Change of Measure Theorems for Semimartingales and Stochastic Processes with Jumps

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Outline

Overview

This research is a comprehensive study on the change of measure theorems for stochastic processes. Specifically, we conducted a systematic review of the following:

- Classical Girsanov Theorem
- Girsanov-Meyer Theorem
- Lenglart-Girsanov Theorem

Lastly, we discussed some applications of these theorems in finance for asset pricing.

Basic Concepts and Definitions

A stochastic process $X = \{X_t\}_{t \in I}$ adapted to the filtration $\{\mathcal{F}_t\}_{t \in I}$ is:

Definition (Martingale)

is a martingale if:

(i) it is measurable; i.e. $\forall t, \quad \mathbb{E}[|X|] < \infty$

(ii) for all $s \leq t$, $\mathbb{E}\{X_t \mid \mathcal{F}_s\} = X_s$

Definition (Local Martingale)

a local martingale if X is càdlàg and there exists a localizing sequence of stopping times T_n such that $X_{t \wedge T_n} 1_{\{T_n > 0\}}$ is a uniformly integrable martingale for each n.

Definition (Finite Variation Process (FV))

a finite variation process if almost all of the paths $t \mapsto A_t(\omega)$ are of finite variation on each compact interval of $\mathbb{R}^+$.

Definition (Semimartingale)

a semimartingale if it has the decomposition

$$X = M + A$$

where M is a local martingale and A is a finite variation (FV) process.

Definition (Stochastic Integration)

stochastic integrals are integrals of the form

$$\int_0^t H_s dX_s$$

where both the integrand (H_t) and the integrator (X_t) are stochastic processes

Protter (2005), Dellacherie & Claude (1980), Daniel & Yor (1999), He, Wan, & Yan (1992)

Theorem (Radon-Nikodym's Theorem)

Let P be a σ-finite measure and Q a signed measure on the filtered space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_t)$, such that Q is absolutely continuous to P ($Q \ll P$). Then there exists a unique $f \in L^1(dP)$ such that

$$Q = \int_\omega f dP, \qquad \omega \in \Omega.$$

f is called the Radon-Nikodym's derivative of Q $w.r.t.$ P denoted by

$$f = \frac{dQ}{dP}.$$

Lemma 2

Let $Q \sim P$, and $Z_t = \mathbb{E}_P\left\{\frac{dQ}{dP}\middle|\mathcal{F}_t\right\}$. An adapted, càdlàg processs $M = \{M_t\}_{t\geq 0}$ is said to be a $Q-$local martingale if and only if $MZ = \{M_t Z_t\}_{t\geq 0}$ is a $P-$local martingale.

Protter (2005), Dellacherie, Claude and Meyer, Paul-Andrè (1978)

Theorem (Novikov's Criterion)

Let M be a continuous local martingale, and suppose that

$$\mathbb{E}\left\{e^{\frac{1}{2}[M,M]_\infty}\right\} < \infty. \tag{1}$$

Then $\mathcal{E}(M)$ is a uniformly integrable martingale.

Lemma 1

For any $\theta \in L^2_{loc}[0, T]$, the process

$$Z_t = e^{\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds} \tag{2}$$

is a martingale, provided that θ satisfies the Novikov criterion

$$\mathbb{E}\left\{e^{\frac{1}{2}\int_0^T \theta_s^2 ds}\right\} < \infty$$

Protter (2005), J. Michael Steele (2001)

Classical Girsanov Theorem

Theorem

Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion on a bounded interval $[0, T]$ defined on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_t)$. Let $\{\theta_t\}_{0\leq t\leq T}$ be an adapted, measurable process such that $\int_0^T \theta_s^2 ds < \infty$ a.s. and satisfies the Novikov's criterion. If there exists Q defined by the Radon-Nikodym's derivative

$$Z_T = \frac{dQ}{dP} = \exp\left(-\int_0^T \theta_s dB_s - \frac{1}{2}\int_0^T \theta_s^2 ds\right) \tag{3}$$

then, Q is a *probability measure* equivalent to P, the process

$$Z_t = \mathbb{E}_P[Z_T|\mathcal{F}_t] \tag{4}$$

is a *P-martingale,* and

$$W_t = B_t + \int_0^t \theta_s ds \tag{5}$$

is a *Brownian motion under* Q.

J. Michael Steele (2001), Fabio (2024)

Generalized Girsanov Theorems

Consider a semimartingale X on a space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_t)$, and a measure Q define by the Radon-Nikodym's derivative

$$Z = \frac{dQ}{dP}$$

.

Z is a right-continuous version of the process defined as

$$Z_t = \mathbb{E}_P \left\{ \frac{dQ}{dP} \middle| \mathcal{F}_t \right\}. \tag{6}$$

Now we present the first major extension of the Girsanov theorem.

Theorem (Girsanov-Meyer Theorem)

Let P and Q be equivalent. Let X be a semimartingale under P with decomposition $X = M + A$. Then, under Q, X is also a semimartingale with decomposition $X = N + B$, where

$$N_t = M_t - \int_0^t \frac{1}{Z_s} d[Z, M]_s \tag{7}$$

is a Q-local martingale, and

$$B_t = A_t + \int_0^t \frac{1}{Z_s} d[Z, M]_s \tag{8}$$

is a Q-FV process.

Proof

Given that M and Z are P-local martingales. By the fundamental theorem of local martingale, they are also a P-semimartingale. Applying the integration by part of stochastic process,

$$ZM = \int Z_- dM + \int M_- dZ + [Z, M] \tag{9}$$

and $\int Z_- dM + \int M_- dZ$ is a local martingale. Hence, $ZM - [Z, M]$ is also a P-local martingale.
Given that $Q \sim P$ then $\frac{1}{Z}$ is a Q-semimartingale and by Lemma 6 we have that the product

$$M - \frac{1}{Z}[Z, M] = \frac{1}{Z}\left(\int Z_- dM + \int M_- dZ\right) \tag{10}$$

is a Q-local martingale. Using integration by parts, we expand the following to get

$$
\begin{aligned}
\frac{1}{Z_t}[Z,M]_t &= \int_0^t \frac{1}{Z_{s-}}\, d[Z,M]_s + \int_0^t [Z,M]_{s-}\, d\left(\frac{1}{Z_s}\right) + [[Z,M],\tfrac{1}{Z}]_t \\
&= \int_0^t \frac{1}{Z_{s-}}\, d[Z,M]_s + \int_0^t [Z,M]_{s-}\, d\left(\frac{1}{Z_s}\right) + \sum_{0<s\le t} \Delta\left(\tfrac{1}{Z_s}\right)\Delta[Z,M]_s \qquad (11) \\
&= \int_0^t \frac{1}{Z_s}\, d[Z,M]_s + \int_0^t [Z,M]_{s-}\, d\left(\frac{1}{Z_s}\right)
\end{aligned}
$$

substituting this expression into the Q-local martingale in (10) above we have

$$
\begin{aligned}
M_t - \frac{1}{Z_t}[Z,M]_t =& M_t - \int_0^t \frac{1}{Z_s} d[Z,M]_s - \int_0^t [Z,M]_{s-} d\left(\frac{1}{Z_s}\right) \\
M_t - \frac{1}{Z_t}[Z,M]_t + \int_0^t [Z,M]_{s-} d\left(\frac{1}{Z_s}\right) =& M_t - \int_0^t \frac{1}{Z_s} d[Z,M]_s .
\end{aligned} \qquad (12)
$$

Finally, $B = X - N$ is a Q-FV because

$$
\begin{aligned}
B_t =& X_t - N_t = M_t + A_t - M_t + \int_0^t \frac{1}{Z_s} d[Z,M]_s \\
=& A_t + \int_0^t \frac{1}{Z_s} d[Z,M]_s
\end{aligned} \qquad (13)
$$

which is a sum of two FV processes and since $P \sim Q$ ensures that the measure Q inherits almost sure properties of processes defined under P.

$\square$

Lemma

Let X be an adapted stochastic process and suppose $Q \ll P$, then

$$\mathbb{E}_P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right]\cdot\mathbb{E}_Q\left[X|\mathcal{F}_t\right]=\mathbb{E}_P\left[X\cdot\frac{dQ}{dP}\middle|\mathcal{F}_t\right]$$

Corollary

Suppose $Q \sim P$, the inverse process $\frac{1}{Z}$ is a cadlag version of

$$\frac{1}{Z_t}=\mathbb{E}_Q\left[\frac{dP}{dQ}\middle|\mathcal{F}_t\right]$$

Theorem (Lenglart-Girsanov Theorem)

Let Q be a probability measure absolutely continuous with respect to P, and let X be P-local martingale with $X_0 = 0$. Let $Z_t = \mathbb{E}_P\left[\frac{dQ}{dP} \mid \mathcal{F}_t\right]$, $R = \inf\{t > 0 : Z_t = 0, Z_{t-} > 0\}$ and define $U_t = \Delta X_R 1_{\{t \geq R\}}$, then the following is a Q-local martingale:

$$X_t - \int_0^t \frac{1}{Z_s} d[X,Z]_s + \tilde{U}_t. \tag{14}$$

Lemma

Let X be a positive right continuous supermartingale. Define

$$T(\omega) = \inf\{t : X_t(\omega) = 0 \text{ or } X_{t-}(\omega) = 0\}.$$

Then, for all ω, $X_t(\omega) = 0$ for $t \geq T(\omega)$.

Corollary

Let Q be absolutely continuous with respect to P, the increasing stopping time $R_n = \inf(t > 0 : Z_t \leq \frac{1}{n})$ converges to $R = \inf\{t > 0 : Z_t = 0 \text{ and } Z_{t-} = 0\}$.

Lemma

If X is continuous (for instance the Brownian motion), the quadratic covariation $\langle X, Z\rangle = [X, Z]^c$ is a predictable finite variation (FV) process and the integrand $\frac{1}{Z_s}$ in Theorem 10 is replaced by the left continuous version $\frac{1}{Z_{s-}}$.

Corollary

If X is a continuous P-local martingale and Q is absolutely continuous with respect to P, then $\langle X, Z\rangle = [X, Z]^c$ exists and there exists a predictable process α such that

$$X_t - \int_0^t \frac{1}{Z_{s-}} d[X, Z]_s^c = X_t - \int_0^t \alpha_s d[X, X]_s \tag{15}$$

is a Q-local martingale.

Protter (2005), Heinrich and Winkler (1980), Paul-Andrè Meyer (1976), Lenglart (1977)

Application to Finance

The Girsanov theorem plays an important role in finance, for example, in asset pricing theory. A major guiding principle in asset pricing is the Fundamental Theory of Asset Pricing (FTAP) which demands that a market is:

- **Viable:** presents no arbitrage opportunity, and
- **Complete:** replicate every contingency claim.

Upholding this principle implies that for a given contingency claim (expected payoff) denoted by $H = f(S_T)$, the discounted portfolio value $V = \{V_{t \geq 0}\}$ must satisfy

$$\tilde{V}_t = \mathbb{E}_Q[\tilde{V}_T|\mathcal{F}_t] = \mathbb{E}_Q[R_T^{-1}H|\mathcal{F}_t] = \mathbb{E}_Q[R_T^{-1}f(S_T)|\mathcal{F}_t] = a_0\tilde{S}_0 + b_0 + \int_0^t a_s d\tilde{S}s \tag{16}$$

In the specific case when $R_t = 1$ this becomes

$$V_t = \mathbb{E}_Q[V_T|\mathcal{F}_t] = \mathbb{E}_Q[H|\mathcal{F}_t] = \mathbb{E}_Q[f(S_T)|\mathcal{F}_t] = a_0S_0 + b_0 + \int_0^t a_s dSs. \tag{17}$$

Reformulation of Price Process Using Girsanov's Theorem

In general, the price process $S = \{S_t\}_{t \geq 0}$ is considered a semimartingale. Assumming S defined under P satisfies the stochastic differential equation

$$dS_s = h(s, S_s)dB_s + b(s, S_r; r \leq s)ds \tag{18}$$

where B_t is a standard P-Brownian motion, with volatility h and drift b.

Applying Girsanov's theorem, construct an equivalent measure $Q \sim P$ such that under Q, the drift term vanishes and S_t becomes a local martingale.

- Under P the process is a semimartingale of the form $S = M + A$.
- Under Q, using Girsanov-Meyer's theorem we have the following expression

$$\int_0^t h(s, S_s)dB_s - \int_0^t \frac{1}{Z_s} d\left[Z, \int_0^{\cdot} h(r, S_r)dB_r\right]_s \tag{19}$$

which is a local martingale.

Protter (2005)

A priori, one may not know the nature of the process Z_t, however it is assumed to be strictly positive, continuous, and a local martingale with $Z_0 = 1$. By the Martingale Representation Theorem, it admits the form:

$$Z_t = 1 + \int_0^t H_s Z_s dB_s. \tag{20}$$

where H_s is predictable.

Equation (20) leads to a stochastic differential equation (SDE)

$$dZ_t = H_t Z_t dB_t.$$

From Equation (19), Computing the covariation:

$$d[Z, \int_0^{\cdot} h(s, S_s) dB_s] = Z_s H_s h(s, S_s) ds,$$

Hence, Equation (19) becomes,

$$\int_0^t h(s, S_s) dB_s - \int_0^t \frac{1}{Z_s} Z_s H_s h(s, S_s) ds = \int_0^t h(s, S_s) dB_s - \int_0^t H_s h(s, S_s) ds \tag{21}$$

if we choose $H_s = -\frac{b(s, S_r; r \leq s)}{h(s, S_s)}$, this will result into a pricing process given by

$$S_t = \int_0^t h(s, S_s) dB_s + \int_0^t b(s, S_r; r \leq s) ds \tag{22}$$

which is a Q-local martingale.
Finally, defining the process

$$W_t = B_t + \int_0^t \frac{b(s, S_r; r \leq s)}{h(s, S_s)} ds, \tag{23}$$

Lévy's theorem implies W_t is a Q-Brownian motion, and under Q, the price dynamics simplify to:

$$dS_t = h(t, S_t) dW_t. \tag{24}$$

Asset Pricing

Suppose that the stock price of an asset S_t satisfies the SDE

$$dS_t = S_t[\mu dt + \sigma dB_t] \tag{25}$$

where $\mu = \mu(t, S_t)$ is the drift term representing the average rate of return of the stock and $\sigma(t, S_t)$ denotes the volatility, which quantifies the randomness in stock price movements. We also define the risk-free bond (riskless asset) R_t satisfying

$$dR_t = rR_t dt \tag{26}$$

that is $R_t = R_0 e^{\int_0^t r(s,S_s)\,ds}$, where $r = r(t, S_t)$ is the risk-free rate at which the riskless asset grows.

Assuming there exists a probability measure Q that is equivalent to P such that under Q, the discounted stock price

$$\tilde{S}_t = \frac{S_t}{R_t} \tag{27}$$

is a martingale. Using the product rule and the Ito's Lemma, one can see that under P, the discounted stock price $\tilde{S}_t = S_t/R_t$ satisfies

$$d\tilde{S}_t = \tilde{S}_t \left[(\mu - r)dt + \sigma dB_t\right]. \tag{28}$$

Therefore, for an option with value $V_T = F(S_T)$ at maturity T, such that $\mathbb{E}_Q[R_T^{-1}] < \infty$ the right arbitrage-free portfolio value at time $t < T$ is

$$V_t = R_t \mathbb{E}_Q \left[R_T^{-1} F(S_T) \mid \mathcal{F}_t\right]. \tag{29}$$

Example

Suppose that S_t satisfies,

$$dS_t = S_t \left[\mu dt + \sigma dB_t\right] \tag{30}$$

and the bond rate r is constant. Suppose that the claim is the average stock price over the time interval $[0, T]$,

$$V = \frac{1}{T} \int_0^T S_t \, dt$$

In the new measure Q, the discounted stock price $\tilde{S}_t = e^{-rt} S_t$ satisfies

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t$$

where W_t is a Q-Brownian motion. From Equation (29), under the risk-neutral condition, the discounted portfolio value must satisfy

$$\tilde{V}_t = \mathbb{E}_Q \left[e^{-rT} \frac{1}{T} \int_0^T S_s \, ds \,\middle|\, \mathcal{F}_t \right]. \tag{31}$$

Given that $\int_0^t S_s ds$ is $\mathcal{F}_t$-measurable, and by the linearity of expectation, we have

$$Te^{rT} \tilde{V}_t = \int_0^t S_s ds + \int_t^T \mathbb{E}_Q \left[S_s \mid \mathcal{F}_t\right] ds \tag{32}$$

and since $\tilde{S}_t$ is a Q- martingale, thus for $s > t$

$$\mathbb{E}_Q\left[S_s \mid \mathcal{F}_t\right] = e^{rs}\mathbb{E}_Q\left[\tilde{S}_s \mid \mathcal{F}_t\right] = e^{rs}\tilde{S}_t = e^{r(s-t)}S_t$$

simplifying this yields,

$$\begin{aligned}\int_0^t S_s ds + \int_t^T \mathbb{E}_Q\left[S_s \mid \mathcal{F}_t\right] ds &= \int_0^t S_s ds + \frac{e^{r(T-t)}-1}{r}S_t \\ \implies \tilde{V}_t &= \frac{e^{-rT}}{T}\int_0^t S_s ds + \frac{1-e^{-r(T-t)}}{rT}\tilde{S}_t\end{aligned} \tag{33}$$

Therefore,

$$V_t = e^{rt}\tilde{V}_t = \frac{e^{-r(T-t)}}{T}\int_0^t S_s ds + \frac{e^{rt}-e^{-rT}}{rT}S_t \tag{34}$$

This shows that at maturity T, $V_T = V$, and at 0,

$$V_0 = \frac{1-e^{-rT}}{rT}S_0 \tag{35}$$

Note: In the special case where the coefficients μ and σ are constants, Equation (30) reduces to the Geometric Brownian Motion (GBM) model.

Black-Scholes Equation

The Black-Scholes Equation model assumed that the risky asset $\{S_t\}_{t\geq 0}$ follows the geometric Brownian motion under the measure P and satisfies

$$dS_t = \sigma S_t dB_t + \mu S_t dt; \qquad S_0 = 1. \tag{36}$$

Applying the Ito's Lemma will yield

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}.$$

The riskless asset (bond) satisfies

$$dR_t = rR_t dt, \qquad R_0 = 1 \quad \Rightarrow \quad R_t = e^{rt}.$$

Then let $\theta = \frac{\mu - r}{\sigma}$, and by Girsanov theorem, we define the Q-Brownian motion $W_t = B_t + \theta t$ so that under Q,

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{37}$$

By FTAP, considering the European call option with strike K, expiration T and expected payout $(S_T - K)_+$,

$$\begin{aligned} V_t &= R_t \mathbb{E}_Q \left[R_T^{-1} F(S_T) \mid \mathcal{F}_t \right] \\ &= e^{-r(T-t)} \left[\mathbb{E}_Q \left[S_T 1_{\{S_T \geq K\}} \mid \mathcal{F}_t \right] - K \mathbb{E}_Q \left[1_{\{S_T \geq K\}} \mid \mathcal{F}_t \right] \right]. \end{aligned} \tag{38}$$

But under the risk-neutral measure Q, the stock price follows Equation (37), thus,

$$S_T = S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)}$$

therefore, we can rewrite the expectation as:

$$V_t = e^{-r(T-t)}\left[S_t \mathbb{E}_Q\left[e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)}1_{\{S_T\geq K\}} \mid \mathcal{F}_t\right] - K\mathbb{E}_Q\left[1_{\{S_T\geq K\}} \mid \mathcal{F}_t\right]\right].$$

Using the lognormal property of S_T, the conditional expectation simplifies further, ultimately leading to the Black-Scholes pricing formula.

$$V_t = V(t, S_t) = S_t\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2), \tag{39}$$

Where $\mathcal{N}(d_i) =$ cumulative standard normal distribution function and the terms d_1 and d_2 are defined as:

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Conclusion

Researchers find the Girsanov theorem challenging to understand, particularly in areas such as the decomposition of semimartingales, the construction Z_t, and the treatment of jump components in cases $Q \ll P$. Furthermore, the theorem is poorly adopted in some areas of research like biology and climate science because of the availability of little physical interpretation of the tool in those areas.

Addressing these challenges motivated this study. This research has established a descriptive analysis of the Girsanov theorems. The classical Girsanov theorem which is based on Brownian motion is the simplest case of the change of measure theorem. In general, the semimartingale may not be a Brownian motion, in which case the general Girsanov theorem is applied.

Despite the poor adoption of the Girsanov theorems, as mentioned earlier, these theorems are widely applicable in various fields. For example:

- **Asset Pricing**: Investment banks and asset managers like Goldman Sachs, J.P. Morgan, Morgan Stanley, and Intercontinental Exchange (ICE) apply the Girsanov theorem to risk-neutral pricing in derivative markets
- **Interest Rate Modeling**: For pricing interest rate swaps or bond options
- **Credit Risk and Default Modeling**: Credit risk analytics teams in major banks apply Girsanov theorems in modeling default times and pricing credit derivatives under a new measure
- **Stochastic Control and Filtering:** It is used in robotics by engineers, and it is used in signal processing by space agencies

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Thank You!