Chapter II. Basic Topology of sets

DEF A function OR MAP BETWEEN TWO (nonempty) sets A & B (denoted $A \to B$, $A \xrightarrow{f} B$, $f(A)$, etc) is a CORRESPONDENCE that set in correspondence to every element $a \in A$ a unique $b \in B$: $b = f(a)$.

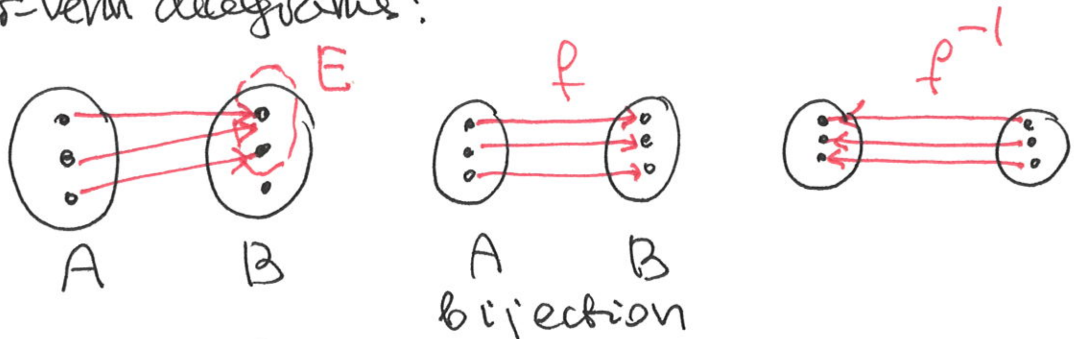
$f(A) = E \subseteq B$, A is the domain of f and E is the range of f.

DEF IF $E = B$ then the map is called surjective (OR mapping ONTO)

IF $f(a_1) \neq f(a_2)$ for $a_1 \neq a_2$, the map is injective.

DEF If a map is surjective and injective it is called a bijection, OR 1-1 correspondence

Euler-Venn diagrams:



If f is a bijection, $\forall y \in B$ $\exists!$ $x \in A$: $f(x) = y$,
then we have an inverse function $f^{-1}(y) = x$. unique

SEQUENCE $f: \mathbb{N} \to \mathbb{R}$. $(x_1, x_2, x_3, \dots)$

$\boxed{\text{Cardinality}}$: "size" of a set: $|A|$

$|A| = |B|$ $\underbrace{\text{iff}}$ $\exists f$ a bijection $A \leftrightarrow B$ $(A \sim B)$

"if and only if"

$|A| \leq |B|$ if $\exists$ an <u>injection</u> $A \to B$ $(A \leftrightarrow E \subseteq B)$

<u>Theorem</u> (hard) $|A| \leq |B| \wedge |B| \leq |A| \Rightarrow |A| = |B|$

[see Appendix]

<u>Classification</u> : let $I_n = 1, 2, \ldots, n \subset \mathbb{N}$ then

(a) A is <u>finite</u> if $A \leftrightarrow I_n$ for some n.
(then $|A| = n$)

(b) A is <u>infinite</u> if not finite

(c) A is <u>countable</u> if $A \sim \mathbb{N}$

(d) A is <u>uncountable</u> if it is not finite nor countable

(e) A is at most countable if A is finite or countable.

<u>Theorem</u> $|A| = |B|$ is an <u>equivalence relation</u> : $A \sim B$

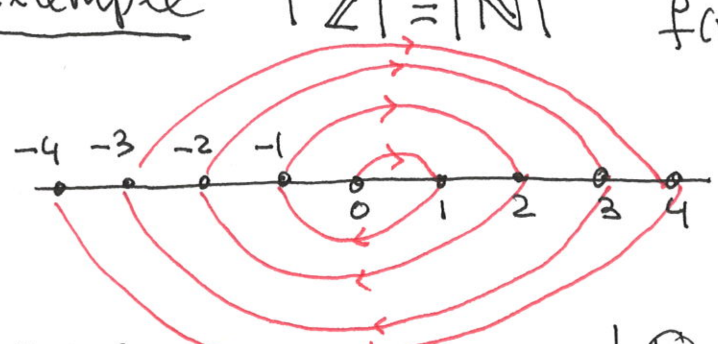
(i) . $A \sim A$

(ii) if $A \sim B$ then $B \sim A$. $B = f(A)$: $\exists f^{-1}$ so $A = f^{-1}(B)$

(iii) if $A \sim B$ and $B \sim C$ then $A \sim C$ by injection

(composite function) $B = f(A)$, $C = g(B)$ then
$C = g(f(A)) = g \circ f(A)$

<u>Example</u> $|\mathbb{Z}| = |\mathbb{N}|$ $\quad f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ -\frac{n-1}{2} & n \text{ odd} \end{cases}$



<u>NOT</u>

because here we <u>cannot</u> assign a specific finite n to, say -1, or 0.

<u>Example</u> (Lab Work) $|\mathbb{Q}| = |\mathbb{N}|$.

The set of $\alpha \in [0,1]$ is <u>uncountable</u>

Assume α is countable; so $\exists f(n) \Leftrightarrow [0,1]$:

Arrange $x \in \alpha$ is an order of increasing n:

n			
1	0.10110110... x_1		0.00110110
2	0.01110010... x_2		0.00110010
3	0.10100101... x_3	$\rightarrow$	0.10000101
4	0.01100100... x_4		0.01110100
	diagonal		

and take

$x_0 = 0.0001011\cdots$ such that each n'th element is the changed (swapped) n^{th} digit of x_n

Then $x_0 \neq x_n \ \forall n \in \mathbb{N}$ (since they differ by n^{th} digit)

So x_0 is NOT in the list of Dedekind cuts $\{x_1, \ldots x_n \ldots\}$

so this list is incomplete – contradiction.

[This is called Cantor's "diagonal process"]

DEF Union of sets $\bigcup_{\alpha\in A} E_\alpha$: all x : $x \in E_\alpha$ for SOME $\alpha \in A$

Intersection of sets $\bigcap_{\alpha\in A} E_\alpha$: all x: $x \in E_\alpha$ for ALL $\alpha \in A$.

~~xxxx~~ A can be finite or infinite.

interesting EXAMPLES: $\{ E_n = (0, \frac{1}{n}] , n = 1, 2, \ldots \}$

$$\bigcup_{n=1}^{\infty} E_n = E_1 = (0,1] \qquad E_1 \supset E_2 \supset E_3 \supset E_4 \cdots$$

$$\bigcap_{n=1}^{\infty} E_n = \{ \text{all } x : \ 0 < x \leq \frac{1}{n} \text{ for all } n \in \mathbb{N} \}$$

but $\forall x > 0 \ \exists n : \ \frac{1}{n} < x \Rightarrow$ no such x exist

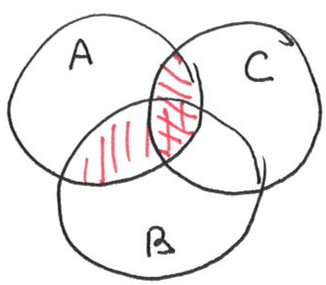
$$\bigcap_{n=1}^{\infty} E_n = \emptyset$$

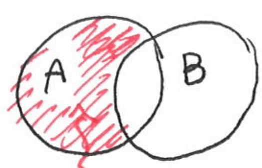
properties $A \cup B = B \cup A \qquad A \cap B = B \cap A$

distributive law:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

what about $A \cup (B \cap C)$?





$A \setminus B$

$A \setminus B = ?$

$\underbrace{A \setminus B = A \cap B^c}$

Complement E^c: implies we HAVE some ambient space

[... let E be a nonempty subset of $\underline{\mathbb{R}}$ ...]

$$E^c = \{ x \in \mathbb{R} : \ x \notin E \}.$$

$$[A \cup B]^c = A^c \cap B^c.$$

Metric spaces (IR)

$d: A \to \mathbb{R}_{+,0}$:

(i) $d(x,x)=0$; $d(x,y)=0 \Leftrightarrow x=y$
(ii) $d(x,y)=d(y,x)$

(iii) $d(x,z) \leq d(x,y)+d(y,z)$. Δ inequality
(on $\mathbb{R}$ $d(x,y)=|x-y|$)
Already on $\mathbb{R}^2 := (x_1,x_2)$-ordered pairs of real numbers we have many different "distances")

$\boxed{\text{DEF}}$ (a) A ball (a neighborhood) $B_r(p)$ centered at p:

$\forall x \in \mathbb{R}$: $|x-p|<r$

(b) A point p is a limit point of the set E if
$\forall r>0$ $\exists q \neq p$, $q \in E$: $q \in B_r(p)$

(c) if $p \in E$ and p is not a limit point, then p is an isolated point of E.

(d) A point p is an interior point of E if $\exists r>0$: $B_r(p) \subseteq E$

$\boxed{\text{DEF}}$ A set E is OPEN iff every point of E is an ~~isolated~~ interior point.

EXAMPLE: $(0,1)$ is open: $\forall x \in (0,1)$, $0<x<1$; take $r=\min\{x, 1-x\}$ then $\forall y \in B_r(x)$

if $y \leq x$ then $x-y < \min\{x,1-x\} \leq x$, so $y>0$ hence $0<y\leq x<1$ and $y \in (0,1)$.

if $y \geq x$ then $y-x < \min\{x,1-x\} \leq 1-x$, so $y<1$ and $x \leq y < 1$ and again $y \in (0,1)$, so proven

<u>Theorem</u> every $B_r(x)$ is open:

<u>proof</u> Take $y \in B_r(x)$, then $|y-x|<r$, so

$r-|y-x| = \varepsilon > 0$. Prove that $B_\varepsilon(y) \subseteq B_r(x)$:

Take $\forall z \in B_\varepsilon(y)$, then $|x-z| \leq |x-y|+|y-z| <$

$<$ ~~[illegible]~~ $|y-x|+\varepsilon = r$ so $z \in B_r(x)$. $\square$

<u>Theorem</u> (a) <u>Any</u> union of open sets is an open set.

$$\left(\bigcup_{\alpha \in A} E_\alpha = E \text{ - open} \right)$$

<u>proof</u> if $x \in \bigcup_{\alpha \in A} E_\alpha \Rightarrow x \in E_\alpha$ for some α, then

$\exists r: B_r(x) \subseteq E_\alpha$, so $B_r(x) \subseteq \bigcup_{\alpha \in A} E_\alpha = E$, so E is open

(b) Intersection of a <u>finite</u> number of open sets is <u>open</u>.

$\bigcap_{j=1}^{n} E_j = E$ is open:

<u>proof</u>. if $x \in \bigcap_{j=1}^{n} E_j$ then $\forall j \; \exists r_j > 0 : B_{r_j}(x) \subseteq E_j$

Take $r = \min\{r_j, j=1,..,n\}$, then the ball ~~[illegible]~~

$B_r(x) \subseteq B_{r_j}(x) \;\; \forall j=1,\dots n$, so $B_r(x) \subseteq \bigcap_{j=1}^{n} E_j$

[Why it does not work for infinite intersections?
Can be that no minimum exist, instedd $\inf\{r_\alpha\} = 0$]

Can be that $\bigcap E_j = \varnothing$, then $\varnothing$ is <u>OPEN</u>.

$\mathbb{R}$ is <u>open</u>; proper open sets are those not equal $\mathbb{R}$, $\varnothing$.

Theorem If p is a limit point, then every $B_r(p)$ contains ∞ many points of E.

proof by contradiction: assume $\exists r>0$:

$$\left(B_r(p)\setminus\{p\}\right)\cap E = \{q_1,\dots,q_n\}\subset E.$$

$q_i \neq p$, so take $r_0 = \min\{|p-q_i|,\ i=1,\dots,n\}$; $r_0>0$

then $\{q_1,\dots,q_n\}\cap B_{r_0}(p)=\emptyset$, so $B_{r_0}(p)$ does not contain any point of E (besides, possibly, p itself)

So p is not a limit point – contradiction

Corollary A finite point set has no limit points.

DEF E is CLOSED if every limit point of E belongs to E.

$\updownarrow$

DEF / THM E is closed if and only if its complement is open

$\Rightarrow$ Assume E^c is open. Then $\forall x\in E^c\ \exists B_r(x)\subseteq E^c$, so $B_r(x)\cap E=\emptyset$, so x is NOT a limit point of E and ALL limit points of E (if any exist) ARE in E.

$\Leftarrow$ Assume E contains ALL its limit points. Then no limit points of E ARE in E^c, so $\forall x\in E^c\ \exists r>0$ such that $B_r(x)\setminus\{x\}$ contains no points of E. Since $x\notin E$, $B_r(x)\cap E=\emptyset$ so $B_r(x)\subseteq E^c$ so E^c is open.

Example (a) $\mathbb{N} \subset \mathbb{R}$ is a closed subset (no limit points)

(b) $\mathbb{Q} \subseteq \mathbb{R}$ is neither closed nor open.

(c) set $\{\frac{1}{n}, n \in \mathbb{N}\}$ is not closed as $B_\varepsilon(0)$ contains ∞ many points. But if we add this limit point then the set ~~$B_\varepsilon(0)$~~ $\{\frac{1}{n}, n \in \mathbb{N}\} \cup \{0\}$ is closed.

Let E_{lim} be the set of all limit points of E.
Then the set $E_{lim} \cup E$ called the <u>closure</u> of E and denoted by $\overline{E}$ is closed.

<u>proof</u>. Assume x is a limit point of $\overline{E}$ and $x \notin \overline{E}$ (therefore $x \notin E$) Then $\forall r > 0$ ~~such that~~ $\exists y \in B_r(x) \cap \overline{E}$. We show that $\exists y' \in E$ such that $y' \in B_r(x)$. If $y \in E$ we done.
If y is a limit point of E then take
$\varepsilon = r - |y-x| > 0$, $\exists y' \in E$: $y' \in B_\varepsilon(y)$ But then $|y'-x| \leq |y'-y| + |y-x| < \varepsilon + |y-x| = r$ so $y' \in B_r(x)$. Thus ~~[illegible]~~ x is a limit point of E contradiction. So $\overline{E}$ is closed.

<u>EXAMPLE</u> ... SOME SETS CAN BE WEIRD...

$$W = \bigcup_{\substack{p/q \in \mathbb{Q} \\ 0 \leq |p/q| \leq 1}} \left(\frac{p}{q} - \frac{1}{2q^3}, \frac{p}{q} + \frac{1}{2q^3}\right)$$: W IS OPEN AS A UNION of open sets.

Therefore W^c is closed TRY TO DESCRIBE IT.

<u>Theorem</u> (continuation)

(b) $E = \overline{E}$ iff E is closed

(c) $\overline{E} \subseteq F$ $\forall$ closed set $F \supseteq E$.

(c) Since F is closed, F^c is open and $\forall x \in F^c$ $\exists B_r(x) \subseteq F^c$, so F^c contains no limit points of E and no points of E. So $F^c \cap \overline{E} = \emptyset$ and $\overline{E} \subseteq F$.

<u>Theorem</u> Let $E \subset \mathbb{R}$ be nonempty, bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. So $y \in E$ if E is closed.

<u>proof</u> If $y \in E$ then $y \in \overline{E}$. If $y \notin E$ then $\forall r > 0$ $\exists x \in E : |y - x| < r$ otherwise $y - r$ is upper bound. So y is a limit point of E. $\Rightarrow y \in \overline{E}$.

$\boxed{\text{DEF}}$ <u>Interior</u> of a set E denoted by E° is the <u>largest</u> open subset of E.
[why exists?]

<u>Theorem</u> $E^\circ = \bigcup$ ALL OPEN subsets of E: $\left(\bigcup_{\alpha \in I} A_\alpha\right)$

Indeed, (i) $\forall$ $A \subseteq E$ A-open, $A \subseteq \bigcup_{\alpha \in I} A_\alpha$

(ii) Union of any number of open sets is open.

<u>Example</u> $[0,1]^\circ = (0,1)$; $\mathbb{Q}^\circ = \emptyset$.

Obviously $E^\circ = \emptyset$ for any countable set (Because any ball is not countable.

Lindelöf ~~Heine-Borel~~ theorem

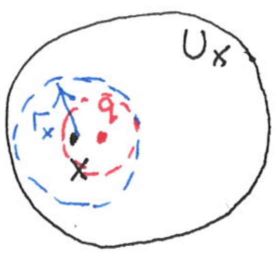
Consider any subset $E \subseteq \mathbb{R}$.

A (open) covering of E is a union of open $A_\alpha \subset \mathbb{R}$:

$E \subseteq \bigcup_{\alpha \in I} A_\alpha$. Prove that any open covering has a countable subcovering (i.e. $\exists\, \alpha_i \in I$, $i = 1, 2, \ldots$ such that $E \subseteq \bigcup_{i=1}^{\infty\ (\text{or } n)} A_{\alpha_i}$.

proof (constructive)

$\forall x \in E \quad \exists\, U_x \in \{A_\alpha\}$ so $\exists\, r_x$: $B_{r_x}(x) \subseteq U_x$



Then INSIDE $B_{r_x}(x)$ we have a rational point q_x such that $|x - q_x| < \frac{1}{3} r_x$. Then take ANY rational number $\frac{1}{3} r_x < s_x < \frac{2}{3} r_x$ AND CONSIDER A BALL $B_{s_x}(q_x)$:

$B_{s_x}(q_x) \ni x$ and $B_{s_x}(q_x) \subseteq B_{r_x}(x)$, so

$\bigcup_x B_{s_x}(q_x) \supseteq E$. But we have at most countable set of $B_{s_x}(q_x)$ (parameterized by $\mathbb{Q} \times \mathbb{Q}_+$)

Taking exactly one U_x for each $B_{s_x}(q_x)$ we have an at most countable subcovering.