Taylor Series Approximations

We use Taylor series to solve differential equations

Objectives

In this dive we use the Taylor series of a function to find a sequence of approximate solutions of a first order differential equation. We start recalling the Taylor series expansion centered at $t = t_0$ of a function $y(t)$, which is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} \frac{1}{n!}\, y^{(n)}(t_0)\,(t - t_0)^n \\ &= y(t_0) + y'(t_0)\,(t - t_0) + \frac{1}{2!}\, y''(t_0)\,(t - t_0)^2 + \cdots , \end{aligned}$$

where $n!$ is the n-th factorial, $y^{(n)}(t_0)$ is the n-th derivative of y evaluated at $t = t_0$, but we also denoted $y^{(0)} = y$ (the zero derivative is the original function), and $y^{(1)} = y'$, $y^{(2)} = y''$ (first and second derivatives are denoted as usual). We also used that $0! = 1$ and $1! = 1$. In this dive we focus on the Taylor formula centered at $t_0 = 0$, which is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} \frac{1}{n!}\, y^{(n)}(0)\, t^n \\ &= y(0) + y'(0)\, t + \frac{1}{2!}\, y''(0)\, t^2 + \cdots . \end{aligned}$$

The first $n + 1$ terms of the expansion above are called the n-th order Taylor approximation.

Definition 1. *The n-th order **Taylor approximation** centered at t_0 of a function y is given by*

$$\tau_n(t) = \sum_{k=0}^{n} \frac{1}{k!}\, y^{(k)}(t_0)\,(t - t_0)^k$$

Notice that the definition above implies a simple relation between τ_n and τ_{n-1},

$$\tau_n(t) = \tau_{n-1}(t) + \frac{1}{n!}\, y^{(n)}(t_0)\,(t - t_0)^n.$$

We will use these polynomials to get approximate solutions of a differential equation.

Further Reading

If the summaries in this Dive are not enough to fully understand any of the transformations in this dive, then students may find useful to read from our textbook the appropriate parts of the following sections or subsections:

- In Section 1.9, Approximate Solutions, see subsection 1.9.2 or help with Taylor Series.

Solving Differential Equations

Let $y(t)$ be a solution of a differential equation with an initial condition,

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0.$$

It turns out that the initial condition and the differential equation is enough to compute all the derivatives of the function $y(t)$ at the time of the initial condition, t_0.

Theorem 2 (Taylor Approximation)**.** *The initial value problem*

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0, \tag{1}$$

with $f(t, y)$ infinitely continuously differentiable in both variables, determines $\tau_n(t)$, the n-th order Taylor approximation of the solution $y(t)$ of (1), for any integer $n \geqslant 0$.

Question 1. *(20 points)* Give an idea of the proof of Theorem 2 by computing the Taylor approximation $\tau_3(t)$. You do not need to compute higher order approximations.

Question 2. *(20 points)* Use the Taylor approximation defined in Theorem 2 to find the first four approximate solutions of the linear initial value problem

$$y'(t) = 3\,y(t) + 2, \qquad y(0) = 1.$$

Question 3. *(20 points)* Use the Taylor approximation defined in Theorem 2 to find the solution formula for all solutions of the initial value problem

$$y'(t) = a\,y(t) + b, \qquad y(0) = y_0,$$

with a, b constants, that is, use the Taylor approximation method to find the formula

$$y(t) = \left(y_0 + \frac{a}{b}\right) e^{at} - \frac{b}{a}.$$

Question 4. *(20 points)* Use the Taylor approximation defined in Theorem 2 to find the first four approximate solutions of the linear initial value problem

$$y' = 2t\,y^2 + t^2 + 3, \qquad y(0) = 1.$$

Question 5. *(20 points)* Let $\tau_n(t)$ be the Taylor approximation given in Theorem 2. Also assume that the limit $n \to \infty$ of $\tau_n(t)$ converges and

$$y_T(t) = \lim_{n\to\infty} \tau_n(t)$$

is a continuously differentiable function. Then, show that this function $y_T(t)$ is a solution of the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0.$$

Hint: Since we assume that $y_T(t)$ is well defined, so is the function $g(t) = f(t, y_T(t))$. Study the relation between $y_T(t)$ and $g(t)$ and their derivatives when we evaluate them at t_0.