Developing intuition for multipole expansion

Distribution 1:

$+q \quad +q$ $\quad Q_{total} = 4q \rightarrow$ monopole term dominates

$+q \quad +q$

Distribution 2:

$+q \quad +q$ $\quad q_{total} = 0 \rightarrow$ no monopole term

$-q \quad -q$

$\begin{matrix} +q \\ -q \end{matrix} \uparrow$ dipole up $\quad \begin{matrix} +q \\ -q \end{matrix} \uparrow$ dipole up $\rightarrow$ they add

$\rightarrow$ dipole term dominates

Distribution 3:

$+q \quad -q$ $\quad q_{total} = 0 \rightarrow$ no monopole term

$-q \quad +q$ $\quad$ Dipoles up/down and left/right cancel

$\rightarrow$ no dipole term

This is a quadrupole, so it dominates

The beauty of the multipole expansion

① $Q_{total} = \sum_i q_i = +q + q - q = q$

$$V_{monopole} = \frac{1}{4\pi\epsilon_0} \frac{Q_{total}}{r}$$

$$\vec{p} = \sum_i \vec{r}_i q_i = q r_0(+\hat{y}) + q r_0(-\hat{y}) + (-q) r_0(-\hat{z}) = q r_0 \hat{z}$$

$$V_{dipole} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{q r_0 (\hat{z}\cdot\hat{r})}{r^2} = \frac{q r_0 \cos\theta}{4\pi\epsilon_0 r^2}$$

$\hat{z}\cdot\hat{r} = \cos\theta$ where θ is the usual polar angle

$$V(r,\theta) \cong \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} + \frac{r_0\cos\theta}{r^2}\right)$$

② $Q_{net} \neq 0$, $\vec{p} \neq 0$.

So, monopole and dipole terms will be the lowest non-vanishing terms.

③ $\vec{E} = -\vec{\nabla} V$

$$E_r = -\frac{\partial V}{\partial r} \cong -\frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{1}{r} + \frac{r_0\cos\theta}{r^2}\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r^2} + \frac{2r_0\cos\theta}{r^3}\right)$$

$$E_\theta = \frac{1}{r}\frac{\partial V}{\partial \theta} \cong \frac{1}{r}\left(-\frac{q}{4\pi\epsilon_0}\right)\frac{\partial}{\partial\theta}\left(\frac{1}{r} + \frac{r_0\cos\theta}{r^2}\right) = \frac{q}{4\pi\epsilon_0}\frac{r_0}{r^3}\sin\theta$$

So, far from charges:

$$\vec{E}(r,\theta) = \frac{q}{4\pi\epsilon_0}\left[\left(\frac{1}{r^2} + \frac{2r_0\cos\theta}{r^3}\right)\hat{r} + \frac{r_0\sin\theta}{r^3}\hat{\theta}\right]$$

A sphere of charge

① $$Q_{total} = \int_{\mathcal{V}} \rho(\vec{r}')\,d\tau' = \int_{\mathcal{V}} \mu r' \sin\left(\frac{3\theta'}{2}\right) \cdot r'^2 \sin\theta'\,dr'\,d\theta'\,d\phi'$$

$$= \mu \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^R r'^3\,dr'}_{\frac{R^4}{4}} \underbrace{\int_0^{\pi} \sin\theta' \sin\left(\frac{3\theta'}{2}\right) d\theta'}_{\frac{4}{5}\underbrace{\left[\sin^3\left(\frac{\theta'}{2}\right) 2\cos\theta' + 3\sin^3\left(\frac{\theta'}{2}\right)\right]\Big|_0^{\pi}}_{1}}$$

$$= 2\pi\mu \frac{R^4}{4} \cdot \frac{4}{5}$$

$$Q_{total} = \frac{2}{5}\pi\mu R^4$$

② $$\vec{p} = \int_{\mathcal{V}} \rho(\vec{r}')\,\vec{r}'\,d\tau'$$

Because the distribution is azimuthally symmetric (no ϕ dependence), we expect $\vec{p} = p_0\hat{z}$; i.e. $\vec{r}' = r'\cos\theta'\,\hat{z}$

$$\vec{p} = \int_{\mathcal{V}} \overbrace{\mu r' \sin\left(\frac{3\theta}{2}\right)}^{\rho} \overbrace{r'\cos\theta}^{|\vec{r}'|} \underbrace{r'^2 \sin\theta'\,dr'\,d\theta\,d\phi}_{d\tau'} \cdot \hat{z}$$

$$= \mu \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} \underbrace{\int_0^R r'^4\,dr'}_{R^5/5} \underbrace{\int_0^{\pi} \sin\left(\frac{3\theta'}{2}\right) \cos\theta' \sin\theta'\,d\theta'}_{\frac{4}{7}}\ \hat{z}$$

$$\vec{p} = \frac{8}{35}\pi\mu R^5\ \hat{z}$$

③ $V(r,\theta) \cong V_{monopole} + V_{dipole} + \dots$

$$V \cong \frac{1}{4\pi\epsilon_0} \frac{Q_{total}}{r} + \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\frac{2}{5}\pi\mu R^4}{r} + \frac{1}{4\pi\epsilon_0} \frac{\frac{8}{35}\pi\mu R^5 \overbrace{\hat{z}\cdot\hat{r}}^{\cos\theta}}{r^2}$$

$$V \simeq \frac{1}{4\pi\epsilon_0} \pi\mu R^4 \left[\frac{2}{5r} + \frac{8R\cos\theta}{35r^2} \right]$$

Python: Electric Dipole Animation

Change the lines in the code to the following

```
    Epx = k*(+q)*(X-xp)/( np.sqrt( (X-xp)**2 + (Y-yp)**2 ) )**3
    Epy = k*(+q)*(Y-yp)/( np.sqrt( (X-xp)**2 + (Y-yp)**2 ) )**3
```

```
    Emx = k*(-q)*(X-xm)/( np.sqrt( (X-xm)**2 + (Y-ym)**2 ) )**3
    Emy = k*(-q)*(Y-ym)/( np.sqrt( (X-xm)**2 + (Y-ym)**2 ) )**3
```

```
    Ex = Epx + Emx
    Ey = Epy + Emy
```