

Advanced Data Structures and Algorithms

Dynamic Programming and
Floyd-Warshall (All-Pairs Shortest Paths: APSP)

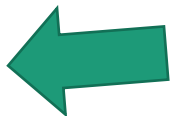
Today

- Bellman-Ford is a special case of ***Dynamic Programming!***
- What is dynamic programming?
 - Warm-up example: Fibonacci numbers
- Another example:
 - Floyd-Warshall Algorithm

Bellman-Ford is an example of...

Dynamic Programming!

Today:

- Example of Dynamic programming: 
 - Fibonacci numbers.
 - (And Bellman-Ford)
- What is dynamic programming, exactly?
 - And why is it called “dynamic programming”?
- Another example: Floyd-Warshall algorithm
 - An “all-pairs” shortest path algorithm

Fibonacci Numbers

- Definition:
 - $F(n) = F(n-1) + F(n-2)$, with $F(0) = F(1) = 1$.
 - The first several are:
 - 1
 - 1
 - 2
 - 3
 - 5
 - 8
 - 13, 21, 34, 55, 89, 144,...

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 - 13, 21, 34, 55, 89, 144,...

- Question:

- Given n , what is $F(n)$?

Candidate algorithm

- **def** Fibonacci(n):
 - **if** n == 0 or n == 1:
 - **return** 1
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Running time?

- $T(n) = T(n-1) + T(n-2) + O(1)$
- $T(n) \geq T(n-1) + T(n-2)$ for $n \geq 2$

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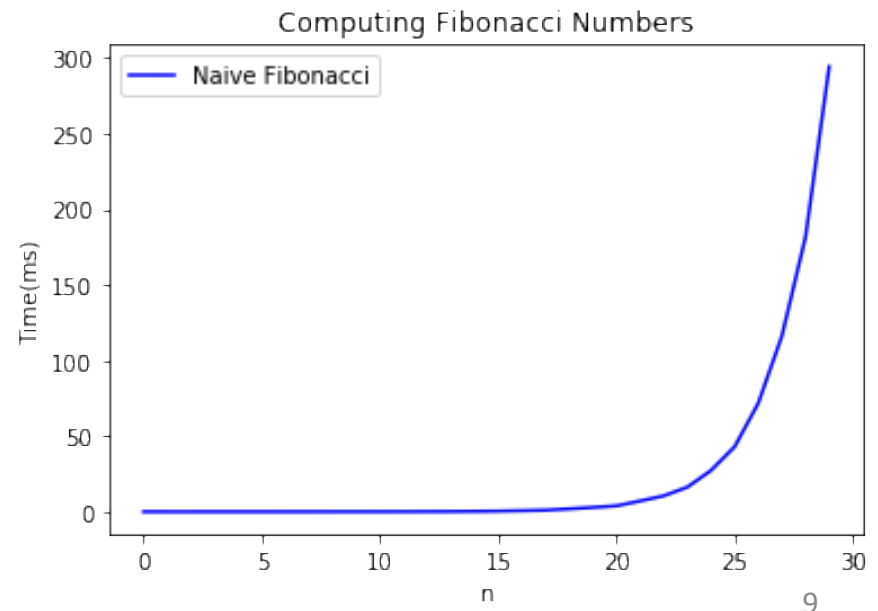
- $T(n) = T(n-1) + T(n-2) + O(1)$
- $T(n) \geq T(n-1) + T(n-2)$ for $n \geq 2$
- So $T(n)$ grows *at least* as fast as the Fibonacci numbers themselves...
- Fun fact, that's like ϕ^n where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Candidate algorithm

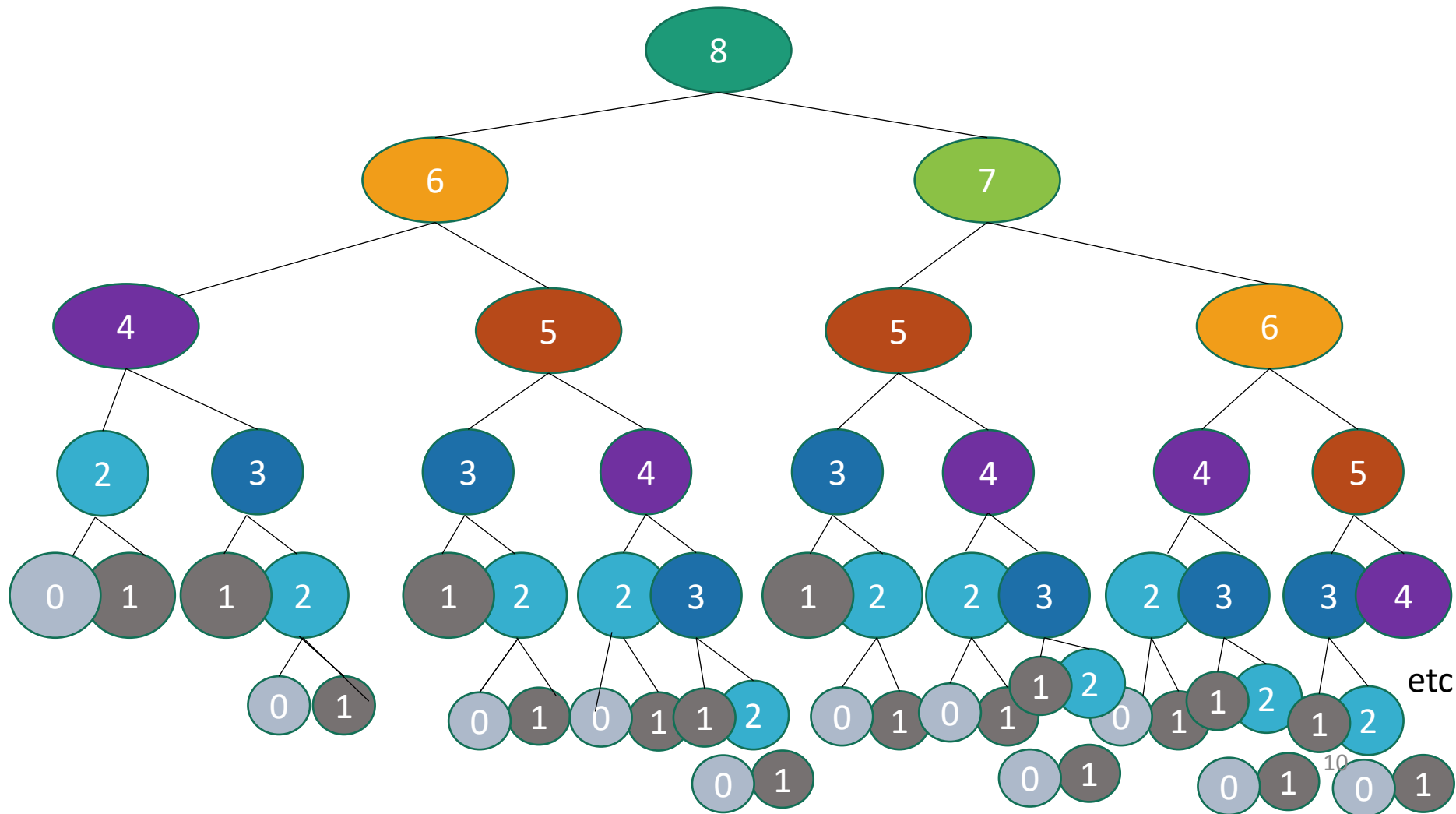
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- So $T(n)$ grows *at least* as fast as the Fibonacci numbers themselves...
- Fun fact, that's like ϕ^n where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.
- aka, **EXPONENTIALLY QUICKLY** 😞

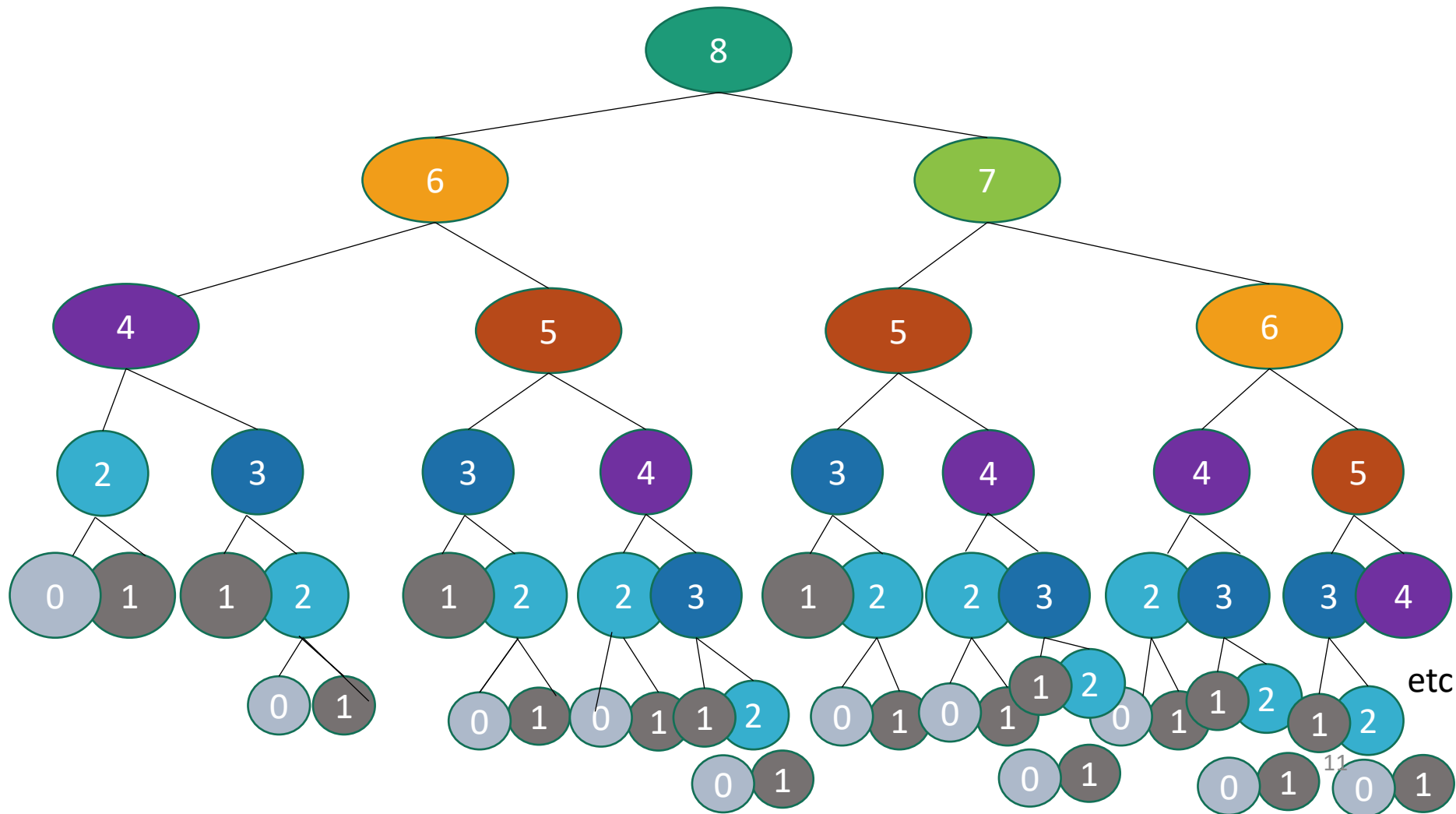


What's going on?
Consider $\text{Fib}(8)$

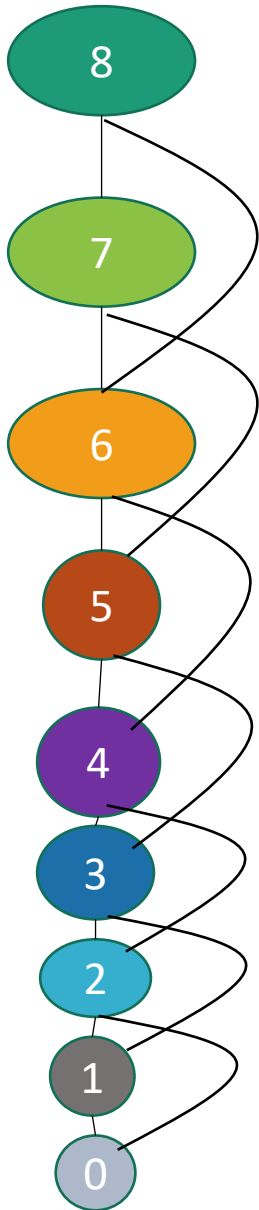


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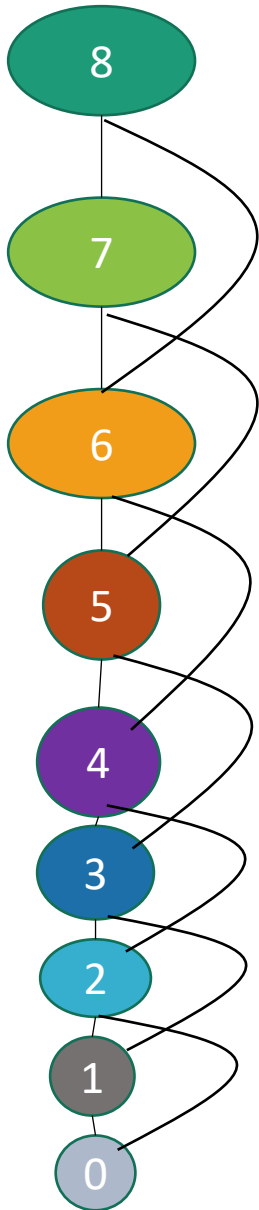
That's a lot of repeated computation!



Maybe this would be better:



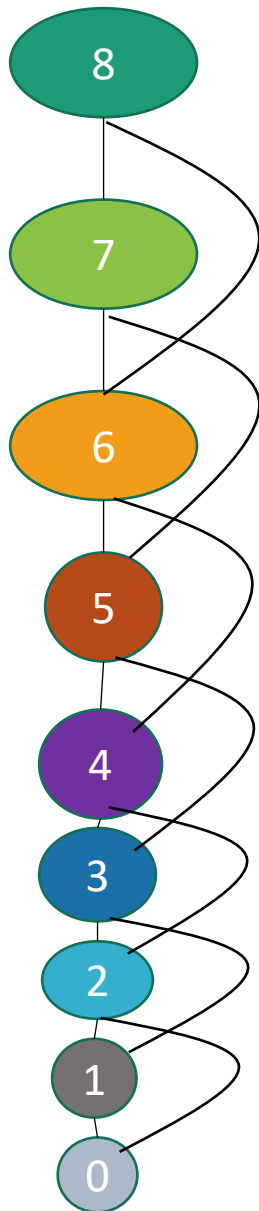
Maybe this would be better:



```
def fasterFibonacci(n):
```

- $F = [1, 1, \text{None}, \text{None}, \dots, \text{None}]$
 - $\backslash F$ has length $n + 1$
- **for** $i = 2, \dots, n$:
 - $F[i] = F[i-1] + F[i-2]$
- **return** $F[n]$

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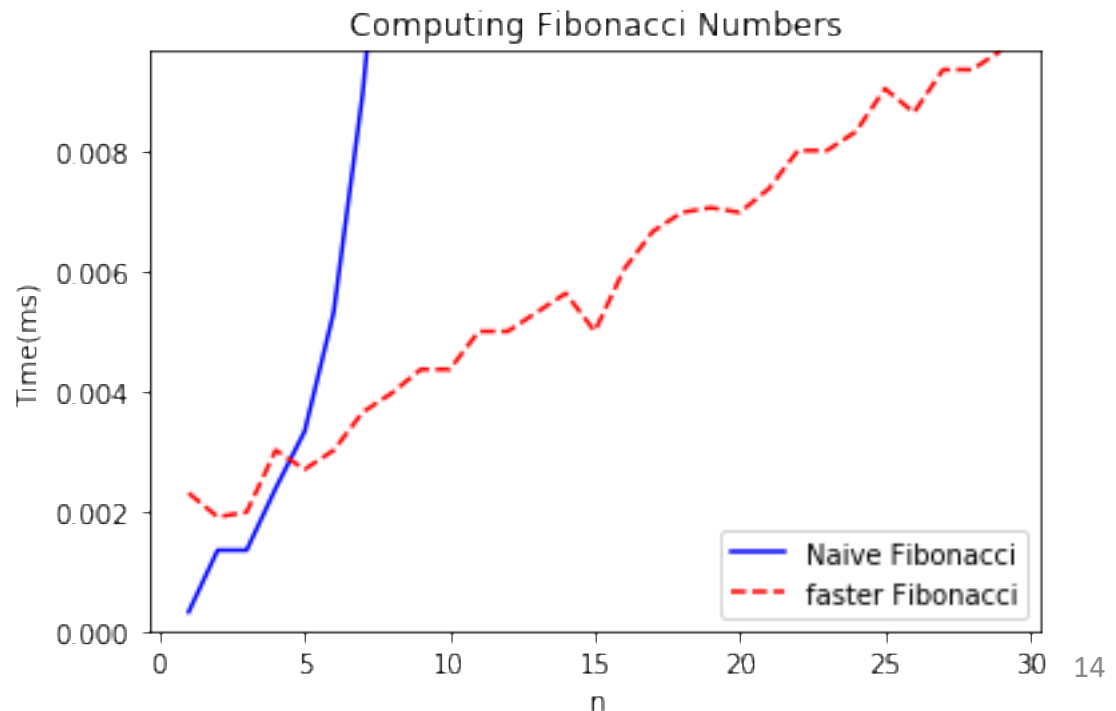


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```

Much better running time!



This was an example of...

***Dynamic
programming!***

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- It is an algorithm design paradigm
 - like divide-and-conquer is an algorithm design paradigm.

What is *dynamic programming*?

- It is an algorithm design paradigm
 - like divide-and-conquer is an algorithm design paradigm.
- Usually it is for solving **optimization problems**
 - eg, *shortest* path
 - (Fibonacci numbers aren't an optimization problem, but they are a good example...)

Elements of dynamic programming

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 - Fibonacci:

$$F(i+1) = F(i) + F(i-1)$$

- Bellman-Ford:

$$d^{(i+1)}[v] \leftarrow \min\{d^{(i)}[v], \min_u \{d^{(i)}[u] + \text{weight}(u,v)\}\}$$

Shortest path with at most i edges from s to v

Shortest path with at most i edges from s to u .

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 - Many different entries of $d^{(i+1)}$ will directly use $d^{(i)}[v]$.
 - And lots of different entries of $d^{(i+x)}$ will indirectly use $d^{(i)}[v]$.
- This means that we can save time by solving a sub-problem just once and storing the answer.

Elements of dynamic programming

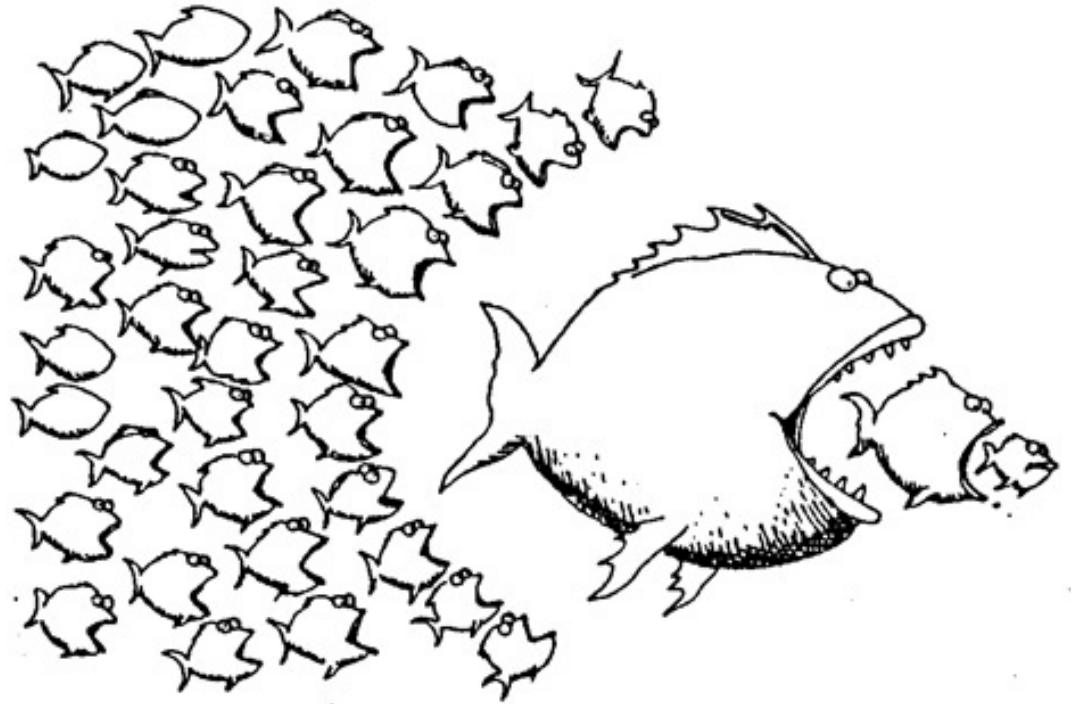
- Optimal substructure.
 - Optimal solutions to sub-problems can be used to find the optimal solution of the original problem.
- Overlapping subproblems.
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- Optimal substructure.
 - Optimal solutions to sub-problems can be used to find the optimal solution of the original problem.
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 - The subproblems show up again and again
- Using these properties, we can design a **dynamic programming** algorithm:
 - Keep a table of solutions to the smaller problems.
 - Use the solutions in the table to solve bigger problems.
 - At the end we can use information we collected along the way to find the solution to the whole thing.

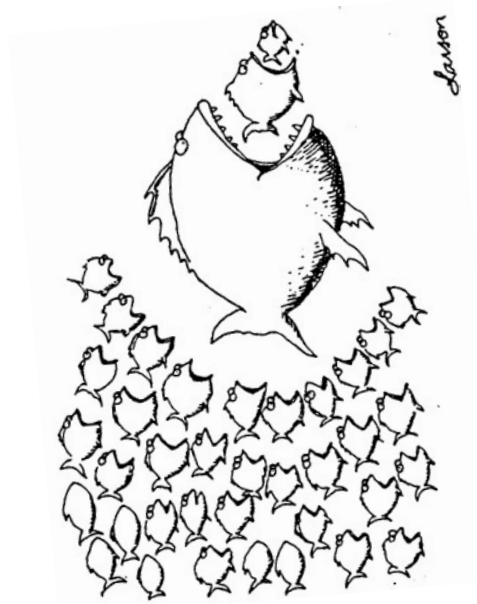
Two ways to think about and/or implement DP algorithms

- Top down
- Bottom up



Bottom up approach

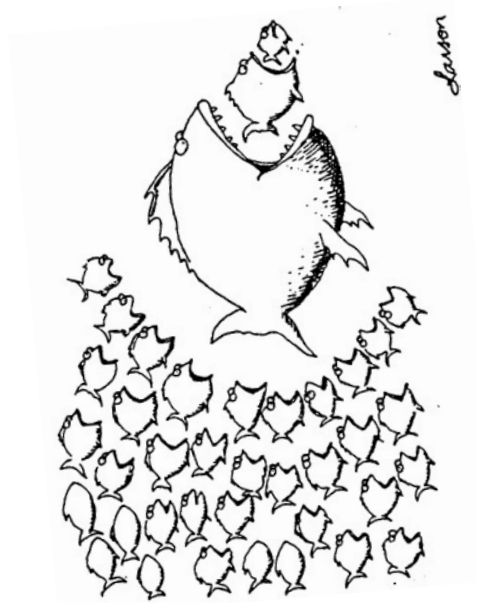
what we just saw.



Bottom up approach

what we just saw.

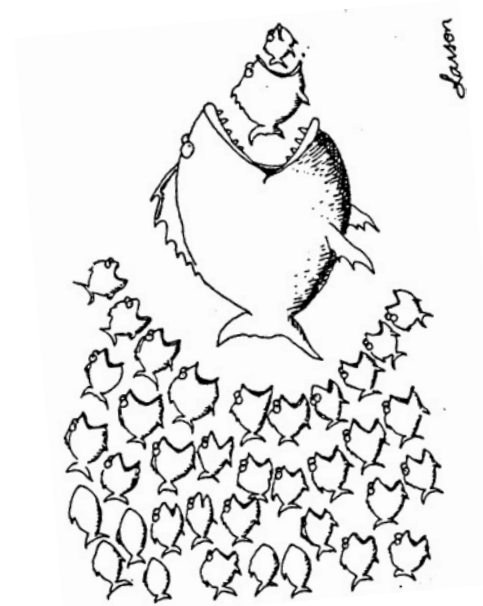
- For Fibonacci:
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 - fill in $F[0], F[1]$



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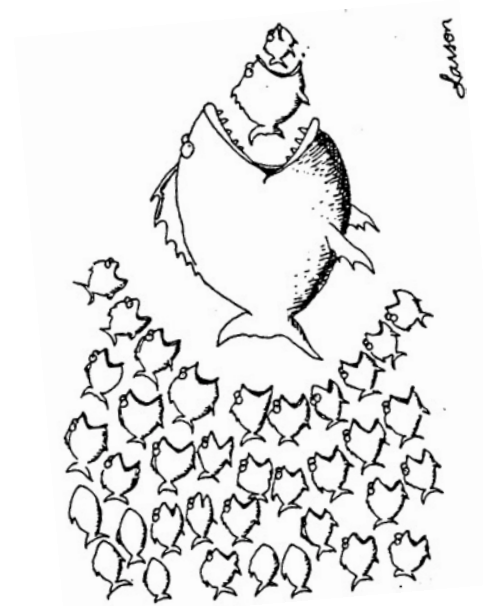
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- Then bigger problems
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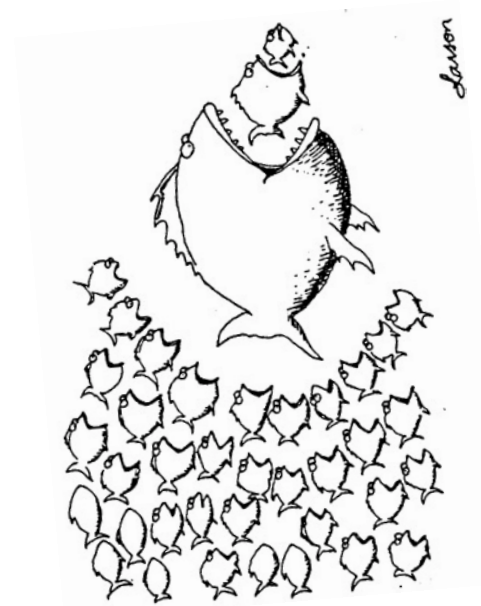
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 - fill in $F[n-1]$



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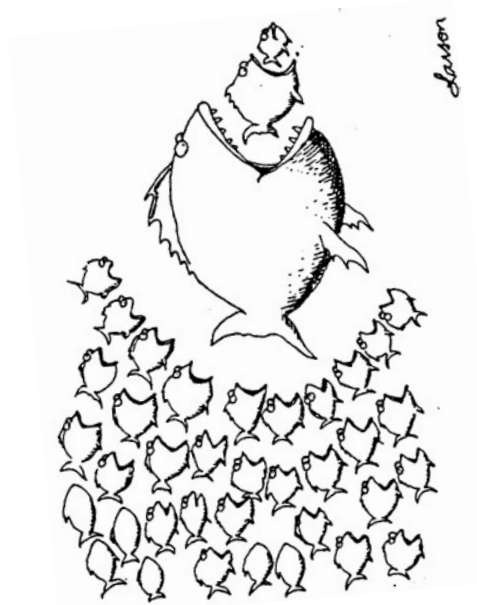
- For Fibonacci:
- Solve the small problems first
 - fill in $F[0], F[1]$
- Then bigger problems
 - fill in $F[2]$
- ...
- Then bigger problems
 - fill in $F[n-1]$
- Then finally solve the real problem.
 - fill in $F[n]$



Bottom up approach

what we just saw.

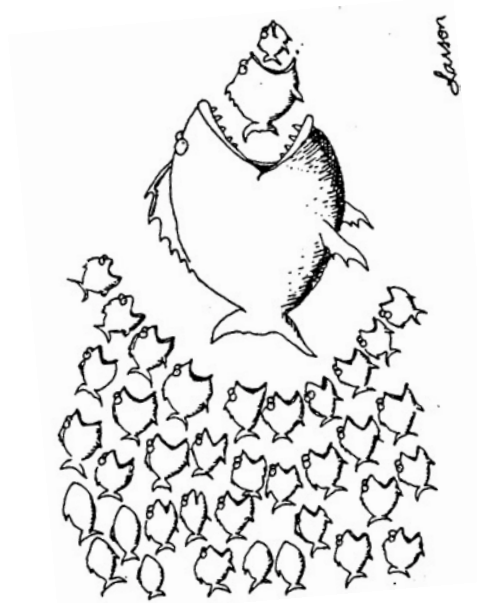
- For Bellman-Ford:
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 - fill in $d^{(0)}$



Bottom up approach

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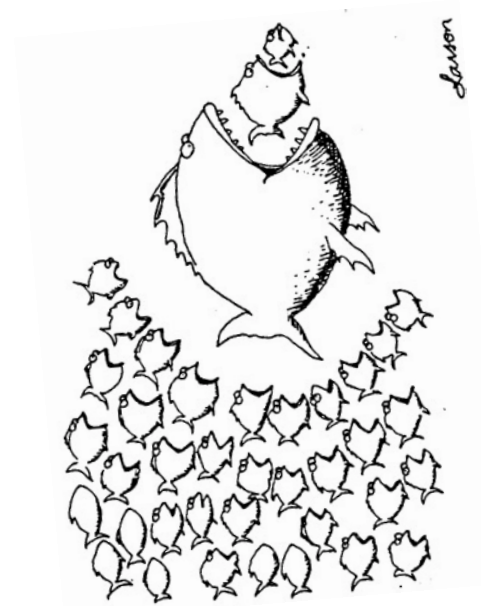
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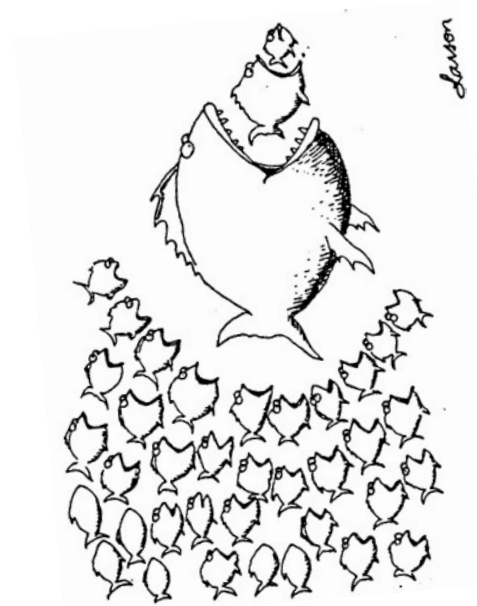
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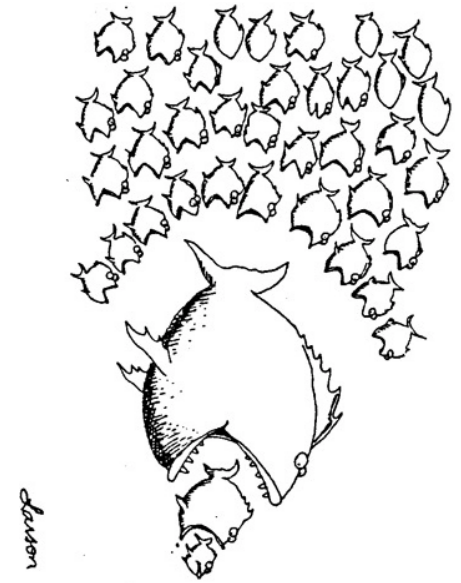
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- For Bellman-Ford:
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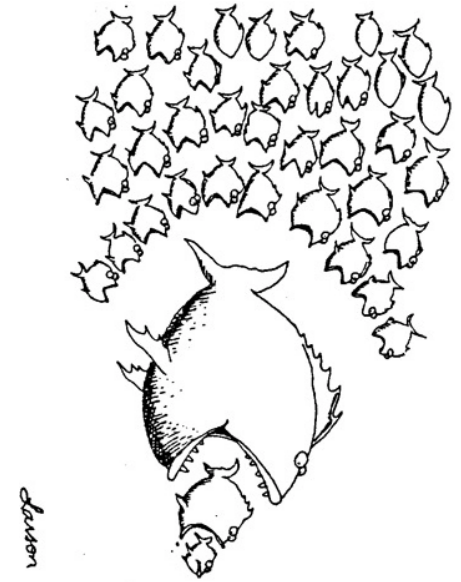


Top down approach



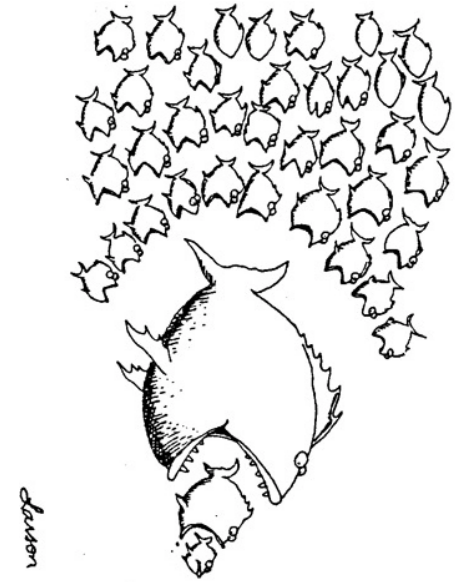
Top down approach

- Think of it like a recursive algorithm.
- To solve the big problem:
 - Recurse to solve smaller problems
 - Those recurse to solve smaller problems
 - etc..



Top down approach

- Think of it like a recursive algorithm.
- To solve the big problem:
 - Recurse to solve smaller problems
 - Those recurse to solve smaller problems
 - etc..
- The difference from divide and conquer:
 - Keep track of what small problems you've already solved to prevent re-solving the same problem twice.
 - Aka, “**memorization**”



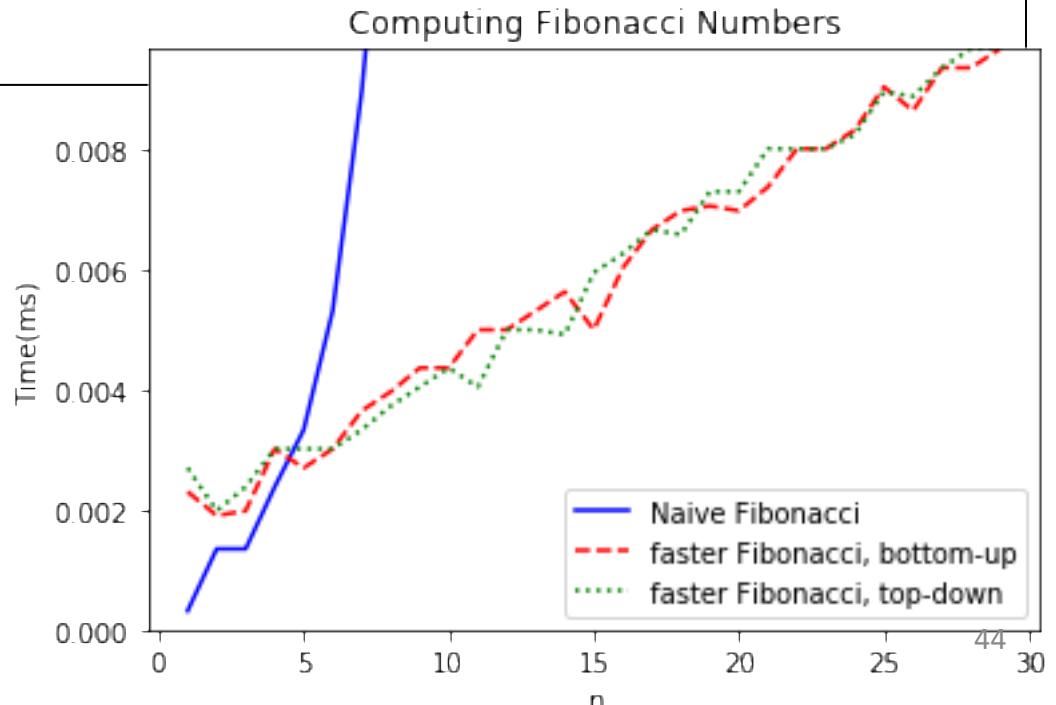
Example of top-down Fibonacci

- define a global list $F = [1, 1, \text{None}, \text{None}, \dots, \text{None}]$
- **def** Fibonacci(n):
 - **if** $F[n] \neq \text{None}$:
 - **return** $F[n]$
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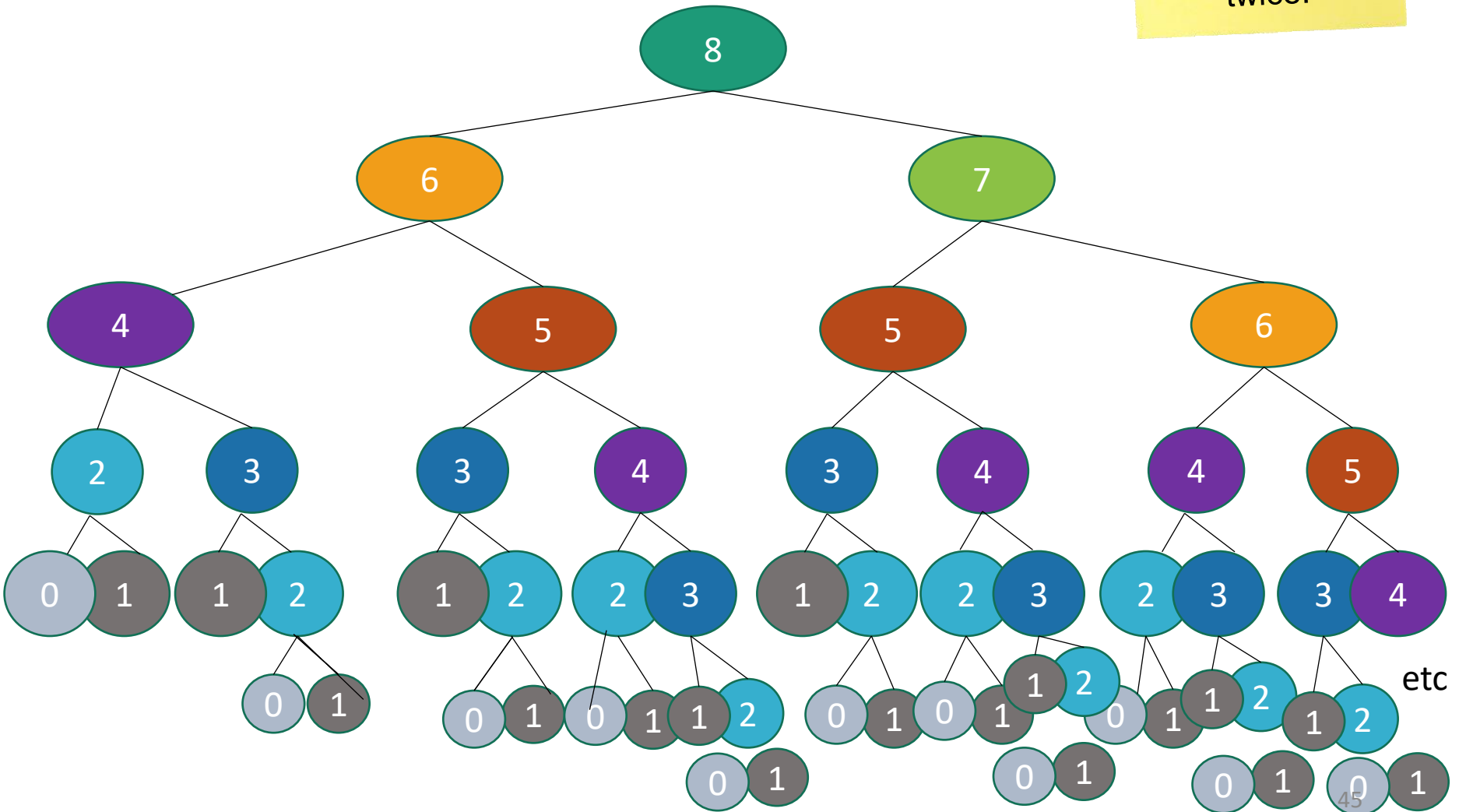
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Memorization:
Keeps track (in F)
of the stuff you've
already done.



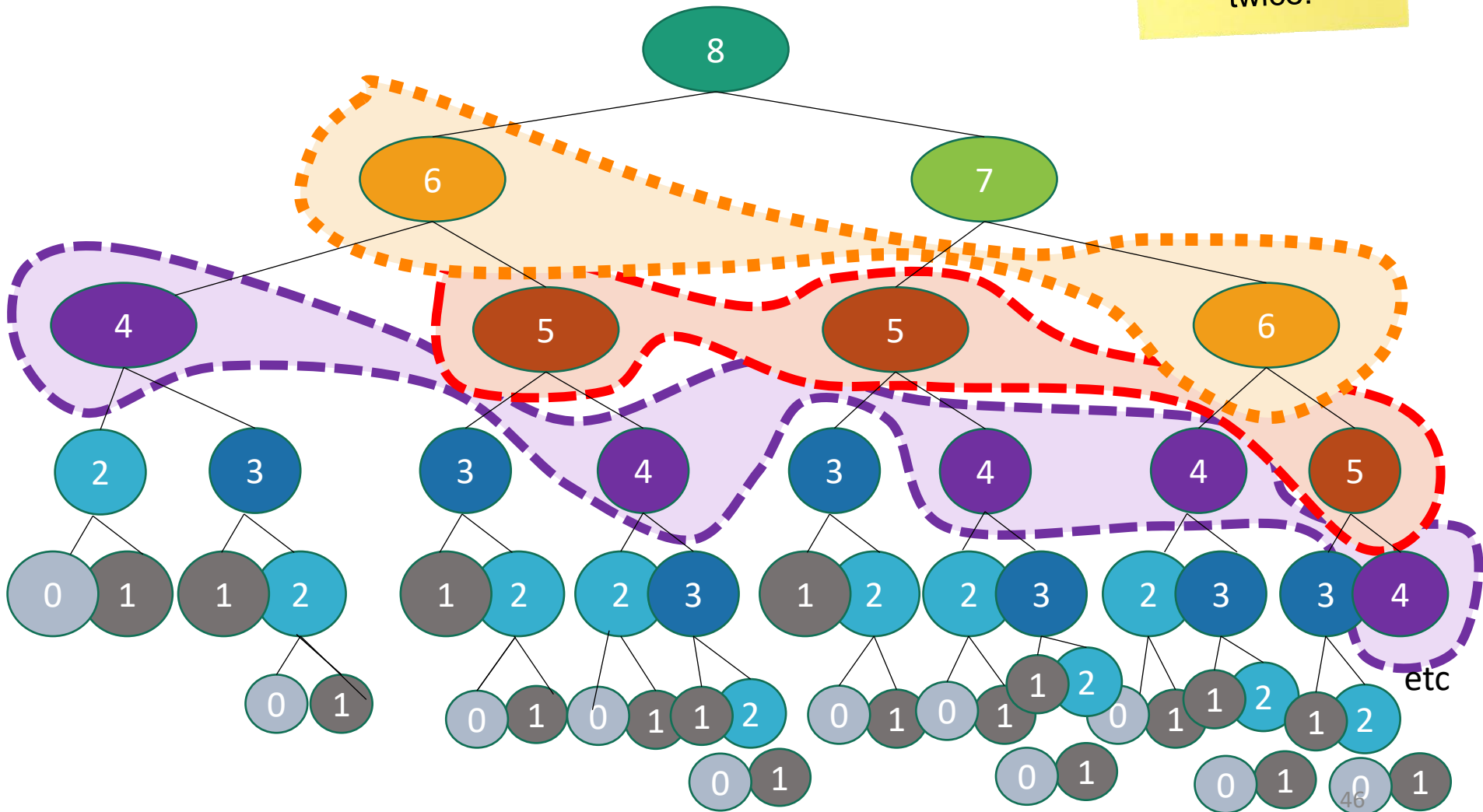
Memorization visualization

**Collapse
repeated nodes
and don't do
the same work
twice!**



Memorization visualization

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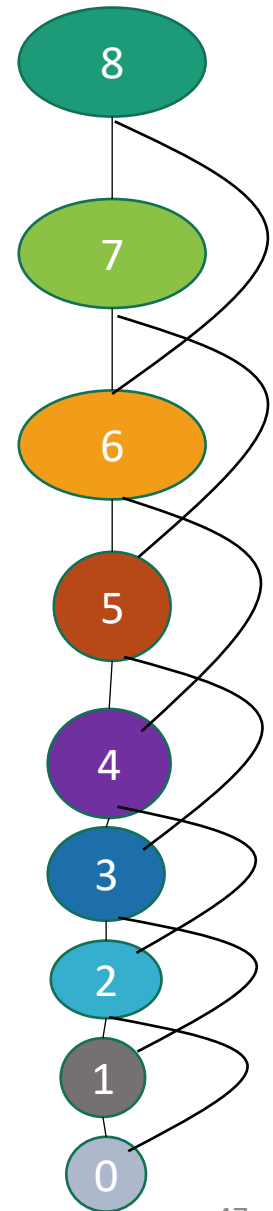
Memorization Visualization

ctd

Collapse
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But otherwise
treat it like the
same old
recursive
algorithm.


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What have we learned?

- ***Dynamic programming:***

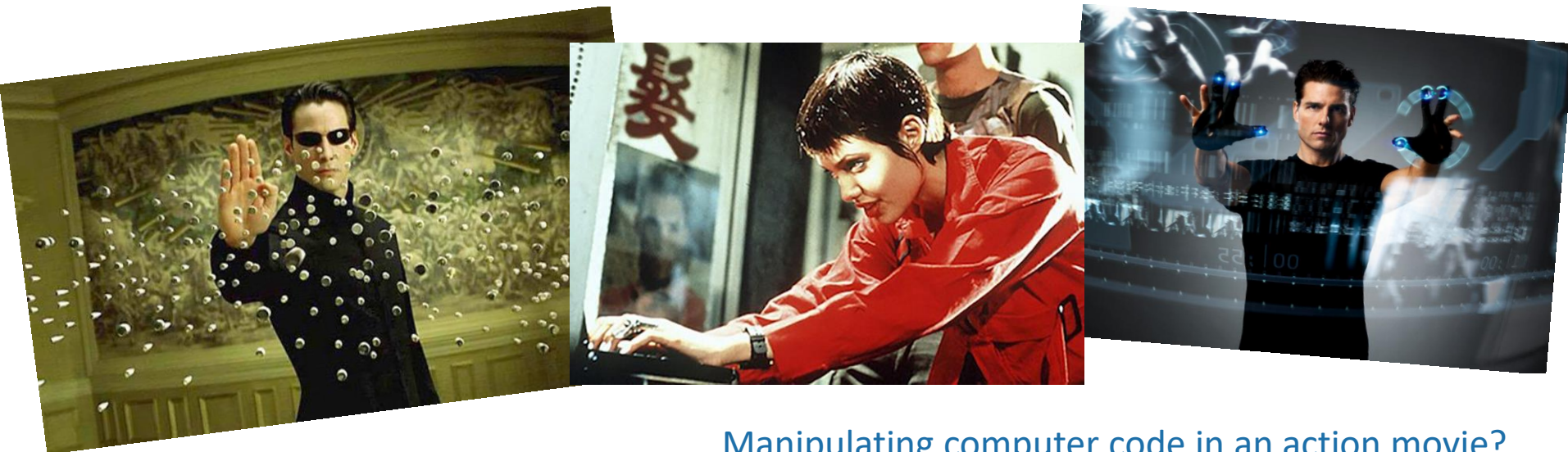
- Paradigm in algorithm design.
- Uses **optimal substructure**
- Uses **overlapping subproblems**
- Can be implemented **bottom-up** or **top-down**.
- It's a fancy name for a pretty common-sense idea:



Don't
duplicate
work if you
don't have to!

Why “*dynamic programming*” ?

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Manipulating computer code in an action movie?

Why “*dynamic programming*” ?

- **Programming** refers to finding the optimal “program.”
 - as in, a shortest route is a *plan* aka a *program*.



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- **Dynamic** refers to the fact that it’s multi-stage.



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- **Programming** refers to finding the optimal “program.”
 - as in, a shortest route is a *plan* aka a *program*.
- **Dynamic** refers to the fact that it’s multi-stage.
- **But also it’s just a fancy-sounding name.**



Manipulating computer code in an action movie?

Why “*dynamic programming*” ?

- Richard Bellman invented the name in the 1950's.
- At the time, he was working for the RAND Corporation, and projects needed flashy names to get funded.

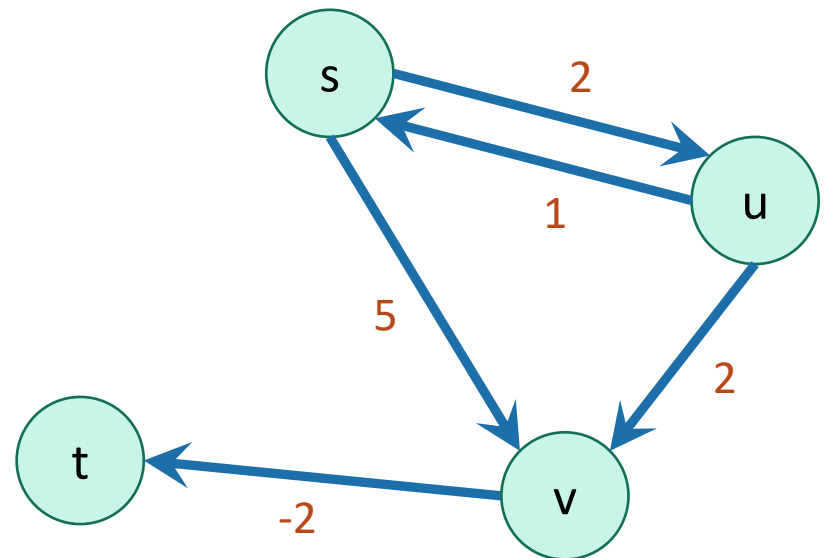
Floyd-Warshall Algorithm

Another example of Dynamic Programming

Floyd-Warshall Algorithm

Another example of Dynamic Programming

- This is an algorithm for **All-Pairs Shortest Paths (APSP)**
 - That is, I want to know the shortest path from u to v for **ALL pairs** u, v of vertices in the graph.
 - Not just from a special single source s .

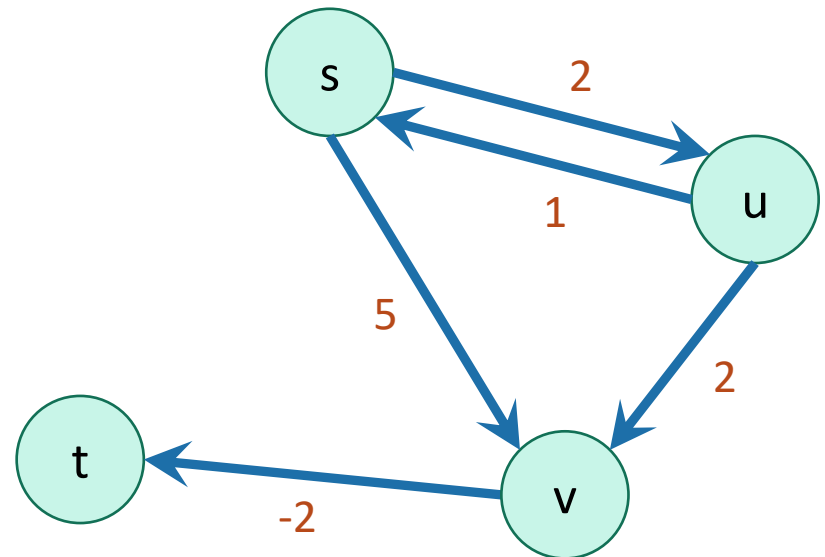


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		Destination			
Source		s	u	v	t
	s	0	2	4	2
	u	1	0	2	0
	v	∞	∞	0	-2
	t	∞	∞	∞	0



Floyd-Warshall Algorithm

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- Naïve solution (if we want to handle negative edge weights):
 - For all s in G :
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- Time ?

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 - may be as bad as n^4 if $m=n^2$

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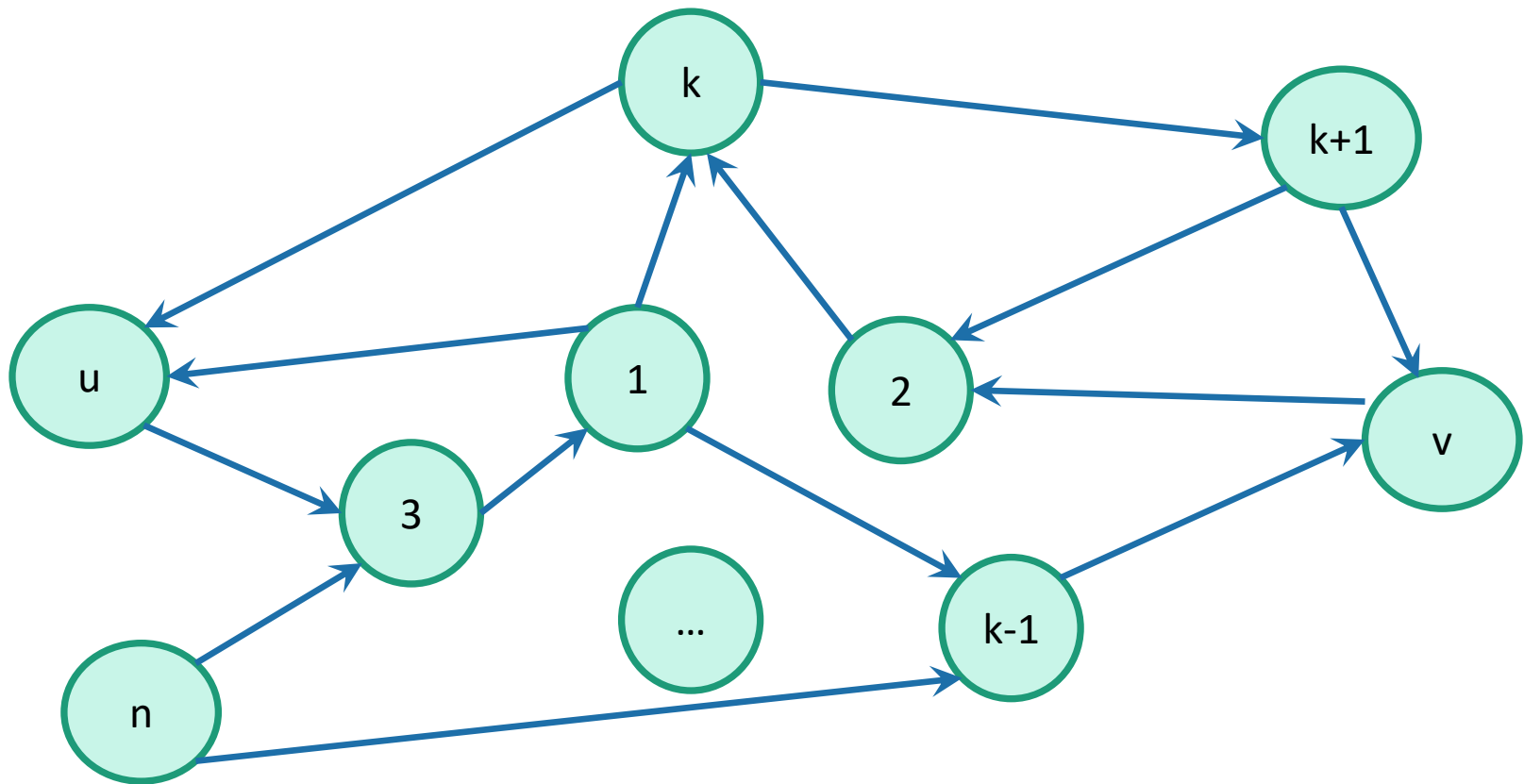
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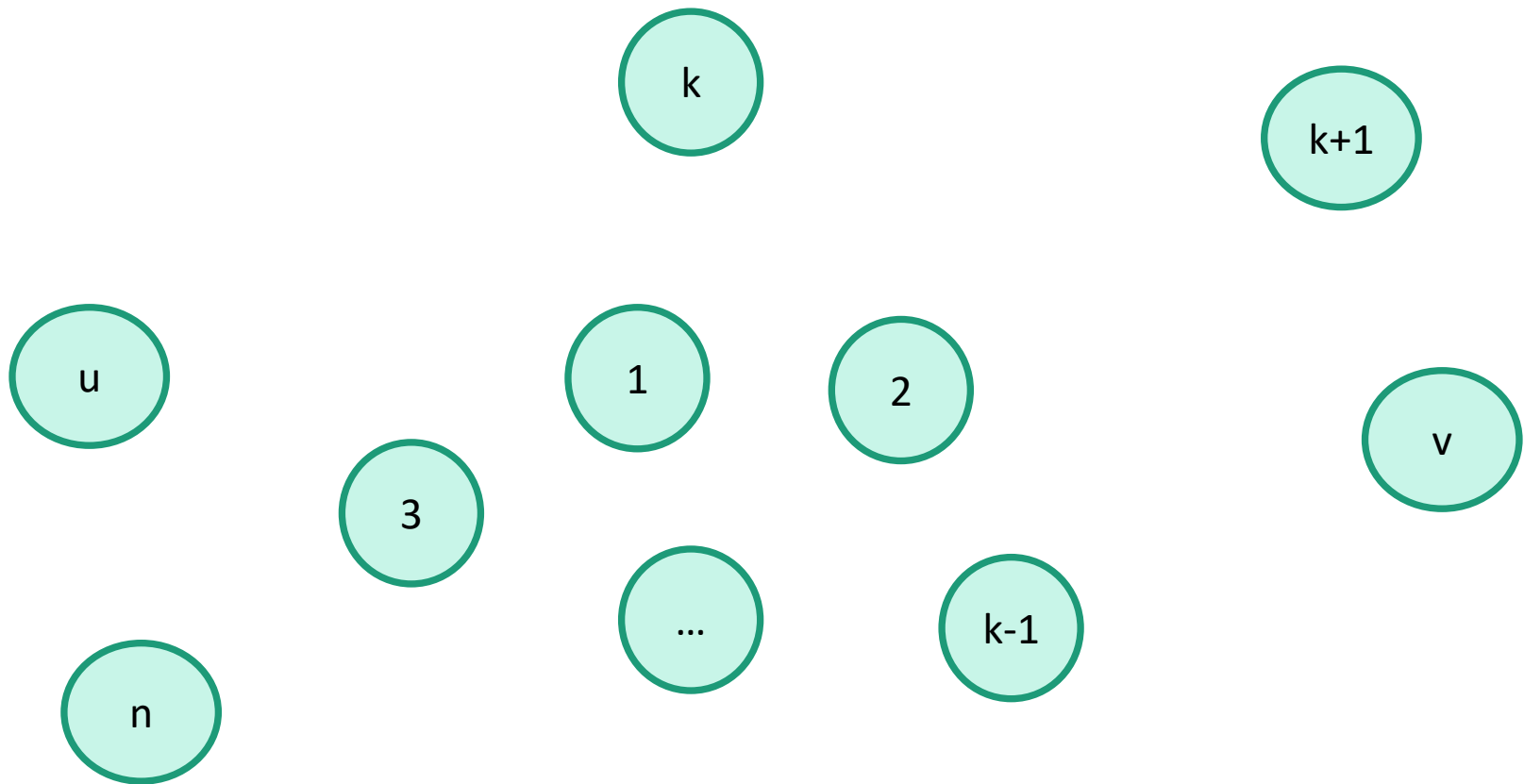
Can we do better?

Optimal substructure



Optimal substructure

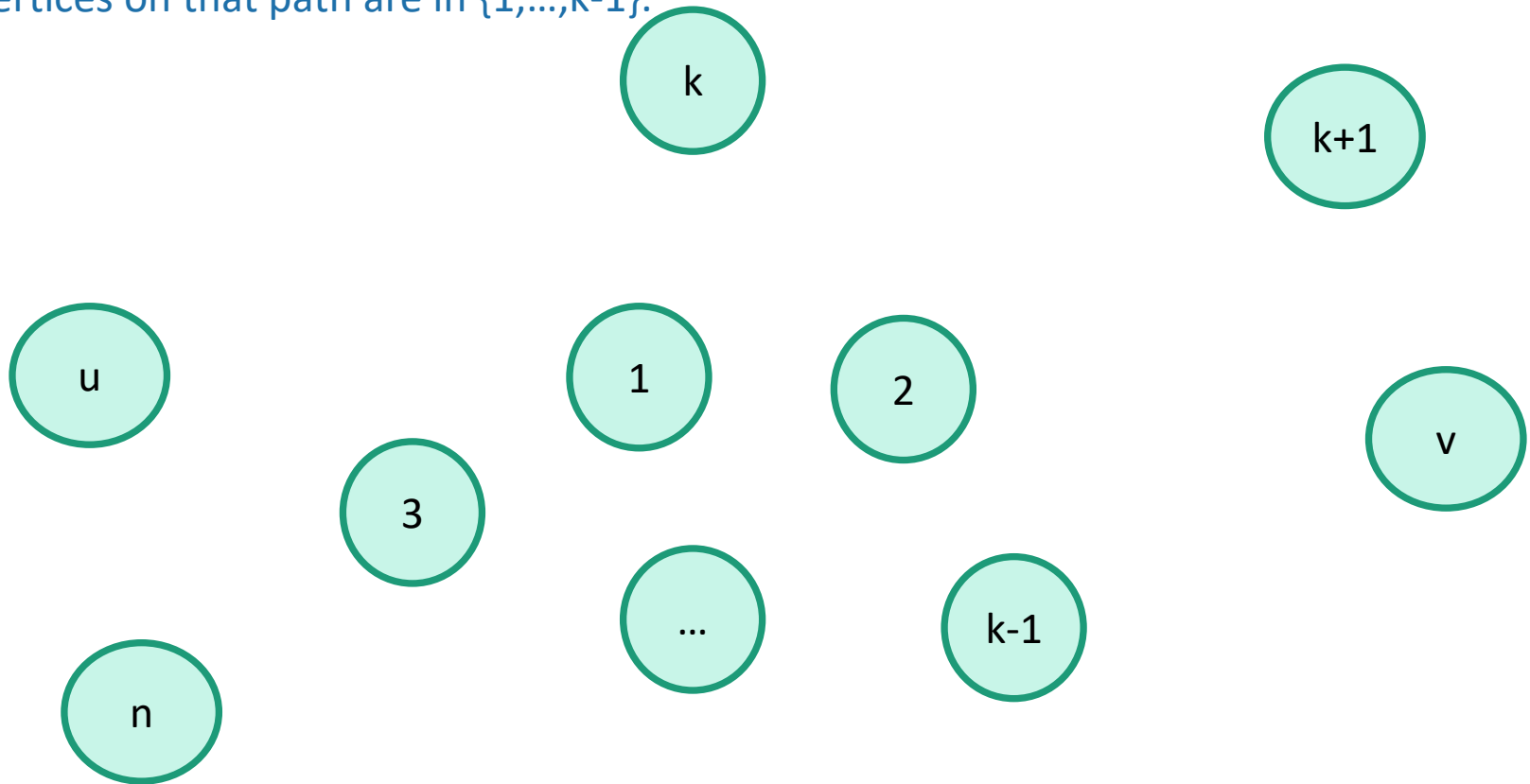
Label the vertices $1, 2, \dots, n$
(We omit some edges in the picture below – meant to be a cartoon, not an example).



Optimal substructure

Sub-problem(k-1):

For all pairs, u, v , find the cost of the shortest path from u to v , so that all the internal vertices on that path are in $\{1, \dots, k-1\}$.



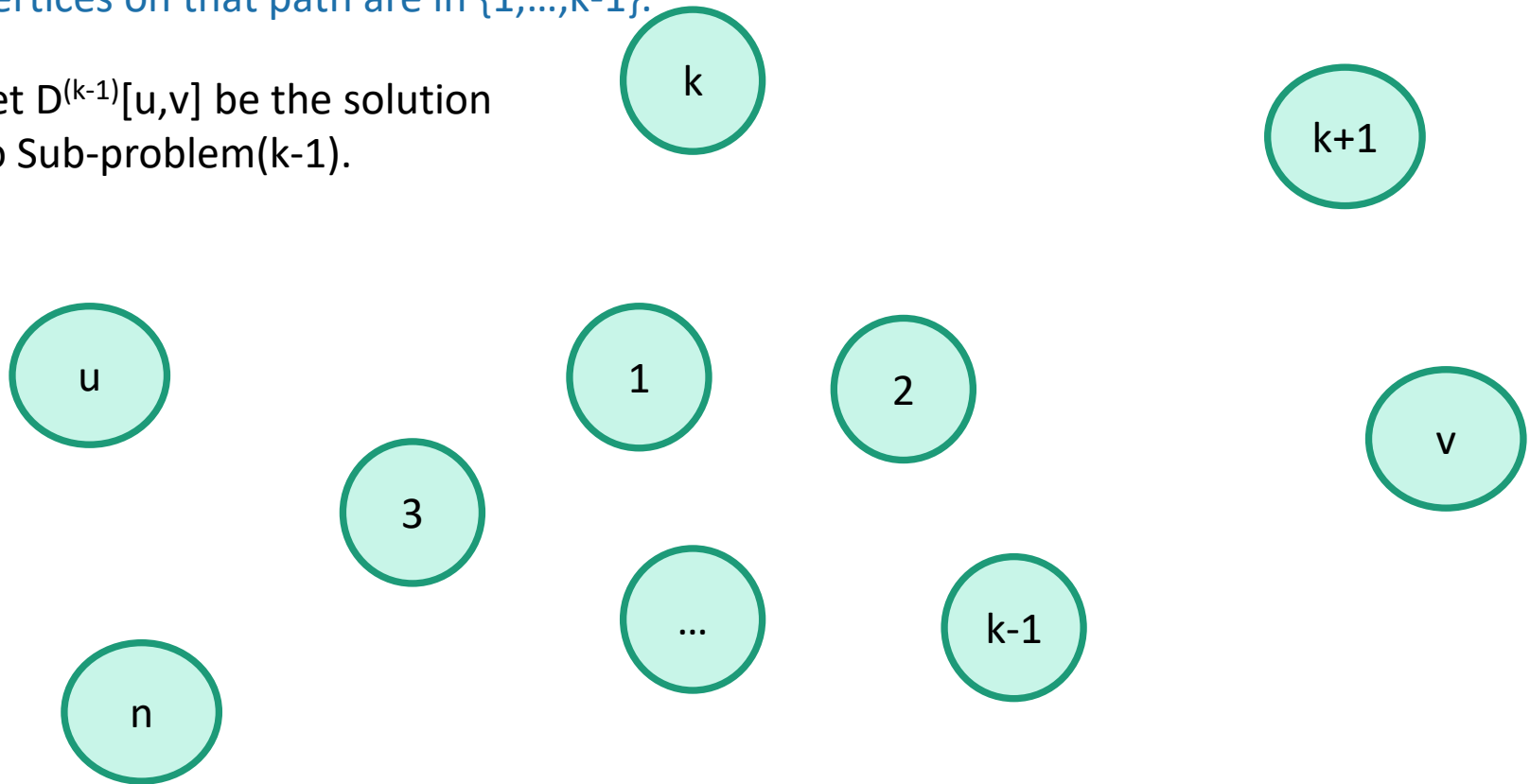
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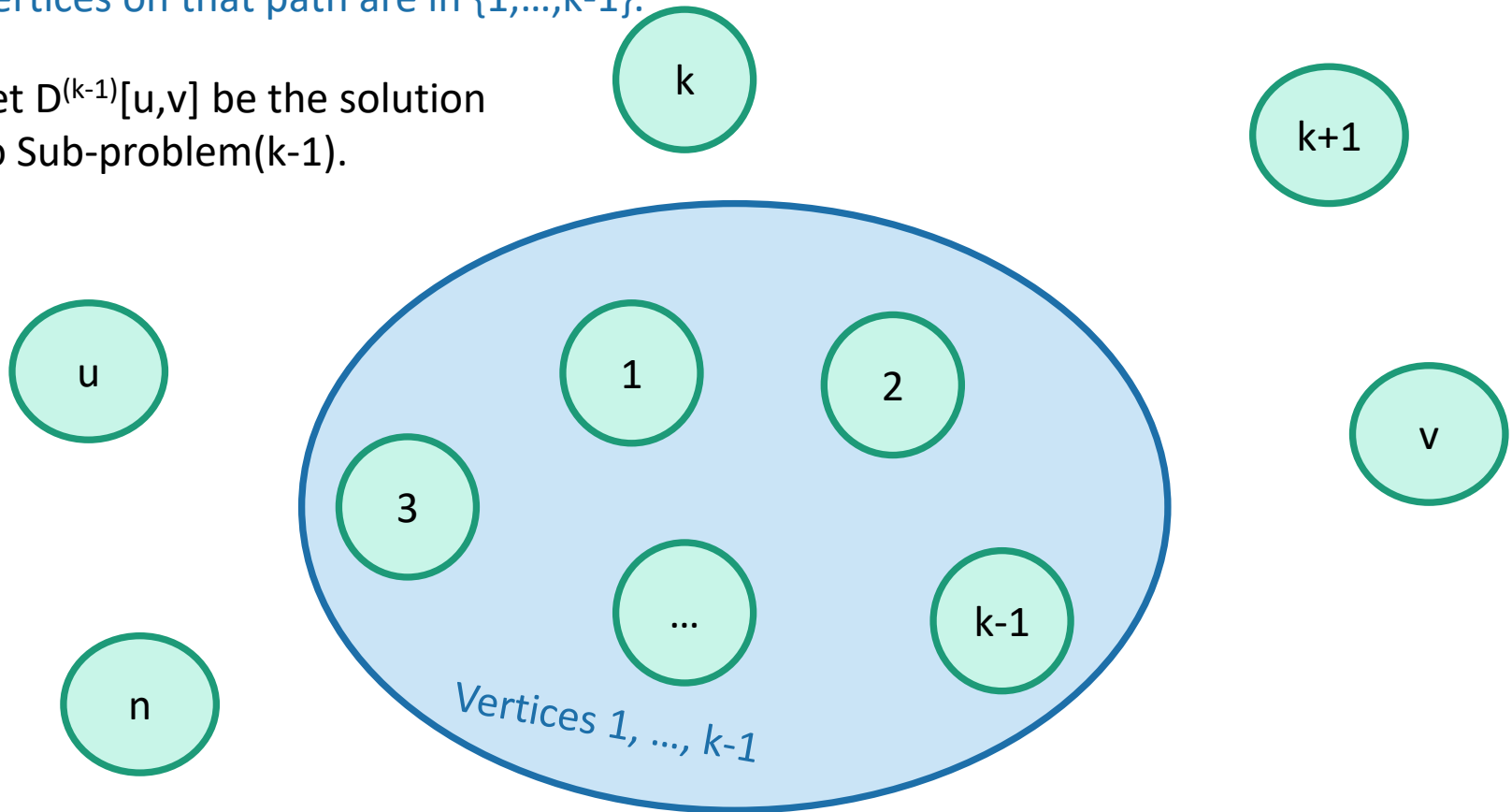
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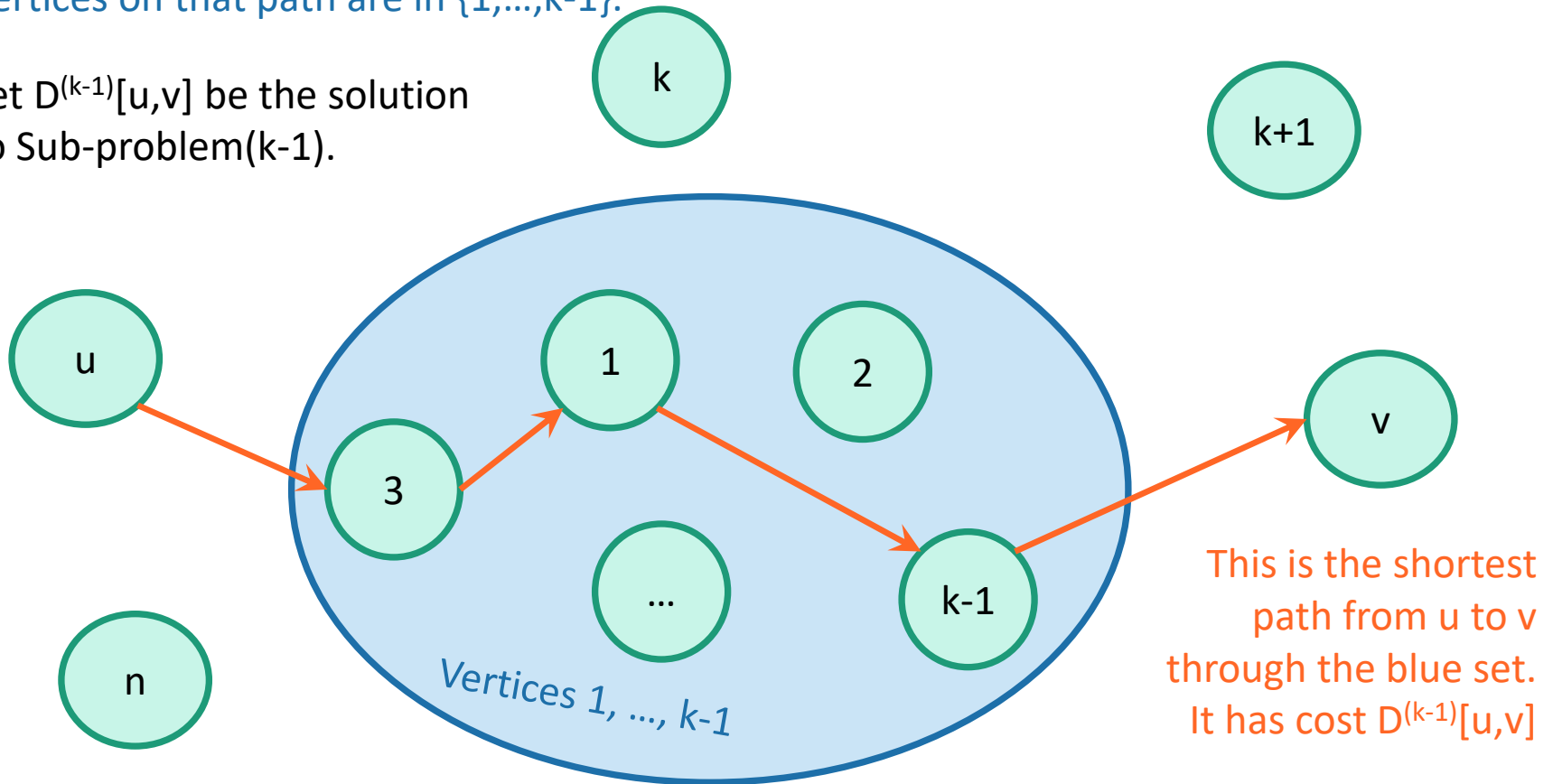
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Sub-problem($k-1$):

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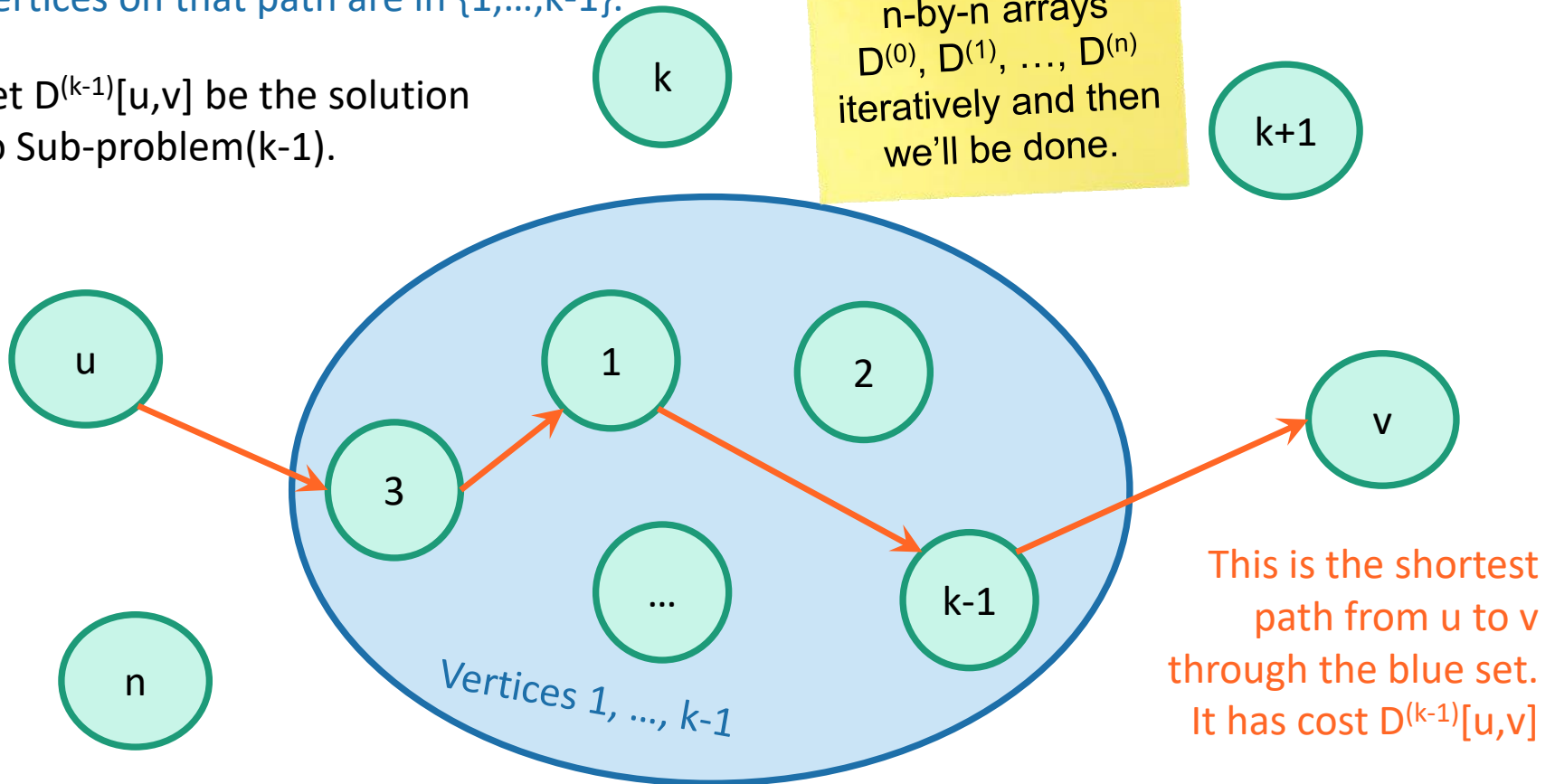
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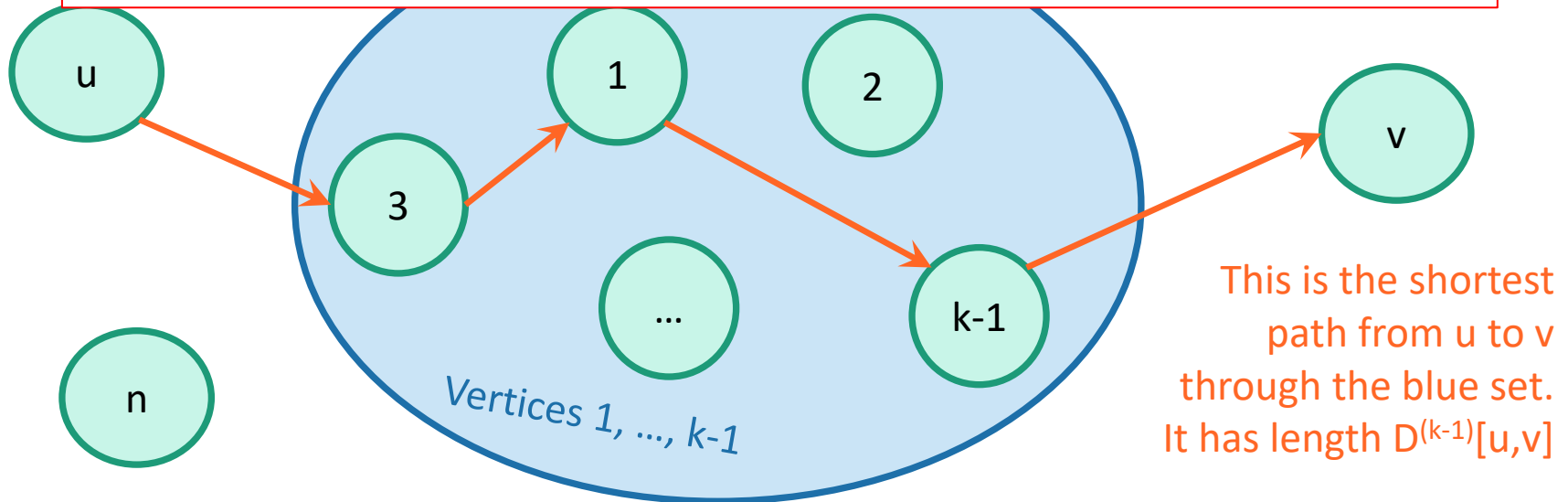
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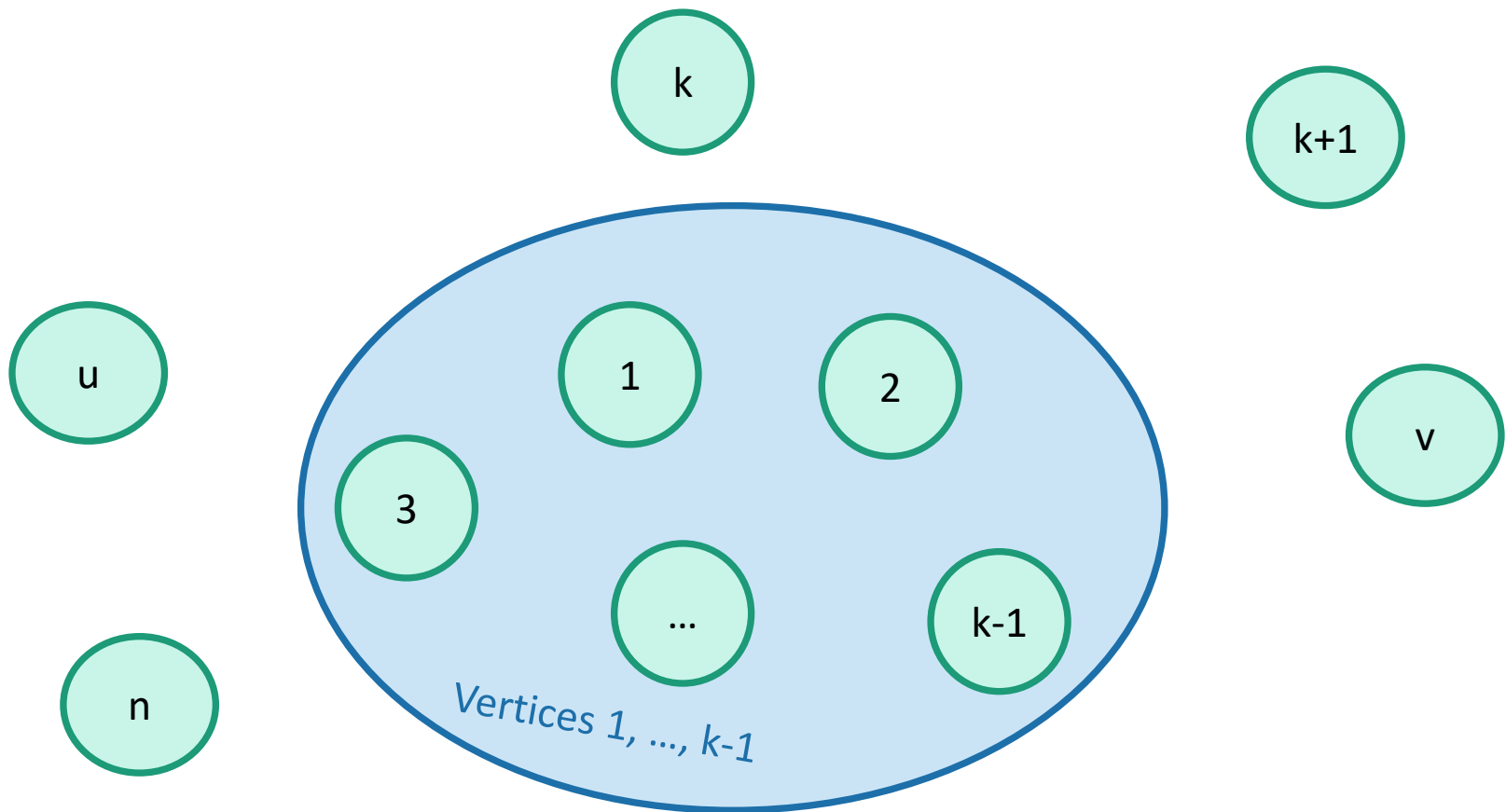
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Question: How can we find $D^{(k)}[u, v]$ using $D^{(k-1)}$?



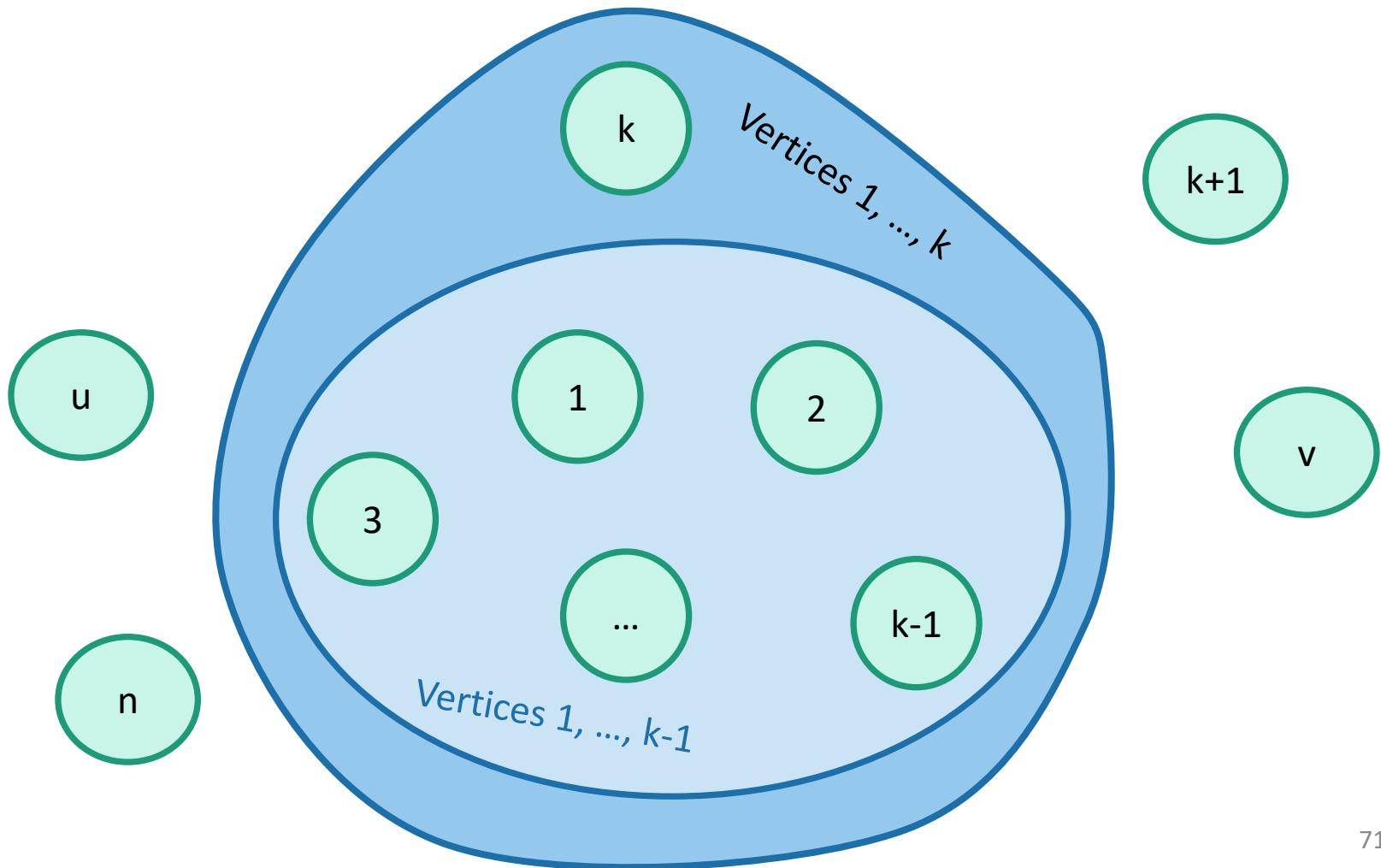
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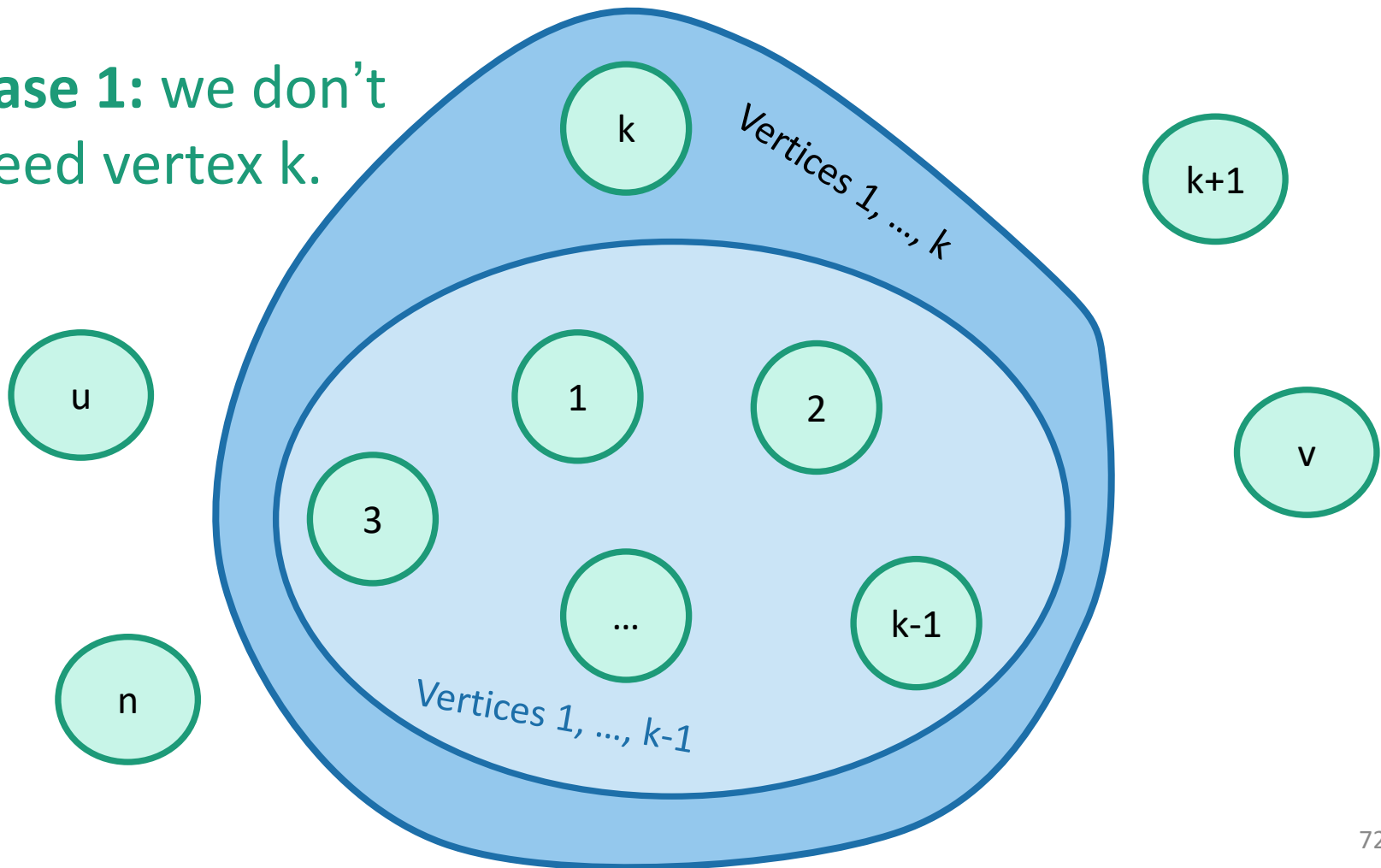
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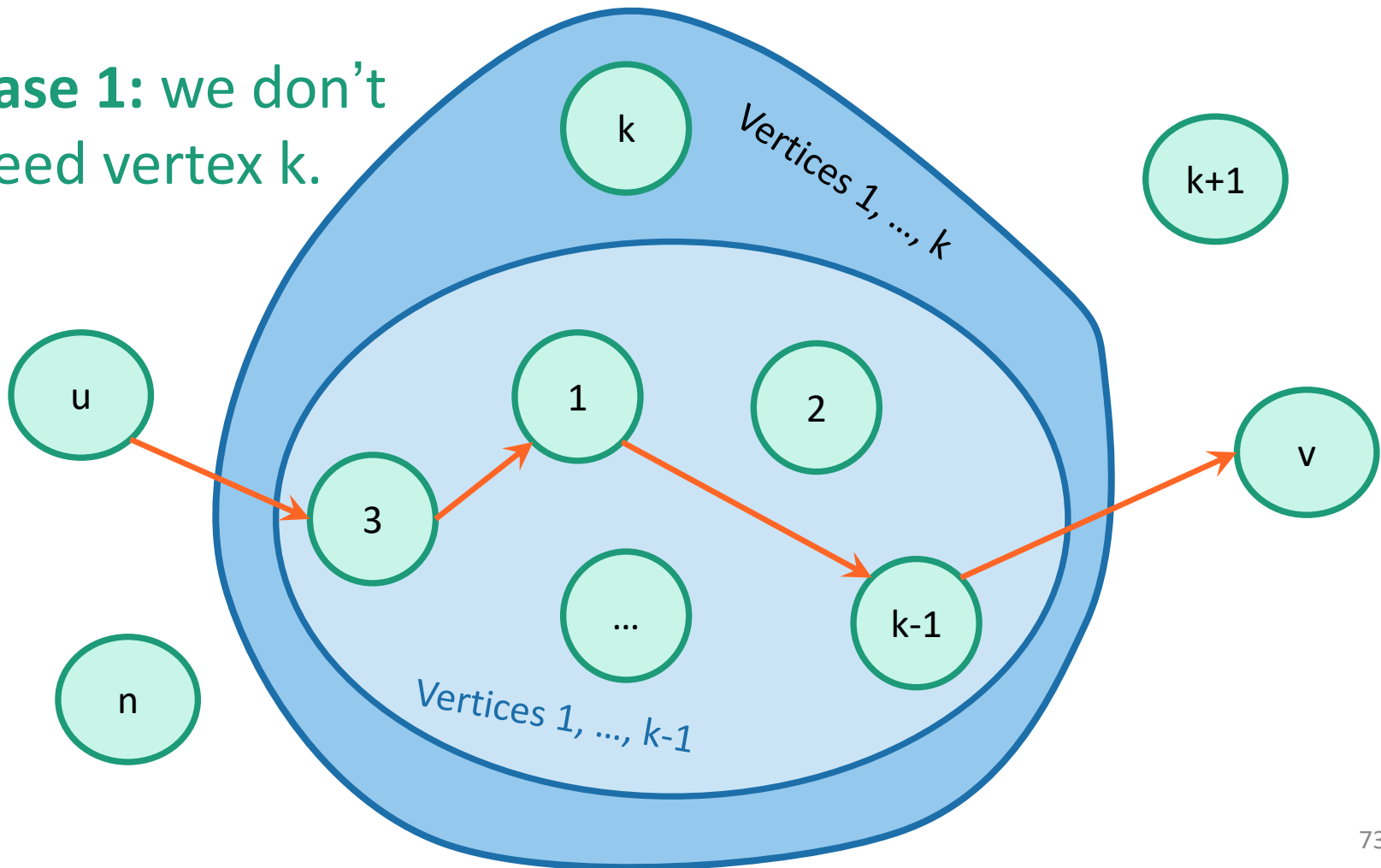
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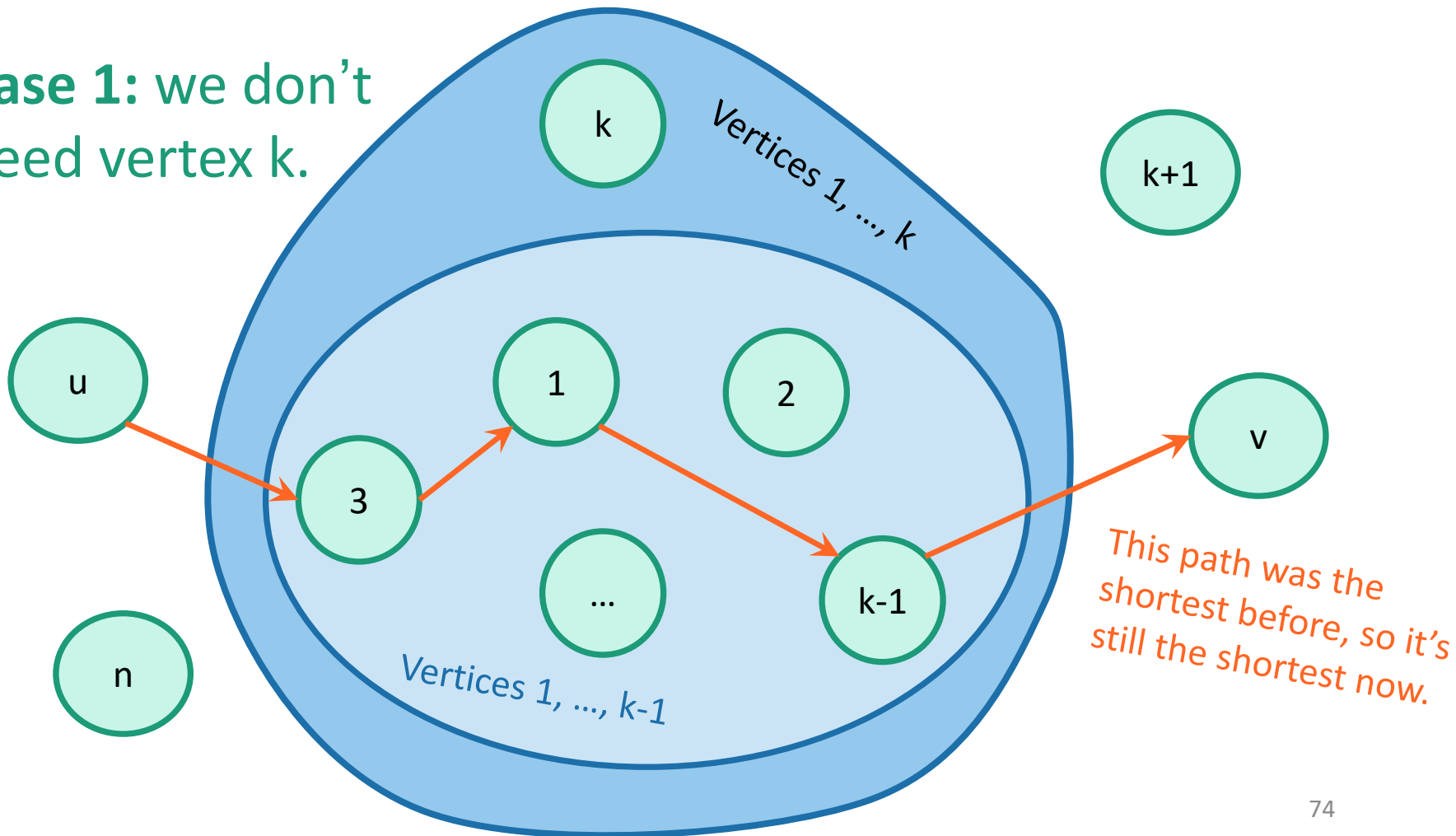
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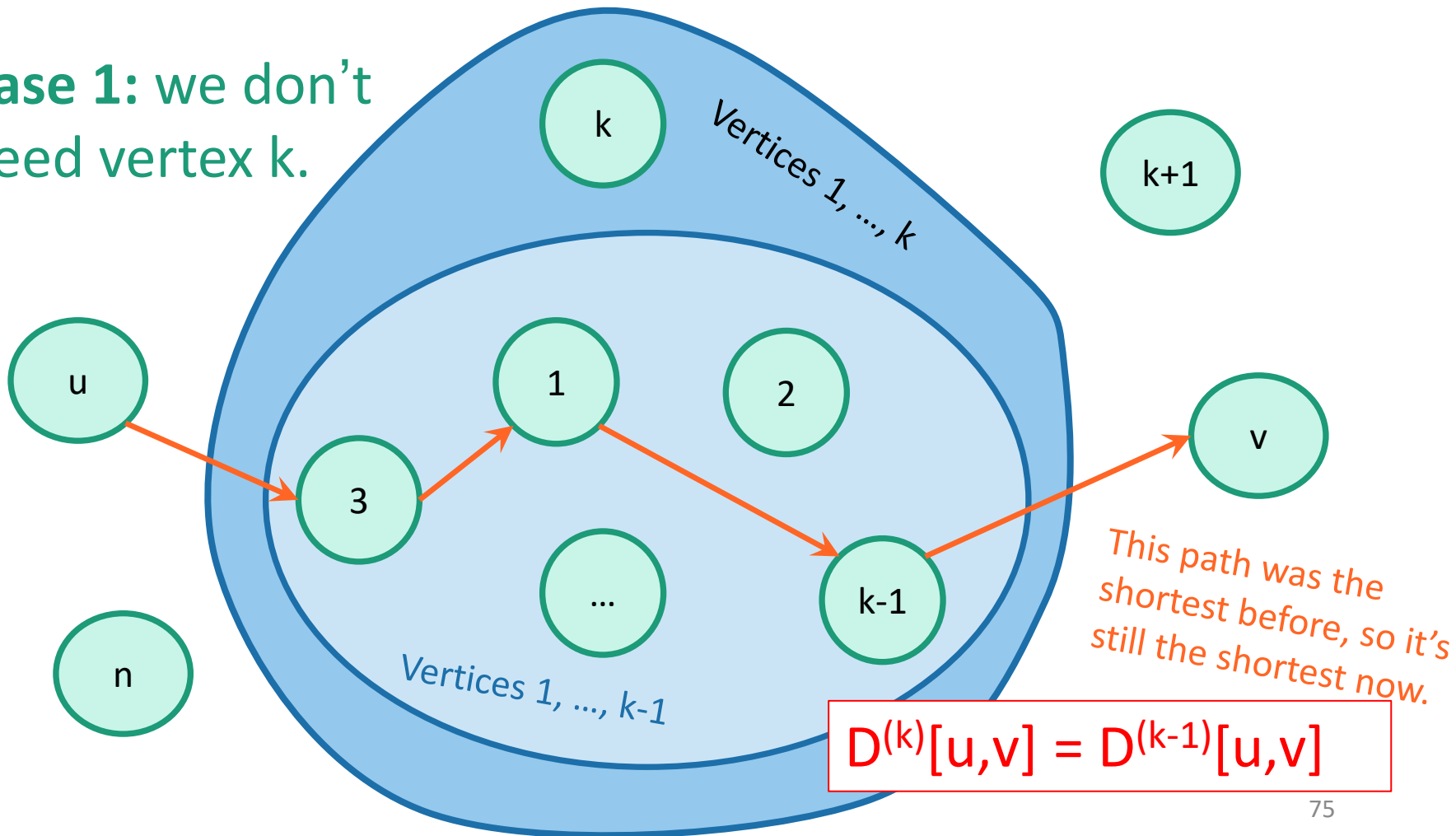
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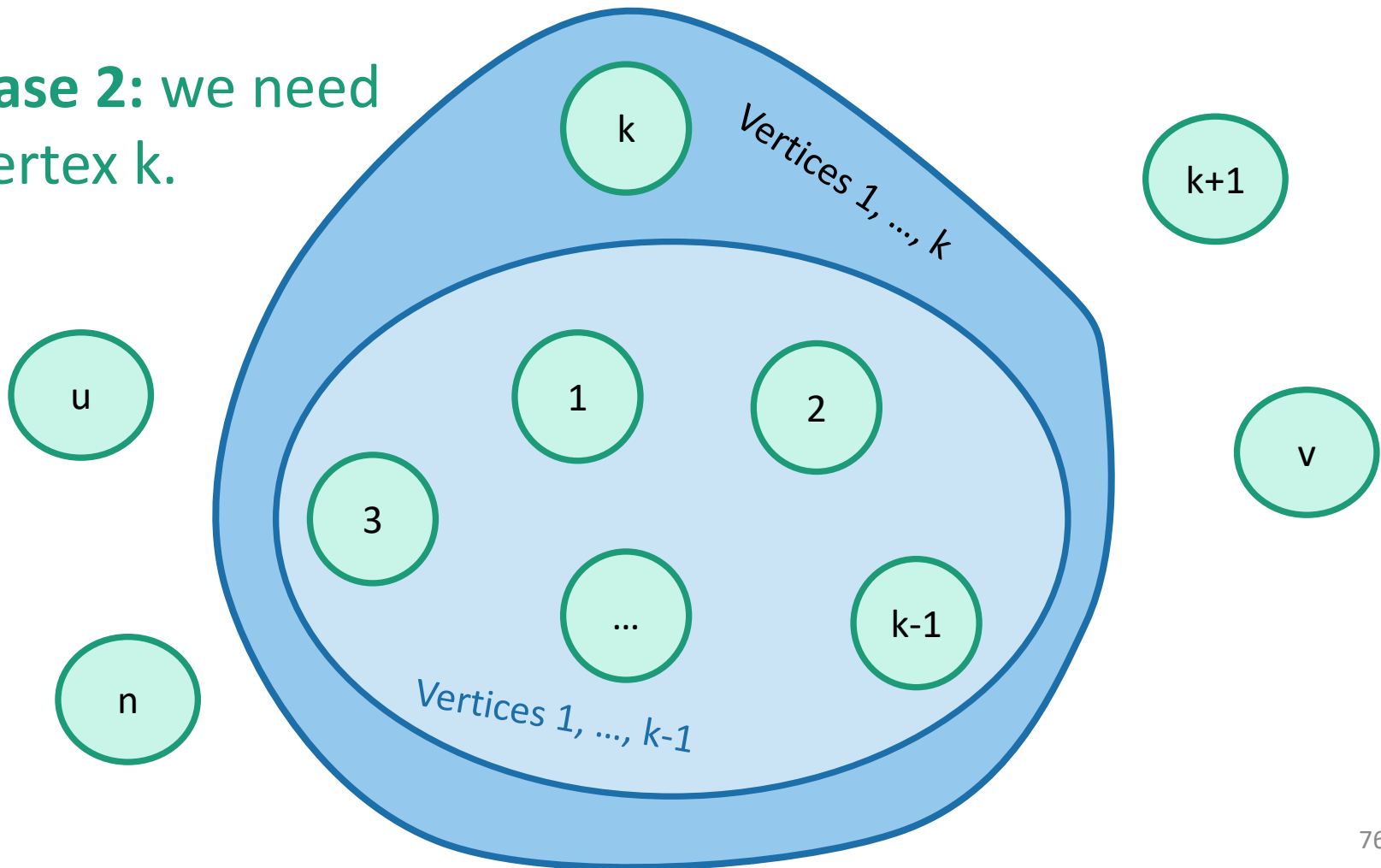
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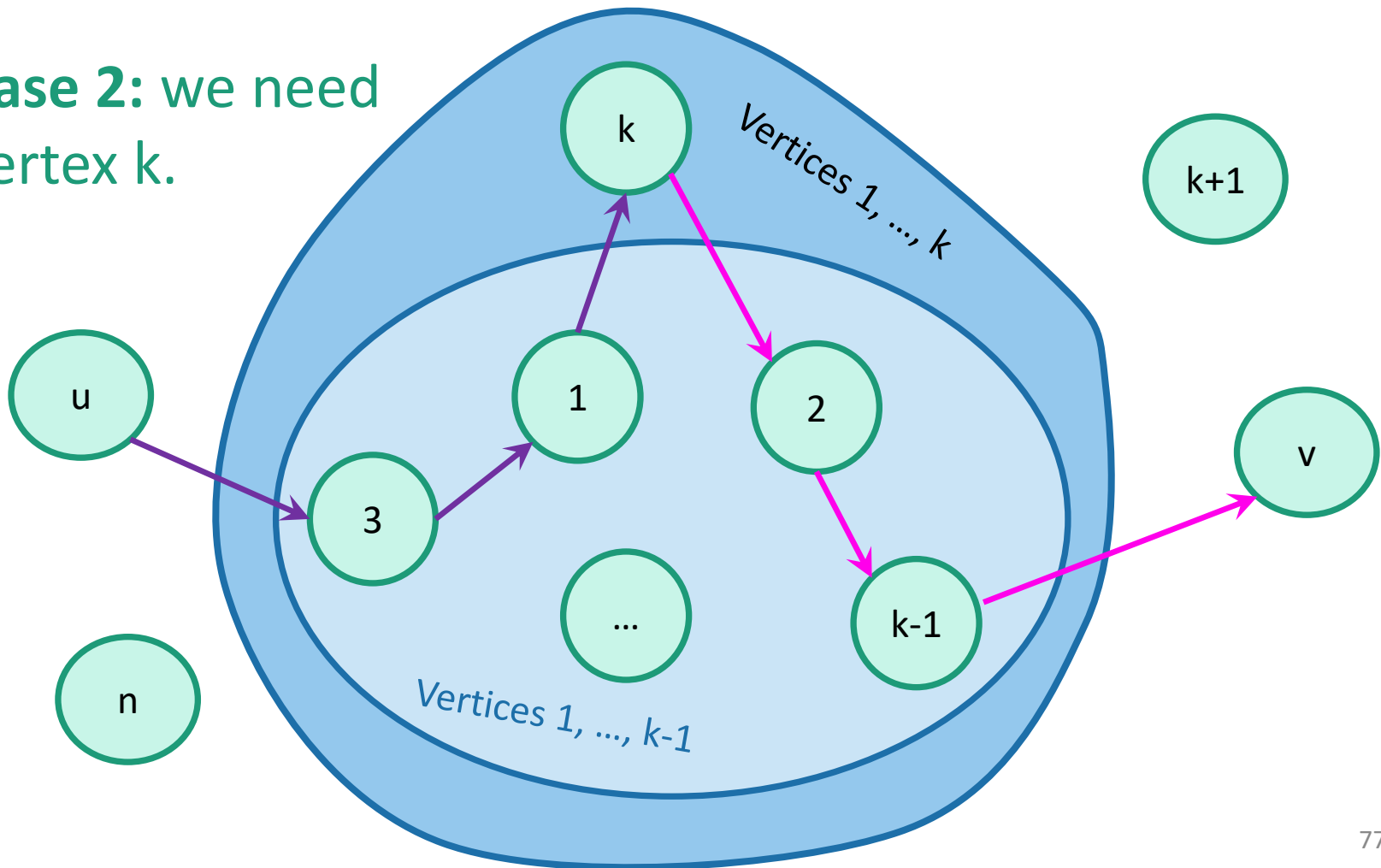
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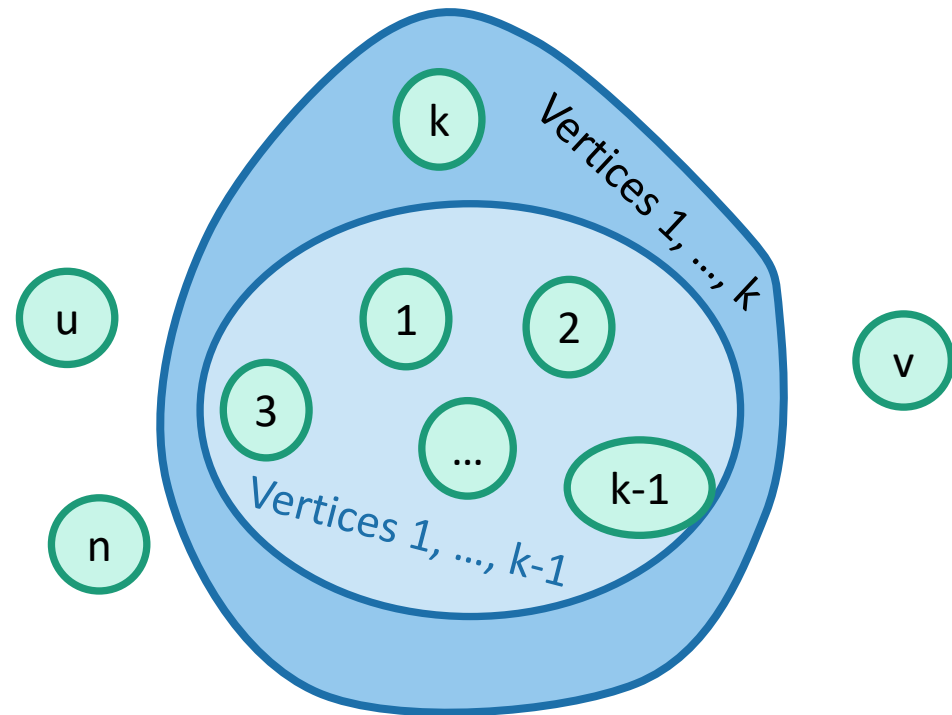
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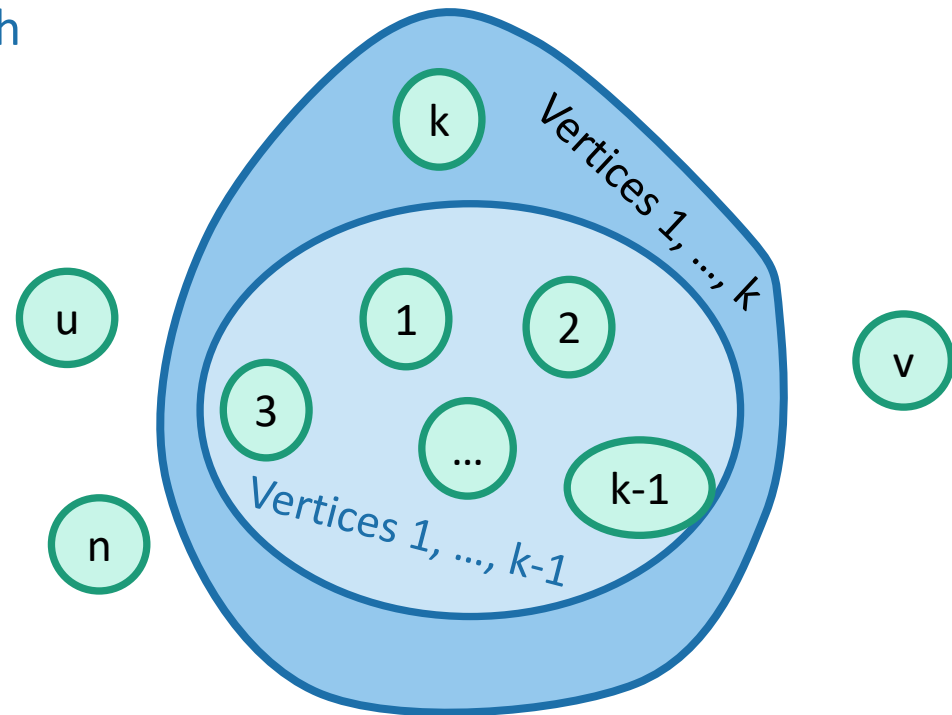
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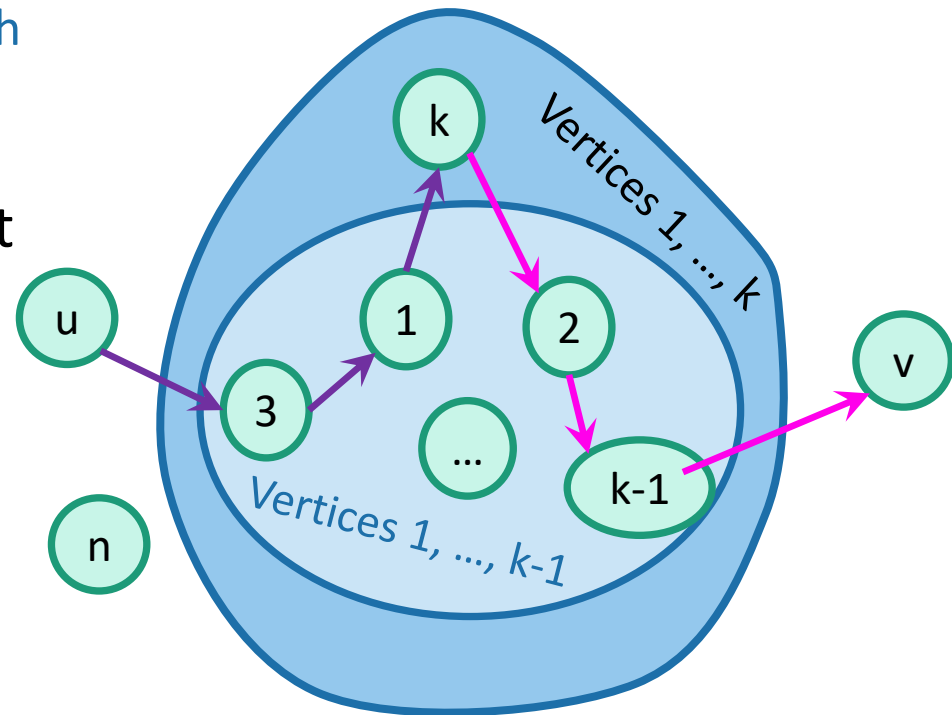
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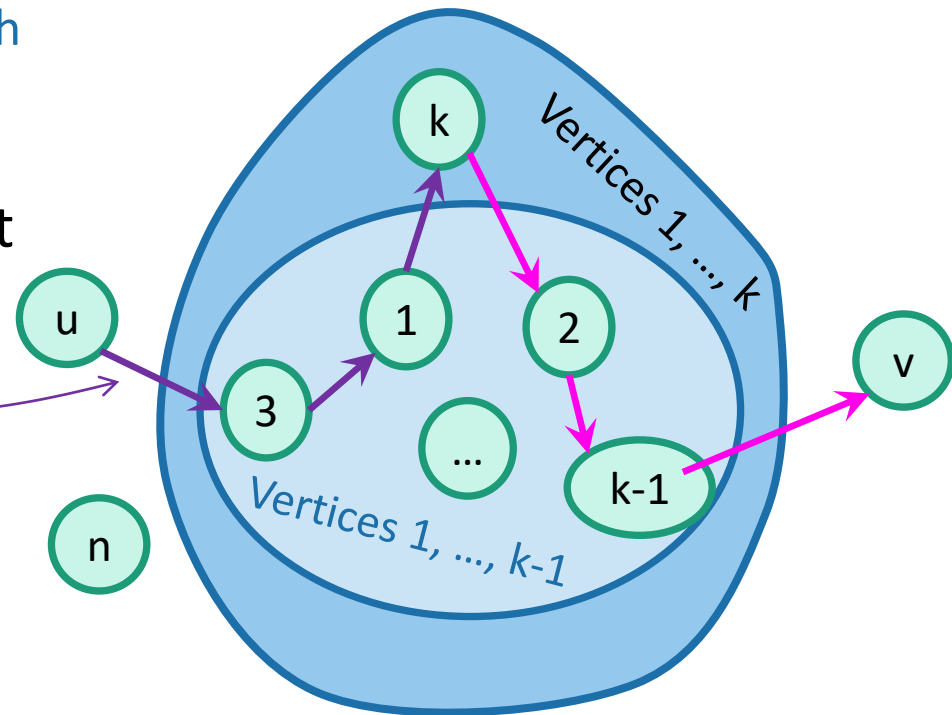
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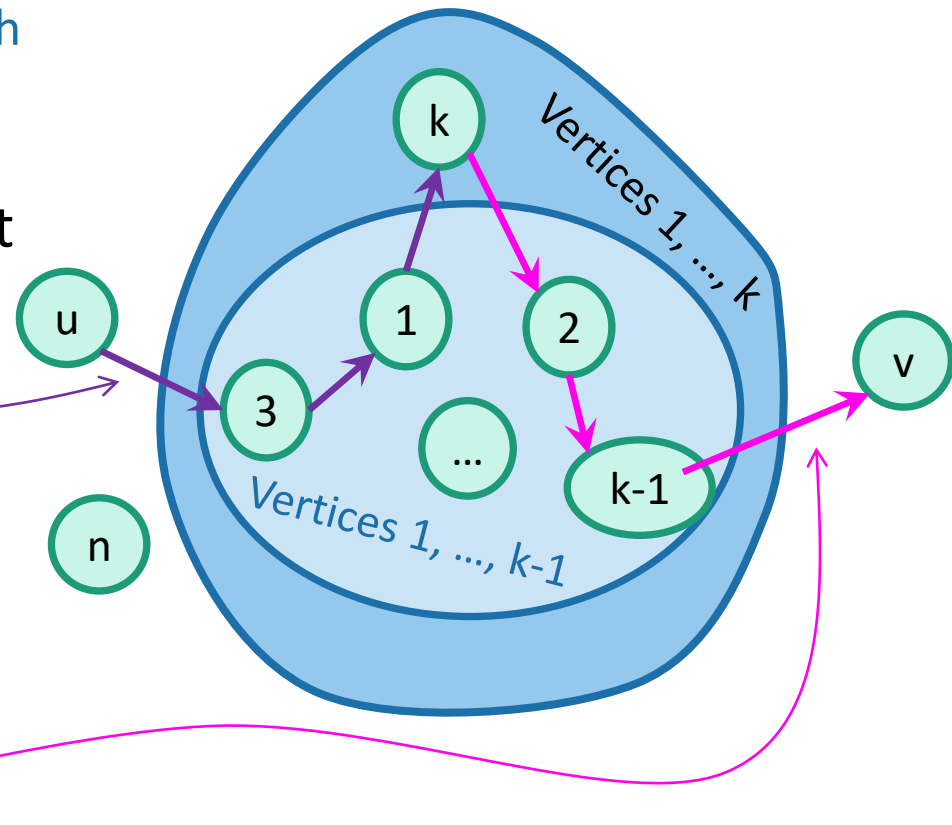
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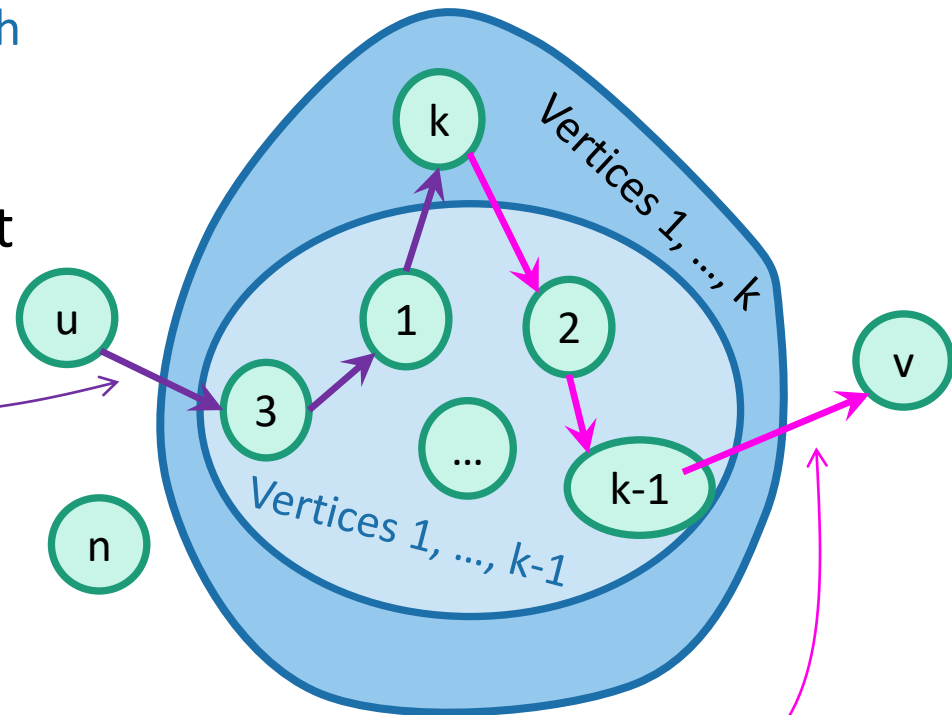
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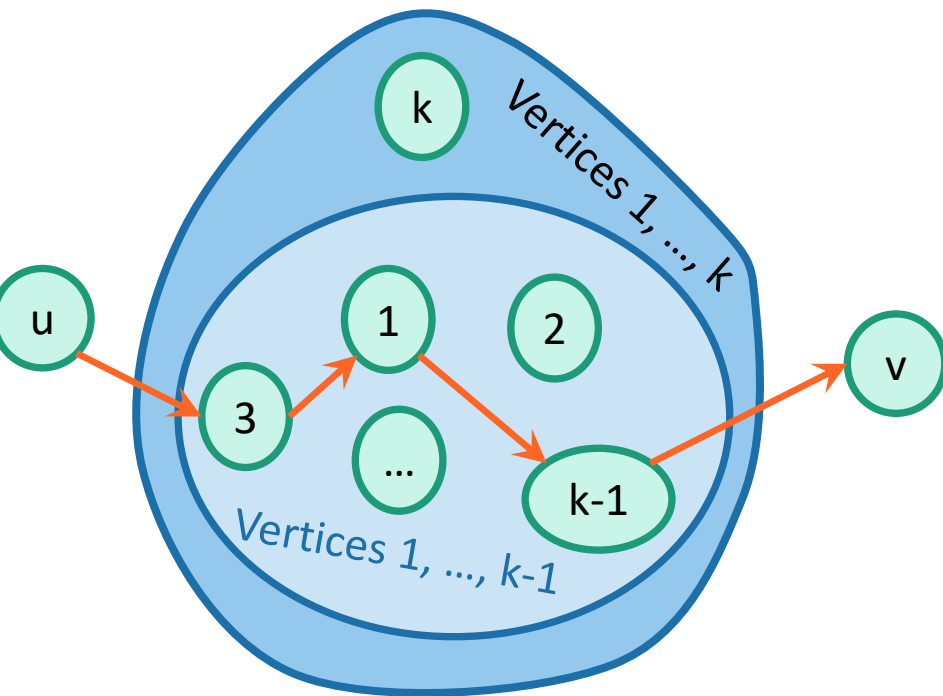
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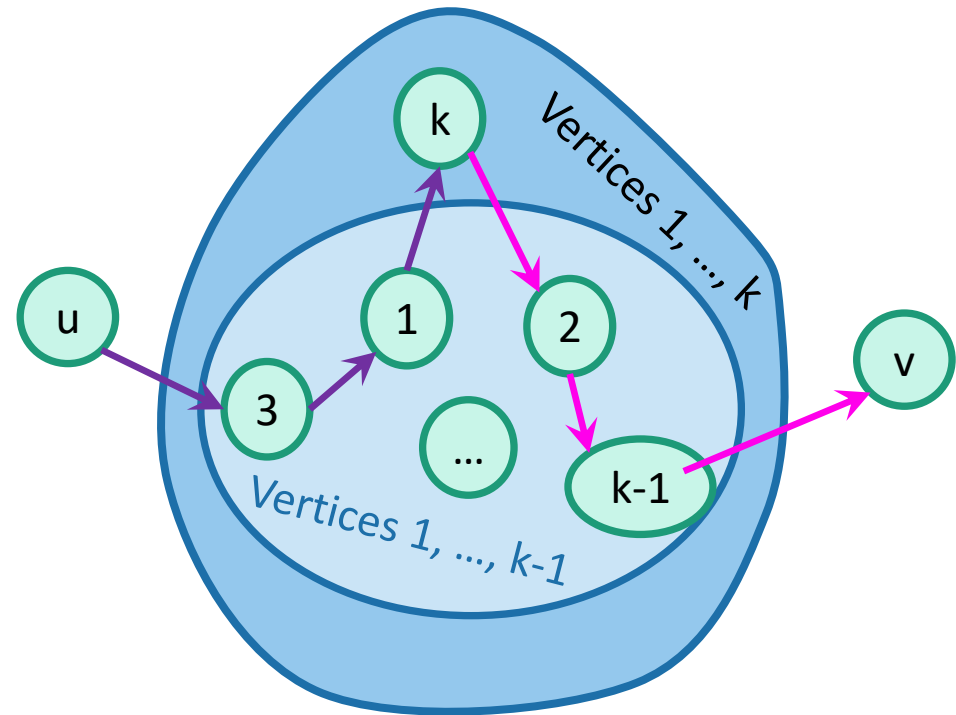
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- Optimal substructure:
 - We can solve the big problem using solutions to smaller problems.
- Overlapping sub-problems:
 - $D^{(k-1)}[k,v]$ can be used to help compute $D^{(k)}[u,v]$ for lots of different u 's.

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- Using our ***Dynamic programming*** paradigm, this immediately gives us an algorithm!



Floyd-Warshall algorithm


- Initialize n -by- n arrays $D^{(k)}$ for $k = 0, \dots, n$
 - $D^{(k)}[u,u] = 0$ for all u , for all k
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This is a bottom-up *Dynamic programming* algorithm.

We've basically just shown

- Theorem:

If there are **no negative cycles** in a weighted directed graph G , then the Floyd-Warshall algorithm, running on G , returns a matrix $D^{(n)}$ so that:

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As with Bellman-Ford, we don't really need to store all n of the $D^{(k)}$.

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- Algorithm:
 - Run Floyd-Warshall as before.
 - If there is some v so that $D^{(n)}[v,v] < 0$:
 - **return** negative cycle.

What have we learned?

- The Floyd-Warshall algorithm is another example of *dynamic programming*.
- It computes All Pairs Shortest Paths in a directed weighted graph in time $O(n^3)$.

Recap

- ***Two shortest-path algorithms:***
 - Bellman-Ford for single-source shortest path
 - Floyd-Warshall for all-pairs shortest path
- ***Dynamic programming!***
 - This is a fancy name for:
 - Break up an optimization problem into smaller problems
 - The optimal solutions to the sub-problems should be sub-solutions to the original problem.
 - Build the optimal solution iteratively by filling in a table of sub-solutions.
 - Take advantage of overlapping sub-problems!

Next time

- More examples of *dynamic programming*!

We will stop bullets with our
action-packed coding skills, and
also maybe find longest
common subsequences.



Acknowledgement

- Stanford University