AIME Handout

Trigonometry in the AIME and USA(J)MO

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For:
AoPS

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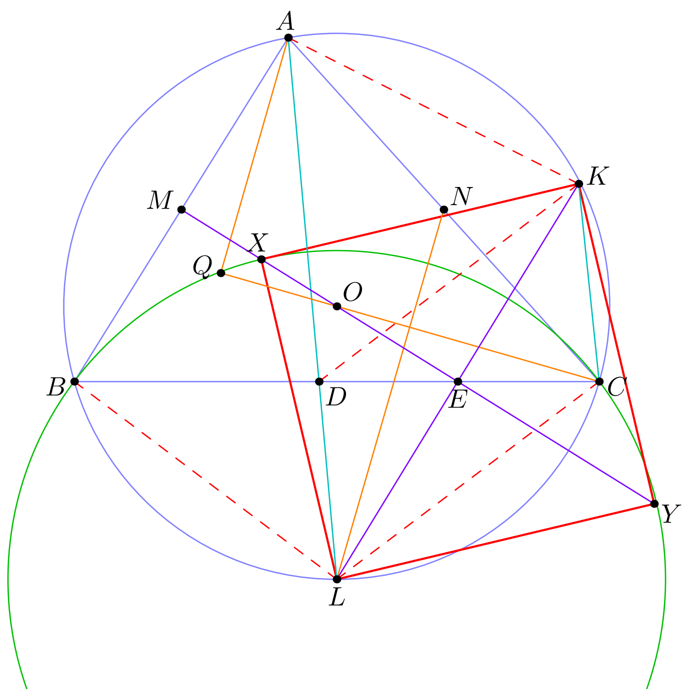


Yet another beauty by Evan

I was trying to unravel the complicated trigonometry of the radical thought that silence could make up the greatest lie ever told. - Pat Conroy

Contents

§0 Acknowledgements

This was made for the Art of Problem Solving Community out there! We would like to thank Evan Chen for his evan.sty code. In addition, all problems in the handout were either copied from the Art of Problem Solving Wiki or made by ourselves.

Art of Problem Solving Community

Evan Chen's Personal Sty File

Say hi!

freeman66

And Evan says he would like this here for evan.sty:

He also helped with the hint formatting. Evan is a LaTeXgod!
And finally, please do not make any copies of this document without referencing this original one. At least cite us when you are using this document.

§1 Introduction

§1.1 Motivation and Goals

Trigonometry is one of the main ways to solve a geometry problem. Although there are synthetic solutions, trigonometry frequently offers an solution that is very easy to find - even in the middle of the AIME or USA(J)MO. Here's a fish we will be trying to chase:

Problem 1 (2016 AIME II Problem 14)
Equilateral $\triangle ABC$ has side length 600. Points P and Q lie outside the plane of $\triangle ABC$ and are on opposite sides of the plane. Furthermore, $PA = PB = PC$, and $QA = QB = QC$, and the planes of $\triangle PAB$ and $\triangle QAB$ form a 120° dihedral angle (the angle between the two planes). There is a point O whose distance from each of A, B, C, P, and Q is d. Find d.

Geometry in three dimensions often is very hard to visualize - that's why it's useful to use algebraic vectors, which will be talked about in the section 3-D Geometry, as a way to easily manipulate such things. A second such problem follows:

Problem 2 (2014 AIME II Problem 12)
Suppose that the angles of $\triangle ABC$ satisfy $\cos(3A) + \cos(3B) + \cos(3C) = 1$. Two sides of the triangle have lengths 10 and 13. There is a positive integer m so that the maximum possible length for the remaining side of $\triangle ABC$ is $\sqrt{m}$. Find m.

Note how it is impossible to solve this problem without knowledge of trigonometry - such problems will be there on the AIME. And finally, here's a third problem:

Problem 3 (2005 AIME II Problem 12)
Square $ABCD$ has center O, $AB = 900$, E and F are on AB with $AE < BF$ and E between A and $F, m\angle EOF = 45^\circ$, and $EF = 400$. Given that $BF = p + q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime, find $p + q + r$.

A word of advice for those who intend to follow this document: almost all problems are from the AIME; a few HMMT and USA(J)MO problems might be scattered in, but remember we go into a fair amount of depth here. Many of the areas will have olympiad-style questions, but the underlying idea is that they could very well show up on the AIME, and most definitely olympiads.

§1.2 Contact

If do you have questions, comments, concerns, issues, or suggestions? Here are two ways to contact naman12 or freeman66:

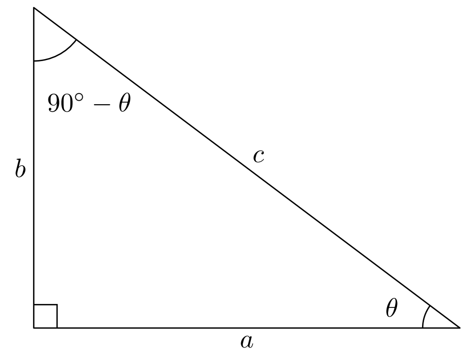
1. Send an email to realnaman12@gmail.com and I should get back to you (unless I am incorporating your suggestion into the document, then it might take a bit more time).

2. Send a private message to naman12 or freeman66 by either clicking the button that says PM or by going here and clicking New Message and typing naman12 or freeman66.

Please include something related to **Trigonometry AIME/USA(J)MO Handout** in the subject line so naman12 or freeman66 knows what you are talking about.

§2 Basic Trigonometry

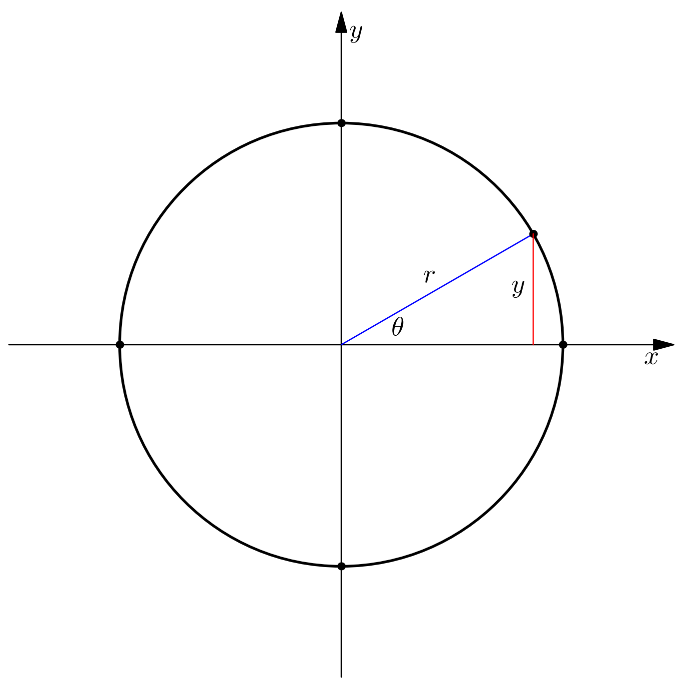
We'll start out with a right triangle. It's a nice triangle - we know an angle of $90°$. What about the other angles? Let's call one θ and the other one will be $90° - \theta$:



The big question arises: how does θ even relate to a, b, c? That's why we introduce trigonometric functions:

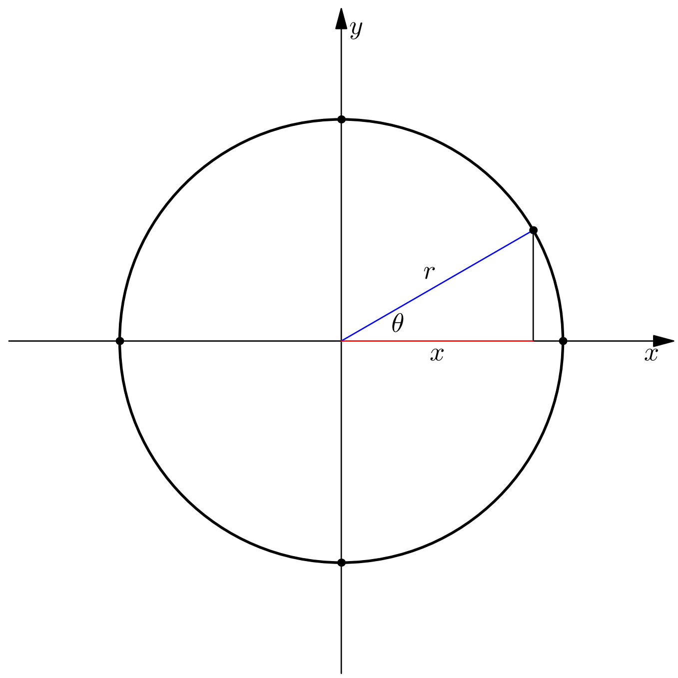
§2.1 Definitions of Trigonometric Functions

> **Definition 2.1** (Sine) — The **sine** of an angle θ is written as $\sin(\theta)$ and is the distance from the intersection of the radius of the unit circle that has been rotated θ counter clockwise about the origin from the x-axis and the diameter of the unit circle to the x-axis (in the diagram below it is $\frac{y}{r}$).



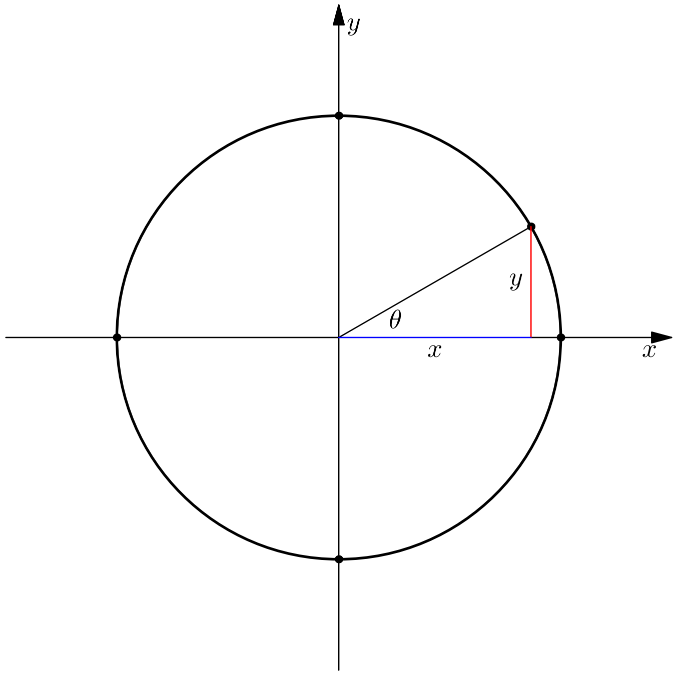
Note that when this altitude to the x-axis is below the x-axis the sine of the angle is negative. When θ is between $0°$ and $180°$ or 0 rad and π rad, then $\sin(\theta)$ is positive. In addition, when θ is between $0°$ and $90°$, $\sin(\theta)$ can be viewed in the context of a right triangle as the ratio of the length side opposite the angle to the length of the hypotenuse (think about how the radius of the unit circle is the hypotenuse of the triangle in the first definition and how from there we can scale it up for larger hypotenuses without changing the value of the sine).

Definition 2.2 (Cosine) — The **cosine** of an angle θ is written as $\cos(\theta)$ and is the distance from the intersection of the radius of the unit circle that has been rotated θ counter clockwise about the origin from the x-axis and the diameter of the unit circle to the x-axis (in the diagram below it is $\frac{x}{r}$).



Similar to the sine, the cosine is negative when the point is to the left of the y axis (i.e. for $90^\circ < \theta < 270^\circ$). In addition, for angles angles between 0° and 90°, the cosine can be seen in the context of a right triangle as the ratio of the lengths of the side adjacent to the angle over the hypotenuse of the triangle (again, think about scaling up the unit circle).

Definition 2.3 (Tangent) — The **tangent** of an angle θ is written as $\tan(\theta)$ equivalent to the ratio of the length of the line segment opposite the angle to the length of the line segment adjacent to the angle (that is not the radius of the circle, i.e. the hypotenuse)



The tangent is negative when exactly one of the sine cosine is negative. The tangent can also be seen as $\frac{\sin\theta}{\cos\theta}$. Thinking about the right triangle definitions of sine and cosine, we can get that for angles between 0° and 180°, the tangent in a right triangle is equal to the ratio of the side opposite the angle to the side adjacent to the angle.

Definition 2.4 (SOH-CAH-TOA) — If a is the length of the side opposite θ in a right triangle, and b is the length of the side adjacent to θ, and c is the length of the hypotenuse, then

$$\begin{aligned}\sin(\theta) &= \tfrac{a}{c}\\ \cos(\theta) &= \tfrac{b}{c}\\ \tan(\theta) &= \tfrac{a}{b}\\ \cot(\theta) &= \tfrac{b}{a}\\ \sec(\theta) &= \tfrac{c}{b}\\ \csc(\theta) &= \tfrac{c}{a}.\end{aligned}$$

This is commonly memorized as SOH-CAH-TOA, where S represents sine, C represents cosine, T represents tangent, all Os represent opposite (the leg opposite the angle), all As represent adjacent (the leg adjacent/-touching the angle), and H represents hypotenuse. Using the above definition of $\sin(\theta)$ and $\cos(\theta)$, we can similarly define

$$\begin{aligned}\tan(\theta) &= \tfrac{\sin(\theta)}{\cos(\theta)}\\ \cot(\theta) &= \tfrac{\cos(\theta)}{\sin(\theta)}\\ \sec(\theta) &= \tfrac{1}{\cos(\theta)}\\ \csc(\theta) &= \tfrac{1}{\sin(\theta)}\end{aligned}$$

§2.2 Trigonometry on the Unit Circle

Although these definitions are accurate, there is a sense in which they are lacking, because the angle θ in a right triangle can only have a measure between 0° and 90°. We need a definition which will allow the domain of the sine function to be the set of all real numbers. Our definition will make use of the unit circle, $x^2 + y^2 = 1$. We first associate every real number t with a point on the unit circle. This is done by "wrapping" the real line around the circle so that the number zero on the real line gets associated with the point $(0, 1)$ on the circle. A way of describing this association is to say that for a given t, if $t > 0$ we simply start at the point $(0, 1)$ and move our pencil counterclockwise around the circle until the tip has moved t units. The point we stop at is the point associated with the number t. If $t < 0$, we do the same thing except we move clockwise. If $t = 0$, we simply put our pencil on $(0, 1)$ and don't move. Using this association, we can now define $\cos(t)$ and $\sin(t)$.

Using the above association of t with a point $(x(t), y(t))$ on the unit circle, we define $\cos(t)$ to be the function $x(t)$, and $\sin(t)$ to be the function $y(t)$, that is, we define $\cos(t)$ to be the x coordinate of the point on the unit circle obtained in the above association, and define $\sin(t)$ to be the y coordinate of the point on the unit circle obtained in the above association.

Exercise 2.5. What point on the unit circle corresponds with $t = \pi$? What therefore is $\cos(\pi)$ and $\sin(\pi)$?

Exercise 2.6. What point on the unit circle correspond with $t = \frac{3\pi}{2}$? What therefore is $\cos(\frac{3\pi}{2})$?

§2.3 Radian Measure

Definition 2.7 (Radian) — A **radian** is defined to be the measure of an angle cut of in the circle of radius one by an arc of length one. Thus, a $90°$ angle corresponds to an angle of radian measure $\frac{\pi}{2}$, since the distance one fourth of the way around the unit circle is $\frac{\pi}{2}$.

It is also useful to note that an angle of measure $1°$ corresponds with an angle of radian measure $\frac{\pi}{180}$, since 90 of these would correspond to a right angle. Also, an angle of radian measure 1 would correspond to an angle of measure $\left(\frac{180}{\pi}\right)^\circ$, since $\frac{\pi}{2}$ of these would correspond to a right angle. These facts are enough to help you convert from degrees to radians and back, when necessary.

Exercise 2.8. What is the degree measure of the angle $\theta = \frac{\pi}{6}$?

Exercise 2.9. What is the radian measure of the angle $225°$?

§2.4 Properties of Trigonometric Functions

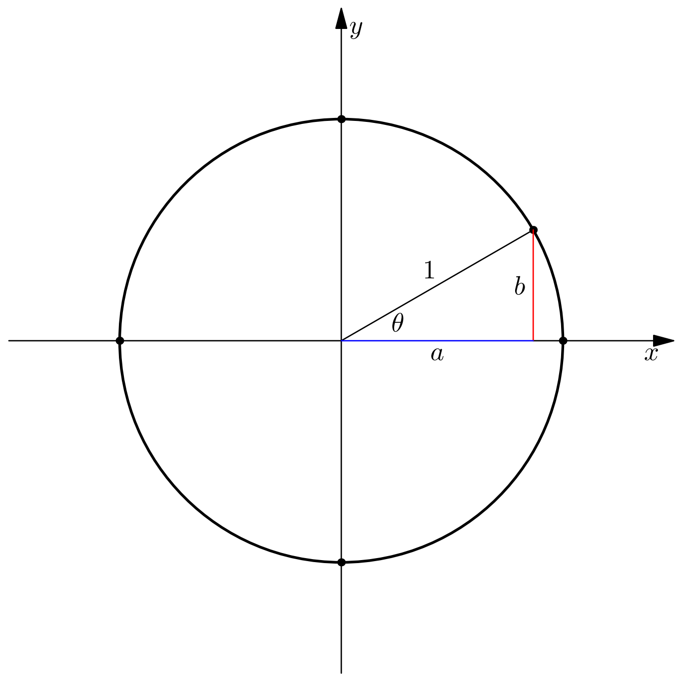
Theorem 2.10 (Trigonometric Properties)

The following are some properties of functions:

1. Range of $\sin(x)$ and $\cos(x)$: $-1 \le \sin(x) \le 1$, $-1 \le \cos(x) \le 1$.
2. $\cos(x)$ is Even: $\cos(-x) = \cos(x)$.
3. $\sin(x)$ is Odd: $\sin(-x) = -\sin(x)$.
4. Periodicity: $\sin(x + 2\pi) = \sin(x)$, $\cos(x + 2\pi) = \cos(x)$.

Remark 2.11. Don't get fooled! $\sin^2(x)$ doesn't mean $\sin(\sin(x))$ - rather, it means $(\sin(x))^2$. But later, you will learn that $\sin^{-1}(x) \neq \frac{1}{\sin(x)}$ - it's actually the angle y such that $\sin(y) = x$. While this seems confusing for now, you will get accustomed to it.

Proofs. 1. Take a look at the unit circle again:



We can see that a and b are fully contained inside the unit circle. However, this means that $|a|$ and $|b|$ are at most 1 (as they are contained in a circle radius 1). Thus, we get that

$$|x| \leq 1 \implies -1 \leq a \leq 1$$

$$|y| \leq 1 \implies -1 \leq b \leq 1$$

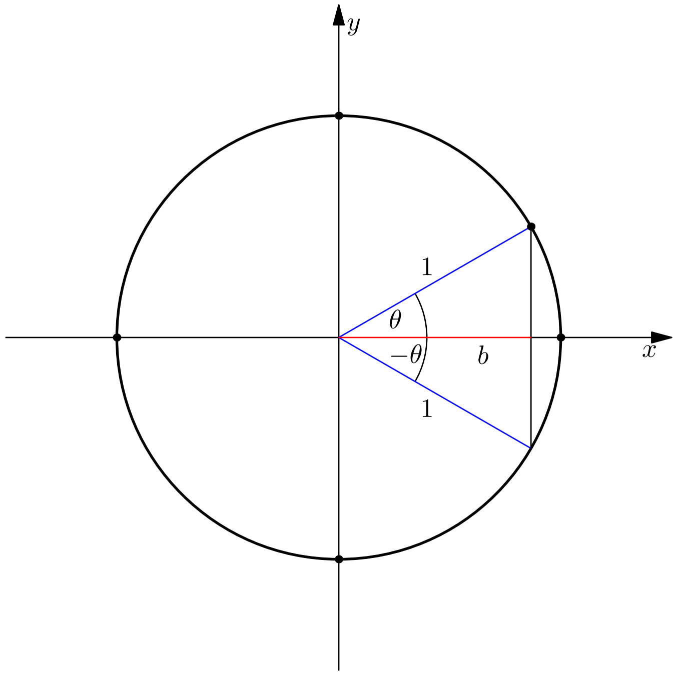
However, we know that $a = \sin x$ and $b = \cos x$, so then we get

$$-1 \leq \sin x \leq 1$$

$$-1 \leq \cos x \leq 1$$

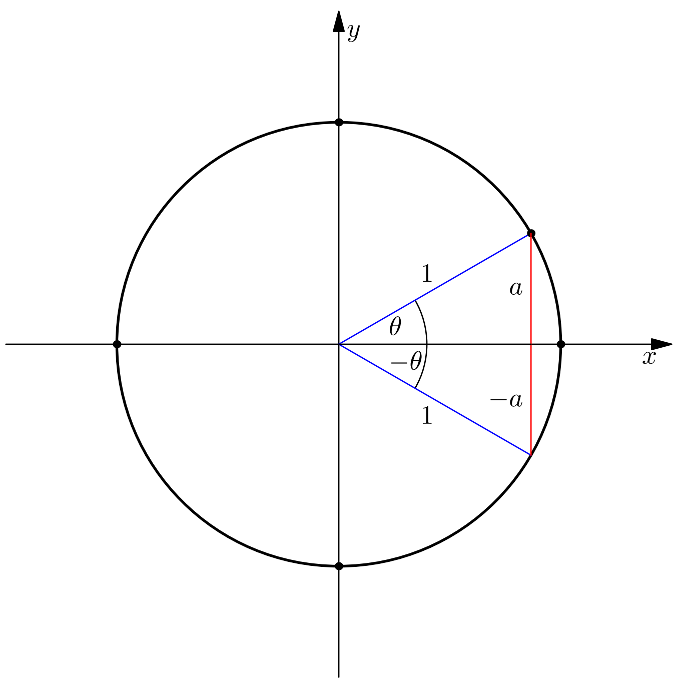
> **Remark 2.12.** Typically, when it is unambiguous, we will resort to writing $\sin x$ instead of $\sin(x)$. However, if there is a chance of misinterpretation, we shall include parenthesis.

2. Once again, we resort to the unit circle:



We see this is just a reflection over the x-axis - in particular, the value of the x-coordinate, b, stays the same. However, we know that this particular value is $\cos\theta$, so we get that $\cos\theta = \cos -\theta = b$.

3. Can you guess what we will use? The unit circle:



We see this is just a reflection over the x-axis - in particular, the value of the y-coordinate, a, becomes negative. However, we know that this particular value is $\sin\theta$, so we get that $\sin\theta = -\sin -\theta = a$.

4. Think of this visually - as $2\pi = 360°$, in reality, we are just going all the way around the circle, so indeed the point corresponding to $(\cos x, \sin x)$ also corresponds to $(\cos(2\pi + x), \sin(2\pi + x))$.

$\square$

§2.5 Graphs of Trigonometric Functions

§2.5.1 Graph of $\sin(x)$ and $\cos(x)$

Note that from the definition of sine and cosine, it is clear that the domain of each of these is the set of all real numbers. Also, from the properties above, we know that the range of both of these is the set of numbers between -1 and 1, and that the functions are periodic. This information, together with a few points plotted as a guide, are enough to graph the two functions. Note that if we shift the graph of the sine function by $\frac{\pi}{2}$ units to the left, we get the graph of the cosine function. This is related to the fact that $\sin(x - \frac{\pi}{2}) = \cos(x)$.

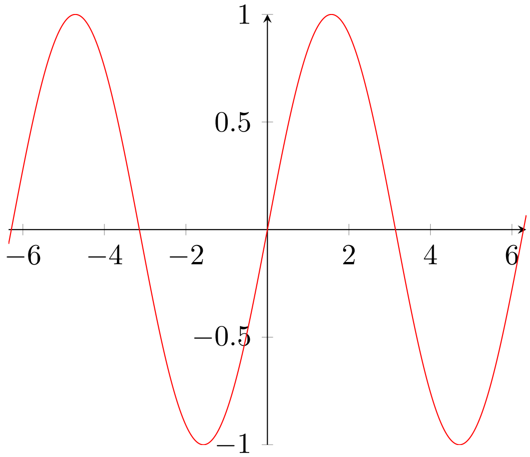


Figure 1: Graph of $\sin x$

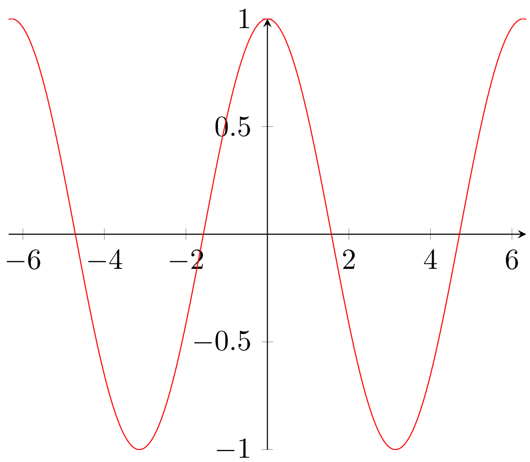


Figure 2: Graph of $\cos x$

§2.5.2 Graph of $\tan(x)$ and $\cot(x)$

Note that the domain of $\tan(x)$ is the set of all real numbers except those at which $\cos(x) = 0$. Thus, the points $\frac{\pi}{2}$, $\frac{3\pi}{2}$, and so on aren't in the domain of $\tan(x)$. An easy way to characterize these points is to say that these are all the points which have the form $\frac{\pi}{2} + k\pi$, where k is any integer. Thus the domain of the tangent function is everything unless $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$.

Exercise 2.13. What is the domain of $\cot(x)$?

We can get a good grasp on the graph of $\tan(x)$ by plotting a few points and doing a careful analysis of the limiting behavior when x is near $\frac{\pi}{2}$ and the other points that aren't in the domain. Note that when x is a little less than $\frac{\pi}{2}$, $\sin(x)$ is close to 1, while $\cos(x)$ is close to zero (but is positive.)

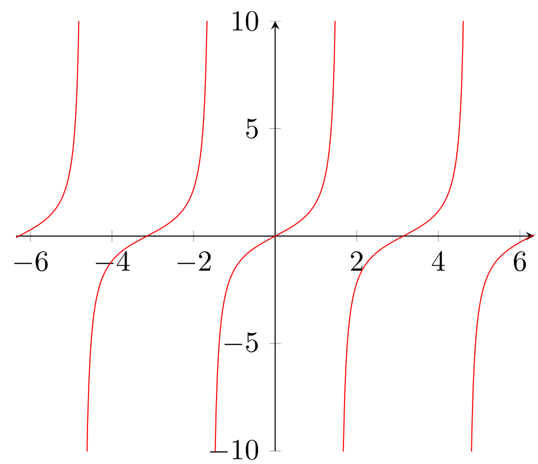


Figure 3: Graph of $\tan x$

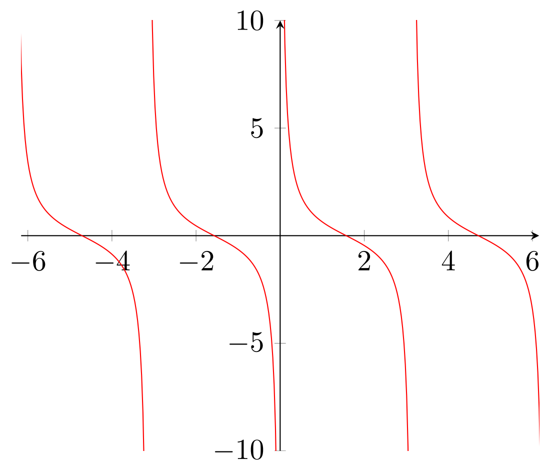


Figure 4: Graph of $\cot x$

§2.5.3 Graph of $\sec(x)$ and $\csc(x)$

Like the tangent function, the domain of the secant function is the set of all real numbers except those which make $\cos(x)$ equal to zero. Thus the domain of the secant function is the same as the domain of the tangent function. Also, the fact that the cosine function always has values between -1 and 1 tells us that $\sec(x) = \frac{1}{\cos(x)}$ always has values less than or equal to -1 or greater than or equal 1. An analysis of the limiting behavior of $\sec(x)$ near $x = \frac{\pi}{2}$ and $\frac{-\pi}{2}$ and a few strategically plotted points leads to the graph of $y = \sec(x)$.

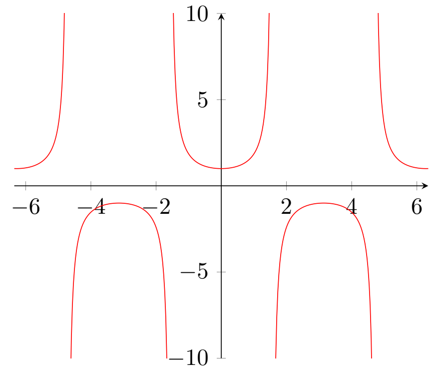


Figure 5: Graph of $\sec x$

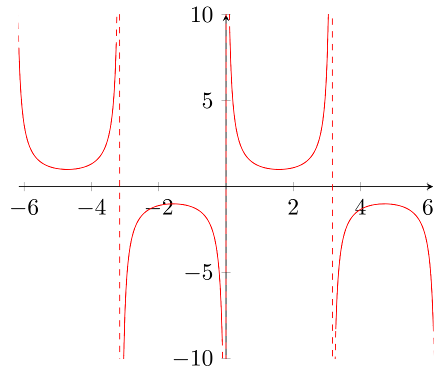


Figure 6: Graph of $\csc x$

§2.6 Bounding Sine and Cosine

The following theorem is extremely trivial but extremely useful. It is analogous to the "Trivial Inequality" of trigonometry:

Theorem 2.14 (Bounds of $\sin\theta$ and $\cos\theta$)
For all angles θ,

$$-1 \le \sin\theta \le 1,$$
$$-1 \le \cos\theta \le 1.$$

Remark 2.15. The angle θ is actually a Greek Letter, theta, and is typically used to represents angles.

Proof. Refer to Property 3 of Trigonometric Properties. □

Exercise 2.16. Bound $\tan\theta, \cot\theta, \sec\theta$, and $\csc\theta$. **Hints**: 1 2

§2.7 Periodicity

From the graphs of $\sin x$ and $\cos x$, one intuitively knows sine and cosine have periods.

Theorem 2.17 (Periods of Trigonometric Functions)

The periods of the following functions are:

1. **sine**: 2π
2. **cosine**: 2π
3. **tangent**: π
4. **cotangent**: π
5. **secant**: 2π
6. **cosecant**: 2π

Notice that both of tan and cot actually have a period of π. That's because (from the graphs) we have $\sin(x+\pi) = -\sin x$ and $\cos(x+\pi) = -\cos x$. Later, we'll also see another way to prove it with algebra.

§2.8 Trigonometric Identities

Let me now list them out:

Theorem 2.18 (Even-Odd Identities)

For all angles θ,

- $\sin(-\theta) = -\sin(\theta)$
- $\cos(-\theta) = \cos(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

Sketch of Proof. We've already seen the proof of the sin and cos. Now, the rest follows by expressing each function in terms of sin and cos. For example,

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

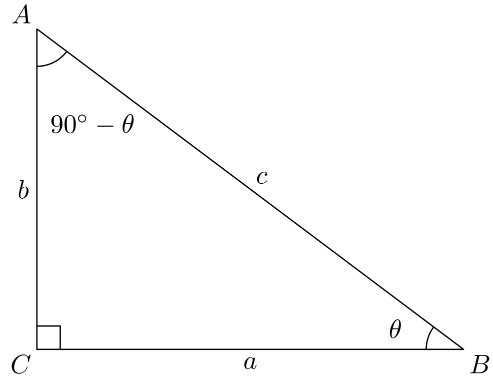
□

Theorem 2.19 (Pythagorean Identities)

For all angles θ,

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \cot^2\theta = \csc^2\theta$
- $\tan^2\theta + 1 = \sec^2\theta$

Proof. We consider the triangle $\triangle ABC$:



The Pythagorean Theorem tells us that

$$a^2 + b^2 = c^2$$

or upon dividing by c^2,

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

We now can use SOH-CAH-TOA. This tells us $\sin\theta = \frac{b}{c}$ and $\cos\theta = \frac{a}{c}$, so we can substitute to get

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

We just use the definition of Tangent and Secant:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

Now, we get

$$1 + \tan^2(\theta) = 1 + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{\sin^2(\theta) + \cos^2(\theta)}{\cos^2(\theta)}$$

However, by the first identity, we have that $\sin^2(\theta) + \cos^2(\theta) = 1$. Thus, we get

$$1 + \tan^2(\theta) = \frac{\sin^2(\theta) + \cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} = \sec^2(\theta)$$

The other one follows similarly. The definitions of Cotangent and Cosecant are:

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

Now, we get

$$1 + \cot^2(\theta) = 1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{\sin^2(\theta) + \cos^2(\theta)}{\sin^2(\theta)}$$

However, by the first identity, we have that $\sin^2(\theta) + \cos^2(\theta) = 1$. Thus, we get

$$1 + \cot^2(\theta) = \frac{\sin^2(\theta) + \cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)} = \csc^2(\theta)$$

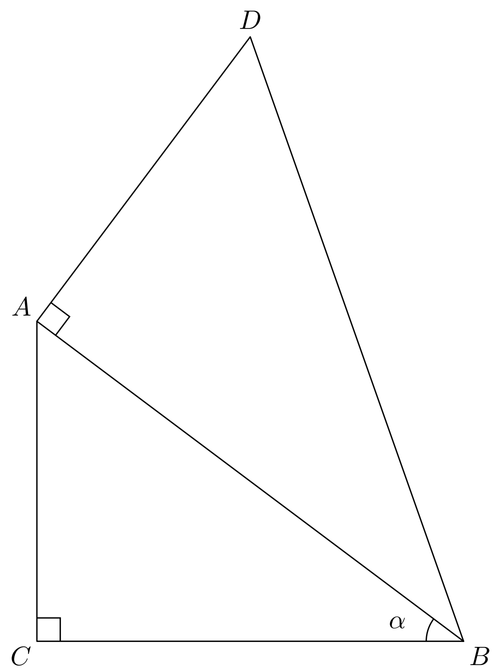
□

Theorem 2.20 (Addition-Subtraction Identities)

For all angles α and β,

- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$
- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Proof. The proof of these will feel pretty magical. That's completely intended:



$\square$

If we let $\alpha = \beta$, then

Theorem 2.21 (Double Angle Identities)

For all angles α,

- $\sin 2\alpha = 2\sin\alpha\cos\alpha$
- $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$
- $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$
- $\csc(2a) = \frac{\csc(a)\sec(a)}{2}$
- $\sec(2a) = \frac{1}{2\cos^2(a)-1} = \frac{1}{\cos^2(a)-\sin^2(a)} = \frac{1}{1-2\sin^2(a)}$
- $\cot(2a) = \frac{1-\tan^2(a)}{2\tan(a)}$

Theorem 2.22 (Half Angle Identities)
For all angles θ,

- $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$
- $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$
- $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$

Theorem 2.23 (Sum to Product Identities)
For all angles θ and γ,

- $\sin\theta + \sin\gamma = 2\sin\dfrac{\theta+\gamma}{2}\cos\dfrac{\theta-\gamma}{2}$
- $\sin\theta - \sin\gamma = 2\sin\dfrac{\theta-\gamma}{2}\cos\dfrac{\theta+\gamma}{2}$
- $\cos\theta + \cos\gamma = 2\cos\dfrac{\theta+\gamma}{2}\cos\dfrac{\theta-\gamma}{2}$
- $\cos\theta - \cos\gamma = -2\sin\dfrac{\theta+\gamma}{2}\sin\dfrac{\theta-\gamma}{2}$

Theorem 2.24 (Trigonometric Laws)
In triangle ABC with $a = BC, b = CA, c = AB$,

- Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cos A$
- Law of Tangents: $\frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)} = \frac{a-b}{a+b}$

Theorem 2.25 (Potpourri)

Some other identities:

- $\sin(90-\theta)=\cos(\theta)$
- $\cos(90-\theta)=\sin(\theta)$
- $\tan(90-\theta)=\cot(\theta)$
- $\sin(180-\theta)=\sin(\theta)$
- $\cos(180-\theta)=-\cos(\theta)$
- $\tan(180-\theta)=-\tan(\theta)$
- $e^{i\theta}=\cos\theta+i\sin\theta$ (This is also written as $\operatorname{cis}\theta$)
- $|1-e^{i\theta}|=2\sin\frac{\theta}{2}$
- $(\tan\theta+\sec\theta)^2=\frac{1+\sin\theta}{1-\sin\theta}$
- $\sin(\theta)=\cos(\theta)\tan(\theta)$
- $\cos(\theta)=\frac{\sin(\theta)}{\tan(\theta)}$
- $\sec(\theta)=\frac{\tan(\theta)}{\sin(\theta)}$
- $\arctan(x)+\arctan(y)=\arctan\left(\dfrac{x+y}{1-xy}\right)$
- $\sin^2(\theta)+\cos^2(\theta)+\tan^2(\theta)=\sec^2(\theta)$
- $\sin^2(\theta)+\cos^2(\theta)+\cot^2(\theta)=\csc^2(\theta)$

Example 2.26 (AIME I 2012/12)

Let $\triangle ABC$ be a right triangle with right angle at C. Let D and E be points on $\overline{AB}$ with D between A and E such that $\overline{CD}$ and $\overline{CE}$ trisect $\angle C$. If $\frac{DE}{BE}=\frac{8}{15}$, then $\tan B$ can be written as $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers, and p is a positive integer not divisible by the square of any prime. Find $m+n+p$.

Solution. Let $CB=1$, and let the feet of the altitudes from D and E to $\overline{CB}$ be D' and E', respectively. Also, let $DE=8k$ and $EB=15k$. We see that $BD'=15k\cos B$ and $BE'=23k\cos B$ by right triangles $\triangle BDD'$ and $\triangle BEE'$. From this we have that $D'E'=8k\cos B$. With the same triangles we have $DD'=23k\sin B$ and $EE'=15k\sin B$. From $30^\circ-60^\circ-90^\circ$ triangles $\triangle CDD'$ and $\triangle CEE'$, we see that $CD'=\frac{23k\sqrt{3}\sin B}{3}$ and $CE'=15k\sqrt{3}\sin B$, so $D'E'=\frac{22k\sqrt{3}\sin B}{3}$. From our two values of $D'E'$ we get:

$$8k\cos B=\frac{22k\sqrt{3}\sin B}{3},$$

$$\frac{\sin B}{\cos B}=\frac{8k}{\frac{22k\sqrt{3}}{3}}=\tan B,$$

$$\tan B = \frac{8}{\frac{22\sqrt{3}}{3}} = \frac{24}{22\sqrt{3}} = \frac{8\sqrt{3}}{22} = \frac{4\sqrt{3}}{11}.$$

Thus, $m = 4, n = 3, p = 11$, so $4 + 3 + 11 = \boxed{018}$. □

Example 2.27 (AIME 1983/4)

A machine-shop cutting tool has the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of AB is 6 cm and that of BC is 2 cm. The angle ABC is a right angle. Find the square of the distance (in centimeters) from B to the center of the circle.

Solution. Draw segment OB with length x, and draw radius OQ such that OQ bisects chord AC at point M. This also means that OQ is perpendicular to AC. By the Pythagorean Theorem, we get that $AC = \sqrt{(BC)^2 + (AB)^2} = 2\sqrt{10}$, and therefore $AM = \sqrt{10}$. Also by the Pythagorean theorem, we can find that $OM = \sqrt{50 - 10} = 2\sqrt{10}$.

Next, find $\angle BAC = \arctan\left(\frac{2}{6}\right)$ and $\angle OAM = \arctan\left(\frac{2\sqrt{10}}{\sqrt{10}}\right)$. Since $\angle OAB = \angle OAM - \angle BAC$, we get

$$\angle OAB = \arctan 2 - \arctan\frac{1}{3}$$

$$\tan\left(\angle OAB\right) = \tan\left(\arctan 2 - \arctan\frac{1}{3}\right)$$

By the subtraction formula for tan, we get

$$\tan\left(\angle OAB\right) = \frac{2 - \frac{1}{3}}{1 + 2 \cdot \frac{1}{3}}$$

$$\tan\left(\angle OAB\right) = 1$$

$$\cos\left(\angle OAB\right) = \frac{1}{\sqrt{2}}$$

Finally, by the Law of Cosines on $\triangle OAB$, we get

$$x^2 = 50 + 36 - 2(6)\sqrt{50}\frac{1}{\sqrt{2}}$$

$$x^2 = \boxed{026}.$$

□

§2.9 Exercises

Exercise 2.28 (AIME 1995/7)**.** Given that $(1+\sin t)(1+\cos t) = \frac{5}{4}$, compute $(1-\sin t)(1-\cos t)$.

Exercise 2.29. If $\cos x + \sin x = 0.2$, compute $\cos^4 x + \sin^4 x$.

Exercise 2.30. Compute $\sin 18^\circ$.

Exercise 2.31 (AIME II 2000/15)**.** Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.$$

Exercise 2.32 (AIME I 2006/12)**.** Find the sum of the values of x such that $\cos^3 3x + \cos^3 5x = 8\cos^3 4x \cos^3 x$, where x is measured in degrees and $100 < x < 200$.

Exercise 2.33. The angle θ has the property that

$$\sin\theta + \cos\theta = \frac{2}{3}.$$

Compute $\sin 2\theta$.

Exercise 2.34. Compute the exact numerical value of

$$\cos\frac{\pi}{9}\cos\frac{3\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}.$$

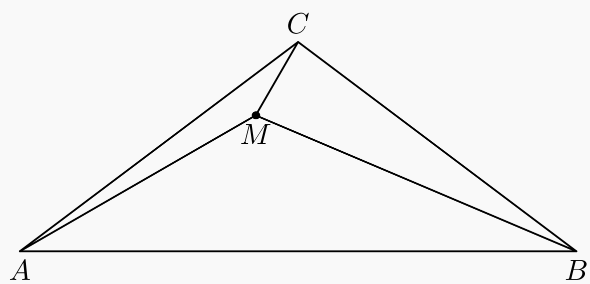
Exercise 2.35. Determine all real $0 \le \theta < 2\pi$ such tath

$$1 + \sin 2\theta = \sin\left(\theta + \frac{\pi}{4}\right).$$

Exercise 2.36. Determine the sum of the values of $\tan\theta$ for which $0 \le \theta < \pi$ and $1 = 2004\cos\theta \cdot (\sin\theta - \cos\theta)$.

Exercise 2.37 (ARML)**.** Compute $\frac{\sin 13^\circ + \sin 47^\circ + \sin 73^\circ + \sin 1{-}7^\circ}{\cos 17^\circ}$.

Exercise 2.38 (AIME I 2003/10)**.** Triangle ABC is isosceles with $AC = BC$ and $\angle ACB = 106^\circ$. Point M is in the interior of the triangle so that $\angle MAC = 7^\circ$ and $\angle MCA = 23^\circ$. Find the number of degrees in $\angle CMB$.



§3 Applications to Complex Numbers

Theorem 3.1 (Euler's Theorem)
For all angles θ,
$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Theorem 3.2 (Properties of Complex Numbers)
Let complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then
$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$
While this is not very impressive, this directly implies
$$\operatorname{cis}\theta_1 \cdot \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1+\theta_2).$$

Theorem 3.3 (Complex Form of Trigonometric Functions)
For some angle θ and constant k,
$$\cos k\theta = \frac{1}{2}\left(z^k + \frac{1}{z^k}\right),$$
and
$$\sin k\theta = \frac{1}{2i}\left(z^k - \frac{1}{z^k}\right).$$

Theorem 3.4 (DeMoivre's Theorem)
Let θ be an angle. Then
$$(\operatorname{cis}\theta)^n = \operatorname{cis}(n\theta).$$

§3.1 Roots of Unity

Definition 3.5 (Root of Unity) — A **root of unity** is a root of the equation
$$\omega^n = 1.$$
We define ω_k as the kth root of unity, ordered by their angle with respect to the positive x-axis counter-clockwise.

Theorem 3.6 (Roots of Unity)
Let ω be a solution to the equation
$$\omega^n = 1.$$
Then
$$\omega = e^{\frac{2k\pi i}{n}},$$
where $k = 0, 1, 2, \ldots, n-1$. This of course implies there exist n solutions to this equation (which should be intuitive from the Fundamental Theorem of Algebra).

Theorem 3.7 (Vieta's Formulas in Roots of Unity)
Let ω_k (where $k = 0, 1, 2, \ldots, n-1$) be the solutions to the equation

$$\omega_k^n - z_0 = 0.$$

By Vieta's Formulas,

$$\sum_{k=0}^{n-1} \omega_k = 0,$$

where ω_k is the kth root of unity. This implies

$$\sum_{k=0}^{n-1} \mathrm{Re}(\omega_k) = \sum_{k=0}^{n-1} \cos\left(\theta_0 + \frac{2k\pi}{n}\right) = 0,$$

and

$$\sum_{k=0}^{n-1} \mathrm{Im}(\omega_k) = \sum_{k=0}^{n-1} \sin\left(\theta_0 + \frac{2k\pi}{n}\right) = 0.$$

Also by Vieta's,

$$\prod_{k=0}^{n-1} \omega_k = (-1)^{n+1} z_0.$$

Theorem 3.8 (Complex Trigonometric Products)
For all $z = re^{i\theta}$,

$$z\overline{z} = |z|^2 = r^2,$$
$$z + \overline{z} = 2r\cos\theta,$$
$$z - \overline{z} = 2ri\sin\theta.$$

Thus, if $\omega^n = 1$ or -1, then for all $\omega_k = e^{i\theta_k}$,

$$(x - \omega_k)(x - \omega_{n-k}) = (x - \omega_k)(x - \overline{\omega_k}) = x^2 - 2x\cos\theta_k + 1,$$
$$(x + \omega_k)(x + \omega_{n-k}) = (x - \omega_k)(x + \overline{\omega_k}) = x^2 - 2xi\sin\theta_k - 1.$$

If we plug in $x = 1$ and take the product over all ω_k, we get

$$\prod_{k=1}^{n-1}(1 - \omega_k)(1 + \overline{\omega_k}) = (-2i)^{n-1}\prod_{k=1}^{n-1}\sin\theta_k.$$

Exercise 3.9. Derive a similar equation for cosines using $x = i$.

Theorem 3.10 (Sine-Unity Relation)
Let $\omega_k = e^{i\theta_k}$ be the nth roots of unity. Then

$$\sin \frac{\omega_k = \omega_0}{2} = \frac{1}{2}|\omega_k - \omega_0|.$$

Note that

$$\sin \frac{\omega_k = \omega_0}{2} = \frac{k\pi}{n}.$$

Exercise 3.11. Derive a formula for

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n},$$

and

$$\prod_{k=1}^{n-1} \cos \frac{k\pi}{n}.$$

Theorem 3.12 (Complex Trigonometric Sums)
For all integer constants $c \neq 0$,

$$\sum_{k=0} \sin ck\pi = \operatorname{Im}\left(\sum_{k=0}^{n-1} \omega^k\right),$$

$$\sum_{k=0} \cos ck\pi = \operatorname{Re}\left(\sum_{k=0}^{n-1} \omega^k\right).$$

Theorem 3.13 (Triple Angle Trig Theorem)
Let A, B, C be angles such that

$$\sin A + \sin B + \sin C = \cos A + \cos B + \cos C = 0.$$

Then $3\cos(A+B+C) = \cos 3A + \cos 3B + \cos 3B$ and $3\sin(A+B+C) = \sin 3A + \sin 3B + \sin 3C$

Example 3.14
Find $2\cos 72°$.

Solution. Let $z = e^{\frac{2k\pi}{5}}$. This implies

$$z^5 = 1,$$

and $z \neq 1$, so

$$(z-1)(z^4 + z^3 + z^2 + z + 1) = 0,$$
$$z^4 + z^3 + z^2 + z + 1 = 0.$$

Note that $2\cos 72° = z + \frac{1}{z}$. If we divide the equation above by z^2, we get

$$z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0,$$

$$\left(z^2+\frac{1}{z^2}\right)+\left(z+\frac{1}{z}\right)+1=0,$$

$$\left(z+\frac{1}{z}\right)^2+\left(z+\frac{1}{z}\right)-1=0,$$

which implies

$$\left(z+\frac{1}{z}\right)=\boxed{\frac{-1+\sqrt{5}}{2}}.$$

Note that we find that the other root doesn't work from bounding $\cos 72^\circ$ (i.e. it is positive from 0° to 90°). $\square$

Exercise 3.15 (Lagrange's Trigonometric Identity)**.** For all angles θ and positive integer n,

$$1+\cos\theta+\cos 2\theta+\ldots+\cos n\theta=\frac{1}{2}+\frac{\sin\left[(2n+1)\frac{\theta}{2}\right]}{2\sin\left(\frac{\theta}{2}\right)},$$

and derive a similar expression for sine.

Exercise 3.16 (Generalized ARML 2013)**.** Let $a=\cos\frac{2\pi}{7}, b=\cos\frac{4\pi}{7}$, and $c=\cos\frac{8\pi}{7}$. Then compute $ab+bc+ca$ and $a^3+b^3+c^3$.

Exercise 3.17 (PUMaC 2010)**.** The expression $\sin 2^\circ \sin 4^\circ \sin 6^\circ \ldots \sin 90^\circ$ is equal to $\frac{p\sqrt{5}}{2^{50}}$, where p is an integer. Find p.

Exercise 3.18. What is the value of $\sin 20^\circ \sin 40^\circ \sin 80^\circ$?

Exercise 3.19. Let $\omega=e^{\frac{2\pi i}{101}}$. Evaluate the product

$$\prod_{0\le p<q\le 100}(\omega^p+\omega^q).$$

Exercise 3.20 (PUMaC 2015)**.** Let $P(x)$ be a polynomial with positive integer coefficients and degree 2015. Given that there exists some $\omega\in\mathbb{C}$ satisfying:

$$\omega^{73}=1 \text{ and}$$

$$P(\omega^{2015})+P(\omega^{2015^2})+P(\omega^{2015^3})+\ldots+P(\omega^{2015^{72}})=0,$$

what is the minimum possible value of $P(1)$?

Exercise 3.21 (CMIMC 2018)**.** Compute the value of

$$\sum_{k=0}^{2017}\frac{5+\cos\left(\frac{k\pi}{1009}\right)}{26+10\cos\left(\frac{k\pi}{1009}\right)}.$$

Exercise 3.22 (HMMT 2014)**.** Evaluate

$$\sum_{k=1}^{1007}\left(\cos\left(\frac{k\pi}{1007}\right)\right)^{2014}.$$

Exercise 3.23. Let ABC be a triangle with inradius r and circumradius R. Show that

1. $4\sin A\sin B\sin C=\sin 2A+\sin 2B+\sin 2C$.
2. if $\sin^2 A+\sin^2 B+\sin^2 C=2$ then ABC is a right triangle.
3. if ABC is a cute then $2\cos A\cos B\cos C+\cos 2A+\cos 2B+\cos 2C=-1$.
4. $[ABC]=2R^2\sin A\sin B\sin C$.
5. $a\cos A+b\cos B+c\cos C=\frac{abc}{2R}$.
6. $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.
7. $a\cos B+b\cos C+c\cos A=\frac{a+b+c}{2}$.

§4 Applications to Planar Geometry

§4.1 Direct Applications

Theorem 4.1 (Extended Law of Sines)
Let ABC be a triangle with sides a, b, and c, and of circumradius R. Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Law of Cosines has been listed before, so to avoid repetition I will not list it again.

Theorem 4.2 (Trig Ceva)
Let ABC be a triangle with points D, E, and F on sides BC, AC, and AB respectively of triangle ABC. Line segments AD, BE, and CF are concurrent if and only if

$$\frac{\sin\angle BAC \sin\angle ACF \sin\angle CBE}{\sin\angle DAC \sin\angle FCB \sin\angle EBA} = 1.$$

Theorem 4.3 (Quadratic Formula of Trigonometry)
Let

$$a\cos\theta + b\sin\theta = c.$$

Then

$$\cos\theta = \frac{ac \pm b\sqrt{a^2+b^2-c^2}}{a^2+b^2},$$

$$\sin\theta = \frac{bc \pm \sqrt{a^2+b^2-c^2}}{a^2+b^2}$$

§4.2 Indirect Applications

§4.3 Trigonometric Functions at Special Values

There are a few special angles for which you should know the values of the trigonometric functions, without having to resort to a table or a calculator. These are summarized in the following table.

θ	$\cos(\theta)$	$\sin(\theta)$
0	1	0
$\frac{\pi}{12}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$
$\frac{\pi}{10}$	$\frac{\sqrt{5}-1}{4}$	$\sqrt{\frac{5+\sqrt{5}}{8}}$
$\frac{\pi}{8}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$
$\frac{\pi}{5}$	$\sqrt{\frac{5-\sqrt{5}}{8}}$	$\frac{\sqrt{5}+1}{4}$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	1

You should also be able to use reference angles along with these values to compute the values of the trigonometric functions at related angles in the second, third and fourth quadrants. For example, the point associated with $t = \frac{5\pi}{6}$ is directly across the unit circle from the point associated with $\frac{\pi}{6}$. (In this case we say that we are using $\frac{\pi}{6}$ as a reference angle.) Thus the coordinates of the point associated with $\frac{5\pi}{6}$ has the same y value and the opposite x value of the point associated with $\frac{\pi}{6}$. Thus $\cos(\frac{5\pi}{6}) = -\cos(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$, and $\sin(\frac{5\pi}{6}) = \sin(\frac{\pi}{6}) = \frac{1}{2}$.

This especially useful for geometry problems in which the angle is given, and the angles are nice. Memorizing special properties of certain triangles is extremely useful. One of the first things you should try when parts of a triangle are given is to look for special angles.

Theorem 4.4 (Blanchet's Theorem)

Let AD, BE, and CF be concurrent cevians in $\triangle ABC$. If $AD \perp BC$, show that ray AD bisects $\angle EDF$.

Proof. Note that

$$\begin{aligned}
\frac{\tan\angle ADE}{\tan\angle ADF} &= \frac{\sin\angle ADE}{\sin\angle ADF}\cdot\frac{\cos\angle ADF}{\cos\angle ADE}\\
&= \frac{\sin\angle ADE}{\sin\angle ADF}\cdot\frac{\sin\angle FDB}{\sin\angle EDC}\\
&= \frac{\frac{AE}{AD}\sin\angle AED}{\frac{AF}{AD}\sin\angle AFD}\cdot\frac{\sin FDB}{\sin EDC}\\
&= \frac{AE}{AF}\cdot\frac{\sin\angle CED}{\sin\angle BFD}\cdot\frac{\sin\angle FDB}{\sin\angle EDC}\\
&= \frac{AE}{AF}\cdot\frac{\sin\angle CED}{\sin\angle EDC}\cdot\frac{\sin\angle FDB}{\sin\angle BFD}\\
&= \frac{AE}{AF}\cdot\frac{CD}{CE}\cdot\frac{FB}{BD}\\
&= \frac{AE}{CE}\cdot\frac{CD}{BD}\cdot\frac{FB}{AF}\\
&= 1
\end{aligned}$$

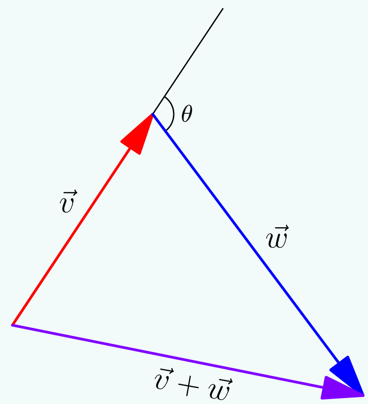
by Ceva's theorem, so $\angle ADE = \angle ADF$. □

§4.4 Vector Geometry

Definition 4.5 (Vector) — A **vector** is a directed line segment. It can also be considered a quantity with magnitude and direction. Every vector $\overrightarrow{UV}$ has a starting point $U\langle x_1, y_1\rangle$ and an endpoint $V\langle x_2, y_2\rangle$.

Theorem 4.6 (Addition of Vectors)

For vectors $\vec{v}$ and $\vec{w}$, with angle θ formed by them, $\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\|\cos\theta$.



Theorem 4.7 (Multiplying Vectors by Constant)

For some constant $c > 0$, $c\vec{v}$ increases the magnitude of $\vec{v}$ by c times in the same direction as $\vec{v}$. If $c < 0$, $c\vec{v}$ increases the magnitude of $\vec{v}$ by c times in the opposite direction as $\vec{v}$. If $c = 0$, the magnitude becomes 0 and there is no direction.

Theorem 4.8 (Vector Identities)

For any vectors $\vec{x}$, $\vec{y}$, $\vec{z}$, and real numbers a, b,

1. **Commutative Property**: $\vec{x}+\vec{y} = \vec{y}+\vec{x}$
2. **Associative Property**: $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$
3. **Additive Identity**: There exists the zero vector $\vec{0}$ such that $\vec{x}+\vec{0} = \vec{x}$
4. **Additive Inverse**: For each $\vec{x}$, there is a vector $\vec{y}$ such that $\vec{x}+\vec{y} = \vec{0}$
5. **Unit Scalar Identity**: $1\vec{x} = \vec{x}$
6. **Associative in Scalar**: $(ab)\vec{x} = a(b\vec{x})$
7. **Distributive Property of Vectors**: $a(\vec{x}+\vec{y}) = a\vec{x}+a\vec{y}$
8. **Distributive Property of Scalars**: $(a+b)\vec{x} = a\vec{x}+b\vec{x}$

Definition 4.9 (Dot Product) — Consider two vectors $\mathbf{a} = \langle a_1, a_2, \ldots, a_n\rangle$ and $\mathbf{b} = \langle b_1, b_2, \ldots, b_n\rangle$ in $\mathbb{R}^n$. The **dot product** is equal to the length of the projection (i.e. the distance from the origin to the foot of the head of **a** to **b**) of **a** onto **b** times the length of **b**.

Theorem 4.10 (Magnitude of Dot Product)
Consider two vectors $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$ and $\mathbf{b} = \langle b_1, b_2, \ldots, b_n \rangle$ in $\mathbb{R}^n$. The dot product is then

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{b}| \cos\theta = a_1b_1 + a_2b_2 + \cdots + a_nb_n,$$

where θ is the angle formed by the two vectors.

Definition 4.11 (Cross Product) — The **cross product** between two vectors **a** and **b** in $\mathbb{R}^3$ is defined as the vector whose length is equal to the area of the parallelogram spanned by **a** and **b** and whose direction is in accordance with the right-hand rule.

Theorem 4.12 (Magnitude of Cross Product)
The magnitude of the cross product is

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta,$$

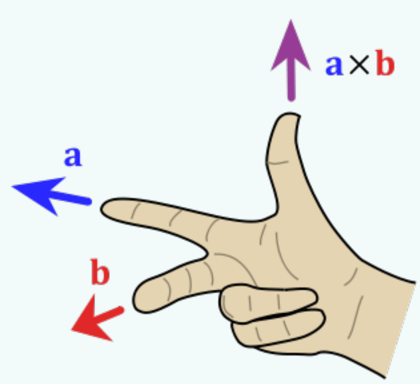
where θ is the angle formed by the two vectors.

Exercise 4.13. Show that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

Exercise 4.14. Show that $|\mathbf{a}|^2|\mathbf{b}|^2 = |\mathbf{a} \cdot \mathbf{b}|^2 + |\mathbf{a} \times \mathbf{b}|^2$.

Exercise 4.15. A ship is travelling at a speed of 4 m/s to the north. A boy on the ship travels to the east at 3 m/s with respect to the ship. What speed does he travel at with respect to the sea (which is not moving)?

Theorem 4.16 (Right Hand Rule)
The **right hand rule** is used to determine the directiton of the cross product. One can see this by holding one's hands outward and together, palms up, with the fingers curled, and the thumb out-stretched. If the curl of the fingers represents a movement from the first or x-axis to the second or y-axis, then the third or z-axis can point along either thumb.



Theorem 4.17 (Triple Scalar Product)

The triple scalar product of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is defined as $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Geometrically, the triple scalar product gives the signed volume of the parallelepiped determined by $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. It follows that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}.$$

Theorem 4.18 (Triple Vector Product)

The vector triple product of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is defined as the cross product of one vector, so that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$, which can be remembered by the mnemonic "BAC-CAB".

While the above theorems are extremely useful, the only crucial piece (for the AIME) is the following:

Theorem 4.19 (AIME Vectors)

Let θ be the angle between $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \cdot \vec{v} = uv \cos\theta,$$

and

$$|\vec{u} \times \vec{v}| = uv \sin\theta.$$

Theorem 4.20 (Properties of Vectors)

Some geometric properties of vectors:

1. If and only if the dot product of two vectors is zero, then those vectors are orthogonal or perpendicular. (The zero vector is orthogonal to every vector.)
2. If and only if the cross product of two vectors is zero (the zero vector), then those vectors are parallel. They can point in the same direction or in opposite directions.
3. The cross product of $\vec{u}$ and $\vec{v}$ is always orthogonal to $\vec{u}$ and $\vec{v}$. As long as $\vec{u}$ and $\vec{v}$ are not parallel, there exists one unique axis perpendicular to both which $\vec{u} \times \vec{v}$ will lie on.

Example 4.21 (AMC 10 A 2012/21)

Let points $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (0, 2, 0)$, and $D = (0, 0, 3)$. Points E, F, G, and H are midpoints of line segments $\overline{BD}$, $\overline{AB}$, $\overline{AC}$, and $\overline{DC}$ respectively. What is the area of $EFGH$?

$\textbf{(A)}\ \sqrt{2} \qquad \textbf{(B)}\ \frac{2\sqrt{5}}{3} \qquad \textbf{(C)}\ \frac{3\sqrt{5}}{4} \qquad \textbf{(D)}\ \sqrt{3} \qquad \textbf{(E)}\ \frac{2\sqrt{7}}{3}$

Solution. Computing the points of $EFGH$ gives $E(0.5, 0, 1.5), F(0.5, 0, 0), G(0, 1, 0), H(0, 1, 1.5)$. The vector EF is $(0, 0, -1.5)$, while the vector HG is also $(0, 0, -1.5)$, meaning the two sides EF and GH are parallel. Similarly, the vector FG is $(-0.5, 1, 0)$, while the vector EH is also $(-0.5, 1, 0)$. Again, these are equal in both magnitude and direction, so FG and EH are parallel. Thus, figure $EFGH$ is a parallelogram.

Computation of vectors EF and HG is sufficient evidence that the figure is a parallelogram, since the vectors are not only point in the same direction, but are of the same magnitude, but the other vector FG is needed to find the angle between the sides.

Taking the dot product of vector EF and vector FG gives $0 \cdot -0.5 + 0 \cdot 1 + -1.5 \cdot 0 = 0$, which means the two vectors are perpendicular. (Alternately, as above, note that vector EF goes directly down on the z-axis, while vector FG has no z-component and lie completely in the xy plane.) Thus, the figure is a parallelogram with a right angle, which makes it a rectangle. With the distance formula in three dimensions, we find that $EF = \frac{3}{2}$ and $FG = \frac{\sqrt{5}}{2}$, giving an area of $\frac{3}{2} \cdot \frac{\sqrt{5}}{2} = \boxed{\textbf{(C) } \frac{3\sqrt{5}}{4}}$. □

§4.5 Parameterization

Parameterization is extremely useful for changing to only one variable, especially for conic sections.

Theorem 4.22 (Parameterizations of Conic Sections)
The following is the parametric equations for conic sections:

1. **circle:** $x = \sin\theta, y = \cos\theta$
2. **ellipse:** $x = a\sin\theta, y = b\cos\theta$
3. **hyperbola:** $x = a\sec\theta, y = b\tan\theta$
4. **parabola:** $x = 2pt^2, y = 2pt$

Note that the parameter for the parabola is t, because using an angle is mostly useless for parabolas.

Parameterization is also heavily influenced by complex numbers.

Theorem 4.23 (Polar Form of Conic Sections)
Let a focal point of a conic section lie at the origin. Then its polar form is

$$r = \frac{l}{1 - e\cos\theta},$$

where e is the eccentricity, and l is a constant. If:

1. $e = \mathbf{0}$: the equation is a circle
2. $\mathbf{0} < e < \mathbf{1}$: the equation is an ellipse
3. $e = \mathbf{1}$: the equation is a parabola
4. $e > \mathbf{1}$: the equation is a hyperbola

The parabola can also be determined by its trajectory:

Theorem 4.24 (Trajectory of a Parabola)
The trajectory of a parabola is given by

$$x \cdot \tan\theta \left(1 - \frac{x}{R}\right),$$

for constants θ and R.

§4.6 Exercises

Exercise 4.25. Evaluate $\sin(\frac{7\pi}{6})$.

Exercise 4.26. Evaluate $\tan(\frac{-3\pi}{4})$.

Exercise 4.27. Solve $\sin(x) + \cos(x) = 0$ for x.

Exercise 4.28. Solve $2\cos(2x) + 1 = 0$ for x.

Exercise 4.29. $ABCDEFG$ is a regular heptagon inscribed in a unit circle. Compute the value of the following expression:

$$AB^2 + AC^2 + AD^2 + AE^2 + AF^2 + AG^2.$$

Exercise 4.30. Given that quadrilateral $ABCD$ has in inscribed circle, show that

$$[ABCD] = \sqrt{abcd}\sin\theta,$$

where a, b, c, d are the side lengths and $\theta = \frac{\angle A + \angle C}{2}$.

Exercise 4.31. Let D and E be the trisection points of segment AB, where D is between A and E. Construct a circle using DE as diameter, and let C be a point on the circle. Find the value of

$$\tan\angle ACD \cdot \tan\angle BCE.$$

Exercise 4.32. In $\triangle ABC, \angle B = 3\angle C$. If $AB = 10$ and $AC + 15$, compute the length of BC.

Exercise 4.33 (ARML 1988)**.** If $0° < x < 180°$ and $\cos x + \sin x = \frac{1}{2}$, then find (p, q) such that $\tan x = -\frac{p+\sqrt{q}}{3}$.

Exercise 4.34. $ARML$ is a convex kite with $A(0, 0), R(1, 3)$, and $M(7, 2)$. Determine the coordinates of L.

Exercise 4.35. Consider a rectangle $ABCD$ such that side AB has length n and side BC has length m. A circle is drawn with center E at the midpoint of side BC such that it is tangent to the diagonal AC. Determine the radius of this circle in terms of n and m.

Exercise 4.36. Find the number of intersections of the parabola $x^2 = 2p(y + \frac{p}{2})$ and the line $x\cos\theta + y\sin\theta = p\sin\theta$.

Exercise 4.37. For $a \neq b$,

$$a^2\sin\theta + a\cos\theta - 1 = 0,$$
$$b^2\sin\theta + b\cos\theta - 1 = 0.$$

Let l be the line determined by (a, a^2) and (b, b^2). Find the number of intersections of l and the unit circle.

§5 3-D Geometry

§5.1 More Vector Geometry

Vectors are very useful, especially for 3D geometry. Consider the distance between a point and a plane. We can find the vector normal to the plane by taking the cross product of two linearly independent vectors lying in the plane. We can then take any vector from a point on the plane to the point of interest and compute its dot product with a unit vector in the direction of the normal. By projecting the arbitrary displacement vector from the plane to the point onto the normal vector, we eliminate the "sideways" portion of the displacement and reduce it to its perpendicular part. The magnitude of the resulting value is the distance we wished to determine.

Theorem 5.1 (Vector on Vector Projection)
Let
$$\text{proj}_{\vec{b}}(\vec{a})$$
be the projection of $\vec{a}$ onto $\vec{b}$. Then
$$\text{proj}_{\vec{b}}(\vec{a}) = a\cos\theta\hat{b},$$
where θ is the angle between the two vectors and $\hat{b}$ is the direction the projection of $\vec{a}$ onto $\vec{b}$ faces (in this case, the direction is the same as $\vec{b}$).

Let us turn to areas now.

Theorem 5.2 (Area-Sine Formula)
Let there exist a triangle ABC such that $BC = a, AC = b$, and $\angle ACB = \theta$. Then the area of $\triangle ABC$ is
$$\frac{1}{2}ab\sin\theta.$$

Notice that this is exactly one half of the expression for the cross product of two vectors in terms of their magnitudes and the angle between them. In the case that the angle involved is not easily determined, such as in a three-dimensional situation, we can directly apply the cross product to vectors representing two sides of the triangle to determine its area. This will eliminate the necessity to find the angle. Similarly, finding the area of parallelogram is simply the cross product of the two vectors that determine it (also note that the area of a parallelogram is simply twice of the triangle).

Now that we have dealt with distances and areas, let us see how we can generalize to volumes. The method is very similar:

Theorem 5.3 (Volume of a Parallelepiped)
A **parallelepiped** (which is basically a shifted box [think 3D parallelogram]) is defined by three vectors $\vec{a}, \vec{b}, \vec{c}$. Then the volume of the parallelepiped is
$$|\vec{a} \times \vec{b}| \cdot \vec{c}.$$
Note that half of this volume is the volume of the tetrahedron defined by $\vec{a}, \vec{b}, \vec{c}$.

Vectors are also great for finding dihedral angles.

Definition 5.4 (Dihedral Angle) — A **dihedral angle** is the angle formed by two intersecting planes.

Definition 5.5 (Normal Vector) — The **normal vector**, often simply called the "normal," to a surface is a vector which is perpendicular to the surface at a given point.

Definition 5.6 (Unit Vector) — A **unit vector** is a vector of magnitude one. We say the unit vector of $\vec{u}$ is $\hat{u}$, and is used to show direction.

Theorem 5.7 (Unit Normal Vector Formula)
Let $\hat{\mathbf{n}}_P$ and $\hat{\mathbf{n}}_Q$ be the unit normal vectors of planes P and Q, respectively. Also, let $\vec{p_1}$ and $\vec{p_2}$ be vectors in the plane P and let $\vec{q_1}$ and $\vec{q_2}$ be vectors in the plane Q. Then

$$\hat{\mathbf{n}}_P = \frac{\vec{p_1} \times \vec{p_2}}{p_1 p_2},$$

and

$$\hat{\mathbf{n}}_Q = \frac{\vec{q_1} \times \vec{q_2}}{q_1 q_2}.$$

Theorem 5.8 (Dihedral Angle Formula)
Let θ be the angle between two planes P and Q, and let $\hat{\mathbf{n}}_P$ and $\hat{\mathbf{n}}_Q$ be the unit normal vectors of P and Q, respectively. Then

$$\cos\theta = \hat{\mathbf{n}}_P \cdot \hat{\mathbf{n}}_Q.$$

§5.2 Exercises

Exercise 5.9. Let PQ be the line passing through the points $P = (-1, 0, 3)$ and $Q = (0, -2, -1)$. Determine the shortest distance from PQ to the origin.

Exercise 5.10. A parallelpiped has a vertex at $(1, 2, 3)$, and adjacent vertices (that form edges with this vertex) at $(3, 5, 7), (1, 6, -2)$, and $(6, 3, 6)$. Find the volume of this parallelpiped.

Exercise 5.11. Find the dihedral angle between adjacent faces of a:

1. regular tetrahedron,
2. regular octahedron,
3. regular dodecahedron, and
4. regular icosahedron.

§6 Trigonometric Substitution

Trigonometry substitution is extremely useful for a variety of problems. Here are a few substitutions to employ.

Theorem 6.1 (Weierstrauss Substitution)
Let $t = \tan \frac{x}{2}$, where $x \in (-\pi, \pi)$. Then

$$\sin \frac{x}{2} = \frac{t}{\sqrt{1+t^2}},$$

and

$$\cos \frac{x}{2} + \frac{1}{\sqrt{1+t^2}}.$$

Similarly,

$$\sin x = \frac{2t}{1+t^2},$$

$$\cos x = \frac{1-t^2}{1+t^2},$$

and

$$\tan x = \frac{2t}{1-t^2}.$$

Theorem 6.2 (Trigonometric Triangle-Angle Condition)
Let α, β, γ be angles in the range $(0, \pi)$. Then α, β, γ are angles of a triangle if and only if

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} =,$$

or

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} + 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = 1.$$

The former is useful for expressions of the form $ab + bc + ca = 1$.

Theorem 6.3 (Triangle-Angle Substitution)
Let α, β, γ be angles of a triangle. Then

$$A = \frac{\pi - \alpha}{2}, B = \frac{\pi - \beta}{2}, C = \frac{\pi - \gamma}{2}$$

transforms the triangle into an acute triangle with angles A, B, C.

Theorem 6.4 ($ab + bc + ca = 1$ Substitution)
Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Then we can substitute

$$a = \frac{\tan \alpha}{2}, b = \frac{\tan \beta}{2}, c = \frac{\tan \gamma}{2},$$

or

$$a = \cot A, b = \cot B, c = \cot C,$$

where α, β, γ and A, B, C are angles of a triangle.

Theorem 6.5 ($a+b+c=abc$ Substitution)
Let a,b,c be positive real numbers such that $a+b+c=abc$. Then we can substitute

$$a=\cot\frac{\alpha}{2}, b=\cot\frac{\beta}{2}, c=\cot\frac{\gamma}{2},$$

or

$$a=\tan A, b=\tan B, c=\tan C,$$

where α,β,γ are angles of a triangle.

Theorem 6.6 ($a^2+b^2+c^2+2abc=1$ Substitution)
Let a,b,c be positive real numbers such that $a^2+b^2+c^2+2abc=1$. Then we can substitute

$$a=\sin\frac{\alpha}{2}, b=\sin\frac{\beta}{2}, c=\sin\frac{\gamma}{2},$$

or

$$a=\cos A, b=\cos B, c=\cos C.$$

Example 6.7 (Darij Grinberg)
Let x,y,z be positive real numbers. Prove that

$$\sqrt{x(y+z)}+\sqrt{y(z+x)}+\sqrt{z(x+y)}\geq 2\sqrt{\frac{(x+y)(y+z)(z+x)}{x+y+z}}.$$

Solution. We can rewrite this inequality as

$$\sum_{\text{cyc}}\sqrt{\frac{x(x+y+z)}{(x+y)(y+z)}}\geq 2.$$

These values can be substituted for $\sin A, \sin B$, and $\sin C$, so it suffices to prove

$$\sin A+\sin B+\sin C\geq 2,$$

where A,B,C are angles of an acute triangle (prove why this substitution is true!). Using Jordan's Inequality, we have

$$\frac{2\alpha}{\pi}\leq\sin\alpha\leq\alpha,$$

and summing cyclically gives us the desired result. □

Example 6.8 (HMMT)
Find the minimum possible value of $\sqrt{58-42x}+\sqrt{149-140\sqrt{1-x^2}}$ where $-1\leq x\leq 1$.

Solution. The $\sqrt{1-x^2}$ is an obvious indicator of trigonometric substiution. Thus, if we let

$$x=\cos\theta,$$

then

$$\sqrt{1-x^2} = \sin\theta.$$

While $\sqrt{58-42x}$ is rather innocent, 149 and 140 should indicate Law of Cosines. In particular,

$$149 = 7^2 + 10^2,$$

$$140 = 2 \cdot 7 \cdot 10.$$

If we turn our attention to 58 and 42, we have

$$58 = 3^2 + 7^2,$$

$$42 = 2 \cdot 3 \cdot 7.$$

Thus, if we have a triangle with side lengths 3 and 7, with angle θ between them, then $\sqrt{58-42x}$ would be the last side. Similarly, if we have a triangle with side lengths 7 and 10, with angle $90^\circ - \theta$ between them, $\sqrt{149-140\sqrt{1-x^2}}$ would be the last side. The θ and $90^\circ - \theta$, paired with the common 7, inspires us to combine these two triangles such that the angles of measure θ and $90^\circ - \theta$ become 90°, and the two sides of length 7 become one side. Thus, we have a triangle with side lengths $3, 10$, and $\sqrt{58-42x} + \sqrt{149-140\sqrt{1-x^2}}$, with a 90-degree angle between 3 and 10. Thus,

$$\sqrt{58-42x} + \sqrt{149-140\sqrt{1-x^2}} \geq \sqrt{3^2+10^2} = \boxed{\sqrt{109}}.$$

□

Theorem 6.9 (Trigonometric Inequalities)

Let A, B, C be angles of triangle ABC. Then

1. $\cos A + \cos B + \cos C \leq \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \leq \frac{3}{2}$
2. $\sin A + \sin B + \sin C \leq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{3\sqrt{3}}{2}$
3. $\cos A \cos B \cos C \leq \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}$
4. $\sin A \sin B \sin C \leq \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \leq \frac{3\sqrt{3}}{8}$
5. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} \geq 3\sqrt{3}$
6. $\cos^2 A + \cos^2 B + \cos^2 C \geq \sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} \geq \frac{3}{4}$
7. $\sin^2 A + \sin^2 B + \sin^2 C \leq \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \leq \frac{9}{4}$
8. $\cot A + \cot B + \cot C \geq \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \geq \sqrt{3}$

Theorem 6.10 (Well-Known Triangle Trigonometric Identities)

Let A, B, C be angles of triangle ABC. Then

1. $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
2. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
3. $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$
4. $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$

Theorem 6.11 (Well-Known Trigonometric Identities)
For arbitrary angles α, β, γ,

$$\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2},$$

and

$$\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\alpha}{2}\cos\frac{\gamma+\alpha}{2}.$$

§7 Worked Through Problems

Example 7.1 (AIME 1989/10)
Let a, b, c be the three sides of a triangle, and let α, β, γ, be the angles opposite them. If $a^2+b^2 = 1989c^2$, find

$$\frac{\cot\gamma}{\cot\alpha+\cot\beta}.$$

Solution. We can draw the altitude h to c, to get two right triangles. $\cot\alpha + \cot\beta = \frac{c}{h}$, from the definition of the cotangent. From the definition of area, $h = \frac{2A}{c}$, so $\cot\alpha+\cot\beta = \frac{c^2}{2A}$.

Now we evaluate the numerator:

$$\cot\gamma = \frac{\cos\gamma}{\sin\gamma}$$

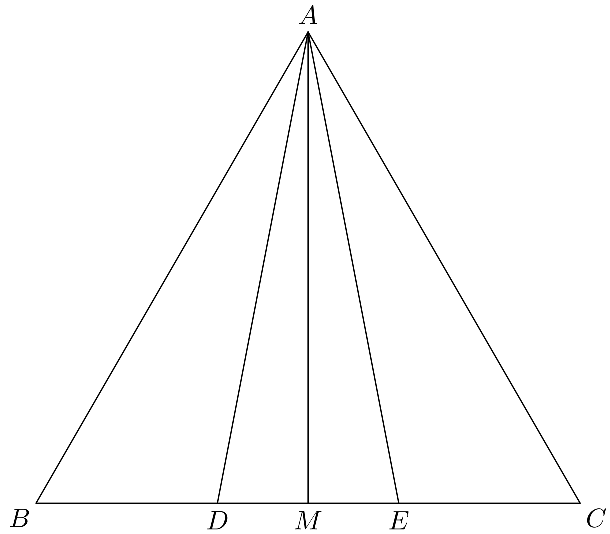
From the Law of Cosines and the sine area formula,

$$\cos\gamma = \frac{1988c^2}{2ab}$$
$$\sin\gamma = \frac{2A}{ab}$$
$$\cot\gamma = \frac{\cos\gamma}{\sin\gamma} = \frac{1988c^2}{4A}$$

Then $\frac{\cot\gamma}{\cot\alpha+\cot\beta} = \frac{\frac{1988c^2}{4A}}{\frac{c^2}{2A}} = \frac{1988}{2} = \boxed{994}$. □

Example 7.2 (AIME II 2013/5)
In equilateral $\triangle ABC$ let points D and E trisect $\overline{BC}$. Then $\sin(\angle DAE)$ can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is an integer that is not divisible by the square of any prime. Find $a+b+c$.

Solution. Without loss of generality, assume the triangle sides have length 3. Then the trisected side is partitioned into segments of length 1, making your computation easier.



Let M be the midpoint of $\overline{DE}$. Then ΔMCA is a 30-60-90 triangle with $MC = \dfrac{3}{2}$, $AC = 3$ and $AM = \dfrac{3\sqrt{3}}{2}$. Since the triangle ΔAME is right, then we can find the length of $\overline{AE}$ by pythagorean theorem, $AE = \sqrt{7}$. Therefore, since ΔAME is a right triangle, we can easily find $\sin(\angle EAM) = \dfrac{1}{2\sqrt{7}}$ and $\cos(\angle EAM) = \sqrt{1-\sin(\angle EAM)^2} = \dfrac{3\sqrt{3}}{2\sqrt{7}}$. So we can use the double angle formula for sine, $\sin(\angle EAD) = 2\sin(\angle EAM)\cos(\angle EAM) = \dfrac{3\sqrt{3}}{14}$. Therefore, $a+b+c = \boxed{020}$. □

Example 7.3 (AIME 1994/10)

In triangle ABC, angle C is a right angle and the altitude from C, meets $\overline{AB}$, at D. The lengths of the sides of $\triangle ABC$, are integers, $BD = 29^3$, , and $\cos B = m/n$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. We will solve for $\cos B$ using $\triangle CBD$, which gives us $\cos B = \frac{29^3}{BC}$. By the Pythagorean Theorem on $\triangle CBD$, we have $BC^2 - DC^2 = (BC+DC)(BC-DC) = 29^6$. Trying out factors of 29^6, we can either guess and check or just guess to find that $BC + DC = 29^4$ and $BC - DC = 29^2$ (The other pairs give answers over 999). Adding these, we have $2BC = 29^4 + 29^2$ and $\frac{29^3}{BC} = \frac{2*29^3}{29^2(29^2+1)} = \frac{58}{842} = \frac{29}{421}$, and our answer is $\boxed{450}$. □

Example 7.4 (AIME 1996/10)

Find the smallest positive integer solution to $\tan 19x^\circ = \dfrac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}$.

Solution. Note that

$$\frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}$$

$$= \frac{\sin 186^\circ + \sin 96^\circ}{\sin 186^\circ - \sin 96^\circ}$$

$$= \frac{\sin(141^\circ + 45^\circ) + \sin(141^\circ - 45^\circ)}{\sin(141^\circ + 45^\circ) - \sin(141^\circ - 45^\circ)}$$

$$= \frac{2\sin 141^\circ \cos 45^\circ}{2\cos 141^\circ \sin 45^\circ} = \tan 141^\circ.$$

The period of the tangent function is 180°, and the tangent function is one-to-one over each period of its domain.

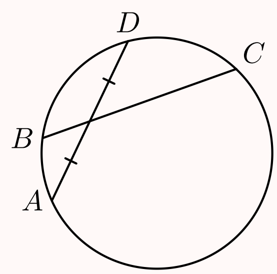
Thus, $19x \equiv 141 \pmod{180}$.

Since $19^2 \equiv 361 \equiv 1 \pmod{180}$, multiplying both sides by 19 yields $x \equiv 141 \cdot 19 \equiv (140+1)(18+1) \equiv 0 + 140 + 18 + 1 \equiv 159 \pmod{180}$.

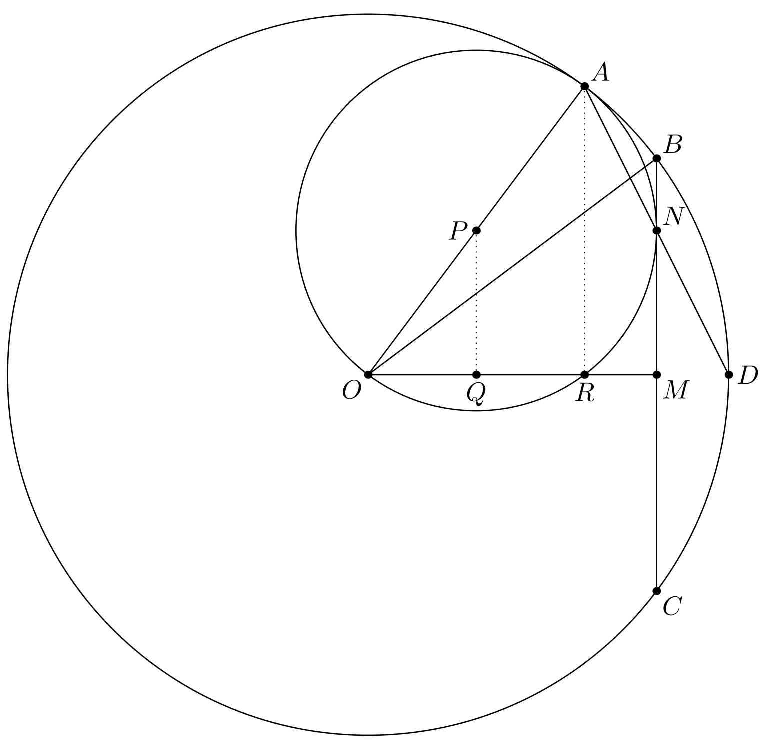
Therefore, the smallest positive solution is $x = \boxed{159}$. □

Example 7.5 (AIME 1983/15)

The adjoining figure shows two intersecting chords in a circle, with B on minor arc AD. Suppose that the radius of the circle is 5, that $BC = 6$, and that AD is bisected by BC. Suppose further that AD is the only chord starting at A which is bisected by BC. It follows that the sine of the central angle of minor arc AB is a rational number. If this number is expressed as a fraction $\frac{m}{n}$ in lowest terms, what is the product mn?



Solution. (Figure by AoPS User Adamz.)



Let A be any fixed point on circle O, and let AD be a chord of circle O. The locus of midpoints N of the chord AD is a circle P, with diameter AO. Generally, the circle P can intersect the chord BC at two points, one point, or they may not have a point of intersection. By the problem condition, however, the circle P is tangent to BC at point N.

Let M be the midpoint of the chord BC. From right triangle OMB, we have $OM = \sqrt{OB^2 - BM^2} = 4$. This gives $\tan \angle BOM = \frac{BM}{OM} = \frac{3}{4}$.

Notice that the distance OM equals $PN + PO \cos \angle AOM = r(1 + \cos \angle AOM)$, where r is the radius of circle P.

Hence

$$\cos \angle AOM = \frac{OM}{r} - 1 = \frac{2OM}{R} - 1 = \frac{8}{5} - 1 = \frac{3}{5}$$

(where R represents the radius, 5, of the large circle given in the question). Therefore, since $\angle AOM$ is clearly acute, we see that

$$\tan \angle AOM = \frac{\sqrt{1 - \cos^2 \angle AOM}}{\cos \angle AOM} = \frac{\sqrt{5^2 - 3^2}}{3} = \frac{4}{3}$$

Next, notice that $\angle AOB = \angle AOM - \angle BOM$. We can therefore apply the subtraction formula for tan to obtain

$$\tan \angle AOB = \frac{\tan \angle AOM - \tan \angle BOM}{1 + \tan \angle AOM \cdot \tan \angle BOM} = \frac{\frac{4}{3} - \frac{3}{4}}{1 + \frac{4}{3} \cdot \frac{3}{4}} = \frac{7}{24}$$

It follows that $\sin \angle AOB = \frac{7}{\sqrt{7^2+24^2}} = \frac{7}{25}$, such that the answer is $7 \cdot 25 = \boxed{175}$. □

Example 7.6 (AIME I 2003/11)

An angle x is chosen at random from the interval $0^\circ < x < 90^\circ$. Let p be the probability that the numbers $\sin^2 x, \cos^2 x$, and $\sin x \cos x$ are not the lengths of the sides of a triangle. Given that $p = d/n$, where d is the number of degrees in $\arctan m$ and m and n are positive integers with $m + n < 1000$, find $m + n$.

Solution. Note that the three expressions are symmetric with respect to interchanging sin and cos, and so the probability is symmetric around 45°. Thus, take $0 < x < 45$ so that $\sin x < \cos x$. Then $\cos^2 x$ is the largest of the three given expressions and those three lengths not forming a triangle is equivalent to a violation of the triangle inequality

$$\cos^2 x > \sin^2 x + \sin x \cos x$$

This is equivalent to

$$\cos^2 x - \sin^2 x > \sin x \cos x$$

and, using some of our trigonometric identities, we can re-write this as $\cos 2x > \frac{1}{2} \sin 2x$. Since we've chosen $x \in (0, 45)$, $\cos 2x > 0$ so

$$2 > \tan 2x \implies x < \frac{1}{2} \arctan 2.$$

The probability that x lies in this range is $\frac{1}{45} \cdot \left(\frac{1}{2} \arctan 2\right) = \frac{\arctan 2}{90}$ so that $m = 2$, $n = 90$ and our answer is $\boxed{092}$. □

Example 7.7 (AIME I 2003/12)

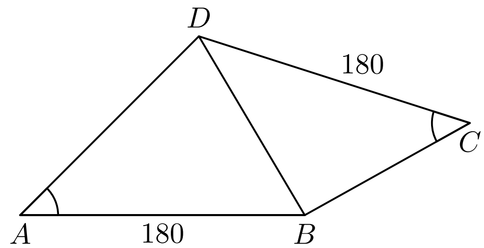
In convex quadrilateral $ABCD, \angle A \cong \angle C, AB = CD = 180$, and $AD \neq BC$. The perimeter of $ABCD$ is 640. Find $\lfloor 1000 \cos A \rfloor$. (The notation $\lfloor x \rfloor$ means the greatest integer that is less than or equal to x.)

Solution. By the Law of Cosines on $\triangle ABD$ at angle A and on $\triangle BCD$ at angle C (note $\angle C = \angle A$),

$$180^2 + AD^2 - 360 \cdot AD \cos A = 180^2 + BC^2 - 360 \cdot BC \cos A$$
$$(AD^2 - BC^2) = 360(AD - BC) \cos A$$
$$(AD - BC)(AD + BC) = 360(AD - BC) \cos A$$
$$(AD + BC) = 360 \cos A$$

We know that $AD + BC = 640 - 360 = 280$. $\cos A = \dfrac{280}{360} = \dfrac{7}{9} = 0.777\ldots$

$\lfloor 1000 \cos A \rfloor = \boxed{777}$.



□

Example 7.8 (AIME I 2014/10)

A disk with radius 1 is externally tangent to a disk with radius 5. Let A be the point where the disks are tangent, C be the center of the smaller disk, and E be the center of the larger disk. While the larger disk remains fixed, the smaller disk is allowed to roll along the outside of the larger disk until the smaller disk has turned through an angle of 360°. That is, if the center of the smaller disk has moved to the point D, and the point on the smaller disk that began at A has now moved to point B, then $\overline{AC}$ is parallel to $\overline{BD}$. Then $\sin^2(\angle BEA) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

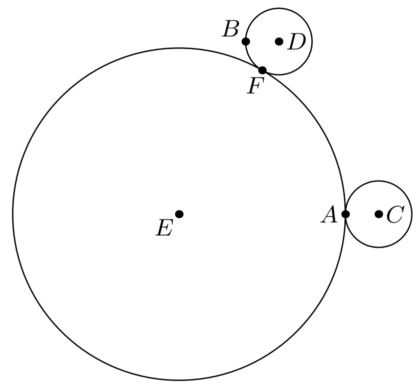
Solution. First, we determine how far the small circle goes. For the small circle to rotate completely around the circumference, it must rotate 5 times (the circumference of the small circle is 2π while the larger one has a circumference of 10π) plus the extra rotation the circle gets for rotating around the circle, for a total of 6 times. Therefore, one rotation will bring point D 60° from C.

Now, draw $\triangle DBE$, and call $\angle BED$ x, in degrees. We know that $\overline{ED}$ is 6, and $\overline{BD}$ is 1. Since $EC||BD$, $\angle BDE = 60^\circ$. By the Law of Cosines, $\overline{BE}^2 = 36 + 1 - 2 \times 6 \times 1 \times \cos 60^\circ = 36 + 1 - 6 = 31$, and since lengths are positive, $\overline{BE} = \sqrt{31}$.

By the Law of Sines, we know that $\frac{1}{\sin x} = \frac{\sqrt{31}}{\sin 60^\circ}$, so $\sin x = \frac{\sin 60^\circ}{\sqrt{31}} = \frac{\sqrt{93}}{62}$. As x is clearly between 0 and 90°, $\cos x$ is positive. As $\cos x = \sqrt{1 - \sin^2 x}$, $\cos x = \frac{11\sqrt{31}}{62}$.

Now we use the angle sum formula to find the sine of $\angle BEA$: $\sin 60^\circ \cos x + \cos 60^\circ \sin x = \frac{\sqrt{3}}{2} \frac{11\sqrt{31}}{62} + \frac{1}{2} \frac{\sqrt{93}}{62} = \frac{11\sqrt{93}+\sqrt{93}}{124} = \frac{12\sqrt{93}}{124} = \frac{3\sqrt{93}}{31} = \frac{3\sqrt{31}\sqrt{3}}{31} = \frac{3\sqrt{3}}{\sqrt{31}}$.

Finally, we square this to get $\frac{9 \times 3}{31} = \frac{27}{31}$, so our answer is $27 + 31 = \boxed{058}$.



$\square$

Example 7.9 (AIME 1997/14)

Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Let $\frac{m}{n}$ be the probability that $\sqrt{2+\sqrt{3}} \le |v+w|$, where m and n are relatively prime positive integers. Find $m+n$.

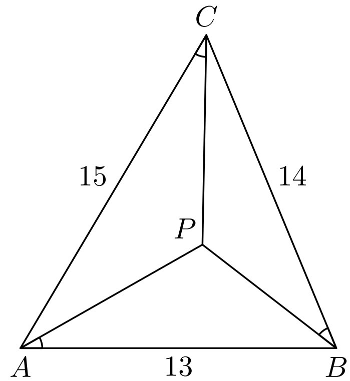
Solution. We know that
$$z^{1997} = 1 = 1(\cos 0 + i \sin 0).$$
By De Moivre's Theorem, we find that ($k \in \{0, 1, \ldots, 1996\}$)
$$z = \cos\left(\frac{2\pi k}{1997}\right) + i \sin\left(\frac{2\pi k}{1997}\right).$$
Now, let v be the root corresponding to $\theta = \frac{2\pi m}{1997}$, and let w be the root corresponding to $\theta = \frac{2\pi n}{1997}$. The magnitude of $v + w$ is therefore:
$$\sqrt{\left(\cos\left(\frac{2\pi m}{1997}\right) + \cos\left(\frac{2\pi n}{1997}\right)\right)^2 + \left(\sin\left(\frac{2\pi m}{1997}\right) + \sin\left(\frac{2\pi n}{1997}\right)\right)^2}$$
$$= \sqrt{2 + 2\cos\left(\frac{2\pi m}{1997}\right)\cos\left(\frac{2\pi n}{1997}\right) + 2\sin\left(\frac{2\pi m}{1997}\right)\sin\left(\frac{2\pi n}{1997}\right)}$$
We need $\cos\left(\frac{2\pi m}{1997}\right)\cos\left(\frac{2\pi n}{1997}\right) + \sin\left(\frac{2\pi m}{1997}\right)\sin\left(\frac{2\pi n}{1997}\right) \ge \frac{\sqrt{3}}{2}$. The cosine difference identity simplifies that to $\cos\left(\frac{2\pi m}{1997} - \frac{2\pi n}{1997}\right) \ge \frac{\sqrt{3}}{2}$. Thus, $|m-n| \le \frac{\pi}{6} \cdot \frac{1997}{2\pi} = \lfloor \frac{1997}{12} \rfloor = 166$.

Therefore, m and n cannot be more than 166 away from each other. This means that for a given value of m, there are 332 values for n that satisfy the inequality; 166 of them $> m$, and 166 of them $< m$. Since m and n must be distinct, n can have 1996 possible values. Therefore, the probability is $\frac{332}{1996} = \frac{83}{499}$. The answer is then $499 + 83 = \boxed{582}$. $\square$

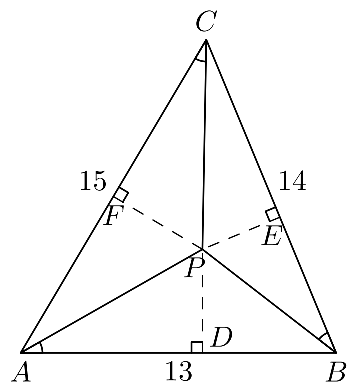
Example 7.10 (AIME 1999/14)

Point P is located inside triangle ABC so that angles PAB, PBC, and PCA are all congruent. The sides of the triangle have lengths $AB = 13, BC = 14$, and $CA = 15$, and the tangent of angle PAB is m/n, where m and n are relatively prime positive integers. Find $m+n$.

Solution. The following is the figure for this problem.



Drop perpendiculars from P to the three sides of $\triangle ABC$ and let them meet $\overline{AB}, \overline{BC}$, and $\overline{CA}$ at D, E, and F respectively.



Let $BE = x, CF = y$, and $AD = z$. We have that

$$\begin{aligned} DP &= z\tan\theta \\ EP &= x\tan\theta \\ FP &= y\tan\theta \end{aligned}$$

We can then use the tool of calculating area in two ways

$$\begin{aligned} [ABC] &= [PAB] + [PBC] + [PCA] \\ &= \frac{1}{2}(13)(z\tan\theta) + \frac{1}{2}(14)(x\tan\theta) + \frac{1}{2}(15)(y\tan\theta) \\ &= \frac{1}{2}\tan\theta(13z + 14x + 15y) \end{aligned}$$

On the other hand,

$$\begin{aligned} [ABC] &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{21\cdot 6\cdot 7\cdot 8} \\ &= 84 \end{aligned}$$

We still need $13z + 14x + 15y$ though. We have all these right triangles and we haven't even touched Pythagoras. So we give it a shot:

$$x^2 + x^2\tan^2\theta = z^2\tan^2\theta + (13-z)^2 \tag{1}$$
$$z^2 + z^2\tan^2\theta = y^2\tan^2\theta + (15-y)^2 \tag{2}$$
$$y^2 + y^2\tan^2\theta = x^2\tan^2\theta + (14-x)^2 \tag{3}$$

Adding (1) + (2) + (3) gives

$$\begin{aligned} x^2+y^2+z^2 &= (14-x)^2+(15-y)^2+(13-z)^2 \\ \Rightarrow 13z+14x+15y &= 295 \end{aligned}$$

Recall that we found that $[ABC] = \frac{1}{2}\tan\theta(13z+14x+15y) = 84$. Plugging in $13z+14x+15y = 295$, we get $\tan\theta = \frac{168}{295}$, giving us $\boxed{463}$ for an answer. □

Example 7.11 (AIME I 2007/12)

In isosceles triangle $\triangle ABC$, A is located at the origin and B is located at $(20, 0)$. Point C is in the first quadrant with $AC = BC$ and angle $BAC = 75^\circ$. If triangle ABC is rotated counterclockwise about point A until the image of C lies on the positive y-axis, the area of the region common to the original and the rotated triangle is in the form $p\sqrt{2}+q\sqrt{3}+r\sqrt{6}+s$, where p, q, r, s are integers. Find $\frac{p-q+r-s}{2}$.

Solution. Let the new triangle be $\triangle AB'C'$ (A, the origin, is a vertex of both triangles). Let $\overline{B'C'}$ intersect with $\overline{AC}$ at point D, $\overline{BC}$ intersect with $\overline{B'C'}$ at E, and $\overline{BC}$ intersect with $\overline{AB'}$ at F. The region common to both triangles is the quadrilateral $ADEF$. Notice that $[ADEF] = [\triangle ADB'] - [\triangle EFB']$, where we let $[\dots]$ denote area.

To find $[\triangle ADB']$: Since $\angle B'AC'$ and $\angle BAC$ both have measures 75°, both of their complements are 15°, and $\angle DAB' = 90 - 2(15) = 60^\circ$. We know that $\angle DB'A = 75^\circ$, so $\angle ADB' = 180 - 60 - 75 = 45^\circ$.

Thus $\triangle ADB'$ is a $45-60-75\triangle$. It can be solved by drawing an altitude splitting the 75° angle into 30° and 45° angles, forming a $30-60-90$ right triangle and a $45-45-90$ isosceles right triangle. Since we know that $AB' = 20$, the base of the $30-60-90$ triangle is 10, the base of the $45-45-90$ is $10\sqrt{3}$, and their common height is $10\sqrt{3}$. Thus, the total area of $[\triangle ADB'] = \frac{1}{2}(10\sqrt{3})(10\sqrt{3}+10) = \boxed{150+50\sqrt{3}}$.

To find $[\triangle EFB']$: Since $\triangle AFB$ is also a $15-75-90$ triangle,

$AF = 20\sin 75 = 20\sin(30+45) = 20\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right) = 5\sqrt{2}+5\sqrt{6}$ and

$FB' = AB' - AF = 20 - 5\sqrt{2} - 5\sqrt{6}$ Since $[\triangle EFB'] = \frac{1}{2}(FB' \cdot EF) = \frac{1}{2}(FB')(FB'\tan 75^\circ)$. With some horrendous algebra, we can calculate

$$\begin{aligned} [\triangle EFB'] &= \frac{1}{2}\tan(30+45)\cdot(20-5\sqrt{2}-5\sqrt{6})^2 \\ &= 25\left(\frac{\frac{1}{\sqrt{3}}+1}{1-\frac{1}{\sqrt{3}}}\right)\left(8-2\sqrt{2}-2\sqrt{6}-2\sqrt{2}+1+\sqrt{3}-2\sqrt{6}+\sqrt{3}+3\right) \\ &= 25(2+\sqrt{3})(12-4\sqrt{2}-4\sqrt{6}+2\sqrt{3}) \\ [\triangle EFB'] &= \boxed{-500\sqrt{2}+400\sqrt{3}-300\sqrt{6}+750}. \end{aligned}$$

To finish,

$$\begin{aligned} [ADEF] &= [\triangle ADB'] - [\triangle EFB'] \\ &= \left(150+50\sqrt{3}\right) - \left(-500\sqrt{2}+400\sqrt{3}-300\sqrt{6}+750\right) \\ &= 500\sqrt{2}-350\sqrt{3}+300\sqrt{6}-600 \end{aligned}$$

Hence, $\frac{p-q+r-s}{2} = \frac{500+350+300+600}{2} = \frac{1750}{2} = \boxed{875}$. □

Example 7.12 (AIME I 2012/12)

Let $\triangle ABC$ be a right triangle with right angle at C. Let D and E be points on $\overline{AB}$ with D between A and E such that $\overline{CD}$ and $\overline{CE}$ trisect $\angle C$. If $\frac{DE}{BE} = \frac{8}{15}$, then $\tan B$ can be written as $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers, and p is a positive integer not divisible by the square of any prime. Find $m+n+p$.

Solution. Without loss of generality, set $CB = 1$. Then, by the Angle Bisector Theorem on triangle DCB, we have $CD = \frac{8}{15}$. We apply the Law of Cosines to triangle DCB to get $1 + \frac{64}{225} - \frac{8}{15} = BD^2$, which we can simplify to get $BD = \frac{13}{15}$.

Now, we have $\cos\angle B = \frac{1+\frac{169}{225}-\frac{64}{225}}{\frac{26}{15}}$ by another application of the Law of Cosines to triangle DCB, so $\cos\angle B = \frac{11}{13}$. In addition, $\sin\angle B = \sqrt{1-\frac{121}{169}} = \frac{4\sqrt{3}}{13}$, so $\tan\angle B = \frac{4\sqrt{3}}{11}$.

Our final answer is $4+3+11 = \boxed{018}$. $\square$

Example 7.13 (AIME II 2014/12)

Suppose that the angles of $\triangle ABC$ satisfy $\cos(3A)+\cos(3B)+\cos(3C) = 1$. Two sides of the triangle have lengths 10 and 13. There is a positive integer m so that the maximum possible length for the remaining side of $\triangle ABC$ is $\sqrt{m}$. Find m.

Solution. Note that $\cos 3C = -\cos(3A+3B)$. Thus, our expression is of the form $\cos 3A + \cos 3B - \cos(3A+3B) = 1$. Let $\cos 3A = x$ and $\cos 3B = y$.

Using the fact that $\cos(3A+3B) = \cos 3A \cos 3B - \sin 3A \sin 3B = xy - \sqrt{1-x^2}\sqrt{1-y^2}$, we get $x + y - xy + \sqrt{1-x^2}\sqrt{1-y^2} = 1$, or $\sqrt{1-x^2}\sqrt{1-y^2} = xy - x - y + 1 = (x-1)(y-1)$.

Squaring both sides, we get $(1-x^2)(1-y^2) = [(x-1)(y-1)]^2$. Cancelling factors, $(1+x)(1+y) = (1-x)(1-y)$.

Notice here that we cancelled out one factor of (x-1) and (y-1), which implies that (x-1) and (y-1) were not 0. If indeed they were 0 though, we would have $\cos(3A) - 1 = 0, \cos(3A) = 1$

For this we could say that A must be 120 degrees for this to work. This is one case. The B case follows in the same way, where B must be equal to 120 degrees. This doesn't change the overall solution though, as then the other angles are irrelevant (this is the largest angle, implying that this will have the longest side and so we would want to have the 120 degreee angle opposite of the unknown side).

Expanding, $1 + x + y + xy = 1 - x - y + xy \rightarrow x + y = -x - y$.

Simplification leads to $x + y = 0$.

Therefore, $\cos(3C) = 1$. So $\angle C$ could be 0° or 120°. We eliminate 0° and use law of cosines to get our answer:

$$m = 10^2 + 13^2 - 2 \cdot 10 \cdot 13 \cos\angle C$$
$$\rightarrow m = 269 - 260\cos 120^\circ = 269 - 260\left(-\frac{1}{2}\right)$$
$$\rightarrow m = 269 + 130 = \boxed{399}.$$

$\square$

Example 7.14 (AIME I 2011/14)

Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a regular octagon. Let M_1, M_3, M_5, and M_7 be the midpoints of sides $\overline{A_1A_2}$, $\overline{A_3A_4}$, $\overline{A_5A_6}$, and $\overline{A_7A_8}$, respectively. For $i = 1, 3, 5, 7$, ray R_i is constructed from M_i towards the interior of the octagon such that $R_1 \perp R_3$, $R_3 \perp R_5$, $R_5 \perp R_7$, and $R_7 \perp R_1$. Pairs of rays R_1 and R_3, R_3 and R_5, R_5 and R_7, and R_7 and R_1 meet at B_1, B_3, B_5, B_7 respectively. If $B_1B_3 = A_1A_2$, then $\cos 2\angle A_3M_3B_1$ can be written in the form $m - \sqrt{n}$, where m and n are positive integers. Find $m + n$.

Solution. Let $\theta = \angle M_1M_3B_1$. Thus we have that $\cos 2\angle A_3M_3B_1 = \cos\left(2\theta + \frac{\pi}{2}\right) = -\sin 2\theta$.

Since $A_1A_2A_3A_4A_5A_6A_7A_8$ is a regular octagon and $B_1B_3 = A_1A_2$, let $k = A_1A_2 = A_2A_3 = B_1B_3$.

Extend $\overline{A_1A_2}$ and $\overline{A_3A_4}$ until they intersect. Denote their intersection as I_1. Through similar triangles and the $45 - 45 - 90$ triangles formed, we find that $M_1M_3 = \frac{k}{2}(2 + \sqrt{2})$.

We also have that$\triangle M_7B_7M_1 = \triangle M_1B_1M_3$ through ASA congruence ($\angle B_7M_7M_1 = \angle B_1M_1M_3$, $M_7M_1 = M_1M_3$, $\angle B_7M_1M_7 = \angle B_1M_3M_1$). Therefore, we may let $n = M_1B_7 = M_3B_1$.

Thus, we have that $\sin\theta = \frac{n+k}{\frac{k}{2}(2+\sqrt{2})}$ and that $\cos\theta = \frac{n}{\frac{k}{2}(2+\sqrt{2})}$. Therefore $\sin\theta - \cos\theta = \frac{k}{\frac{k}{2}(2+\sqrt{2})} = \frac{2}{2+\sqrt{2}} = 2 - \sqrt{2}$.

Squaring gives that $\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = 6 - 4\sqrt{2}$ and consequently that $-2\sin\theta\cos\theta = 5 - 4\sqrt{2} = -\sin 2\theta$ through the identities $\sin^2\theta + \cos^2\theta = 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

Thus we have that $\cos 2\angle A_3M_3B_1 = 5 - 4\sqrt{2} = 5 - \sqrt{32}$. Therefore $m + n = 5 + 32 = \boxed{037}$. □

Example 7.15 (AIME II 2013/15)

Let A, B, C be angles of an acute triangle with

$$\cos^2 A + \cos^2 B + 2\sin A\sin B\cos C = \frac{15}{8} \text{ and}$$
$$\cos^2 B + \cos^2 C + 2\sin B\sin C\cos A = \frac{14}{9}$$

There are positive integers p, q, r, and s for which

$$\cos^2 C + \cos^2 A + 2\sin C\sin A\cos B = \frac{p - q\sqrt{r}}{s},$$

where $p + q$ and s are relatively prime and r is not divisible by the square of any prime. Find $p + q + r + s$.

Solution. Let's draw the triangle. Since the problem only deals with angles, we can go ahead and set one of the sides to a convenient value. Let $BC = \sin A$.

By the Law of Sines, we must have $CA = \sin B$ and $AB = \sin C$.

Now let us analyze the given:

$$\begin{aligned}\cos^2 A + \cos^2 B + 2\sin A\sin B\cos C &= 1 - \sin^2 A + 1 - \sin^2 B + 2\sin A\sin B\cos C \\ &= 2 - (\sin^2 A + \sin^2 B - 2\sin A\sin B\cos C)\end{aligned}$$

Now we can use the Law of Cosines to simplify this:

$$= 2 - \sin^2 C$$

Therefore:

$$\sin C = \sqrt{\frac{1}{8}}, \cos C = \sqrt{\frac{7}{8}}.$$

Similarly,

$$\sin A = \sqrt{\frac{4}{9}}, \cos A = \sqrt{\frac{5}{9}}.$$

Note that the desired value is equivalent to $2 - \sin^2 B$, which is $2 - \sin^2(A + C)$. All that remains is to use the sine addition formula and, after a few minor computations, we obtain a result of $\frac{111 - 4\sqrt{35}}{72}$. Thus, the answer is $111 + 4 + 35 + 72 = \boxed{222}$.

Note that the problem has a flaw because $\cos B < 0$ which contradicts with the statement that it's an acute triangle. Would be more accurate to state that A and C are smaller than 90. Also note that the identity $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$ would have easily solved the problem. □

§8 Parting Words and Final Problems

So with this, you should be able to solve almost any AIME Problem on trigonometry and its applications. We hope this document helped you learn a bit about how to use trigonometry in all kinds of contexts, even ones that aren't obviously apparent. In addition, we hope that this will boost your geometry skills, as trigonometry is very commonly used to solve problems. Any suggestion would be extremely helpful, whether it would be problem suggestions, mistakes we made, or stuff we should explain better. Here's a final problem set that should incorporate (almost) every AIME Problem which requires trigonometry (that hasn't been solved above):

§9 Hints

1. Try to use our bounds on $\sin\theta$ and $\cos\theta$ instead of rederiving them.
2. Just because the question said to bound these functions does not mean they have a bound. Think about the graphs of the functions.

§A Appendix A: List of Theorems and Definitions

List of Theorems

List of Definitions