Repulsive Force in the Point Charge – Neutral Metal System

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We investigate the interaction between a point charge and a neutral metallic object in electrostatics. The two-dimensional system consists of a infinitesimally thin metallic semicircle concave up, centered at the origin, and a point charge lies on the center axis of the semicircle. When the charge is still far away from the metallic object, the electrostatic force is attractive and will pull down the charge towards the metal. However, once the distance between the charge and the center of the semicircle is small enough, the force becomes repulsive and will push the charge back to its initial position. This special phenomenon can provide an invisible barrier, restricting the movement of particles.

I. INTRODUCTION

One classic problem in electrostatics is the interaction between a charged particle and a perfectly conducting, neutral metallic object. Let's say we have a system consists of a point charge q situated a distance a from the center of a conducting sphere of radius R. This configuration is just the same as lining up the source charge with two image charges situated inside the imaginary sphere – one at the center providing an equivalent potential at the surface of the sphere, and the other one playing the role of induced charges, located at a distance R^2/a from the center with its quantity $-Rq/a$. Such a problem can be cleverly solved using the method of images, and as expected, after summing all the forces exerted by the two image charges, the net force is attractive. Generally speaking, the electric force will be attractive, which make sense intuitively: the opposite charge will be induced closer to the source charge, resulting an attractive force. It seems to be a general result of this kind of system. Amazingly, it is not the whole story.

For precision, we need to specify what is an *attractive* force exactly. We use an imaginary plane $z = 0$ to separate these two objects, where the source charge q is in the upper space of the plane, and the conductor lies in the lower one. With this assumption, it is obvious to see that when the source charge is attracted, it moves *down* and the force F_z acting on it is therefore *negative*.

This time, we consider a special geometry of a infinitesimally thin metallic semicircle concave up and centered at the origin. A point charge is placed on the z-axis above the $z = 0$ plane. This geometry will cause a repulsive force when the point charge lies at a small distance just above the cross section.

It is useful to analyse the system from its electric field lines and the electrostatic energy. First, when the charged particle is located at origin, i.e. $z = 0$, its electric field lines radiate out and are perpendicular to the metallic semicircle. Since the equipotential surface at the distance R from the point charge forms a circle which perfectly overlaps the extremely thin metal, it solves the relevant Poisson equation with corresponding boundary conditions. Uniqueness theorem guarantees that there will be only one specific solution to this boundary condition, which is totally the same as the electric field and potential for a point charge at infinity. Therefore, both fields as well as the energy are identical, i.e. $U(z = 0) = U(z = \infty)$, since the energy in electrostatics is given by

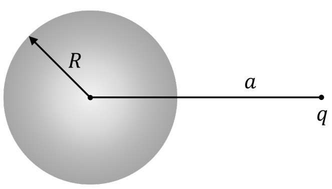

FIG. 1: Example of a point charge – neutral metal system which can be solved using the method of images.

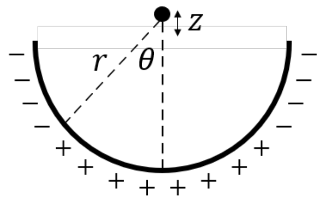

FIG. 2: The side view of a point charge resides just next to an infinitesimally thin semicircle with induced charge.

$$U = \frac{1}{8\pi} \int E^2 d\tau \tag{1}$$

which only depends on the field itself. We note that what we really care about is the difference in the energy between two points, $\Delta U = U(z) - U(\infty)$, instead of the exact values, for that Eq.(1) will diverge for a point charge. Without loss of generality, we may set the difference $\Delta U = 0$ at $z = 0$ and the difference will vary non-monotonically from the origin to infinity.

So where does the repulsive force come from? We know that the relation between the force and the energy is

$$\mathbf{F} = -\nabla U \tag{2}$$

and particularly, $F_z = -dU/dz$ for a cylindrically symmetric system. It is obvious that there must be a specific regime which gives us a repulsive force, namely, F_z being *positive*. When we put a point charge q at a distance z above the origin of an infinitesimally thin metallic semicircle, the induced surface charges density is shown in FIG. 2. The forces F_z, attractive or repulsive, exert on the source charge is proportional to $\cos\theta/r^2$. If the point charge is very close to origin, $R \gg z$, then $\cos\theta$ dominates. With the distances from the point charge to the induced positive and negative charges being $r \approx R$, we conclude that the positive induced charges account for larger $\cos\theta$ value, which contribute to the repulsive force.

II. METHODS

In this section, we give the analytical solution of the configuration discussed above. Also, we use Matlab for simulation. Since we are treating a two-dimensional electrostatics problem, we should adopt the 2D Maxwell's equation instead of 3D's. In Cartesian coordinate (x_1, x_2), the geometry consists of a source charge at position $\mathbf{y} = (0, z)$ where $z > 0$, and the semicircle C is described by $|x_1|^2 + |x_2|^2 = R^2$ with $x_2 \leq 0$. Therefore, we must solve a 2D Poisson equation,

$$\nabla^2 V_{\mathbf{y}}(\mathbf{x}) = -2\pi q \delta(\mathbf{x} - \mathbf{y}), \tag{3}$$

subject to the boundary conditions

$$\left.\begin{aligned} &V_{\mathbf{y}}(\mathbf{x}) = \text{const., when } \mathbf{x} \in \partial C \\ &V_{\mathbf{y}}(\mathbf{x}) + q \ln|\mathbf{x} - \mathbf{y}| = 0, \text{ for } \mathbf{x} \to \infty \\ &\textstyle\int_{\partial C} \mathbf{n} \cdot \nabla V_{\mathbf{y}}(\mathbf{x}) d\mathbf{x} = 0 \end{aligned}\right\} \tag{4}$$

The force that the metallic object exerts on the charge is given by

$$\mathbf{F}(\mathbf{y}) = -q \nabla \tilde{V}_{\mathbf{y}}(\mathbf{x})|_{\mathbf{x}=\mathbf{y}} \tag{5}$$

where the potential

$$\tilde{V}_{\mathbf{y}}(\mathbf{x}) = V_{\mathbf{y}}(\mathbf{x}) + q \ln|\mathbf{x} - \mathbf{y}| \tag{6}$$

is attribute to the induced charges on the metallic object. The energy of the system ΔU is

$$\begin{aligned} \Delta U &= U(\mathbf{y}) - U(\infty) \\ &= -\int_{\infty}^{\mathbf{y}} \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = \frac{q}{2} \tilde{V}_{\mathbf{x}}(\mathbf{x}) \Big|_{\infty}^{\mathbf{y}} = \frac{q}{2} \tilde{V}_{\mathbf{y}}(\mathbf{y}) \end{aligned} \tag{7}$$

Now, we treat the system as a problem on the Argand plane (Complex plane) and manage to get the energy difference through conformal mapping. Instead of using the original coordinate $\mathbf{x}$ and $\mathbf{y}$, we define the complex coordinates $u = x_1 + ix_2$, $v = y_1 + iy_2$. The mapping function

$$w(u) = \frac{iR + u + i\sqrt{R^2 - u^2}}{2} \tag{8}$$

will map the region outside the semicircle C to the region outside the disk D with radius $R/\sqrt{2}$ centered at origin. The original boundary condition problem is now transformed into another form: a source charge interacting with a metallic disk D. We can solve the later problem using the method of images mentioned in the introduction paragraph. From the principle of superposition, the potential of this geometry is

$$V_v^D(u) = -q \ln|u - v| + q \ln\left|u - \frac{R^2}{2\bar{v}}\right| - q \ln|u|, \tag{9}$$

one from the source charge, and two from the image charges inside the disk D. By using the mapping function $w(u)$, finally, we obtain the potential

$$\tilde{V}_v(v) = q \ln\left(\frac{2 - \frac{4R^2}{\left|iR+v+i\sqrt{R^2-v^2}\right|^2}}{\left|1 - \frac{iv}{\sqrt{R^2-v^2}}\right|}\right) \tag{10}$$

Since we have made the point charge move along the z axis (horizontal components of forces are eliminated due to symmetry), which means $\mathbf{y} = (0, z)$, we obtain the energy in Eq.(7)

$$U(z) = \frac{q^2}{2} \ln\left(\frac{2 - \frac{4R^2}{\left(R+z+\sqrt{R^2+z^2}\right)^2}}{1 + \frac{z}{\sqrt{R^2+z^2}}}\right) \tag{11}$$

if we set $U(z = \infty) = 0$.

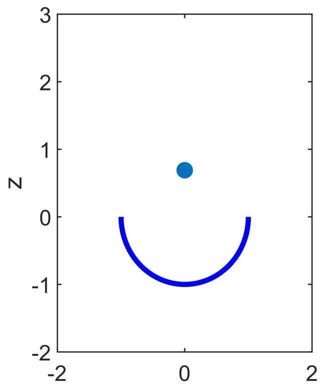

FIG. 3: Matlab numerical simulation screenshot. The point charge is released at a long distance, attracted by the neutral metal. The repulsive animation begins when the charge is very close to the origin.

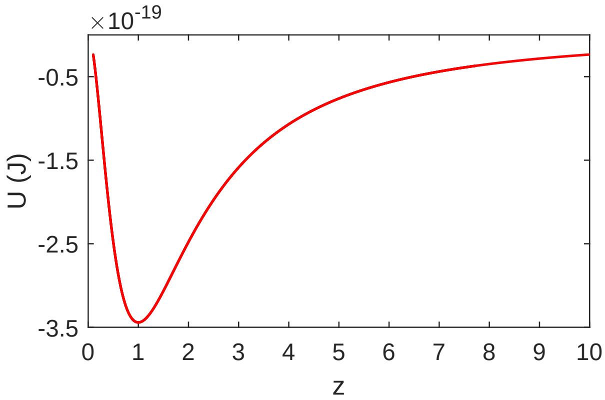

FIG. 4: Relation between the electrostatic energy and the position of the point charge. Strictly speaking, the parameter in the vertical axis is the difference $U(z) - U(\infty)$, and we had set $U(\infty) = 0$ at the beginning.

III. RESULTS

The simulation result are shown from FIG. 3 to FIG. 5. We release a point charge from $z = 10$ with no initial velocity. From the relation between the electrostatic energy and position (FIG. 4), we see that $z = R = 1$ is a local minimum, given that the semicircle has radius of 1. When the point charge is still at large z (where $z > 1$), it is attracted by the metallic object, i.e. the force is *negative* (illustrated by FIG. 5). However, once the distance between the point charge and the origin becomes smaller than R, the electroststic energy surges, which means that the sign of force changes to *positive* and pushes the point charge upwards.

IV. DISCUSSION

We construct the numerical simulation using the analytical solution of electrostatic force. For convenience, we set the metallic object with no thickness, and the simulation result does meet our expectation. During our research process, we found that this phenomenon worked not only for an extremely thin semicircle, but also for a hemispherical metal shell with finite thickness. Nevertheless, the effect

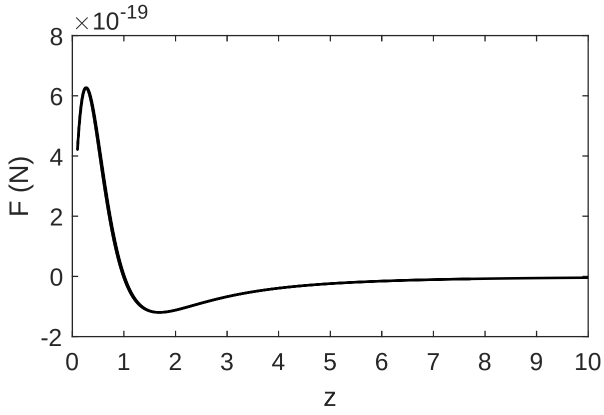

FIG. 5: Relation between the electrostatic force and the position of the point charge. The sign of the force changes when the point charge reaches $z = R = 1$.

of repulsive force exerting on the source charge reduces, since the induced charges only lie on the surface of conductors. Thus, the surface charge density distribution will not arrange as FIG. 2 shows exactly. Opposite charges lie mostly on the inner surface of the shell, while like charges lie on the outer surface. Qualitatively speaking, the effect of $\cos\theta/r^2$ caused by like charges decreases, which means the repulsive force may disappear when the hemispherical shell is thick enough. One way to simulate such a system is to apply the Boundary Element Method (BEM). BEM is best suited to reproduce accurately high surface stress gradients that are generally a modeling issue, though it is a pity that we are not familiar with this technique. On a practical aspect, this special phenomenon can also provide as an invisible barrier, restricting the movement of particles.

V. CONCLUSION

We studied one special problem in electrostatics and showed that the force between a point charge and a nearby uncharged conductor is not always attractive. The system consists of a two-dimensional metallic semicircle concave up, centered at the origin, and a point charge lies at position $(0, 0, z)$ on the positive z axis. The repulsive force appears when the point charge and the origin is close enough. From the simulation result, we can see the point charge being attracted down by the conductor at the first release. After reaching $z = R = 1$, it will be repelled to the initial position, provided that the total energy is conserved.

VI. REFERENCES

1. D. J. Griffiths, Introduction to Electrodynamics, 4th ed. 2017.
2. Spencer Tamagni and Costas Efthimiou 2021 Eur. J. Phys. 42 015206
3. Fiolhais, M. C., Essen, H. ,& Gouveia, T. M. (2016). A variational proof of Thomson’s theorem. Physics Letters A, 380(35), 2703-2705.
4. Levin M and Johnson S G 2011 Am. J. Phys. 79 843-9