SOLUTIONS

Physics 471 – Quiz #5

Friday, November 3, 2023

Name:____ (v1)

This quiz has questions on both sides of the paper!

$S_x \doteq \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\vert+\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$	$\vert-\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$
$S_y \doteq \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$	$\vert+\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$	$\vert-\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$
$S_z \doteq \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$\vert+\rangle \doteq \begin{pmatrix}1\\ 0\end{pmatrix}$	$\vert-\rangle \doteq \begin{pmatrix}0\\ 1\end{pmatrix}$

1) [4] An operator $\widehat{\boldsymbol{A}}$ (representing observable A) has two normalized eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$, with eigenvalues a_1 and a_2, respectively. Operator $\widehat{\boldsymbol{B}}$ (representing observable B) has two normalized eigenstates $|\varphi_1\rangle$ and $|\varphi_2\rangle$, with eigenvalues b_1 and b_2. Suppose these eigenstates are related by the following:

$$|\psi_1\rangle = \frac{1}{\sqrt{5}}|\varphi_1\rangle + \frac{2i}{\sqrt{5}}|\varphi_2\rangle, \quad |\psi\rangle_2 = \frac{2}{\sqrt{5}}|\varphi_1\rangle - \frac{i}{\sqrt{5}}|\varphi_2\rangle$$

a) [2] Our quantum system starts in some unspecified random state, and then observable A is measured. The result of the measurement is a_1. Immediately after, the observable B is measured. What is the probability that the result of the measurement will be b_2?

After A measurement, system is in state $|\psi_1\rangle$

$$P(b_2) = |\langle\varphi_2|\psi_1\rangle|^2 = \left|\frac{2i}{\sqrt{5}}\right|^2 = \underline{\underline{4/5}}$$

b) [2] After the previous measurement of B with result b_2, the observable A is measured again. What is the probability that the result will be a_2?

After B measurement, system is in state $|\varphi_2\rangle$

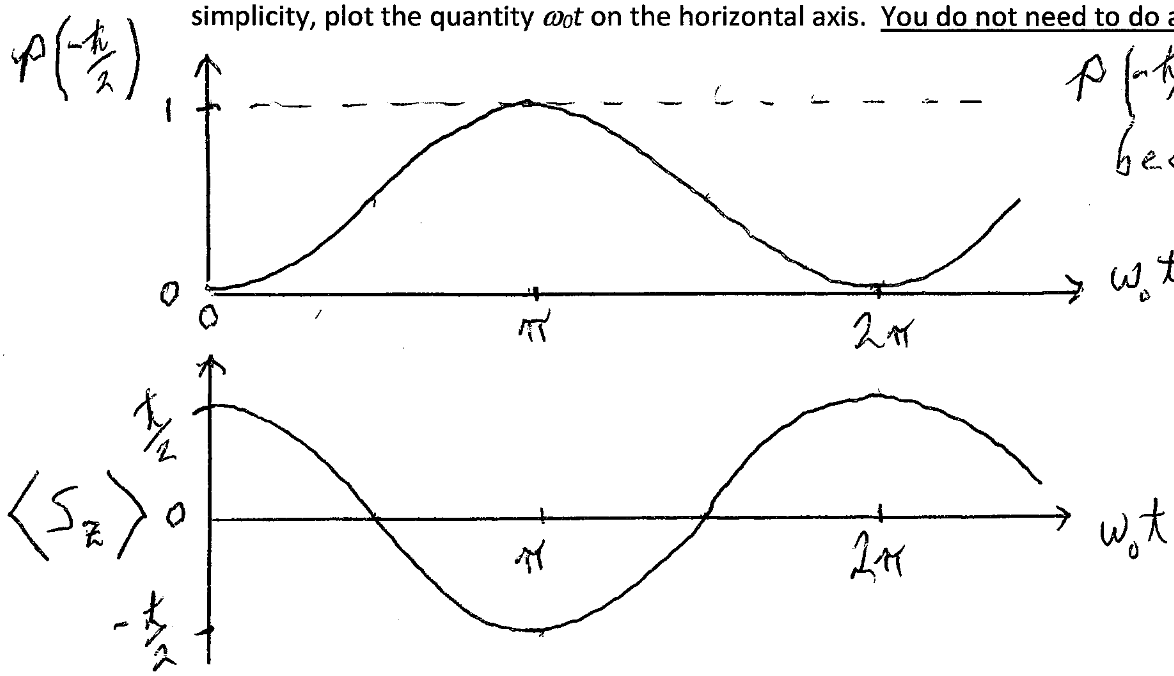
$$P(a_2) = |\langle\psi_2|\varphi_2\rangle|^2 = |\langle\varphi_2|\psi_2\rangle^*|^2 = \left|\frac{+i}{\sqrt{5}}\right|^2 = \underline{\underline{\frac{1}{5}}}$$

VI

2) [6] At time $t = 0$, the state of an electron spin is $|\psi(t=0)\rangle = |+\rangle$ in our usual z-spin basis. The system is allowed to time evolve in a uniform magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{x}}$. After a time t, the state of the system is:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}|+\rangle_x + e^{+i\omega_0 t/2}|-\rangle_x\right),\ \text{where } \omega_0 = eB_0/m.$$

a) [3] At time t, we measure the z-component of the electron spin. Draw a graph of the probability vs time to obtain the result $\left(-\frac{\hbar}{2}\right)$ from that measurement. Just below that, draw a second graph of $\langle \hat{S}_z\rangle$ vs time. Label the axes on both graphs: the graphs should show the largest and smallest values of the functions being plotted, as well as the time scale. For simplicity, plot the quantity $\omega_0 t$ on the horizontal axis. <u>You do not need to do any calculations!</u>



$P(-\hbar/2)$ starts at 0 because initial state is $|+\rangle$

Full period is $T = \frac{2\pi}{\omega_0}$ for both quantities

b) [3] Choose <u>one of the two</u> quantities you plotted above and calculate it.

Warning: If you choose to calculate the time dependence of $\langle \hat{S}_z\rangle$, remember that you will first have to express $|\psi(t)\rangle$ in the standard z-basis.

$$P\left(-\frac{\hbar}{2}\right) = \left|\langle - |\psi(t)\rangle\right|^2 = \frac{1}{2}\left| e^{-i\omega_0 t/2}\langle - |+\rangle_x + e^{+i\omega_0 t/2}\langle - | - \rangle_x\right|^2$$

$$\langle - |+\rangle_x = \begin{pmatrix}0 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}} \qquad \langle - | - \rangle_x = \begin{pmatrix}0 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = -\frac{1}{\sqrt{2}}$$

$$P\left(-\hbar/2\right) = \frac{1}{4}\left|e^{-i\omega_0 t/2} - e^{+i\omega_0 t/2}\right|^2 = \frac{1}{4}\left|-2i\sin\left(\omega_0 t/2\right)\right|^2$$

$$= \underline{\underline{\sin^2\left(\omega_0 t/2\right)}} = \underline{\underline{\frac{1}{2}\left(1 - \cos(\omega_0 t)\right)}}$$

Physics 471 – Quiz #5

Friday, November 3, 2023

Name:____ (v2)

This quiz has questions on both sides of the paper!

$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\lvert+\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\lvert-\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
$S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\lvert+\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$	$\lvert-\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$
$S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\lvert+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\lvert-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

1) [4] An operator $\hat{\boldsymbol{A}}$ (representing observable A) has two normalized eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$, with eigenvalues a_1 and a_2, respectively. Operator $\hat{\boldsymbol{B}}$ (representing observable B) has two normalized eigenstates $|\varphi_1\rangle$ and $|\varphi_2\rangle$, with eigenvalues b_1 and b_2. Suppose these eigenstates are related by the following:

$$|\psi_1\rangle = \frac{2}{\sqrt{5}}|\varphi_1\rangle + \frac{i}{\sqrt{5}}|\varphi_2\rangle, \quad |\psi\rangle_2 = \frac{1}{\sqrt{5}}|\varphi_1\rangle - \frac{2i}{\sqrt{5}}|\varphi_2\rangle$$

a) [2] Our quantum system starts in some unspecified random state, and then observable A is measured. The result of the measurement is a_1. Immediately after, the observable B is measured. What is the probability that the result of the measurement will be b_2?

After A measurement, system is in state $|\psi_1\rangle$

$$p(b_2) = |\langle\varphi_2|\psi_1\rangle|^2 = \left|\frac{i}{\sqrt{5}}\right|^2 = \underline{\underline{\frac{1}{5}}}$$

b) [2] After the previous measurement of B with result b_2, the observable A is measured again. What is the probability that the result will be a_2?

After B measurement, system is in state $|\varphi_2\rangle$

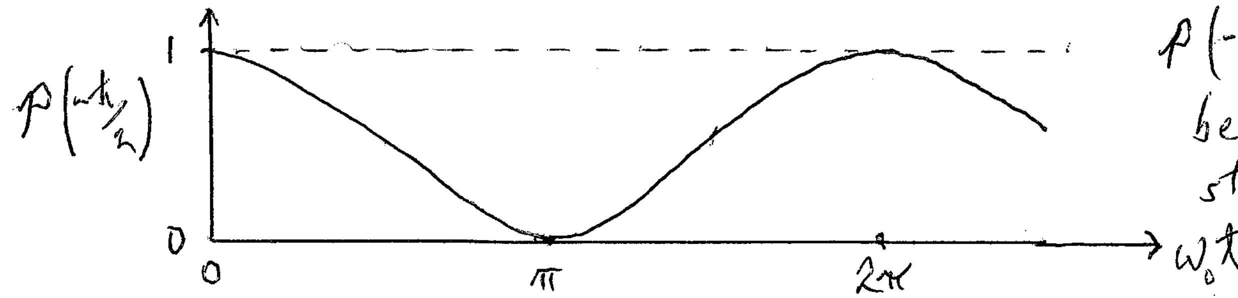
$$p(a_2) = |\langle\psi_2|\varphi_2\rangle|^2 = |\langle\varphi_2|\psi_2\rangle^*|^2 = \left|+\frac{2i}{\sqrt{5}}\right|^2 = \underline{\underline{\frac{4}{5}}}$$

V2

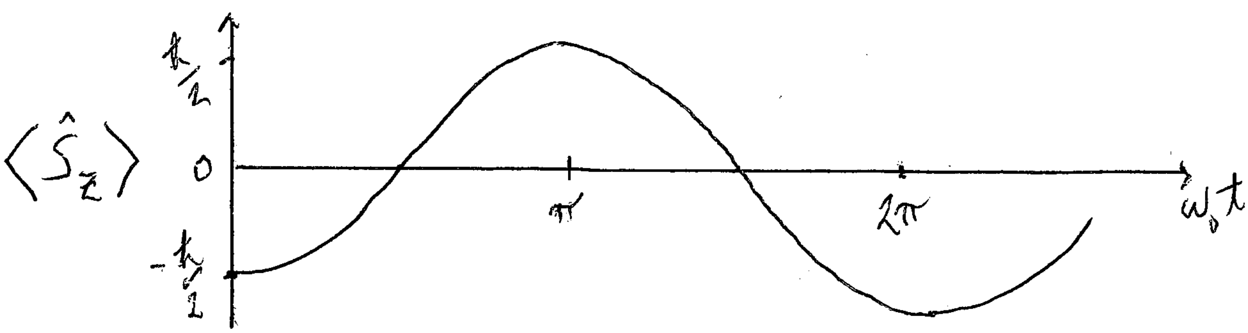
2) [6] At time $t = 0$, the state of an electron spin is $|\psi(t=0)\rangle = |-\rangle$ in our usual z-spin basis. The system is allowed to time evolve in a uniform magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{x}}$. After a time t, the state of the system is:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}|+\rangle_x - e^{+i\omega_0 t/2}|-\rangle_x\right), \text{ where } \omega_0 = eB_0/m.$$

a) [4] At time t, we measure the z-component of the electron spin. Draw a graph of the probability vs time to obtain the result $\left(-\frac{\hbar}{2}\right)$ from that measurement. Just below that, draw a second graph of $\langle \hat{S}_z \rangle$ vs time. Label the axes on both graphs: the graphs should show the largest and smallest values of the functions being plotted, as well as the time scale. For simplicity, plot the quantity $\omega_0 t$ on the horizontal axis. <u>You do not need to do any calculations!</u>



$P(-\hbar/2)$ starts at 1 because initial state is $|-\rangle$



Full period is $T = \frac{2\pi}{\omega_0}$

b) [2] Choose <u>one of the two</u> quantities you plotted above and calculate it.

Warning: If you choose to calculate the time dependence of $\langle \hat{S}_z \rangle$, remember that you will first have to express $|\psi(t)\rangle$ in the standard z-basis.

$$P(-\tfrac{\hbar}{2}) = \left|\langle - | \psi(t)\rangle\right|^2 = \frac{1}{2}\left| e^{-i\omega_0 t/2}\langle - | + \rangle_x - e^{+i\omega_0 t/2}\langle - | - \rangle_x \right|^2$$

$$\langle - | + \rangle_x = \begin{pmatrix} 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \qquad \langle - | - \rangle_x = \begin{pmatrix} 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{1}{\sqrt{2}}$$

$$P(-\tfrac{\hbar}{2}) = \frac{1}{4}\left| e^{-i\omega_0 t/2} + e^{+i\omega_0 t/2}\right|^2 = \frac{1}{4}\left|2\cos(\omega_0 t/2)\right|^2$$

$$= \underline{\underline{\cos^2(\omega_0 t/2)}} = \underline{\underline{\frac{1}{2}\left(1 + \cos(\omega_0 t)\right)}}$$

To calculate $\langle \hat{S}_z \rangle = \langle \psi(t) | \hat{S}_z | \psi(t) \rangle$, we need to express $|\psi(t)\rangle$ in the z basis.

v1:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{+i\omega_0 t/2} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$$

$$= \frac{1}{2}\begin{pmatrix} e^{-i\omega_0 t/2} + e^{+i\omega_0 t/2} \\ e^{-i\omega_0 t/2} - e^{+i\omega_0 t/2} \end{pmatrix} = \begin{pmatrix} \cos(\omega_0 t/2) \\ -i\sin(\omega_0 t/2) \end{pmatrix}$$

$$\langle \hat{S}_z \rangle = \left(\cos\left(\frac{\omega_0 t}{2}\right) \quad +i\sin\left(\frac{\omega_0 t}{2}\right) \right) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\omega_0 t/2) \\ -i\sin\left(\frac{\omega_0 t}{2}\right) \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\cos\left(\frac{\omega_0 t}{2}\right) \quad +i\sin\left(\frac{\omega_0 t}{2}\right) \right) \begin{pmatrix} \cos(\omega_0 t/2) \\ +i\sin(\omega_0 t/2) \end{pmatrix}$$

$$= \frac{\hbar}{2}\left[\cos^2\left(\frac{\omega_0 t}{2}\right) - \sin^2\left(\frac{\omega_0 t}{2}\right) \right] = \frac{\hbar}{2}\cos(\omega_0 t)$$

v2: Change sign in $|\psi(t)\rangle \Rightarrow |\psi(t)\rangle = \begin{pmatrix} -i\sin\left(\frac{\omega_0 t}{2}\right) \\ \cos\left(\frac{\omega_0 t}{2}\right) \end{pmatrix}$

Follow steps shown above

$$\langle \hat{S}_z \rangle = \frac{\hbar}{2}\left[\sin^2\left(\frac{\omega_0 t}{2}\right) - \cos^2\left(\frac{\omega_0 t}{2}\right) \right] = -\frac{\hbar}{2}\cos(\omega_0 t)$$

Note: You can also calculate the z-spin measurement probability using $|\psi(t)\rangle$ expressed in the z-basis:

v1: $$P\left(-\frac{\hbar}{2}\right) = \left|\langle - | \psi(t)\rangle\right|^2 = \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\omega_0 t}{2}\right) \\ -i\sin\left(\frac{\omega_0 t}{2}\right) \end{pmatrix} \right|^2$$

$$= \left| -i\sin\left(\frac{\omega_0 t}{2}\right)\right|^2 = \sin^2\left(\frac{\omega_0 t}{2}\right)$$

v2: $$P\left(-\frac{\hbar}{2}\right) = \left|\langle - | \psi(t)\rangle\right|^2 = \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -i\sin\left(\frac{\omega_0 t}{2}\right) \\ \cos\left(\frac{\omega_0 t}{2}\right) \end{pmatrix} \right|^2$$

$$= \left| \cos\left(\frac{\omega_0 t}{2}\right)\right|^2 = \cos^2\left(\frac{\omega_0 t}{2}\right)$$

but I think it's easier to use the first method I showed.