Real Space Renormalization Group

Assignment 8

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Abstract

The quantum Ising model in a transverse field serves as a foundational framework for studying quantum phase transitions and many-body correlations. This work computes the ground state energy density ($\frac{E_0}{N}$) as a function of the transverse field strength (λ) using the Real-Space Renormalization Group (RSRG) algorithm and compares the results with mean-field approximations. The RSRG method, which retains low-energy states iteratively, demonstrates superior accuracy near the critical region ($\lambda \in [-1, 1]$), capturing the quadratic behavior of the energy around ($\lambda = 0$). In contrast, the mean-field approximation aligns with RSRG for ($|\lambda| \gtrsim 2$), where the paramagnetic phase dominates. The divergence between methods peaks at ($\lambda = 0$) and ($\lambda = \pm 2$), revealing limitations of mean-field in correlated regimes and potential RSRG truncation errors. These results underscore the importance of non-perturbative methods for studying quantum criticality and highlight the interplay between correlations and external fields in low-dimensional systems.

Introduction

The quantum Ising model is one of the simplest non-trivial many-body quantum systems. It is a simple yet powerful tool to study interacting quantum systems that exhibit a rich array of phenomena, making it an ideal model for studying quantum phase transitions and many-body quantum states. Spin-$\frac{1}{2}$ systems, which form the basis of the quantum Ising model, are fundamental building blocks in quantum mechanics. These systems are represented as two-level quantum states, commonly referred to as "up" $|\uparrow\rangle$ and "down" $|\downarrow\rangle$, with applications ranging from magnetic materials to qubits in quantum computing. The interactions between these spins, as well as their response to external fields, reveal insights into the collective behaviour of quantum systems.

In this report, the focus is on computing the ground state energy as a function of λ, the strength of the transverse field, considering the Hamiltonian of the $1D$ quantum Ising model. This will be realised using the real space renormalization group (RSRG) algorithm, and the results will be compared with the ones found using the mean-field approximation.[1]

Methodology

Formalism

Quantum Ising model To study the quantum Ising model, we consider N spin-$\frac{1}{2}$ particles on a one-dimensional lattice with nearest neighbour interaction, that are described by the Hamiltonian $\hat{H}$:

$$\hat{H} = \lambda \sum_{i}^{N} \sigma_i^z + \sum_{i}^{N-1} \sigma_i^x \sigma_{i+1}^x$$

where :

- N is the number of particles,
- λ is the strength of the external field,
- σ_i^z and σ_i^x are the Pauli matrices acting on the i-th particle, i.e.

$$\sigma_i^z = I_1 \otimes I_2 \otimes ... \otimes \sigma_i^z \otimes I_{i+1} \otimes ... \otimes I_N \quad (1)$$

and

$$\sigma_i^x \otimes \sigma_{i+1}^x = I_1 \otimes I_2 \otimes ... \otimes \sigma_i^x \otimes \sigma_{i+1}^x \otimes I_{i+2} \otimes ... \otimes I_N \quad (2)$$

with I_i the identity matrices and the Pauli matrices are :
$\sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The first term σ_i^z describes the interaction of each spin with the external field, which aligns spins in the z-direction, while the second term $\sigma_i^x \sigma_{i+1}^x$ describes the interaction between neighbouring spins, favouring alignment in the x-direction.

RSRG algorithm Real Space Renormalization Group is a method to approximate the ground state of a many-body quantum system (via coarse-graining modelling). The general idea is starting from an initial system size n_0 whose Hamiltonian H_{n_0} can be diagonalized exactly and iteratively increasing the system size while keeping the (approximated) Hilbert space dimension fixed. This is achieved by performing a truncation, projecting onto a fixed number of states considered to be appropriate for a low-energy description of the system.

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[1] The optional task, consisting in computing the solutions using the infinite density matrix renormalization group algorithm, was not performed.

If we let d be the local Hilbert space dimension and $n_0 = 2m$, with $N = n_0$ set as initial condition, the algorithm should follow these steps :

- Find the d_m lowest energy eigenvectors $|v\rangle$ of H_N and build the projector $P = \sum |v\rangle\langle v|$,
- Truncate the system description (states and operators) using P ; compute $\tilde{H} = P^\dagger H P$,
- Double the system $n \to 2n$ using the truncated basis for each side ; in particular $H_{2N} = H_N \otimes \mathbb{I} + \mathbb{I} \otimes H_N + H_{\text{int}}$.

Finally, the expression to compute the solutions with the mean-field approximation is taken from [1] (page 39, equation 4.8).

$$\begin{cases} e = -\dfrac{\lambda^2}{4} - 1 & \text{if } \lambda \in [-2\,;\,2], \\ e = -|\lambda| & \text{if } \lambda \notin [-2\,;\,2] \end{cases} \tag{3}$$

Implementation

Quantum Ising model In order to numerically implement this formalism, we need first to define the Pauli and identity matrices, σ^z = `sigma_z`, σ^x = `sigma_x` and I = `identity`, as described previously, as well as a function `tensor_product_operators` that takes as inputs a list of operators (represented by 2×2 matrices) and returns the tensor products of these operators. We use the `scipy.sparse` package for more efficient computations, with sparse matrices and their built-in optimized functions.

To compute the matrix representation of the Hamiltonian H, we define the function `hamiltonian(N, S)` that takes as inputs `N`, the number of spin-$\frac{1}{2}$ particles, and `S`, the strength of the external field (λ). It first initializes with zeros a matrix of size $2^N \times 2^N$ (since $\hat{H}$ is a tensor product of N 2×2 matrices). The function first fills the matrix with all the N terms related to the external field, and then adds all the N terms related to interaction between the particles, using the function `tensor_product_operators` in order to compute the operators described in equations 1 and 2.

RSRG algorithm The RSRG algorithm is implemented with Python following these steps :

Initialization :

- N spin Hamiltonian that can be solved exactly and the interaction operators :
  $H_N^{(0)} = \lambda \sum_i^N \sigma_i^z + \sum_i^{N-1} \sigma_i^x \sigma_{i+1}^x$
  $A^{(0)} = \mathbb{I}_{N-1} \otimes \sigma^x$
  $B^{(0)} = \sigma^x \otimes \mathbb{I}_{N-1}$

Algorithm :

1. Build the Hamiltonian and state for the system of size $2N$ from the previous step :
$$H_{2N}^{(n)} = H_N^{(n)} \otimes \mathbb{I}_N + \mathbb{I}_N \otimes H_N^{(n)} + A_N^{(n)} \otimes B_N^{(n)}$$
2. Compute the d^N lowest eigenvectors of H_{2N} and construct the projector :
$$P = \sum_{i=1}^{d^N} |v_i\rangle \langle v_i|$$
3. Check the convergence of the energy density $\epsilon = E_0/N$ with a threshold criterion :
$$\Delta_n = \left| \epsilon^{(n)} - \epsilon^{(n-1)} \right| < \tau$$
4. If it is not converged, update the Hamiltonian and the interaction operators for the next iteration :
$$H_N^{n+1} \leftarrow \frac{1}{2} P^\dagger H_{2N}^n P$$
$$A_N^{n+1} \leftarrow \frac{1}{\sqrt{2}} P^\dagger \left(\mathbb{I}_N \otimes A_N^n \right) P$$
$$B_N^{n+1} \leftarrow \frac{1}{\sqrt{2}} P^\dagger \left(B_N^n \otimes \mathbb{I}_N \right) P$$

In our case, we choose as initial parameters :

- `N = 14`, the number of particles,
- `m = 5`, where 2^m is the number of low-energy states to keep during the RSRG process,
- `n = N`, the current Hilbert space dimension (that starts with N),
- `iteration_energy = 1e6`, the initial ground state energy for the first iteration,
- `convergence = 1e-14`, the threshold criterion τ.

The RSRG algorithm is defined as a function and called for different values of λ (set by the user).

Results

The results of the computations made with the RSRG algorithm and the mean-field approximation are represented on the figure 1. The solutions are computed for 100 values of $\lambda \in [-3\,;\,3]$ (Fig. 1a), and a more detailed view of the solutions around $\lambda = 0$ is represented on the figure 1b (200 points for $\lambda \in [-1.2\,;\,1.2]$). The difference of these solutions, $\frac{E_0}{N}_{MF} - \frac{E_0}{N}_{RSRG}$, is represented on the figure 2. $\frac{E_0}{N}$ seems to follow the behaviour of a quadratic function, with a maximum at $\lambda = 0$. The RSRG and mean-field solutions are matching around $\lambda = \pm 1$. The RSRG algorithm seems to describe more precisely the phenomenon around $\lambda = 0$ where the solutions are

perturbed, as the mean-field approximation does not take into account the correlations.

The figure 2a shows us that the difference between the solutions from the RSRG algorithm and the mean-field approximation is the largest around $\lambda = \pm 2$ and for $\lambda = 0$. The RSRG solutions are greater than the ones from the mean-field approximations only for $\lambda \in [-1\ ;\ 1]$. It seems that for $|\lambda| \gtrsim 2$, the difference $\frac{E_0}{N}_{MF} - \frac{E_0}{N}_{RSRG} \to 0$, meaning for large values of $|\lambda|$, the solutions of the RSRG algorithm follow a linear trend. This exhibits the paramagnetic phase where the transverse field dominates.

In order to evaluate the performance and good execution of the implemented RSRG algorithm, the number of iterations needed to reach the threshold criterion τ is represented on the figure 3 for 100 values of $\lambda \in [-3\ ;\ 3]$. Additionally, the values of $\frac{E_0}{N}$ through the iterations leading to the convergence are represented on the figure 4 for $\lambda = \{-3, -2, -1,\ 0,\ 1,\ 2,\ 3\}$.

Conclusion

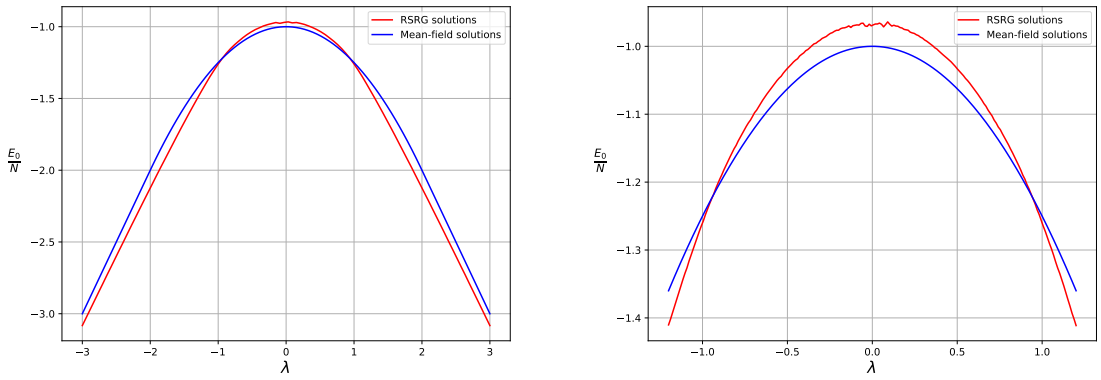
The RSRG algorithm successfully captures the ground state energy of the 1D quantum Ising model, particularly excelling in the critical region ($\lambda \in [-1, 1]$) where quantum correlations dominate. Here, the energy density $\frac{E_0}{N}$ exhibits a quadratic dependence on λ, a feature neglected by the mean-field approximation due to its inability to account for spatial fluctuations. Outside this region ($|\lambda| \gtrsim 2$), both methods converge to a linear trend, reflecting the dominance of the transverse field in the paramagnetic phase.

Notably, the largest discrepancies occur at $\lambda = 0$ and $\lambda = \pm 2$. The former arises from mean-field's failure to describe strong correlations, while the latter may stem from RSRG truncation limits or numerical instabilities.

Future work should optimize RSRG parameters (e.g., the 2^m retained states) to improve accuracy for $|\lambda| \gg 1$ and validate results against exact solutions. Additionally, extending this approach to higher dimensions or dynamical properties could deepen insights into quantum critical systems. This study reaffirms RSRG as a powerful tool for strongly correlated systems while emphasizing the nuanced trade-offs between computational complexity and physical accuracy.

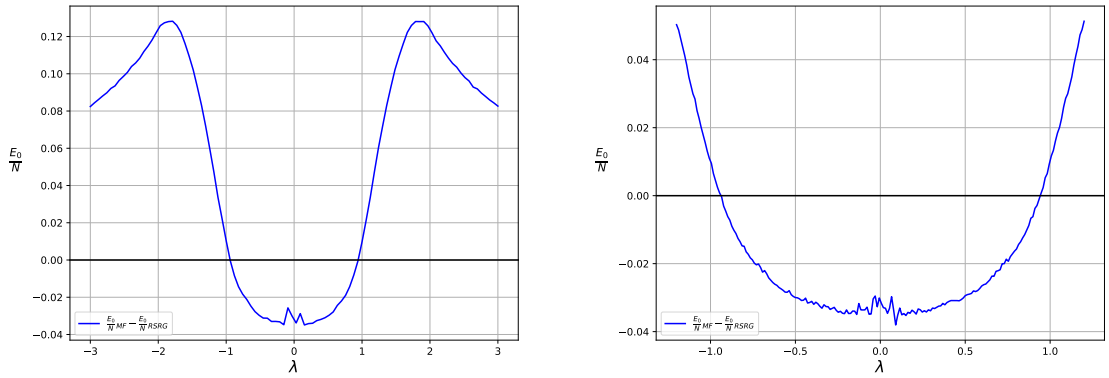
References

[1] S. Montangero. *Introduction to Tensor Network Methods: Numerical simulations of low-dimensional many-body quantum systems*. Springer International Publishing, 2018. ISBN: 9783030014094. URL: https://link.springer.com/book/10.1007/978-3-030-01409-4.



(a) $\lambda \in [-3\ ;\ 3]$ *(100 points)* **(b)** $\lambda \in [-1.2\ ;\ 1.2]$ *(200 points)*

Figure 1: $\frac{E_0}{N}$ *solutions from the RSRG algorithm and the Mean-field approximation, for different values of* λ



(a) $\lambda \in [-3\ ;\ 3]$ *(100 points)* **(b)** $\lambda \in [-1.2\ ;\ 1.2]$ *(200 points)*

Figure 2: $\frac{E_0}{N}_{MF} - \frac{E_0}{N}_{RSRG}$ *difference, for different values of* λ

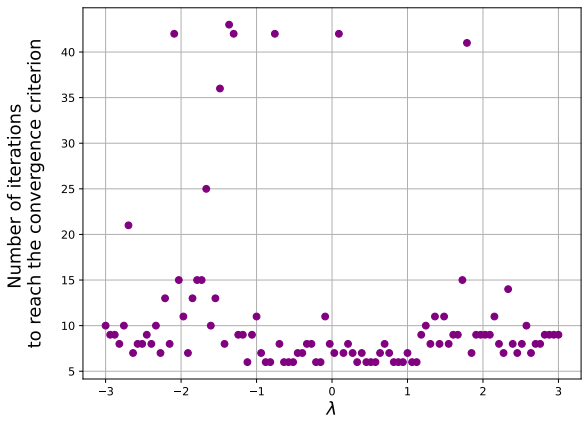
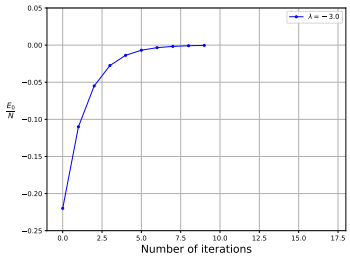
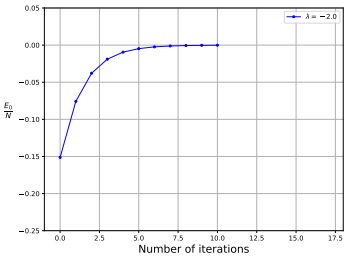


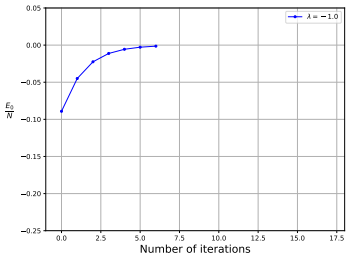
Figure 3: *Number of iterations needed to reach the convergence criterion* $\lambda \in [-3\ ;\ 3]$ *(100 points)*



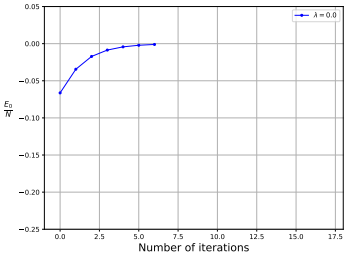
(a) $\lambda = -3$



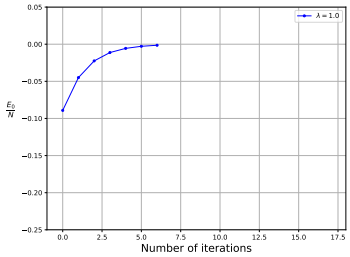
(b) $\lambda = -2$



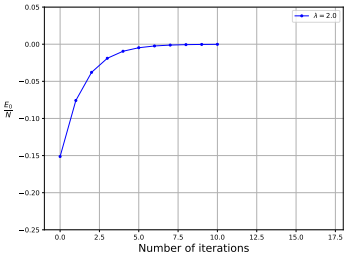
(c) $\lambda = -1$



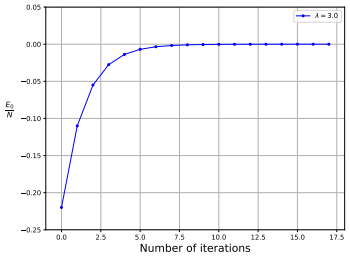
(d) $\lambda = 0$



(e) $\lambda = 1$



(f) $\lambda = 2$



(g) $\lambda = 3$

Figure 4: $\frac{E_0}{N}$ *through the iterations up to convergence, for different values of* λ