Deep Dive #1

Part 1: Scale Invariant Equations

Definition 1 (Scale Invariant Equations). *A differential equation for $y(t)$ is scale invariant iff holds,*

$$y' = F(y/t).$$

Remark: That is, the right-hand side of the differential equation written in normal form depends only on the quotient y/t.

Question 1. *(10 points)* Use simple algebraic transformations to show that the differential equation

$$2ty\,y' - 5\,t^2 - 3\,y^2 = 0, \qquad t > 0,$$

is a scale invariant equation and write it in the normal form

$$y' = F(y/t),$$

that is, find the function F.

1. Show $2tyy' - 5t^2 - 3y^2 = 0$ is scale invariant for $t > 0$ by finding the funtion F such that $y' = F(y/t)$

$$2tyy' = 5t^2 + 3y^2$$

$$y' = \frac{5t^2 + 3y^2}{2ty}$$

$$= \frac{3y}{2t} + \frac{5t}{2y} = \frac{3}{2}\frac{y}{t} + \frac{5}{2}\frac{t}{y}$$

$$\frac{3}{2}(y/t) + \frac{5}{2}(y/t)^{-1} = F(y/t)$$

> Now we show our main result. We can transform a scale invariant equation on a function $y(t)$ into a separable equation for a function
>
> $$v(t) = \frac{y(t)}{t}.$$
>
> **Theorem 2** (Scale Invariant into Separable). *The scale invariant equation for the function $y(t)$ given by*
>
> $$y' = F\left(\frac{y}{t}\right)$$
>
> *determines a separable equation for the function $v(t) = y(t)/t$, given by*
>
> $$\frac{v'}{(F(v) - v)} = \frac{1}{t}.$$
>
> **Question 2.** *(12 points)* Prove Theorem 2 above.
>
> **Hint:** The relation $v(t) = y(t)/t$ implies a relation between the derivatives v' and y'.

2. Pf. Show the separable equation is true

$$v' = \frac{d}{dt}\frac{y(t)}{t} = \frac{ty'(t) - y(t)}{t^2}$$

$$F(v) - v = F(\tfrac{y}{t}) - \tfrac{y}{t}$$

Substitute these into the seperable equation

$$\frac{v'}{F(v) - v} = \frac{\frac{ty'(t)-y(t)}{t^2}}{F(\frac{y}{t}) - \frac{y}{t}}$$

Substitute and simplify

$$= \frac{ty'(t) - y(t)}{t^2(F(\frac{y}{t}) - \frac{y}{t})} = \frac{(y' - \frac{y}{t})}{t(y' - \frac{y}{t})} = \frac{1}{t}$$

QED

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Question 3. *(12 points)* Use the result in Theorem 2 to find all solutions of the scale invariant (and non-separable) differential equation in Question 1.

3. Close the loop

$$y' = F(y/t) = \frac{3}{2}(y/t) + \frac{5}{2}(y/t)^{-1} = F(v)$$

Since $y = tv$, then $y' = v + tv'$, which implies

$$v + tv' = \frac{3}{2}v + \frac{5}{2}v^-1$$

We rewrite and integrate it

$$\frac{2v}{v^2+5}v' = \frac{1}{t} \implies \int \frac{2v}{v^2+5}v'dt = \int \frac{dt}{t} + c_0$$

We use $u = v^2 + 5$ which implies $du = 2vv'$ and get something familiar from the textbook

$$\int \frac{du}{u} = \int dtt + c_0 \implies \ln(u) = \ln(t) + c_0 \implies u = tc_1$$

where $c_1 = e^{c_0}$

Subistituting back in v, and then y/t, we have $$ v^2 +5 = tc_1 \implies \dfrac{y^2}{t^2} = tc_1 - 5 \implies y(t) = \pm t\sqrt{tc_1-5}

Part 2: Variation of parameters for linear equations

Question 4.

(4a) *(5 points)* Find one solution $y_h(t)$ of the separable equation

$$y_h' = a(t)\, y_h.$$

Denote by $A(t)$ any antiderivative of $a(t)$, that is, $A(t) = \int a(t)\, dt$.

4. a)

Any solution would be $y_h(t) = ce^{A(t)}$, so I'll choose $y_h(t) = e^{A(t)}$

(4b) *(10 points)* Write any solution $y(t)$ of the linear equation (1) as

$$y(t) = v(t)\, y_h(t),$$

where $y_h(t)$ is a solution found in part (1a). Then show that the differential equation for $v(t)$ is

$$v' = e^{-A(t)}\, b(t),$$

where $A(t)$ is the function defined in part (1a).

b)

Any solution for $y(t)$ is

$$y(t) = v(t)e^{A(t)}$$

$$y'(t) = \frac{d}{dt}v(t)e^{A(t)} = v'(t)e^{A(t)} + v(t)A(t)e^{A(t)}$$

We can rewrite this in the form of linear equations for y'

$$y'(t) = v'(t)e^{A(t)} + y(t)A(t)$$

Since all solutions of y'(t) can be written as $y' = a(t)\, y + b(t)$, the components associated with $b(t)$ are those that aren't multiplied by y

$$b(t) = v'(t)e^{A(t)} \implies v' = e^{-A(t)}b(t)$$

(4c) *(5 points)* Solve this differential equation for $v(t)$ obtained in part (1b) and show that all solutions $y(t)$ of the linear equation (1) are given by the formula given in the textbook, Section 1.4, Theorem 1.4.3, Equation (1.4.7), that is,

$$y(t) = c\, e^{A(t)} + e^{A(t)} \int e^{-A(t)}\, b(t)\, dt,$$

where c is an arbitrary constant.

c)

Let's integrate v'(t)

$$\int v' dt = \int e^{-A(t)} b(t) dt + c$$

Since $v(t) = \dfrac{y(t)}{e^{A(t)}}$ we get

$$\int v' dt = v(t) = \frac{y(t)}{e^{A(t)}} = c + \int e^{-A(t)} b(t) dt \implies y(t) = ce^{A(t)} + e^{A(t)} \int e^{-A(t)} b(t) dt$$

Question 5. *(12 points)* Use the Variation of Parameters Method to find all solutions, y, of the equation

$$y' = \frac{3}{t} y + t^5, \qquad t > 0.$$

Hint: First find y_h, solution of the homogeneous linear equation, as in Question (4a), then find $v(t)$ as defined in Question (4b), and then get the solution $y(t) = v(t)\, y_h(t)$.

5. First we'll define $a(t)$ and $b(t)$

$$a(t) = \frac{3}{t} \implies A(t) = 3\ln(t) + c$$

$$b(t) = t^5$$

Now to the hint.

$$y(t) = v(t) y_h(t) \implies y_h' = a(t) y_h \implies y_h(t) = e^{A(t)}$$

This leads us to

$$y(t) = v(t) e^{A(t)}$$

All we need now is $v(t)$, and not in terms of y. From our result in (4b) and (4c) we have

$$\int v' dt = v(t) = \frac{y(t)}{e^{A(t)}} = c + \int e^{-A(t)} b(t) dt \implies y(t) = ce^{A(t)} + e^{A(t)} \int e^{-A(t)} b(t) dt$$

Lets fill in and integrate

$$y(t) = c_0 e^{3\ln(t) + c_1} + e^{3\ln(t) + c_1} \int e^{-3\ln(t) - c_1} t^5 dt$$

$$= c_0 t^3 e^{c_1} + t^3 e^{c_1} \int \frac{t^2}{e^{c_1}} dt$$

Lets define $c_2 = c_0 e^{c_1}$

$$y(t) = c_2 t^3 + \frac{t^6}{3}$$

6. The Bernoulli Equation

Energy per unit volume before = Energy per unit volume after

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

from gsu hyperphysics

Wait. Not Daniel.

$$y' = p(t)y + q(t)y^n$$

Theorem 4. *The function y is a solution of the Bernoulli equation*

$$y' = p(t)\,y + q(t)\,y^n, \qquad n \neq 1,$$

iff the function $v = 1/y^{(n-1)}$ is solution of the linear differential equation

$$v' = -(n-1)p(t)\,v - (n-1)q(t).$$

Question 6. *(12 points)* Prove Theorem 4.

Okay I guess I'll just go and do that then. I'll be following the proof in the text book

We start by getting v into the first equation

$$\frac{y'}{y^n} = \frac{p(t)y}{y^n} + q(t) = vp(t) + q(t)$$

From $v = \frac{1}{y^{(n-1)}}$ we have

$$v' = (-n+1) * \frac{y'(t)}{y^n(t)}$$

The fraction on the RHS can be substituted with the equation we found earlier

$$v' = (-n+1) * vp(t) + q(t)$$

which becomes the final equation of the proof

$$v' = -(n-1)p(t)v - (n-1)q(t)$$

Question 7. *(10 points)* Find every nonzero solution of the differential equation

$$y' = y + 3\,y^4.$$

By doing a deep dive into the textbook, I find Theorem 1.5.2 which yields

$$y(t) = e^{P(t)}\Big(c + (-n+1)\int e^{(n-1)P(t)}\,q(t)\,dt\Big)^{\frac{1}{(-n+1)}}$$

$p(t) = 1,\ q(t) = 3,\ n = 4, P(t) = t$

$$y(t) = e^t(c - 3\int 3e^{(-3t)}dt)^{-\frac{1}{3}} = \frac{e^t}{\sqrt[3]{(c + 3e^{(-3t)})}}$$

Question 8. *(12 points)* Find every nonzero solution of the constant coefficients Bernoulli equation

$$y' = p\,y + q\,y^n, \qquad n \neq 0, 1, \qquad p \neq 0,$$

where p, q are constants. Write the implicit form of the solution as

$$\frac{1}{y^{n-1}} = f(t, n, p, q, c)$$

where c is an integration constant. Find the right-hand side above, $f(t, n, p, q, c)$.

Well... I already used this result to solve question 7.

Since $\dfrac{1}{y^{n-1}} = v$, lets start with the result we found in Theorem 4.

$v' = -(n-1)p(t)v - (n-1)q(t)$ is a linear equation, so I'll solve this using the solution we found in (4c).

$a(t) = -(n-1)p(t) \implies A(t) = -(n-1)P(t)$
$b(t) = -(n-1)q(t)$

$$v(t) = ce^{-(n-1)P(t)} + e^{-(n-1)P(t)}\int e^{(n-1)P(t)}(-n+1)q(t)dt = f(t, n, p, q, c)$$

$$y(t) = c\,e^{A(t)} + e^{A(t)}\int e^{-A(t)}\,b(t)\,dt,$$