Applied Advanced Optimisation iRAT 2

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January 23, 2025

Question 1

Define a randomized detector and a deterministic detector.

A randomized detector T is $t_{ik} = \textbf{prob}(\hat{\theta} = i | x = k)$, where, if we observe $x = k$, then the detector returns the hypothesis $\hat{\theta} = i$ with probability t_{ik}.

A deterministic detector is a detector whose behavior does not involve any randomness, ie. always gives the same result if the same input is processed. So for a deterministic detector, $t_{ik} = 1$ if $\hat{\theta} = i$ and 0 otherwise.

Question 2

Define the detection probability matrix.

The detection probability matrix can be defined as $D = TP$.

$$d_{ij} = (TP)_{ij} = \textbf{prob}(\hat{\theta} = i | \theta = j).$$

Question 3

Define a convex combination and use it to define a convex set.

A convex combination of the points $x_1, \ldots, x_k \in C$ is a point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$, with $\theta_1 + \cdots + \theta_k = 1$ and $\theta_i \geq 0$, for all $i = 1, \ldots, k$ (ie. a linear combination $\sum_{i=1}^{k} \lambda_i x_i$ where each $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$).

A set C is a convex set if and only if it contains every convex combination of its points. (ie. "every point can be seen by every other point in the set")

Question 4

Make a sketch of a convex set with exactly two corners (this is not in the handouts, think).

The shaded region in the figure below:

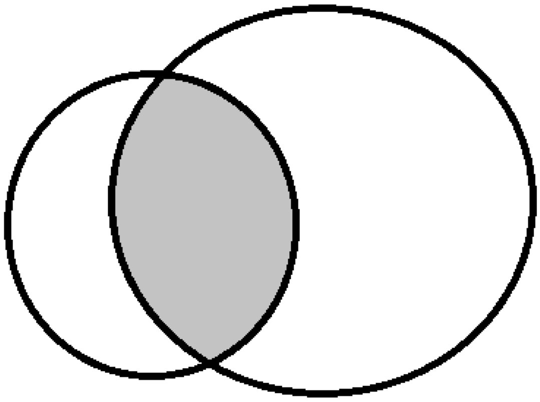

Figure 1: A convex set with exactly two corners (moon shape)

Question 5

Explain the difference between an affine set, a convex set and a conic set.

An affine set is closed under affine combinations: if every affine combination $\theta_1 x_1 + \cdots + \theta_k x_k$, with $\theta_1 + \cdots + \theta_k = 1$, of its points $x_1, \ldots, x_k \in C$ belongs to C.

A convex set is closed under concex combinations: it requires that for any $x, y \in C$ and $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda) y \in C$. This ensures the line segment between any two points is entirely within the set.

A conic set is closed under positive scalar multiplication: for any $x \in C$ and $\alpha > 0$, $\alpha x \in C$. This makes it a ray of cone region from the origin.

Question 6

Prove that the positive semidefinite cone is a convex cone using the definition of convex cone (as done in the video of Section 3.3).

A set C is a convex cone if for any $A, B \in C$ and any non-negative scalars $\alpha, \beta \geq 0$, the combination $\alpha A + \beta B \in C$.

Let A and B be positive semidefinite matrices, i.e., $A \succeq 0$ and $B \succeq 0$. Let $\alpha, \beta \geq 0$.

Since for any vector $x \in \mathbb{R}^n$,

$$x^T(\alpha A + \beta B)x = \alpha x^T A x + \beta x^T B x \geq 0,$$

givent that $A \succeq 0$ and $B \succeq 0$.

Therefore $\alpha A + \beta B$ is positive semidefinite. Hence the positive semidefinite cone is a convex cone.

Question 7

Prove that the positive semidefinite cone is convex using the intersection property (as done in Section 3.4).

A matrix $A \in \mathbb{R}^{n\times n}$ is positive semidefinite if and only if for all vectors $x \in \mathbb{R}^n$, $x^T Ax \geq 0$.

This condition can be expressed as an intersection of convex sets. For each $x \in \mathbb{R}^n$, define the set:

$$C_x = \{A \in \mathbb{R}^{n\times n} \mid x^T Ax \geq 0\}$$

Each C_x is a convex set because it is defined by a linear inequality in terms of A. The positive semidefinite cone S^n_+ is the intersection of all such sets C_x. Since convexity is preserved under intersection, S^n_+ is convex. Hence the positive semidefinite cone is convex.