Honors Analysis: Notes

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Problem 1

Note: Linear maps require additivity $T(u+v) = Tu + Tv : u, v \in V$ and homogeneity $T(\lambda v) = \lambda(Tv) : \lambda \in F, v \in V$

(i) Define $T : \mathcal{P}_4 \to \mathcal{P}_4$ by

$$(Tp)(x) = x^2 p''(x).$$

Let $p, q \in \mathcal{P}_4$. Show that $T(p+q)(x) = (Tp)(x) + (Tq)(x)$ and $T(\lambda p)(x) = x^2(\lambda p)(x) = \lambda(Tp)(x)$

Pf. Show T is a linear map.

$$T(p+q)(x) = x^2(p+q)''(x)$$

by definition of T

$$x^2(p+q)''(x) = x^2(p''+q'')(x) = x^2(p'')(x) + x^2(q'')(x)$$

by definition of differentiation, distribution in $\mathcal{P}_4$.

$$x^2(p'')(x) + x^2(q'')(x) = (Tp)(x) + (Tq)(x)$$

by definition of T.

So $T(p+q)(x) = (Tp)(x) + (Tq)(x)$ and thus T has additivity

$$T(\lambda p)(x) = x^2(\lambda p)''(x)$$

by definition of T

$$x^2(\lambda p)''(x) = \lambda x^2 p''(x) = \lambda(Tp)(x)$$

by definition of T, differentiation

So $T(\lambda p)(x) = \lambda(Tp)(x)$ and thus T has homogeneity.

Since T has additivity and homogeneity, T is linear $\square$

(ii) Define $S : \mathcal{P}_4 \to \mathcal{P}_4$ by

$$(Sp)(x) = p''(x) + x^2$$

Pf. Show that S is not a linear map by example. Let $p(x) = x$ and $q(x) = x$

$$(Sp)(x) = x'' + x^2 = 0 + x^2 = x^2$$

By definition of S, differentiation, additive identity in $\mathcal{P}_4$

$$(Sq)(x) = x'' + x^2 = 0 + x^2 = x^2$$

By definition of S, differentiation, additive identity in $\mathcal{P}_4$

$$(Sp)(x) + (Sq)(x) = x^2 + x^2 = 2x^2$$

However

$$(S(p+q))(x) = (x+x)'' + x^2 = 0 + x^2 = x^2$$

By definition of S, differentiation, additive identity in $\mathcal{P}_4$, and

$$x^2 \neq 2x^2$$

So $(S(p+q))(x) \neq (Sp)(x) + (Sq)(x)$, and S does not have additivity, so S is not linear. $\square$

(iii) $H \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ x_1 \\ 5x_3 + x_4 \end{pmatrix}$ Pf. Show H does not have additivity and is thus not linear through example.

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Let $\vec{p} \in \mathbb{R}^4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{q} \in \mathbb{R}^4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$$H\vec{p} = H \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 * 1 \\ 1 \\ 5 * 1 + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix}$$

$$H\vec{q} = H \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 * 1 \\ 1 \\ 5 * 1 + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix}$$

$$H(\vec{p}) + H(\vec{q}) = \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 1+1 \\ 6+6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 12 \end{pmatrix}$$

$$H(\vec{p} + \vec{q}) = H \begin{pmatrix} 1+1 \\ 1+1 \\ 1+1 \\ 1+1 \end{pmatrix} = H \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 * 2 \\ 2 \\ 5(2) + 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix}$$

However $\begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 2 \\ 12 \end{pmatrix}$, so $H\vec{p} + H\vec{q} \neq H(\vec{p} + \vec{q})$, so H does not have additivity and is not linear. $\square$

(iv) Pf. Show I is not linear since I does not have homogeneity.

Let $\vec{o} \in \mathbb{R}^4$ be the zero vector $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\lambda \in \mathbb{R} = 0$

$$H\vec{o} = \lambda H \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 - 3 * 0 \\ 0 \\ 0 + 7 \\ 5 * 0 + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix}$$

Note that

$$H(\lambda\vec{o}) = H(0\vec{o}) = H\vec{o}$$

Since any vector multiplied by zero is zero

$$\lambda H\vec{o} = \lambda H \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 - 3 * 0 \\ 0 \\ 0 + 7 \\ 5 * 0 + 0 \end{pmatrix} = 0 * \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix}$, $\lambda H\vec{o} \neq H(\lambda\vec{o})$, H is not homogeneous and thus not linear. $\square$

(v) Pf. Show that J has both additivity and homogeneity, and is thus linear.

Let $\vec{v} \in \mathbb{R}^4 = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$ Additivity.

$$J(\vec{v}+\vec{w}) = J\begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3 \\ v_4+w_4 \end{pmatrix} = \begin{pmatrix} v_1+w_1-3(v_2+w_2) \\ v_1+w_1 \\ 5(v_3+w_3)+v_4+w_4 \end{pmatrix}$$

$$J\vec{v} = J\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} v_1-3v_2 \\ v_1 \\ 5v_3+v_4 \end{pmatrix}$$

$$J\vec{w} = J\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} w_1-3w_2 \\ w_1 \\ 5w_3+w_4 \end{pmatrix}$$

$$J\vec{w}+J\vec{v} = \begin{pmatrix} w_1-3w_2 \\ w_1 \\ 5w_3+w_4 \end{pmatrix} + \begin{pmatrix} v_1-3v_2 \\ v_1 \\ 5v_3+v_4 \end{pmatrix} = \begin{pmatrix} w_1-3w_2+v_1-3v_2 \\ w_1+w2 \\ 5w_3+w_4+5v_3+w_4 \end{pmatrix} = \begin{pmatrix} v_1+w_1-3(v_2+w_2) \\ v_1+w_1 \\ 5(v_3+w_3)+v_4+w_4 \end{pmatrix}$$

So $J\vec{w}+J\vec{v} = J\vec{w}\vec{v}$ and J has additivity.
Homogeneity.

$$\lambda J\vec{v} = \lambda\begin{pmatrix} v_1-3v_2 \\ v_1 \\ 5v_3+v_4 \end{pmatrix} = \begin{pmatrix} \lambda v_1-\lambda 3v_2 \\ \lambda v_1 \\ \lambda 5v_3+\lambda v_4 \end{pmatrix}$$

$$J(\lambda\vec{v}) = J(\lambda\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}) = J\begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \\ \lambda v_4 \end{pmatrix} = \begin{pmatrix} \lambda v_1-3\lambda v_2 \\ \lambda v_1 \\ 5\lambda v_3+\lambda v_4 \end{pmatrix} = \begin{pmatrix} \lambda v_1-\lambda 3v_2 \\ \lambda v_1 \\ \lambda 5v_3+\lambda v_4 \end{pmatrix}$$