Change of Measure Theorems for Semimartingales and Stochastic Processes with Jumps

Amos Tochukwu Ezeh

Department of Information Engineering,
Computer Science and Mathematics
University of L'Aquila, Italy

July 23, 2025

Supervisor: Prof. Tobias Kuna

Outline

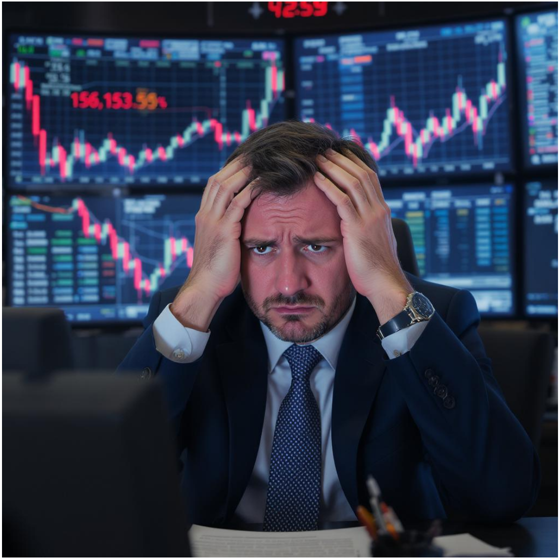
Overview

This study is a comprehensive study on the change of measure theorems for stochastic processes. Specifically, we conducted a systematic review of the following:

- Classical Girsanov Theorem
- Girsanov-Meyer Theorem
- Lenglart-Girsanov Theorem

Lastly, we discussed some applications of these theorems in finance for asset pricing.

Introduction



Generated from openart.ai

Basic Concepts and Definitions

Martingale Theory

A stochastic process $X = \{X_t\}_{t \in I}$ adapted to the filtration $\{\mathcal{F}_t\}_{t \in I}$ is a martingale if:

(i) it is measurable; i.e. $\forall t, \quad \mathbb{E}[|X|] < \infty$

(ii) for all $s \leq t$, $\mathbb{E}\left\{X_t \mid \mathcal{F}_s\right\} = X_s$

Semimartingale Theory

A stochastic process $X = \{X_t\}_{t \in I}$ is a semimartingale if it has the decomposition

$$X = M + A$$

where M is a local martingale (a process that behave like a martingale when it is stopped at a random time) and A is a finite variation (FV) process.

Stochastic Integration

stochastic integrals are integrals of the form

$$\int_0^t H_s dX_s$$

where both the integrand (H_t) and the integrator (X_t) are stochastic processes

Theorem (Radon-Nikodym's Theorem)

Let P be a σ-finite measure and Q a signed measure on the filtered space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_t)$, such that Q is absolutely continuous to P ($Q \ll P$). Then there exists a unique $f \in L^1(dP)$ such that

$$Q = \int_{\omega} f dP, \qquad \omega \in \Omega.$$

f is called the Radon-Nikodym's derivative of Q $w.r.t.$ P denoted by

$$f = \frac{dQ}{dP}.$$

Theorem (Novikov's Criterion)

Let M be a continuous local martingale, and suppose that

$$\mathbb{E}\left\{e^{\frac{1}{2}[M,M]_\infty}\right\} < \infty. \tag{1}$$

Then $\mathcal{E}(M)$ is a uniformly integrable martingale.

Protter (2005), Dellacherie, Claude, and Meyer (1978), Steele (2001)

Classical Girsanov Theorem

Theorem

Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion on a bounded interval $[0, T]$ defined on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_t)$. Let $\{\theta_t\}_{0\leq t\leq T}$ be an adapted, measurable process such that $\int_0^T \theta_s^2 ds < \infty$ a.s. and satisfies the Novikov's criterion. If there exists Q defined by the Radon-Nikodym's derivative

$$Z_T = \frac{dQ}{dP} = \exp\left(-\int_0^T \theta_s dB_s - \frac{1}{2}\int_0^T \theta_s^2 ds\right), \tag{2}$$

then, Q is a *probability measure* equivalent to P, the process

$$Z_t = \mathbb{E}_P[Z_T|\mathcal{F}_t] \tag{3}$$

is a *P-martingale,* and

$$W_t = B_t + \int_0^t \theta_s ds \tag{4}$$

is a *Brownian motion under Q*.

Steele (2001), Anthonelli (2024)

Generalized Girsanov Theorems

Now we present the first major extension of the Girsanov theorem.

Theorem (Girsanov-Meyer Theorem)

Let P and Q be equivalent. Let X be a semimartingale under P with decomposition $X = M + A$. Then, under Q, X is also a semimartingale with decomposition $X = N + B$, where

$$N_t = M_t - \int_0^t \frac{1}{Z_s} d[Z, M]_s \tag{5}$$

is a Q-local martingale, and

$$B_t = A_t + \int_0^t \frac{1}{Z_s} d[Z, M]_s \tag{6}$$

is a Q-FV process.

Lemma

Let X be an adapted stochastic process and suppose $Q \ll P$, then

$$\mathbb{E}_P \left[\frac{dQ}{dP} \middle| \mathcal{F}_t \right] \cdot \mathbb{E}_Q \left[X | \mathcal{F}_t \right] = \mathbb{E}_P \left[X \cdot \frac{dQ}{dP} \middle| \mathcal{F}_t \right]$$

Corollary

Suppose $Q \sim P$, the inverse process $\frac{1}{Z}$ is a cadlag version of

$$\frac{1}{Z_t} = \mathbb{E}_Q \left[\frac{dP}{dQ} \middle| \mathcal{F}_t \right]$$

Theorem (Lenglart-Girsanov Theorem)

Let Q be a probability measure absolutely continuous with respect to P, and let X be P-local martingale with $X_0 = 0$. Let $Z_t = \mathbb{E}_P \left[\frac{dQ}{dP} \mid \mathcal{F}_t \right]$, $R = \inf\{t > 0 : Z_t = 0, Z_{t-} > 0\}$ and define $U_t = \Delta X_R 1_{\{t \geq R\}}$, then the following is a Q-local martingale:

$$X_t - \int_0^t \frac{1}{Z_s} d[X, Z]_s + \tilde{U}_t. \tag{7}$$

Lemma

If X is continuous (for instance the Brownian motion), the quadratic covariation $\langle X, Z \rangle = [X, Z]^c$ is a predictable finite variation (FV) process and the integrand $\frac{1}{Z_s}$ in Theorem 9 is replaced by the left continuous version $\frac{1}{Z_{s-}}$.

Corollary

If X is a continuous P-local martingale and Q is absolutely continuous with respect to P, then $\langle X, Z \rangle = [X, Z]^c$ exists and there exists a predictable process α such that

$$X_t - \int_0^t \frac{1}{Z_{s-}} d[X, Z]_s^c = X_t - \int_0^t \alpha_s d[X, X]_s \tag{8}$$

is a Q-local martingale.

Protter (2005), Weizsäcker & Winkler (1980), Meyer (1976), Lenglart (1977)

Conclusion

Researchers find the Girsanov theorem challenging to understand, particularly in areas such as the decomposition of semimartingales, the construction Z_t, and the treatment of jump components in cases $Q \ll P$. Furthermore, the theorem is poorly adopted in some areas of research like biology because of the availability of little physical interpretation of the tool in those areas.

Addressing these challenges motivated this thesis. This study has established a descriptive analysis of the Girsanov theorems. The classical Girsanov theorem which is based on Brownian motion is the simplest case of the change of measure theorem. In general, the semimartingale may not be a Brownian motion, in which case the general Girsanov theorem is applied.

Despite the poor adoption of the Girsanov theorems, as mentioned earlier, these theorems are widely applicable in various fields. For example:

- **Asset Pricing**: Investment banks and asset managers like Goldman Sachs, J.P. Morgan, Morgan Stanley, and Intercontinental Exchange (ICE) apply the Girsanov theorem to risk-neutral pricing in derivative markets
- **Interest Rate Modeling**: For pricing interest rate swaps or bond options
- **Credit Risk and Default Modeling**: Credit risk analytics teams in major banks apply Girsanov theorems in modeling default times and pricing credit derivatives under a new measure
- **Stochastic Control and Filtering:** It is used in robotics by engineers, and it is used in signal processing by space agencies

References I

[1] Antonelli, F. (2024). *Stochastic Financial Market Models*. Lecture Notes, a.a. 2023–24.

[2] Dellacherie, C., & Meyer, P.-A. (1978). *Probability and Potential*. North Holland Publishing Company, Vol. 29.

[3] Lenglart, E. (1977). Transformation des Martingales locales par changement absolument continu de probabilités. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*.

[4] Meyer, P.-A. (1976). Un cours sur les intégrales stochastiques. *Séminaire de probabilités (Strasbourg)*, Tome 10, pp. 245–400.

[5] Protter, P. E. (2005). *Stochastic Integration and Differential Equations* (2nd ed.). Springer.

[6] Steele, J. M. (2001). *Stochastic Calculus and Financial Applications*. Springer.

[7] Weizsäcker, H., & Winkler, G. (1990). *Stochastic Integrals: An Introduction*. Vieweg+Teubner Verlag Wiesbaden.

Thank You!