Computer Vision I
CSE 252A, Winter 2007
David Kriegman

Homography Estimation*

1. From 3D to 2D Coordinates

Under homography, we can write the transformation of points in 3D from camera 1 to camera 2 as:

$$\mathbf{X}_2 = H\mathbf{X}_1 \quad \mathbf{X}_1, \mathbf{X}_2 \in \mathbb{R}^3 \tag{1}$$

In the image planes, using homogeneous coordinates, we have

$$\lambda_1 \mathbf{x}_1 = \mathbf{X}_1, \quad \lambda_2 \mathbf{x}_2 = \mathbf{X}_2, \quad \text{therefore} \quad \lambda_2 \mathbf{x}_2 = H\lambda_1 \mathbf{x}_1 \tag{2}$$

This means that $\mathbf{x}_2$ is equal to $H\mathbf{x}_1$ up to a scale (due to universal scale ambiguity). Note that $\mathbf{x}_2 \sim H\mathbf{x}_1$ is a *direct mapping* between points in the image planes. If it is known that some points all lie in a plane in an image[1], the image can be rectified directly without needing to recover and manipulate 3D coordinates.

2. Homography Estimation

To estimate H, we start from the equation $\mathbf{x}_2 \sim H\mathbf{x}_1$. Written element by element, in homogenous coordinates we get the following constraint:

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \Leftrightarrow \mathbf{x}_2 = H\mathbf{x}_1 \tag{3}$$

In inhomogenous coordinates ($x'_2 = x_2/z_2$ and $y'_2 = y_2/z_2$),

$$x'_2 = \frac{H_{11}x_1 + H_{12}y_1 + H_{13}z_1}{H_{31}x_1 + H_{32}y_1 + H_{33}z_1} \tag{4}$$

$$y'_2 = \frac{H_{21}x_1 + H_{22}y_1 + H_{23}z_1}{H_{31}x_1 + H_{32}y_1 + H_{33}z_1} \tag{5}$$

Without loss of generality, set z_1 = 1 and rearrange:

$$x'_2(H_{31}x_1 + H_{32}y_1 + H_{33}) = H_{11}x_1 + H_{12}y_1 + H_{13} \tag{6}$$

$$y'_2(H_{31}x_1 + H_{32}y_1 + H_{33}) = H_{21}x_1 + H_{22}y_1 + H_{23} \tag{7}$$

We want to solve for H. Even though these inhomogeneous equations involve the coordinates nonlinearly, the coefficients of H appear linearly. Rearranging equations 6 and 7 we get,

$$\mathbf{a}_x^T \mathbf{h} = \mathbf{0} \tag{8}$$

$$\mathbf{a}_y^T \mathbf{h} = \mathbf{0} \tag{9}$$

*Adapted from lecture notes from CSE252b Spring 2004 with permission from Serge Belongie.
[1]For cameras related by a pure rotation, every scene point can be thought of as lying on a plane at infinity.

where

$$\begin{aligned}
\mathbf{h} &= (H_{11}, H_{12}, H_{13}, H_{21}, H_{22}, H_{23}, H_{31}, H_{32}, H_{33})^T && (10)\\
\mathbf{a}_x &= (-x_1, -y_1, -1, 0, 0, 0, x_2' x_1, x_2' y_1, x_2')^T && (11)\\
\mathbf{a}_y &= (0, 0, 0, -x_1, -y_1, -1, y_2' x_1, y_2' y_1, y_2')^T\,. && (12)
\end{aligned}$$

Given a set of corresponding points, we can form the following linear system of equations,

$$A\mathbf{h} = \mathbf{0} \tag{13}$$

where

$$A = \begin{pmatrix} \mathbf{a}_{x1}^T \\ \mathbf{a}_{y1}^T \\ \vdots \\ \mathbf{a}_{xN}^T \\ \mathbf{a}_{yN}^T \end{pmatrix}. \tag{14}$$

Equation 13 can be solved using homogeneous linear least squares, described in the next section.

3. Homogeneous Linear Least Squares

We will frequently encounter problems of the form

$$A\mathbf{x} = \mathbf{0} \tag{15}$$

known as the Homogeneous Linear Least Squares problem. It is similar in appearance to the inhomogeneous linear least squares problem

$$A\mathbf{x} = \mathbf{b} \tag{16}$$

in which case we solve for $\mathbf{x}$ using the pseudoinverse or inverse of A. This won't work with Equation 15. Instead we solve it using Singular Value Decomposition (SVD).

Starting with equation 13 from the previous section, we first compute the SVD of A:

$$A = U\Sigma V^\top = \sum_{i=1}^{9} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top \tag{17}$$

When performed in Matlab, the singular values σ_i will be sorted in descending order, so σ_9 will be the smallest. There are three cases for the value of σ_9:

- If the homography is *exactly determined*, then $\sigma_9 = 0$, and there exists a homography that fits the points exactly.
- If the homography is *overdetermined*, then $\sigma_9 \geq 0$. Here σ_9 represents a "residual" or goodness of fit.
- We will not handle the case of the homography being *underdetermined*.

From the SVD we take the "right singular vector" (a column from V) which corresponds to the smallest singular value, σ_9. This is the solution, $\mathbf{h}$, which contains the coefficients of the homography matrix that best fits the points. We reshape $\mathbf{h}$ into the matrix H, and form the equation $\mathbf{x}_2 \sim H\mathbf{x}_1$.

4. Homogeneous Linear Least Squares Alternate Derivation

Starting again with the equation $A\mathbf{h} = \mathbf{0}$, the sum squared error can be written as,

$$\begin{aligned}
f(\mathbf{h}) &= \frac{1}{2}(A\mathbf{h} - \mathbf{0})^T (A\mathbf{h} - \mathbf{0}) && (18)\\
f(\mathbf{h}) &= \frac{1}{2}(A\mathbf{h})^T (A\mathbf{h}) && (19)\\
f(\mathbf{h}) &= \frac{1}{2}\mathbf{h}^T A^T A\mathbf{h}. && (20)
\end{aligned}$$

Taking the derivative of f with respect to $\mathbf{h}$ and setting the result to zero, we get

$$\begin{aligned}\frac{d}{d\mathbf{h}}f = 0 &= \frac{1}{2}\left(A^TA + (A^TA)^T\right)\mathbf{h} \qquad (21)\\ 0 &= A^TA\mathbf{h}. \qquad (22)\end{aligned}$$

Looking at the eigen-decomposition of A^TA, we see that $\mathbf{h}$ should equal the eigenvector of A^TA that has an eigenvalue of zero (or, in the presence of noise the eigenvalue closest to zero).

This result is identical to the result obtained using SVD, which is easily seen from the following fact,

Fact 1 *Given a matrix A with SVD decomposition $A = U\Sigma V^T$, the columns of V correspond to the eigenvectors of A^TA.*