Deep Dive #3

Taylor Series Approximations

We use Taylor series to solve differential equations

$$y(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\, y^{(n)}(t_0)\,(t-t_0)^n$$
$$= y(t_0) + y'(t_0)\,(t-t_0) + \frac{1}{2!}\, y''(t_0)\,(t-t_0)^2 + \cdots,$$

Definition 1. *The n-th order* ***Taylor approximation*** *centered at t_0 of a function y is given by*

$$\tau_n(t) = \sum_{k=0}^{n} \frac{1}{k!}\, y^{(k)}(t_0)\,(t-t_0)^k$$

Notice that the definition above implies a simple relation between τ_n and τ_{n-1},

$$\tau_n(t) = \tau_{n-1}(t) + \frac{1}{n!}\, y^{(n)}(t_0)\,(t-t_0)^n.$$

Question 1

It turns out that the initial condition and the differential equation is enough to compute all the derivatives of the function $y(t)$ at the time of the initial condition, t_0.

Theorem 2 (Taylor Approximation). *The initial value problem*

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0, \tag{1}$$

with $f(t,y)$ infinitely continuously differentiable in both variables, determines $\tau_n(t)$, the n-th order Taylor approximation of the solution $y(t)$ of (1), for any integer $n \geqslant 0$.

Question 1. *(20 points)* Give an idea of the proof of Theorem 2 by computing the Taylor approximation $\tau_3(t)$. You do not need to compute higher order approximations.

I will simply pull up $\tau_3(t)$

$$\tau_3(t) = \sum_{k=0}^{3} \frac{1}{k!} y^{(k)}(t_0)(t-t_0)^k$$
$$= y(t_0) + y'(t_0)(t-t_0) + \frac{1}{2} y^{(2)}(t_0)(t-t_0)^2 + \frac{1}{6} y^{(3)}(t_0)(t-t_0)^3$$

Since we know $t - 0, y(t_0),$ and have $y'(t)$, which is infinitely differentiable, we have every value in τ_3 and any τ_n

Question 2

Question 2. *(20 points)* Use the Taylor approximation defined in Theorem 2 to find the first four approximate solutions of the linear initial value problem

$$y'(t) = 3\,y(t) + 2, \qquad y(0) = 1.$$

$t_0 = 0, y(t_0) = 1, y'(t) = 3y(t) + 2, y''(t) = 3y'(t), \ldots$

$$\tau_0(t) = 1$$
$$\tau_1(t) = 1 + (3y(0) + 2)(t) = 1 + 5t$$
$$\tau_2(t) = 1 + 5t + \frac{1}{2!}3(5)t^2 = 1 + 5t + \frac{15}{2}t^2$$
$$\tau_3(t) = 1 + 5t + \frac{15}{2}t^2 + \frac{1}{6}3(3(5))t^3 = 1 + 5t + \frac{15}{2}t^2 + \frac{45}{6}t^3$$

Question 3

Question 3. *(20 points)* Use the Taylor approximation defined in Theorem 2 to find the solution formula for all solutions of the initial value problem

$$y'(t) = a\,y(t) + b, \qquad y(0) = y_0,$$

with a, b constants, that is, use the Taylor approximation method to find the formula

$$y(t) = \left(y_0 + \frac{a}{b}\right)e^{at} - \frac{b}{a}.$$

$$\begin{cases} t_0 = 0 \\ y(0) = y_0 \\ y'(0) = ay_0 + b \\ y'' = ay'(0) = a(ay_0 + b) \\ y^{(3)} = \ldots \end{cases}$$

Thus $y^{(n)}(0) = (y_0 + \frac{b}{a})a^n$ for $n > 0$

Remember the taylor expansion for $e^{at} = 1 + at + \frac{(at)^2}{2} + \frac{(at)^3}{6} + \cdots = \sum_{k=0}^{n} \frac{(at)^n}{n!}$

$$\tau_n = \sum_{k=0}^{n} \frac{1}{k!} y^{(k)}(t_0)(t - t_0)^k$$
$$= y_0 + \sum_{k=1}^{n} \frac{1}{k!}(y_0 + \frac{b}{a})a^k t^k$$
$$= y_0 + \sum_{k=1}^{n} \frac{(at)^k}{k!}(y_0 + \frac{b}{a})$$

To get back to $k = 0$ we add back in $\frac{(at)^0}{0!}(y_0 + \frac{b}{a}) = y_0 + \frac{b}{a}$

$$= y_0 + \sum_{k=0}^{n} \frac{(at)^k}{k!}(y_0 + \frac{b}{a}) - y_0 - \frac{b}{a}$$

$$= (y_0 + \frac{b}{a}) \sum_{k=0}^{n} \frac{(at)^k}{k!} - \frac{b}{a}$$

At τ_∞ this becomes

$$= (y_0 + \frac{b}{a})e^{at} - \frac{b}{a}$$

Question 4

Question 4. *(20 points)* Use the Taylor approximation defined in Theorem 2 to find the first four approximate solutions of the linear initial value problem

$$y' = 2t\,y^2 + t^2 + 3, \qquad y(0) = 1.$$

$$\begin{cases} t_0 = 0 \\ y(0) = 1 \\ y'(0) = 2(0)(1)^2 + (0)^2 + 3 = 5 \\ y''(t) = 4tyy' + 2y^2 + 2t + 3 \\ y''(0) = 0 + 2(1)^2 + 0 + 3 = 5 \\ y^{(3)}(t) = 4yy' + 4t(yy')' + 4yy' + 2 \\ y^{(3)}(0) = 8yy' + 2 = 8(1)(5) + 2 = 42 \end{cases}$$

$$\tau_0 = 1$$

$$\tau_1 = 1 + y'(0)t = 1 + 5t$$

$$\tau_2 = 1 + 5t + \frac{5}{2}t^2$$

$$\tau_3 = 1 + 5t + \frac{5}{2}t^2 + \frac{42}{6}t^3$$

Question 5

Question 5. *(20 points)* Let $\tau_n(t)$ be the Taylor approximation given in Theorem 2. Also assume that the limit $n \to \infty$ of $\tau_n(t)$ converges and

$$y_\tau(t) = \lim_{n\to\infty} \tau_n(t)$$

is a continuously differentiable function. Then, show that this function $y_\tau(t)$ is a solution of the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0.$$

Hint: Since we assume that $y_\tau(t)$ is well defined, so is the function $g(t) = f(t, y_\tau(t))$. Study the relation between $y_\tau(t)$ and $g(t)$ and their derivatives when we evaluate them at t_0.

From theorem #2 we know that $\tau_n(t)$ is an approximate solution of $y'(t) = f(t, y(t))$. We're asked to show that $y_T(t)$ is an actual solution.

For $y_T(t_0)$ we have all factors in the summation equal to zero for $k > 0$, since it includes the factor $(t - t_0) = 0$ for $t = t_0$. Thus $y_T(t_0) = \frac{1}{1}y(t_0) = y_0$

As such $g(t_0) = f(t_0, y_T(t_0)) = f(t_0, y(t_0)) = y'(t_0)$. However the hint suggests we take the derivative of $y_T(t)$ and $g(t)$ at $t = t_0$.

Since all of these functions are well defined, we know that

$$g'(t_0) = f'(t_0, y_T(t_0)) = f'(t_0, y(t_0)) = y''(t_0)$$

and thus

$$g^{[n]}(t_0) = y^{[n+1]}(t_0).$$

As for the derivative of $y_T(t)$, since $\lim_{n\to\infty} \tau_n(t)$ converges, $y_T(t)$ becomes

$$y_T(t) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} y^{(k)}(t_0)(t-t_0)^k = \sum_{k=0}^{\infty} \frac{1}{k!} y^{(k)}(t_0)(t-t_0)^k$$

and the derivative $y_T'(t)$ becomes

$$\begin{aligned} y_T'(t) = \sum_{k=0}^{\infty} \frac{d}{dt} \frac{1}{k!} y^{(k)}(t_0)(t-t_0)^k &= \sum_{k=0}^{\infty} \frac{1}{k!} y^{(k)}(t_0) \frac{d}{dt}(t-t_0)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} y^{(k)}(t_0) k (t-t_0)^{k-1} \\ &= \sum_{k=1}^{\infty} \frac{1}{k-1!} y^{(k)}(t_0)(t-t_0)^{k-1} \end{aligned}$$

The $k = 0$ becomes zero, allowing us to kick down by k

$$= \sum_{k=0}^{\infty} \frac{1}{k!} y^{(k+1)}(t_0)(t-t_0)^k$$

For $t = t_0$ this becomes

$$\begin{aligned} y_T'(t_0) &= \sum_{k=0}^{\infty} \frac{1}{k!} y^{(k+1)}(t_0)(t_0-t_0)^k \\ y_T'(t_0) &= y'(t_0) \\ y_T'(t_0) &= g(t_0) \end{aligned}$$

Thus from the relationship between $g(t_0)$ and $y'(t_0)$, we have

$$y_T^{[n]}(t_0) = y^{[n]}(t_0)$$

for all n.

Considering the fact that $y_T(t_0)$ and all of its derivatives are equal to $y(t_0)$ and its derivatives, all of $y_T(t)$ is defined and is a solution to $y(t)$