Math 2120
Homework 2

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Question 1

Consider the following differential equation.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = t^2 y, y(0) = 1.$$

(a) Use Euler's Method with $\triangle t = .1$ to approximate y(1).

Euler's Method: $y_{i+1} = y_i + \triangle t f(y_i, t_i)$

$y_0 = 1$
$y_1 = 1 + 0.1 \times 0^2 \times 1 = 1$
$y_2 = 1 + 0.1 \times 0.1^2 \times 1 = 1.001$
.
.
.

0	1
0.1	1
0.2	1.001
0.3	1.005004
0.4	1.014049036
0.5	1.030273821
0.6	1.056030666
0.7	1.09404777
0.8	1.147656111
0.9	1.221106102
1	1.320015696

Solution: $y(1) = 1.320015696$

(b) Use Euler's Method with $\triangle t = .05$ to approximate y(1).
Euler's Method: $y_{i+1} = y_i + \triangle t f(y_i, t_i)$

$y_0 = 1$
$y_1 = 1 + 0.05 \times 0^2 \times 1 = 1$
$y_2 = 1 + 0.05 \times 0.05^2 \times 1 = 1.000125$
.
.
.

0	1
0.05	1
0.1	1.000125
0.15	1.000625063
0.2	1.001750766
0.25	1.003754267
0.3	1.006890999
0.35	1.011422009
0.4	1.017616969
0.45	1.025757904
0.5	1.036143703
0.55	1.049095499
0.6	1.064963069
0.65	1.084132404
0.7	1.107034701
0.75	1.134157051
0.8	1.166055218
0.85	1.203368985
0.9	1.24684069
0.95	1.297337738
1	1.355880103

Solution: $y(1) = 1.355880103$

(c) Find the exact Solution to the probelm. Use this solution to compare the error for different values of $\triangle t$. What does this say about this method?
Write $\frac{\mathrm{d}x}{\mathrm{d}y}$ as y' :

$$y' = t^2 y$$
$$\frac{1}{y} y' = t^2$$
$$ln(y) = \frac{t^3}{3} + C_1$$

plug in: $y(0) = 1$,we can get $C_1 = 0$

$$ln(y) = \frac{t^3}{3}$$
$$y = e^{\frac{t^3}{3}}$$

plug in t=1

$$y(1) = e^{\frac{1^3}{3}} = e^{\frac{1}{3}} = 1.395612425$$

Conclusion: The error when $\triangle t = 0.1$ is 0.075596729, the error when $\triangle t = 0.05$ is 0.039732322 which is $\frac{1}{2}$ of the error when is it is 0.1. So this method's complexity is O(N).

Question 2

Consider the following differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy+3x^2}{x^2}, \quad y(1) = 2.$$

(a)Show the equation is exact

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = -(2xy+3x^2)$$

$$(2xy+3x^2) + x^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}f}{\mathrm{d}y} = 2x, \quad \frac{\mathrm{d}g}{\mathrm{d}x} = 2x$$

$$\because 2x = 2x \therefore \text{the equation is exact}$$

(b)Find the solution

$$F_x = 2xy + 3x^2 \quad , \quad F = yx^2 + x^3 + g(x)$$

$$F_y = x^2 \quad , \quad F = x^2y + h(x)$$

So far we can get :

$$F(x,y) = x^2y + x^3$$

Plug in y(1)=2, which is (1,2):

$$F(1,2) = 1^2 \times 2 + 1^3 = 3$$

Solution is:

$$x^2y + x^3 = 3$$

Question 3

Consider the following differential equation

$$2xy + (y^2 - x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

(a)Show the equation is not exact.

$$\frac{\mathrm{d}f}{\mathrm{d}y} = 2x, \quad \frac{\mathrm{d}g}{\mathrm{d}x} = -2x$$

$$\because 2x = -2x \therefore \text{the equation is not exact}$$

(b)Find an integrating factor to make the equation exact.

$$\frac{M_y - N_x}{N} = \frac{4x}{y^2 - x^2}, \quad \frac{M_y - N_x}{N} = \frac{4x}{y^2 - x^2}$$

$$U(y)\frac{\mathrm{d}u}{\mathrm{d}y} = -\frac{2}{y}\mathrm{d}y$$

$$\int \frac{\mathrm{d}u}{u} = -\int \frac{2}{y}\mathrm{d}y$$

$$lnu = -2lny$$

$$u = \frac{1}{y^2}$$

(c)Find the general solution.

$$\frac{1}{y^2}2xy + \frac{y^2 - x^2}{y^2}\frac{dy}{dx} = 0$$

$$M = \frac{2x}{y}, \quad M_y = -\frac{2x}{y^2}$$

$$N = 1 - \frac{x^2}{y^2}, \quad N_x = -\frac{2x}{y^2}$$

$$N_x = M_y \Rightarrow EXACT$$

$$F_x = M = \frac{2x}{y} \Rightarrow \quad F = \frac{x^2}{y} + g(y)$$

$$F_y = N = 1 - \frac{x^2}{y^2} \Rightarrow \quad F = y + \frac{x^2}{y} + h(x)$$

$$\begin{cases} g(y) = y \\ h(x) = 0 \end{cases}$$

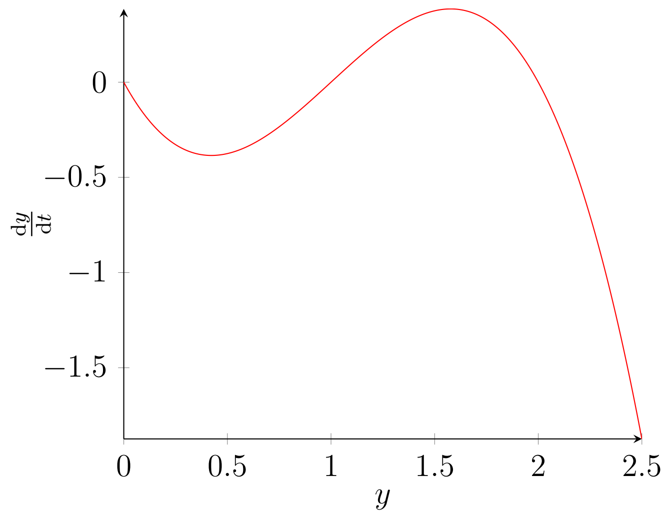
$$\therefore F = \frac{x^2}{y} + y$$

$$\therefore \frac{x^2}{y} + y = C$$

Question 4

Consider the following differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y(1-y)(2-y)$$



$$y = 0, 2 \quad : \textit{stable}$$

$$y = 1 \quad : \textit{unstable}$$

(a)If y(0) $= 0.5$ find $\lim_{t\to\infty} y(t)$ without solving the equation. Justify your answer.

$$\text{y(0)=0.5 , start at 0.5}$$

$$y' < 0 \Rightarrow \textit{decrease}$$

$$\textit{stable}$$

$$\lim_{t\to\infty} y(t) = 0$$

(b)If y(0) $= 1.5$ find $\lim_{t\to\infty} y(t)$ without solving the equation. Justify your answer.

$$y(0) = 1.5, y' > 0 \Rightarrow \textit{increase till } y = 2$$

$$\lim_{t\to\infty} y(t) = 2$$

(c)If y(0) $= 15$ find $\lim_{t\to\infty} y(t)$ without solving the equation. Justify your answer.

$$y(0) = 15, y' < 0 \Rightarrow \textit{decrease till } y = 2$$

$$\lim_{t\to\infty} y(t) = 2$$

Question 5

Consider a wild fish population which is harvest. We assume the fish poplulation is governed by a logistic growth when there is no harvest. The level of the harvest will be directly proportional to the population. The equation governing the system is then given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP(1 - \frac{P}{N}) - EP$$

where P(t) is the fish population, r is the growth rate, N is the carrying capacity and E is the fishing effort.

(a)Show if E < r , there are two equilibria $P_1 = 0$ and P_2 Find P_2 .

$$rP(1 - \frac{P}{N}) - EP = 0.$$

$$P(r - r\frac{P}{N} - E) = 0.$$

$P_1 = 0$

$$(r - r\frac{P}{N}) - E = 0$$

$$P = \frac{r - E}{r}N$$

Thus:

$$P_2 = \frac{r - E}{r}N$$

if E < r, both equilibria are physical .

(b)Determine the stability of the equilibria.

$$P'' = r - 2\frac{rP}{N} - E.$$

$$P''(0) = r - E$$

$\therefore$ When E>r , P_1 is stable; When E < r, P_1 is unstable.

$$P''(P_2) = E - r$$

so when $r > E$,P_2 is stable; when $r < E$,P_2 is unstable (c)The yield (or catch) is given by Y = EP. The sustainable yield is the yield's value at a stable equilibrium. In this problem the sustainable yield is a function of E the amount of effort put in to fishing. If E is very small, the catch will be small. If E is too large, the equilibrium population will go down. Find the optimum value of E that will maximize the catch.

$$stable\ equilibrium\ will\ be\ P_2 = N(1 - \frac{E}{r})$$

$$Y = N(E - \frac{E^2}{r})$$

FInd the derivative of yield Y:

$$Y' = N(1 - \frac{2E}{r})$$

$$N(1 - \frac{2E}{r}) = 0$$

$$E = \frac{r}{2}$$