PHY 471 : Homework 6 Solutions

1. $$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

In the energy basis $|E_1\rangle, |E_2\rangle$, $\hat{H}$ is diagonal

$$|\psi(t)\rangle \doteq \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} \qquad \hat{H} \doteq \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$i\hbar \frac{d}{dt} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} E_1 c_1(t) \\ E_2 c_2(t) \end{pmatrix}$$

1st component: $$i\hbar \frac{dc_1}{dt} = E_1 c_1$$

$$\frac{dc_1}{dt} = \frac{-iE_1}{\hbar} c_1 \quad ✓$$

2nd $$i\hbar \frac{dc_2}{dt} = E_2 c_2$$

$$\frac{dc_2}{dt} = \frac{-iE_2}{\hbar} c_2 \quad ✓$$

2. a) $\hat{A} \doteq \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}$ Find eigenvalues + eigenstates

$$\det\left(\lambda \hat{I} - \hat{A}\right) = \begin{vmatrix} \lambda & -\mu \\ -\mu & \lambda \end{vmatrix} = \lambda^2 - \mu^2 = 0 \Rightarrow \underline{\underline{\lambda = \pm\mu}}$$

$$\begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = +\mu\begin{pmatrix} a \\ b \end{pmatrix}$$ top row: $\mu b = \mu a \Rightarrow a = b$

Normalize to get $|a_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

This result is not surprising, because $\hat{A} \propto \hat{S}_x$ $\qquad a_1 = +\mu$

The second eigenstate is $|a_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad a_2 = -\mu$

b) $|\psi(t=0)\rangle = |a_1\rangle$

To get $|\psi(t)\rangle$, we need to express $|a_1\rangle$ as a linear superposition of energy eigenstates $|E_1\rangle$ and $|E_2\rangle$. This is easy because $\hat{H}$ is already diagonal in the z-basis:

$$\hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \Rightarrow |E_1\rangle = |+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |E_2\rangle = |-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$|E_1\rangle$ and $|E_2\rangle$ have the standard time dependence for energy eigenstates $\sim e^{-iEt/\hbar}$.

2. b) $|\psi(t=0)\rangle = |a_1\rangle = \frac{1}{\sqrt{2}}\left(|E_1\rangle + |E_2\rangle\right)$

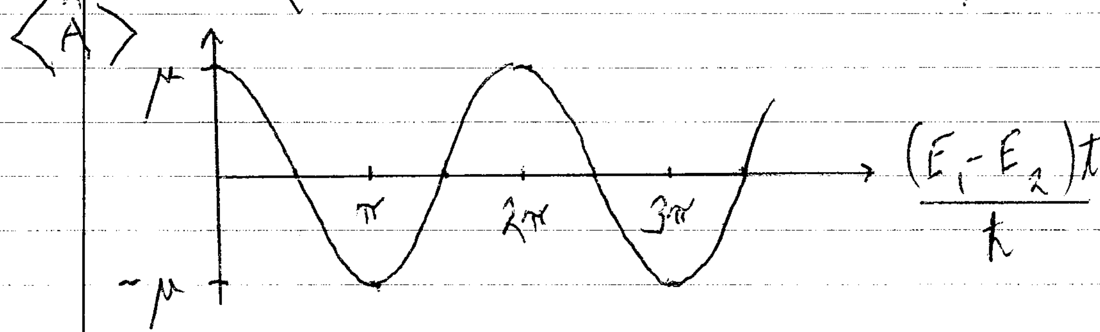
$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_1t/\hbar}|E_1\rangle + e^{-iE_2t/\hbar}|E_2\rangle\right) \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-iE_1t/\hbar} \\ e^{-iE_2t/\hbar} \end{pmatrix}$$

c) $\langle \hat{A} \rangle = \langle \psi(t)|\hat{A}|\psi(t)\rangle$

$$= \frac{1}{2}\begin{pmatrix} e^{+iE_1t/\hbar} & e^{+iE_2t/\hbar} \end{pmatrix}\begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}\begin{pmatrix} e^{-iE_1t/\hbar} \\ e^{-iE_2t/\hbar} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{iE_1t/\hbar} & e^{iE_2t/\hbar} \end{pmatrix}\begin{pmatrix} \mu e^{-iE_2t/\hbar} \\ \mu e^{-iE_1t/\hbar} \end{pmatrix}$$

$$= \frac{\mu}{2}\left(e^{i(E_1-E_2)t/\hbar} + e^{i(E_2-E_1)t/\hbar}\right) = \mu\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)$$



d) $\langle \hat{H} \rangle = \langle \psi(t)|\hat{H}|\psi(t)\rangle$ does not change with time. The probabilities of measuring E_1 or E_2 are both $\frac{1}{2}$, so

$$\langle \hat{H} \rangle = \frac{1}{2}E_1 + \frac{1}{2}E_2$$

You can confirm this using matrix multiplication.

3. $|\psi(t=0)\rangle = |+\rangle_n$ where $\hat{n}$ has $\theta = \frac{\pi}{2}$, $\phi = -\frac{\pi}{4}$

$\vec{B} = B_0 \hat{z}$

a) $|+\rangle_n = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2} e^{i\phi}|-\rangle$

$$= \cos\frac{\pi}{4}|+\rangle + \sin\frac{\pi}{4} e^{-i\pi/4}|-\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle + e^{-i\pi/4}|-\rangle\right) \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{-i\pi/4} \end{pmatrix}$$

(For this problem, it's better not to convert $e^{-i\pi/4} = \frac{1-i}{\sqrt{2}}$)

b) $\vec{B} = B_0 \hat{z}$ $\quad \hat{H} = \frac{eB_0}{m_e}\hat{S}_z$ $\quad$ McIntyre (3.25)

Define $\omega_0 \equiv \frac{eB_0}{m_e}$ so $\hat{H} = \omega_0 \hat{S}_z \doteq \frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Energy eigenstates are $|E_+\rangle = |+\rangle$ with energy $E_+ = \frac{\hbar\omega_0}{2}$

and $|E_-\rangle = |-\rangle$ with energy $E_- = -\frac{\hbar\omega_0}{2}$.

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_+t/\hbar}|+\rangle + e^{-i\pi/4} e^{-iE_-t/\hbar}|-\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}|+\rangle + e^{+i\omega_0 t/2} e^{-i\pi/4}|-\rangle\right)$$

You might choose to factor out $e^{-i\omega_0 t/2}$:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_0 t/2}\left(|+\rangle + e^{i(\omega_0 t - \pi/4)}|-\rangle\right)$$

c) $\mathcal{P}(y\ up) = \left| {}_y\langle + | \psi(t) \rangle \right|^2 \qquad {}_y\langle + | \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix}$

$$= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\omega_0 t/2} \\ e^{+i\omega_0 t/2} e^{-i\pi/4} \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{2} \left(e^{-i\omega_0 t/2} - i e^{-i\pi/4} e^{+i\omega_0 t/2} \right) \right|^2$$

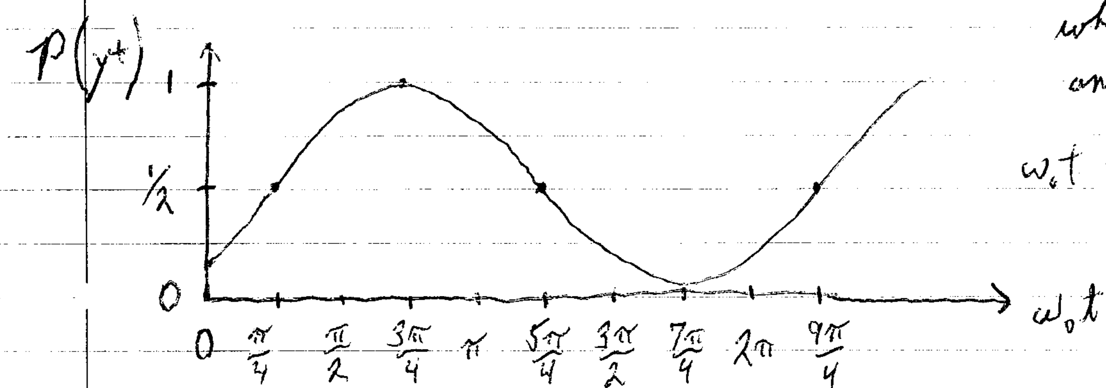
$$= \frac{1}{4} \left(e^{+i\omega_0 t/2} + i e^{i\pi/4} e^{-i\omega_0 t/2} \right) \left(e^{-i\omega_0 t/2} - i e^{-i\pi/4} e^{+i\omega_0 t/2} \right)$$

$$= \frac{1}{4} \left[1 + 1 - i e^{-i\pi/4} e^{i\omega_0 t} + i e^{+i\pi/4} e^{-i\omega_0 t} \right]$$

$$= \frac{1}{4} \left[2 - i \underbrace{\left(e^{i(\omega_0 t - \pi/4)} - e^{-i(\omega_0 t - \pi/4)} \right)}_{2i \sin(\omega_0 t - \pi/4)} \right]$$

$$= \frac{1}{2} + \frac{1}{2} \sin\left(\omega_0 t - \frac{\pi}{4}\right)$$

$\sin\left(\omega_0 t - \frac{\pi}{4}\right) = 0$ when $\omega_0 t = +\frac{\pi}{4}$ and again when $\omega_0 t = \frac{5\pi}{4}$ and $\frac{9\pi}{4}$



$\mathcal{P}(y+)$ stays between 0 and 1, as it must.

3. d) $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}|+\rangle + e^{i\omega_0 t/2}\, e^{-i\pi/4}|-\rangle\right)$

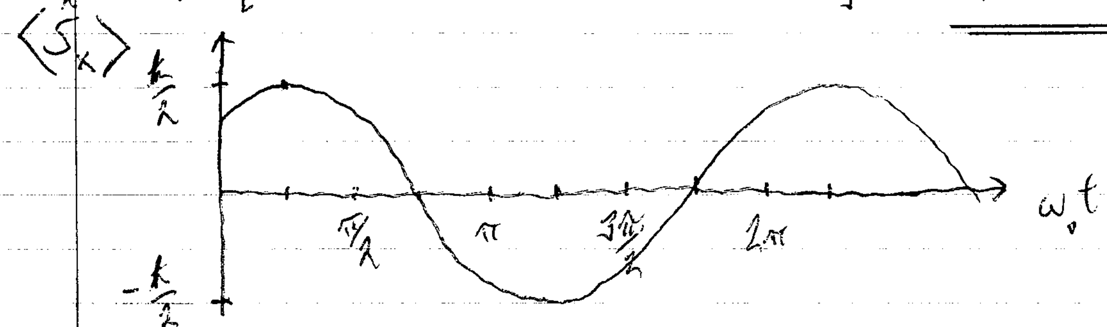
When $\omega_0 t = 2\pi$, both terms are multiplied by $e^{\pm i\pi} = -1$ so $|\psi(t = \frac{2\pi}{\omega_0})\rangle$ is identical to $|\psi(0)\rangle$, but with an overall sign change. The spin precesses around the z axis at frequency ω_0

e) $\langle \hat{S}_x \rangle = \langle \psi(t)|\hat{S}_x|\psi(t)\rangle$

$$= \frac{1}{2}\begin{pmatrix} e^{+i\omega_0 t/2} & e^{-i\omega_0 t/2}\, e^{+i\pi/4} \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega_0 t/2} \\ e^{+i\omega_0 t/2}\, e^{-i\pi/4} \end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix} e^{i\omega_0 t/2} & e^{-i\omega_0 t/2}\, e^{i\pi/4} \end{pmatrix}\begin{pmatrix} e^{i\omega_0 t/2}\, e^{-i\pi/4} \\ e^{-i\omega_0 t/2} \end{pmatrix}$$

$$= \frac{\hbar}{4}\left[e^{i\omega_0 t}\, e^{-i\pi/4} + e^{-i\omega_0 t}\, e^{i\pi/4}\right] = \underline{\underline{\frac{\hbar}{2}\cos\left(\omega_0 t - \frac{\pi}{4}\right)}}$$

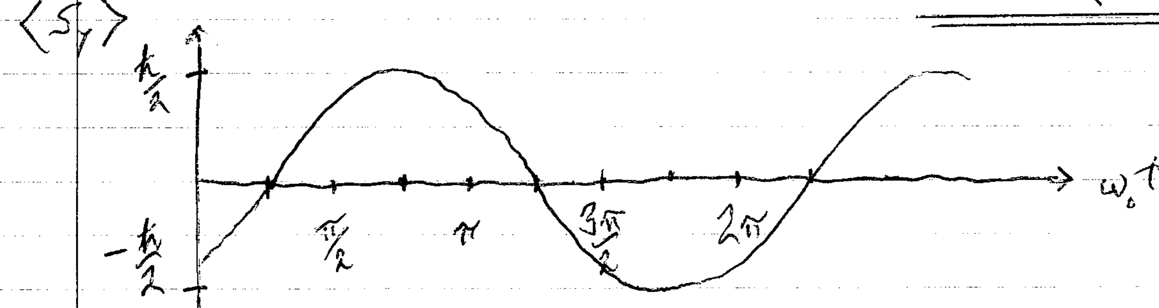


$$\langle S_y \rangle = \langle \psi(t) | \hat{S}_y | \psi(t) \rangle$$

$$= \frac{1}{2} \begin{pmatrix} e^{i\omega_0 t/2} & e^{-i\omega_0 t/2} e^{i\pi/4} \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_0 t/2} \\ e^{i\omega_0 t/2} e^{-i\pi/4} \end{pmatrix}$$

$$= \frac{\hbar}{4} \begin{pmatrix} e^{i\omega_0 t/2} & e^{-i\omega_0 t/2} e^{i\pi/4} \end{pmatrix} \begin{pmatrix} -i e^{i\omega_0 t/2} e^{-i\pi/4} \\ i e^{-i\omega_0 t/2} \end{pmatrix}$$

$$= \frac{\hbar}{4} \left(-i e^{i\omega_0 t} e^{-i\pi/4} + i e^{-\omega_0 t} e^{i\pi/4} \right) = +\frac{\hbar}{2} \sin\left(\omega_0 t - \frac{\pi}{4}\right)$$



$$\langle S_z \rangle = \frac{\hbar}{4} \begin{pmatrix} e^{i\omega_0 t/2} & e^{-i\omega_0 t/2} e^{i\pi/4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-i\omega_0 t/2} \\ e^{i\omega_0 t/2} e^{-i\pi/4} \end{pmatrix}$$

$$= \frac{\hbar}{4} (1 - 1) = 0$$

The spin precesses in the x-y plane, so it never acquires a nonzero $\langle \hat{S}_z \rangle$.

Notice that, since $\hat{\mathcal{H}} = \omega_0 \hat{S}_z$, $\langle \hat{\mathcal{H}} \rangle = \omega_0 \langle \hat{S}_z \rangle = 0$

The energy starts at zero and stays that way.