PHY 481 - Fall 2023
Homework 02

Due Friday September 15, 2023

In this course, you will perform lots of different kinds of integrals, some of which you might have done in previous courses. In this homework set, we will just dust off some of those integration techniques. To earn full credit, you will need to show the key steps of the integration for each one.

1 Line (or path) integrals

These integrals are important for thinking about energy (and in our case electric potential). Determine the work done by the vector force $\mathbf{F} = y^3\ \hat{x} - 2x^2\ \hat{y}$ along the path $y = x^2 + 1$ from (0,1) to (2,5). This path is restricted to the x-y plane and recall that $W = \int \mathbf{F} \cdot d\mathbf{l}$.
Is the result of this line integral path-independent (i.e., is $\mathbf{F}$ a conservative vector field? Explain why or why not.

2 Surface integrals

Calculating the flux over a particular surface is a very common way of determine the electric field. Evaluate the integral $\int_S \mathbf{v} \cdot d\mathbf{A}$ where $\mathbf{v}(x, y, z) = 3zx\ \hat{x} + 5x\ \hat{y} + 2y\ \hat{z}$ and S is the rectangular surface lying in x-z plane from (0,0,0) to (2,0,3). Choose the direction of $+\hat{y}$ to be indicative of positive flux.
Explain how the resulting sign of the flux makes sense. You may use sketches or diagrams.

3 Volume integrals

It will be common for you to determine the amount of total charge in a situation where the charge is distributed in space according to some function. You might be familiar with this concept from the perspective of distributed mass in Classical Mechanics. Consider two different spheres: one with uniform mass density, ρ_0, and the other with a radially varying density, $\rho(r) = \frac{4\rho_0}{5R} r$. If both spheres have the same radius R, which has more mass?

4 Some vector proofs

In electromagnetism, developing a deep understanding of vector mathematics can facilitate a deeper understanding of the physical systems that we will investigate. While we will rarely ask you to prove relationships outright, knowing how certain proofs are done can often help you simplify a complicated problem.

In Griffiths, you read about a few integral theorems: the gradient theorem, Gauss's theorem (for divergences), and Stokes' theorem (for curls). You will make use of those theorems to prove a few things to develop some intuition about vector calculus.

1. From vector calculus, you know that the curl of any gradient of any scalar field is zero: $\nabla \times \nabla T(x, y, z) = 0$. Use the corollary of the gradient theorem, namely that closed loop integral of any gradient of a scalar field is zero, $\oint \nabla T \cdot d\mathbf{l} = 0$, along with Stokes' theorem, $\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_C \mathbf{v} \cdot d\mathbf{l}$, to demonstrate that the curl of a gradient is zero.

2. From vector calculus, you know that the divergence of the curl of any vector field is zero, $\nabla \cdot (\nabla \times \mathbf{v}) = 0$. Use the corollary of Stokes' theorem, namely that the closed surface integral of the curl of a vector field is zero, $\oint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$, along with the divergence theorem, $\int (\nabla \cdot \mathbf{v}) d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a}$, to demonstrate that the divergence of a curl is zero.

5 Test Stokes' theorem

Show $\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_C \mathbf{v} \cdot d\mathbf{l}$ using the vector function $\mathbf{v} = xy\ \hat{x} + 2yz\ \hat{y} + 3zx\ \hat{z}$ and the triangular surface in yz plane, with coordinates (0,0,0), (0,2,0), (0,0,2).

6 Python: An odd charge distribution

Up till now, most of your experience with integration has likely been integrating functions that have anti-derivatives (i.e. can be solved analytically). While this kind of integration is problematic when you have data that must be integrated (e.g., using measures of position to determine potential energy), we will focus on functions for which there are no anti-derivatives.

Consider a line of charge that lives on the x-axis. It exists from $x = -1$ to $x = 2$, and distribution of that charge is given by the Gaussian,

$$\lambda(x) = 3e^{-x^2} \tag{1}$$

In this problem, you will work through this Jupyter notebook to determine the total charge on this line by performing the integral,

$$Q = \int_{-1}^{2} 3e^{-x^2} dx \tag{2}$$

You can download the notebook HW2-AnOddChargeDistribution.ipynb from D2L. As you work through this problem, you will work through the following activities:
(In working through the first 4 questions, some scaffolded code has been provided for an analytically integrable function. It can be re-used.)

1. Plot the distribution of charge $\lambda(x)$ between $-1 \leq x \leq 2$.

2. Use "sympy" to compute the integral Q of the function. What does it return? Discuss the function. Is it analytic?

3. Apply the trapezoidal rule to compute the integral of this function. Use 10 equal width steps. To receive full credit, you must write the code to do this and not use built-in integration functions of "scipy".

4. Apply the trapezoidal rule to compute the integral again but this time use 100 equal width steps.

5. Apply Simpson's rule to compute the integral of this function. Use 10 equal width steps. To receive full credit, you must write the code to do this and not use built-in integration functions of "scipy".
   Hint: for the summations that require you to distinguish between even and odd terms, review the "range" function, which allows you to specify how many steps to take in between each term in a sum. Also you will need to use two "for" statements; one for the odd sum and one for the even sum in the Simpson's rule equation

6. Apply Simpson's rule to compute the integral of this function again. This time use 100 equal width steps.

7. Look up "scipy.integrate" built-in quadrature function, "quad." Use it to compute the same integral and compare its result to what your code produced. What are the two variables that "quad" gives you by default?

8. Explain in a few sentences how the trapezoidal rule and Simpson's rule are different. How do they compute the integrals? Explain how the Gaussian quadrature, which is what "quad" and its cousins in the "scipy.integrate" library do, is different from both of them.