Applied Advanced Optimisation iRAT 5

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Question 1

Write a quadratic programme.

A quadratic programme:

$$\begin{array}{ll} \min & \frac{1}{2}x^\top Px + q^\top x + r \\ s.t. & Gx \preccurlyeq h \\ & Ax = b \end{array}$$

where $P \in \mathbb{S}^n_+$, $G \in \mathbb{R}^{m\times n}$ and $A \in \mathbb{R}^{p\times n}$.

Question 2

Write a geometric programme in posynomial form. Explain why this problem is not convex. Hence, write the same problem in convex form.

A geometric programme:

$$\begin{array}{lll} \min & f_0(x) & \\ s.t. & f_i(x) \leq 1, & i = 1,\ldots,m \\ & h_i(x) = 1, & i = 1,\ldots,p \end{array}$$

where f_0, ..., f_m are posynomials and h_1, ..., h_p are monomials, thus the programme is not convex.

We can introduce $y_i = \log x_i$ to rewrite the posynomials as monomials:

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}} = \sum_{k=1}^{K} e^{a_k^\top y + b_k}$$

where $a_k = (a_{1k}, \ldots, a_{nk})$ and $b_k = \log c_k$.

The geometric programme can be rewritten as:

$$
\begin{array}{lll}
\min & \sum_{k=1}^{K_0} e^{a_{0k}^\top y + b_{0k}} & \\
s.t. & \sum_{k=1}^{K_i} e^{a_{ik}^\top y + b_{ik}} \leq 1, & i = 1, \ldots, m \\
& e^{g_i^\top y + h_i} = 1, & i = 1, \ldots, p
\end{array}
$$

Take the logarithm of all the functions, we get the convex form:

$$
\begin{array}{lll}
\min & \log\left(\sum_{k=1}^{K_0} e^{a_{0k}^\top y + b_{0k}}\right) & \\
s.t. & \log\left(\sum_{k=1}^{K_i} e^{a_{ik}^\top y + b_{ik}}\right) \leq 0, & i = 1, \ldots, m \\
& g_i^\top y + h_i = 0, & i = 1, \ldots, p
\end{array}
$$

Question 3

By varying λ in the scalarised problem, you will find different optimal points for the scalarised problem. Explain what these points are for the original convex vector optimisation problem if

- $\lambda \succ_{K*} 0$
- $\lambda \succcurlyeq_{K*} 0$

If $\lambda \succ_{K*} 0$, every solution of the scalarised problem is Pareto optimal for the original optimisation problem. However, if a Pareto optimal point lies on the boundary of the feasible region and thus is not strictly dominated by any other points, we cannot find it with this method.

If $\lambda \succcurlyeq_{K*} 0$, we can find all Pareto optimal points. In this case, our solution includes the non-Pareto optimal points that lie on the boundary of the feasible region and are not strictly dominated by any other points.

Question 4

Define the conjugate function and use it to write the dual function of the problem

$$
\begin{array}{ll}
\textbf{min} & \|x\| \\
\textbf{s.t.} & Ax = b
\end{array}
$$

The conjugate function:

$$
f^*(y) = \sup_{x \in \textbf{dom}\, f} \left(y^\top x - f(x)\right) = \begin{cases} 0 & \|y\|_* \leq 1 \\ \infty & \text{otherwise.} \end{cases}
$$

where $\|y\|_*$ is the dual norm.

The dual function:

$$
g(\nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = -b^\top \nu - f_0^*(-A^\top \nu) = \begin{cases} -b^\top \nu & \|A^\top \nu\|_* \leq 1 \\ \infty & \text{otherwise.} \end{cases}
$$

Question 5

Define weak duality, and strong duality. Give conditions which guarantee strong duality for a convex optimisation problem.

Weak duality: the optimal value of the Lagrange dual problem d^* is always less than or equal to the optimal value of the primal problem p^* ($d^* \leq p^*$).

Strong duality: the optimal value of the Lagrange dual problem d^* is equal to the optimal value of the primal problem p^* ($d^* = p^*$).

Strong duality holds if Slater's condition is satisfied: if there exists a feasible $x \in$ **int** $\mathcal{D}$ such that the inequality constraints hold strictly, and the problem is convex.

Appendix

1. Write the epigraph form of a convex optimisation problem in standard form.

$$\begin{array}{ll} \min & f_0(x) \\ s.t. & f_i(x) \leq 0,\ i = 1, \ldots, m \\ & Ax = b \end{array}$$

2. Write a linear programme.

$$\begin{array}{ll} \min & c^\top x \\ s.t. & Gx \preccurlyeq h \\ & Ax = b \end{array}$$

where $G \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{p \times n}$.

3. Define a convex vector optimisation problem. Write its scalarised version.

$$\begin{array}{ll} \min & f_0(x) \\ s.t. & f_i(x) \leq 0,\ i = 1, \ldots, m \\ & Ax = b \end{array}$$

The scalarised version:

$$\begin{array}{ll} \min & \lambda^\top f(x) \\ s.t. & Ax = b \\ & \lambda \geq 0 \end{array}$$

4. Define the dual function.

$$g(\nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu)$$

Feedback

Question 1

Some students did not clearly define the matrix P, e.g. they did not say that this is a symmetric matrix or, for convex problems, that this is positive semidefinite.

Question 2

Some did not justify why the GP in posynomial form is not convex. Some gave the following incorrect reason: the domain is R_{++}. As R_{++} is a convex set, it has no impact on the convex status of the problem. The real reason is that monomials/posynomials are not convex functions (e.g. x^3) and that the equality constraints are not linear.

Question 3

Some did not explain that the first condition does not give all Pareto optimal points, while the second condition give all plus non-Pareto optimal points. The correct answer is given in the paragraph starting with "In summary" above Example 5.14

Question 4

Some gave the generic dual function instead of giving the one for the problem with the norm