Insurance Risk Control with Reinsurance

A Physics-Informed Neural Network (PINN) Approach

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Outline

Introduction: Problem Setting

- An insurance company manages its surplus while facing:
  - financial risk from investing in
    - a risky asset (with drift μ and volatility σ)
    - a risk free asset that grows at rate r.
  - risk arising from random claim arrivals.
- To control these risks, the insurer chooses two time–dependent decisions:
  - **Investment control** π_t: amount allocated to the risky asset.
  - **Reinsurance control**: fraction of each claim to transfer or maximum loss to retain.
- The goal is to determine optimal investment and reinsurance policies that maximize the expected utility of terminal surplus.

Key Definitions

- **Insurer's Surplus**: Denoted as X_t is difference between the insurer's assets and liabilities at time t.
- **Claim Process**: Claims arrive according to a Poisson process with intensity λ. Each claim has random size Z, drawn independently from a chosen distribution.
- **Cumulative Claim Process**: We let C_t be the cumulative claim process modeled by a compound Poisson process:

$$C_t = \sum_{i=1}^{N_t} Z_i \qquad \text{where} \qquad N_t \sim \text{Poisson}(\lambda)$$

- **Reinsurance**: A contract transferring part of each claim to a reinsurer.
    - **Proportional**: retain $(1-\theta_t)Z$ where θ is the proportion of risk ceded.
    - **Excess-of-loss (XoL)**: retain $\min(Z, M_t)$, where M_t is the reinsurance deductible.

Introduction: Two Part Structure

Presentation has two parts:

Part 1 - Proportional Reinsurance

- Develop the SDE for insurer surplus.
- Formulate and interpret the HJB.
- Introduce the PINN methodology.
- Present results for numerical experiments.

Part 2 - Excess of Loss (XoL) Reinsurance

- Modify the reinsurance mechanism.
- Reformulate the HJB.
- Compare XoL results with proportional reinsurance.

PART 1: Insurance Risk Control with Proportional Reinsurance

Insurer's Surplus SDE

The insurer's surplus X_t satisfies the jump-diffusion SDE

$$dX_t = rX_t + \pi_t(\mu - r)dt - (1 - \theta_t)ZdC_t + \sigma dW_t$$

where

- π_t: investment in risky asset.
- $\theta_t \in (0, 1)$: proportion of claim ceded (to reinsurance)
- $(1 - \theta_t)Z$: retained loss after reinsurance
- C_t is the cumulative claim process.
- W_t is the standard Brownian motion.

Control Problem

The insurer's controls are:

- π_t: how much is invested in risky assets.
- θ_t proportion of risk ceded.

The insurer's objective is

$$\max_{\pi_t, \theta_t} \mathbb{E}[U(X_T)], \qquad U(x) = -e^{-x}$$

- Maximize expected utility of terminal wealth.
- Admissible over all controls.
- Utility function is convex and easily differentiable.

The HJB Equation

Define the value function as

$$V(t,x) = \sup_{\{\pi_s,\theta_s\}_{s\in[t,T]}} \mathbb{E}[\, U(X_T) \mid X_t = x \,]$$

The value function satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \sup_{\pi,\theta} \left\{ V_t + (rx + \pi(\mu - r))V_x + \frac{1}{2}\sigma^2\pi^2 V_{xx} + \lambda(\mathbb{E}[V(t, x - (1-\theta)Z)] - V) \right\}$$

Terminal condition: $V(T,x) = U(x)$

The jump expectation term:

$$\mathbb{E}[V(t,x-(1-\theta)Z)] = \int_0^\infty V(t,x-(1-\theta)z) f_Z(z)\, dz$$

- Captures loss of surplus from random claims.
- Nonlocal integration term, which cannot be solved analytically.

Why PINNs for HJB with Jumps

- Automatic differentiation.
- Mesh-free
- Handles non-local terms
- Learns value function and optimal controls simultaneously.
- Stable for high dimensional PDEs.

PINN Architecture

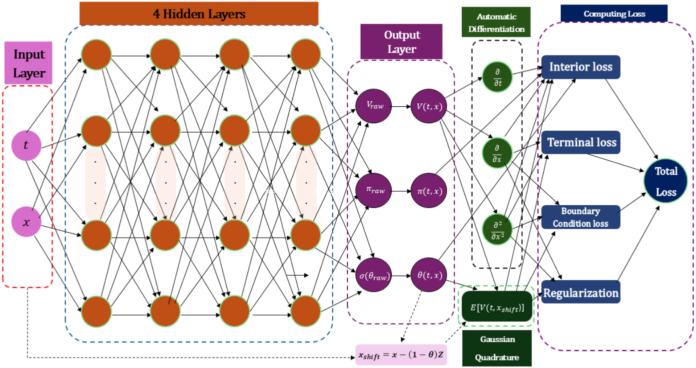

Figure 1: Forward Propagation of a Fully Interconnected feed-forward MLP

PINN Architecture: Input Layer

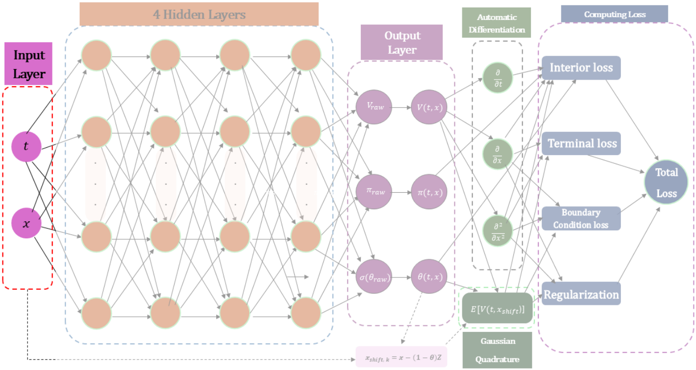

Figure 1: Forward Propagation of a Fully Interconnected feed-forward MLP

- Time $t \in [0, T]$
- Insurer surplus $x \in [x_{\min}, x_{\max}]$

PINN Architecture: Hidden Layer

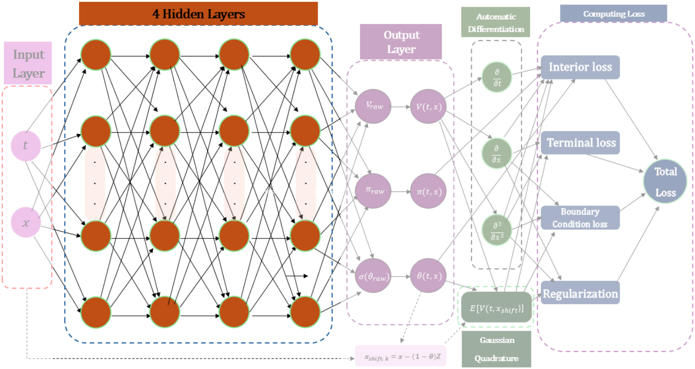

Figure 1: Forward Propagation of a Fully Interconnected feed-forward MLP

- 4 layers, 64 neurons per layer
- $z^{[l]} = \tanh\left(W^{[l]} z^{[l-1]} + b^{[l]}\right), \qquad l = 1, \ldots, L.$

PINN Architecture: Output Layer

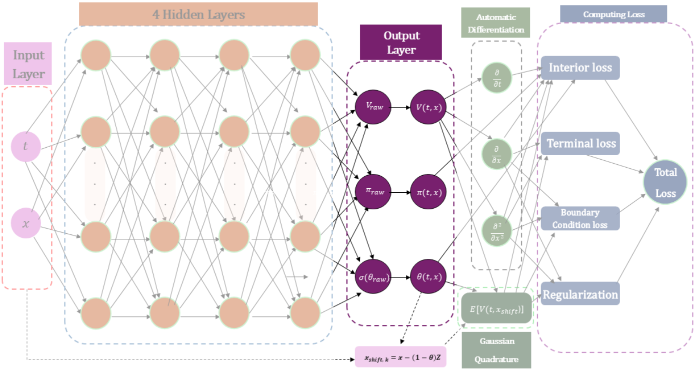

Figure 1: Forward Propagation of a Fully Interconnected feed-forward MLP

- Raw outputs: $(V_{\text{raw}}(t,x), \pi_{\text{raw}}(t,x), \theta_{\text{raw}}(t,x))$
- Process only reinsurance control: $\theta(t,x) = \sigma(\theta_{\text{raw}}(t,x)) \in (0,1)$

PINN Architecture: Autograd and Quadrature

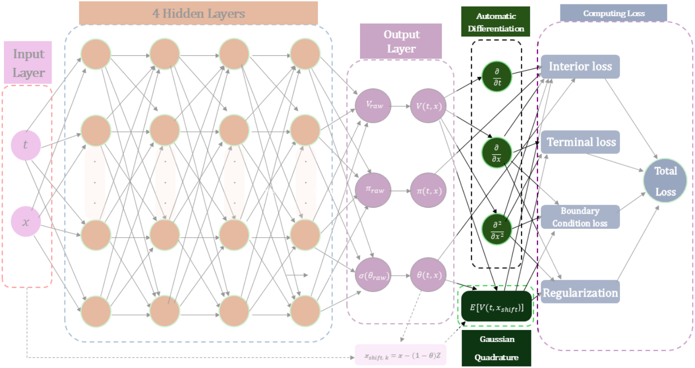

Figure 1: Forward Propagation of a Fully Interconnected feed-forward MLP

- PyTorch AutoGrad computes: V_t, V_x and V_{xx}
- The expectation term $\mathbb{E}[V(t, x - (1-\theta)Z)$ is approximated by Gaussian-Quadrature.

PINN Architecture: HJB Residual and Loss Computation

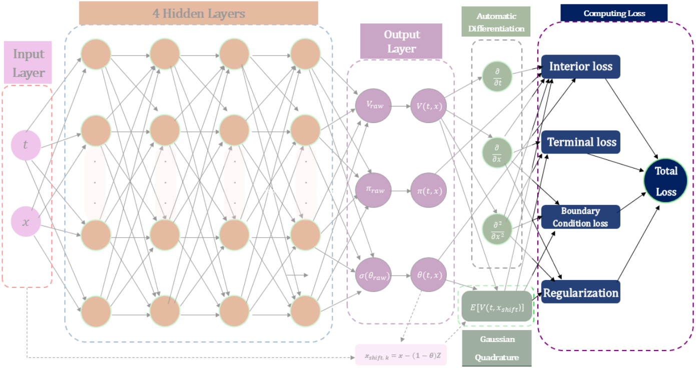

Figure 1: Forward Propagation of a Fully Interconnected feed-forward MLP

- HJB Residual

$$\mathcal{R} = V_t + (rx + \pi(\mu - r))V_x + \frac{1}{2}\sigma^2\pi^2 V_{xx} + \lambda(\mathbb{E}[V(t, x - (1-\theta)Z)] - V)$$

- Total Loss is made up of 4 losses.

PINN Architecture: HJB Residual and Loss Computation

Total loss is given by

$$\mathcal{L} = w_{PDE}\mathcal{L}_{PDE} + w_T\mathcal{L}_{\text{terminal}} + w_{BC}\mathcal{L}_{BC} + \mathcal{L}_{\text{reg}}$$

- PDE Loss:

$$\mathcal{L}_{PDE} = \frac{1}{N_{int}} \sum_{i=1}^{N_{int}} \mathcal{R}(t_i, x_i; \Theta_{NN})^2.$$

- Terminal loss:

$$\mathcal{L}_{\text{terminal}} = \frac{1}{N_T} \sum_{j=1}^{N_T} \big(V_\Theta(T, x_j) - U(x_j)\big)^2.$$

- Boundary Loss

$$\mathcal{L}_{BC} = \frac{1}{N_{BC}} \sum_{k=1}^{N_{BC}} \Big[(V(t_k, x_{\min}) - U(x_{\min}))^2 + (V(t_k, x_{\max}) - U(x_{\max}))^2\Big].$$

PINN Architecture: HJB Residual and Loss Computation

- Regularization of Controls

$$\mathcal{L}_{\text{reg}} = \eta_\pi \frac{1}{N_{int}} \sum_i \pi(t_i, x_i)^2 + \eta_\theta \frac{1}{N_{int}} \sum_i \big(\theta(t_i, x_i) - \theta_0\big)^2.$$

- Loss weights

$$w_{\text{PDE}} = 1, \quad w_{\text{T}} = 10, \quad w_{\text{BC}} = 1, \quad \eta_\pi = 10^{-6}, \quad \eta_\theta = 10^{-4}$$

Experiment 1: Impact of Claim Distributions on Optimal Policies

Goal: To investigate how different claim distributions affect optimal investment and proportional reinsurance policies.

- Fix:

$$r = 0.03, \quad \mu = 0.08, \quad \lambda = 1.0, \quad T = 10, \quad x \in [0, 10]$$

- Vary only claim distributions:
  - **Experiment 1A: Exponential Claims.**

$$Z \sim \mathrm{Exponential}(\beta), \qquad f_Z(z) = \beta e^{-\beta z}, \quad \beta = 1.$$

  - **Experiment 1B: Pareto Claims.**

$$Z \sim \mathrm{Pareto}(\alpha, k), \qquad f_Z(z) = \frac{\alpha k^\alpha}{(z+k)^{\alpha+1}}, \quad \alpha = 3,\ k = 1.$$

  - **Experiment 1C: Lognormal Claims.**

$$Z \sim \mathrm{Lognormal}(\mu_Z, \sigma_Z^2), \quad f_Z(z) = \frac{1}{z\sigma_Z\sqrt{2\pi}} \exp\left(-\frac{(\ln z - \mu_Z)^2}{2\sigma_Z^2}\right)$$

with $\mu_Z = 0, \sigma_Z = 0.5$

Experiment 1 Results: Value Function $V(t, x)$

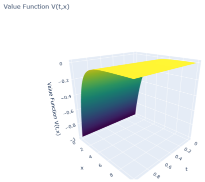

Figure: (a) Exponential

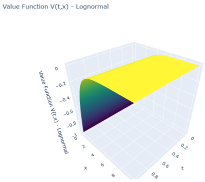

Figure: (b) Lognormal

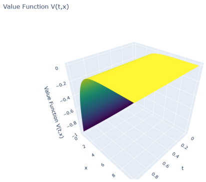

Figure: (c) Pareto

Figure: Learned value function $V(t, x)$ for the three claim-size distributions.

We observe that

- $V(t, x)$ increases in x and decreases in t across all claim distributions.
- V increases in negative in order: Exponential $\rightarrow$ lognormal $\rightarrow$ Pareto
- Distributions with heavier tails capture the most risk.

Experiment 1 Results: Investment Policy $\pi(t,x)$

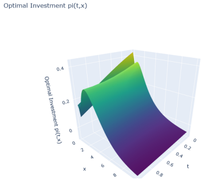

Figure: (a) Exponential

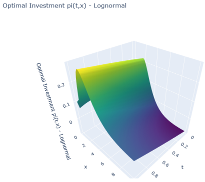

Figure: (b) Lognormal

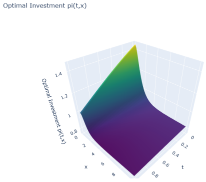

Figure: (c) Pareto

Figure: Optimal investment policy $\pi(t,x)$ under different claim-size distributions.

Key observations

- Heavier tail distributions are more conservative.
- Spikes are as a result of numerical approximations.
- Optimal investment strategy shifts toward smaller and safer positions in the risky asset.

Experiment 1 Results: Reinsurance Policy $\theta(t, x)$

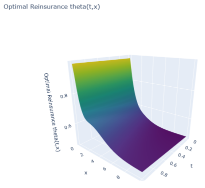

Figure: (a) Exponential

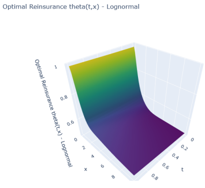

Figure: (b) Lognormal

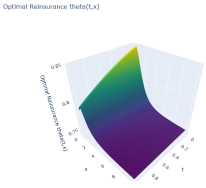

Figure: (c) Pareto

Figure: Optimal reinsurance retention $\theta(t, x)$ across the three claim-size models.

- **Exponential:** Moderate reinsurance demand.
- **Lognormal:** Nearly identical to the exponential case except for slightly higher retention near $x = 0$.
- **Pareto:** Significantly higher reinsurance across almost the entire domain.

PART 2: Insurance Risk Control with Excess-of-Loss (XoL) Reinsurance

Model Setup

- Under XoL reinsurance, the insurer retains only the first $M(t,x)$ units of the claim.

$$\text{Retained Loss} = \min(Z, M_t) = Z \wedge M_t$$

- The insurer's surplus X_t follows the SDE

$$dX_t = rX_t + \pi_t(\mu - r)dt - (Z \wedge M_t)dC_t + \sigma dW_t$$

- The value function $V(t,x)$ satisfies the HJB:

$$0 = \sup_{\pi, M} \left\{ V_t + (rx + \pi(\mu - r))V_x + \frac{1}{2}\sigma^2\pi^2 V_{xx} + \lambda(\mathbb{E}[V(t, x - (Z \wedge M))] - V) \right\}$$

where

$$\mathbb{E}[V(t, x - (Z \wedge M))] = \int_0^\infty V(t, x - (z \wedge M)) f_Z(z)\, dz$$

- We want to find optimal investment strategy $\pi(t,x)$ and reinsurance deductible $M(t,x)$ by solving the HJB.

PINN Architecture: XoL Reinsurance

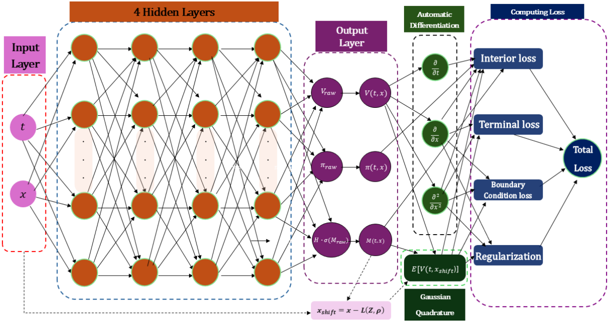

Figure 1: Forward Propagation of a Fully Interconnected feed-forward MLP

Change from proportional case:

- Output reinsurance control is now $M(t,x)$.
- Enforced from:

$$M(t,x) = M_{\text{max}} \cdot \sigma(M_{\text{raw}}(t,x))$$

Experiment 2: Impact of Reinsurance Structures on Optimal Policies

Goal: To compare how different reinsurance structures (proportional and excess-of-loss) affect the value function and optimal investment and reinsurance policies.

- **Claim distribution:** Exponential

$$Z \sim \mathrm{Exponential}(\beta), \qquad f_Z(z) = \beta e^{-\beta z}, \quad \beta = 1.$$

- Common financial parameters

$$r = 0.03, \quad \mu = 0.08, \quad \lambda = 1.0, \quad T = 10, \quad x \in [0, 10]$$

- We only vary reinsurance mechanism.

Experiment 2 Results: Value Function Comparison

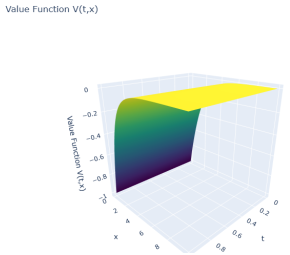

Figure: (a) Proportional reinsurance

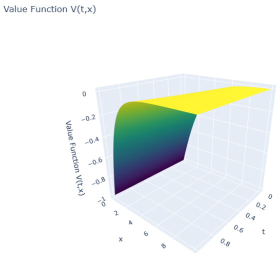

Figure: (b) XoL Reinsurance

Figure: Optimal value function $V(t,x)$ under different reinsurance types

Main differences:

- XoL produces a lower value function.
- Insurer is better off under proportional reinsurance than under XoL reinsurance.

Experiment 2 Results: Investment Policy Comparison

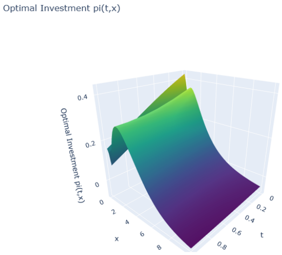

Figure: (a) Proportional reinsurance

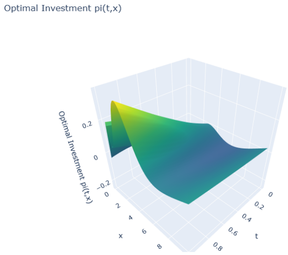

Figure: (b) XoL Reinsurance

Figure: Optimal investment policy $\pi(t, x)$ under different reinsurance types

Main differences:

- XoL pushes the insurer to take much smaller risky positions, with $\pi(t, x)$ even slightly negative in some regions.
- The negative investment positions signifies increased hedging motive.

Experiment 2 Results: Deductible Surface $M(t, x)$

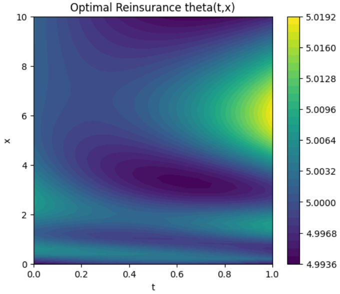

Figure: Heatmap: $M(t, x)$ under XoL

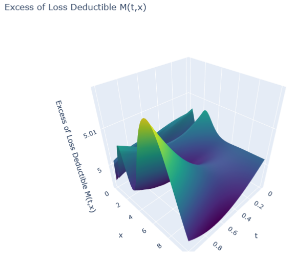

Figure: 3D Surface: $M(t, x)$ under XoL

Key features:

- Deductible converges to an interior constant level
- Not pushed to constraint boundaries
- Weak dependence on t and x.

Conclusion

- PINNs provide a flexible framework for solving nonlinear HJBs with jumps
- Claim distribution affects optimal policies primarily through tail risk
- Reinsurance structure matters as much as distribution
- XoL vs proportional leads to qualitatively different investment behavior

THANK YOU