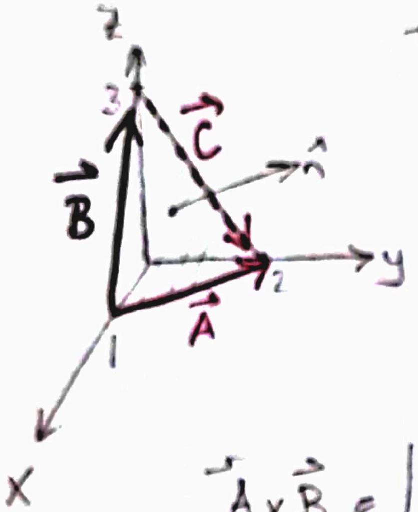
Griffiths 1.4



To find a vector perpendicular to a plane, one can take two vectors on the plane and perform cross-product. For example, let's choose $\vec{A} = -1\hat{x} + 2\hat{y} + 0\hat{z}$

and $\vec{B} = -1\hat{x} + 0\hat{y} + 3\hat{z}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\hat{x} + 3\hat{y} + 2\hat{z} \quad \text{in } \hat{n} \text{ direction}$$

We need to normalize to get $\hat{n} = \dfrac{6\hat{x} + 3\hat{y} + 2\hat{z}}{\sqrt{6^2+3^2+2^2}}$

Hence, $\boxed{\hat{n} = \frac{6}{7}\hat{x} + \frac{3}{7}\hat{y} + \frac{2}{7}\hat{z}}$.

Similarly, we could have done $\hat{n} = \dfrac{\vec{C}\times\vec{B}}{|\vec{C}\times\vec{B}|}$

or $\hat{n} = \dfrac{\vec{C}\times\vec{A}}{|\vec{C}\times\vec{A}|}$ where $\vec{C} = 0\hat{x} + 2\hat{y} - 3\hat{z}$

Griffiths 1.7

$\vec{r} = 4\hat{x} + 6\hat{y} + 8\hat{z}$, $\vec{r}\,' = 2\hat{x} + 8\hat{y} + 7\hat{z}$

$\boxed{\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = 2\hat{x} - 2\hat{y} + \hat{z}}$ $\quad \mathscr{r} = |\vec{\mathscr{r}}| = \sqrt{2^2+(-2)^2+1^2} = 3$

$\boxed{\hat{\mathscr{r}} = \dfrac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{2}{3}\hat{x} - \frac{2}{3}\hat{y} + \frac{1}{3}\hat{z}}$

What operations can be done to different functions?

① $T(x, y, z)$ scalar function

- only gradient

$$\vec{\nabla} T = \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z}$$

- $\vec{\nabla} T$ point in the direction of maximum increase of T. Its magnitude, $|\vec{\nabla} T|$, gives the slope along this maximal direction
- $\vec{\nabla} T$ is a vector function.

② $\vec{V}(x, y, z)$ vector function

- Gradient of a vector is a second-order tensor. In Cartesian coordinate system

$$\vec{\nabla} \vec{V} = \begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_y}{\partial x} & \frac{\partial V_z}{\partial x} \\ \frac{\partial V_x}{\partial y} & \frac{\partial V_y}{\partial y} & \frac{\partial V_z}{\partial y} \\ \frac{\partial V_x}{\partial z} & \frac{\partial V_y}{\partial z} & \frac{\partial V_z}{\partial z} \end{pmatrix}$$

However, this will not be part of how we treat vectors in the context of this course.

It is OK if you just mentioned div and curl for vectors.

- Divergence of $\vec{V}(x,y,z)$

- $\vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

- It is a measure of how much $\vec{V}$ spreads out from the point in question
- $\vec{\nabla} \cdot \vec{V}$ results in a scalar function

- Curl of $\vec{V}(x,y,z)$

- $\vec{\nabla} \times \vec{V} = \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{pmatrix}$

$$= \hat{x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \hat{y}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \hat{z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

- It is a measure of how much $\vec{V}$ curls around the point in question

- $\vec{\nabla} \times \vec{V}$ results in a vector function

Determine the gradient of a scalar function.

1 $\vec{\mathscr{r}} = \vec{r} - \vec{r}' = (x-x')\,\hat{x} + (y-y')\,\hat{y} + (z-z')\,\hat{z}$

$$\mathscr{r} = |\vec{\mathscr{r}}| = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

$$\vec{\nabla}\mathscr{r} = \frac{\partial \mathscr{r}}{\partial x}\hat{x} + \frac{\partial \mathscr{r}}{\partial y}\hat{y} + \frac{\partial \mathscr{r}}{\partial z}\hat{z}$$

$$\frac{\partial \mathscr{r}}{\partial x} = \frac{\partial}{\partial x}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

$$= \frac{1}{2}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2} \cdot 2(x-x')$$

$$= \frac{(x-x')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

Similarly for $\frac{\partial \mathscr{r}}{\partial y}$ and $\frac{\partial \mathscr{r}}{\partial z}$.

Hence

$$\boxed{\vec{\nabla}\mathscr{r} = \frac{(x-x')\,\hat{x} + (y-y')\,\hat{y} + (z-z')\,\hat{z}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}}$$

② $$\frac{1}{r} = \left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}$$

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial x} = -\frac{1}{2}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}\cdot 2(x-x')$$

$$= -\frac{(x-x')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}$$

Similarly for $\frac{\partial\left(\frac{1}{r}\right)}{\partial y}$ and $\frac{\partial\left(\frac{1}{r}\right)}{\partial z}$.

Hence,

$$\boxed{\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{(x-x')\,\hat{x}+(y-y')\,\hat{y}+(z-z')\,\hat{z}}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}}$$

③ Looking at our result in part ①, we see $\boxed{\vec{\nabla} r = \frac{\vec{r}}{r} = \hat{r}}$ Similarly from part ②, we see $\boxed{\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}}$

This actually can be generalized:

$$\vec{\nabla}(r^n) = n\, r^{n-1}\,\hat{r}.$$