

# Study guide: Finite difference methods for vibration problems

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## A simple vibration problem

$$u''(t) + \omega^2 u = 0, \quad u(0) = l, \quad u'(0) = 0, \quad t \in (0, T]$$

Exact solution:

$$u(t) = l \cos(\omega t)$$

$u(t)$  oscillates with constant amplitude  $l$  and (angular) frequency  $\omega$ . Period:  $P = 2\pi/\omega$ .

## A centered finite difference scheme; step 1 and 2

- Strategy: follow the **four steps** of the finite difference method.
- Step 1: Introduce a time mesh, here uniform on  $[0, T]$ :  
 $t_n = n\Delta t$
- Step 2: Let the ODE be satisfied at each mesh point:

$$u''(t_n) + \omega^2 u(t_n) = 0, \quad n = 1, \dots, N_t$$

## A centered finite difference scheme; step 3

Step 3: Approximate derivative(s) by finite difference approximation(s). Very common (standard!) formula for  $u''$ :

$$u''(t_n) \approx \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}$$

Use this discrete initial condition together with the ODE at  $t = 0$  to eliminate  $u^{-1}$ :

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} = -\omega^2 u^n$$

## A centered finite difference scheme; step 4

Step 4: Formulate the computational algorithm. Assume  $u^{n-1}$  and  $u^n$  are known, solve for unknown  $u^{n+1}$ :

$$u^{n+1} = 2u^n - u^{n-1} - \Delta t^2 \omega^2 u^n$$

Nick names for this scheme: Störmer's method or **Verlet integration**.

# Computing the first step

- The formula breaks down for  $u^1$  because  $u^{-1}$  is unknown and outside the mesh!
- And: we have not used the initial condition  $u'(0) = 0$ .

Discretize  $u'(0) = 0$  by a centered difference

$$\frac{u^1 - u^{-1}}{2\Delta t} = 0 \quad \Rightarrow \quad u^{-1} = u^1$$

Inserted in the scheme for  $n = 0$  gives

$$u^1 = u^0 - \frac{1}{2}\Delta t^2 \omega^2 u^0$$

# The computational algorithm

- ❶  $u^0 = I$
- ❷ compute  $u^1$
- ❸ for  $n = 1, 2, \dots, N_t - 1$ :
  - ❶ compute  $u^{n+1}$

More precisely expressed in Python:

```
t = linspace(0, T, Nt+1)  # mesh points in time
dt = t[1] - t[0]          # constant time step.
u = zeros(Nt+1)           # solution

u[0] = I
u[1] = u[0] - 0.5*dt**2*w**2*u[0]
for n in range(1, Nt):
    u[n+1] = 2*u[n] - u[n-1] - dt**2*w**2*u[n]
```

Note:  $w$  is consistently used for  $\omega$  in my code.



With  $[D_t D_t u]^n$  as the finite difference approximation to  $u''(t_n)$  we can write

$$[D_t D_t u + \omega^2 u = 0]^n$$

$[D_t D_t u]^n$  means applying a central difference with step  $\Delta t/2$  twice:

$$[D_t(D_t u)]^n = \frac{[D_t u]^{n+\frac{1}{2}} - [D_t u]^{n-\frac{1}{2}}}{\Delta t}$$

which is written out as

$$\frac{1}{\Delta t} \left( \frac{u^{n+1} - u^n}{\Delta t} - \frac{u^n - u^{n-1}}{\Delta t} \right) = \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}.$$

$$[u = I]^0, \quad [D_{2t}u = 0]^0$$

where  $[D_{2t}u]^n$  is defined as

$$[D_{2t}u]^n = \frac{u^{n+1} - u^{n-1}}{2\Delta t}.$$

$u$  is often displacement/position,  $u'$  is velocity and can be computed by

$$u'(t_n) \approx \frac{u^{n+1} - u^{n-1}}{2\Delta t} = [D_{2t}u]^n$$

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# Core algorithm

```
import numpy as np
import matplotlib.pyplot as plt

def solver(I, w, dt, T):
    """
    Solve  $u'' + w^2 u = 0$  for  $t$  in  $(0, T]$ ,  $u(0)=I$  and  $u'(0)=0$ ,
    by a central finite difference method with time step  $dt$ .
    """
    dt = float(dt)
    Nt = int(round(T/dt))
    u = np.zeros(Nt+1)
    t = np.linspace(0, Nt*dt, Nt+1)

    u[0] = I
    u[1] = u[0] - 0.5*dt**2*w**2*u[0]
    for n in range(1, Nt):
        u[n+1] = 2*u[n] - u[n-1] - dt**2*w**2*u[n]
    return u, t

def solver_adjust_w(I, w, dt, T, adjust_w=True):
    """
    Solve  $u'' + w^2 u = 0$  for  $t$  in  $(0, T]$ ,  $u(0)=I$  and  $u'(0)=0$ ,
    by a central finite difference method with time step  $dt$ .
    """
    dt = float(dt)
    Nt = int(round(T/dt))
    u = np.zeros(Nt+1)
    t = np.linspace(0, Nt*dt, Nt+1)
```

# Plotting

```
def u_exact(t, I, w):  
    return I*np.cos(w*t)  
  
def visualize(u, t, I, w):  
    plt.plot(t, u, 'r--o')  
    t_fine = np.linspace(0, t[-1], 1001) # very fine mesh for u_e  
    u_e = u_exact(t_fine, I, w)  
    plt.hold('on')  
    plt.plot(t_fine, u_e, 'b-')  
    plt.legend(['numerical', 'exact'], loc='upper left')  
    plt.xlabel('t')  
    plt.ylabel('u')  
    dt = t[1] - t[0]  
    plt.title('dt=%g' % dt)  
    umin = 1.2*u.min(); umax = -umin  
    plt.axis([t[0], t[-1], umin, umax])  
    plt.savefig('tmp1.png'); plt.savefig('tmp1.pdf')
```

# Main program

```
I = 1
w = 2*pi
dt = 0.05
num_periods = 5
P = 2*pi/w      # one period
T = P*num_periods
u, t = solver(I, w, dt, T)
visualize(u, t, I, w, dt)
```

## User interface: command line

```
import argparse
parser = argparse.ArgumentParser()
parser.add_argument('--I', type=float, default=1.0)
parser.add_argument('--w', type=float, default=2*pi)
parser.add_argument('--dt', type=float, default=0.05)
parser.add_argument('--num_periods', type=int, default=5)
a = parser.parse_args()
I, w, dt, num_periods = a.I, a.w, a.dt, a.num_periods
```



# Running the program

`vib_undamped.py`:

```
Terminal> python vib_undamped.py --dt 0.05 --num_periods 40
```

Generates frames `tmp_vib%04d.png` in files. Can make movie:

```
Terminal> ffmpeg -r 12 -i tmp_vib%04d.png -c:v flv movie.flv
```

Can use `avconv` instead of `ffmpeg`.

Format	Codec and filename
Flash	<code>-c:v flv movie.flv</code>
MP4	<code>-c:v libx264 movie.mp4</code>
Webm	<code>-c:v libvpx movie.webm</code>
Ogg	<code>-c:v libtheora movie.ogg</code>

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# First steps for testing and debugging

- **Testing very simple solutions:**  $u = \text{const}$  or  $u = ct + d$  do not apply here (without a force term in the equation:  $u'' + \omega^2 u = f$ ).
- **Hand calculations:** calculate  $u^1$  and  $u^2$  and compare with program.

# Checking convergence rates

The next function estimates convergence rates, i.e., it

- performs  $m$  simulations with halved time steps:  $2^{-k} \Delta t$ ,  $k = 0, \dots, m - 1$ ,
- computes the  $L_2$  norm of the error,  
$$E = \sqrt{\Delta t_i \sum_{n=0}^{N_t-1} (u^n - u_e(t_n))^2}$$
 in each case,
- estimates the rates  $r_i$  from two consecutive experiments  $(\Delta t_{i-1}, E_{i-1})$  and  $(\Delta t_i, E_i)$ , assuming  $E_i = C \Delta t_i^{r_i}$  and  $E_{i-1} = C \Delta t_{i-1}^{r_i}$ :

# Implementational details

```
def convergence_rates(m, solver_function, num_periods=8):  
    """  
    Return  $m-1$  empirical estimates of the convergence rate  
    based on  $m$  simulations, where the time step is halved  
    for each simulation.  
    solver_function( $I, w, dt, T$ ) solves each problem, where  $T$   
    is based on simulation for num_periods periods.  
    """  
    from math import pi  
    w = 0.35; I = 0.3          # just chosen values  
    P = 2*pi/w                 # period  
    dt = P/30                  # 30 time step per period 2*pi/w  
    T = P*num_periods  
  
    dt_values = []  
    E_values = []  
    for i in range(m):  
        u, t = solver_function(I, w, dt, T)  
        u_e = u_exact(t, I, w)  
        E = np.sqrt(dt*np.sum((u_e-u)**2))  
        dt_values.append(dt)  
        E_values.append(E)  
        dt = dt/2  
  
    r = [np.log(E_values[i-1]/E_values[i])/  
          np.log(dt_values[i-1]/dt_values[i])  
          for i in range(1, m, 1)]  
    return r, E_values, dt_values
```

# Unit test for the convergence rate

Use final  $r[-1]$  in a unit test:

```
def test_convergence_rates():
    r, E, dt = convergence_rates(
        m=5, solver_function=solver, num_periods=8)
    # Accept rate to 1 decimal place
    tol = 0.1
    assert abs(r[-1] - 2.0) < tol
    # Test that adjusted w obtains 4th order convergence
    r, E, dt = convergence_rates(
        m=5, solver_function=solver_adjust_w, num_periods=8)
    print 'adjust w rates:', r
    assert abs(r[-1] - 4.0) < tol

def plot_convergence_rates():
    r2, E2, dt2 = convergence_rates(
        m=5, solver_function=solver, num_periods=8)
    plt.loglog(dt2, E2)
    r4, E4, dt4 = convergence_rates(
        m=5, solver_function=solver_adjust_w, num_periods=8)
    plt.loglog(dt4, E4)
    plt.legend(['original scheme', r'adjusted  $\omega$ '],
               loc='upper left')
    plt.title('Convergence of finite difference methods')
    from plotslopes import slope_marker
    slope_marker((dt2[1], E2[1]), (2,1))
    slope_marker((dt4[1], E4[1]), (4,1))
    plt.savefig('tmp_convrate.png'); plt.savefig('tmp_convrate.pdf')
```

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# Effect of the time step on long simulations



- The numerical solution seems to have right amplitude.
- There is an angular frequency error (reduced by reducing the time step).
- The total angular frequency error seems to grow with time.



# Using a moving plot window

- In long time simulations we need a plot window that follows the solution.
- Method 1: `scitools.MovingPlotWindow`.
- Method 2: `scitools.avplotter` (ASCII vertical plotter).

Example:

```
Terminal> python vib_undamped.py --dt 0.05 --num_periods 40
```

Movie of the moving plot window.

!splot

- [Bokeh](#) is a Python plotting library for fancy web graphics
- Example here: long time series with many coupled graphs that can move simultaneously



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# Analysis of the numerical scheme

Can we understand the frequency error?



# Movie of the angular frequency error

$$u'' + \omega^2 u = 0, \quad u(0) = 1, \quad u'(0) = 0, \quad \omega = 2\pi, \quad u_e(t) = \cos(2\pi t), \\ \Delta t = 0.05 \text{ (20 intervals per period)}$$

*Movie 1: [https:](https://raw.githubusercontent.com/hplgit/fdm-book/master/doc/.src/chapters/vib/[[[mov-vib/vib_undamped_movie_dt0.05/movie.ogg)*

*[//raw.githubusercontent.com/hplgit/fdm-book/  
master/doc/.src/chapters/vib/\[\[\[mov-vib/  
vib\\_undamped\\_movie\\_dt0.05/movie.ogg](https://raw.githubusercontent.com/hplgit/fdm-book/master/doc/.src/chapters/vib/[[[mov-vib/vib_undamped_movie_dt0.05/movie.ogg)*

## We can derive an exact solution of the discrete equations

- We have a linear, homogeneous, difference equation for  $u^n$ .
- Has solutions  $u^n \sim IA^n$ , where  $A$  is unknown (number).
- Here:  $u_e(t) = I \cos(\omega t) \sim I \exp(i\omega t) = I(e^{i\omega\Delta t})^n$
- Trick for simplifying the algebra:  $u^n = IA^n$ , with  $A = \exp(i\tilde{\omega}\Delta t)$ , then find  $\tilde{\omega}$
- $\tilde{\omega}$ : unknown *numerical frequency* (easier to calculate than  $A$ )
- $\omega - \tilde{\omega}$  is the angular *frequency error*
- Use the real part as the physical relevant part of a complex expression

## Calculations of an exact solution of the discrete equations

$$u^n = lA^n = l \exp(\tilde{\omega} \Delta t n) = l \exp(\tilde{\omega} t) = l \cos(\tilde{\omega} t) + il \sin(\tilde{\omega} t).$$

$$\begin{aligned} [D_t D_t u]^n &= \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} \\ &= l \frac{A^{n+1} - 2A^n + A^{n-1}}{\Delta t^2} \\ &= l \frac{\exp(i\tilde{\omega}(t + \Delta t)) - 2 \exp(i\tilde{\omega} t) + \exp(i\tilde{\omega}(t - \Delta t))}{\Delta t^2} \\ &= l \exp(i\tilde{\omega} t) \frac{1}{\Delta t^2} (\exp(i\tilde{\omega}(\Delta t)) + \exp(i\tilde{\omega}(-\Delta t)) - 2) \\ &= l \exp(i\tilde{\omega} t) \frac{2}{\Delta t^2} (\cosh(i\tilde{\omega} \Delta t) - 1) \\ &= l \exp(i\tilde{\omega} t) \frac{2}{\Delta t^2} (\cos(\tilde{\omega} \Delta t) - 1) \\ &= -l \exp(i\tilde{\omega} t) \frac{4}{\Delta t^2} \sin^2\left(\frac{\tilde{\omega} \Delta t}{2}\right) \end{aligned}$$

## Solving for the numerical frequency

The scheme with  $u^n = l \exp(i\omega\tilde{\Delta}t n)$  inserted gives

$$-l \exp(i\tilde{\omega}t) \frac{4}{\Delta t^2} \sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right) + \omega^2 l \exp(i\tilde{\omega}t) = 0$$

which after dividing by  $l \exp(i\tilde{\omega}t)$  results in

$$\frac{4}{\Delta t^2} \sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right) = \omega^2$$

Solve for  $\tilde{\omega}$ :

$$\tilde{\omega} = \pm \frac{2}{\Delta t} \sin^{-1}\left(\frac{\omega\Delta t}{2}\right)$$

- Frequency error because  $\tilde{\omega} \neq \omega$ .
- Note: dimensionless number  $p = \omega\Delta t$  is the key parameter (i.e., no of time intervals per period is important, not  $\Delta t$  itself)
- But how good is the approximation  $\tilde{\omega}$  to  $\omega$ ?

# Polynomial approximation of the frequency error

Taylor series expansion for small  $\Delta t$  gives a formula that is easier to understand:

```
>>> from sympy import *
>>> dt, w = symbols('dt w')
>>> w_tilde = asin(w*dt/2).series(dt, 0, 4)*2/dt
>>> print w_tilde
(dt*w + dt**3*w**3/24 + O(dt**4))/dt  # note the final "/dt"
```

$$\tilde{\omega} = \omega \left( 1 + \frac{1}{24} \omega^2 \Delta t^2 \right) + \mathcal{O}(\Delta t^3)$$

The numerical frequency is too large (to fast oscillations).



# Plot of the frequency error



Recommendation: 25-30 points per period.

## Exact discrete solution

$$u^n = l \cos(\tilde{\omega} n \Delta t), \quad \tilde{\omega} = \frac{2}{\Delta t} \sin^{-1} \left( \frac{\omega \Delta t}{2} \right)$$

The error mesh function,

$$e^n = u_e(t_n) - u^n = l \cos(\omega n \Delta t) - l \cos(\tilde{\omega} n \Delta t)$$

is ideal for verification and further analysis!

$$e^n = l \cos(\omega n \Delta t) - l \cos(\tilde{\omega} n \Delta t) = -2l \sin \left( t \frac{1}{2} (\omega - \tilde{\omega}) \right) \sin \left( t \frac{1}{2} (\omega + \tilde{\omega}) \right)$$

# Convergence of the numerical scheme

Can easily show *convergence*:

$$e^n \rightarrow 0 \text{ as } \Delta t \rightarrow 0,$$

because

$$\lim_{\Delta t \rightarrow 0} \tilde{\omega} = \lim_{\Delta t \rightarrow 0} \frac{2}{\Delta t} \sin^{-1} \left( \frac{\omega \Delta t}{2} \right) = \omega,$$

by L'Hopital's rule or simply asking sympy: or [WolframAlpha](#):

```
>>> import sympy as sym
>>> dt, w = sym.symbols('x w')
>>> sym.limit((2/dt)*sym.asin(w*dt/2), dt, 0, dir='+')
w
```

## Observations:

- Numerical solution has constant amplitude (desired!), but an angular frequency error
- Constant amplitude requires  $\sin^{-1}(\omega\Delta t/2)$  to be real-valued  $\Rightarrow |\omega\Delta t/2| \leq 1$
- $\sin^{-1}(x)$  is complex if  $|x| > 1$ , and then  $\tilde{\omega}$  becomes complex

What is the consequence of complex  $\tilde{\omega}$ ?

- Set  $\tilde{\omega} = \tilde{\omega}_r + i\tilde{\omega}_i$
- Since  $\sin^{-1}(x)$  has a **\*negative\* imaginary part** for  $x > 1$ ,  $\exp(i\omega\tilde{t}) = \exp(-\tilde{\omega}_i t) \exp(i\tilde{\omega}_r t)$  leads to exponential growth  $e^{-\tilde{\omega}_i t}$  when  $-\tilde{\omega}_i t > 0$
- This is *instability* because the qualitative behavior is wrong

# The stability criterion

Cannot tolerate growth and must therefore demand a *stability criterion*

$$\frac{\omega \Delta t}{2} \leq 1 \quad \Rightarrow \quad \Delta t \leq \frac{2}{\omega}$$

Try  $\Delta t = \frac{2}{\omega} + 9.01 \cdot 10^{-5}$  (*slightly* too big!):



# Summary of the analysis

We can draw three important conclusions:

- ❶ The key parameter in the formulas is  $p = \omega \Delta t$  (dimensionless)
  - ❶ Period of oscillations:  $P = 2\pi/\omega$
  - ❷ Number of time steps per period:  $N_P = P/\Delta t$
  - ❸  $\Rightarrow p = \omega \Delta t = 2\pi/N_P \sim 1/N_P$
  - ❹ The smallest possible  $N_P$  is 2  $\Rightarrow p \in (0, \pi]$
- ❷ For  $p \leq 2$  the amplitude of  $u^n$  is constant (stable solution)
- ❸  $u^n$  has a relative frequency error  $\tilde{\omega}/\omega \approx 1 + \frac{1}{24}p^2$ , making numerical peaks occur too early

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## Rewriting 2nd-order ODE as system of two 1st-order ODEs

The vast collection of ODE solvers (e.g., in [Odespy](#)) cannot be applied to

$$u'' + \omega^2 u = 0$$

unless we write this higher-order ODE as a system of 1st-order ODEs.

Introduce an auxiliary variable  $v = u'$ :

$$u' = v, \tag{1}$$

$$v' = -\omega^2 u. \tag{2}$$

Initial conditions:  $u(0) = I$  and  $v(0) = 0$ .



# The Forward Euler scheme

We apply the Forward Euler scheme to each component equation:

$$\begin{aligned} [D_t^+ u &= v]^n, \\ [D_t^+ v &= -\omega^2 u]^n, \end{aligned}$$

or written out,

$$u^{n+1} = u^n + \Delta t v^n, \tag{3}$$

$$v^{n+1} = v^n - \Delta t \omega^2 u^n. \tag{4}$$

# The Backward Euler scheme

We apply the Backward Euler scheme to each component equation:

$$[D_t^- u = v]^{n+1}, \quad (5)$$

$$[D_t^- v = -\omega u]^{n+1}. \quad (6)$$

Written out:

$$u^{n+1} - \Delta t v^{n+1} = u^n, \quad (7)$$

$$v^{n+1} + \Delta t \omega^2 u^{n+1} = v^n. \quad (8)$$

This is a *coupled*  $2 \times 2$  system for the new values at  $t = t_{n+1}$ !

# The Crank-Nicolson scheme

$$[D_t u = \bar{v}^t]^{n+\frac{1}{2}}, \quad (9)$$

$$[D_t v = -\omega \bar{u}^t]^{n+\frac{1}{2}}. \quad (10)$$

The result is also a coupled system:

$$u^{n+1} - \frac{1}{2}\Delta t v^{n+1} = u^n + \frac{1}{2}\Delta t v^n, \quad (11)$$

$$v^{n+1} + \frac{1}{2}\Delta t \omega^2 u^{n+1} = v^n - \frac{1}{2}\Delta t \omega^2 u^n. \quad (12)$$

# Comparison of schemes via Odespy

Can use `Odespy` to compare many methods for first-order schemes:

```
import odespy
import numpy as np

def f(u, t, w=1):
    # v, u numbering for EulerCromer to work well
    v, u = u # u is array of length 2 holding our [v, u]
    return [-w**2*u, v]

def run_solvers_and_plot(solvers, timesteps_per_period=20,
                        num_periods=1, I=1, w=2*np.pi):
    P = 2*np.pi/w # duration of one period
    dt = P/timesteps_per_period
    Nt = num_periods*timesteps_per_period
    T = Nt*dt
    t_mesh = np.linspace(0, T, Nt+1)

    legends = []
    for solver in solvers:
        solver.set(f_kwargs={'w': w})
        solver.set_initial_condition([0, I])
        u, t = solver.solve(t_mesh)
```

# Forward and Backward Euler and Crank-Nicolson

```
solvers = [  
    odespy.ForwardEuler(f),  
    # Implicit methods must use Newton solver to converge  
    odespy.BackwardEuler(f, nonlinear_solver='Newton'),  
    odespy.CrankNicolson(f, nonlinear_solver='Newton'),  
]
```

Two plot types:

- $u(t)$  vs  $t$
- Parameterized curve  $(u(t), v(t))$  in *phase space*
- Exact curve is an ellipse:  $(I \cos \omega t, -\omega I \sin \omega t)$ , closed and periodic

# Phase plane plot of the numerical solutions



Note: CrankNicolson in Odespy leads to the name MidpointImplicit in plots.

# Plain solution curves

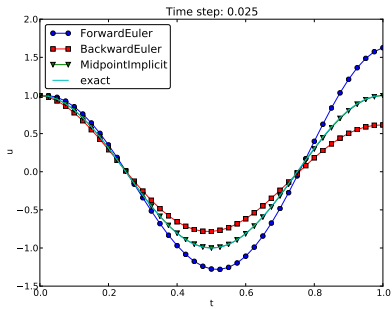


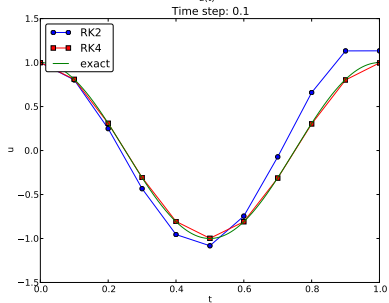
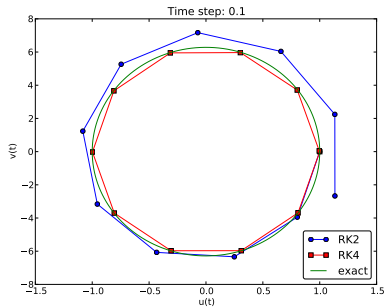
Figure: Comparison of classical schemes.

# Observations from the figures

- Forward Euler has growing amplitude and outward  $(u, v)$  spiral - pumps energy into the system.
- Backward Euler is opposite: decreasing amplitude, inward spiral, extracts energy.
- **Forward and Backward Euler are useless for vibrations.**
- Crank-Nicolson (MidpointImplicit) looks much better.



# Runge-Kutta methods of order 2 and 4; short time series



# Runge-Kutta methods of order 2 and 4; longer time series



# Crank-Nicolson; longer time series



(MidpointImplicit means CrankNicolson in Odespy)

# Observations of RK and CN methods

- 4th-order Runge-Kutta is very accurate, also for large  $\Delta t$ .
- 2th-order Runge-Kutta is almost as bad as Forward and Backward Euler.
- Crank-Nicolson is accurate, but the amplitude is not as accurate as the difference scheme for  $u'' + \omega^2 u = 0$ .

# Energy conservation property

The model

$$u'' + \omega^2 u = 0, \quad u(0) = I, \quad u'(0) = V,$$

has the nice *energy conservation property* that

$$E(t) = \frac{1}{2}(u')^2 + \frac{1}{2}\omega^2 u^2 = \text{const.}$$

This can be used to check solutions.

# Derivation of the energy conservation property

Multiply  $u'' + \omega^2 u = 0$  by  $u'$  and integrate:

$$\int_0^T u'' u' dt + \int_0^T \omega^2 u u' dt = 0.$$

Observing that

$$u'' u' = \frac{d}{dt} \frac{1}{2} (u')^2, \quad u u' = \frac{d}{dt} \frac{1}{2} u^2,$$

we get

$$\int_0^T \left( \frac{d}{dt} \frac{1}{2} (u')^2 + \frac{d}{dt} \frac{1}{2} \omega^2 u^2 \right) dt = E(T) - E(0),$$

where

$$E(t) = \frac{1}{2} (u')^2 + \frac{1}{2} \omega^2 u^2$$

## Remark about $E(t)$

$E(t)$  does not measure energy, energy per mass unit.

Starting with an ODE coming directly from Newton's 2nd law  $F = ma$  with a spring force  $F = -ku$  and  $ma = mu''$  ( $a$ : acceleration,  $u$ : displacement), we have

$$mu'' + ku = 0$$

Integrating this equation gives a physical energy balance:

$$E(t) = \underbrace{\frac{1}{2}mv^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}ku^2}_{\text{potential energy}} = E(0), \quad v = u'$$

Note: the balance is not valid if we add other terms to the ODE.

# The Euler-Cromer method; idea

2x2 system for  $u'' + \omega^2 u = 0$ :

$$v' = -\omega^2 u$$

$$u' = v$$

Forward-backward discretization:

- Update  $v$  with Forward Euler
- Update  $u$  with Backward Euler, using latest  $v$

$$[D_t^+ v = -\omega^2 u]^n \tag{13}$$

$$[D_t^- u = v]^{n+1} \tag{14}$$



# The Euler-Cromer method; complete formulas

Written out:

$$u^0 = l, \quad (15)$$

$$v^0 = 0, \quad (16)$$

$$v^{n+1} = v^n - \Delta t \omega^2 u^n \quad (17)$$

$$u^{n+1} = u^n + \Delta t v^{n+1} \quad (18)$$

Names: Forward-backward scheme, [Semi-implicit Euler method](#), symplectic Euler, semi-explicit Euler, Newton-Stormer-Verlet, and *Euler-Cromer*.

# Euler-Cromer is equivalent to the scheme for $u'' + \omega^2 u = 0$

- Forward Euler and Backward Euler have error  $\mathcal{O}(\Delta t)$
- What about the overall scheme? Expect  $\mathcal{O}(\Delta t)$ ...

We can eliminate  $v^n$  and  $v^{n+1}$ , resulting in

$$u^{n+1} = 2u^n - u^{n-1} - \Delta t^2 \omega^2 u^n$$

which is the centered finite difference scheme for  $u'' + \omega^2 u = 0$ !

The schemes are not equivalent wrt the initial conditions

$$u' = v = 0 \quad \Rightarrow \quad v^0 = 0,$$

so

$$v^1 = v^0 - \Delta t \omega^2 u^0 = -\Delta t \omega^2 u^0$$

$$u^1 = u^0 + \Delta t v^1 = u^0 - \Delta t \omega^2 u^0 = \underbrace{u^0 - \frac{1}{2} \Delta t \omega^2 u^0}_{\text{from } [D_t D_t u + \omega^2 u = 0]^n \text{ and } [D_{2t} u = 0]^0}$$

The exact discrete solution derived earlier does not fit the Euler-Cromer scheme because of mismatch for  $u^1$ .

- 1 A simple vibration problem
- 2 Implementation
- 3 Verification
- 4 Long time simulations
  - Long time simulations visualized with aid of Bokeh: coupled panning of multiple graphs
  - How does Bokeh plotting code look like?
- 5 Analysis of the numerical scheme
- 6 Alternative schemes based on 1st-order equations
- 7 Generalization: damping, nonlinear spring, and external excitation

# Generalization: damping, nonlinear spring, and external excitation

$$mu'' + f(u') + s(u) = F(t), \quad u(0) = I, \quad u'(0) = V, \quad t \in (0, T]$$

Input data:  $m$ ,  $f(u')$ ,  $s(u)$ ,  $F(t)$ ,  $I$ ,  $V$ , and  $T$ .

Typical choices of  $f$  and  $s$ :

- linear damping  $f(u') = bu$ , or
- quadratic damping  $f(u') = bu'|u'|$
- linear spring  $s(u) = cu$
- nonlinear spring  $s(u) \sim \sin(u)$  (pendulum)

## A centered scheme for linear damping

$$[mD_t D_t u + f(D_{2t} u) + s(u) = F]^n$$

Written out

$$m \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + f\left(\frac{u^{n+1} - u^{n-1}}{2\Delta t}\right) + s(u^n) = F^n$$

Assume  $f(u')$  is linear in  $u' = v$ :

$$u^{n+1} = \left( 2mu^n + \left(\frac{b}{2}\Delta t - m\right)u^{n-1} + \Delta t^2(F^n - s(u^n)) \right) (m + \frac{b}{2}\Delta t)^{-1}$$

$$u(0) = I, \quad u'(0) = V:$$

$$\begin{aligned}[u = I]^0 &\Rightarrow u^0 = I \\ [D_{2t} u = V]^0 &\Rightarrow u^{-1} = u^1 - 2\Delta t V\end{aligned}$$

End result:

$$u^1 = u^0 + \Delta t V + \frac{\Delta t^2}{2m}(-bV - s(u^0) + F^0)$$

Same formula for  $u^1$  as when using a centered scheme for  $u'' + \omega u = 0$ .

# Linearization via a geometric mean approximation

- $f(u') = bu'|u'|$  leads to a quadratic equation for  $u^{n+1}$
- Instead of solving the quadratic equation, we use a geometric mean approximation

In general, the geometric mean approximation reads

$$(w^2)^n \approx w^{n-\frac{1}{2}} w^{n+\frac{1}{2}}.$$

For  $|u'|u'$  at  $t_n$ :

$$[u'|u'|]^n \approx u'(t_n + \frac{1}{2})|u'(t_n - \frac{1}{2})|.$$

For  $u'$  at  $t_{n\pm 1/2}$  we use centered difference:

$$u'(t_{n+1/2}) \approx [D_t u]^{n+\frac{1}{2}}, \quad u'(t_{n-1/2}) \approx [D_t u]^{n-\frac{1}{2}}$$



## A centered scheme for quadratic damping

After some algebra:

$$u^{n+1} = (m + b|u^n - u^{n-1}|)^{-1} \times \\ (2mu^n - mu^{n-1} + bu^n|u^n - u^{n-1}| + \Delta t^2(F^n - s(u^n)))$$

## Initial condition for quadratic damping

Simply use that  $u' = V$  in the scheme when  $t = 0$  ( $n = 0$ ):

$$[mD_t D_t u + bV|V| + s(u) = F]^0$$

which gives

$$u^1 = u^0 + \Delta t V + \frac{\Delta t^2}{2m} (-bV|V| - s(u^0) + F^0)$$

# Algorithm

- ①  $u^0 = I$
- ② compute  $u^1$  (formula depends on linear/quadratic damping)
- ③ for  $n = 1, 2, \dots, N_t - 1$ :
  - ① compute  $u^{n+1}$  from formula (depends on linear/quadratic damping)

# Implementation

```
def solver(I, V, m, b, s, F, dt, T, damping='linear'):
    dt = float(dt); b = float(b); m = float(m) # avoid integer div.
    Nt = int(round(T/dt))
    u = zeros(Nt+1)
    t = linspace(0, Nt*dt, Nt+1)

    u[0] = I
    if damping == 'linear':
        u[1] = u[0] + dt*V + dt**2/(2*m)*(-b*V - s(u[0]) + F(t[0]))
    elif damping == 'quadratic':
        u[1] = u[0] + dt*V + \
            dt**2/(2*m)*(-b*V*abs(V) - s(u[0]) + F(t[0]))

    for n in range(1, Nt):
        if damping == 'linear':
            u[n+1] = (2*m*u[n] + (b*dt/2 - m)*u[n-1] +
                dt**2*(F(t[n]) - s(u[n])))/(m + b*dt/2)
        elif damping == 'quadratic':
            u[n+1] = (2*m*u[n] - m*u[n-1] + b*u[n]*abs(u[n] - u[n-1])
                + dt**2*(F(t[n]) - s(u[n])))/\
                (m + b*abs(u[n] - u[n-1]))

    return u, t
```

- Constant solution  $u_e = I$  ( $V = 0$ ) fulfills the ODE problem and the discrete equations. Ideal for debugging!
- Linear solution  $u_e = Vt + I$  fulfills the ODE problem and the discrete equations.
- Quadratic solution  $u_e = bt^2 + Vt + I$  fulfills the ODE problem and the discrete equations with linear damping, but not for quadratic damping. A special discrete source term can allow  $u_e$  to also fulfill the discrete equations with quadratic damping.

# Demo program

`vib.py` supports input via the command line:

```
Terminal> python vib.py --s 'sin(u)' --F '3*cos(4*t)' --c 0.03
```

This results in a **moving window** following the function on the screen.



We rewrite

$$mu'' + f(u') + s(u) = F(t), \quad u(0) = I, \quad u'(0) = V, \quad t \in (0, T]$$

as a first-order ODE system

$$u' = v$$

$$v' = m^{-1} (F(t) - f(v) - s(u))$$

# Staggered grid

- $u$  is unknown at  $t_n$ :  $u^n$
- $v$  is unknown at  $t_{n+1/2}$ :  $v^{n+\frac{1}{2}}$
- All derivatives are approximated by centered differences

$$[D_t u = v]^{n-\frac{1}{2}}$$

$$[D_t v = m^{-1} (F(t) - f(v) - s(u))]^n$$

Written out,

$$\frac{u^n - u^{n-1}}{\Delta t} = v^{n-\frac{1}{2}}$$
$$\frac{v^{n+\frac{1}{2}} - v^{n-\frac{1}{2}}}{\Delta t} = m^{-1} (F^n - f(v^n) - s(u^n))$$

Problem:  $f(v^n)$



With  $f(v) = bv$ , we can use an arithmetic mean for  $bv^n$  a la Crank-Nicolson schemes.

$$u^n = u^{n-1} + \Delta t v^{n-\frac{1}{2}},$$
$$v^{n+\frac{1}{2}} = \left(1 + \frac{b}{2m}\Delta t\right)^{-1} \left(v^{n-\frac{1}{2}} + \Delta t m^{-1} \left(F^n - \frac{1}{2}f(v^{n-\frac{1}{2}}) - s(u^n)\right)\right)$$

## Quadratic damping

With  $f(v) = b|v|v$ , we can use a geometric mean

$$b|v^n|v^n \approx b|v^{n-\frac{1}{2}}|v^{n+\frac{1}{2}},$$

resulting in

$$\begin{aligned} u^n &= u^{n-1} + \Delta t v^{n-\frac{1}{2}}, \\ v^{n+\frac{1}{2}} &= \left(1 + \frac{b}{m}|v^{n-\frac{1}{2}}|\Delta t\right)^{-1} \left(v^{n-\frac{1}{2}} + \Delta t m^{-1}(F^n - s(u^n))\right). \end{aligned}$$

$$u^0 = I$$

$$v^{\frac{1}{2}} = V - \frac{1}{2}\Delta t \omega^2 I$$