Chapter 7

Basics of Information Theory

When we talk about *information*, we often use the term in qualitative sense. We say things like *This is valuable information* or *We have a lack of information*. We can also make statements about some information being more helpful than other. For a long time, however, people have been unable to quantify information. The person who succeeded in this endeavour was Claude E. Shannon who with his famous 1948 article *A Mathematical Theory of Communication* single-handedly created a new discipline: Information Theory! He also revolutionised digital communication and can be seen as one of the main contributors to our modern communication systems like the telephone, the internet etc.

The beauty about information theory is that it is based on probability theory and many results from probability theory seamlessly carry over to information theory. In this chapter, we are going to discuss the bare basics of information theory. These basics are often enough to understand many information theoretic arguments that researchers make in fields like computer science, psychology and linguistics.

7.1 Basic Information-Theoretic Quantities

Shannon's idea of information is as simple as it is compelling. Intuitively, if we are observing a realisation of a random variable, this realisation is surprising if it is unlikely to occur according to the distribution of that random variable. However, if the probability for the realisation is very low, than on average it does not occur very often, meaning that if we sample from the RV repeatedly, we are not surprised very often. We are not surprised when the probability mass of the distribution is concentrated on only a small subset of its support.

On the other hand, we quite often are surprised, if we cannot predict what the outcome of our next draw from the RV might be. We are surprised when the distribution over values of the RV is (close to) uniform. Thus, we

are going to be most surprised on average if we are observing realisations of a uniformly distributed RV.

Shannon's idea was that observing RVs that cause a lot of surprises is informative because we cannot predict the outcomes and with each new outcome we have effectively learned something (namely that the ith outcome took on the value that it did). Observing RVs with very concentrated distributions is not very informative under this conception because by just choosing the most probable outcome we can correctly predict most actually observed outcomes. Obviously, if I manage to predict an outcome beforehand, it's occurrence is not teaching me anything.

The goal of Shannon was to find a function that captures this intuitive idea. He eventually found it and showed that it is the only function to have properties that encompass the intuition. This function is called the **entropy** of a RV and it is simply the expected **surprisal** value, expressed in bits.

Definition 7.1 (Surprisal) *The surprisal (value) of an outcome $x \in \mathrm{supp}(X)$ of some RV X is defined as $-\log_2(P(X = x)) = \log_2(\frac{1}{P(X=x)})$.*

Notice that we are using the logarithm of base 2 here. This is because surprisal and entropy are standardly measured in bits. Intuitively, the surprisal measures how many bits one needs to encode an observed outcome given that one knows the distribution underlying that outcome. The entropy measures how many bits one will need on average to encode an outcome that is generated by the distribution P_X.

Definition 7.2 (Entropy) *The entropy $H(P_X)$ of a RV X with distribution P_X is defined as*

$$H(P_X) := \mathbb{E}[-\log_2(P(X = x))] = - \sum_{x \in \mathrm{supp}(X)} P(X = x) \log_2(P(X = x)).$$

For the ease of notation, we often write $H(X)$ instead of $H(P_X)$.

Figure 7.1 shows the entropy of the Bernoulli distribution as a function of the parameter θ. The entropy function of the Bernoulli is often called the **binary entropy**. It measures the information of a binary decision, like a coin flip or an answer to a yes/no-question. The entropy of the Bernoulli is 1 bit when the distribution is uniform, i.e. when both choices are equally probable.

From the plot is it also easy to see that entropy is never negative. It holds in general that entropy is non-negative, because entropy is defined as expectation of surprisal and surprisal is the negative logarithm of probabilities. Because $\log(x) \leq 0$ for $x \in (0, 1]$, it is clear that $-\log(x) \geq 0$ for x in the same interval. Notice that from here on we drop the subscript and by convention let $\log = \log_2$.
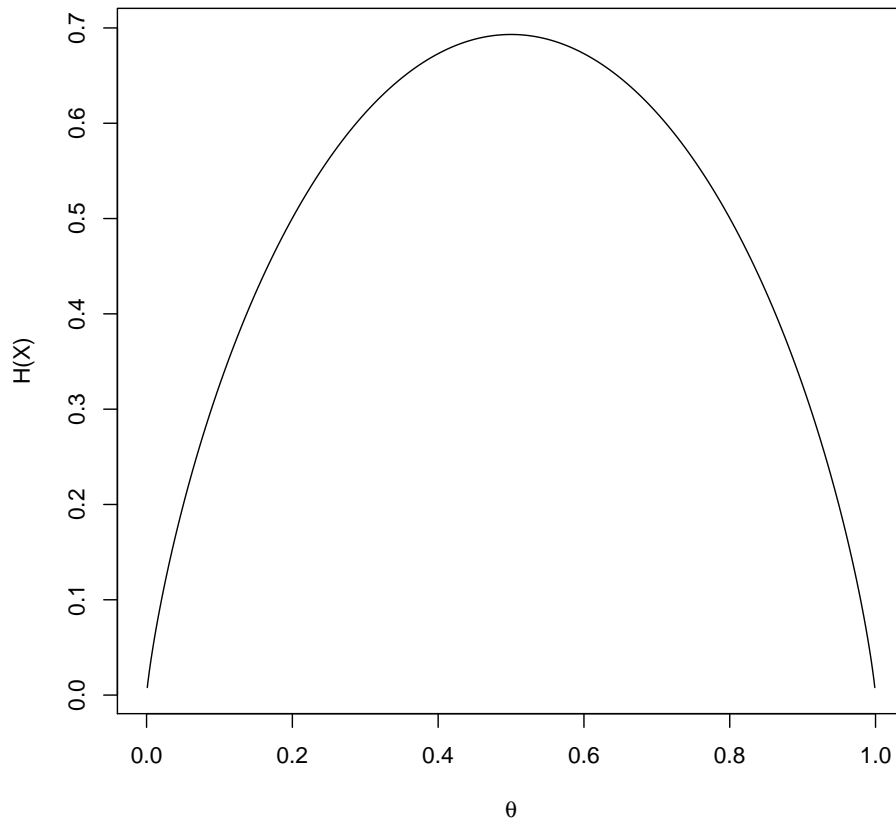
Figure 7.1: Binary entropy function

A standard interpretation of the entropy is that it quantifies uncertainty. As we have pointed out before, a uniform distribution means that you are most uncertain and indeed the uniform distribution maximizes the entropy. However, the more choices you have to pick from, the more uncertain you are going to be. The entropy function also captures this intuition. Notice that if a discrete distribution is uniform, all probabilities are $\frac{1}{|\operatorname{supp}(X)|}$. Clearly, as we increase $|\operatorname{supp}(X)|$, we decrease the probabilities. By decreasing the probabilities, we increase their negative logarithms, and hence their surprisal. Let us make this intuition more formal.

> **Theorem 7.3** *A discrete RV X with uniform distribution and support of size n has entropy $H(X) = \log(n)$.*

Proof:

$$(7.1) \qquad H(X) = \sum_{x \in \text{supp}(X)} -\log(P(X = x))P(X = x)$$

$$(7.2) \qquad = \sum_{x \in \text{supp}(X)} -\log(\frac{1}{|\text{supp}(X)|})P(X = x)$$

$$(7.3) \qquad = \sum_{x \in \text{supp}(X)} \log(n)P(X = x) = \log(n) . \qquad \square$$

Exercise 7.4 *You are trying to learn chess and you start by studying where chess grandmasters move their king when it is positioned in one of the middle fields of the board. The king can move to any of the adjoining 8 fields. Since you do not know a thing about chess yet, you assume that each move is equally probable. In this situation, what is the entropy of moving the king?*

At the outset of this section we promised you that you could easily transfer results from probability theory to information theory. We will not be able to show any kind of linearity for entropy because it contains log-terms and the logarithm is not linear. We can however find alternative expressions for joint entropy (where the joint entropy is simply the entropy of a joint RV). Before we do so, let us also define the notion of conditional entropy. We have seen in Section **??** that $P_{X|Y=y}$ is a valid probability distribution for any $y \in \text{supp}(Y)$ such that $P(Y = y) > 0$. Hence, we can also define its conditional entropy.

Definition 7.5 (Conditional Entropy) *For two jointly distributed RVs X, Y and $y \in \text{supp}(Y)$ such that $P(Y = y) > 0$, the conditional entropy of X given that $Y = y$ is defined as*

$$H(X \mid Y = y) := \mathbb{E}_X[-\log_2(P(X = x \mid Y = y))]$$
$$= -\sum_{x \in \text{supp}(X)} P(X = x \mid Y = y) \log_2(P(X = x \mid Y = y)) .$$

The conditional entropy of X given Y is defined as

$$H(X \mid Y) := \mathbb{E}_Y[H(X \mid Y)] = \sum_{y \in \text{supp}(Y)} P(Y = y)H(X \mid Y = y) .$$

With this definition at hand we show that the joint entropy decomposes according to the chain rule.

$$H(X,Y) = \sum_{\substack{x\in\mathrm{supp}(X) \\ y\in\mathrm{supp}(Y)}} -\log(P(X=x, Y=y)) \times P(X=x, Y=y)$$

$$= \sum_{\substack{x\in\mathrm{supp}(X) \\ y\in\mathrm{supp}(Y)}} -\log(P(X=x \mid Y=y)) \times P(X=x, Y=y)$$

$$- \sum_{y\in\mathrm{supp}(Y)} \log(P(Y=y)) \times \sum_{x\in\mathrm{supp}(X)} P(X=x, Y=y)$$

$$= \sum_{y\in\mathrm{supp}(Y)} P(Y=y) \times \sum_{x\in\mathrm{supp}(X)} -\log(P(X=x \mid Y=y)) \times P(X=x \mid Y=y)$$

$$- \sum_{y\in\mathrm{supp}(Y)} \log(P(Y=y)) \times P(Y=y)$$

$$= H(X \mid Y) + H(Y)$$

Exercise 7.6 *Prove that* $H(X,Y \mid Z) = H(X \mid Z) + H(Y \mid Z)$ *if* $X \perp Y \mid Z$.

Now that we have seen some information-theoretic concepts, you may be happy to hear that there is an information-theoretic interpretation of EM. This interpretation helps us to get a better intuition for the algorithm. To formulate that interpretation we need one more concept, however.

Definition 7.7 (Relative Entropy) *The relative entropy of RVs* X, Y *with distributions* P_X, P_Y *and* $\mathrm{supp}(X) \subseteq \mathrm{supp}(Y)$ *is defined as*

$$D(P_X || P_Y) := \sum_{x\in\mathrm{supp}(X)} P(X=x) \log \frac{P(X=x)}{P(Y=x)} \ .$$

If $P(Y=x) = 0$ *for any* $x \in \mathrm{supp}(X)$ *we define* $D(P_X || P_Y) = \infty$*. As with entropy, we often abbreviate* $D(P_X || P_Y)$ *with* $D(X || Y)$*.*

The relative entropy is commonly known as **Kullback-Leibler (KL) divergence**. It measures the entropy of X as scaled to Y. Intuitively, it gives a measure of how "far away" P_X is from P_Y. To understand "far away", recall that entropy is a measure of uncertainty. This uncertainty is low if both distributions place most of their mass on the same outcomes. Since $\log(1) = 0$ the relative entropy is 0 if $P_X = P_Y$.

It is worthwhile to point out the difference between relative and conditional entropy. Conditional entropy is the average entropy of X given that you know what value Y takes on. In the case of relative entropy you do not know the value of Y, only its distribution.

7.2 An Information-Theoretic View on EM

Let us start by remembering why we need EM. We have a model that defines a joint distribution over observed (x) and latent data (z). Such a model generally looks as follows:

(7.4)
$$P(X = x, Z = z \mid \Theta = \theta) = P(X = x \mid Z = z, \Theta = \theta)P(Z = z \mid \Theta = \theta)$$

where we have chosen a factorization that provides a separate term for a distribution over only the latent data.

Recall that the goal of the EM algorithm is to iteratively increase the likelihood through consecutive updates of parameter estimates. These updates are achieved through maximum-likelihood estimation based on expected sufficient statistics. We are now going to show that a) EM computes a lower bound on the marginal log-likelihood of the data in each iteration and b) that this lower bound becomes tight when the expected sufficient statistics are taken with respect to the model posterior. The latter implies that EM performs the optimal update in each iteration.

Let us start by expanding the data log-likelihood and then lower-bounding it.

(7.5) $\quad \log(P(X = x \mid \Theta = \theta)) = \log(\sum_{y} P(X = x, Y = y \mid \Theta = \theta))$

(7.6) $\quad = \log \left(\sum_{y} Q(Y = y \mid \Phi = \phi) \dfrac{P(X = x, Y = y \mid \Theta = \theta)}{Q(Y = y \mid \Phi = \phi)} \right)$

(7.7) $\quad \geq \sum_{y} Q(Y = y \mid \Phi = \phi) \log \left(\dfrac{P(X = x, Y = y \mid \Theta = \theta)}{Q(Y = y \mid \Phi = \phi)} \right)$

Here, we have used Jensen's Inequality to derive the lower bound. Observe that the log is indeed a concave function.

We also have introduced an auxiliary distribution Q over the latent variables with parameters ϕ. For reasons that we will explain shortly, this distributions is often called the **variational distribution** and its parameters the **variational parameters**. The letter Q is slightly non-standard to denote distributions but we are are following conventions from the field of **variational inference** here.

In the next step, we factorise the model distribution in order to recover a KL divergence term between the variational distribution and the model

posterior over latent variables.

(7.8)
$$\sum_y Q(Y = y \mid \Phi = \phi) \log \left(\frac{P(X = x, Y = y \mid \Theta = \theta)}{Q(Y = y \mid \Phi = \phi)} \right)$$

(7.9)
$$= \sum_y Q(Y = y \mid \Phi = \phi) \log \left(\frac{P(Y = y \mid X = x, \Theta = \theta) P(X = x \mid \Theta = \theta)}{Q(Y = y \mid \Phi = \phi)} \right)$$

(7.10)
$$= \sum_y Q(Y = y \mid \Phi = \phi) \log \left(\frac{P(Y = y \mid X = x, \Theta = \theta)}{Q(Y = y \mid \Phi = \phi)} \right) + \log(P(X = x \mid \Theta = \theta))$$

(7.11)
$$= -D(Q||P) + \log(P(X = x \mid \Theta = \theta))$$

Equation (7.11) gives us two insights. First it quantifies the gap between the lower bound and the actual data likelihood. This gap is equal to the KL divergence between the variational distribution and the model posterior over latent variables. Second, since KL divergence is always positive, the bound only becomes tight when $P = Q$. But this is exactly what is happening in the E-step! The E-step sets $P = Q$ and then computes expectations under that distribution (see Equation (7.7)). Thus, the E-step increases the lower bound on the marginal log-likelihood.

Looking back at Equation (7.7), we also see that the M-step increases the lower bound because it maximises $\mathbb{E}\left[P(X = x, Y = y \mid \Theta = \theta)\right]$. We conclude that both steps are increasing the lower bound on the log-likelihood. We therefore conclude that EM increases the data likelihood in every iteration (or leaves it unchanged at worst).

We will finish with a quick rejoinder on variational inference. EM is a special case of variational inference. Variational inference is any inference procedure which uses an auxiliary distribution Q to compute a lower bound on the likelihood. In the general setting, the auxiliary distribution can be different from the model posterior. This means that the bound never gets tight. However, in models in which the exact posterior is hard (read: impossible) to compute, using a non-tight lower bound instead can be incredibly useful!

The reason this inference procedure is called *variational* is because it is based on the calculus of variations. This works mostly like normal calculus except that standard operations like differentiation are done with respect to functions instead of variables.

Further Material

At the ILLC, there is a whole course about information theory, currently taught by Christian Schaffner. David MacKay also offers a free book on the subject. Finally, Coursera also offers an online course on information theory.

The information-theoretic formulation of EM was pioneered in this paper. A very recent and intelligible tutorial on variational inference can be found on the archive.