MTH 327h Honors Analysis I, Fall 2023 Homework 4

Problem 1 Let $P_k(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$ be a polynomial with nonnegative coefficients a_i and with $a_k > 0$. Prove that

$$\lim_{n\to\infty} \left[P_k(n)\right]^{1/n} = 1.$$

You may use without the proof that $[a_k]^{1/n} \to 1$ and $n^{1/n} \to 1$ as $n \to \infty$.

Problem 2 Find $\lim\limits_{n\to\infty} \left(\sqrt{n^2+2n} - \sqrt{n^2-1}\right)$. [Hint: multiply both numerator and denominator by $\left(\sqrt{n^2+2n} + \sqrt{n^2-1}\right)$ and analyze the obtained expression.]

Problem 3 Prove using only definition of the limit that $\lim\limits_{n\to\infty} n^{-1/3} = 0$.

Problem 4 Assume $\limsup a_n = A$ and $\limsup b_n = B$ with A and B real numbers (not $+\infty$).

(a) prove that $\limsup(a_n + b_n) \le A + B$;

(b) can it be that $\limsup(a_n + b_n) = -\infty$? [As always, if yes, provide an example, if no, prove it!]

Problem 5 Consider $(r_n) \subset \mathbb{Q}$ defined by the recurrence relation: $r_1 = 1$, $r_{n+1} = r_n + \dfrac{1}{r_n}$.

(a) show that (r_n) is monotonically increasing;

(b) Prove that this sequence has no limit. (Assume $r_n \to p < +\infty$ and come to a contradiction.) [Hint: if $\forall n > N$ $|r_n - p| < \varepsilon$, then what is $|r_{n+1} - r_n|$? Obviously, part (a) implies that p must be greater than one.]

(c) describe the behavior of (r_n) as $n \to \infty$ (just general properties, no details necessary);

Problem 6 Consider $(a_n) \subset \mathbb{Q}$ and such that

$\forall \varepsilon > 0\ \exists N : \forall n, m > N,\ |a_n - a_m| < \varepsilon$ (the **Cauchy property**).

Prove that (a_n) may have at most one subsequential limit in $\mathbb{Q}$.

Problem 7 Does a sequence (a_n) exist such that

(i) considered as a set $S = \{a_n\}$ in which coinciding terms of (a_n) are mapped into a single real number, we have that $S = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$;

(ii) the set of subsequential limits of (a_n) is equal to $\overline{S} = \{0\} \cup S$?

Problem 8* Construct a sequence (a_n) whose set of subsequential limits is exactly the Cantor set or prove that such sequence does not exist.