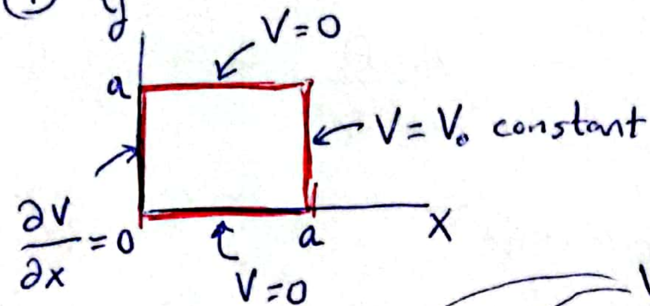
Rectangular pipe: Separation of Variables
Cartesian - 2D

① 

Infinitely long pipe in z-direction
$\Rightarrow$ no z dependence

$$V(x,y,z) = X(x)\,Y(y)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \underbrace{\frac{\partial V}{\partial z^2}}_{0 \text{ here}} = 0$$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0 \xrightarrow[\text{by } V=XY]{\text{divide}} \underbrace{\frac{1}{X^2}\frac{d^2X}{dx^2}}_{c_1} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{c_2} = 0$$

Because $V=0$ at $y=0$ and $y=a$, we expect a periodic behaviour in y. So, choose $c_2 = -k^2$. Therefore, $c_1 = k^2$

$$\frac{1}{X}\frac{d^2X}{dx^2} = k^2 \rightarrow X(x) = Ae^{kx} + Be^{-kx}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2 \rightarrow Y(y) = C\cos(ky) + D\sin(ky)$$

- Always a good idea to start with periodic boundary conditions.

$$Y(0) = 0 = C\cos 0 + D\sin 0 \Rightarrow C = 0$$

$$Y(a) = 0 = D\sin(ka) \Rightarrow ka = n\pi \Rightarrow k = \frac{n\pi}{a}$$

where $n \geq 1$ is an integer

- Apply the boundary condition at $x=0$

$$\frac{\partial V}{\partial x} = Y \cdot \frac{d}{dx}\left[A e^{kx} + B e^{-kx}\right] = Y \cdot k\left[A e^{kx} - B e^{-kx}\right]$$

$$\left.\frac{\partial V}{\partial x}\right|_{x=0} = 0 = Y \cdot k\left[A e^{0} - B e^{0}\right] \Rightarrow A - B = 0 \Rightarrow A = B$$

So, $X(x) = A\left[e^{kx} + e^{-kx}\right] = 2A\cosh(kx)$

- Now, $V_n = 2A\cosh(kx) \cdot D_n \sin(ky)$; $k = \frac{n\pi}{a}$ $n \geq 1$ is an integer

Absorb $2AD_n$ into one C_n.

$$\boxed{V = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi}{a}x\right) \sin\left(\frac{n\pi}{a}y\right)}$$

- Apply Fourier's trick for the boundary condition at $x=a$

$$V(x=a, y) = V_0 = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi}{\cancel{a}}\cancel{a}\right) \sin\left(\frac{n\pi}{a}y\right)$$

by multiplying both sides by $\sin\left(\frac{n'\pi}{a}y\right)$ and integrating for $0 \leq y \leq a$

$$\int_0^a \underset{\text{constant here}}{V_0} \sin\left(\frac{n'\pi}{a}y\right)dy = \int_0^a \sum_{n=1}^{\infty} C_n \cosh(n\pi) \sin\left(\frac{n\pi}{a}y\right) \sin\left(\frac{n'\pi}{a}\right)y \, dy \quad \circledast$$

Left hand side: $V_0 \int_0^a \sin\left(\frac{n'\pi}{a}y\right)dy = V_0 \left.\frac{\cos\left(\frac{n'\pi y}{a}\right)}{-\frac{n'\pi}{a}}\right|_0^a = -\frac{aV_0}{n'\pi}\left[\underbrace{\cos(n'\pi)}_{(-1)^{n'}} - 1\right]$

$$= -\frac{aV_0}{n'\pi}\left[\underbrace{(-1)^{n'} - 1}\right]$$

→ -2 if n' is odd
0 if n' is even

So the left hand side is $\frac{2aV_0}{n'\pi}$ for odd n'

0 for even n'

Right hand side of ⊛ :

$$= \sum_{n=1}^{\infty} C_n \cosh(n\pi) \underbrace{\int_0^a \sin\left(\frac{n\pi}{a}y\right) \cdot \sin\left(\frac{n'\pi}{a}\right) y \, dy}_{\substack{0 \text{ if } n \neq n' \\ \frac{a}{2} \text{ if } n = n'}} = C_{n'} \cosh(n'\pi) \frac{a}{2}$$

LHS = RHS

Odd n' $\quad \frac{2aV_0}{n'\pi} = C_{n'} \cosh(n'\pi) \frac{a}{2} \Rightarrow C_{n'} = \frac{4V_0}{n'\pi \cosh(n'\pi)}$

Even n' $\quad 0 = C_{n'} \cosh(n'\pi) \frac{a}{2} \rightarrow C_{n'} = 0$

Label n' to n for convenience and put it into V.

$$\boxed{V(x,y,z) = \frac{4V_0}{\pi} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{\cosh\left(\frac{n\pi}{a}x\right)}{\cosh(n\pi)} \cdot \frac{\sin\left(\frac{n\pi}{a}y\right)}{n}}$$

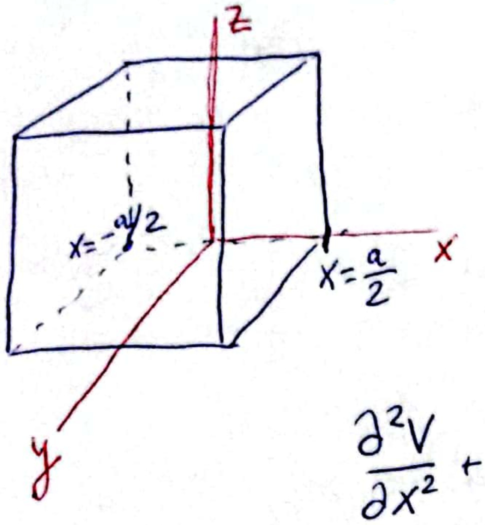
2) Use $\left.\frac{\partial V}{\partial n}\right|_{\text{above}} - \left.\frac{\partial V}{\partial n}\right|_{\text{below}} = -\frac{\sigma}{\epsilon_0}$

Bottom plate is grounded $(V=0)$. So $\left.\frac{\partial V}{\partial n}\right|_{\text{below}} = 0$

$$\Rightarrow \sigma = -\epsilon_0 \left.\frac{\partial V}{\partial n}\right|_{\text{above}} = -\epsilon_0 \left.\frac{\partial V}{\partial y}\right|_{y=0} = -\epsilon_0 \cdot \frac{4V_0}{\cancel{\pi}} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{\cosh\left(\frac{n\pi x}{a}\right)}{\cosh(n\pi)} \cdot \frac{\frac{\cancel{n}\cancel{\pi}}{a}\cos(0)}{\cancel{n}}$$

$$\Rightarrow \boxed{\sigma_{\text{bottom plate}} = -\frac{4\epsilon_0 V_0}{a} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{\cosh\left(\frac{n\pi}{a}x\right)}{\cos(n\pi)}}$$

Cubical Box : Separation of Variables - Cartesian - 3D



All surfaces are at $V=0$ except the surfaces at $x=-\frac{a}{2}$ and $x=\frac{a}{2}$.

$$V(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \Rightarrow YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0$$

Then divide by XYZ

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{c_1} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{c_2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{c_3} = 0$$

$V=0$ at $y=0$, $y=a$, $z=0$, and $z=a$ implies periodicity.

So choose $c_2 = -l^2$, $c_3 = -p^2$, $\Rightarrow$ $c_1 = \boxed{l^2 + p^2 = k^2}$

$$\frac{1}{X}\frac{d^2X}{dx^2} = k^2 \Rightarrow X(x) = A e^{kx} + B e^{-kx}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -l^2 \Rightarrow Y(y) = C\cos(ly) + D\sin(ly)$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -p^2 \Rightarrow Z(z) = E\cos(pz) + F\sin(pz)$$

- Apply periodic boundary conditions first.

$V=0$ at $y=0$ $\Rightarrow Y(0)=0=C\cos(0)+D\sin(0) \Rightarrow C=0$

$V=0$ at $y=a$ $\Rightarrow Y(a)=0=D\sin(la) \Rightarrow la = n\pi$ where $n\geq 1$ is an integer

Similarly in z direction : $E=0$ and $pa = m\pi$ where $m \geq 1$ is an integer

Hence, $Y(y) = D \sin\left(\frac{n\pi}{a} y\right)$ and $Z(z) = F \sin\left(\frac{m\pi}{a} z\right)$

and $X(x) = A e^{kx} + B e^{-kx}$; $k^2 = l^2 + p^2 = \frac{\pi^2}{a^2}(n^2+m^2)$

~~Apply the boundary Condition V = V₀ for x = ±a/2 and 0 ≤ y ≤ a, 0 ≤ z ≤ a~~

So far we have

$$V_{n,m} = \left[A e^{kx} + B e^{-kx}\right] D_n \sin\left(\frac{n\pi}{a} y\right) F_m \cdot \sin\left(\frac{m\pi}{a} z\right)$$

Absorb into A and B and apply the boundary condition at $x = \frac{a}{2}$ $(V = V_0)$

$$V_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[A_{n,m} e^{k\frac{a}{2}} + B_{n,m} e^{-k\frac{a}{2}}\right] \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} z\right)$$

Apply Fourier's trick by multiplying both sides by $\sin\left(\frac{n'\pi}{a} y\right) \cdot \sin\left(\frac{m'\pi}{a} z\right)$ and integrating.

$$\int_0^a \int_0^a V_0 \sin\left(\frac{n'\pi}{a} y\right) \cdot \sin\left(\frac{m'\pi}{a} z\right) dy\, dz = \iint_0^a \sum_{n,m=1}^{\infty} \left[A_{n,m} e^{k\frac{a}{2}} + B_{n,m} e^{-k\frac{a}{2}}\right] \sin\left(\frac{n\pi}{a} y\right)$$

$$\times \sin\left(\frac{m\pi}{a}\right) z \cdot \sin\left(\frac{n'\pi}{a} y\right) \cdot \sin\frac{m'\pi}{a} z$$

$$\times\, dy\, dz$$

Let's solve the left-hand side first.

$$V_0 \int_0^a \sin\left(\frac{n'\pi}{a}y\right) dy \int_0^a \sin\left(\frac{m'\pi}{a}z\right) dz = V_0 \left[\frac{\cos\left(\frac{n'\pi}{a}y\right)}{-\frac{n'\pi}{a}} \Big|_0^a \right] \left[\frac{\cos\left(\frac{m'\pi}{a}\right)z}{-\frac{m'\pi}{a}} \Big|_0^a \right]$$

$$= \frac{V_0 a^2}{\pi^2 n' m'} \underbrace{\left[\underbrace{\cos(n'\pi)}_{(-1)^{n'}} - 1 \right]}_{\substack{-2 \text{ if } n' \text{ odd} \\ 0 \text{ if } n' \text{ even}}} \underbrace{\left[\underbrace{\cos(m'\pi)}_{(-1)^{m'}} - 1 \right]}_{\substack{-2 \text{ if } m' \text{ odd} \\ 0 \text{ if } m' \text{ even}}}$$

$$= \frac{4V_0 a^2}{\pi^2 n' m'} \text{ if both } n' \text{ and } m' \text{ odd. } 0 \text{ otherwise.}$$

Now the right-hand side:

$$\sum_{\substack{n,m=1 \\ odd}}^{\infty} \left[A_{n,m} e^{\frac{ka}{2}} + B_{n,m} e^{-\frac{ka}{2}} \right] \left[\int_0^a \sin\left(\frac{n\pi y}{a}\right) \cdot \sin\left(\frac{n'\pi}{a}y\right) dy \right] \times \underbrace{\left[\int_0^a \sin\left(\frac{m\pi}{a}z\right) \cdot \sin\left(\frac{m'\pi}{a}z\right) dz \right]}_{\substack{0 \text{ if } m \neq m' \\ \frac{a}{2} \text{ if } m = m'}}$$

$$= \begin{cases} 0 & \text{if } n \neq n' \text{ or } m \neq m' \\ \left[A_{n',m'} e^{k\frac{a}{2}} + B e^{-k\frac{a}{2}}_{n,m'} \right] \cdot \left(\frac{a}{2}\right)^2 & \text{if } n = n' \text{ and } m = m' \end{cases}$$

Combine LHS and RHS.

$$\frac{4V_0 a^2}{\pi^2 n' m'} = \frac{a^2}{4} \left[A_{n',m'} e^{k\frac{a}{2}} + B_{n'm'} e^{-k\frac{a}{2}} \right]$$

Re-label n' to n and m' to m for convenience.

$$A_{n,m}\, e^{k\frac{a}{2}} + B_{n,m}\, e^{-k\frac{a}{2}} = \frac{16 V_0}{\pi\, n \cdot m} \qquad (*)$$

We can now use $V = V_0$ at $x = -\frac{a}{2}$ in a similar way.
The only change from the equation above is $\frac{a}{2} \to -\frac{a}{2}$.

$$A_{n,m}\, e^{-k\frac{a}{2}} + B_{n,m}\, e^{k\frac{a}{2}} = \frac{16 V_0}{\pi\, n \cdot m} \qquad (**)$$

Now $(*)$ and $(**)$ are two equations with two unknowns.

Subtracting: $(*) - (**) = (A_{n,m} - B_{n,m})\, e^{k\frac{a}{2}} - (A_{n,m} - B_{n,m})\, e^{-k\frac{a}{2}}$

$= 0$

$$\Rightarrow \quad (A_{n,m} - B_{n,m})\left(e^{k\frac{a}{2}} - e^{-k\frac{a}{2}}\right) = 0$$

$\Rightarrow \quad A_{n,m} = B_{n,m}$. Put this into $*$

$$A_{n,m}\underbrace{\left(e^{k\frac{a}{2}} + e^{-k\frac{a}{2}}\right)}_{2\cosh\left(k\frac{a}{2}\right)} = \frac{16 V_0}{\pi\, nm} \quad \Rightarrow \quad A_{n,m} = B_{n,m} = \frac{8 V_0}{\pi \cdot n \cdot m \cosh\left(k\frac{a}{2}\right)}$$

Putting it altogether.

$$V = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \sum_{\substack{m=1 \\ \text{odd}}}^{\infty} \left[A_{n,m}\, e^{kx} + B_{n,m}\, e^{-kx}\right] \sin\left(\frac{n\pi}{a} y\right) \cdot \sin\left(\frac{m\pi}{a} z\right)$$

$$= \frac{8 V_0}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{e^{kx} + e^{-kx}}{\cosh\left(k\frac{a}{2}\right)} \, \frac{\sin\left(\frac{n\pi}{a} y\right)}{n} \cdot \frac{\sin\left(\frac{m\pi}{a} z\right)}{m}$$

Use again $e^{kx} + e^{-kx} = 2\cosh(kx)$

and $k = \sqrt{l^2 + p^2} = \frac{\pi}{a}\sqrt{n^2+m^2}$ to write

$$\boxed{V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{\substack{n=1\\ \text{odd}}}^{\infty} \sum_{\substack{m=1\\ \text{odd}}}^{\infty} \frac{\cosh\left[\frac{\sqrt{n^2+m^2}\,\pi}{a} x\right]}{\cosh\left[\sqrt{n^2+m^2}\cdot\frac{\pi}{2}\right]} \cdot \frac{\sin\left(\frac{n\pi}{a}y\right)}{n} \cdot \frac{\sin\left(\frac{m\pi}{a}z\right)}{m}}$$

② $$V_{center} = \frac{\sum_{sides} V_{sides}}{6} = \frac{0+0+0+0+3+3}{6} = 1\,V$$

Check by setting $n=1$, $m=1$ in V. for $x=0, y=\frac{a}{2}, z=\frac{a}{2}$ (center)

$$V \approx \frac{16\overset{3}{V_0}}{\pi^2} \cdot \underbrace{\frac{\overset{\to 1}{\cosh(0)}}{\cosh\left(\frac{\sqrt{2}\,\pi}{2}\right)}}_{\to 4.664\ldots} \cdot \frac{\overset{\to 1}{\sin\frac{\pi}{2}}}{1} \cdot \frac{\overset{\to 1}{\sin\frac{\pi}{2}}}{1} = 1.0426\ldots \text{ Volt.}$$

Including more terms from the sum will get us closer and closer to $1V$.

③ Method 1 : By inspection:

Four boundaries with 0 potential have the same charge distribution by symmetry. Similarly, for sides where $V=V_0$.
So, exactly at the center of the box, $\vec{E}$ field due to all charges cancels out. $\vec{E}_{center} = 0$

Method 2: Use $\vec{E} = -\vec{\nabla} V$

$$\vec{E} = -YZ\left(\frac{dX}{dx}\right)\hat{x} - XZ\left(\frac{dY}{dy}\right)\hat{y} - XY\left(\frac{dZ}{dz}\right)\hat{z}$$

$\frac{dX}{dx} = k \sinh(kx)$: For $x=0$, $\sinh(0)=0$

$\frac{dY}{dy} = \frac{n\pi}{a}\cos\left(\frac{n\pi}{a}y\right)$: For $y = \frac{a}{2}$; $\cos\left(\frac{n\pi}{2}\right) = 0$ (remember n is odd)

$\frac{dZ}{dz} = \frac{m\pi}{a}\cos\left(\frac{m\pi}{a}z\right)$: For $z = \frac{a}{2}$, $\cos\left(\frac{m\pi}{2}\right) = 0$ (m is also odd)

Hence at $(0, \frac{a}{2}, \frac{a}{2})$; $\vec{E} = 0$.

Sphere with a known potential

① $V_0 = k\cos(3\theta)$

Use $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$:

$\cos(3\theta) = \cos(2\theta+\theta) = \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta$

and

$\cos(2\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$

Using $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$:

$\sin(2\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\cos\theta\sin\theta$

Using $\sin^2\theta = 1-\cos^2\theta$: $\cos(2\theta) = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$

Putting altogether: $\cos(3\theta) = (2\cos^2\theta - 1)\cos\theta - (2\cos\theta \cdot \sin\theta)\cdot\sin\theta$

$$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta)$$

$$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta = 4\cos^3\theta - 3\cos\theta$$

Using

$P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, $P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$

and $P_3(\cos\theta) = \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$; we have

$$\cos^3\theta = \frac{2}{5}\Big(P_3 + \frac{3}{2}\underbrace{\cos\theta}_{P_1}\Big) = \underbrace{\frac{2}{5}P_3 + \frac{3}{5}P_1}$$

$$V_0 = k\cos(3\theta) = k\Big[4\cos^3\theta - 3\overbrace{\cos\theta}^{P_1}\Big] = k\Big[4\cdot\Big(\frac{2}{5}P_3 + \frac{3}{5}P_1\Big) - 3P_1\Big]$$

$$= k\Big(\frac{8}{5}P_3 - \frac{3}{5}P_1\Big)$$

② Inside the sphere : V finite as $r \to 0$

$\Rightarrow$ all $B_\ell = 0$

$$V_{in} = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)$$

$$V(r=R) = k\left[\frac{8}{5}P_3(\cos\theta) - \frac{3}{5}P_1(\cos\theta)\right] = \sum_\ell A_\ell \cdot R^\ell P_\ell(\cos\theta)$$

$\Rightarrow$ $A_\ell = 0$ for all ℓ except $\ell = 3$ and $\ell = 1$

$$k\left[\frac{8}{5}P_3 - \frac{3}{5}P_1\right] = A_1 R^1 P_1 + A_3 R^3 P_3$$

$$\Rightarrow \quad A_1 = -\frac{3k}{5R} \quad , \quad A_3 = \frac{8k}{5R^3}$$

$$\boxed{\underset{(r \leq R)}{V_{in}(r,\theta)} = -\frac{3k}{5}\frac{r}{R}P_1(\cos\theta) + \frac{8k}{5}\left(\frac{r}{R}\right)^3 P_3(\cos\theta)}$$

③ Outside the sphere : $V \to 0$ as $r \to \infty$

$\Rightarrow$ all $A_\ell = 0$

$$V_{out} = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$V(r=R) = k\left[\frac{8}{5}P_3(\cos\theta) - \frac{3}{5}P_1(\cos\theta)\right] = \sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$\Rightarrow$ $B_\ell = 0$ for all ℓ except $\ell = 3$ and $\ell = 1$

$$k\left[\frac{8}{5}P_3 - \frac{3}{5}P_1\right] = \frac{B_1}{R^2}P_1 + \frac{B_3}{R^4}P_3$$

$$\Rightarrow \quad B_1 = -\frac{3}{5}kR^2 \quad , \quad B_3 = \frac{8}{5}kR^4$$

$$\boxed{V_{out}(r,\theta) = -\frac{3}{5}k\left(\frac{R}{r}\right)^2 P_1(\cos\theta) + \frac{8}{5}k\left(\frac{R}{r}\right)^4 P_3(\cos\theta) \quad (r \geq R)}$$

④ $V_{in}(R,\theta) \overset{?}{=} V_{out}(R,\theta)$

$$-\frac{3k}{5}\frac{R}{R}P_1(\cos\theta) + \frac{8k}{5}\left(\frac{R}{R}\right)^3 P_3(\cos\theta) \overset{?}{=} -\frac{3}{5}k\left(\frac{R}{R}\right)^2 P_1(\cos\theta) + \frac{8}{5}k\left(\frac{R}{R}\right)^4 P_3(\cos\theta)$$

✓

⑤
$$\frac{\partial V_{out}}{\partial r} = -\frac{3k}{5}P_1(\cos\theta)\cdot R^2\frac{dr^{-2}}{dr} + \frac{8k}{5}P_3(\cos\theta)\,R^4\frac{dr^{-4}}{dr}$$
$$= -\frac{3k}{5}P_1(\cos\theta)\cdot R^2\left(-\frac{2}{r^3}\right) + \frac{8k}{5}P_3(\cos\theta)\,R^4\cdot\left(-\frac{4}{r^5}\right)$$

Evaluate at $r=R$: $\left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} = \frac{6k\,P_1(\cos\theta)}{5R} - \frac{32k}{5R}P_3(\cos\theta)$

$$\frac{\partial V_{in}}{\partial r} = -\frac{3k}{5R}P_1(\cos\theta)\frac{dr}{dr} + \frac{8k}{5R^3}P_3(\cos\theta)\frac{dr^3}{dr}$$
$$= -\frac{3k}{5R}P_1(\cos\theta) + \frac{24k\,r^2}{5R^3}P_3(\cos\theta)$$

Evaluate at $r=R$: $\left.\frac{\partial V_{in}}{\partial r}\right|_{r=R} = -\frac{3k}{5R}P_1(\cos\theta) + \frac{24k}{5R}P_3(\cos\theta)$

$$\sigma = -\epsilon_0\left.\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\right|_{r=R} = \frac{\epsilon_0 k}{5R}\left[56P_3(\cos\theta) - 9P_1(\cos\theta)\right]$$

(6)

$$Q = \int \sigma \, da = \int_0^{2\pi}\int_0^{\pi} \sigma \, R^2 \sin\theta \, d\theta \, d\phi$$

$$= R^2 \Big[\underbrace{\int_0^{2\pi} d\phi}_{2\pi}\Big] \Big[\int_0^{\pi} \frac{\epsilon_0 k}{5R} \left[56\, P_3(\cos\theta) - 9\, P_1(\cos\theta)\right] \sin\theta \, d\theta$$

$$= \frac{2\pi \epsilon_0 k R}{5} \int_0^{\pi} \left(56\, P_3(\cos\theta) - 9\, P_1(\cos\theta)\right) \sin\theta \, d\theta$$

Multiply it by $1 = P_0(\cos\theta)$

$$Q = \frac{2\pi \epsilon_0 k R}{5} \Big[\int_0^{\pi} \left[56\, P_3(\cos\theta) - 9 P_1(\cos\theta)\right] \cdot P_0 \sin\theta \, d\theta$$

Using $\displaystyle \int_0^{\pi} P_l(\cos\theta)\, P_m(\cos\theta) \sin\theta \, d\theta = \begin{cases} 0 & \text{if } l \neq m \\ \frac{2}{2l+1} & \text{if } l = m \end{cases}$

we get $\boxed{Q = 0}$ because $3 \neq 0$ and $1 \neq 0$.

Separation of Variables in Cylindrical Coordinates

① Laplacian : $\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$

$V = S(s)\,\Phi(\phi)\,Z(z)$

cylindrial symmetry implies that $Z(z) = 1 \Rightarrow \frac{\partial^2 V}{\partial z^2} = 0$

Substitute $V = S(s)\Phi(\phi)$ in $\nabla^2 V = 0$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial(S\Phi)}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2(S\Phi)}{\partial \phi^2} + \cancel{\frac{\partial^2 V}{\partial z^2}}^{0} = 0$$

$$\frac{\Phi}{s}\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{S}{s^2}\frac{d^2\Phi}{d\phi^2} = 0$$

Divide by $V = S\Phi \rightarrow \frac{1}{sS}\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{1}{s^2\Phi}\frac{d^2\Phi}{d\phi^2} = 0 \quad \Big| \times s^2$

$$\underbrace{\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right)}_{c_1} + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}}_{c_2} = 0$$

② $\Phi(\phi)$ needs to be periodic in ϕ ; i.e. $\Phi(\phi) = \Phi(2\pi + \phi)$

$$= \Phi(2\pi \cdot m + \phi)$$

where m is an integer. So, choose $c_2 = -k^2$

Therefore, $c_1 = k^2$

③ $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -k^2 \quad \rightarrow \quad \Phi(\phi) = A\cos(k\phi) + B\sin(k\phi)$

Periodicity: $\Phi(2\pi+\phi) = A\cos(k\cdot 2\pi + k\phi) + B\sin(k\cdot 2\pi + k\phi) = \ldots$

$\Rightarrow$ k must be an integer.

So, $\boxed{\Phi_k(\phi) = A_k\cos(k\phi) + B_k\sin(k\phi) \qquad k \geq 0 \text{ integer}}$

④ $\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = k^2$; $k \geq 0$ integer

Case I: $k = 0 \Rightarrow \frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = 0 \Rightarrow \frac{d}{ds}\left(s\frac{dS}{ds}\right) = 0$

has two cases: $\rightarrow$ Ia) $\frac{dS}{ds} = 0 \rightarrow S(s) = c$ (nonzero constant)

Ib) $s\frac{dS}{ds} = \text{const.} \rightarrow \frac{dS}{ds} = \frac{\text{const.}}{s} \Rightarrow S(s) = \text{const.} \ln s$

Case II: $k \neq 0$: $\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = k^2$

$\frac{d}{ds}\left(s\frac{dS}{ds}\right) = \frac{k^2 S}{s} \rightarrow s\frac{d^2S}{ds^2} + \frac{dS}{ds} - k^2\frac{S}{s} = 0$ ⊛

Guess: $S = s^m$ (power law) $\Rightarrow \frac{d^2S}{ds^2} = m(m-1)s^{m-2}$

$\frac{dS}{ds} = m s^{m-1}$

Substitute in (*)

$$s \cdot m(m-1) s^{m-2} + m s^{m-1} - k^2 \frac{s^m}{s} = 0$$

$$m(m-1) s^{m-1} + m s^{m-1} - k^2 s^{m-1} = 0 \quad \rightarrow (m^2 - k^2) s^{m-1} = 0$$

$\Rightarrow \quad m = \pm k \qquad k \geq 0$ integer

Hence, $\boxed{\begin{array}{lll} S(s) = C + D\ln(s) & \text{for } k=0 & k \text{ integer} \\ S(s) = E s^k + F s^{-k} & \text{for } k>0 & \end{array}}$

(5) For $k=0$; B/c $\Phi_k(\phi) = A_k \cos(k\phi) + B_k \sin(k\phi)$

$$\Phi_0 = A_0$$

and $\quad S = C + D\ln(s)$

$$V_{k=0} = [C + D\ln(s)] \cdot A_0 = a_0 + b_0 \ln(s)$$

(just relabeled $CA_0 = a_0$, $DA_0 = b_0$)

For $k>0$: $\quad V_k = \underbrace{[E s^k + F s^{-k}]}_{S} \times \underbrace{[A_k \cos(k\phi) + B_k \sin(k\phi)]}_{\Phi}$

$$= s^k [a_k \cos(k\phi) + b_k \sin(k\phi)] + s^{-k} [c_k \cos(k\phi) + d_k \sin(k\phi)]$$

(relabeled $EA_k = a_k$, $EB_k = b_k$, $FA_k = c_k$, $FB_k = d_k$)

Hence, $\boxed{V = a_0 + b_0 \ln(s) + \sum_{k=1}^{\infty} s^k (a_k \cos(k\phi) + b_k \sin(k\phi)) + s^{-k} (c_k \cos(k\phi) + d_k \sin(k\phi))}$

(6) If $V(s) = \frac{2\lambda}{4\pi\epsilon_0} \ln(s) + \text{const}$, then

$\boxed{a_0 = \text{const.}, \; b_0 = \frac{2\lambda}{4\pi\epsilon_0}, \; \text{all } a_k, b_k, c_k, d_k \text{ zero for } k \geq 1}$