Physics 471: Homework 2 Solutions

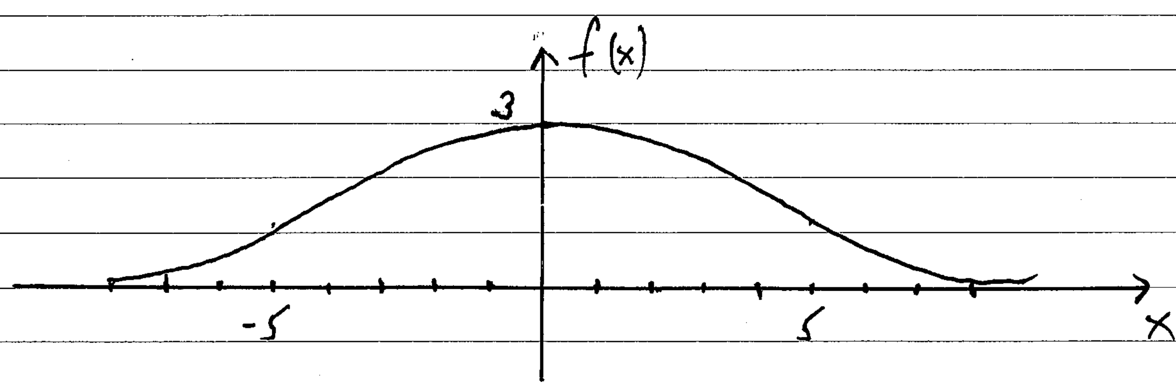
1. Since the exact sequence of heads and tails is specified, the probability is

$$P = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

If I had asked for any combination of 3 heads and 2 tails, then the probability would be larger by a factor $\frac{5!}{3!\,2!} = 10$.

2. $$f(x) = 3e^{-x^2/25} = 3e^{-\left(\frac{x}{5}\right)^2}$$

This is a Gaussian centered at $x=0$ with a peak value of 3. It falls off by a factor $\frac{1}{e}$ when $x = \pm 5$.



This should be perfectly symmetric, but my hand isn't steady enough to draw that!

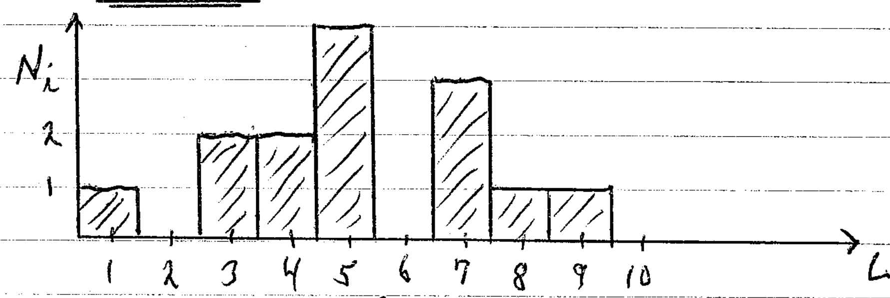
3. Objects have lengths (in cm): 1, 3, 3, 4, 4, 5, 5, 5, 5, 7, 7, 7, 8, 9

a) Prob. of 4 cm is $\frac{2}{14} = \underline{\underline{0.14}}$

b) Average length $\langle L \rangle = \frac{\sum_i L_i}{N} = \frac{1 + 2\cdot 3 + 2\cdot 4 + 4\cdot 5 + 3\cdot 7 + 8 + 9}{14}$

$$\langle L \rangle = \frac{73}{14} = \underline{\underline{5.21\ cm}}$$

Optional histogram



There are no objects with $L = 5.2$ cm so probability is zero to get that result.

c) $\langle L^2 \rangle = \frac{\sum_i L_i^2}{N} = \frac{1 + 2\cdot 9 + 2\cdot 16 + 4\cdot 25 + 3\cdot 49 + 64 + 81}{14}$

$$\langle L^2 \rangle = \frac{443}{14} = \underline{\underline{31.6\ cm^2}}$$

d) McIntyre Eqn. (A.11):

$$\sigma_L = \sqrt{\langle L^2 \rangle - \langle L \rangle^2} = \sqrt{31.6 - 27.1}\ cm = \underline{\underline{2.1\ cm}}$$

e) $\langle L \rangle - \sigma_L = (5.2 - 2.1)\,cm = 3.1\ cm$

$\langle L \rangle + \sigma_L = (5.2 + 2.1)\,cm = 7.3\ cm$

9 of the 14 objects lie in this range

Probability $= \frac{9}{14} = \underline{\underline{0.64}}$ or about $\frac{2}{3}$, which is reasonable

4. $|\psi_1\rangle = \frac{2}{\sqrt{5}}|+\rangle + \frac{1}{\sqrt{5}}|-\rangle$

$|\psi_2\rangle = \sqrt{\frac{2}{3}}|+\rangle + \frac{i}{\sqrt{3}}|-\rangle$

$|\psi_3\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\pi/4}|-\rangle$

a) Find $|\phi_1\rangle$ that is orthogonal to $|\psi_1\rangle$

i.e. $\langle\psi_1|\phi_1\rangle = 0$

Let's first write down the bra's $\langle\psi_i|$ that correspond to the given kets, remembering to take the complex conjugates of the coefficients:

$$\langle\psi_1| = \frac{2}{\sqrt{5}}\langle+| + \frac{1}{\sqrt{5}}\langle-|$$

$$\langle\psi_2| = \sqrt{\frac{2}{3}}\langle+| - \frac{i}{\sqrt{3}}\langle-|$$

$$\langle\psi_3| = \frac{1}{\sqrt{2}}\langle+| + \frac{1}{\sqrt{2}}e^{-i\pi/4}\langle-|$$

Write $|\phi_1\rangle = a|+\rangle + b|-\rangle$

then $\langle\psi_1|\phi_1\rangle = \frac{2}{\sqrt{5}}a + \frac{1}{\sqrt{5}}b = 0 \Rightarrow \underline{b = -2a}$

We need $|\phi_1\rangle$ to be normalized, so choose

$a = \frac{1}{\sqrt{5}}$, $b = \frac{-2}{\sqrt{5}}$ $\qquad \underline{|\phi_1\rangle = \frac{1}{\sqrt{5}}|+\rangle - \frac{2}{\sqrt{5}}|-\rangle}$

You could multiply $|\phi_1\rangle$ by -1 or any other phase factor and you would get the same state.

$$\langle \psi_2 | \phi_2 \rangle = \sqrt{\frac{2}{3}}\, a - \frac{i}{\sqrt{3}}\, b = 0 \Rightarrow b = \frac{\sqrt{2}}{i}\, a = -i\sqrt{2}\, a$$

I'll choose a to be real and positive, so

$$a = \frac{1}{\sqrt{3}}, \quad b = -i\sqrt{\frac{2}{3}}, \quad \underline{\underline{|\phi_2\rangle = \frac{1}{\sqrt{3}}|+\rangle - i\sqrt{\frac{2}{3}}|-\rangle}}$$

$$\langle \psi_3 | \phi_3 \rangle = \frac{1}{\sqrt{2}}\, a + \frac{1}{\sqrt{2}} e^{-i\pi/4}\, b = 0 \Rightarrow b = -e^{i\pi/4}\, a$$

$$a = \frac{1}{\sqrt{2}}, \quad b = -\frac{1}{\sqrt{2}} e^{i\pi/4}, \quad \underline{\underline{|\phi_3\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}} e^{i\pi/4}|-\rangle}}$$

Note: In all 3 cases, to get $|\phi_i\rangle$ from $|\psi_i\rangle$ you swap the two coefficients, add one minus sign and take complex conjugates. It's easy to make sign mistakes!

b) $$\langle \psi_2 | \psi_3 \rangle = \left(\sqrt{\frac{2}{3}}\langle +| - \frac{i}{\sqrt{3}}\langle -|\right)\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}} e^{i\pi/4}|-\rangle\right)$$

$$= \underline{\underline{\frac{1}{\sqrt{3}} - \frac{i}{\sqrt{6}} e^{i\pi/4}}} = \underline{\underline{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}} e^{-i\pi/4}}} \quad \text{using } -i = e^{-i\pi/2}$$

$$\langle \psi_3 | \psi_2 \rangle = \left(\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}} e^{-i\pi/4}\langle -|\right)\left(\sqrt{\frac{2}{3}}|+\rangle + \frac{i}{\sqrt{3}}|-\rangle\right)$$

$$= \underline{\underline{\frac{1}{\sqrt{3}} + \frac{i}{\sqrt{6}} e^{-i\pi/4}}} = \underline{\underline{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}} e^{i\pi/4}}} \quad \text{using } i = e^{i\pi/2}$$

Notice that $\langle \psi_3 | \psi_2 \rangle = \langle \psi_2 | \psi_3 \rangle^*$

i.e. they are complex conjugates of each other.

5. Not normalized: $|\psi_1\rangle = 3|+\rangle - 4|-\rangle$

$$|\psi_2\rangle = 2|+\rangle + i|-\rangle$$

$$|\psi_3\rangle = |+\rangle - 2e^{-i\pi/4}|-\rangle$$

a) A normalized ket $|\psi\rangle = a|+\rangle + b|-\rangle$ has $|a|^2 + |b|^2 = 1$, so we to divide through by $\sqrt{|a|^2 + |b|^2}$.

$$|\psi_{1\,norm}\rangle = \frac{3}{5}|+\rangle - \frac{4}{5}|-\rangle$$

$$|\psi_{2\,norm}\rangle = \frac{2}{\sqrt{5}}|+\rangle + \frac{i}{\sqrt{5}}|-\rangle$$

$$|\psi_{3\,norm}\rangle = \frac{1}{\sqrt{5}}|+\rangle - \frac{2}{\sqrt{5}}e^{-i\pi/4}|-\rangle$$

b) $P^1_{up} = |\langle + | \psi_{1\,norm}\rangle|^2 = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$

$P^2_{up} = |\langle + | \psi_{2\,norm}\rangle|^2 = \left(\frac{2}{\sqrt{5}}\right)^2 = \frac{4}{5}$

$P^3_{up} = \underline{\underline{\frac{1}{5}}}$

c) $$P^2_{up\,x} = |{}_x\langle + | \psi_{2\,norm}\rangle|^2$$

$$= \left| \left(\frac{1}{\sqrt{2}}\langle + | + \frac{1}{\sqrt{2}}\langle - |\right)\left(\frac{2}{\sqrt{5}}|+\rangle + \frac{i}{\sqrt{5}}|-\rangle\right)\right|^2$$

$$= \left|\frac{2}{\sqrt{10}} + \frac{i}{\sqrt{10}}\right|^2 = \left(\frac{2}{\sqrt{10}} + \frac{i}{\sqrt{10}}\right)\left(\frac{2}{\sqrt{10}} - \frac{i}{\sqrt{10}}\right)$$

$$= \frac{4}{10} + \frac{1}{10} = \frac{5}{10} = \underline{\underline{\frac{1}{2}}}$$

d) $P^3_{up\,y} = \left| {}_y\langle + | \psi_{3\,norm} \rangle \right|^2$

Before we dive in, let's be sure to get the signs right:

$$|+\rangle_y = \frac{1}{\sqrt{2}} \left(|+\rangle + i |-\rangle \right)$$

$$\Rightarrow \quad {}_y\langle + | = \frac{1}{\sqrt{2}} \left(\langle + | - i \langle - | \right)$$

$$P^3_{up\,y} = \left| \frac{1}{\sqrt{2}} \left(\langle + | - i \langle - | \right) \left(\frac{1}{\sqrt{5}} |+\rangle - \frac{2}{\sqrt{5}} e^{-i\pi/4} |-\rangle \right) \right|^2$$

$$= \left| \frac{1}{\sqrt{10}} + i \frac{2}{\sqrt{10}} e^{-i\pi/4} \right|^2$$

$$= \left(\frac{1}{\sqrt{10}} + i \frac{2}{\sqrt{10}} e^{-i\pi/4} \right) \left(\frac{1}{\sqrt{10}} - i \frac{2}{\sqrt{10}} e^{+i\pi/4} \right)$$

$$= \frac{1}{10} + i \frac{2}{10} e^{-i\pi/4} - i \frac{2}{10} e^{+i\pi/4} + \frac{4}{10}$$

For the middle 2 terms, use $\dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$

$$P^3_{up\,y} = \frac{5}{10} - i \frac{2}{10} \left(e^{i\pi/4} - e^{-i\pi/4} \right) = \frac{1}{2} - i \frac{1}{5} \left(2i \sin \frac{\pi}{4} \right)$$

$$= \frac{1}{2} + \frac{2}{5} \cdot \frac{1}{\sqrt{2}}$$

$$= \underline{\underline{\frac{1}{10} \left(5 + 2\sqrt{2} \right)}} = \underline{\underline{0.783}}$$

Notice up here we had $|z_1 + z_2|^2 \neq |z_1|^2 + |z_2|^2$.

We <u>had</u> to use $|z_1 + z_2|^2 = (z_1 + z_2)(z_1^* + z_2^*)$.

6. Probability to measure a spin in direction $|\chi\rangle$ is

$$P = |\langle \chi | \psi \rangle|^2$$

If we change $|\psi\rangle$ to $e^{i\beta}|\psi\rangle$ we get

$$P_{new} = |\langle \chi | e^{i\beta} | \psi \rangle|^2 = |e^{i\beta} \langle \chi | \psi \rangle|^2$$

$$= |e^{i\beta}|^2 \cdot |\langle \chi | \psi \rangle|^2 = |\langle \chi | \psi \rangle|^2 = P$$

So changing the overall phase of $|\psi\rangle$ did not change the measurement result.

$\Rightarrow$ If we write $|\psi\rangle = a|+\rangle + b|-\rangle$, we can always make a real by multiplying $|\psi\rangle$ by a phase factor.

Phase is important anytime there is interference!

7. $|\psi\rangle = a|+\rangle + b|-\rangle$ with $|a|^2 = 3|b|^2$

and $|a|^2 + |b|^2 = 1$

I'll choose $a = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$, $b = \sqrt{\frac{1}{4}} = \frac{1}{2}$

so $|\psi\rangle = \frac{\sqrt{3}}{2}|+\rangle + \frac{1}{2}|-\rangle$

but we could have multiplied b by any complex phase factor such as $i, -1, -i$, or any $e^{i\theta}$ with θ real. So $|\psi\rangle$ is <u>not</u> unique.