally, M is *really big*. Examples:

- U = the set of all ASCII strings of length 20. M = 26^{20}
- U = the set of all IPv4 addresses. M = 2^{32}
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Our >> n

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NOTE:

For this lecture, I'm assuming that the # of elements I receive is the same as the # of buckets (both are n). This doesn't have to be the case, but we usually aim for
#buckets = O(# elements that show up)
(otherwise, we're using "too much" space)

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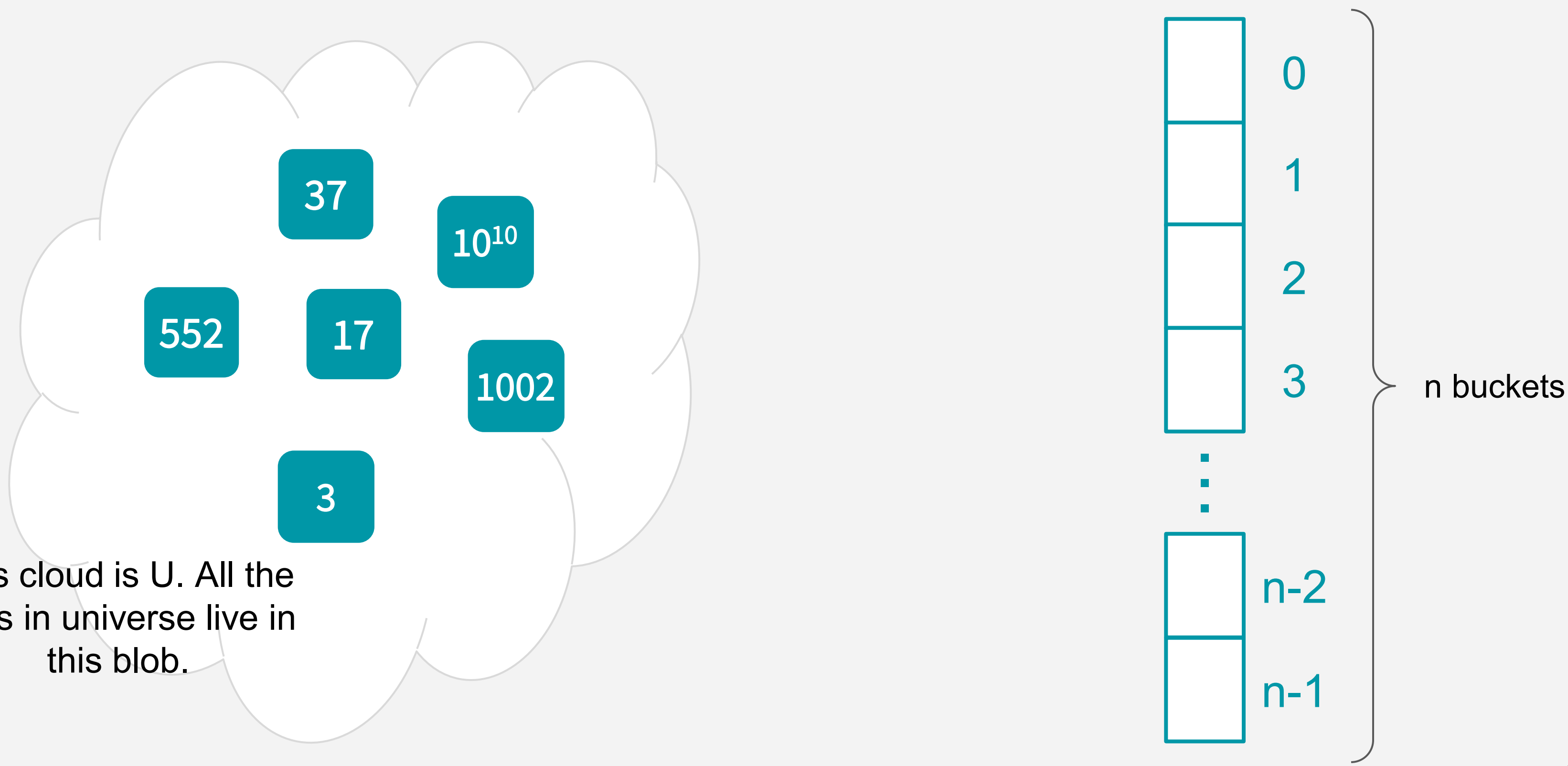

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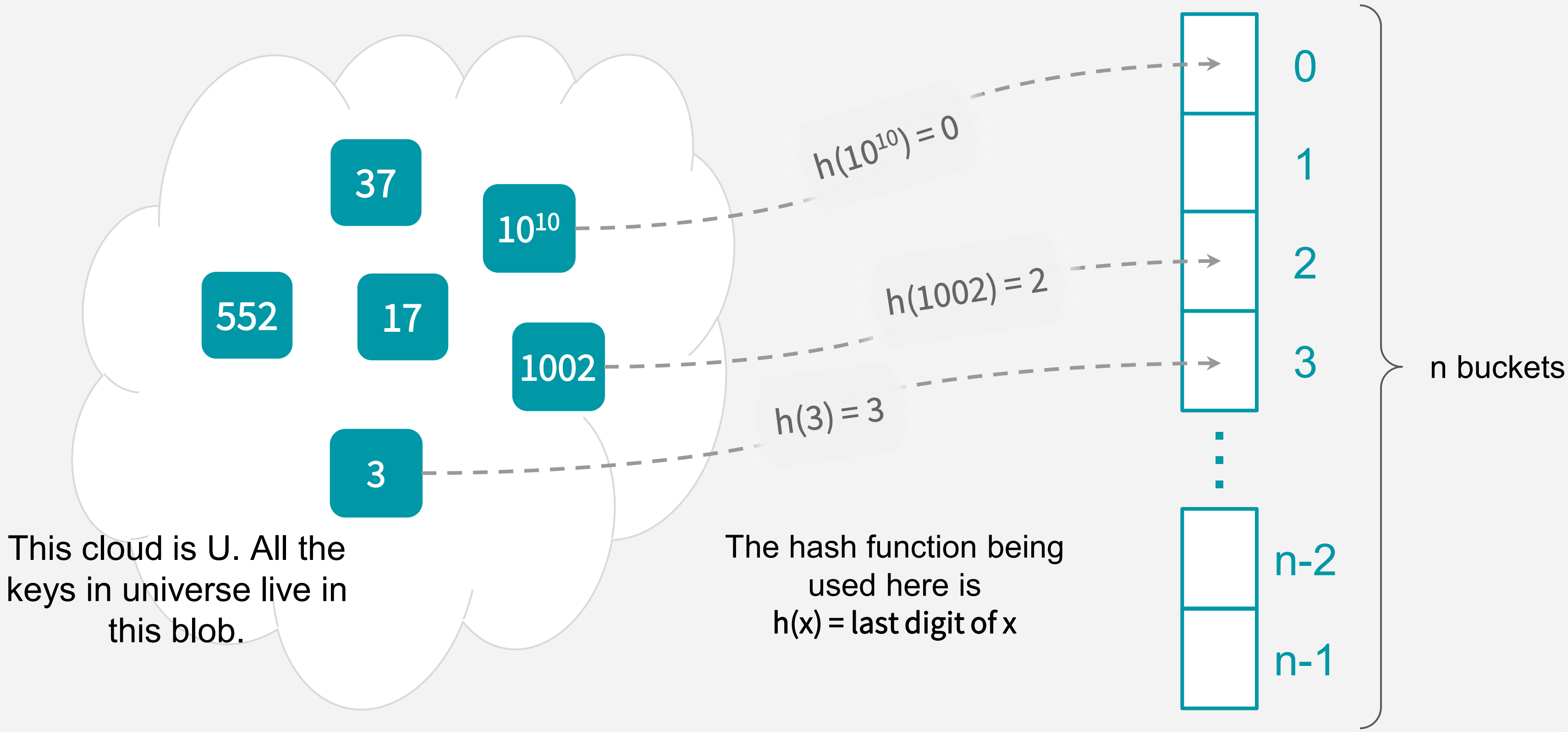

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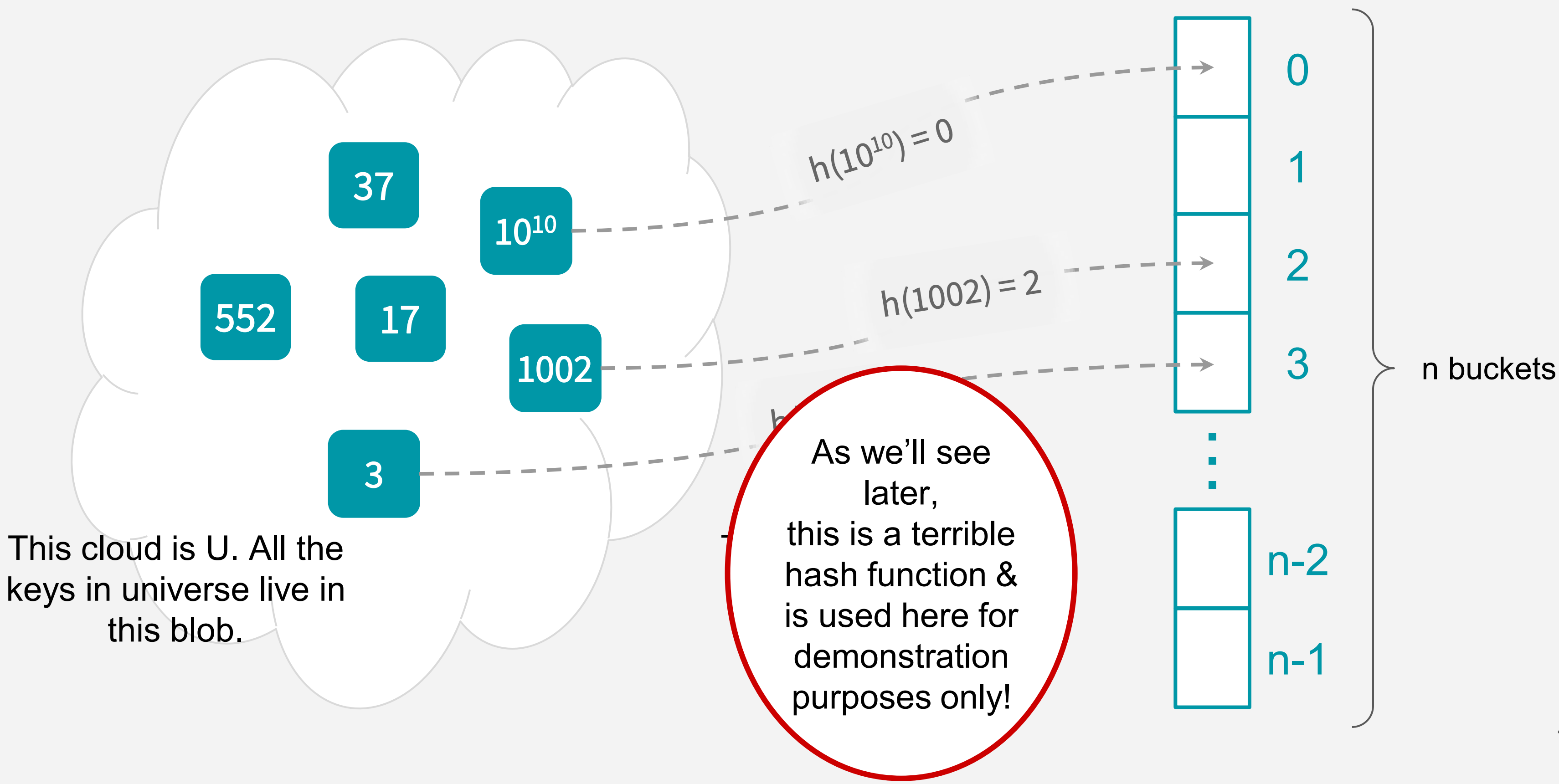

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A hash function tells you where to start looking for an object.

For example, if a particular hash function h has **h(1002) =** 2 , then we say "1002 *hashes* to 2", and we go to bucket 2 to search for 1002, or insert 1002, or delete 1002.

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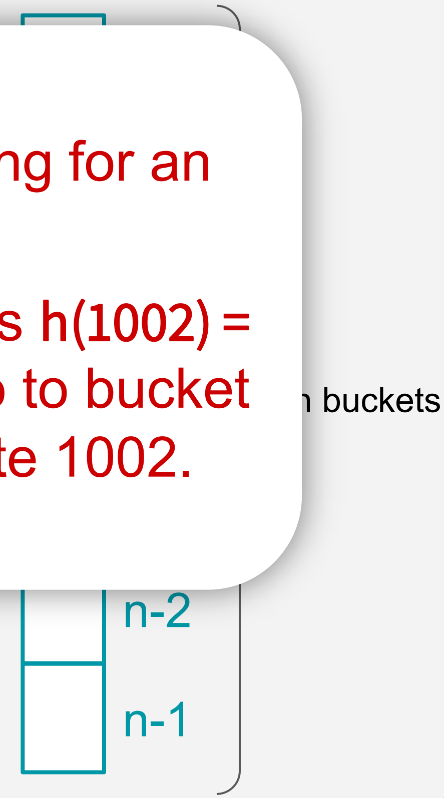
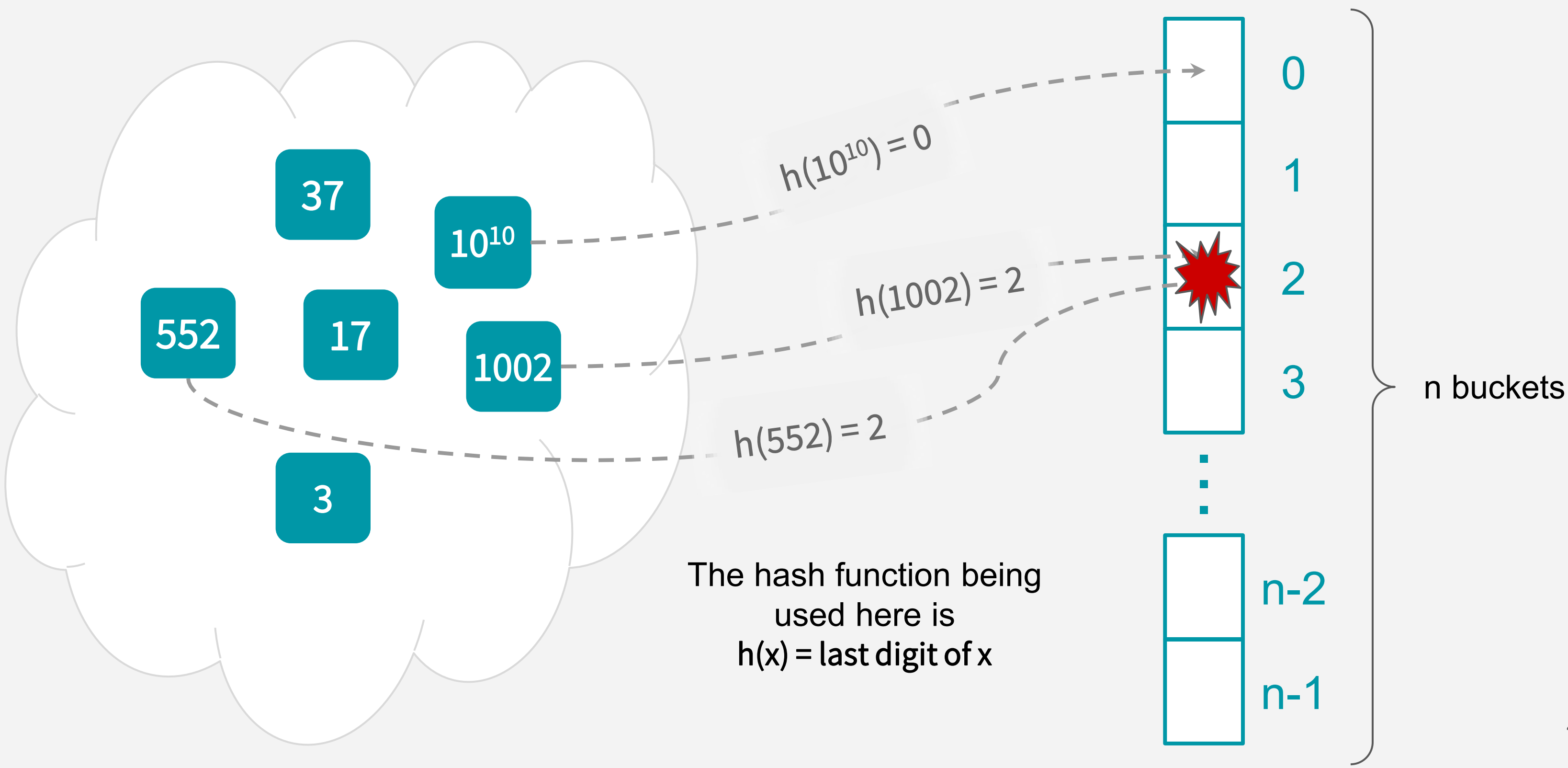


COLLISIONS

Collisions (when a hash function would map 2 different keys to the same bucket) **are inevitable!**

This is because of the *Pigeonhole Principle*.
Since the size of universe U > # of buckets, every hash function (no matter how clever), suffers from at least one collision.



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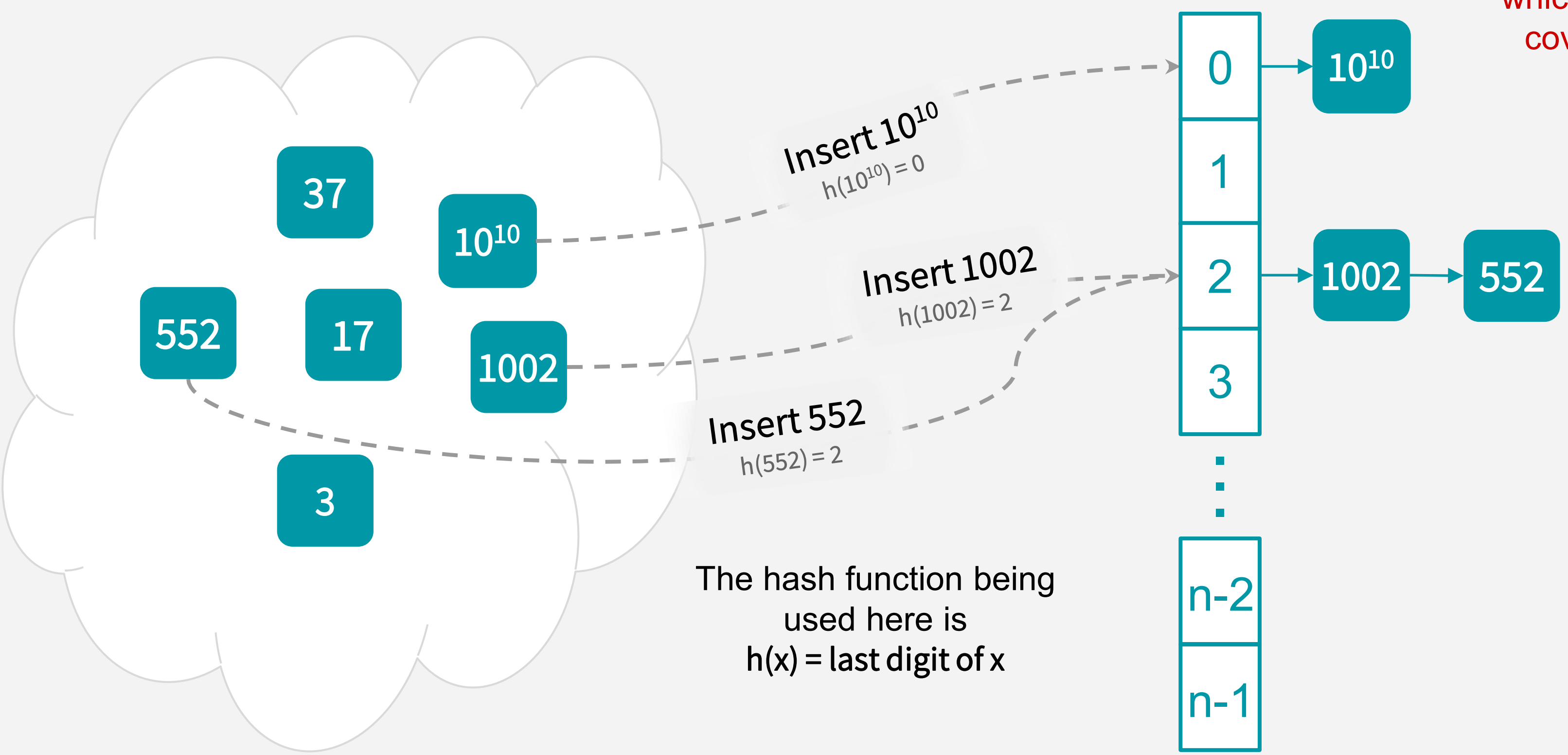

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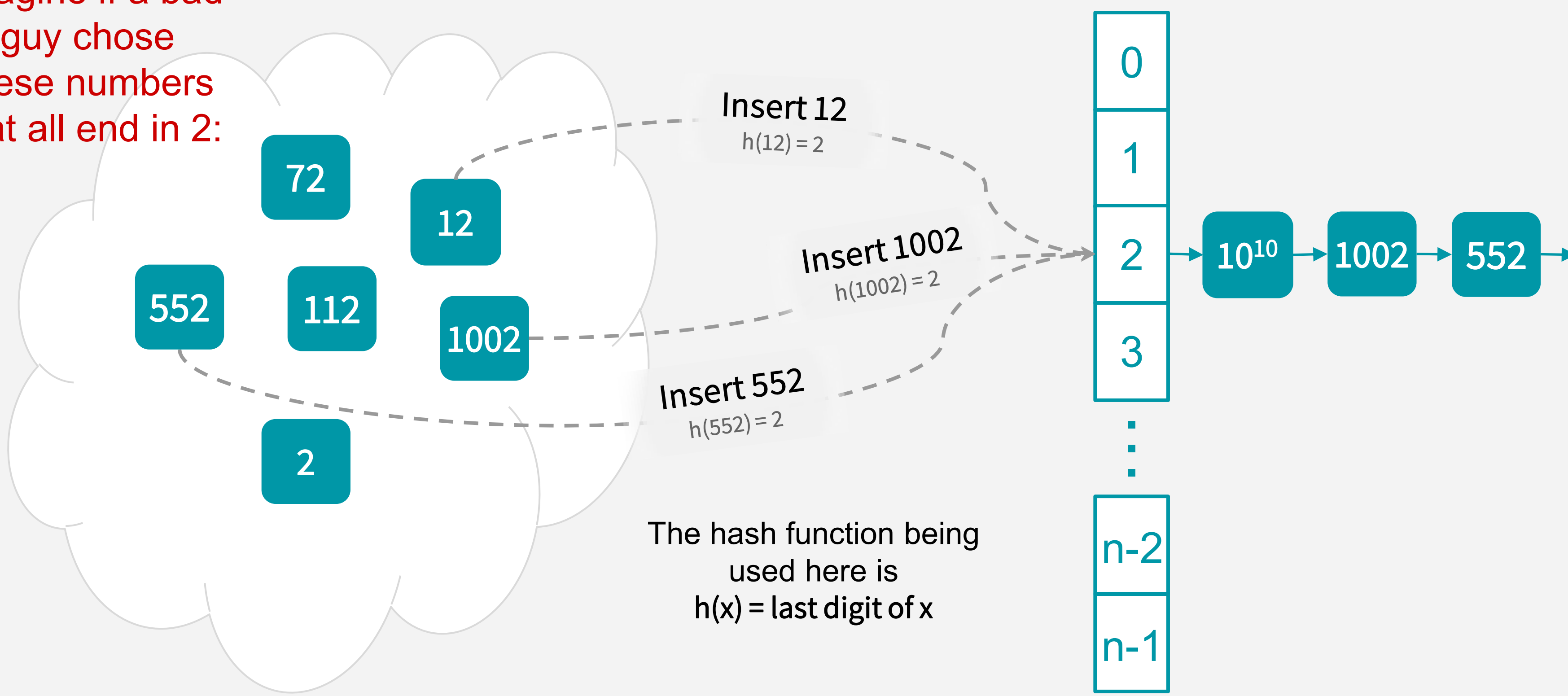
Imagine if a bad guy chose these numbers that all end in 2:



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HASH TABLE GOALS

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OUR GOAL: Design a function $h: U \rightarrow \{1, \ldots, n\}$ so that no matter what n items of U a bad guy chooses & the operations they choose to perform, the buckets will be balanced.

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Maybe there's a way to weaken the adversary...

LET'S BRING IN SOME

RANDOMNESS!

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HASH FUNCTIONS & RANDOMNESS

What it means to weaken the adversary & ways to do it

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Which goal better represents what we want?

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U → {1, …, n}
if we chose a
H and after an
hooses n items
, u_n} to hash,

ny item u_i,
cted # of items in
bucket is O(1)

SUPER IMPORTANT:

The randomness is over the choice of hash function h from a set of hash functions H.

You should *not* think of it as if you've chosen a fixed hash function and are thinking about randomness over possible items the adversary could choose, or randomness over the n possible buckets in your table, or randomness over the M possible items, or anything like that.

Which goal b what we want?

WHAT DOES "GOOD" MEAN?

Design a set
$H = \{h_1, h_2, h_{3,} \ldots, h_k\}$
where $h_i : U \to \{1, \ldots, n\}$
such that if we chose a random **h** in H and after an adversary chooses **n** items to hash,

for any bucket,
its **expected** size is O(1)

Not what we want!

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Not what we want!

Why is this goal not a good one?

Well, this *bad* set of hash functions (which always results in chains of length **n** in a single bucket) would meet this goal:

WHAT DOES "GOOD" MEAN?

Consider these two goals:

Design a set
$H = \{h_1, h_2, h_3, \dots, h_k\}$
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for any item u_i,
the **expected** # of items in u_i's bucket is **O(1)**

We want the one on the right! It tries to control the expected number of collisions (which is what contributes to the linked-list traversal runtime)

WHAT DOES "GOOD" MEAN?

An analogy to explain the difference between the two:

Suppose a university offers 10 classes.
9 classes have only 1 student in them, and
1 class has 491 students.

Using the reasoning on the left, the university might say "Average class size is 50", but in reality,
it should instead report class sizes experienced by the average student (~482).

number of collisions (which is what contributes to the linked-list traversal runtime)

WHAT WE WANT

Design a set **H** = $\{h_1, h_2, h_3, \ldots, h_k\}$ where $h_i : U \rightarrow \{1, \ldots, n\}$, such that if we chose a random **h** in **H** and after an adversary chooses **n** items $\{u_1, u_2, \ldots, u_n\}$ to hash,

for any item u_i,
the expected # of items in u_i's bucket is $O(1)$

Let's see an example of a set of hash functions H that achieves this goal!

H = EXHAUSTIVE SET OF ALL HASH FNs

WHAT WE WANT:

Design a set $H = \{h_1, h_2, h_{3,} \ldots, h_k\}$ where $h_i : U \rightarrow \{1, \ldots, n\}$, such that if we chose a uniformly random **h** in **H** and after an adversary chooses **n** items $\{u_1, u_2, \ldots, u_n\}$ to hash,

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H contains a total of n^M hash functions.

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Here is an example where $U = \{“a”, “b”, “c”\}$ so $M = 3$. Also, we have $n = 2$.

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
“a”	0	0	0	0	1	1	1	1
“b”	0	0	1	1	0	0	1	1
“c”	0	1	0	1	0	1	0	1

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"a"	0	0	0	0	1	1	1	1
"b"	0	0	1	1	0	0	1	1
"c"	0	1	0	1	0	1	0	1

The 0's and 1's represent the binary buckets i.e. h_8 will hash "b" to bucket 1.

H = EXHAUSTIVE SET OF ALL HASH FNs

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$$\mathbb{E}[\text{\# of items in } u_i \text{ 's bucket}] =$$

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This probability is taken over the random choice of hash function!

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$$= P[h(u_i) = h(u_i)] + \sum_{j \neq i} P[h(u_i) = h(u_j)]$$

$$= 1 + \sum_{j \neq i} P[h(u_i) = h(u_j)]$$

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&= 1 + \sum_{j \neq i} P[h(u_i) = h(u_j)] \\
&= 1 + \sum_{j \neq i} \frac{1}{n}
\end{aligned}
$$

How do we know that $P[h(u_i) = h(u_j)] = 1/n$?

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&= 1 + \sum_{j \neq i} \frac{1}{n} \\
&= 1 + \frac{n-1}{n} \leq 2
\end{aligned}
$$

How do we know that $P[h(u_i) = h(u_j)] = 1/n$?

O(1)
This is what we wanted!

H = EXHAUSTIVE SET OF ALL HASH FNs

If the hash function we use is chosen randomly from the exhaustive set of all hash functions, then on expectation, every time we visit a bucket during an operation, there will be O(1) other things that could have also collided there!

(on avg, each student would find O(1) other students in the course!)

$j \neq i$

$$= 1 + \frac{n-1}{n} \leq 2$$

This is what we wanted!

GOOD NEWS!

WHAT WE WANT:

Design a set $H = \{h_1, h_2, h_{3,} \dots, h_k\}$ where $h_i : U \rightarrow \{1, \dots, n\}$, such that if we chose a uniformly random h in H and after an adversary chooses n items $\{u_1, u_2, \dots, u_n\}$ to hash,

for any item u_i,
the **expected** # of items in u_i's bucket is $O(1)$

H = the exhaustive set of all hash functions that map elements in the universe U to buckets 1 to n.
H contains a total of n^M hash functions.

H achieves our goal! If we choose a *uniformly random hash function,* then INSERT/DELETE/SEARCH on any n elements will have **expected runtime of O(1)**.

BAD NEWS

WHAT WE WANT:

Design a set $H = \{h_1, h_2, h_3, \ldots, h_k\}$ where $h_i : U \to \{1, \ldots, n\}$, such that if we chose a uniformly random h in H and after an adversary chooses n items $\{u_1, u_2, \ldots, u_n\}$ to hash,

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How many bits does it take to store a uniformly random hash function?

A lot!

BAD NEWS

How many bits does it take to store a uniformly random hash function?

We'd use a lookup table: one entry per element of U, each storing which bucket to hash that element to.

(M elements) * (log(n) bits to write down a bucket #) = M log n bits

This is HUGE… (& enough to do direct addressing!)

Another way to see this:

There are n^M total hash functions. To uniquely identify every single hash function (each one *is* indeed unique), you'd need n^M different identifiers.

Thus, a single identifier would take up $\log(n^M) = M \log n$ bits.

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This ing!)

How do we fix this size issue?

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Thus, a single identifier would take up $\log(n^M) = M \log n$ bits.

UNIVERSAL HASH FAMILIES

“Good” sets of hash functions that aren’t as large!

WHAT WE WANTED

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H contains a total of n^M hash functions.

$$\mathbb{E}[\text{\# of items in } u_i \text{ 's bucket}] = \sum_{j=1}^{n} P[h(u_i) = h(u_j)]$$

$$= P[h(u_i) = h(u_i)] + \sum_{j \neq i} P[h(u_i) = h(u_j)]$$

$$\boxed{\begin{aligned} &= 1 + \sum_{j \neq i} P[h(u_i) = h(u_j)] \\ &= 1 + \sum_{j \neq i} \frac{1}{n} \end{aligned}}$$

$$= 1 + \frac{n-1}{n} \leq 2$$

The fact that
$P[h(u_i)=h(u_j)] = 1/n$
did all the work
here

O(1)
This is what we wanted!

WHAT WE WANTED

H = the exhaustive set of all hash functions that map elements in the universe U to buckets 1 to n.

$\mathbb{E}[\#$ o

The exhaustive set of all hash functions achieved our goal but was way too big, so let's pick h from a *smaller* hash family where

$$P[h(u_i) = h(u_j)] \leq 1/n$$

$(u_j)]$

The fac

P[h(u_i)=h(

did all th

here

$$\sum_{j \neq i} \quad n$$

$$= 1 + \frac{n-1}{n} \leq 2$$

O(1)

This is what we wanted!

UNIVERSAL HASH FAMILY

A **hash family** is a fancy name for a set of hash functions.

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A hash family **H** is a **universal hash family** if,
when **h** is chosen uniformly at random from **H**,

$$\text{for all } u_i, u_j \in U \text{ with } u_i \neq u_j,$$

$$P_{h \in H}\left[h(u_i) = h(u_j)\right] \leq \frac{1}{n}$$

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Then if we randomly choose **h** from a universal hash family **H**,
we'll be guaranteed that:

E[# of items in u_i's bucket] ≤ 2 = O(1)

(FLASHBACK OF THE MATH)

A hash family H is a universal hash family if,
when h is chosen uniformly at random from H,

for all $u_i, u_j \in U$ **with** $u_i \neq u_j$,

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$$\mathbb{E}[\text{\# of items in } u_i \text{ 's bucket}] = \sum_{j=1}^{n} P[h(u_i) = h(u_j)]$$

$$= P[h(u_i) = h(u_i)] + \sum_{j \neq i} P[h(u_i) = h(u_j)]$$

$$= 1 + \sum_{j \neq i} P[h(u_i) = h(u_j)]$$

This inequality is now what a universal hash family guarantees!

$$\leq 1 + \sum_{j \neq i} \frac{1}{n}$$

$$= 1 + \frac{n-1}{n} \leq 2$$

O(1)

This is what we wanted!

A SMALL UNIVERSAL HASH FAMILY?

Our H = exhaustive set of all hash functions is a universal hash family!

It is a universal hash family, but unfortunately, as we saw earlier, this H is very very large. Are there smaller ones universal hash families?

A NON-EXAMPLE

$H = \{h_0, h_1\}$ where

h_0 = MOST_SIGNIFICANT_DIGIT
h_1 = LEAST_SIGNIFICANT_DIGIT

Why is this not a universal hash family?

A NON-EXAMPLE

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$$P_{h \in H}\left[h(153) = h(173)\right] = 1 > \frac{1}{n}$$

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Why is this not a universal hash family?

$$P_{h \in H}\left[h(153) = h(173)\right] = 1 > \frac{1}{n}$$

There's a ½ probability of choosing h_0, and $h_0(153) = h_0(173)$ = bucket 1

There's a ½ probability of choosing h_1, and $h_1(153) = h_1(173)$ = bucket 3

Probability that a randomly chosen h from H collides 153 & 173 is 1!

AN EXAMPLE

Here is one of the more well-studied universal hash families:

Pick a prime $p \geq M$

Define $h_{a,b}(x) = ((ax + b) \bmod p) \bmod n$

$H = \{ h_{a,b} : a \in \{1, \ldots, p - 1\}, b \in \{0, \ldots, p - 1\} \}$

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Example: Suppose n = 3, and p = 5. Here's $h_{2,4}$:

$h_{2,4}(1) = ((2*1 + 4) \bmod 5) \bmod 3 = (6 \bmod 5) \bmod 3 = 1 \bmod 3 = 1$

$h_{2,4}(4) = ((2*4 + 4) \bmod 5) \bmod 3 = (12 \bmod 5) \bmod 3 = 2 \bmod 3 = 2$

$h_{2,4}(3) = ((2*3 + 4) \bmod 5) \bmod 3 = (6 \bmod 5) \bmod 3 = 1 \bmod 3 = 1$

AN EXAMPLE

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Pick a prime $p \geq M$

Define $h_{a,b}(x) = ((ax + b) \bmod p) \bmod n$

$$H = \{\, h_{a,b} : a \in \{1, \ldots, p - 1\}, b \in \{0, \ldots, p - 1\} \,\}$$

To draw a hash function h from H:

Pick a random a in $\{1, \ldots, p - 1\}$. & Pick a random b in $\{0, \ldots, p - 1\}$.

AN EXAMPLE

Here is one of the more well-studied universal hash families:

To store $h_{a,b}$, you just need to store two numbers: **a** and **b**!

Since **a** and **b** are at most p-1, we need **~2·log(p) bits.**

p is a prime that's close-ish to M, so this means the space needed =

O(log M)

This is so much better than O(M log n)!

Pick a random a in {1, …, p - 1}.

&

Pick a random b in {0, …, p - 1}.

AN EXAMPLE

Claim: This H is a universal hash family!

The proof is a bit complicated, and relies on number theory. See CLRS (Theorem 11.5) for details if you're curious, but **YOU ARE NOT RESPONSIBLE** for the proof in this class.

What you should know:

There exists a small universal hash family! A hash function from this universal hash family is quick to compute, lightweight to store, and relies on number theory to achieve our expected O(1) operation costs!

HASH TABLES

Putting everything together, what's the scheme?

THE WHOLE SCHEME

You choose your set of hash functions H, a universal hash family like H = mod p mod n.

H

THE WHOLE SCHEME

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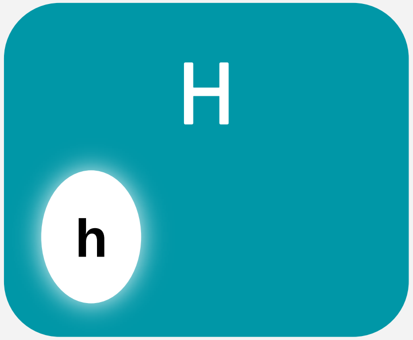

When the client initializes a hash table, randomly pick a hash function **h** from **H** to use in the hash table to hash the items.

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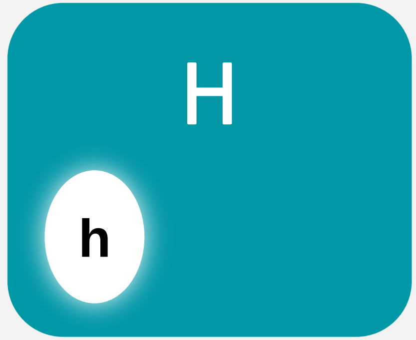

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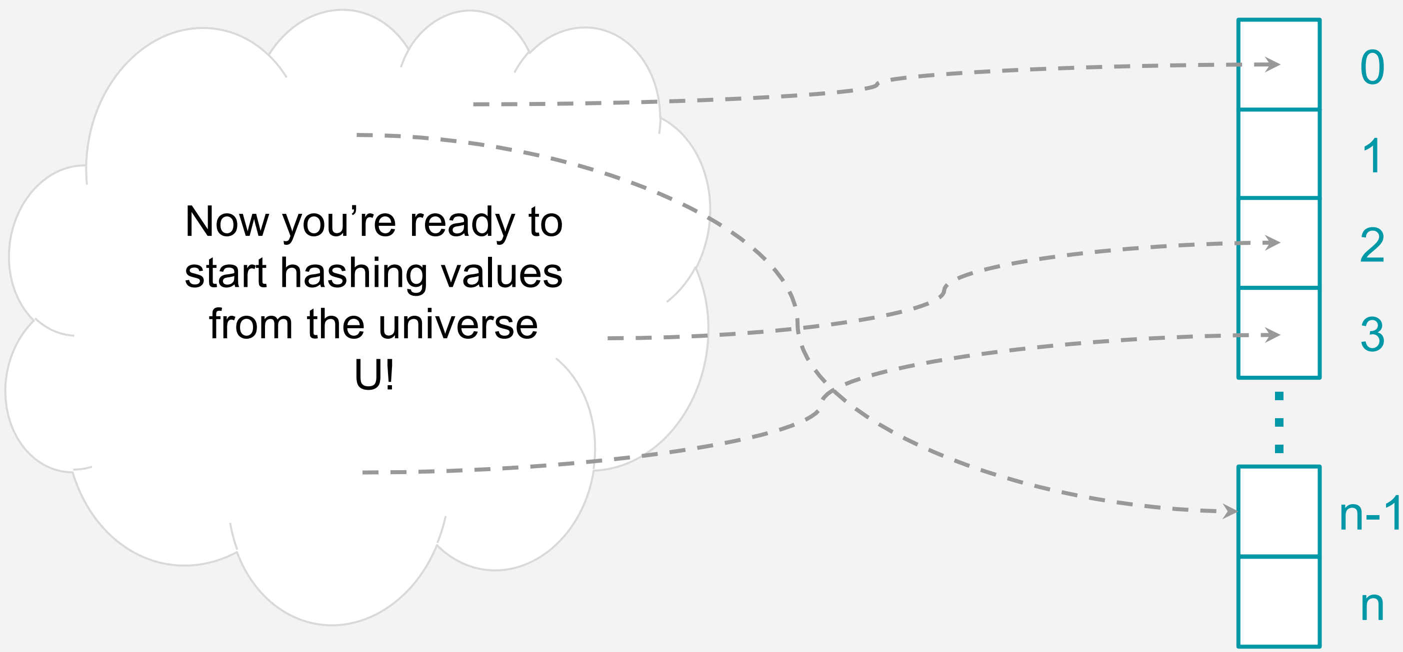

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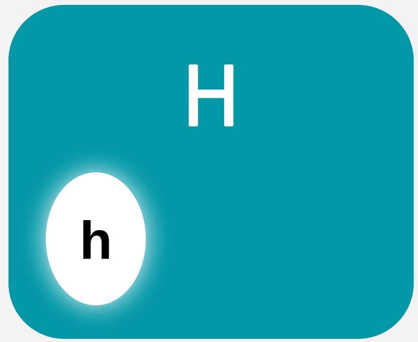

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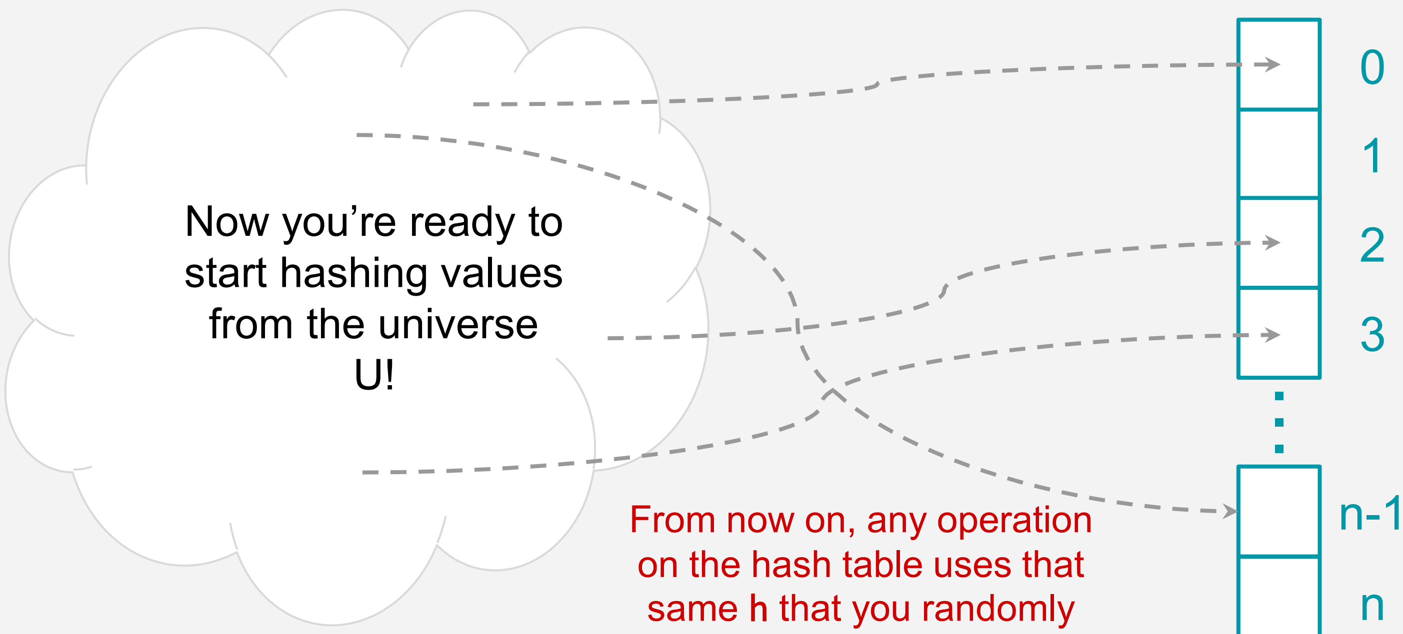

We can now expect that these buckets will be pretty balanced

From now on, any operation on the hash table uses that same **h** that you randomly selected from **H**

HASH TABLE MOTIVATION

OPERATION	SORTED ARRAY	UNSORTED LINKED LIST	HASH TABLES (WORST-CASE)	HASH TABLES (EXPECTED) *
SEARCH	O(log(n))	O(n)	O(n)	O(1)
DELETE	O(n)	O(n)	O(n)	O(1)
INSERT	O(n)	O(1)	O(1)	O(1)

* Assuming we implement it cleverly with a “good” hash function

RECAP OF HASHING

- We want a data structure that supports ***fast*** INSERT/SEARCH/DELETE
- We considered this setting:
  - Come up with a set of hash functions (a hash family)
  - Bad guy chooses any n items from U & some series of operations
  - You randomly choose a hash function from your set to use
- **UNIVERSAL HASH FAMILIES**: a "GOOD" hash family
  - H = exhaustive set of all hash functions
    - Good because it is a universal hash family
    - Bad because you need so much space!
  - H = {{ $h_{a,b}$: a ∈ {1, …, p - 1}, b ∈ {0, …, p - 1} }} where $h_{a,b}(x)$ = ((ax + b) mod p) mod n
    - Good because it is still a universal hash family!!! (& quick to compute)
    - Good because storing an $h_{a,b}$ doesn't take up much space!!!

A hash family H is a universal hash family if, when h is chosen uniformly at random from H,

for all $u_i, u_j \in U$ with $u_i \neq u_j$,

$$P_{h \in H}\left[h(u_i) = h(u_j)\right] \leq \frac{1}{n}$$

RECAP OF HASHING

- We want a data structure that supports *fast* INSERT/SEARCH/DELETE
- W
-

f,
m

CONCLUSION:

We can build a hash table that supports
INSERT/DELETE/SEARCH in O(1) expected time.

Requires O(n log M) bits of space:

- O(n) buckets
- O(n) items with log(M) bits per item
- O(log(M)) to store the hash function

Acknowledgement

- Stanford University