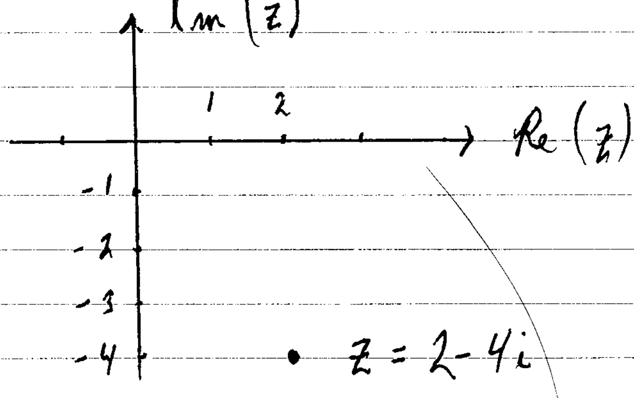
Physics 471 : Homework 1 Solutions

1. $z = 2 - 4i$

a) $|z|^2 = z^* z = (2+4i)(2-4i) = 4 + 16 = 20$

b)



c) $z = Ae^{i\theta}$

$$A = \sqrt{|z|^2} = \sqrt{20} = \underline{\underline{4.47}} \quad \text{from part a)}$$

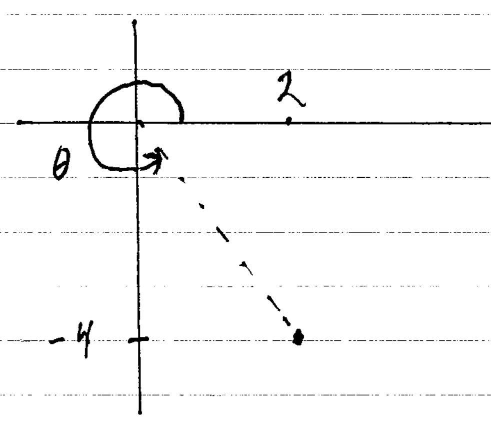
$$\theta = \arctan\left(\frac{-4}{2}\right) = -\underline{\underline{1.107}} \text{ radians}$$

If we want $0 \le \theta < 2\pi$, then $\theta = -1.107 + 2\pi = \underline{\underline{5.176}}$

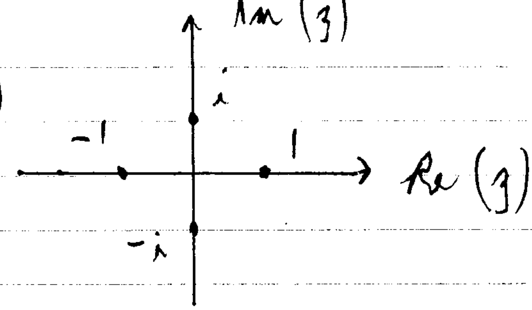
It may be convenient to know how θ compares with π:

$$\theta = \frac{5.176}{\pi} \cdot \pi = 1.648\,\pi$$

Since θ is between $\frac{3}{2}\pi$ and 2π, we know z is in the 4^{th} quadrant.



2. 4 points : $1, i, -1, -i$ a)



$$\left.\begin{aligned} i &= e^{i\pi/2} \\ -1 &= e^{i\pi} \\ -i &= e^{i3\pi/2} \end{aligned}\right\} \text{In all 3 cases, } A = 1$$

b)

$$\left.\begin{aligned} i \cdot 1 &= i \\ i \cdot i &= -1 \\ i \cdot -1 &= -i \\ i \cdot -i &= 1 \end{aligned}\right\}$$

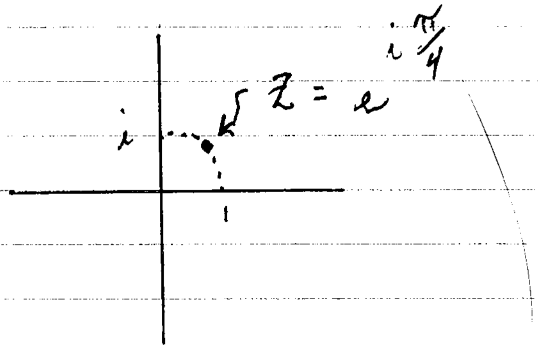
Multiplying by i rotates the position counter-clockwise by $\pi/2$ in the complex plane.

3. $z = i\,e^{-i\frac{\pi}{4}}$

a) Write the "i" in front as $i = e^{i\frac{\pi}{2}}$

$$z = e^{i\frac{\pi}{2}}\,e^{-i\frac{\pi}{4}} = \underline{\underline{e^{+i\frac{\pi}{4}}}}$$

b)



z has a modulus of 1, so it lies on the unit circle in the complex plane.

Multiply by i = multiply by $e^{i\pi/2}$

= add $\frac{\pi}{2}$ to θ = rotate CCW by $\frac{\pi}{2}$

4. a) $AB = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{i}{2} \\ -i & 0 \end{pmatrix}$

$AC = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} i \\ -\frac{3}{5} i \end{pmatrix}$

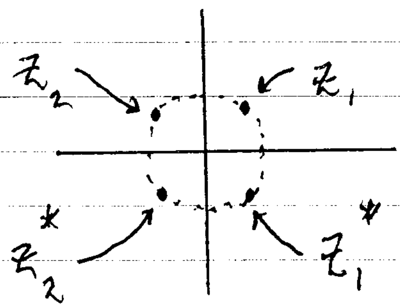
b) To find eigenvalues λ, solve

$$\det(\lambda\hat{I} - \hat{A}) = \begin{vmatrix} \lambda - 0 & -i \\ i & \lambda - 0 \end{vmatrix} = \lambda^2 - 1 = 0$$

$\underline{\underline{\lambda = \pm 1}}$

c) $CA = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$ This cannot be done because it doesn't follow the rules for matrix multiplication

5. $z_1 = e^{i\pi/4}$ $\quad z_2 = e^{i\frac{3\pi}{4}}$ $\quad z_1^* = e^{-i\frac{\pi}{4}}$ $\quad z_2^* = e^{-i\frac{3\pi}{4}}$

a) 

b) $z_1 z_2 = e^{i\frac{\pi}{4}} \cdot e^{i\frac{3\pi}{4}} = e^{i\pi} = \underline{\underline{-1}}$

$$\left.\begin{aligned} \frac{z_2}{z_1} &= \frac{e^{i3\pi/4}}{e^{i\pi/4}} = e^{i\pi/2} = \underline{\underline{i}} \\ z_2 z_1^* &= e^{i\frac{3\pi}{4}} e^{-i\frac{\pi}{4}} = e^{i\pi/2} = \underline{\underline{i}} \end{aligned}\right\} \text{same answer!}$$

Note for z_1 on the unit circle, $\frac{1}{z_1} = z_1^*$.

c) $z_1 = \frac{1+i}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, $\quad z_2 = \frac{-1+i}{\sqrt{2}} = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$z_1 z_2 = \left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{-1+i}{\sqrt{2}}\right) = \frac{1}{2}\left(-1+i-i-1\right) = \underline{\underline{-1}}$

agrees with part b)

6. a) $\left.\begin{array}{l} z = a + bi \\ z^* = a - bi \end{array}\right\}$ Add: $z + z^* = 2a \Rightarrow \mathrm{Re}\{z\} = a = \dfrac{z+z^*}{2}$

Subtract: $z - z^* = 2bi \Rightarrow \mathrm{Im}\{z\} = b = \dfrac{z-z^*}{2i}$

b) Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$

$z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 - b_1 b_2 + i(a_1 b_2 + b_1 a_2)$

$\Rightarrow \mathrm{Re}\{z_1 z_2\} = a_1 a_2 - b_1 b_2 = \mathrm{Re}\{z_1\}\mathrm{Re}\{z_2\} - \mathrm{Im}\{z_1\}\mathrm{Im}\{z_2\}$

$\mathrm{Im}\{z_1 z_2\} = a_1 b_2 + b_1 a_2 = \mathrm{Re}\{z_1\}\mathrm{Im}\{z_2\} + \mathrm{Im}\{z_1\}\mathrm{Re}\{z_2\}$

c) $z = Ae^{i\theta} \Rightarrow |z|^2 = A^2$ but $z^2 = A^2 e^{2i\theta}$

$\mathrm{Re}\{|z|^2\} = A^2$, $\mathrm{Re}\{z^2\} = A^2 \cos 2\theta$ from Euler

Alternative: $z = a + bi \Rightarrow |z|^2 = a^2 + b^2$ but $z^2 = (a^2 - b^2) + 2abi$

d) $z = Ae^{i\theta}$ $\quad \mathrm{Re}\{z\} = A\cos\theta \quad \mathrm{Im}\{z\} = A\sin\theta$

$z^* = Ae^{-i\theta}$, $|z| = A$ $\quad$ All results from Euler formula.

e) $\left.\begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{array}\right\}$ Add: $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\Rightarrow \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

Subtract:

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \Rightarrow \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$