M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 16: More on the Wave Equation

The Damped Wave Equation

Consider the equation

$$u_{tt} + 2cu_t = \beta^2 u_{xx}$$

where c is a small positive constant. The term $2cu_t$ represents a damping force proportional to the velocity u_t. For simplicity, we will assume that the length of the string is $L = \pi$, and the constant $\beta^2 = 1$.

If we let $u(x,t) = X(x)T(t)$, then

$$\frac{XT'' + 2cXT'}{XT} = \frac{X''T}{XT} \Rightarrow \frac{T'' + 2cT'}{T} = \frac{X''}{X} = -\lambda.$$

This gives us the two ordinary differential equations

$$X'' + \lambda X = 0, \quad T'' + 2cT' + \lambda T = 0.$$

If we assume the two ends of the string are fixed, so $u(0,t) = u(\pi,t) = 0$ for all $t > 0$, then we again have the Sturm-Liouville problem

$$X'' + \lambda X = 0, \quad X(0) = X(\pi) = 0$$

with eigenvalues $\lambda_n = \frac{n^2\pi^2}{\pi^2} = n^2$ and corresponding eigenfunctions $X_n(x) = b_n \sin(nx)$. The equation for $T_n(t)$ is

$$T_n'' + 2cT_n' + n^2 T_n = 0,$$

with characteristic polynomial $r^2 + 2cr + n^2 = 0$. We will assume that $c < 1$, and then

$$r = \frac{-2c \pm \sqrt{4c^2 - 4n^2}}{2} = -c \pm \sqrt{c^2 - n^2}.$$

The discriminant $c^2 - n^2 < 0$ for any integer $n \geq 1$, so the roots are complex and the general solution is

$$T_n(t) = a_n e^{-ct} \cos(\sqrt{n^2 - c^2}\ t) + b_n e^{-ct} \sin(\sqrt{n^2 - c^2}\ t).$$

This means that

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx) e^{-ct} \left[a_n \cos(\sqrt{n^2 - c^2}\ t) + b_n \sin(\sqrt{n^2 - c^2}\ t) \right].$$

If the initial conditions are $u(x,0)=f(x),\ \ u_t(x,0)=g(x)$, then

$$u(x,0)=\sum_{n=1}^{\infty} a_n \sin(nx) = f(x)$$

implies that $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$. The partial of u with respect to t is

$$\begin{aligned} u_t(x,t) = \sum_{n=1}^{\infty} & \sin(nx)\left[-ce^{-ct}\left(a_n\cos(\sqrt{n^2-c^2}\ t)+b_n\sin(\sqrt{n^2-c^2}\ t)\right)\right] \\ & +\sin(nx)\left[e^{-ct}\sqrt{n^2-c^2}\left(-a_n\sin(\sqrt{n^2-c^2}\ t)+b_n\cos(\sqrt{n^2-c^2}\ t)\right)\right]; \end{aligned}$$

therefore,

$$u_t(x,0)=\sum_{n=1}^{\infty}\sin(nx)\left[-ca_n+\sqrt{n^2-c^2}\ b_n\right]\equiv g(x).$$

This is a Fourier Sine Series for $g(x)$ with coefficients

$$-ca_n+\sqrt{n^2-c^2}\ b_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)dx;$$

therefore,

$$b_n=\frac{1}{\sqrt{n^2-c^2}}\left(ca_n+\frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)dx\right).$$

The solution of $u_{tt}+2cu_t=u_{xx},\ \ u(0,t)=u(\pi,t)=0$, for all $t>0$ with the given initial conditions is

$$\boxed{\begin{aligned} & u(x,t)=\sum_{n=1}^{\infty}\sin(nx)e^{-ct}\left[a_n\cos(\sqrt{n^2-c^2}\ t)+b_n\sin(\sqrt{n^2-c^2}\ t)\right], \\ & a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx,\ \ b_n=\frac{1}{\sqrt{n^2-c^2}}\left(ca_n+\frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)dx\right) \end{aligned}}$$

Example 1 *We will redo the "plucked" string problem from Lecture 15, letting*

$$f(x)=\begin{cases} \frac{2}{\pi}x & 0\le x<\frac{\pi}{2} \\ -\frac{2}{\pi}(x-\pi) & \frac{\pi}{2}\le x\le\pi \end{cases}$$

and $g(x)\equiv 0$.

A graph of $u(x,t)$ at $t = 0, \frac{\pi}{8}, \frac{\pi}{4}, \cdots \pi$, and a 3-dimensional plot of $u(x,t)$ for $0 \le t \le 2\pi$ are shown in Figure 1.

It can be seen that with c equal to 0.2, the string moves down and back once as t goes from 0 to approximately 2π. However the solution is no longer a sum of exact harmonics; that is, the sines and cosines in successive terms do not have frequencies that are exact integral multiples of a fundamental frequency. In addition, the oscillations in the string damp out to zero as $t \to \infty$.

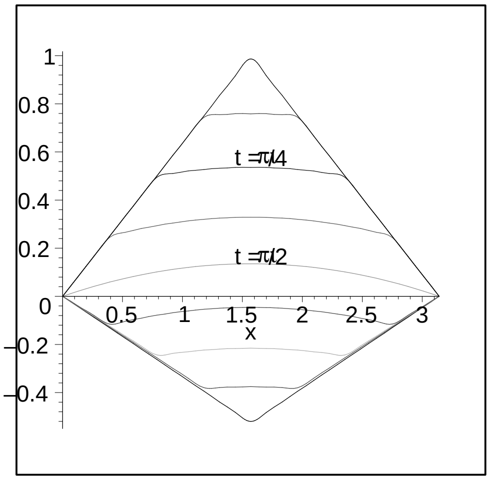

Figure 1: (a) Position of the string at times $t = 0, \frac{\pi}{8}, \frac{\pi}{4} \cdots \pi$

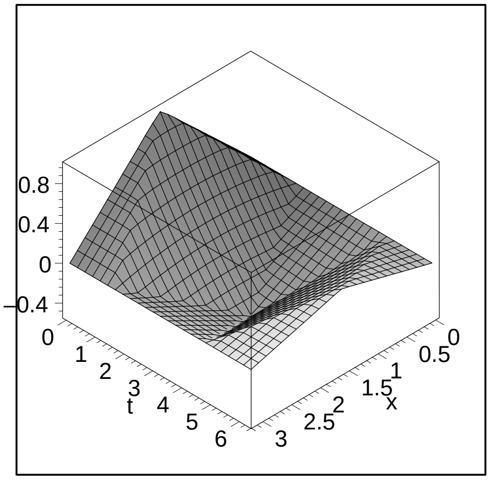

Figure 1: (b) A 3-dimensional plot of the function $u(x,t)$ over 1 period

D'Alembert's Solution of the Wave Equation

To solve the wave equation $u_{tt} = \beta^2 u_{xx}$ on the infinite line $-\infty < x < \infty, t > 0$, it is easier to make the change of independent variables

$$r = x + \beta t, \quad s = x - \beta t,$$

and let $u(x,t) = w(r,s) = w(r(x,t), s(x,t))$.

To find u_{xx} and u_{tt} in terms of the new variables r and s, we will need the four derivatives

$$\frac{\partial r}{\partial x} = 1, \quad \frac{\partial s}{\partial x} = 1, \quad \frac{\partial r}{\partial t} = \beta, \quad \frac{\partial s}{\partial t} = -\beta.$$

Then, using the Chain Rule for differentiation of a function of a function

$$u_t \equiv \frac{\partial u}{\partial t} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial t} = \beta(w_r - w_s)$$

and

$$u_{tt} = \frac{\partial}{\partial t}(\beta(w_r - w_s)) = \beta(w_{rr}\frac{\partial r}{\partial t} + w_{rs}\frac{\partial s}{\partial t} - w_{sr}\frac{\partial r}{\partial t} - w_{ss}\frac{\partial s}{\partial t})$$
$$= \beta(\beta w_{rr} - \beta w_{rs} - \beta w_{sr} + \beta w_{ss}) = \beta^2(w_{rr} - 2w_{rs} + w_{ss}).$$

Differentiating with respect to x,

$$u_x \equiv \frac{\partial u}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} = w_r + w_s$$

and

$$u_{xx} = \frac{\partial}{\partial x}(w_r + w_s) = w_{rr}\frac{\partial r}{\partial x} + w_{rs}\frac{\partial s}{\partial x} + w_{sr}\frac{\partial r}{\partial x} + w_{ss}\frac{\partial s}{\partial x} = w_{rr} + 2w_{rs} + w_{ss}.$$

If the formulas for u_{tt} and u_{xx} are substituted into the wave equation $u_{tt} = \beta^2 u_{xx}$,

$$\beta^2(w_{rr} - 2w_{rs} + w_{ss}) = \beta^2(w_{rr} + 2w_{rs} + w_{ss})$$

and cancelling like terms on each side gives $4\beta^2 w_{rs} = 0$. Since β is a non-zero constant, this implies that

$$w_{rs}(r, s) \equiv 0.$$

This is easily solved by integrating twice, first with respect to s:

$$w_r(r, s) = p(r)$$

where p is an arbitrary function of r (since for any such function $\frac{\partial p}{\partial s} = 0$). A second integration with respect to r gives:

$$w(r, s) = \int p(r)dr + G(s),$$

where G is an arbitrary function of s. Now we can write $w(r, s) = F(r) + G(s)$, and therefore

$$\boxed{u(x, t) = F(x + \beta t) + G(x - \beta t)}$$

where F and G can be any functions of a **single** variable. This is the **general solution** of the wave equation on the infinite line.

Now suppose we are given initial conditions $u(x, 0) = f(x), \quad u_t(x, 0) = g(x)$. The partial of u with respect to t is

$$u_t(x, t) = \frac{\partial}{\partial t}\left(F(x + \beta t) + G(x - \beta t)\right) = F'(x + \beta t) \cdot \beta + G'(x - \beta t) \cdot (-\beta).$$

Remember that F and G are functions of *one* variable. At $t = 0$ we have

$$u(x,0) = F(x) + G(x) \equiv f(x) \tag{1}$$

The initial velocity condition gives

$$u_t(x,0) = \beta F'(x) - \beta G'(x) = \beta \frac{d}{dx}(F(x) - G(x)) \equiv g(x).$$

Integrating the last equality,

$$F(x) - G(x) = \frac{1}{\beta} \int_0^x g(\tau) d\tau + C. \tag{2}$$

If we add equations (1) and (2) we get

$$2F(x) = f(x) + \frac{1}{\beta} \int_0^x g(\tau) d\tau + C \Rightarrow F(x) = \frac{1}{2} \left(f(x) + \frac{1}{\beta} \int_0^x g(\tau) d\tau + C \right).$$

Subtracting equations (1) and (2) gives

$$2G(x) = f(x) - \frac{1}{\beta} \int_0^x g(\tau) d\tau - C \Rightarrow G(x) = \frac{1}{2} \left(f(x) - \frac{1}{\beta} \int_0^x g(\tau) d\tau - C \right).$$

Using these formulas for $F(x)$ and $G(x)$, the function u can be written as

$$u(x,t) = F(x + \beta t) + G(x - \beta t)$$

$$= \frac{1}{2} \left(f(x + \beta t) + \frac{1}{\beta} \int_0^{x+\beta t} g(\tau) d\tau + C \right) + \frac{1}{2} \left(f(x - \beta t) - \frac{1}{\beta} \int_0^{x-\beta t} g(\tau) d\tau - C \right),$$

and the exact *analytic* solution of the heat equation is

$$\boxed{u(x,t) = \frac{1}{2} \left(f(x + \beta t) + f(x - \beta t) \right) + \frac{1}{2\beta} \int_{x-\beta t}^{x+\beta t} g(\tau) d\tau.}$$

Example 2 *To study the behavior of a wave on an infinite line, we will assume the initial position is given by*

$$u(x,0) \equiv f(x) = \begin{cases} 1 & -1 \le x \le 1 \\ 0 & \textit{otherwise} \end{cases}$$

and the initial velocity $g(x)$ is identically zero. Let the constant velocity $\beta = 1$.

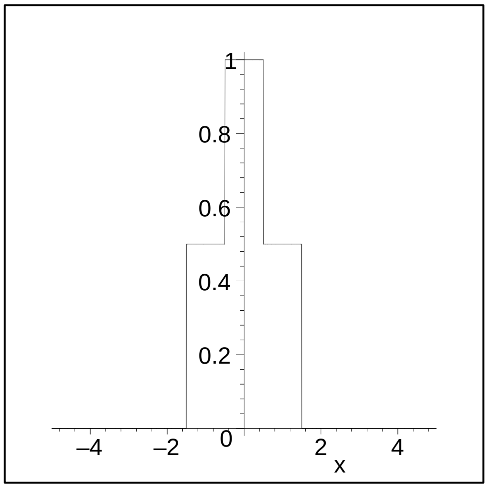

Figure 1: Graph of u at time $t = 0.5$

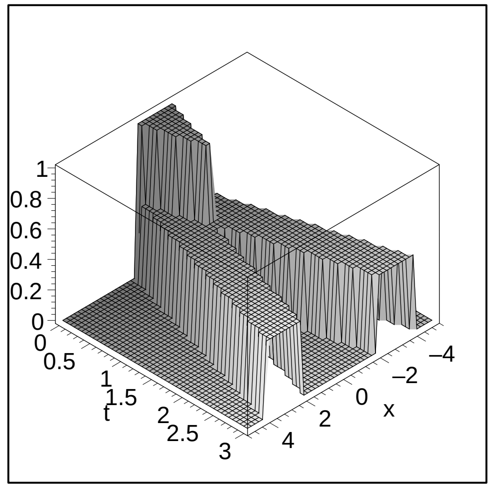

Figure 2: 3-dimensional plot of u

In Figure (2), the position of the wave at time $t = 0.5$ is shown on the left, and the 3-dimensional plot shows the wave moving out along the infinite line. At any time $\bar{t}$ the function $u(x, \bar{t})$ is equal to $\frac{f(x+\bar{t})+f(x-\bar{t})}{2}$. This can be graphed by moving the graph of f $\bar{t}$ units to the right and $\bar{t}$ units to the left, adding the two functions and dividing by two. When $t > 1$ the two graphs are completely separated, and the graph of u consists of two copies of $f(t)/2$ moving to the right and left with speed determined by β.

Practice Problems:

1. * Find the solution of $u_{tt} + 0.4u_t = u_{xx}$ if $u(x, 0) \equiv 0$ and

$$u_t(x, 0) = \begin{cases} 0 & 0 \le x < 0.4\pi \\ 0.2 & 0.4\pi \le x \le 0.6\pi \\ 0 & 0.6\pi < x \le \pi \end{cases}$$

   This is the damped version of the piano string, struck at time $t = 0$.

2. * Solve the equation $u_{tt} = \beta^2 u_{xx}$ on the infinite line, given the initial condition $u(x, 0) = \sin(x)$, $u_t(x, 0) \equiv 0$. Describe the behavior of $u(x, t)$ for $t > 0$. This is a model of a standing wave with no boundaries. How does the value of β affect the wave?

3. * Use MAPLE to make a 3-dimensional plot of the solution of Problem #2, for $-20 \le x \le 20$, $0 \le t \le 6\pi$. Do the plot for $\beta = 1.0$ and for $\beta = 0.5$.