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Change of Measure Theorems for Semimartingales and Stochastic Processes with Jumps

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Abstract

This research is an intensive study of the change of measure theorems and hypotheses of stochastic processes like martingales, processes with jumps and finite variations processes, and semimartingales in general. Specifically, we conducted a systematic review of the Girsanov theorems, starting from the classical Girsanov theorem to the generalized Girsanov-Meyer and Lenglart-Girsanov theorems for stochastic processes. In addition, we discussed some major properties of semimartingales, martingales, variation processes, processes with jumps, and stochastic integration. Lastly, in the discussion of the result, some applications of the change of measure theorems in finance and asset pricing were discussed. We presented an introduction to asset pricing and highlighted the role of the change of measure theorems in financial modeling.

1 Introduction

The concept of change of measure in mathematics has become of high interest to many mathematicians especially those in stochastic analysis. The change of measure theorem evolved following the studies of stochastic processes, their properties, and analysis. First proposed in the 1940s by Cameron-Martin, and later formalized by Igor Girsanov in 1960, it has since been called the Girsanov Theorem.

Initially, the Girsanov theorem was demonstrated for Brownian motions but was later shown for generalized semimartingale by Paul-André Meyer using Doob-Meyer's decomposition of semimartingale approach [1]. Unlike the earlier Girsanov theorems, which relied on the equivalence of the probability measures, in 1977, E. Lenglart in [2] established this result for a more restrained condition where the new measures is only absolutely continuous w.r.t. the initial measure.

The change of measure theorem for stochastic processes, popularly called the Girsanov theorem, has played a significant role in several fields of study; it is an important tool in finance used for transforming the measure on a risky asset to a risk-neutral measure. The risk-neutral measure is what the market relies on to be viable under a self-financing condition, because one will need to identify an equivalent probability measure that makes a discounted price process a martingale. The Girsanov condition is important in financial applications since we have to deal with the same set of asset price movements even when we switch to a new probability measure.

In stochastic integration, the study of the Girsanov theorem has become increasingly important as it highlights the connections between absolutely continuous transformation of measures associated with Ito's processes. Specifically, this theorem has highlighted the existence of measures $Q \ll P$ for which a class of Ito's process analyzed under a probability space $(\Omega, \mathcal{F}, P)$ has the same properties. The Girsanov theorem also plays an important role in stochastic partial differential equations (SPDEs); for example, it is use in establishing the laws (equivalence) between white-noise-driven SPDEs.

In this study, we followed [12] and established the change of measure theorems using a more descriptive approach. Many researchers find the Girsanov theorem challenging to understand, particularly in areas such as the decomposition of semimartingales (or local martingale), the construction of the right-continuous version of the Radon-Nikodym density function Z_t, and the treatment of jump components. In order to address these difficulties, we have provided a more comprehensive analysis and detailed arguments that makes the exposition more clearer and easier to understand and utilize.

The structure of the study is as follows: Section 2, introduces preliminary definitions and key results arising from probability and measure theory used in this study. Section 3 presents useful results from martingale theory, which include studies on local martingales, mutual and finite variation processes, semimartingales, and stochastic integrations with their properties. The main result of this study is presented in Section 4, where we begin with the classical Girsanov theorem based on the Brownian motion and later extend it to the more general cases established by Meyer and Lenglart. Section 5 presents a discussion of the established result and demonstrates some of their applications, and lastly, the conclusion of the study.

2 Preliminary

For this study we assumed and uphold all classical studies of measure theory. We present some basic definitions and properties used often in establishing the results. In most cases, our study is based on the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ unless stated otherwise.

Given the probability space $(\Omega, \mathcal{F}, P)$ and a real-valued integrable random variable f, the mathematical expectation of f is denoted and defined by

$$\mathbb{E}[f] = \int_\Omega f(\omega)P(dw), \quad \omega \in \Omega.$$

Definition 2.1 (**Conditional Expectation**). Given $X \in L^1(\Omega, \mathcal{F}, P)$, and let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$, then the random variable $\mathbb{E}[X|\mathcal{G}]$ is called a conditional expectation of X given $\mathcal{G}$ if the following condition holds: for all $G \in \mathcal{G}$,

$$\int_G \mathbb{E}[X|\mathcal{G}]dP = \int_G XdP.$$

Moreover, if a random variable Y is $\mathcal{G}$ measurable and it holds that for all $G \in \mathcal{G}$,

$$\int_G YdP = \int_G XdP.$$

Then $Y = \mathbb{E}[X|\mathcal{G}]$, $P - a.s.$

In the study, we made references to some of properties of conditional expectations stated below (the proofs can be found for example in [14] Theorem 2.10 (Page 38))

Theorem 2.2 (**Properties of Conditional Expectations**). *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{G}, \mathcal{D}$ sub-σ-algebras of $\mathcal{F}$, and $X, Y \in L^1(\Omega, \mathcal{F}, P)$, then the following holds $P - a.s.$:*

(a) $\mathbb{E}[X \mid \{\Omega, \emptyset\}] = \mathbb{E}[X]$

(b) If X is $\mathcal{G}$-measurable, then $\mathbb{E}[X \mid \mathcal{G}] = X$

(c) (Linearity). $\mathbb{E}[aX + bY \mid \mathcal{G}] = a\mathbb{E}[X \mid \mathcal{G}] + b\mathbb{E}[Y \mid \mathcal{G}]$ for all constants a and b

(d) (Order). If $X \leq Y \quad a.s.$, then $\mathbb{E}[X \mid \mathcal{G}] \leq \mathbb{E}[Y \mid \mathcal{G}]$

(e) (Smoothing). if $\mathcal{D} \subset \mathcal{G}$, then $\mathbb{E}[\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{D}] = \mathbb{E}[X \mid \mathcal{D}]$ (Tower property)

(f) if $XY \in L^1$ and X is $\mathcal{G}$-measurable, then $\mathbb{E}[XY \mid \mathcal{G}] = X\mathbb{E}[Y \mid \mathcal{G}]$

(g) if X is $\mathcal{G}$ independent, $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$

*(h) **(Conditional Jensen's Inequality).** Let ψ be a convex function on an interval J such that ψ has finite right- (or left-) hand derivative(s) at left (or right) endpoint(s) of J if J is not open. If $P(X \in J) = 1$, and if $\psi(X) \in L^1$, then*

$$\psi(\mathbb{E}(X|\mathcal{G})) \leq \mathbb{E}(\psi(X)|\mathcal{G}).$$

*(i) **(Contraction).** For $X \in L^p(\Omega, \mathcal{F}, P)$, $p \geq 1$,*

$$\|\mathbb{E}(X|\mathcal{G})\|_p \leq \|X\|_p \quad \forall p \geq 1.$$

*(j) **(Convergences).***

(i) If $X_n \to X$ in L^p, then $\mathbb{E}(X_n|\mathcal{G}) \to \mathbb{E}(X|\mathcal{G})$ in L^p $(p \geq 1)$.

(ii) ***(Conditional Monotone Convergence)*** *If $0 \leq X_n \uparrow X$ a.s., X_n and $X \in L^1$ $(n \geq 1)$, then $\mathbb{E}(X_n|\mathcal{G}) \uparrow \mathbb{E}(X|\mathcal{G})$ a.s. and $\mathbb{E}(X_n|\mathcal{G}) \to \mathbb{E}(X|\mathcal{G})$ in L^1.*

(iii) ***(Conditional Dominated Convergence)*** *If $X_n \to X$ a.s. and $|X_n| \leq Y \in L^1$, then $\mathbb{E}(X_n|\mathcal{G}) \to \mathbb{E}(X|\mathcal{G})$ a.s.*

Next, we introduce integration. First, we start with the integration with respect to a measure.

Lebesgue Integral

Let μ be an integrable measure on $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to [0, \infty]$ a measurable function with respect to the Lebesgue σ-algebra $\mathcal{B}(\Omega)$, the integral

$$\int f d\mu$$

is called the Lebesgue integral of f. The Lebesgue integral is essential for stochastic calculus as it accommodates randomness and irregular path behaviors through its reliance on the respective measure ([3]).

Definition 2.3. If μ be a measure on $(\Omega, \mathcal{F})$, $f : \Omega \to [0, \infty)$ is a measurable map. If we define the measure $\mathcal{V}$ by

$$\mathcal{V}(A) = \int_A f d\mu \qquad \text{for all } A \in \mathcal{F}$$

the function f is called the **density** of $\mathcal{V}$ with respect to μ. In simpler terms, the density function f describes the rate of change or the relationship between the two measures.

Definition 2.4 (**Absolutely Continuous Measures**)**.** Given two measures P and Q on the space $(\Omega, \mathcal{F})$, Q is said to be absolutely continuous *w.r.t.* P if for any $A \in \Omega, P(A) = 0 \implies Q(A) = 0$ and is denoted by $Q \ll P$.

Theorem 2.5. ***(Radon-Nikodym's Theorem)***
Let P be a σ-finite measure and Q a signed measure on the filtered space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_t)$, such that Q is absolutely continuous to P $(Q \ll P)$. Then there exists a unique $f \in L^1(dP)$ such that

$$Q = \int_\omega f dP, \qquad \omega \in \Omega.$$

f is called the Radon-Nikodym's derivative of Q w.r.t. P denoted by

$$f = \frac{dQ}{dP}.$$

Definition 2.6 (**Equivalent Measures**)**.** Two probability laws P and Q on $(\Omega, \mathcal{F})$ are said to be equivalent if $P \ll Q$ and $Q \ll P$. We write $Q \sim P$ to denote equivalence.

Note that if Q is absolutely continuous with respect to P, with density f, then P and Q are equivalent if and only if $P(f > 0) = 1$.

Definition 2.7 (**Stochastic Process**)**.** A stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ is a family of $\mathbb{R}^d$-valued random variables $\{X_t\}_{t \in I}$ for $I \subset \mathbb{R}^+$ and $d \geq 1$ such that for each fixed $t \in I$, $X_t(.)$ is a random variable and for each fixed $\omega \in \Omega$, the map $X_.(\omega) : I \to \mathbb{R}$ is a Borel function called the path of the stochastic process.

The stochastic process can be either continuous or discrete depending on the nature of the time interval of the process. If the time interval $t \in I \subset \mathbb{R}^+$ is discrete, the corresponding process is a discrete stochastic process. While in the case where $t \in I$ is a continuous time interval, the corresponding process is a continuous stochastic process.

Definition 2.8 (**Filtration**)**.** Filtration is a family of sub σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ of $\mathcal{F}$ such that for all $0 \leq s \leq t$

$$\mathcal{F}_s \subseteq \mathcal{F}_t$$

Moreover, filtration could be seen as the set of information or observations available up-to time t. The quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is a called a **Filtered space**. Lastly, the stochastic process is said to be **adapted** if for each $t \in I$, X_t is $\mathcal{F}_t$ measurable.

Definition 2.9. A filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is said to satisfy the **usual hypotheses** if:

(i) $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$

(ii) $\mathcal{F}_t = \bigcap_{n>t} \mathcal{F}_n$, for $0 \leq t < \infty$: This condition simply implies that the filteration is right continuous

Note: the usual hypothesis is assumed throughout.

Definition 2.10. Two stochastic processes X and Y defined on probability space $(\Omega, \mathcal{F}, P)$ are said to be:

(a) **version** of each other if for all $t \in I$,

$$P(\omega \in \Omega : X_t(\omega) = Y_t(\omega)) = 1$$

(b) **indistinguishable** if

$$P(\omega \in \Omega : X_t(\omega) = Y_t(\omega), \forall t \in I) = 1$$

Definition 2.11. A stochastic process X is said to be **Càdlàg** if it has a sample path that is a.s. right continuous with a left limit. While, its called **Càdlàd** if it has a sample path that is a.s. left continuous with right limit.

3 Martingales and Semimartingale Theorems

In this section, we follow closely the definitions and essential results of continuous time martingales as outlined in [12]. While most results are stated without proof, detailed proofs can be found in sources such as [12], [5], [6], or [7]. It is important to note that we always assume that the usual hypothesis defined with respect to the filtration holds. The usual hypothesis on the filtration guarantees the measurability of a right continuous version of a stochastic process. In some literature, the usual hypothesis on the filtration aids the progressive measureability assumption of the stochastic process; specifically, when the usual hypotheses hold, one knows that every martingale has a version which is càdlàg. Right-continuity of processes is a standard assumption in stochastic processes because it ensures that processes are well-adapted to the filtration.

Definition 3.1 (**Martingale**)**.** An adapted stochastic process $X = \{X_t\}_{t\in I}$ is a martingale (submartingale or supermartingale respectively) with respect to the filtration $\{\mathcal{F}_t\}_{t\in I}$ if

(i) it is measurable; i.e. $\mathbb{E}[|X|] < \infty$

(ii) for all $s \leq t$,

$$\mathbb{E}\{X_t \mid \mathcal{F}_s\} = X_s$$

(With $>$ or $<$ for submartingale or supermartingale respectively)

It is important to note that the time interval $I \subset \mathbb{R}^+$ can be a finite interval $[0, T]$ or an infinite interval $[0, \infty)$. In a finite interval case, one can extend the definition of a martingale to $[0, \infty)$, we typically consider the process at $t = \infty$, which corresponds to the limiting random variable $X_\infty = \lim_{t\to\infty} X_t$, if it exists. For the martingale X_∞ to be well-defined, $\mathbb{E}[|X_\infty|] < \infty$ must be satisfied. Also, a less probabilistic definition of a martingale can be given in relation to a Hilbert space. Since any filtration $\{\mathcal{F}_t\}_{t\geq 0} \subset \mathcal{F}$ induces an increasing family of subspaces $\{L^2(\Omega, \mathcal{F}_t, P)\}_{t\geq 0}$ within the Hilbert space $L^2(\Omega, \mathcal{F}, P)$.

Definition 3.2. A martingale is said to be **closed** by a random variable X_∞, if $\mathbb{E}[|X_\infty|] < \infty$ and

$$X_t = \mathbb{E}[X_\infty \mid \mathcal{F}_t]$$

In this definition, we use the term **closed** under the assumption that $I \subset \mathbb{R}^+$. However, in some contexts, the time interval can be $I \subset \mathbb{R}$. When I has a finite supremum $(\sup I \leq \infty)$, then the process expressed as $X_t = \mathbb{E}[X_{\sup I} \mid \mathcal{F}_t]$ is called right closed. Conversely, if I has a finite infimum $(-\infty \leq \inf I)$, the process $X_t = \mathbb{E}[X_{\inf I} \mid \mathcal{F}_t]$ is a left closed process.

The random variable X_∞ that closes a martingale $\{X_t\}_{t\geq 0}$ is measurable *w.r.t* $\mathcal{F}_\infty$. A key property related to X_∞ and the martingale $\{X_t\}$ is uniform integrability (UI). Uniform integrability ensures that the random variables $\{X_t\}$ do not exhibit excessive growth in expectation, even at the tails. The uniform integrability property plays a vital role in the martingale convergence theorem.

Definition 3.3 (Uniform Integrability). A family of random variable $\{Y_\alpha\}_{\alpha\in\mathcal{A}}$ for $\mathcal{A} \subset \mathbb{R}^+$ is said to be uniformly integrable if for any $\epsilon > 0$, there exists a constant $C > 0$ such that

$$\sup_{\alpha\in\mathcal{A}} \int_{\{|Y_\alpha|\geq C\}} |Y_\alpha| d\mathbb{P} < \epsilon$$

Furthermore, uniform integrability of a family of random variables $\{Y_\alpha\}_{\alpha\in\mathcal{A}}$ is often ensured if for some $\beta > 0$

$$\sup_{\alpha\in\mathcal{A}} \mathbb{E}(|Y_\alpha|^{1+\beta}) < \infty.$$

Theorem 3.4 (Martingale Convergence Theorem). *Let X be a right continuous martingale which is uniformly integrable. Then $Y = \lim_{t\to\infty} X_t$ a.s. exists, $\mathbb{E}[|Y|] < \infty$ and X is closed by Y.*

Clearly, one can see that If X is a uniformly integrable martingale, then X_t converges to $X_\infty = Y$ in L^1 as well as almost surely. Analogously, as demonstrated by Doob's Optional Sampling Theorem, if given two bounded stopping times $S \leq T$ one can define a stopped martingale $X_{S\wedge.}$ closed by $X_{T\wedge.}$. This is certainly possible because a stopped martingale is a martingale.

Furthermore, it is worth mentioning that certain properties of martingales are more conveniently studied for L^2-martingales. By definition, an L^2-martingale is a martingale $\{X_t\}_{t\geq 0}$ satisfying the additional condition of square-integrability: $\mathbb{E}[X_t^2] < \infty$ for all t. The concept of L^2-martingales does not introduce a fundamentally new class of processes but ensures square-integrability, which allows the use of tools from Hilbert space theory.

An important consequence is that L^2-convergence implies convergence in probability, which simplifies certain proofs and applications in stochastic analysis.

Definition 3.5. A martingale X with $X_0 = 0$ and $\mathbb{E}[X_t^2] < \infty$ for each $t > 0$ is called a **squared integrable martingale**. If X is uniformly integrable martingale as well, then it is called an L^2**-martingale**

The additional square-integrability condition simplifies various aspects of analysis, such as proving convergence or bounding expectations in L^2-spaces. For example, as noted by [16], when working with stochastic integrals defined with respect to an L^2-bounded martingale, the L^2-convergence ensures that the integral inherits the martingale properties in the limit. However, it can be challenging to establish these results for the entire process without additional tools, such as localization. Localization plays a crucial role in extending the validity of L^2-convergence results to broader cases, thereby ensuring the convergence of the stochastic integral and the preserving of martingale properties.

Definition 3.6 (Fundamental Sequence). An increasing sequence of stopping time T_n such that $\lim_{n\to\infty} T_n = \infty$ *a.s.* is called a fundamental sequence.

In some literature like [7], this sequence of stopping times is called a **localizing sequence of stopping time**, and it plays an important role in extending some properties of stochastic processes like being a local martingale or satisfying certain integrability conditions. The localizing sequence of stopping time is very important in analysing processes that are unbounded and whose behaviour and analysis can only be done locally.

Since it has been established that a stopped martingale is a martingale, to establish this for a more general process led us to the definition of local martingale. This are processes that becomes martingales after they are stopped.

Definition 3.7 (Local Martingale). An Adapted, càdlàg process X is called a local martingale if there exists a localizing sequence of stopping times T_n such that $X_{t\wedge T_n} 1_{\{T_n>0\}}$ is uniformly intgrable martingale for each n.

Recall that we are assumming the usual hypothesis holds, which guarantees the measurability of a càdlàg version of the process X. And since the localizing sequence of stopping T_n is used to show certain results of a stochastic process in a specific random times or localized segments, w.l.o.g., it is sufficient to deduce and show certain results of a stochastic process for a randomly selected stopping time. Hence, for a process X and a *a.s.* finite stopping time T, we can define the **stopped process** as

$$X_{t\wedge T} = X_t 1_{\{t<T\}} + X_T 1_{\{t\geq T\}}$$

The definition of a local martingale above by [12] emphasized the need for $X_{t\wedge T} 1_{\{T>0\}}$ to be a martingale with a vanishing tail; that is a uniformly integrable martingale. Although some definition of a local martingale may not explicitly emphasize on the uniform integrability condition, however, they for sure will imply that a local martingale is a process that behaves like a martingale in a given random stopping time. [12] uses the uniform integrability condition, a key component of martingale convergence theorems, to define local martingale. In other words, a local martingale can be seen as a càdlàg adapted process that becomes a true martingale when stopped at an appropriate stopping time. Also, any local martingale can be stopped in such a way that the stopped process is uniformly integrable. The stopped process is sometimes called a **reduced process**.

As stated before, many results in stochastic analysis are first proven for L^2-martingales because of their nice properties (e.g., boundedness in L^2-norm). In the same way, local

L^2-martingales generalize the concept by allowing L^2-integrability to hold locally (up to a stopping time). This enables us to work with a broader class of processes and still maintain the useful properties of L^2-martingales.

Definition 3.8. An adapted càdlàg process X is called an local L^2-martingale, if there exists a localizing sequence of stopping times $\{T_k\}_k$ such that the stopped process $(X_{t\wedge T_k})_t$ are L^2-bounded martingales

Stochastic Integrations
Now, we present useful results involving stochastic integrals. Stochastic integration is an extension of the classical integral to the integration of stochastic processes. One can consult for example [7], [8] and [11] For a more comprehensive note of stochastic integration. The stochstic integral we used are of the form

$$\int_0^t H_s dX_s$$

where both the integrand (H_t) and the integrator (X_t) are stochastic processes

Lemma 3.9. *Let M be a local martingale and let $H \in \mathcal{H}(]0, \infty])$ (the space of predictable processes). Then the stochastic integral*

$$\int_0^t H_s dM_s$$

is again a local martingale.

Definition 3.10 (Finite Variation Process). A càdlàg process $A = (A_t)_{t\geq 0}$ is called a process of **finite variation (FV)** if almost all of the paths $t \mapsto A_t(\omega)$ are of finite variation on each compact interval of $\mathbb{R}^+$; that is, for every $T > 0$ and for each $\omega \in \Omega$,

$$\sup_{\pi} \sum_{i=1}^{n} |A_{t_i}(\omega) - A_{t_{i-1}}(\omega)| < \infty$$

where the supremum is taken over all finite partitions $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$.

Theorem 3.11 (Fundamental Theorem of Local Martingales). *Let M be a local martingale and $\beta > 0$. Then there exist local martingales N, A such that A is a finite variation process, the jumps of N is bounded by 2β, and $M = N + A$*

Proof. *see [12]*

Definition 3.12. A stochastic process X is said to be a **semimartingale** with respect to the filtered space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_t)$ if it has the decomposition

$$X = M + A$$

where M a local martingale and A a Finite Variation (FV) process.

The class of semimartingales is broad and includes various types of processes. Also, as demonstrated by [4], the decomposition of a semimartingale into a local martingale and finite variation process may not be unique in general, except for special semimartingale, which has a predictably finite variation part.

Please note that only a few results and properties of semimartingales are highlighted in this study. More comprehensive analysis of semimartingales can be found for example in [13], [5] and [12] among others. the class of semimartingales include local martingales, finite variation processes, arbitrary processes with stationary independent increments, etc.

Although the class of semmartingales is wide, it is also important to mention that not all processes are semimartingales; for example, an arbitrary deterministic function of unbounded variation is not a semimartingal. It was shown, for instance in [9] that the square root of a Brownian process is not a semimartingale.

In the major results of this study, we have used the decomposition definition of semimartingale above. However, it was first established by Bichteler-Dellacherie theorems and later verified, for instance, in [12] that semimartingales are processes that are "good integrators". In the earlier chapters of Protter [12], semimartingales were described as good integrators, meaning they serve as stochastic processes that, when used as integrators in stochastic integrals, ensure proper convergence in probability, uniform convergence, and other desirable forms of convergence.

Definition 3.13 (Predictability). A process H is said to be simple predictable process if H can be written in the form

$$H_t = H_0 1_{\{0\}} + \sum_{i=0}^{n} H_i 1_{]T_i T_{i+1}]}(t)$$

where $0 = T_0 \leq T_1 \leq T_2 \leq \cdots \leq T_{n+1} < \infty$ is a finite sequence of stopping times, $H_i \in \mathcal{F}_{T_i}$ with $H_i < \infty$ a.s., $0 \leq i \leq n$. Lets denote the collection of simple predictable processes as $\mathcal{H}$.

Let's denote the space of $\mathcal{H}$ that is endowed with the topology of uniform convergence in (t, ω) by $\mathbb{H}_u$, and let denote $\mathbb{L}^0$ denote the space of finite-valued random variables endowed with convergence in probability.

Definition 3.14 (Semimartingale as Proper Integrator). A càdlàg, adapted process X is called a semimartingale if for each $t \in [0, \infty)$, the linear map

$$I_X : \mathbb{H}_u \to \mathbb{L}^0$$

defined for $H \in \mathcal{H}$ as,

$$I_X(H) = H_0 X_0 + \sum_{i=0}^{n} H_i \left(X_{t \wedge T_{i+1}} - X_{t \wedge T_i} \right) \tag{1}$$

is continuous.

This definition of a semimartingale ensures the well-posedness of stochastic integration and facilitates certain types of convergence, particularly bounded convergence within the framework of stochastic processes. From this definition of semimartingale, the following argument is easily verifiable:

Theorem 3.15 (Bichteler-Dellacherie Theorem). *An adapted càdlàg process X is a semimartingale if and only if it is a proper integrator. That is, a proper integrator X is a semimartingale if and only if it can be written $X = M + A$, where M is a local martingale and A is an FV proeess.*

Proof. *see [12] Theorem 47 (page 146).*

Theorem 3.16. *If Q is a probability measure absolutely continuous with respect to P, then every P-semimartingale is a Q-semimartingale.*

Proof. *This argument holds true since convergence in P implies convergence in Q.*

We will see later in detail the proof of this theorem for the decomposable semimartingale, which is the Girsanov theorem. The following arguments proved by [12] identifies some useful examples of processes that are semimartingales based on the satisfaction of the definition of semimartingale and their properties.

Theorem 3.17. *Every adapted process with càdlàg paths of finite variation on compact is a semimartingale*

Theorem 3.18. *Every càdlàg local martingale is a semimartingale. In particular:*

1. *Every L^2-martingale with càdlàg paths is a semimartingale.*
2. *Every càdlàg, locally square-integrable local martingale is a semimartingale.*
3. *A local martingale with continuous paths is a semimartingale.*

When considering process that has an increasing component (such as in the decomposition $X_t = M_t + A_t$, where M_t is a local martingale and A_t is an increasing process), the concept of quadratic variation is crucial for understanding the behavior of the random fluctuations arising from M_t. The increasing component A_t itself has a quadratic variation of zero because it has no jump, but the quadratic variation of M_t captures the stochastic nature of the overall process X_t . Hence, the quadratic variation is a tool for analyzing the randomness of a system. In general, the term covariation or mutual variation is used to describe the variation of two processes X, Y and it is denoted by $[Y, X]$.

For this study, we adopt [12] definition of quadratic variation and covariation given as follows:

Definition 3.19. Let X and Y be semimartingales, the **quadratic variation** of X is denoted and defined as

$$[X, X]_t = X_t^2 - 2\int_0^t X_- dX$$

recall that $X_{0^-} = 0$. The bracket process denoted by [X, Y] is called the **quadratic covariation** of the semimartingales X and Y, and its defined as

$$[X, Y]_t = X_t Y_t - \int_0^t X_{s^-} dY - \int_0^t Y_{s^-} dX$$

The continuous part of the quadratic variation processes is denoted by $[X, X]_t^c = \langle X, X\rangle_t$. The Doob-Meyer's decomposition theorem established that any submartingale can be uniquely decomposed into a martingale and a predictable, integrable, increasing process. The existence of this predictable path of the quadratic variation is due to the fact that the quadratic variation is non-decreasing and has right continuous path. This allows the quadratic and mutual variation to be decomposed, path-by-path into its continuous part and pure jump part.

Definition 3.20. Let X and Y be two semimartingale, and $[X, Y]^c$ denotes the path by path continuous part of $[X, Y]$, we write

$$[X, Y]_t = [X, Y]_t^c + X_0 Y_0 + \sum_{0<s\leq t} \Delta X_s \Delta Y_s$$

Definition 3.21. A semimartingale X is called a quadratic pure jump if

$$[X, X]^c = 0$$

An example of a quadratic pure jump process is the Poisson process. Several results involving quadratic variations and covariation processes were extensively discussed and proved in [5], [7] and [12]. However, here we mentioned some notable results that we used in the study.

Lemma 3.22. *Given two semimartingales X and Y, the bracket $[X,Y]$ has paths of finite variation on compact, and hence it is a semimartingale.*

Lemma 3.23 (Integration of Parts). *Given two semimartingales X,Y, the product process $XY = \{(XY)_t\}_{t\geq 0}$ given by*

$$XY = \int X_- dY + \int Y_- dX + [X,Y]$$

is a semimartingale.

Corollary 3.23.1. *Let Y and X be two semimartingales then a.s. $\Delta[X,Y] = \Delta X \Delta Y$. Where the operator Δ is given by $\Delta X_t = X_{t^+} - X_{t^-}$*

Theorem 3.24. *Let X be a quadratic pure jump semimartingale, (i.e. $[X,X]^c = 0$). Then for a semimartingale Y*

$$[X,Y]_t = X_0 Y_0 + \sum_{0<s\leq t} \Delta X_s \Delta Y_s$$

Theorem 3.25. *Let M be a local martingale.*

(a) If A is a predictable process of finite variation, then product MA is a special semimartingale with canonical decomposition

$$M_t A_t = \int_0^t M_{s^-} dA_s + N_t$$

where N is a local martingale which is zero at 0.

(b) If B is a process of finite variation, then product MB is a semimartingale with decomposition

$$M_t B_t = \int_0^t M_s dB_s + \bar{N}_t$$

where $\bar{N}$ is a local martingale which is zero at 0.

Proof. *see [5] Chapter VII.35.*

Definition 3.26 (Stochastic Exponetial). For a semimartingale X, with $X_0 = 0$, the stochastic exponential of X, denoted as $\mathcal{E}(X)$, is the unique semimartingale Z that satisfies the stochastic differential equation $Z_t = 1 + \int_0^t Z_{s^-}\, dX_s$ and is given by:

$$\mathcal{E}(X)_t = \exp\left\{X_t - \frac{1}{2}[X,X]_t\right\} \prod_{0<s\leq t} (1 + \Delta X_s) \exp\{-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\}, \qquad (2)$$

The stochastic exponential is also known as the DoIeans-Dade exponential.

4 Change of Measure Theorems

4.1 Classical Girsanov Theorem

Let $\{B_t\}_{t\geq 0}$ be a Brownian motion on a space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_t)$. For some drift term θ, Girsanov established conditions for which an equivalent measure Q exists so that the process

$$W_t = B_t + \theta t$$

becomes a Q-Brownian motion. Thus, we would want to find this equivalent measure Q for which the stochastic process $\{W_t\}_{0\leq t\leq T}$ is a Q-Brownian motion for some fixed T but arbitrary large, and also derive the relationship (density) between the measures Q and P.

As demonstrated in [15], the Girsanov theorem established the existence of Q and also define the density function of Q with respect to P for the Brownian motion $\{B_t\}_{t\geq 0}$. To establish the relationship between P and Q, we let Ω be the set of all continuous paths such that for $\omega \in \Omega$, $\omega_0 = 0$. Also, for $t \geq 0$ and for an interval $I = [a, b] \subset \mathbb{R}$, we consider a cylinder subset of Ω define by

$$C(t; I) = \{\omega \in \Omega : B_t(\omega) \in I\}.$$

Supposing that for every bounded measurable function $f : \mathbb{R} \to \mathbb{R}$, there exists an equivalent measure Q such that

$$\int_\Omega f(B_t(\omega))dP(\omega) = \int_\Omega f(W_t(\omega))dQ(\omega) = \int_\Omega f(B_t(\omega) + \theta t)dQ(\omega).$$

If we take $f(x) = \mathbf{1}_I(x)$ then we have

$$\int_\Omega f(B_t(\omega))dP = \int_{C(t;I)} dP = P(C(t, I))$$

$$\int_\Omega f(W_t(\omega))dQ = \int_\Omega f(B_t(\omega) + \theta t)dQ = \int_{C(t;I-\theta t)} dQ = Q(C(t, I - \theta t))$$

hence

$$P(C(t, I)) = Q(C(t, I - \theta t))$$

or

$$P(C(t, I + \theta t)) = Q(C(t, I)) \tag{3}$$

for every $t \geq 0$ and every I. By the definition of the measure P of the Bownian motion $\{B_t\}_{t\geq 0}$

$$\begin{aligned} P(C(t, I + \theta t)) &= \int_{a+\theta t}^{b+\theta t} \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} dy \\ &= \int_a^b \frac{1}{\sqrt{2\pi t}} e^{-(x+\theta t)^2/2t} dx \end{aligned}$$

To establish the nature of the density function of Q with respect to P, let Z be the Radon-Nikodym derivative that is $Z = \frac{dQ}{dP}$ and let denote and define its restriction to t by $Z_t = \mathbb{E}[Z \mid \mathcal{F}_t]$. Then $\{Z_t\}_{t\in[0,T]}$ is a P-martingale; specifically, by definition 3.2 and Lemma 4.5, for $T < \infty$ and on the interval $0 \leq t \leq T$, Z_t is a martingale closed by Z_T. This process $\{Z_t\}$ is known as the density process, and under standard assumptions it is strictly positive, $Z_0 = 1$, and it satisfies: $\mathbb{E}_P[Z_T] = 1$.

Assuming there exist a function of a Brownian motion that is equal to Z_t i.e. for $g_t : \mathbb{R} \to \mathbb{R}$, $Z_t = g_t(B_t)$ then,

$$
\begin{aligned}
Q(C(t,I)) &= \mathbb{E}_Q[\mathbf{1}_{C(t,I)}] \\
&= \mathbb{E}_P[\mathbf{1}_{C(t,I)}Z] \qquad \text{Radon-Nikodym theorem} \\
&= \mathbb{E}_P[\mathbb{E}_P[\mathbf{1}_{C(t,I)}Z \mid \mathcal{F}_t]] \qquad \text{by using the Tower principle} \\
&= \mathbb{E}_P[\mathbf{1}_{C(t,I)}Z_t] \\
&= \int_{C(t,I)} g_t(B_t)dP \\
&= \int_a^b g_t(x)\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}dx
\end{aligned}
$$

hence by (3)

$$g_t(x)e^{-x^2/2t} = e^{-(x+\theta t)^2/2t}$$

therefore,

$$g_t(x) = e^{-\theta x - \frac{1}{2}\theta^2 t}$$

This demonstration illustrates how the equivalent measure could possibly exist and how it relates to the original measure for the specific case of Brownian motion. However, unlike Brownian motion, which has certain already known properties (such as being Gaussian with a known distribution) that can help in establishing the equivalent measure, for a more general process, say a semimartingale, the nature of the function relating Q and P is quite different and sometimes tedious to compute.

Lemma 4.1. *Given a Brownian motion B_t and real constant θ for $0 \leq s < t$,*

$$\mathbb{E}\left[e^{\theta B_t} \mid \mathcal{F}_s\right] = e^{\frac{1}{2}\theta^2(t-s)}e^{\theta B_s}$$

in other word, if we let

$$Z_t = e^{-\frac{1}{2}\theta^2 t - \theta B_t}$$

then Z_t is a martingale.

Proof. *Since B_t has independent increament, i.e. $B_t - B_s$ is independent of $\mathcal{F}_s$ and $B_t - B_s \sim \mathcal{N}(0, t-s)$ we have*

$$\mathbb{E}\left[e^{\theta(B_t - B_s)} \mid \mathcal{F}_s\right] = \mathbb{E}\left[e^{\theta(B_t - B_s)}\right] = e^{\frac{1}{2}\theta^2(t-s)}$$

hence,

$$\mathbb{E}\left[e^{-\theta B_t - \frac{1}{2}\theta^2 t} \mid \mathcal{F}_s\right] = e^{-\frac{1}{2}\theta^2 t}\mathbb{E}\left[e^{-\theta B_t} \mid \mathcal{F}_s\right] = e^{-\frac{1}{2}\theta^2 s}e^{-\theta B_s} = e^{-\frac{1}{2}\theta^2 s - \theta B_s}.$$

In general, processes in this form Z_t are called exponential processes. Moreover, the term θ considered in Lemma 4.1 is constant but in general $\theta = \{\theta_t\}_{t\geq 0}$ is a variable. This variable θ is the major ingredient that determines whether the process is indeed a martingale. In particular, a sufficient condition for such a process to be a martingale is that the term θ is bounded. This boundedness requirement is what ensures that the exponential process is a martingale. However, one can weaken the boundedness requirement by a more general alternative sufficient condition, widely known as Novikov's criterion.

Theorem 4.2 (Novikov's Criterion). *Let M be a continuous local martingale, and suppose that*

$$\mathbb{E}\left\{e^{\frac{1}{2}[M,M]_\infty}\right\} < \infty.$$

Then $\mathcal{E}(M)$ is a uniformly integrable martingale.

Proof. *see e.g. [12]*

Lemma 4.3. *For any $\theta \in L^2_{loc}[0,T]$, the process*

$$Z_t = e^{\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds} \tag{4}$$

is a martingale, provided that θ satisfies the Novikov criterion

$$\mathbb{E}\left\{e^{\frac{1}{2}\int_0^T \theta_s^2 ds}\right\} < \infty$$

Theorem 4.4 (**Girsanov theorem**). *Given a filtered space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_t)$, Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion on a bounded interval $[0,T]$ under the probability measure P with a given filtration $\mathcal{F}_t$. Suppose $\{\theta_t\}_{0\leq t\leq T}$ is an adapted, measurable process such that $\int_0^T \theta_s^2 ds < \infty$ a.s. and satisfies the Novikov's criterion. Let Q be a measure on $\mathcal{F}_T$ defined by the Radon-Nikodym's derivative*

$$Z_T = \frac{dQ}{dP} = \exp\left(-\int_0^T \theta_s dB_s - \frac{1}{2}\int_0^T \theta_s^2 ds\right).$$

Then, Q is a probability measure equivalent to P, the process

$$Z_t = \mathbb{E}_P[Z_T|\mathcal{F}_t]$$

is a P-martingale, and the process

$$W_t = B_t + \int_0^t \theta_s ds$$

is a Brownian motion under Q.

Proof. *Since θ is assumed to satisfy the Novikov's condition, then by Lemma 4.3, the process*

$$Z_t = \exp(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds) = \mathbb{E}_P\left[\exp(-\int_0^T \theta_s dB_s - \frac{1}{2}\int_0^T \theta_s^2 ds) \mid \mathcal{F}_t\right]$$

is a P-martingale. Using Ito's rule, one can write

$$Z_t = 1 - \int_0^t Z_s \theta_s dB_s \tag{5}$$

Since Z_t is a P-martingale, we have

$$\mathbb{E}_p[Z_t] = 1.$$

To show that Q is a probability measure, let $A \in \mathcal{F}_T$, by Radon-Nikodym theorem,

$$Q(A) = \int_A Z_T dP \geq 0 \text{ and } Q(\Omega) = \int_\Omega Z_T dP = \mathbb{E}_p[Z_T] = 1 = \mathbb{E}_p[Z_t].$$

Lastly, it remains to show that

$$W_t = B_t + \int_0^t \theta_s ds \tag{6}$$

is a standard Brownian motion. To show this, we use the Levy's characterization theorem. This theorem opined that a continuous process W_t with $W_0 = 0$, which is a local martingale with $[W, W]_t = t$ is a Brownian motion. Since $W_0 = 0$, and (W_t) is a.s. continuous process, due to the fact that (B_t) is continuous a.s. and the integral of an adapted process is continuous, and

$$[W, W]_t = [B, B]_t = t.$$

Then by Levy's theorem, it remains to show that if W_t is a Q-(local) martingale, then it is a Brownian motion under Q. To show that W_t is a Q-(local) martingale, by Lemma 4.6 it is equivalent to showing that $(WZ)_t$ is a P-(local) martingale: by applying the Ito's rule to equations (5) and (6), that is $dZ_t = -Z_t\theta_t dB_t$, $dW_t = dB_t + \theta_t dt$, and using product rule,

$$\begin{aligned}
Z_t W_t &= \int_0^t W_s dZ_s + \int_0^t Z_s dW_s + \int_0^t d[Z, W]_s \\
&= -\int_0^t W_s Z_s \theta_s dB_s + \int_0^t Z_s dB_s + \int_0^t Z_s \theta_s ds - \int_0^t Z_s \theta_s ds \qquad (7) \\
&= \int_0^t Z_s(1 - W_s\theta_s) dB_s
\end{aligned}$$

being a stochastic integral with respect to Brownian motion (martingale) and, by Lemma (3.9), $(ZW)_t$ is a P-(local) martingale. Therefore, W_t is a Q-(local) martingale, and the result follows. □

Remark 1. One of the most important tasks in establishing the Girsanov Theorem is in determining the relationship between Q and P, that is by defining the Radon-Nikodym derivative Z (the density function) such that the corresponding Z_t is a positive P-martingale satisfying $Z_0 = 1$ and $\mathbb{E}_P[Z_t] = 1$ for all t. In general, by Definition 3.26 given a semimartingale X with $X_0 = 0$, the Doléans-Dade exponential defines the unique choice of the Z_t satisfies equation of the form $Z_t = 1 + \int_0^t Z_{s^-}\, dX_s$ and it is given by:

$$Z_t = \exp\left\{X_t - \frac{1}{2}[X, X]_t\right\} \prod_{0<s\le t} (1 + \Delta X_s) \exp\left\{-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right\},$$

(see [12] Theorem 37).

Note: In situations where the drift term θ is deterministic and constant, the $Q-$Brownian motion is of the form:

$$W_t = B_t + \theta t.$$

In this case, the Radon-Nikodym derivative $Z_t = \frac{dQ}{dP}\big|_{\mathcal{F}_t}$ relating the measure Q to P is given by:

$$Z_t = \exp\left(-\theta B_t - \frac{1}{2}\theta^2 t\right).$$

This expression is the Doléans-Dade exponential (or stochastic exponential) of $-\theta B_t$, and it satisfies the stochastic differential equation: $dZ_t = -\theta Z_t\, dB_t, \quad Z_0 = 1$. This process is related to the dynamics of the Geometric Brownian Motion (GBM), as it satisfies the SDE $dZ_t = \theta Z_t dB_t$, similar in form to the GBM equation.

4.2 Generalized Girsanov Theorem

The generalized Girsanov theorem extends the Girsanov theorem to a more braoder class of process; semimartingales. The class of semimartingales has the property of being invariant with respect to an equivalent transformation of measure. Given a semimartingale X on a space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_t)$. If Q is another probability measure on $(\Omega, \mathcal{F})$ such that $Q \ll P$, then by the Radon-Nikodym's theorem (Theorem 2.5), there exists a random variable $Z \in L^1(dP)$ such that

$$Z = \frac{dQ}{dP}$$

and $E_P\{Z\} = 1$.

If Q is equivalent to P $(Q \sim P)$, then

$$Z^{-1} = \frac{dP}{dQ} \in L^1(dQ)$$

Additionally, Z is a right-continuous version of the process defined as

$$Z_t = \mathbb{E}_P\left\{\frac{dQ}{dP}\middle|\mathcal{F}_t\right\}.$$

This right continuous version exists since the filtration $\mathcal{F}_t$ satisfied the usual hypothesis assumption, which claimed the filtration is right continuous.

Lemma 4.5. *The process Z_t defined above is a martingale (and hence a local martingale).*

Lemma 4.6. *Let $Q \sim P$, and $Z_t = \mathbb{E}_P\left\{\frac{dQ}{dP}\middle|\mathcal{F}_t\right\}$. An adapted, càdlàg processs $M = \{M_t\}_{t\geq 0}$ is said to be a $Q-$local martingale if and only if $MZ = \{M_t Z_t\}_{t\geq 0}$ is a $P-$local martingale.*

Proof. *w.l.o.g. let $\{T_n\}_n$ be the sequence of bounded stopping times for which $M_{.\wedge T_n}$ is a local martingales. Hence it is sufficient for an arbitrarily chosen bounded stopping time T, we will show that the stopped process $M_{.\wedge T}$ being a Q-martingale implies $\{MZ\}_{.\wedge T}$ is a P-martingale and vice versa. That is, for all $s \leq t$,*

$$\mathbb{E}_Q\left\{M_{t\wedge T}|\mathcal{F}_s\right\} = M_{s\wedge T} \iff \mathbb{E}_P\left\{\{MZ\}_{t\wedge T}|\mathcal{F}_s\right\} = \{MZ\}_{s\wedge T}$$

Starting from the left on the assumption that $M_{.\wedge T}$ a Q-martingale, it implies that

$$\mathbb{E}_Q\left\{M_{t\wedge T}|\mathcal{F}_s\right\} = M_{s\wedge T} \tag{8}$$

Note that $s \wedge T \leq s$ and this implies that $\mathcal{F}_{s\wedge T} \subset \mathcal{F}_s$.
hence, $\forall A \in \mathcal{F}_{s\wedge T}$

$$\int_A M_{s\wedge T} dQ = \int_A \mathbb{E}_Q\left\{M_{t\wedge T}|\mathcal{F}_s\right\} dQ = \int_A M_{t\wedge T} dQ \tag{9}$$

Where the first equality is based on the fact that $M_{.\wedge T}$ is a Q- martingale and the second equality is given by the definition of conditional expectation $\forall A \in \mathcal{F}_{s\wedge T} \subset \mathcal{F}_s$.

Considering the third term in equation (9), and applying the Radon-Nikodym's theorem (Theorem 2.5), we have

$$\int_A M_{t\wedge T} dQ = \int_A M_{t\wedge T}\frac{dQ}{dP} dP = \mathbb{E}_P\left[\mathbf{1}_A M_{t\wedge T}\frac{dQ}{dP}\right]$$

applying the Towers principle and conditioning on $\mathcal{F}_{t\wedge T}$ we have:

$$\mathbb{E}_P\left[\mathbf{1}_A M_{t\wedge T}\frac{dQ}{dP}\right] = \mathbb{E}_P\left[\mathbb{E}_P\left[\mathbf{1}_A M_{t\wedge T}\frac{dQ}{dP}\middle|\mathcal{F}_{t\wedge T}\right]\right] = \mathbb{E}_P\left[\mathbf{1}_A M_{t\wedge T} Z_{t\wedge T}\right] \tag{10}$$

Note that $\mathbf{1}_A \in \mathcal{F}_{s\wedge T} \subset \mathcal{F}_{t\wedge T}$ since $s\wedge T \leq t\wedge T$ and equation (10) becomes:

$$\mathbb{E}_P\left[\mathbf{1}_A M_{t\wedge T} Z_{t\wedge T}\right] = \int_A M_{t\wedge T} Z_{t\wedge T} dP = \int_A \mathbb{E}_P\left[M_{t\wedge T} Z_{t\wedge T}|\mathcal{F}_{s\wedge T}\right] dP. \tag{11}$$

Where the last equality is implied by the definition of conditional expectation. Similarly, the first term of equation (9) will gives:

$$\int_A M_{s\wedge T} dQ = \int_A M_{s\wedge T}\frac{dQ}{dP} dP = \mathbb{E}_P\left[\mathbf{1}_A M_{s\wedge T}\frac{dQ}{dP}\right]$$

applying the Towers principle again and this time conditioning on $\mathcal{F}_{s\wedge T}$ we have:

$$\mathbb{E}_P\left[\mathbf{1}_A M_{s\wedge T}\frac{dQ}{dP}\right] = \mathbb{E}_P\left[\mathbb{E}_P\left[\mathbf{1}_A M_{s\wedge T}\frac{dQ}{dP}\middle|\mathcal{F}_{s\wedge T}\right]\right] = \mathbb{E}_P\left[\mathbf{1}_A M_{s\wedge T} Z_{s\wedge T}\right] = \int_A M_{s\wedge T} Z_{s\wedge T} dP \tag{12}$$

By equation (9), we have that

$$\int_A M_{s\wedge T} dP = \int_A M_{t\wedge T} dP$$

equating (11) and (12) yield:

$$\int_A \mathbb{E}_Q\left[M_{t\wedge T} Z_{t\wedge T}|\mathcal{F}_{s\wedge T}\right] dQ = \int_A M_{s\wedge T} Z_{s\wedge T} dQ$$

Hence,

$$\mathbb{E}_Q\left[M_{t\wedge T} Z_{t\wedge T}|\mathcal{F}_{s\wedge T}\right] = M_{s\wedge T} Z_{s\wedge T}$$

Additionally, the "only if" direction follows similarly to the above, simply working from the end to the beginning. □

Now we present the first major extension of the Girsanov theorem.

Theorem 4.7 (Girsanov-Meyer Theorem). *Let P and Q be equivalent. Let X be a semimartingale under P with decomposition $X = M + A$. Then, under Q, X is also a semimartingale with decomposition $X = N + B$, where*

$$N_t = M_t - \int_0^t \frac{1}{Z_s} d[Z, M]_s$$

is a Q-local martingale, and

$$B_t = A_t + \int_0^t \frac{1}{Z_s} d[Z, M]_s$$

is a Q-FV process.

Proof. *Given that M and Z are P-local martingales. By the fundamental theorem of local martingale and by Theorem 3.18, they are also a P-semimartingale. Applying the integration by part of stochastic process,*

$$ZM = \int Z_- dM + \int M_- dZ + [Z, M] \tag{13}$$

and from the Lemma 3.9,

$$\int Z_- dM + \int M_- dZ$$

is a local martingale. Hence,

$$ZM - [Z, M]$$

is a P-local martingale. Since Z is a version of $Z_t = \mathbb{E}_P \left\{ \frac{dQ}{dP} \middle| \mathcal{F}_t \right\}$ *we have* $\frac{1}{Z}$ *is a cadlag version of* $\frac{1}{Z_t} = \mathbb{E}_Q \left\{ \frac{dP}{dQ} \middle| \mathcal{F}_t \right\}$ *(see Lemma 4.8.1), therefore* $\frac{1}{Z}$ *is a Q-semimartingale. Given that* $P \ll Q$ *then* $\frac{1}{Z}$ *is also a P-Semimartingale.*

By Lemma 4.6 we have that the product

$$M - \frac{1}{Z}[Z, M] = \frac{1}{Z}\left(\int Z_- dM + \int M_- dZ \right) \tag{14}$$

is a Q-local martingale and thence a Q-Semimartingale. Using integration by part we expand the following to get

$$\frac{1}{Z_t}[Z, M]_t = \int_0^t \frac{1}{Z_{s-}} d[Z, M]_s + \int_0^t [Z, M]_{s-} d\left(\frac{1}{Z_s}\right) + [[Z, M], \frac{1}{Z}]_t \tag{15}$$

since $\frac{1}{Z}$ *is a Q local martingale and* $[Z, M]$ *is a FV process, then by Lemma 3.9,* $\int_0^t [Z, M]_{s-} d\left(\frac{1}{Z_s}\right)$ *is a Q local martingale as well. Also by Theorem 3.24 we can rewrite*

$$[[Z, M], \frac{1}{Z}]_t = \sum_{0<s\le t} \Delta\left(\frac{1}{Z_s}\right) \Delta[Z, M]_s$$

Equation (15) becomes:

$$\begin{aligned}
\frac{1}{Z_t}[Z, M]_t &= \int_0^t \frac{1}{Z_{s-}} d[Z, M]_s + \int_0^t [Z, M]_{s-} d\left(\frac{1}{Z_s}\right) + [[Z, M], \frac{1}{Z}]_t \\
&= \int_0^t \frac{1}{Z_{s-}} d[Z, M]_s + \int_0^t [Z, M]_{s-} d\left(\frac{1}{Z_s}\right) + \sum_{0<s\le t} \Delta\left(\frac{1}{Z_s}\right) \Delta[Z, M]_s \\
&= \int_0^t \frac{1}{Z_s} d[Z, M]_s + \int_0^t [Z, M]_{s-} d\left(\frac{1}{Z_s}\right)
\end{aligned}$$

Where the last substitution is made based on the equality:

$$\int_0^t \frac{1}{Z_s} d[Z, M]_s = \int_0^t \frac{1}{Z_{s^-}} d[Z, M]_s + \sum_{0<s\le t} \Delta\left(\frac{1}{Z_s}\right) \Delta[Z, M]_s$$

which arises from decomposing the stochastic integral into its continuous part and jump part. The term $\int_0^t \frac{1}{Z_{s^-}} d[Z, M]_s$ *corresponds to the continuous part, while* $\sum_{0<s\le t} \Delta\left(\frac{1}{Z_s}\right) \Delta[Z, M]_s$ *captures the jumps.*

Next, substituting this expression into the Q-local martingale in (14) *above we have*

$$M_t - \frac{1}{Z_t}[Z, M]_t = M_t - \int_0^t \frac{1}{Z_s} d[Z, M]_s - \int_0^t [Z, M]_{s-} d\left(\frac{1}{Z_s}\right)$$

$$M_t - \frac{1}{Z_t}[Z, M]_t + \int_0^t [Z, M]_{s-} d\left(\frac{1}{Z_s}\right) = M_t - \int_0^t \frac{1}{Z_s} d[Z, M]_s.$$

This is a Q-local martingale since it is a sum of the two Q-local martingales $M_t - \frac{1}{Z_t}[Z,M]_t$ *and* $\int_0^t [Z,M]_{s-} d\left(\frac{1}{Z_s}\right)$. *With this, we have shown that the process N defined by*

$$N_t = M_t - \int_0^t \frac{1}{Z_s} d[Z,M]_s$$

is indeed a Q-local martingale.

Finally, $B = X - N$ *is a Q-FV because*

$$\begin{aligned} B_t =& X_t - N_t = M_t + A_t - M_t + \int_0^t \frac{1}{Z_s} d[Z,M]_s \\ =& A_t + \int_0^t \frac{1}{Z_s} d[Z,M]_s \end{aligned}$$

which is a sum of two FV processes and since $P \sim Q$ *ensures that the measure Q inherits almost sure properties of processes defined under P. Hence, a finite variation process defined under P almost surely will be well defined under Q. Therefore, every P-FV process is a Q-FV process.*

□

Remark 2. Now, we will show how the classical Girsanov theorem arises as a special case of the Girsanov-Meyer theorem; that is, to reduce the Girsanov-Meyer theorem (4.7) to the clasical Girsanov theorem (4.4). Let B_t be a P-Brownian motion, if we let $X = B$ (i.e. $A = 0$) be the semimartingale under P, and define

$$Z_t = \exp\left(-\int_0^t \theta_s \, dB_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds\right).$$

One can see that under P, the local martingale part is simply $M = B$, and by Theorem (4.7) under Q the local martingale is given by

$$\begin{aligned} N_t &= M_t - \int_0^t \frac{1}{Z_s} d[Z,M]_s \\ &= B_t - \int_0^t \frac{1}{Z_s} d[Z,B]_s \end{aligned} \tag{16}$$

Applying Itô's formula to Z_t, we know that $dZ_t = -\theta_t Z_t \, dB_t$, and thus $d[Z,B]_s = -\theta_s Z_s ds$. Substituting this into Equation (16), we obtain:

$$\begin{aligned} N_t &= B_t + \int_0^t \frac{1}{Z_s} \theta_s Z_s ds \\ &= B_t + \int_0^t \theta_s ds. \end{aligned} \tag{17}$$

Therefore, we recover the classical Girsanov result where the process

$$W_t := B_t + \int_0^t \theta_s \, ds$$

is a Q-Brownian motion. This confirms that Theorem 4.4 is a special case of the more general Girsanov-Meyer result in Theorem 4.7, where $X = B$ is a Brownian motion and $Z_t = \exp\left(-\int_0^t \theta_s \, dB_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds\right)$.

Lemma 4.8. *Let X be an adapted stochastic process and suppose $Q \ll P$, then*

$$\mathbb{E}_P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right]\cdot\mathbb{E}_Q\left[X|\mathcal{F}_t\right]=\mathbb{E}_P\left[X\cdot\frac{dQ}{dP}\middle|\mathcal{F}_t\right]$$

Proof. *In this result, we wish to show the product representation of two conditional expectations under measures $Q \ll P$ with respect to the same filtration.*

First consider the left hand side:

$$\mathbb{E}_P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right]\cdot\mathbb{E}_Q\left[X|\mathcal{F}_t\right]=\mathbb{E}_P\left[\frac{dQ}{dP}\cdot\mathbb{E}_Q\left[X|\mathcal{F}_t\right]\middle|\mathcal{F}_t\right]$$

The equality is possible since both conditional expectations in P and Q are $\mathcal{F}_t$-measurable. Thus, by the definition of conditional expectation, $\forall A \in \mathcal{F}_t$

$$\begin{aligned}\int_A \mathbb{E}_P\left[\frac{dQ}{dP}\cdot\mathbb{E}_Q\left[X|\mathcal{F}_t\right]\middle|\mathcal{F}_t\right]dP &= \int_A \frac{dQ}{dP}\cdot\mathbb{E}_Q\left[X|\mathcal{F}_t\right]dP\\ &= \int_A \mathbb{E}_Q\left[X|\mathcal{F}_t\right]dQ\\ &= \int_A X dQ\end{aligned}$$

Next, applying the definition of conditional expectation to the right hand side

$$\mathbb{E}_P\left[X\cdot\frac{dQ}{dP}\middle|\mathcal{F}_t\right]$$

we have $\forall A \in \mathcal{F}_t$

$$\int_A \mathbb{E}_P\left[X\cdot\frac{dQ}{dP}\middle|\mathcal{F}_t\right]dP=\int_A X\cdot\frac{dQ}{dP}dP=\int_A XdQ$$

This completes the proof, as we have shown that the two terms on the left-hand side and the right-hand side are indeed equal. □

This established a product of conditional expectations under different measures, Q and P, when $Q \ll P$. This result is further extended to the particular case where we assumed $Q \sim P$, this case played a vital role in the proof of Theorem 4.7 above.

Corollary 4.8.1. *Suppose $Q \sim P$, the inverse process $\frac{1}{Z}$ is a cadlag version of*

$$\frac{1}{Z_t}=\mathbb{E}_Q\left[\frac{dP}{dQ}\middle|\mathcal{F}_t\right]$$

Proof. *By definition, Z is a right continuous version of*

$$Z_t=\mathbb{E}_P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right].$$

Hence, to prove this lemma is sufficient to show that

$$Z\cdot\frac{1}{Z}=1 \implies Z_t\cdot\frac{1}{Z_t}=1$$

Using Lemma 4.8,

$$\mathbb{E}_P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right]\cdot\mathbb{E}_Q\left[\frac{dP}{dQ}\middle|\mathcal{F}_t\right]=\mathbb{E}_P\left[\frac{dQ}{dP}\cdot\mathbb{E}_Q\left[\frac{dP}{dQ}\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_t\right]$$

by the definition of conditional expectation, $\forall A\in\mathcal{F}_t$

$$\begin{aligned}\int_A\mathbb{E}_P\left[\frac{dQ}{dP}\cdot\mathbb{E}_Q\left[\frac{dP}{dQ}\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_t\right]dP&=\int_A\frac{dQ}{dP}\cdot\mathbb{E}_Q\left[\frac{dP}{dQ}\middle|\mathcal{F}_t\right]dP\\&=\int_A\mathbb{E}_Q\left[\frac{dP}{dQ}\middle|\mathcal{F}_t\right]dQ\\&=\int_A\frac{dP}{dQ}dQ\\&=\int_A dP\end{aligned}$$

therefore,

$$1=\mathbb{E}_P\left[\frac{dQ}{dP}\cdot\mathbb{E}_Q\left[\frac{dP}{dQ}\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_t\right]=\mathbb{E}_P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right]\cdot\mathbb{E}_Q\left[\frac{dP}{dQ}\middle|\mathcal{F}_t\right]$$

□

Theorem 4.7 established above is a more flexible one, due to the equivalence assumption on the measure. This ensures the density function Z of the measure with respect to each other exists. Next, we introduce a more rigid extension of the theorem where the assumption of equivalence is altered. In this case the density of the measure is only guaranteed with respect to one measure.

Theorem 4.9 (Lenglart-Girsanov Theorem). *Let Q be a probability measure absolutely continuous with respect to P, and let X be P-local martingale with $X_0=0$. Let $Z_t=\mathbb{E}_P\left[\frac{dQ}{dP}\mid\mathcal{F}_t\right]$, $R=\inf\{t>0:Z_t=0,Z_{t^-}>0\}$ and define $U_t=\Delta X_R\mathbf{1}_{\{t\geq R\}}$, then*

$$X_t-\int_0^t\frac{1}{Z_s}d[X,Z]_s+\tilde{U}_t$$

is a Q-local martingale.

Remark 3. The stopping time $R=\inf\{t>0:Z_t=0\text{ and }Z_{t^-}>0\}$ represents the first time the process Z_t jumps to zero from a positive value. This is significant because $Z_t=0$ introduces a singularity in the measure Q, thereby contradicting the absolute continuity assumption of the measures Q w.r.t. P. Since $Q\ll P$, and Z_t is a P local martingale, this means the measure Q is such that, $Q(R=\infty)=1$, it is possible to have $P(R<\infty)>0$, (see corrolary 4.10.1). This guarantees a well-define measure Q by ensuring that under Q, the stopping time R almost surely does not occur. However, R can occur under the original measure P.

Lemma 4.10. *Let X be a positive right continuous supermartingale. Define*

$$T(\omega)=\inf\{t:X_t(\omega)=0\ \text{or}\ X_{t^-}(\omega)=0\}.$$

Then, for all ω, $X_t(\omega)=0$ for $t\geq T(\omega)$.

Proof. *Let $X_\infty=0$ be the r.v. closing the positive supermartingale X by the right. We define the stopping time T_n by:*

$$T_n=\inf\{t:X_t(\omega)\leq\frac{1}{n}\}.$$

Note that $T_{n-1} \le T_n$ *and* $T_n \le T$. *Also,* $X_{T_n} \le \frac{1}{n}$ *on* $\{T_n < \infty\}$ *and* $X_{T_n} = 0$ *on* $\{T_n = \infty\}$ *since* X *is right closed by* $X_\infty = 0$. *By the Doob's optional sampling theorem, and applying the definition of conditional expectation, we have that for every stopping time* $S \ge T_n$,

$$\frac{1}{n} \ge \mathbb{E}[X_{T_n}] \ge \mathbb{E}[X_S].$$

Taking $S = T + s$, *for all* $s \ge 0$, *we obtain* $X_{T+s} = 0$. *Therefore, by the right continuity of the process,* $X_t(\omega) = 0$ *for all* $t \ge T(\omega)$.

□

Corollary 4.10.1. *Let* Q *be absolutely continuous with respect to* P, *the increasing stopping time* $R_n = \inf(t > 0 : Z_t \le \frac{1}{n})$ *converges to* $R = \inf\{t > 0 : Z_t = 0$ *and* $Z_{t^-} = 0\}$.

Proof. *Since* Z_t *is a* P*-positive martingale, which is right continuous and close by* $Z_\infty = 0$, *satisfying the hypothesis of Lemma 4.10, hence, it is* $P - a.s.$ *identically zero on* $[R, \infty)$ *this also holds under* Q; *that is* $Z_t = 0$ Q*-a.s. for all* $t \ge R$. *However, since* $\frac{1}{Z_t}$ *is a* Q*-local martingale, we have that* $Q - a.s.$ $\frac{1}{Z_t} = \infty$ *on* $[R, \infty)$, *But* $\mathbb{E}_Q[\frac{1}{Z_t}] = P\{Z_t > 0\} \le 1$; *that is for any* $\epsilon > 0$

$$\begin{aligned}
\frac{1}{\epsilon} Q(t \ge R) &= \mathbb{E}_Q \left[\frac{1}{Z_t + \epsilon} \mathbf{1}_{[R,\infty)}(t) \right] \\
&= \mathbb{E}_P \left[Z \frac{1}{Z_t + \epsilon} \mathbf{1}_{[R,\infty)}(t) \right] \\
&= \mathbb{E}_P \left[\frac{1}{Z_t + \epsilon} \mathbf{1}_{[R,\infty)}(t) \mathbb{E}\left[Z \mid \mathcal{F}_t \right] \right] \\
&= \mathbb{E}_P \left[\frac{Z_t}{Z_t + \epsilon} \mathbf{1}_{[R,\infty)}(t) \right] \xrightarrow{\epsilon \to 0} \mathbb{E}_P \left[\mathbf{1}_{\{Z_t > 0\}} \mathbf{1}_{[R,\infty)}(t) \right] \le 1
\end{aligned}$$

This shows that

$$R = \infty, \quad Q - a.s.,$$

and since the stopping times $R_n \to R$, *as* $n \to \infty$. *Hence* $Q - a.s.$, $R_n \to \infty$, *as* $n \to \infty$ *and* $Z_t > 0$ $Q - a.s.$ *for all finite* t. □

Remark 4. The process $U_t = \Delta X_R \mathbf{1}_{\{t \ge R\}}$ is an increasing FV process, in particular, it captures the jumps in X that may arise in scenerios where the process touches R. Furthermore, being a difference of locally integrable martingale, U itself is also locally integrable. Also, there is an increasing predictable FV process $\tilde{U}$ called the compensator, defined in such a way that $U - \tilde{U}$ is a P-local martingale.

Proof (Lenglart-Girsanov Theorem). *To avoid the problem that may be encountered at* R, *we define a sequence of stopping time* $R_n = \inf(t > 0 : Z_t \le \frac{1}{n})$ *as in Corollary 4.10.1. Since* X *and* Z *are* P *local martingale, and by the Doob's theorem,* $X_{.\wedge R_n}$ *and* $Z_{.\wedge R_n}$ *are also* P*-local martingales. Also note that*

$$A_{t \wedge R_n} = \int_0^t \frac{1}{Z_{s \wedge R_n}} \mathbf{1}_{\{Z_{s\wedge R_n} > 0\}} d[X, Z]_{s \wedge R_n}$$

and $\tilde{U}_{.\wedge R_n}$ *are well defined under* P. *Because,* $\frac{1}{Z_{s \wedge R_n}} < \infty$ *for all* $s \wedge R_n < R$. *While on* $s \ge R_n = R$ *we populate the integral by* $\mathbf{1}_{\{Z_{s\wedge R_n} > 0\}}$ *and the integral is zero.*

Hence, we can define,

$$A_t = \int_0^t \frac{1}{Z_s} \mathbf{1}_{\{Z_s > 0\}} d[X, Z]_s$$

on $[0, R)$. This is well defined because from Corrolary 4.10.1, $Q-a.s.$, $R_n \to R \to \infty$, as $n \to \infty$, and by Lemma 4.10, the process A_t vanishes in the interval $[R, \infty)$. Note that in general, the process A_t may not be well defined under P, because one may not know for sure if $Z_{s^-} > 0$. But this limitation has no impact on our results since the primary interest is in having a well-defined A_t under Q.

Now we wish to show for a fixed n that the process

$$Y_{\cdot \wedge R_n} = X_{\cdot \wedge R_n} - A_{\cdot \wedge R_n} + \tilde{U}_{\cdot \wedge R_n}$$

is a Q-local martingale. By Lemma 4.6, it is sufficient to show that $\{ZY\}_{\cdot \wedge R_n}$ is a P-local martingale. Since we are considering a fixed n for a chosen stopping time, say R_n. For simplicity, we drop R_n from the notation. Thus, considering the expansion,

$$ZY = ZX - ZA + Z\tilde{U}$$

by applying the integration by part formular in Lemma 3.23 we have

$$ZX = \int Z_- dX + \int X_- dZ + [Z, X]$$

and by Lemma 3.9

$$ZX = \textit{local martingale} + [Z, X] \tag{18}$$

Similarly, and applying Theorem 3.25,

$$\begin{aligned} ZA &= \int A_- dZ + \int Z dA \\ ZA &= \textit{local martingale} + \int Z d\left(\int \frac{1}{Z}\mathbf{1}_{\{Z>0\}} d[X, Z]\right) \\ ZA &= \textit{local martingale} + \int \mathbf{1}_{\{Z>0\}} d[X, Z] \end{aligned} \tag{19}$$

Here we replaced in the integration by part formular $Z_- dA$ by ZdA also, one can observe that $[Z, A] = 0$ these are all direct consequences of Theorem 3.25.

Again, by Theorem 3.25,

$$\begin{aligned} Z\tilde{U} &= \int \tilde{U} dZ + \int Z_- d\tilde{U} \\ &= \textit{local martingale} + \int Z_- dU \end{aligned} \tag{20}$$

Again, here $\tilde{U}$ is the predictable process that makes $U - \tilde{U}$ a local martingale, hence, the integration by part also follows as in equation (41). Also note that we replace $d\tilde{U}$ by dU in the last equality of equation (20) because given that $U - \tilde{U}$ is a local martingale, by Lemma 3.9, the stochastic integral $\int Z_- d(U - \tilde{U})$ is also local martingale. Leveraging on the linearity property of stochastic integrals gives

$$\begin{aligned} \int Z_- d\tilde{U} &= \int Z_- dU - \int Z_- d(U - \tilde{U}) \\ &= \int Z_- dU - \textit{local martingale} \end{aligned}$$

Substituting this for $\int Z_- d\tilde{U}$ in the first line of Equation (20) and considering the fact that the difference of two local martingales is again a local martingale yields the required representation.

Therefore, combining Equations (18), (41) and (21), we obtain;

$$ZY = \text{local martingale} + [Z, X] - \int \mathbf{1}_{\{Z>0\}} d[X, Z] + \int Z_- dU \tag{21}$$

hence, to finalize the prove that ZY is a P-local martingale, it is enough to show that the term

$$\widetilde{ZY} = [Z, X] - \int 1_{\{Z>0\}} d[X, Z] + \int Z_- dU = 0$$

Also we know that $d\widetilde{ZY} = 0$ implies $\widetilde{ZY} = 0$. Therefore,

$$\begin{aligned} d\widetilde{ZY} &= d[X, Z] - \mathbf{1}_{\{Z>0\}} d[X, Z] + Z_- dU \\ \widetilde{ZY} = \int d\widetilde{ZY} &= \Delta X_R \Delta Z_R \mathbf{1}_{\{t \wedge R_n \geq R\}} + Z_{R^-} \Delta X_R \mathbf{1}_{\{t \wedge R_n \geq R\}} \\ &= \Delta X_R (Z_R - Z_{R^-}) \mathbf{1}_{\{t \wedge R_n \geq R\}} + Z_{R^-} \Delta X_R \mathbf{1}_{\{t \wedge R_n \geq R\}} \\ &= -Z_{R^-} \Delta X_R \mathbf{1}_{\{t \wedge R_n \geq R\}} + Z_{R^-} \Delta X_R \mathbf{1}_{\{t \wedge R_n \geq R\}} \\ &= 0. \end{aligned} \tag{22}$$

Note the integral of $d[X, Z] - \mathbf{1}_{\{Z>0\}} d[X, Z]$ is reduced to the jump expression $\Delta X_R \Delta Z_R \mathbf{1}_{\{t \wedge R_n \geq R\}}$. This is due to the right-continuity and non-negativity properties of Z which leads to the following cases:

$$\begin{cases} Z > 0, & \text{If } t \wedge R_n < R \\ \\ Z = 0, & \text{If } t \wedge R_n \geq R. \end{cases}$$

This implies

$$d[X, Z] - \mathbf{1}_{\{Z>0\}} d[X, Z] = \begin{cases} 0, & \text{If } t \wedge R_n < R \\ \\ d[X, Z], & \text{If } t \wedge R_n \geq R \end{cases}$$

thus the expression is reduced to

$$d[X, Z] - \mathbf{1}_{\{Z>0\}} d[X, Z] = d[X, Z] \mathbf{1}_{t \wedge R_n \geq R}.$$

At $t \wedge R_n \geq R$, $Z = 0$ this means Z is a.s. continuous and even better, it is of finite variation at $t \wedge R_n \geq R$, therefore integrating what is left, we see that the continuous part of Definition 3.20 vanishes, that is $[X, Z]^c = 0$. Consequently, we are left with only the jump parts of $[X, Z]$, which is equal to zero since $Z = 0$ at $t \wedge R_n \geq R$ except the jump at R.

Finally, on the third equality of equation 22, since Z is càdlàg, $Z_{R^+} = Z_R$, and Z vanishes at R which resulted in $Z_R = 0$. This comfirms the result. □

Remark 5. Note that in the case where X is continuous (for instance the Brownian motion), the quadratic covariation $\langle X, Z \rangle = [X, Z]^c$ is a predictable finite variation (FV) process. Hence, the integrand $\frac{1}{Z_s}$ in Theorem 4.9 is replaced by the left continuous version $\frac{1}{Z_{s^-}}$.

Corollary 4.10.2. *If X is a continuous P-local martingale and Q is absolutely contuous with respect to P, then $\langle X, Z \rangle = [X, Z]^c$ exists and there exists a predictable process α such that*

$$X_t - \int_0^t \frac{1}{Z_{s^-}} d[X, Z]_s^c = X_t - \int_0^t \alpha_s d[X, X]_s \tag{23}$$

is a Q-local martingale.

Proof. *This is a special case for when $\tilde{U} = 0$. This is true since it is stated that X is a continuous local martingale; hence, it has no jump and $U = 0$.*

Moreover, this result can be derived as a direct consequence of the Kunita-Watanabe theorem: since X, Z are continuous local martingales, applying the Kunita-Watanabe Inequality,

$$\int_0^t \frac{1}{Z_{s^-}} d[Z,X]_s^c \leq \left(\int_0^t \left(\frac{1}{Z_{s^-}} \right)^2 d[Z,Z]_s \right)^{\frac{1}{2}} \left(\int_0^t d[X,X]_s \right)^{\frac{1}{2}}$$

we have, $|d[X,Z]_t^c| \leq \sqrt{d[X,X]_t} \cdot \sqrt{d[Z,Z]_t}$ and thus, $d[Z,X]_s^c \ll d[X,X]_s$ a.s. by the Radon-Nikodym theorem, there exists a predictable process β_t defined so that $\beta_t = \alpha_t Z_{t^-}$ for some predictable process α_t such that

$$d[X,Z]_t^c = \alpha_t Z_{t^-}\, d[X,X]_t$$

thus,

$$\frac{1}{Z_{t^-}} d[X,Z]_t^c = \alpha_t\, d[X,X]_t$$

substituting this to the left-hand side of Equation (23) yield the required result. □

As mentioned in Remark 1, in most cases Z is the Doléans-Dade exponential which is the solution of the stochastic exponential equation of the form $dZ_s = Z_{s^-} H_s dX_s$, where $H_s = \alpha_s$.

5 Discussion of Result and Application

Although this study did not explicitly address Itô's Lemma, it is important to note that both Itô's Lemma and the Itô integral are inherently embedded in the results. This is because solving or analyzing a stochastic differential equation typically involves applying Itô's formula. The Itò formula plays a vital role in the study, for example, in the derivation of Z_t because the Doléans-Dade exponential (or stochastic exponent) is a direct consequence of the Itò formula.

5.1 Application to Finance

The Girsanov theorem plays an important role in financial modeling and asset pricing. In this section, we present an introduction to financial modeling. Specifically, we present basic theories of asset pricing and the application of Girsanov theorems in asset pricing. In this session, we followed closely [10], [18] and [17] and study the European options.

In general, we assume the risky asset pricing process $S = \{S_t\}_{t \geq 0}$ is a semimartingale. And for a variable interest rate $r(t)$, the riskless asset given by

$$R_t = R_0 e^{\int_0^t r_s ds},$$

satisfies $dR_t = r(t) R_t dt$. In many cases, the interest rate is fixed, in which case the riskless asset is reduced to $R_t = R_0 e^{rt}$ satisfying the ODE $dR_t = rR_t dt$. The bond rate is an interest rate compounded continuously over the time interval.

We let V_t denote the portfolio value at time t defined by

$$V_t = a_t S_t + b_t R_t$$

where a_t, b_t denote the number of units of stocks or trading strategy for the risky assets and bonds (riskless assets), respectively. The goal of an option buyer is to have a self-financing system such that

$$V_T = (S_T - K)_+$$

For a self-financing system, the portfolio value must satisfy

$$dV_t = a_t dS_t + b_t dR_t$$

and by integrating, we obtain

$$V_t = a_0 S_0 + b_0 R_0 + \int_0^t a_s dS_s + \int_0^t b_s dR_s.$$

Note, in most models of financial markets (e.g Black-scholes), the price process and the trading strategy are considered predictable and continuous which leads to the representation given above in which the left limit and actual value coincide: $a_{t^-} = a_t$. However, in general, considering jump processes, the representation must be in terms of the left limit a_{t^-}, this is essential to ensure that the integral

$$\int_0^t a_{s^-}\, dS_s$$

is well-defined.

One of the most important conditions in asset pricing that ensures fairness in the market is the "no arbitrage or arbitrage-free" condition. In order to meet this crucial condition, one will want the portfolio value $V = \{V_t(a_t, b_t)\}_{t\geq 0}$ to be a martingale so that the value will remain constant and at no time present a "free lunch" or a chance of making profit without taking risks. We note that the portfolio value V is considered a càdlàg process and a_t which denotes a trading startegy for the risky asset at time t is considered predictable.

In order to ensure fairness and consistency in the market, the fundamental theorems of asset pricing (FTAP) highlighted the need for a trading strategy to be **viable** (presents no arbitrage opportunity) and **complete** (replicate every contingency claim).

For the viability of the market to be attained, there should exist a measure Q that is equivalent (or absolutely continuous) w.r.t the underline measure P, such that the discounted portfolio value is a martingale under Q (the existence of this equivalent measure is established by the Girsanov theorems). i.e

$$\tilde{V}_t = \frac{V_t}{R_t} = \mathbb{E}_Q[\tilde{V}_T|\mathcal{F}_t]$$

The discounted portfolio value is given by

$$\tilde{V}_t = \frac{V_t}{R_t} = a_t\frac{S_t}{R_t} + b_t = a_t\tilde{S}_t + b_t$$

and in a self-financing system, using integration by part, and assuming $S_0 = 0$ and

$R_0 = 1$, the discounted portfolio value satisfies

$$\begin{aligned}
d\tilde{V}_t = d(\frac{V_t}{R_t}) &= R_{t-}^{-1} dV_t + V_{t-} d(R_t^{-1}) + d[V_t, R_t^{-1}] \\
&= R_{t-}^{-1}(a_t dS_t + b_t dR_t) + (a_t S_{t^-} + b_t R_{t^-}) d(R_t^{-1}) + d[a_t S_t + b_t R_t, R_t^{-1}] \\
&= a_t (R_{t-}^{-1} dS_t + S_{t^-} dR_t^{-1} + d[S_t, R_t^{-1}]) + b_t (R_{t-}^{-1} dR_t + R_{t^-} dR_t^{-1} + d[R_t, R_t^{-1}]) \\
&= a_t d(S_t R_t^{-1}) + b_t d(R_t R_t^{-1}) \\
&= a_t d(\tilde{S}_t)
\end{aligned} \tag{24}$$

Note that the trading strategies are assumed predictable and continuous, thus, $a_{t-} = a_t$ and $b_{t-} = b_t$. But as mentioned before, this may not be the case for processes with jump in which case we use the left limits.

Next, for the completeness of the market, let H be the contingency claim (expected payoff) with the form $H = f(S_T)$, for the market to be complete, one is tasked with finding a self-financing strategy (a_t, b_t) that replicates the contingency claim H such that the value of the corresponding portfolio at maturity replicates the payoff, i.e. $V_T = H$.

Therefore, by FTAP, the viability of the market is in finding the risk-neutral measure (Q), which makes the discounted portfolio value a martingale, and the completeness of the market lies in the replication of the contingency claim. That is:

$$\tilde{V}_t = \mathbb{E}_Q[\tilde{V}_T|\mathcal{F}_t] = \mathbb{E}_Q[R_T^{-1} H|\mathcal{F}_t] = \mathbb{E}_Q[R_T^{-1} f(S_T)|\mathcal{F}_t] = a_0 \tilde{S}_0 + b_0 + \int_0^t a_s d\tilde{S}s \tag{25}$$

where the last equality is due to the self-financing assumption of the trading strategy. In the specific case when $R_t = 1$ this becomes

$$V_t = \mathbb{E}_Q[V_T|\mathcal{F}_t] = \mathbb{E}_Q[H|\mathcal{F}_t] = \mathbb{E}_Q[f(S_T)|\mathcal{F}_t] = a_0 S_0 + b_0 + \int_0^t a_s dSs. \tag{26}$$

For the following analysis, we assume a zero interest rate bond, that is, $r(t) = 0$ and consequently $R_t = 1$ for all $t \geq 0$. This assumption allows us to focus entirely on the stochastic dynamics of the risky asset without the additional complexity introduced by deterministic interest rate terms from R_t. In other words it effectively means that all quantities are considered discounted.

Simplifying Hypothesis in Risk-Neutral Pricing

As highlighted in e.g [10] and considering the case for Equation (26) the following hypothesis are used in analyzing and establishing the price dynamics:

(1.) S is a Markov process under Q, i.e.

$$V_t = \mathbb{E}_Q[f(S_T)|\mathcal{F}_t] = \mathbb{E}_Q[f(S_T)|S_t]$$

and by Doob-Dynkin's lemma, there exist $\varphi(t, S_t)$ such that $\mathbb{E}_Q[f(S_T)|S_t] = \varphi(t, S_t)$.

(2.) $\varphi(t, x)$ is C^1 in t and C^2 in x. This enables us to apply Ito's formula to $\varphi(t, S_t)$.

$$\begin{aligned}
V_t = \varphi(t, S_t) = \varphi(0, S_0) &+ \int_0^t \varphi'_x(s, S_{s^-}) dS_s + \int_0^t \varphi'_s(s, S_{s^-}) ds + \frac{1}{2}\int_0^t \varphi''_{xx}(s, S_{s^-}) d[S,S]_s^c \\
&+ \sum_{0<s\leq t} \left(\varphi(s, S_s) - \varphi(s, S_{s^-}) - \varphi_{x'}(s, S_{s^-})\Delta S_s\right).
\end{aligned}$$

(3.) S has a continuous path.

(4.) $[S,S]_t = \int_0^t h(s,S_s)^2 ds$, for some measurable function $h : R_+ \times R \to R$.

Note, that applying hypothesis (3) to hypothesis (2) reduces the Ito's representation to:

$$V_t = \varphi(t,S_t) = \varphi(0,S_0) + \int_0^t \varphi'_x(s,S_{s^-})dS_s + \int_0^t \varphi'_s(s,S_{s^-})ds + \frac{1}{2}\int_0^t \varphi''_{xx}(s,S_{s^-})d[S,S]_s \tag{27}$$

Since V is required to be a Q-martingale, then we need

$$\int_0^t \varphi'_s(s,S_{s^-})ds + \frac{1}{2}\int_0^t \varphi''_{xx}(s,S_{s^-})d[S,S]_s = 0 \tag{28}$$

Applying hypothesis (4), for Equation (28) to hold, it is sufficient that φ satisfies the partial differential equation

$$\frac{1}{2}h^2(t,x)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial\varphi}{\partial t} = 0 \tag{29}$$

with boundary condition $\varphi(T,x) = f(x)$. This construction is a consequence of the Feynman-Kac formula. This ensures that $V_t = \varphi(t,S_t)$ is a Q-Martingale provided that φ satisfies the PDE (29).

Alternatively, we can consider a more general price process $\{S_t\}_{t\geq 0}$ defined under P satisfying the stochastic differential equation

$$dS_s = h(s,S_s)dB_s + b(s,S_r;r\leq s)ds \tag{30}$$

with drift term $b(s,S_r;r\leq s)$ and volatility $h(s,S_s)$, where B_t is a standard P-Brownian motion.

In this case, using Girsanov's theorem we construct a new measure $Q \sim P$ under which the drift vanishes and the process becomes a local martingale. Considering Theorem 4.7, we deduced from Equation (30) a price process of the form $S = M + A$ which is a semimartingale under P. Therefore,

$$\int_0^t h(s,S_s)dB_s - \int_0^t \frac{1}{Z_s}d\left[Z, \int_0^{\cdot} h(r,S_r)dB_r\right]_s \tag{31}$$

is a local martingale under Q.

A priori, one may not know the nature of the process Z_t, however, since Z_t is assumed to be a local martingale with $Z_0 = 1$ and which remains strictly positive, using the martingale representation theorem, one can propose the representation:

$$Z_t = 1 + \int_0^t H_s Z_s dB_s. \tag{32}$$

This formulation is well-defined and reasonable because Z_t is a local martingale, and it is adapted to a filtration generated by Brownian motion. By the Martingale Representation Theorem (see, e.g., [12], Chapter IV), any local martingale $\{M_t\}_{t\geq 0}$ adapted to such a filtration can be represented as a stochastic integral with respect to the Brownian motion. Also, given that Z_t is continuous (by [12] Chapter IV, Corollary 1 of Theorem 43), the process $J_s = H_s Z_s$ is predictable, since both H_s and Z_s are adapted and càdlàg, and H_s is predictable.

Equation (32) leads to a stochastic differential equation (SDE)

$$dZ_t = H_t Z_t dB_t.$$

The solution to SDE in this form is given by the Doléans-Dade exponential (see Definition 3.26) and in this case,

$$Z_t = \mathcal{E}\left(\int_0^t H_s dB_s\right).$$

It remains to determine H.

From Equation (31),

$$\begin{aligned} d[Z, \int_0^{\cdot} h(r, S_r) dB_r] &= d[1 + \int_0^{\cdot} H_s Z_s dB_s, \int_0^{\cdot} h(s, S_s) dB_s] \\ &= d[\int_0^{\cdot} H_s Z_s dB_s, \int_0^{\cdot} h(s, S_s) dB_s] \\ &= H_s Z_s h(s, S_s) d[B, B]_s \\ &= Z_s H_s h(s, S_s) ds \end{aligned}$$

This expression is as a result of a combine properties of stochastic integral and mutual variation.

Hence, Equation (31) becomes,

$$\int_0^t h(s, S_s) dB_s - \int_0^t \frac{1}{Z_s} Z_s H_s h(s, S_s) ds = \int_0^t h(s, S_s) dB_s - \int_0^t H_s h(s, S_s) ds \tag{33}$$

if we choose $H_s = -\frac{b(s, S_r; r \leq s)}{h(s, S_s)}$, this will result into a pricing process given by

$$S_t = \int_0^t h(s, S_s) dB_s + \int_0^t b(s, S_r; r \leq s) ds \tag{34}$$

which is a Q-local martingale. Lastly, if we Let W_t be the local martingale under Q defined by

$$W_t = B_t + \int_0^t \frac{b(s, S_r; r \leq s)}{h(s, S_s)} ds$$

with quadratic variation $[W, W]_t = [B, B]_t = t$. By Lévy's Theorem, it follows that W_t is a Q-Brownian motion. Thus, under Q, the process S_t satisfies the stochastic differential equation:

$$dS_t = h(t, S_t) dW_t. \tag{35}$$

In conclussion, for $R_t = 1$, we have shown that under the risk-neutral measure Q, the price process S_t satisfies Equations (35). And the discounted value V_t required to be a martingale under Q given by the conditional expectation $\mathbb{E}_Q[f(S_T)|\mathcal{F}_t]$, can be written as $\varphi(t, S_t)$, where φ solves the PDE:

$$\frac{1}{2} h^2(s, x) \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial \varphi}{\partial s} = 0, \quad \varphi(T, x) = f(x).$$

Moreover, We get the expression for the portfolio needed to hedge the option; the required trading strategy for the price of the risky and riskless asset respectively are

$$a_t = \frac{d\varphi(t, S_t)}{dS_t} \quad \text{and} \qquad b_t = V_t - a_t S_t \tag{36}$$

Specifically, a_t is the **delta hedge**, which represents the number of units of S_t needed to replicate the claim.

5.2 Asset Pricing

Suppose that the stock price of an asset S_t satisfies the SDE

$$dS_t = S_t[\mu dt + \sigma dB_t] \tag{37}$$

where $\mu = \mu(t, S_t)$ is the drift term representing the average rate of return of the stock and $\sigma(t, S_t)$ denotes the volatility, which quantifies the randomness in stock price movements. We also define the risk-free bond (riskless asset) R_t satisfying

$$dR_t = rR_t dt$$

that is $R_t = R_0 e^{\int_0^t r(s,S_s)\,ds}$, where $r = r(t, S_t)$ is the risk-free rate at which the riskless asset grows. Let T be the future time of maturity at which one has the option to buy (or sale) a stock for strike price K, The payoff of the option at time T is given by:

$$F(S_T) = (S_T - K)_+ = \begin{cases} S_T - K, & \text{if } S_T > K, \\ 0, & \text{if } S_T \le K. \end{cases}$$

the goal would be to a fair price $f(t, S_t)$ for the option at strike time $t < T$.

For the illustration, we demonstrated this for the buying of European **Call**, the process is similar for the case of the European **Put** except that the payoff of the option at maturity will be expected to be $(K - S_T)_+$. Also note that in general stochastic modelling, the terms r, μ, and σ can depend on either or both of the stock price S_t and time t. For example the stochastic volatility models (e.g., Heston) or stochastic interest rate models (e.g., Vasicek, CIR), they evolve dynamically as functions of time and the asset price. In simpler models like Black-Scholes, these parameters are assumed to be constant.

Following the asset price dynamic in Equation (37), where the Brownian motion B_t is defined on the probability space $(\Omega, \mathcal{F}, P)$, assuming there exists a probability measure Q that is equivalent to P such that under Q, the discounted stock price

$$\tilde{S}_t = \frac{S_t}{R_t}$$

is a martingale. Using the product rule and the Ito's Lemma, one can see that under P, the discounted stock price $\tilde{S}_t = S_t/R_t$ satisfies

$$d\tilde{S}_t = \tilde{S}_t\left[(\mu - r)dt + \sigma dB_t\right]. \tag{38}$$

Therefore, for an option with value $V_T = F(S_T)$ at maturity T, such that $\mathbb{E}_Q[R_T^{-1}] < \infty$ the right arbitrage-free portfolio value at time $t < T$ is

$$V_t = R_t \mathbb{E}_Q\left[R_T^{-1} F(S_T) \mid \mathcal{F}_t\right]. \tag{39}$$

Remark 6. The Girsanov theorem is used to establish the risk-neutral measure Q that is equivalent to P for which the discounted stock price is a Martingale. In particular, Under Q we have a Brownian motion W_t which satisfies:

$$dW_t = dB_t + \frac{\mu - r}{\sigma} dt. \tag{40}$$

Rewriting this gives $dB_t = dW_t - \frac{\mu-r}{\sigma} dt$ and substituting this into $\tilde{S}_t$ in Equation (38) yields

$$
\begin{aligned}
d\tilde{S}_t &= \tilde{S}_t \left[(\mu - r)dt + \sigma dB_t\right] \\
d\tilde{S}_t &= \tilde{S}_t \left[(\mu - r)dt + \sigma \left(dW_t - \frac{\mu - r}{\sigma} dt\right)\right] \\
d\tilde{S}_t &= \tilde{S}_t \left[(\mu - r)dt + \sigma dW_t - (\mu - r)dt\right] \\
d\tilde{S}_t &= \tilde{S}_t \sigma dW_t.
\end{aligned}
\tag{41}
$$

This shows that $\tilde{S}_t$ is a local martingale under Q, and in this case, it is also a true martingale (by Equation (3.9)).

Example 5.1. Suppose that S_t satisfies,

$$
dS_t = S_t \left[\mu dt + \sigma dB_t\right] \tag{42}
$$

and the bond rate r is constant. Suppose that the claim is the average stock price over the time interval $[0, T]$,

$$
V = \frac{1}{T} \int_0^T S_t \, dt
$$

In the new measure Q, the discounted stock price $\tilde{S}_t = e^{-rt} S_t$ satisfies

$$
d\tilde{S}_t = \sigma \tilde{S}_t dW_t
$$

where W_t is a Q-Brownian motion. From Equation (39), under the risk-neutral condition, the discounted portfolio value must satisfy

$$
\tilde{V}_t = \mathbb{E}_Q \left[e^{-rT} \frac{1}{T} \int_0^T S_s \, ds \,\middle|\, \mathcal{F}_t \right].
$$

Given that $\int_0^t S_s ds$ is $\mathcal{F}_t$-measurable, and by the linearity of expectation, we have

$$
\begin{aligned}
T e^{rT} \tilde{V}_t &= \int_0^t S_s ds + \mathbb{E}_Q \left[\int_t^T S_s ds \mid \mathcal{F}_t \right] \\
&= \int_0^t S_s ds + \int_t^T \mathbb{E}_Q \left[S_s \mid \mathcal{F}_t\right] ds \qquad \text{(Fubini theorem)}
\end{aligned}
\tag{43}
$$

By the Remark 6 above, $\tilde{S}_t$ is a Q- martingale, thus for $s > t$

$$
\mathbb{E}_Q \left[S_s \mid \mathcal{F}_t\right] = e^{rs} \mathbb{E}_Q \left[\tilde{S}_s \mid \mathcal{F}_t\right] = e^{rs} \tilde{S}_t = e^{r(s-t)} S_t
$$

Hence,

$$
\begin{aligned}
\int_0^t S_s ds + \int_t^T \mathbb{E}_Q \left[S_s \mid \mathcal{F}_t\right] ds &= \int_0^t S_s ds + S_t \int_t^T e^{r(s-t)} ds \\
&= \int_0^t S_s ds + \frac{e^{r(T-t)} - 1}{r} S_t \\
\implies \tilde{V}_t &= \frac{e^{-rT}}{T} \int_0^t S_s ds + \frac{e^{-rt} - e^{-rT}}{rT} S_t \\
&= \frac{e^{-rT}}{T} \int_0^t S_s ds + \frac{1 - e^{-r(T-t)}}{rT} \tilde{S}_t
\end{aligned}
\tag{44}
$$

Therefore,

$$
V_t = e^{rt} \tilde{V}_t = \frac{e^{-r(T-t)}}{T} \int_0^t S_s ds + \frac{e^{rt} - e^{-rT}}{rT} S_t \tag{45}
$$

This shows that at maturity T, $V_T = V$, and at 0,

$$V_0 = \frac{1 - e^{-rT}}{rT} S_0$$

Note: In the special case where the coefficients μ and σ are constants, Equation (42) reduces to the Geometric Brownian Motion (GBM) model.

5.2.1 Black-Scholes Equation

The Geometric Brownian Motion (GBM) has been widely used in financial mathematics in pricing theory. Specifically, the classical GBM is used in the Black-Scholes model for asset pricing. Suppose the price of an risky asset $\{S_t\}_{t\geq 0}$ follows the geometric Brownian motion under the measure P and satisfies

$$dS_t = \sigma S_t dB_t + \mu S_t dt; \qquad S_0 = 1. \tag{46}$$

Applying the Ito's Lemma will yield

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

Suppose the riskless asset (bond) satisfies

$$dR_t = rR_t dt, \qquad R_0 = 1 \quad \Rightarrow \quad R_t = e^{rt}$$

From Equation (6) we let $\theta = \frac{\mu - r}{\sigma}$, and by Girsanov theorem, we define the Q-Brownian motion $W_t = B_t + \theta t$ so that under Q,

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{47}$$

By FTAP, in a complete market, every contingent claim can be perfectly replicated by a trading strategy, and the discounted price of the asset (European option) should be a Q-martingale. Therefore, from Equation (39), for any $t \in [0, T]$, the price of a European call option with strike K and expiration T is given by:

$$\begin{aligned}
V_t &= R_t \mathbb{E}_Q \left[R_T^{-1} F(S_T) \mid \mathcal{F}_t \right] \\
V_t &= e^{-r(T-t)} \mathbb{E}_Q [F(S_T) \mid \mathcal{F}_t] \\
&= e^{-r(T-t)} \mathbb{E}_Q [(S_T - K)_+ \mid \mathcal{F}_t] \\
&= e^{-r(T-t)} \mathbb{E}_Q \left[(S_T - K) \mathbf{1}_{\{S_T \geq K\}} \mid \mathcal{F}_t \right] \\
&= e^{-r(T-t)} \left[\mathbb{E}_Q \left[S_T \mathbf{1}_{\{S_T \geq K\}} \mid \mathcal{F}_t \right] - K \mathbb{E}_Q \left[\mathbf{1}_{\{S_T \geq K\}} \mid \mathcal{F}_t \right] \right].
\end{aligned} \tag{48}$$

But under the risk-neutral measure Q, the stock price follows Equation (47), thus,

$$S_T = S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}$$

therefore, we can rewrite the expectation as:

$$V_t = e^{-r(T-t)} \left[S_t \mathbb{E}_Q \left[e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)} \mathbf{1}_{\{S_T \geq K\}} \mid \mathcal{F}_t \right] - K \mathbb{E}_Q \left[\mathbf{1}_{\{S_T \geq K\}} \mid \mathcal{F}_t \right] \right].$$

Using the lognormal property of S_T, the conditional expectation simplifies further, ultimately leading to the Black-Scholes pricing formula.

$$V_t = V(t, S_t) = S_t \mathcal{N}(d_1) - K e^{-r(T-t)} \mathcal{N}(d_2), \tag{49}$$

Where $\mathcal{N}(d_i)$ = cumulative standard normal distribution function and the terms d_1 and d_2 are defined as:

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Similarly, another alternative approach to viewing this construction is by simply establishing $\varphi(t,x)$ that satisfies the fundamental assumptions above and solves the Equation (29), (see e.g. [10]). However, for $R_t = 1$ (i.e $r = 0$), under the risk-neutral measure Q, the risky asset S_t satsisfying Equation (46) follows the SDE:

$$dS_t = \sigma S_t \, dW_t, \quad S_0 = x > 0,$$

where W_t is a Brownian motion under the risk-neutral measure Q. Let the terminal payoff at maturity T be given by $f(S_T)$, then, the option price at time t is:

$$V_t = \varphi(t,x) = \mathbb{E}_{\mathbb{Q}}[f(S_T) \mid S_t = x].$$

Using the Feynman-Kac theorem, the $\varphi(t,x)$ that solves the equation (29) for $h(t,x) = \sigma x$ is:

$$\varphi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(xe^{\sigma u\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)}\right) e^{-u^2/2} du.$$

Therefore, considering the European call option, $f(x) = (x-K)_+$ this yields

$$\begin{aligned} V_t = \varphi(x,t) &= x\mathcal{N}\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{x}{K} + \frac{1}{2}\sigma^2(T-t)\right)\right) \\ &\quad - K\mathcal{N}\left(\frac{1}{\sigma\sqrt{T-t}}\log\frac{x}{K} - \frac{1}{2}\sigma^2(T-t)\right) \\ &= S_t\mathcal{N}(d_1^0) - K\mathcal{N}(d_2^0), \end{aligned} \tag{50}$$

Again, for $i = 1, 2$, $\mathcal{N}(d_i^0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_i^0} e^{-u^2/2} \, du$ is the cumulative standard normal distribution function of d_i.

6 Conclusion

This research has established a more descriptive analysis of the Girsanov theorems. The classical Girsanov theorem (4.4) is the simplest case of the change of measure theorem. In the classical theorem, Girsanov established the change of measure theorem for the process of Brownian motion and also explicitly defined the density function Z for which the equivalent measure exists and is defined.

The demonstrations for applications of the Girsanov theorem in the context of financial mathematics that we have presented adopt the classical Girsanov theorem. Because in pricing theory, the asset prices are typically modeled as a stochastic process defined on the Brownian motion. In accordance with the FTAP, the Girsanov theorem allows one to change the real probability measure that the actual price dynamics follow to an equivalent measure in such a way that the drift contained in the price process under the original (real-world) measure P, is modified (removed) under an equivalent measure Q.

In a wider prospect, the semimartingale may not be a Brownian motion, in which case the general Girsanov theorem is applied. The general theorem, as shown in Remark (2) can be reduced to the classical Girsanov theorem. There are various applications of stochastic processes that require a change of measure; these include applications in stochastic control and signal processes, stochastic differential equations with drift, and others. For such broader studies of stochastic methods, the generalized Girsanov theorem in (4.7) and (4.9) can be adopted.

A major aspect of this study is the behavior of processes with jumps. Unlike continuous processes, jump processes require careful treatment—particularly in the context of stochastic integration. In Itô calculus, the integrand is typically evaluated at the left limits because the stochastic integral is defined as the limit of simple predictable processes. This construction ensures that the integrand is known just before the jump, preserving the adaptedness and measurability of the integral.

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