Advanced Data Structures and Algorithms

More dynamic programming!

Knapsack

Rest of Dynamic Programming

- Examples of dynamic programming:
  1. Longest common subsequence
  2. Knapsack problem
     - Two versions!

Knapsack Problem

Knapsack Problem

- We have n items with weights and values:

Knapsack Problem

- We have n items with weights and values:

Item:

Weight:

Value:

Knapsack Problem

- We have n items with weights and values:

Item: 

Weight: 6

Value: 20

Knapsack Problem

- We have n items with weights and values:

Item:		
Weight:	6	2
Value:	20	8

Knapsack Problem

- We have n items with weights and values:

Item:			
Weight:	6	2	4
Value:	20	8	14

Knapsack Problem

- We have n items with weights and values:

Item:				
Weight:	6	2	4	3
Value:	20	8	14	13

Knapsack Problem

- We have n items with weights and values:

Mexican Taco

Item:				
Weight:	6	2	4	3
Value:	20	8	14	13

Knapsack Problem

- We have n items with weights and values:

Item:					
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

Knapsack Problem

- We have n items with weights and values:

Item:					
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- And we have a knapsack:

Knapsack Problem

- We have n items with weights and values:

Item:					
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- And we have a knapsack:
  - it can only carry so much weight:

Capacity: 10

Capacity: 10

Item:	🐢	💡	🍉	🌮	🚒
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

Capacity: 10

Item:	🐢	💡	🍉	🌮	🚒
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
  - What's the most valuable way to fill the knapsack?

Capacity: 10

Item:	🐢	💡	🍉	🌮	🚒
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
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🌮 🌮 💡 💡

Total weight: 10
Total value: 42

Capacity: 10

Item:	🐢	💡	🍉	🌮	🚒
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
  - What's the most valuable way to fill the knapsack?

🌮 🌮 💡 💡

Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What's the most valuable way to fill the knapsack?

Capacity: 10

Item:	🐢	💡	🍉	🌮	🚒
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
  - What's the most valuable way to fill the knapsack?

🌮 🌮 💡 💡

Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What's the most valuable way to fill the knapsack?

💡 🍉 🌮

Total weight: 9
Total value: 35

Capacity: 10

Item:	🐢	💡	🍉	🌮	🚒
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
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🌮 🌮 💡 💡

Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What's the most valuable way to fill the knapsack?

💡 🍉 🌮

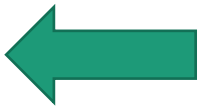
Total weight: 9
Total value: 35

Some notation

Item:				…	
Weight:	w_1	w_2	w_3	…	w_n
Value:	v_1	v_2	v_3		v_n

Capacity: W

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure. 
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
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Optimal substructure

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.

Optimal substructure

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First solve the problem for small knapsacks

Optimal substructure

- Sub-problems:
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First solve the problem for small knapsacks

Then larger knapsacks

Optimal substructure

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.

First solve the problem for small knapsacks

Then larger knapsacks

Then larger knapsacks

Optimal substructure

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.
  - K[x] = value you can fit in a knapsack of capacity x

First solve the problem for small knapsacks

Then larger knapsacks

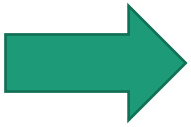
Then larger knapsacks

Optimal substructure

item i

- Suppose this is an optimal solution for capacity x:

Weight w_i
Value v_i



Capacity x
Value V

Optimal substructure

item i

- Suppose this is an optimal solution for capacity x:

Say that the optimal solution contains at least one copy of item i.

Weight w_i
Value v_i

Capacity x
Value V

Optimal substructure

item i

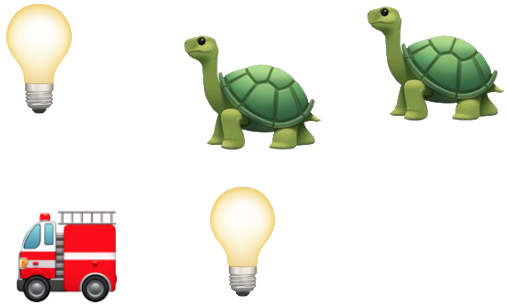
- Suppose this is an optimal solution for capacity x:

Say that the optimal solution contains at least one copy of item i.

Weight w_i
Value v_i

Capacity x
Value V

- Then this optimal for capacity x - w_i:



Capacity x – w_i
Value V - v_i

Optimal substructure

item i

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Say that the optimal solution contains at least one copy of item i.

Weight w_i
Value v_i

Capacity x
Value V

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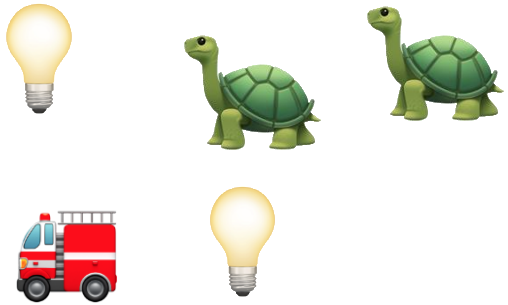
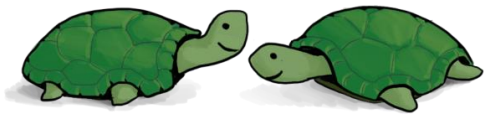


Capacity x – w_i
Value V - v_i

Why?



Optimal substructure

item i

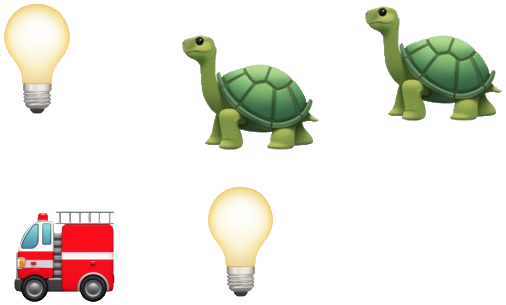
- Suppose this is an optimal solution for capacity x:

Say that the optimal solution contains at least one copy of item i.

Weight w_i
Value v_i

Capacity x
Value V

- Then this optimal for capacity x - w_i:



If I could do better than the second solution, then adding a turtle to that improvement would improve the first solution.

Capacity x – w_i
Value V - v_i

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Recursive relationship

Recursive relationship

- Let K[x] be the optimal value for capacity x.

$$K[x] = \max_i \{ \quad + \quad \}$$

The maximum is over all i so that $w_i \leq x$.

Optimal way to fill the smaller knapsack

The value of item i.

Recursive relationship

- Let K[x] be the optimal value for capacity x.

$K[x] = \max_i \{$  $+$  $\}$

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Optimal way to fill the smaller knapsack

The value of item i.

$$K[x] = \max_i \{ K[x - w_i] + v_i \}$$

Recursive relationship

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$K[x] = \max_i \{$  $+$  $\}$

The maximum is over all i so that $w_i \leq x$.

Optimal way to fill the smaller knapsack

The value of item i.

$$K[x] = \max_i \{ K[x - w_i] + v_i \}$$

- (And K[x] = 0 if the maximum is empty).
  - That is, if there are no i so that $w_i \leq x$

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Let's write a bottom-up DP algorithm

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
  - **return** K[W]

$K[x] = \max_i \{$  +  $\}$

$= \max_i \{ K[x - w_i] + v_i \}$

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  - **return** K[W]

Running time: O(nW)

K[x] = max$_i$ {  +  }

= max$_i$ { K[x – w$_i$] + v$_i$ }

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Why does this work?

Let's write a bottom-up DP algorithm

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Running time: O(nW)

K[x] = max$_i$ {  +  }

= max$_i$ { K[x – w$_i$] + v$_i$ }

Why does this work?

Because our recursive relationship makes sense.

Can we do better?

Can we do better?

- Writing down W takes log(W) bits.
- Writing down all n weights takes at most nlog(W) bits.
- Input size: nlog(W).
  - Maybe we could have an algorithm that runs in time O(nlog(W)) instead of O(nW)?
  - Or even O($n^{1000000}$ $\log^{1000000}(W)$)?

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- Open problem!

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  - Or even O($n^{1000000}$ $\log^{1000000}(W)$)?

- Open problem!
  - (But probably the answer is **no**…otherwise P = NP)

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Let's write a bottom-up DP algorithm

- UnboundedKnapsack(W, n, weights, values):
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K[x] = max$_i$ {  +  }

= max$_i$ { K[x – w$_i$] + v$_i$ }

Let's write a bottom-up DP algorithm

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
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      - **if** $w_i \leq x$:
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        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

K[x] = $\max_i$ {  +  }

= $\max_i$ { K[x – w_i] + v_i }

Example

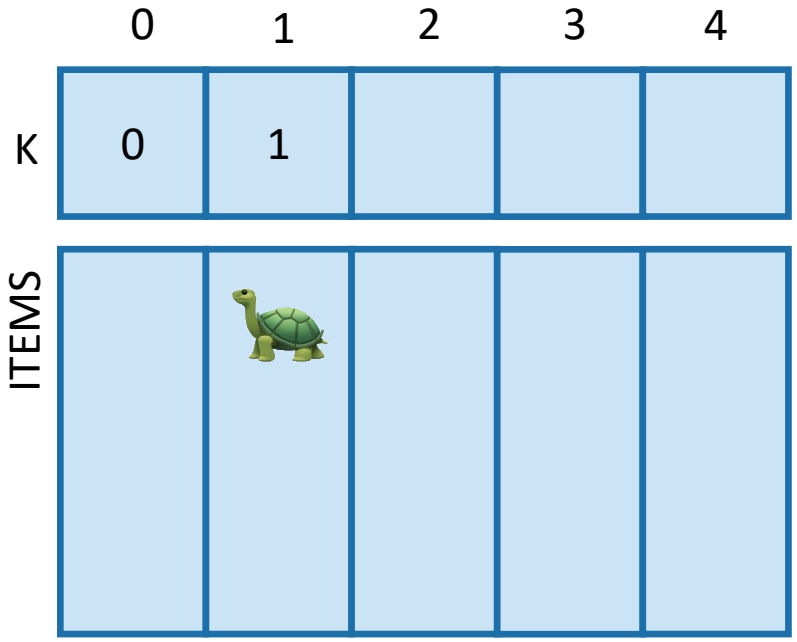
	0	1	2	3	4
K	0				
ITEMS					

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example



ITEMS[1] = ITEMS[0] + 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	2		
ITEMS			 		

ITEMS[2] = ITEMS[1] + 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
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  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4		
ITEMS		🐢	💡		

ITEMS[2] = ITEMS[0] + 💡

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
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        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4	5	
ITEMS		🐢	💡	💡 🐢	

ITEMS[3] = ITEMS[2] + 🐢

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
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        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4	6	
ITEMS		🐢	💡	🍉	

ITEMS[3] = ITEMS[0] + 🍉

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4	6	7
ITEMS					 

ITEMS[4] = ITEMS[3] + 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

- UnboundedKnapsack(W, n, weights, values):
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  - **return** ITEMS[W]

	0	1	2	3	4
K	0	1	4	6	8
ITEMS		🐢	💡	🍉	💡 💡

ITEMS[4] = ITEMS[2] + 💡

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

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(Pass)

What have we learned?

- We can solve unbounded knapsack in time O(nW).
  - If there are n items and our knapsack has capacity W.

- We again went through the steps to create DP solution:
  - We kept a one-dimensional table, creating smaller problems by making the knapsack smaller.

Capacity: 10

Item:	🐢	💡	🍉	🌮	🚒
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
  - What's the most valuable way to fill the knapsack?

🌮 🌮 💡 💡

Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What's the most valuable way to fill the knapsack?

💡 🍉 🌮

Total weight: 9
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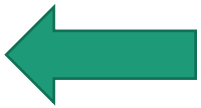
- **Step 1:** Identify optimal substructure. 
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Optimal substructure: try 1

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.

First solve the problem for small knapsacks

Then larger knapsacks

Then larger knapsacks

This won’t quite work…

- We are only allowed **one copy of each item**.
- The sub-problem needs to “know” what items we’ve used and what we haven’t.

Optimal substructure: try 2

- Sub-problems:
  - 0/1 Knapsack with fewer items.

Optimal substructure: try 2

- Sub-problems:
  - 0/1 Knapsack with fewer items.

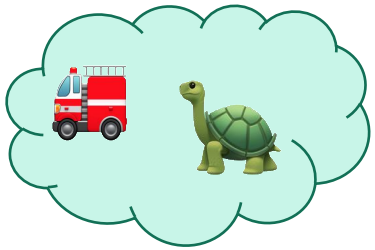
First solve the problem with few items

Optimal substructure: try 2

- Sub-problems:
  - 0/1 Knapsack with fewer items.

First solve the problem with few items

Then more items

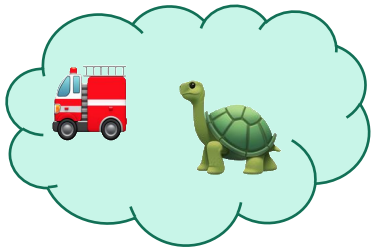


Optimal substructure: try 2

- Sub-problems:
  - 0/1 Knapsack with fewer items.

First solve the problem with few items

Then more items



Then yet more items

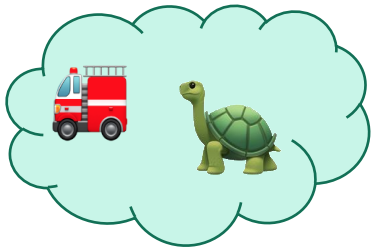
Optimal substructure: try 2

- Sub-problems:
  - 0/1 Knapsack with fewer items.

First solve the problem with few items

We'll still increase the size of the knapsacks.

Then more items



Then yet more items

Optimal substructure: try 2

- Sub-problems:
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First solve the problem with few items

We'll still increase the size of the knapsacks.

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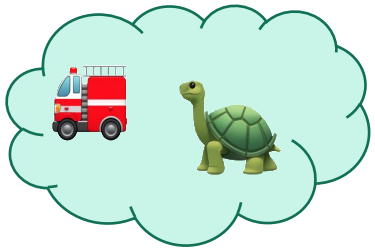
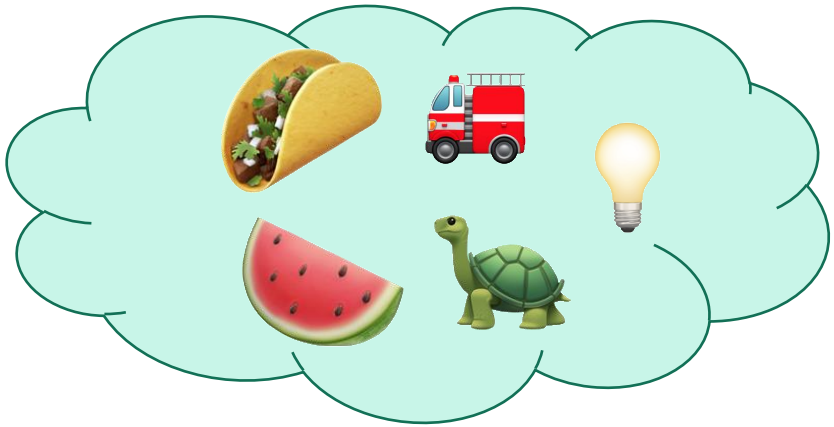


Then yet more items

(We'll keep a two-dimensional table).

Our sub-problems:

- Indexed by x and j



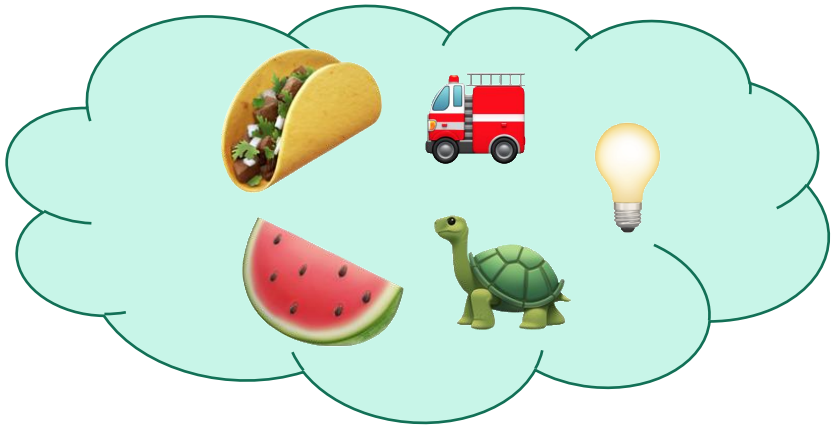
First j items

Capacity x

K[x,j] = optimal solution for a knapsack of size x using only the first j items.

Relationship between sub-problems

- Want to write K[x,j] in terms of smaller sub-problems.



First j items

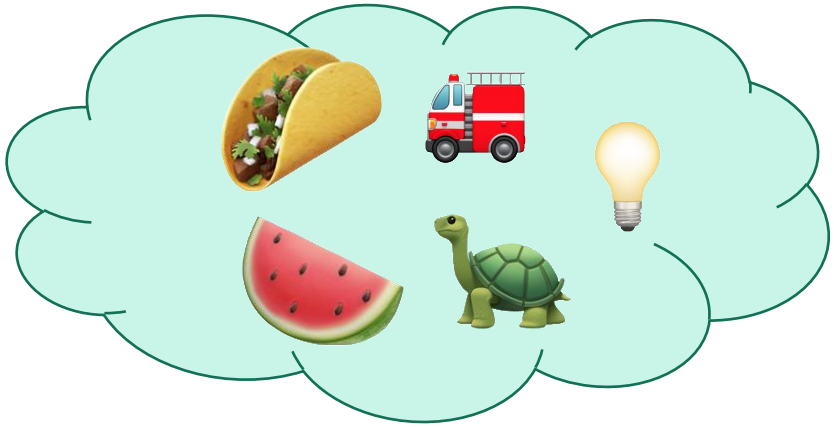
Capacity x

K[x,j] = optimal solution for a knapsack of size x using only the first j items.

Two cases

item j

- **Case 1**: Optimal solution for j items does not use item j.
- **Case 2**: Optimal solution for j items does use item j.



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Capacity x

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Two cases

item j

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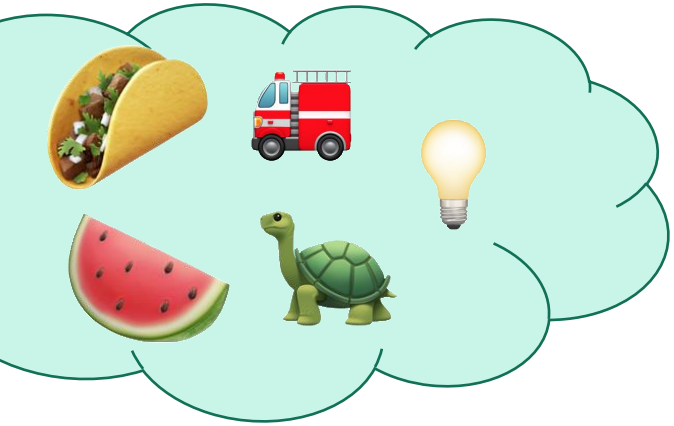
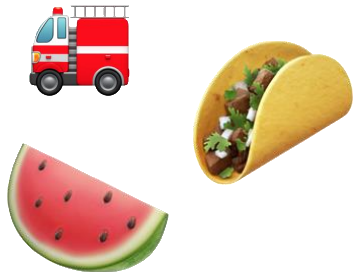


First j items

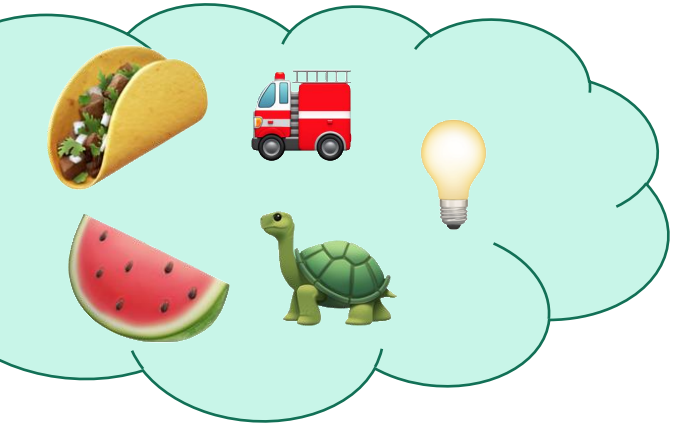


Capacity x
Value V
Use only the first j items

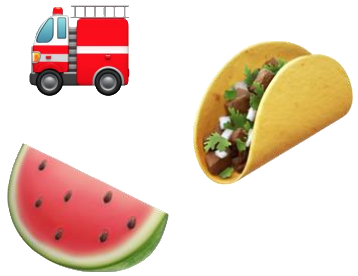
Two cases

item j

- **Case 1**: Optimal solution for j items does not use item j.

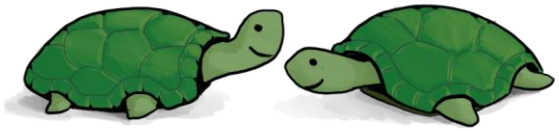


First j items



Capacity x
Value V
Use only the first j items

What lower-indexed problem should we solve to solve this problem?



Two cases

item j

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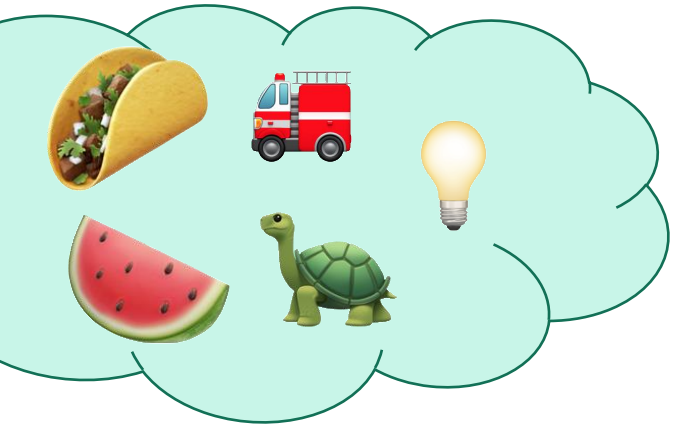
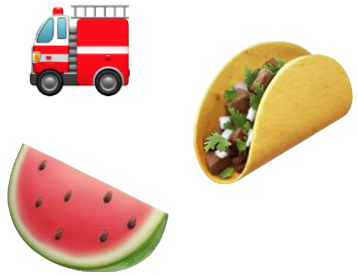
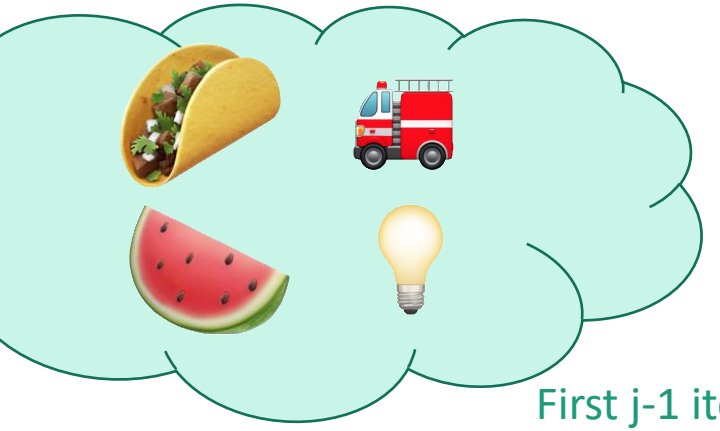


First j items

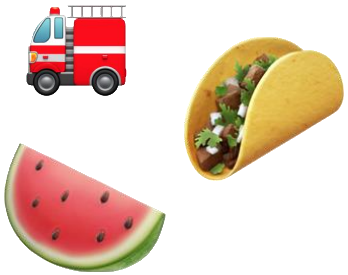


Capacity x
Value V
Use only the first j items

- Then this is an optimal solution for j-1 items:



First j-1 items



Capacity x
Value V
Use only the first j-1 items.

Two cases

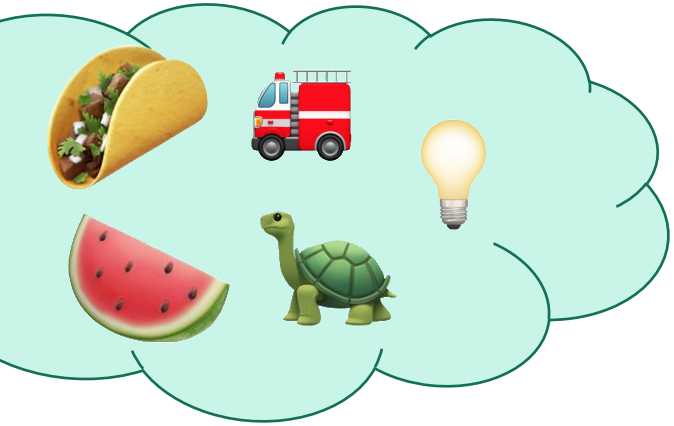
item j

- **Case 2**: Optimal solution for j items uses item j.

Two cases

item j

- **Case 2**: Optimal solution for j items uses item j.



First j items

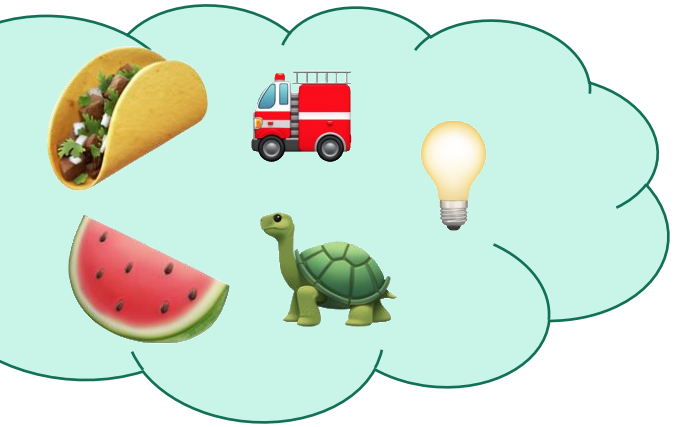
Weight w_j
Value v_j

Capacity x
Value V
Use only the first j items

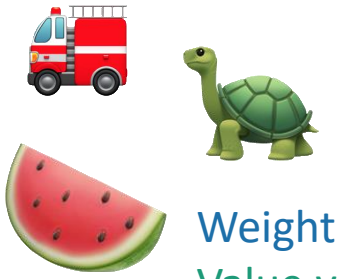
Two cases

item j

- **Case 2**: Optimal solution for j items uses item j.



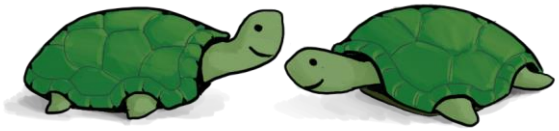
First j items



Weight w_j
Value v_j

Capacity x
Value V
Use only the first j items

What lower-indexed problem should we solve to solve this problem?



Two cases

item j

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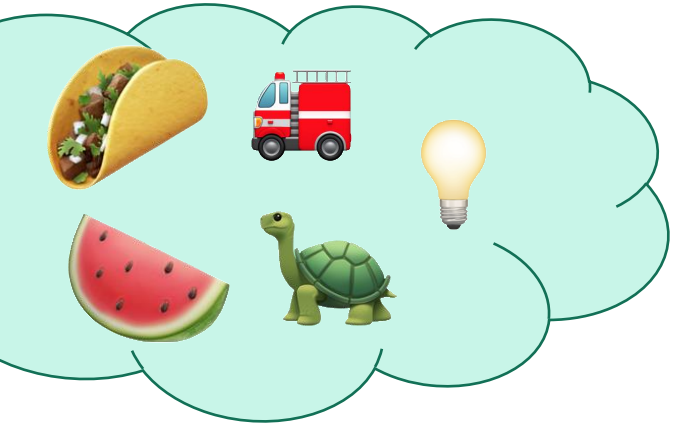
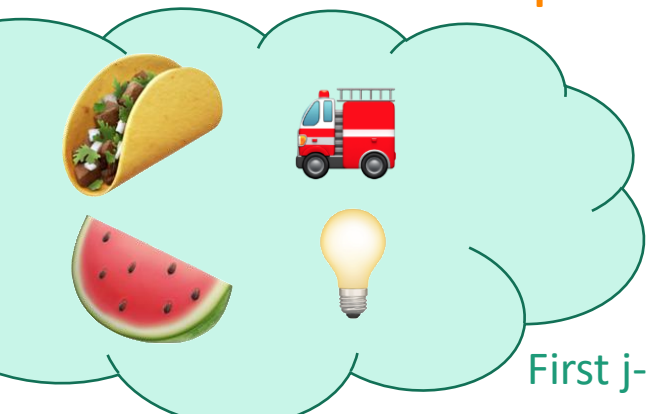


First j items

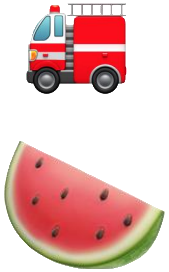
Weight w_j
Value v_j

Capacity x
Value V
Use only the first j items

- Then this is an optimal solution for j-1 items and a smaller knapsack:



First j-1 items



Capacity $x - w_j$
Value $V - v_j$
Use only the first j-1 items.

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Recursive relationship

- Let K[x,j] be the optimal value for:
  - capacity x,
  - with j items.

$$K[x,j] = \max\{ K[x, j-1] , K[x - w_j, j-1] + v_j \}$$

Case 1 Case 2

- (And K[x,0] = 0 and K[0,j] = 0).

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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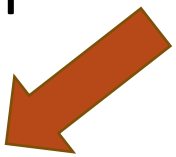

Bottom-up DP algorithm

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1] Case 1
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ } Case 2
  - **return** K[W,n]

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Running time O(nW)

Example

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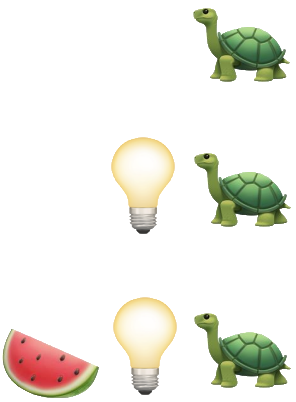
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0			
j=2	0			
j=3	0			



current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
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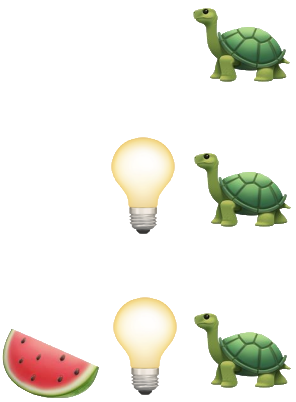
	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	0		
j=2	0			
j=3	0			



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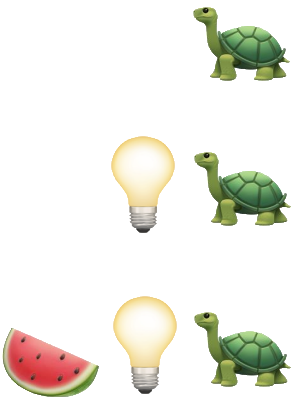
	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1		
j=2	0			
j=3	0			



current entry

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	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1		
j=2	0	1		
j=3	0			

current entry

relevant previous entry

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Capacity: 3

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	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1		
j=2	0	1		
j=3	0	1		

current entry

relevant previous entry

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  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	0	
j=2	0	1		
j=3	0	1		

current entry

relevant previous entry

Item:			
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Value:	1	4	6

Capacity: 3

Example

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	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1		
j=3	0	1		

current entry

relevant previous entry

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	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1	1	
j=3	0	1		





current entry

relevant previous entry

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  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1	4	
j=3	0	1		






current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
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	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1	4	
j=3	0	1	4	

current entry

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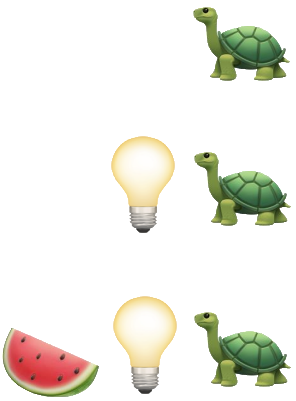
j=0	0	0	0	0
j=1	0	1	1	0
j=2	0	1	4	
j=3	0	1	4	



current entry

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j=1	0	1	1	1
j=2	0	1	4	
j=3	0	1	4	





current entry

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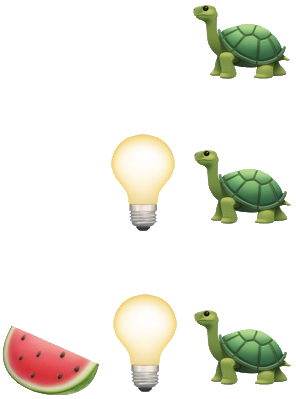
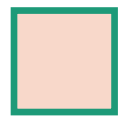
	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	5
j=3	0	1	4	

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relevant previous entry

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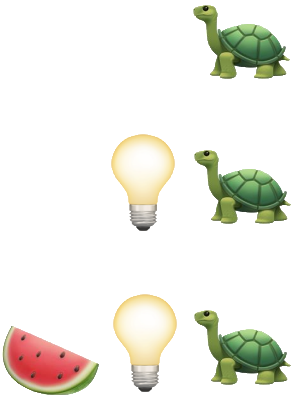
j=2	0	1	4	5
j=3	0	1	4	5



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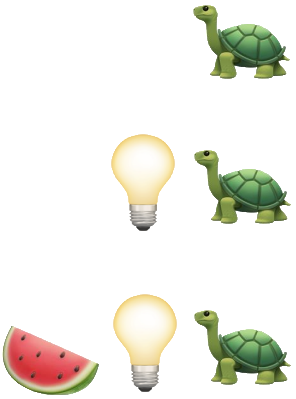
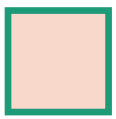
j=2	0	1	4	5
j=3	0	1	4	6




current entry


relevant previous entry

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j=1	0	1	1	1
j=2	0	1	4	5
j=3	0	1	4	6

So the optimal solution is to put one watermelon in your knapsack!

current entry

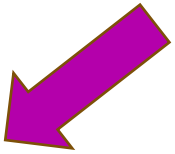
relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.



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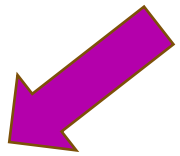


You do this one!
(We did it on the slide in the previous example, just not in the pseudocode!)

What have we learned?

- We can solve 0/1 knapsack in time O(nW).
  - If there are n items and our knapsack has capacity W.

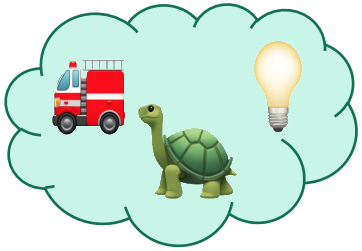
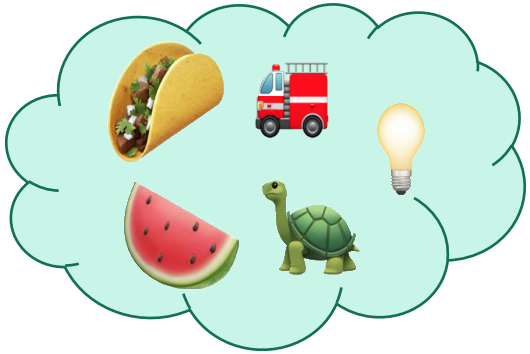
- We again went through the steps to create DP solution:
  - We kept a two-dimensional table, creating smaller problems by restricting the set of allowable items.

Question

- How did we know which substructure to use in which variant of knapsack?



vs.

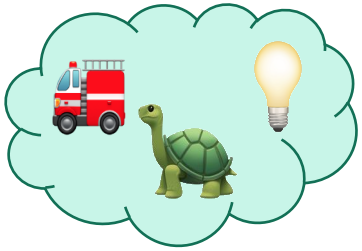
Question

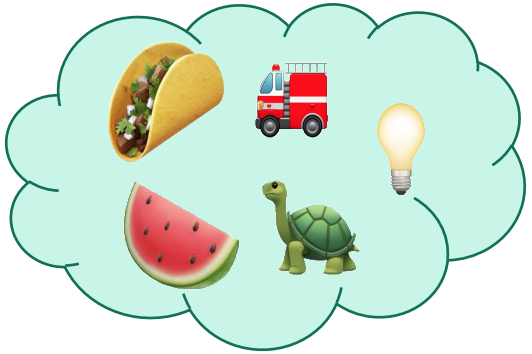
- How did we know which substructure to use in which variant of knapsack?

Answer in retrospect:

This one made sense for unbounded knapsack because it doesn't have any memory of what items have been used.

vs.





In 0/1 knapsack, we can only use each item once, so it makes sense to leave out one item at a time.

Operational Answer: try some stuff, see what works!

Recap

- We saw examples of how to come up with dynamic programming algorithms.
  - Longest Common Subsequence
  - Knapsack two ways
- There is a **recipe** for dynamic programming algorithms.

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
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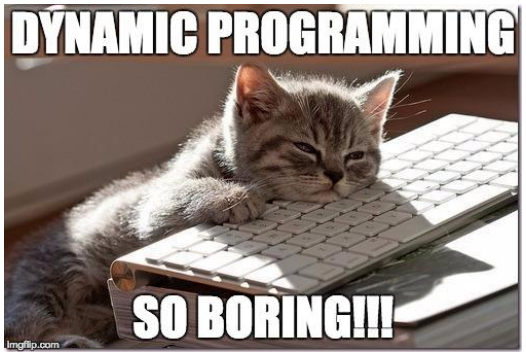
Recap



- We saw examples of how to come up with dynamic programming algorithms.
  - Longest Common Subsequence
  - Knapsack two ways
- There is a **recipe** for dynamic programming algorithms.
- Sometimes coming up with the right substructure takes some creativity

Acknowledgement

- Stanford University