Deep Dive #7 - Matrix exponentials

The Matrix Exponential

We prove several properties of the exponential function of a matrix

What's next, Matrix Taylor expansion? ...wait

Question 1: *(10 points)* Find a closed expression (without the infinite sum) for the exponential of a diagonal matrix $D = \text{diag}\left[d_{11}, \cdots, d_{nn}\right]$.

Find e^D in terms of $D = \text{diag}[d_{11}, \cdots, d_{nn}] = \text{diag}[\lambda_1, \cdots, \lambda_n]$

Well, we can use the infinite sum to find the closed expression.

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + \frac{A}{1} + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!}$$

We'll also use

$$D^n = \text{diag}[d_{11}^n, \cdots, d_{nn}^n]$$

$$e^D = \sum_{k=0}^{\infty} \frac{D^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \text{diag}[d_{11}^k, \cdots, d_{nn}^k]$$
$$= \text{diag}\left[\sum_{k=0} \frac{d_{11}^k}{k!}, \cdots, \sum_{k=0} \frac{d_{nn}^k}{k!}\right] = \text{diag}\left[e^{d_{11}}, \cdots, e^{d_{nn}}\right]$$

Question 2: *(10 points)* Find a closed expression (without the infinite sum) of the exponential of a diagonalizable matrix $A = PDP^{-1}$, where D is diagonal.

$$e^A = e^{PDP^{-1}} = \sum_{k=0}^{\infty} \frac{(PDP^{-1})^k}{k!}$$

This yields a chain of PDP^{-1} s which yields

$$(PDP^{-1})^k = PDP^{-1}PDP^{-1}PDP^{-1}\dots$$
$$= PDIDIDP^{-1}\cdots = PD^kP^{-1}$$

Which gives us

$$\sum_{k=0}^{\infty} \frac{(PDP^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{PD^kP^{-1}}{k!} = P\sum_{k=0}^{\infty} \frac{D^k}{k!}P^{-1} = Pe^DP^{-1}$$

Question 3:

(3a) *(5 points)* If $M^2 = M$, then show that

$$e^M = I + (e-1)\,M.$$

(3b) *(5 points)* If $M^2 = 0$ then compute e^M.

3a

Using the infinite sum

$$e^M = I + \frac{M}{1} + \frac{M^2}{2!} + \cdots + \frac{M^n}{n!}$$

but $M^2 = M$ and $M^3 = M^2M = MM = M$ and $M^n = M$, giving us

$$\begin{aligned} e^M &= I + \sum_{k=1}^{\infty} \frac{M}{k!} = I + \sum_{k=0}^{\infty} \frac{M}{k!} - M \\ &= I + \sum_{k=0}^{\infty} (\frac{1}{k!} - 1)M \\ &= I + (e-1)M \end{aligned}$$

3b

$$e^M = I + M + 0 + \frac{0 * M}{3!} + \ldots + \frac{0 * M^{n-2}}{n!} = I + M$$

Question 4: *(10 points)* By direct computation on the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ show that

$$e^{(A+B)} \neq e^A e^B.$$

Hints: Use **Question (3a)** for one side of the equation and **Question (3b)** for the other side.

$$\begin{aligned} (A+B)^2 &= (A+B) \\ A^2 &= A \\ B^2 &= 0 \\ e^{(A+B)} &= I + (e-1)(A+B) \\ e^A e^B &= (I + (e-1)A)(I+B) \end{aligned}$$

I'll leave distribution as an exercise for the reader. These aren't equivalent

Question 5: *(10 points)* Prove the following: If A is an $n \times n$, diagonalizable matrix, then

$$\det\left(e^A\right) = e^{\operatorname{tr}(A)}.$$

Hint: Use that the determinant on $n \times n$ matrices B, C satisfies $\det(BC) = \det(B)\det(C)$. Use this equation to relate the determinant of an invertible matrix P with the determinant of P^{-1}.

Pf. Show $\det(e^A) = e^{\operatorname{tr}(A)}$

$$\det\left(e^A\right) = \det\left(Pe^D P^{-1}\right) = \det(P)\det(e^D)\det(P^{-1})$$
$$= \det(P)\det(P^{-1})\det(e^D) = \det(PP^{-1})\det(e^D)$$
$$= \det(e^D) = \det(\operatorname{diag}\left[e^{d_{11}}, \cdots, e^{d_{nn}}\right]) = e^{d_{11}}e^{d_{22}} \cdots e^{d_{nn}} = e^{d_{11}+d_{22}+\cdots+d_{nn}}$$
$$= e^{\operatorname{tr}D} = e^{\operatorname{tr}A}$$

Since tr(A) is equal to the sum of its eigenvalues, and the elements of D are the eigenvalues of A

Question 6: *(10 points)* Prove the following: If λ and $\boldsymbol{v}$ are an eigenvalue and eigenvector of a matrix A, that is, $A\boldsymbol{v} = \lambda\boldsymbol{v}$, then $\boldsymbol{v}$ is an eigenvector of the matrix e^A with eigenvalue e^λ, that is,

$$e^A \boldsymbol{v} = e^\lambda \, \boldsymbol{v}.$$

Note: for $Av = \lambda v,\ A^k v = \lambda^k v$. This comes from $A^2 v = A(Av) = A(\lambda v) = \lambda A v = \lambda^2 v$. The full proof can be done via induction.

Pf.

$$e^A v = Iv + Av + \frac{A^2 v}{2} + \cdots + \frac{A^n v}{n!}$$
$$= Iv + \lambda v + \frac{\lambda^2 v}{2} + \cdots + \frac{\lambda^n v}{n!} = e^\lambda v$$

Question 7: *(10 points)* Prove the following: If A, B are $n \times n$ matrices,

$$AB = BA \quad \Rightarrow \quad e^A e^B = e^B e^A.$$

Hints:

- First, prove that $AB = BA$ implies $A\,B^n = B^n\,A$.
- Second, prove that $AB = BA$ implies $A\,e^B = e^B\,A$.

Part 1: $AB = BA \implies AB^n = B^n A$

$$BAB = B^2 A$$
$$(BA)B = (AB)B$$
$$BAB = AB^2 = B^2 A$$

Using this process gives us

$$B^{k-1}AB^1 = B^kA$$
$$B^{k-2}AB^2 = B^kA$$
$$B^{k-3}AB^3 = B^kA$$
$$\vdots$$
$$B^{k-k}AB^k = IAB^k = AB^k = B^kA$$

Part 2: $AB = BA \implies Ae^B = e^BA$

$$\begin{aligned} Ae^B &= A\left(I + \frac{B}{1} + \frac{B^2}{2!} + \cdots + \frac{B^n}{n!}\right) \\ &= AI + \frac{AB}{1} + \frac{AB^2}{2!} + \cdots + \frac{AB^n}{n!} \\ &= IA + \frac{BA}{1} + \frac{B^2A}{2!} + \cdots + \frac{B^nA}{n!} = \left(I + \frac{B}{1} + \frac{B^2}{2!} + \cdots + \frac{B^n}{n!}\right)A \\ &= e^BA \end{aligned}$$

Pf. Show $AB = BA \implies e^Ae^B = e^Be^A$

$$\begin{aligned} e^Ae^B &= e^A\left(I + \frac{B}{1} + \frac{B^2}{2!} + \cdots + \frac{B^n}{n!}\right) \\ &= e^AI + \frac{e^AB}{1} + \frac{e^AB^2}{2!} + \cdots + \frac{e^AB^n}{n!} \end{aligned}$$

Since $e^AB = Be^A$ we have $e^AB^k = B^ke^A$

$$\begin{aligned} &= Ie^A + \frac{Be^A}{1} + \frac{B^2e^A}{2!} + \cdots + \frac{B^ne^A}{n!} = \left(I + \frac{B}{1} + \frac{B^2}{2!} + \cdots + \frac{B^n}{n!}\right)e^A \\ &= e^Be^A \end{aligned}$$