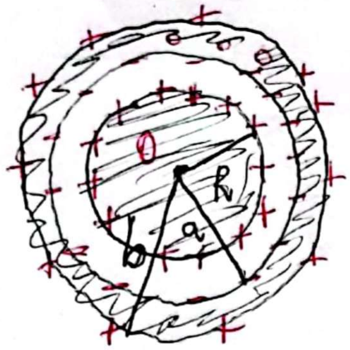
Gauss' Law and Cavities

①



Inside sphere $(r < R)$: no charge

On outer surface of inner sphere $(r = R)$: $+q$

Between sphere and shell $(R < r < a)$: no charge

On inner surface of shell $(r = a)$: $-q$

Inside shell $(a < r < b)$: no charge

On outer surface of shell $(r = b)$: $+q$

$r > b$: no charge

②

- At $r = R$, $\sigma = \dfrac{+q}{4\pi R^2}$

- At $r = a$, $\sigma = \dfrac{-q}{4\pi a^2}$

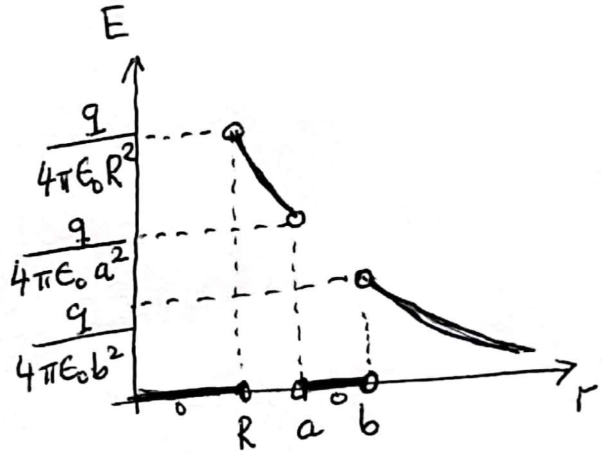
- At $r = b$, $\sigma = \dfrac{+q}{4\pi b^2}$

③ Use Gauss' law: $\oint \vec{E}\cdot d\vec{a} = \dfrac{Q_{enclosed}}{\epsilon_0}$

$$E \cdot 4\pi r^2 = \frac{Q_{encl.}}{\epsilon_0} \quad \text{where} \quad Q_{encl.} = \begin{cases} 0, & r<R \\ q, & R<r<a \\ 0, & a<r<b \\ q, & r>b \end{cases}$$

So:

$$E = \begin{cases} 0, & r<R \\ \dfrac{q}{4\pi\epsilon_0 r^2}, & R<r<a \\ 0, & a<r<b \\ \dfrac{q}{4\pi\epsilon_0 r^2}, & r>b \end{cases}$$



Discontinuities:

- At $r=R$ boundary: $\dfrac{q}{4\pi\epsilon_0 R^2} - 0 = \dfrac{\sigma_{at\ r=R}}{\epsilon_0}$ ✓
- At $r=a$ boundary $0 - \dfrac{q}{4\pi\epsilon_0 a^2} = \dfrac{\sigma_{at\ r=a}}{\epsilon_0}$ ✓
- At $r=b$ boundary $\dfrac{q}{4\pi\epsilon_0 b^2} - 0 = \dfrac{\sigma_{at\ r=b}}{\epsilon_0}$ ✓

Comparing with σ's in part 2

(4) $V = 0$ as $r \to \infty$

$$V(r) - V(\infty) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l} = -\int_{\infty}^{r} (E_r \hat{r}) \cdot (dr' \hat{r}) = -\int_{\infty}^{r} E_r(r')\, dr'$$

- For $r > b$: $V(r) = -\int_{\infty}^{r} \frac{q}{4\pi\epsilon_0 r'^2} dr' = -\frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r'}\right)\Big|_{\infty}^{r} = \frac{q}{4\pi\epsilon_0}\frac{1}{r}$

- For $a < r < b$: $V(r) = -\int_{\infty}^{b} \frac{q}{4\pi\epsilon_0 r'^2} dr' - \int_{b}^{r} 0 \cdot dr' = \frac{q}{4\pi\epsilon_0 b}$

- For $R < r < a$: $V(r) = -\int_{\infty}^{b} \frac{q}{4\pi\epsilon_0 r'^2} dr' - \int_{b}^{a} 0 \cdot dr' - \int_{a}^{r} \frac{q}{4\pi\epsilon_0 r'^2} dr'$

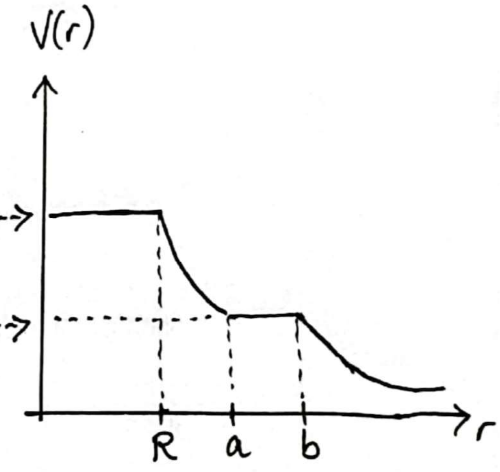
$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{a} + \frac{1}{r}\right)$$

- For $r < R$: $V(r) = -\int_{\infty}^{b} \frac{q}{4\pi\epsilon_0 r'^2} dr' - \int_{b}^{a} 0 \cdot dr' - \int_{a}^{R} \frac{q}{4\pi\epsilon_0 r'^2} dr' - \int_{R}^{r} 0 \cdot dr'$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{a} + \frac{1}{R}\right)$$

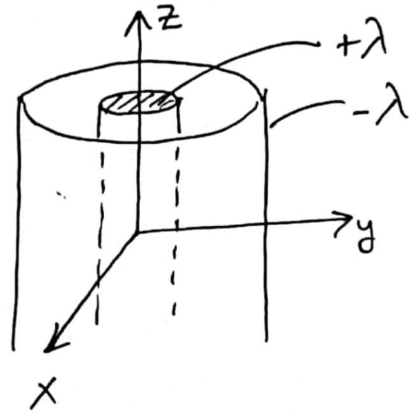
So

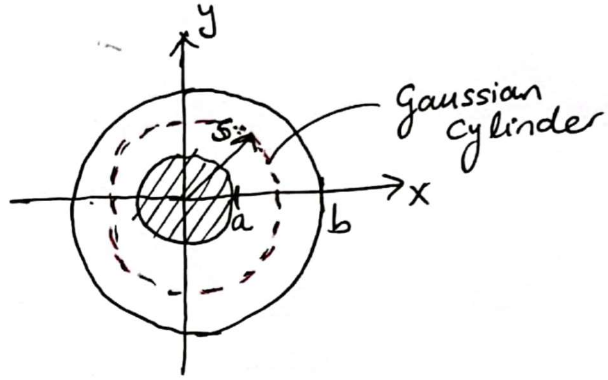
$$V(r) = \frac{q}{4\pi\epsilon_0} \cdot \begin{cases} 1/r & ; r \geq b \\ 1/b & ; a \leq r \leq b \\ 1/b - 1/a + 1/r & ; R \leq r \leq a \\ 1/b - 1/a + 1/R & ; r < R \end{cases}$$



Coax Capacitors

①





Gauss' Law: $\oint \vec{E}\cdot d\vec{a} = \dfrac{Q_{enclosed}}{\epsilon_0}$

- For $s<a$, $Q_{encl} = 0$ because inner cylinder is a conductor.

  So, $\vec{E} = 0$

- For $a<s<b$, $Q_{encl.} = \lambda L$ where L is the length of the Gaussian cylinder

$$\oint \vec{E}\cdot d\vec{a} = \underbrace{\int_{top} \vec{E}\cdot d\vec{a}}_{0} + \int_{side} \vec{E}\cdot d\vec{a} + \underbrace{\int_{bottom} \vec{E}\cdot d\vec{a}}_{0}$$

$$= \int_{side} (E_s\,\hat{s})\cdot(\hat{s}\; s\,d\phi\, dz) = E_s \cdot 2\pi s L$$

So, $E\cdot 2\pi s L = \dfrac{\lambda L}{\epsilon_0} \Rightarrow \vec{E} = \dfrac{\lambda}{2\pi\epsilon_0 s}\,\hat{s}$

- For $s>b$, $Q_{encl} = (\lambda-\lambda)L = 0 \Rightarrow \vec{E} = 0$

$$\Delta V = V(b) - V(a) = -\int_a^b \vec{E}\cdot d\vec{l} = -\int_a^b \left(\frac{\lambda}{2\pi\epsilon_0 s'}\,\hat{s}\right)\cdot(ds'\,\hat{s}) = -\frac{\lambda}{2\pi\epsilon_0}\int_a^b \frac{ds'}{s'}$$

$$= -\frac{\lambda}{2\pi\epsilon_0} \ln s'\Big|_a^b = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{a}{b}\right)$$ → negative because $V(b) < V(a)$

② $C \equiv \dfrac{Q}{|\Delta V|}$ Capacitance is always positive. So we will use $|\Delta V|$

$$\frac{C}{L} = \left(\frac{Q}{L}\right) \Big/ |\Delta V| = \lambda / |\Delta V| = \frac{\lambda}{\frac{\lambda}{2\pi\epsilon_0} \ln\frac{b}{a}} \rightarrow \text{made it positive}$$

$$\frac{C}{L} = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}$$

$$W = \frac{1}{2} C |\Delta V|^2 \Rightarrow \frac{W}{L} = \frac{1}{2}\frac{C}{L}|\Delta V|^2 = \frac{1}{2}\frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}\left[\frac{\lambda}{2\pi\epsilon_0}\ln\frac{a}{b}\right]$$

$$= \frac{1}{2}\frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)} \cdot \frac{\lambda^2}{(2\pi\epsilon_0)^2} \underbrace{\left|\ln\frac{a}{b}\right|^2}_{\left(\ln\frac{b}{a}\right)^2}$$

$$\Rightarrow \frac{W}{L} = \frac{\lambda^2}{4\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$$

③ $W = \dfrac{\epsilon_0}{2}\left[\int |\vec{E}|^2 d\tau\right]_{\text{all space}}$

$E = 0$ except for $a < s < b$

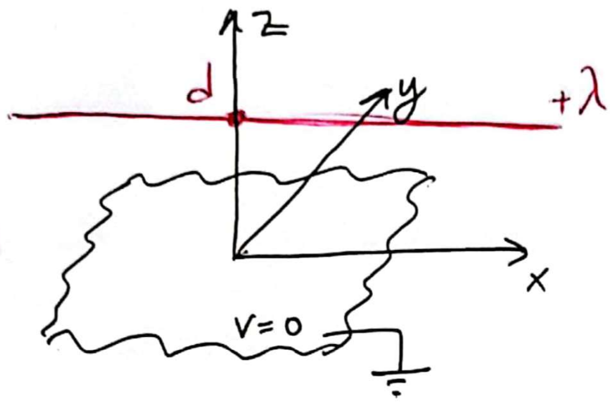
So, $$\frac{W}{L} = \frac{1}{L}\frac{\epsilon_0}{2}\int_{\mathcal{V}} \left(\frac{\lambda}{2\pi\epsilon_0 s'}\right)^2 \overbrace{s'\,d\phi'\,ds'\,dz'}^{d\tau}$$

$$= \frac{1}{L}\frac{\epsilon_0}{2}\int_0^L\int_a^b\int_0^{2\pi}\left(\frac{\lambda}{2\pi\epsilon_0}\right)^2\frac{1}{s'}\,d\phi'\,ds'\,dz'$$

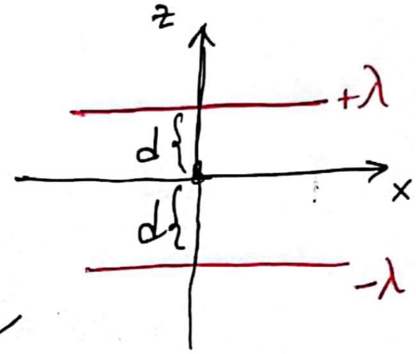
$$= \frac{\epsilon_0}{2L}\frac{\lambda^2}{4\pi^2\epsilon_0^2}\underbrace{\left[\int_0^{2\pi}d\phi'\right]}_{2\pi}\underbrace{\left[\int_a^b\frac{ds'}{s'}\right]}_{\ln\left(\frac{b}{a}\right)}\underbrace{\left[\int_0^L dz'\right]}_{L}$$

$$= \frac{\lambda^2}{4\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$$

The method of images

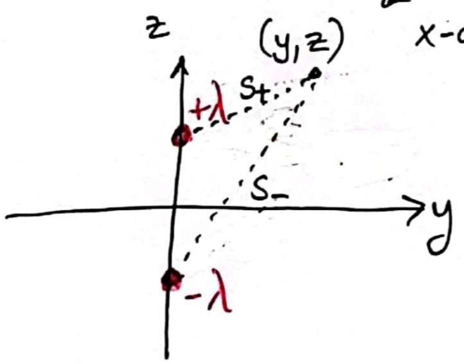


Let's use method of images by removing the grounded infinite plane and adding an infinite line of charge below xy-plane with $-\lambda$.

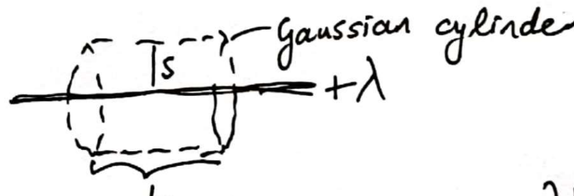


rotate view

x-axis out of page



Start with finding potential due to a line of charge. To do that, we need E-field.



$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{encl.}}{\epsilon_0} \rightarrow E \cdot 2\pi s L = \frac{\lambda L}{\epsilon_0}$$

$$\Rightarrow \vec{E} = \frac{\lambda}{2\pi\epsilon_0 s} \hat{s}$$

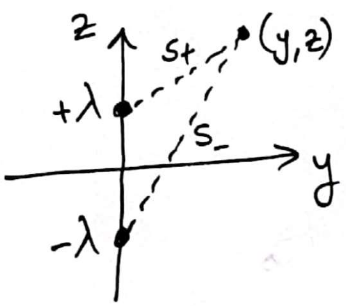
To calculate ΔV, we will use $-\int \vec{E}\cdot d\vec{l} = -\int \left(\frac{\lambda}{2\pi\epsilon_0 s'}\hat{s}\right)\cdot(ds'\hat{s})$

$$= -\frac{\lambda}{2\pi\epsilon_0}\ln s' + \text{const.}$$

Potential blows up if we set $s \to \infty$. So instead we will take an arbitrary distance $s = a$ from the line of charge as our $V=0$ point; i.e. $V(a) = 0$

Now. $$V(s) - V(a) = -\int_a^s \vec{E}\cdot d\vec{l} = -\frac{\lambda}{2\pi\epsilon_0}\ln s'\Big|_a^s = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{a}{s}$$

Returning back to our image charge problem, the potential



at (y,z) (for any x) is the superposition of V_+ and V_- where

$$V_+ = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{a}{s_+} \quad \text{and} \quad V_- = \frac{-\lambda}{2\pi\epsilon_0}\ln\frac{a}{s_-}$$

$$V = V_+ + V_- = \frac{\lambda}{2\pi\epsilon_0}\left[\ln\frac{a}{s_+} - \ln\frac{a}{s_-}\right] = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{a/s_+}{a/s_-}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\ln\frac{s_-}{s_+} = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{\sqrt{y^2+(z+d)^2}}{\sqrt{y^2+(z-d)^2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right] \quad \text{for } z > 0.$$

② $\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$ or $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$

Here $\hat{n} = \hat{z}$ and we evaluate at $z=0$

$$\sigma = -\epsilon_0 \frac{\lambda}{4\pi\epsilon_0} \frac{\partial}{\partial z}\left\{ \ln \frac{y^2+(z+d)^2}{y^2+(z-d)^2} \right\}\Bigg|_{z=0}$$

$$= -\frac{\lambda}{4\pi}\left\{ \frac{2(z+d)}{y^2+(z+d)^2} - \frac{2(z-d)}{y^2+(z-d)^2} \right\}\Bigg|_{z=0} = -\frac{\lambda}{4\pi}\left(\frac{2d}{y^2+d^2} - \frac{-2d}{y^2+d^2}\right)$$

$$= -\frac{\lambda d}{\pi(y^2+d^2)}$$