The Matrix Exponential

We prove several properties of the exponential function of a matrix

Objectives

We introduce the exponential of a square matrix and we show several of its properties.

Introduction

When we multiply two square matrices the result is another square matrix. This property allow us to define power functions and polynomials of a square matrix. W go one step further and define the exponential of a square matrix. The ***exponential of a square matrix*** is the square matrix given by the infinite sum

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots .$$

It can be shown that this infinite sum converges for any real or complex square matrix A. The main reason is the $n!$ in the denominator. Let's recall a couple of definitions we will need in this dive. An $n \times n$ matrix D is called ***diagonal*** iff all the matrix components outside the diagonal vanish, that is,

$$D = \begin{bmatrix} d_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_{nn} \end{bmatrix} = \operatorname{diag}\left[d_{11}, \cdots, d_{nn}\right].$$

An $n \times n$ matrix A is ***diagonalizable*** iff there exist a diagonal matrix D and an invertible matrix P so that

$$A = PDP^{-1}.$$

Recall there is an important relation between the matrices P, D of a diagonalizable matrix A and the eigenvalues and eigenvectors of A.

Theorem 1 (Eigenvectors and Diagonalizability)**.** *A 2×2 matrix, A, has two eigenvectors, $\boldsymbol{v}_1, \boldsymbol{v}_2$ not proportional to each other iff matrix A is diagonalizable, $A = PDP^{-1}$, with*

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad P = [\boldsymbol{v}_1, \boldsymbol{v}_2],$$

where λ_i is the eigenvalue of the eigenvector $\boldsymbol{v}_i$, for $i = 1, 2$.

Also recall the ***trace*** of an $n \times n$ matrix is the sum of its diagonal elements, that is,

$$\operatorname{tr} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{11} + \cdots + a_{nn}.$$

Further Reading

Students may need to read Section 5.4 “Diagonalizable Matrices” and Section 5.5 “The Matrix Exponential” in our textbook.

The Exponential of Diagonal and of Diagonalizable Matrices

Question 1:*(10 points)* Find a closed expression (without the infinite sum) for the exponential of a diagonal matrix $D = \mathrm{diag}\left[d_{11}, \cdots, d_{nn}\right]$.

Question 2: *(10 points)* Find a closed expression (without the infinite sum) of the exponential of a diagonalizable matrix $A = PDP^{-1}$, where D is diagonal.

The Exponential of Particular Matrices

Question 3:

(3a) *(5 points)* If $M^2 = M$, then show that

$$e^M = I + (e-1)\,M.$$

(3b) *(5 points)* If $M^2 = 0$ then compute e^M.

Question 4: *(10 points)* By direct computation on the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ show that

$$e^{(A+B)} \neq e^A \, e^B.$$

Hints: Use **Question (3a)** for one side of the equation and **Question (3b)** for the other side.

General Properties of the Exponential Matrix

Question 5: *(10 points)* Prove the following: If A is an $n \times n$, diagonalizable matrix, then

$$\det\left(e^{A}\right) = e^{\operatorname{tr}(A)}.$$

Hint: Use that the determinant on $n \times n$ matrices B, C satisfies $\det(BC) = \det(B)\,\det(C)$. Use this equation to relate the determinant of an invertible matrix P with the determinant of P^{-1}.

Question 6: *(10 points)* Prove the following: If λ and $\boldsymbol{v}$ are an eigenvalue and eigenvector of a matrix A, that is, $A\boldsymbol{v} = \lambda\boldsymbol{v}$, then $\boldsymbol{v}$ is an eigenvector of the matrix e^{A} with eigenvalue e^{λ}, that is,

$$e^{A}\boldsymbol{v} = e^{\lambda}\,\boldsymbol{v}.$$

Question 7: *(10 points)* Prove the following: If A, B are $n \times n$ matrices,

$$AB = BA \quad \Rightarrow \quad e^A \, e^B = e^B \, e^A.$$

Hints:

- First, prove that $AB = BA$ implies $A\,B^n = B^n\,A$.
- Second, prove that $AB = BA$ implies $A\,e^B = e^B\,A$.

Question 8: *(10 points)* Prove the following: If A is an $n \times n$ matrix and s, t are real constants, then

$$e^{As}\, e^{At} = e^{A(s+t)}.$$

Hints:

- Write $e^{As} = \left(\sum_{j=0}^{\infty} \frac{A^j\, s^j}{j!}\right)$ and $e^{At} = \left(\sum_{k=0}^{\infty} \frac{A^k\, t^k}{k!}\right)$, then compute their product.
- Switch from indices j and k to indices n and k, where $n = j + k$.
- Recall the binomial formula $(s+t)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} s^{n-k} t^k$.

Question 9: *(10 points)* Use the result in **Question 8** to prove the following: If A is an $n \times n$ matrix, then

$$\left(e^{A}\right)^{-1} = e^{-A}.$$

Question 10: *(10 points)* Prove the following: If A is an $n \times n$ matrix, and $t \in \mathbb{R}$, then

$$\frac{d}{dt}e^{At} = A\,e^{At}.$$