Applied Advanced Optimisation iRAT 4

Zhao, Amor (CID: 02019680)

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Question 1

Define quasiconvexity and give the modified Jensen's inequality for quasiconvex functions.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called quasiconvex if its domain and all its sublevel sets $S_\alpha = \{x \in \textbf{dom}\, f : f(x) \leq \alpha\}$ for $\alpha \in \mathbb{R}$ are convex.

Question 2

Define a convex optimisation problem.

The general form of a convex optimisation problem can be defined as:

$$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & a_i(x) = b_i, \quad i = 1, \ldots, p \end{array}$$

where $f_0, \ldots, f_m$ are convex functions.

Question 3

Write the bisection algorithm to solve a quasiconvex optimisation problem.

First we define the convex feasibility problem as follows:

$$\begin{array}{ll} \text{find} & x \\ \text{s.t.} & \phi_t(x) \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & a_i^\top x = b_i, \quad i = 1, \ldots, p \end{array}$$

We can solve the quasiconvex optimisation problem by solving a sequence of convex feasibility problems:

given $l \leq p^*$, $u \geq p^*$, tolerance $\varepsilon > 0$
repeat
1. $t := (l + u)/2$
2. Solve the convex feasibility problem
3. **if** the problem is feasible
   $u := t$
   else
   $l := t$

until $u - l \leq \varepsilon$

Question 4

Consider a general convex optimisation problem and suppose that f_0 is differentiable. State a necessary and sufficient condition for x to be optimal.

With feasible set X, x is optimal if and only if $x \in X$ and $\nabla f_0(x)^\top (y - x) \geq 0$ for all $y \in X$.

Question 5

Consider the convex optimisation problem

$$\begin{array}{ll} \textbf{min} & f_0(x) \\ \textbf{s.t.} & Ax = b. \end{array}$$

State necessary and sufficient conditions for x to be optimal.

$x \in \textbf{dom } f_0$ is optimal if and only if there exists a v such that $Ax = b$ and $\nabla f_0(x) + A^\top v = 0$.

Question 6

Consider the convex optimisation problem

$$\begin{array}{ll} \textbf{min} & f_0(x) \\ \textbf{s.t.} & x < 0. \end{array}$$

State necessary and sufficient conditions for x to be optimal.

$x \in \textbf{dom } f_0$ is optimal if and only if there exists a $x < 0$ such that $\nabla f_0(x) = 0$.

Appendix

The following questions were part of the test in previous years. I removed them to make the test faster to complete. These questions are not marked, but you can use them as extra exercise material.

1. Consider the function $f(x) = \frac{1}{g(x)}$, where $g(x)$ is positive for all $x \in \mathbb{R}$. You want to use the composition rule to establish whether f (x) is convex, concave or neither. To this end:

(a) **Identify the function $h(x)$ (you should consider the function only for $x > 0$).**

The function $h(x)$ is defined as $h(x) = \frac{1}{x}$, where $x > 0$.

(b) **Find $\tilde{h}$ and show whether this is non-decreasing or non-increasing.**

The function $\tilde{h}$ is defined as $\tilde{h}(x) = \frac{1}{x^2}$, which is non-decreasing.

(c) **Add a missing assumption on $g(x)$ to apply the composition rule.**

The missing assumption: $g(x)$ is convex.

2. Define log-concavity and give the modified Jensen's inequality for log-concave functions.

Log-concavity is a property of functions that satisfy the inequality $f(\lambda x+(1-\lambda)y) \geq f(x)^{\lambda} f(y)^{1-\lambda}$ for all x, y in the domain of f and $\lambda \in [0,1]$.

The modified Jensen's inequality for log-concave functions is given by $f(\lambda x + (1 - \lambda)y) \geq f(x)^{\lambda} f(y)^{1-\lambda}$.

Feedback

Performance was generally strong, with one student losing 2 points and one student losing 3 points. For the rest, there was at most one error.

Question 1

some did not provide the definition of quasiconvex function (i.e. domain and all its sublevel sets must be convex) but just described some of its intuitive property (e.g. only one dip). Some only stated the condiction on the sublevel set, but forgot to say that the domain must be convex. One student did not provide the Jensen's inequality

Question 2

some did not say that the f_0 is must be convex functions, without which the problem is not convex.

Question 4

one student gave the condition for unconstrained problems, i.e. equation (5) in Chapter 5, instead of the condition for the general problem as requested, i.e. (4) in Chapter 5.

Question 5

some students forgot to include Ax=b (see (7) in Chapter 5)

Question 6

one student (who lost 3 points here) gave a completely incorrect answer. The correct answer was (8) in Chapter 5.

Appendix

For those who attempted the questions in the Appendix, the solution to the first question is as follows:

(a) $h(x) = 1/x$ for $x > 0$

(b) $h(x)$ is a convex function. Thus, its extended version is +infinity when $x <= 0$ and $1/x$ when $x > 0$. The fact that $1/x$ is nonincreasing can be shown in different ways: for instance with a plot, or reasoning on the sign of the derivative, or reasoning on the value of the function for two values $x1 < x2$.

(c) Since h is convex and h extended is nonincreasing, we can only apply version two of the composition rule. This requires that g is concave.