Magnetic Forces and Motion

Understanding Griffiths Example 5.2 :

Let's write velocity vector as $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$ where $\dot{x} = \frac{dx}{dt}$, etc.

We have $\vec{E} = E\hat{z}$ and $\vec{B} = B\hat{x}$

So, $\vec{v} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = \hat{x}(\dot{y}\cdot 0 - \dot{z} 0) - \hat{y}(\dot{x}\, 0 - \dot{z} B) + \hat{z}(\dot{x}\cdot 0 - \dot{y} B)$

$$= \dot{z} B \hat{y} - \dot{y} B \hat{z}$$

$$\vec{F} = m\vec{a} = Q(\vec{E} + \vec{v} \times \vec{B})$$

$$m(\ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}) = Q(E\hat{z} + \dot{z}B\hat{y} - \dot{y}B\hat{z})$$

$\Rightarrow \hat{x}$: $\ddot{x} = 0 \Rightarrow \dot{x} = \text{const} \rightarrow x(t) = \dot{x}(0)\cdot t + x_0$

$\hat{y}$: $m\ddot{y} = Q\dot{z}B$

$\hat{z}$: $m\ddot{z} = Q(E - \dot{y}B)$

Set $\omega = \frac{QB}{m}$.

$\Rightarrow \ddot{y} = \omega \dot{z}$ and $\ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$

differentiate ↓ insert

$$\dddot{y} = \omega \ddot{z} = \omega^2\left(\frac{E}{B} - \dot{y}\right)$$

integrate ↓

$\int \dddot{y}\, dt = \int \omega^2\left(\frac{E}{B} - \dot{y}\right) dt \Rightarrow \ddot{y} = \omega^2 \frac{E}{B} t - \omega^2 y + c_0$

constant of integration

$\ddot{y} = -w^2 y + w^2 \frac{E}{B} t + c_0$ has a homogeneous plus a particular solution: $y(t) = y_h(t) + y_p(t)$ where

the general solution for y_h is $y_h(t) = c_1 \cos(wt) + c_2 \sin(wt)$

For the particular solution, we can use method of undetermined coefficients: $y_p = a_1 t + a_2$.

Putting into the differential equation:

$\ddot{y}_h + \ddot{y}_p = -w^2(y_h + y_p) + w^2 \frac{E}{B} t + c_0$. Using

$\ddot{y}_h = -w^2 y_h$ and $\ddot{y}_p = 0$, we get

$$0 = w^2 \frac{E}{B} t + c_0 - w^2(a_1 t + a_2)$$

$$\Rightarrow \quad w^2 \frac{E}{B} t = w^2 a_1 t \quad \Rightarrow \quad a_1 = \frac{E}{B}$$

$$\text{and} \quad c_0 = w^2 a_2 \quad \Rightarrow \quad a_2 = \frac{c_0}{w^2} \equiv c_3$$

Therefore, $y_p(t) = \frac{E}{B} t + c_3$

The full solution is: $y(t) = c_1 \cos(wt) + c_2 \sin(wt) + \frac{E}{B} t + c_3$

Now, let's do z part: $\ddot{z} = w\left(\frac{E}{B} - \dot{y}\right)$

$$\dot{y} = -w c_1 \sin(wt) + c_2 w \cos(wt) + \frac{E}{B}$$

$$\Rightarrow \quad \ddot{z} = w^2 c_1 \sin(wt) - w^2 c_2 \cos(wt)$$

integrate $\hookrightarrow \int \ddot{z}\, dt = \dot{z} = \int \left[w^2 c_1 \sin(wt) - w^2 c_2 \cos(wt)\right] dt$

$$= -w c_1 \cos(wt) - w c_2 \sin(wt) + c_5$$

integrate one more time:

$$\int \dot{z}\, dt = \int \left[-\omega c_1 \cos(\omega t) - \omega c_2 \sin(\omega t) + c_5 \right] dt$$

$$z(t) = -c_1 \sin(\omega t) + c_2 \cos(\omega t) + c_5 t + c_4$$

Check for consistency with $\ddot{y} = \omega \dot{z}$

$$\ddot{y} = -\omega^2 c_1 \cos(\omega t) - \omega^2 c_2 \sin(\omega t)$$

$$\omega \dot{z} = \omega(-\omega c_1 \cos(\omega t) - \omega c_2 \sin(\omega t) + c_5)$$

equal

For them to be equal: $c_5 = 0$

So $$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{E}{B} t + c_3$$

$$z(t) = c_2 \cos(\omega t) - c_1 \sin(\omega t) + c_4$$

where $\omega = \frac{QB}{m}$ and c_1, c_2, c_3, c_4 are determined by initial conditions.

1 Initial conditions : at $t=0$

$y_0=0, \quad z_0=0, \quad \dot{y}_0 = v_0, \quad \dot{z}_0 = 0$

$$y(0) = c_1 \cos 0 + c_2 \sin 0 + \frac{E}{B}\cdot 0 + c_3 = c_1 + c_3 = 0 \qquad (1)$$
$$\dot{y}(0) = -\omega c_1 \sin 0 + \omega c_2 \cos 0 + \frac{E}{B} = \omega c_2 + \frac{E}{B} = v_0 \qquad (2)$$
$$z(0) = c_2 \cos 0 - c_1 \sin 0 + c_4 = c_2 + c_4 = 0 \qquad (3)$$
$$\dot{z}(0) = -\omega c_2 \sin 0 - \omega c_1 \cos 0 = -\omega c_1 = 0 \qquad (4)$$

(4) gives $c_1 = 0$. Then using (1), we get $c_3 = 0$.

(2) gives $c_2 = \frac{1}{\omega}\left(v_0 - \frac{E}{B}\right)$. Then using (3), we get $c_4 = \frac{1}{\omega}\left(\frac{E}{B} - v_0\right)$

Now we have,

$$y(t) = \frac{1}{\omega}\left(v_0 - \frac{E}{B}\right)\sin(\omega t) + \frac{E}{B}t \quad \text{and}$$
$$z(t) = \frac{1}{\omega}\left(v_0 - \frac{E}{B}\right)\cos(\omega t) + \frac{1}{\omega}\left(\frac{E}{B} - v_0\right)$$

To get a straight-line trajectory, sin & cos terms should vanish.

So, $\boxed{v_0 = \frac{E}{B}}$ in which case we have $y(t) = \frac{E}{B}t$ and $z(t)=0$

"PHY 184 argument" : constant speed linear motion $\rightarrow a=0 \rightarrow F_{net}=0$

$$\vec{F} = Q(\vec{E} + \vec{v}\times\vec{B}) = 0 \rightarrow \vec{E} = -\vec{v}\times\vec{B}$$
$$E\hat{z} = -(\dot{z}B\hat{y} + \dot{y}B\hat{z})$$

implies $\dot{z}=0$ and $E = \dot{y}B \rightarrow \dot{y} = E/B$

$$\Rightarrow \vec{v} = \frac{E}{B}\hat{y} = v_0\hat{y}$$

(2) For a parallel plate capacitor: $E = \frac{\Delta V}{d}$

$$E = \frac{2\,kV}{2\,cm} = \frac{2\times10^{3}\,V}{2\times10^{-2}\,m} = 10^{5}\,\frac{V}{m}$$

$$v = \frac{c}{3} = \frac{3\times10^{8}\,m/s}{3} = 10^{8}\,m/s$$

$$B = \frac{E}{v} = \frac{10^{5}\,V/m}{10^{8}\,m/s} = 10^{-3}\,T = 1\,mT$$

$|\vec{B}_{earth}| \approx 50\,mT$ or about 5% of B.

Reasonable to neglect B_{earth}.

(3) If there is no E-field, the charged particle will start to move in a circular orbit.

magnetic force = centripetal force

$$qvB = \frac{mv^2}{R} \rightarrow \frac{q}{m} = \frac{v}{RB} \quad \text{and if } v = \frac{E}{B},$$

then $\boxed{\frac{q}{m} = \frac{E}{RB^2}}$

(4) Starting from the trajectory equations found using the initial conditions:

$$y(t) = \frac{1}{\omega}\left(v_0 - \frac{E}{B}\right)\sin(\omega t) + \frac{E}{B}t \quad \text{and}$$

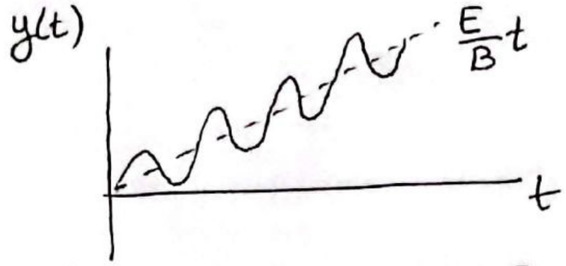
$$z(t) = \frac{1}{\omega}\left(v_0 - \frac{E}{B}\right)\cos(\omega t) + \frac{1}{\omega}\left(\frac{E}{B} - v_0\right)$$

Let $v_0 = \frac{1}{2}\frac{E}{B}$.

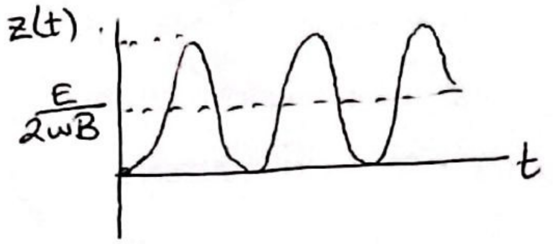
Then
$$y(t) = -\frac{E}{2\omega B}\sin\omega t + \frac{E}{B}t \quad *$$

$$z(t) = -\frac{E}{2\omega B}\cos\omega t + \frac{E}{2\omega B} \quad **$$

So $y(t)$ oscillates around the line $\frac{E}{B}t$



$z(t)$ oscillates around $\frac{E}{2\omega B}$



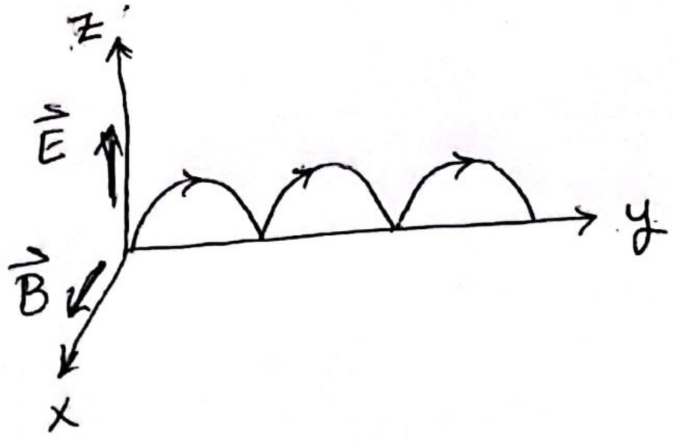
Let's re-write equations * and ** on the previous page as

$y(t) - \frac{E}{B}t = -\frac{E}{2\omega B}\sin(\omega t)$ and $z(t) - \frac{E}{2\omega B} = -\frac{E}{2\omega B}\cos(\omega t)$

Squaring both and adding and using $\sin^2(\omega t) + \cos^2(\omega t)$, we get

$$\left(y(t) - \frac{E}{B}t\right)^2 + \left(z(t) - \frac{E}{2\omega B}\right)^2 = \left(\frac{E}{2\omega B}\right)^2$$

This is the formula of a circle of radius $\frac{E}{2\omega B}$ whose center is at $\left(0, \frac{E}{B}t, \frac{E}{2\omega B}\right)$. Note that the y-component of the circle travels at constant speed $\frac{E}{B}$. This is a cycloid



The kinetic energy $K = \frac{1}{2} m \left(\dot{y}^2 + \dot{z}^2 \right)$

$$\dot{y}(t) = -\frac{E}{2B} \cos(\omega t) + \frac{E}{B}$$

$$\dot{z}(t) = \frac{E}{2B} \sin(\omega t)$$

$$K = \frac{1}{2} m \left(\frac{E^2}{4B^2} \cos^2(\omega t) - \frac{E^2}{B^2} \cos(\omega t) + \frac{E^2}{B^2} + \frac{E^2}{4B^2} \sin^2(\omega t) \right)$$

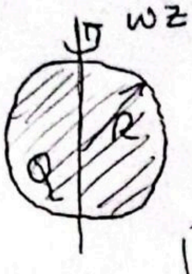
$$= \frac{1}{2} m \left(\frac{5E^2}{4B^2} - \frac{E^2}{B^2} \cos(\omega t) \right) = \frac{1}{2} m \frac{E^2}{B^2} \left(\frac{5}{4} - \cos(\omega t) \right)$$

Time dependent !

Energy must be exchanged with the field.

Current Densities

(1)



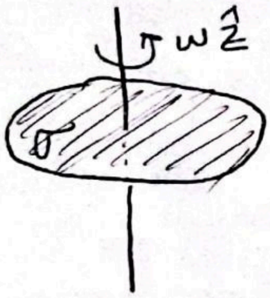
$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega \hat{z} \times r\hat{r} = \omega r \underbrace{\hat{z} \times \hat{r}}_{\sin\theta\,\hat{\phi}}$$ in sph. coord.

$$\boxed{\vec{J} = \rho \vec{v} = \frac{Q}{\frac{4}{3}\pi R^3}\, \omega r \sin\theta\, \hat{\phi}}$$

in spherical coordinates

(2)



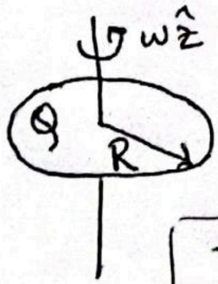
$$\vec{v} = \vec{\omega} \times \vec{r} = \omega \hat{z} \times s\hat{s}$$ in cylindrical coordinates

$$= \omega s \underbrace{\hat{z} \times \hat{s}}_{\hat{\phi}}$$ in cyl. coord.

$$\boxed{\vec{K} = \sigma \vec{v} = \sigma \omega s\, \hat{\phi}}$$

For the volume charge density, the current is constrained to xy-plane at $z=0$. $\Rightarrow$ $\boxed{\vec{J} = \sigma \omega s\, \delta(z)\, \hat{\phi}}$

(3)



$$\lambda = \frac{Q}{2\pi R}$$

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega \hat{z} \times R\hat{s}$$ in cylindrical coordinates

$$= \omega R\, \hat{\phi}$$

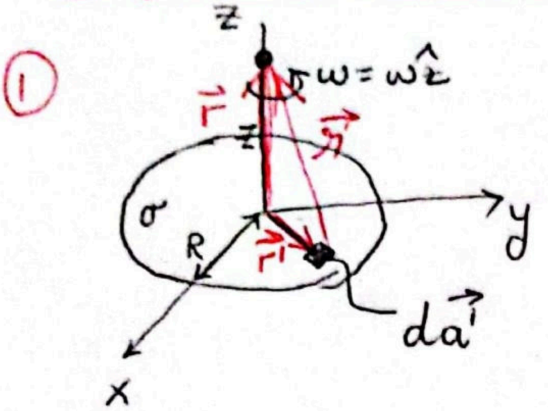
$$\boxed{\vec{I} = \lambda \vec{v} = \frac{Q}{2\pi \cancel{R}}\, \omega \cancel{R}\, \hat{\phi} = \frac{Q\omega}{2\pi}\, \hat{\phi}}$$

The current is constrained to $z=0$ and $s=R$. So

$$\boxed{\vec{J} = \frac{\omega Q}{2\pi}\, \delta(s-R)\, \delta(z)\, \hat{\phi}}$$

Magnetic field of distributed currents

①



$$\vec{r} = z\,\hat{z}$$

$$\vec{r}\,' = s'\hat{s}$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = z\,\hat{z} - s'\hat{s}$$

$$\hat{\mathscr{r}} = \frac{z\,\hat{z} - s'\hat{s}}{\sqrt{z^2 + s'^2}}$$

$$da' = s'ds'd\phi'$$

$$\vec{K} \times \vec{\mathscr{r}} = \sigma\omega s'\hat{\phi} \times (z\,\hat{z} - s'\hat{s}) = \sigma\omega s' z \underbrace{\hat{\phi}\times\hat{z}}_{\hat{s}} - \sigma\omega s'^2 \underbrace{\hat{\phi}\times\hat{s}}_{-\hat{z}}$$

$$= \sigma\omega s' z\,\hat{s} + \sigma\omega s'^2\,\hat{z}$$

② $$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_S \frac{\vec{K}(\vec{r}\,') \times \vec{\mathscr{r}}}{\mathscr{r}^3}\,da' = \frac{\mu_0}{4\pi}\int_S \frac{\sigma\omega s' z\,\hat{s} + \sigma\omega s'^2\,\hat{z}}{(z^2+s'^2)^{3/2}}\,s'ds'd\phi'$$

Using $\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$, we get

x-component: $$B_x = \frac{\mu_0}{4\pi}\int_S \frac{\sigma\omega s'^2 z\cos\phi'}{(z^2+s'^2)^{3/2}}\,ds'd\phi' = 0$$

because $$\int_0^{2\pi}\cos\phi'\,d\phi' = 0$$

y-component: $$B_y = \frac{\mu_0}{4\pi}\int \frac{\sigma\omega s'^2 z\sin\phi'}{(z^2+s'^2)^{3/2}}\,ds'd\phi' = 0$$

because $$\int_0^{2\pi}\sin\phi'\,d\phi' = 0$$

z-component: $$B_z = \frac{\mu_0}{4\pi} \underbrace{\int_0^R \frac{\sigma w s'^3}{(z^2+s'^2)^{3/2}}\, ds'}_{\sigma w \left.\frac{2z^2+s^2}{\sqrt{z^2+s^2}}\right|_{s=0}^{R}} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi}$$

$$\sigma w \left.\frac{2z^2+s^2}{\sqrt{z^2+s^2}}\right|_{s=0}^{R} = \sigma w \left[\frac{2z^2+R^2}{\sqrt{z^2+R^2}} - \frac{2z^2}{|z|}\right]$$

$$\vec{B} = \frac{\mu_0 \sigma w}{2}\left[\frac{R^2+2z^2}{\sqrt{R^2+z^2}} - \frac{2z^2}{|z|}\right]\hat{z}$$

For upper half of plane $z>0$: $\vec{B} = \frac{\mu_0\sigma w}{2}\left[\frac{R^2+2z^2}{\sqrt{R^2+z^2}} - 2z\right]\hat{z}$

(2) Unit check:

$[\mu_0] = \left[\frac{N}{A^2} = \frac{kg\, m}{A s^2}\right]$ $\quad [\sigma] = \frac{C}{m^2}$ $\quad [w] = \frac{1}{s}$ $\quad [\ \ ] = m$

All : $\frac{kg\, \cancel{m}}{\underbrace{A^2 s^2}_{(C/s)^2}} \cdot \frac{C}{\cancel{m^2}} \cdot \frac{1}{s} \cdot \cancel{m} = \frac{kg}{C \cdot s} = T = [B]$ ✓

As $R \to 0$, $B \to 0$ no disk $B = \frac{\mu_0 \sigma w}{2}\left[\frac{0+2z^2}{\sqrt{0^2+z^2}} - 2z\right] = 0$ ✓

As $w \to 0$, $B \to 0$ no spin $B \propto w \to 0$ ✓

As $z \to \infty$, $R/z \ll 1$. Let's re-write B as

$$B = \frac{\mu_0 \sigma w}{\cancel{2}}\left[\frac{\cancel{2} z^{\cancel{2}}\left(1+\frac{1}{2}\frac{R^2}{z^2}\right)}{\cancel{z}\left(1+\frac{R^2}{z^2}\right)^{1/2}} - \cancel{2}z\right]$$

$$\mu_0 \sigma w z\left[\left(1+\frac{1}{2}\frac{R^2}{z^2}\right)\left(1-\frac{1}{2}\frac{R^2}{z^2}+\frac{3}{8}\frac{R^4}{z^4} - \dots\right) - 1\right]$$

$$= \mu_0 \sigma w z\left[\cancel{1} - \cancel{\frac{1}{2}\frac{R^2}{z^2}} + \frac{3}{8}\frac{R^4}{z^4} + \cancel{\frac{1}{2}\frac{R^2}{z^2}} - \frac{1}{4}\frac{R^4}{z^4} + \frac{3}{16}\frac{R^6}{z^6} - \cancel{1}\right]$$

$$= \mu_0 \sigma w z \left(\frac{1}{8}\frac{R^4}{z^4} + \dots\right) \simeq \frac{\mu_0 \sigma w z R^4}{8}\frac{1}{z^3} + \dots$$