# Functions Defined by Differential Equations

*A differential equation can be used to define new functions*

## Objectives

We use a differential equation to determine properties of its solutions.

## Introduction

More often than not, solutions of differential equations cannot be written in terms of previously known functions. When that happens the we say that such solutions of a differential equation define new functions. Then, it is useful to characterize such functions, that is, to find as many properties as possible of these new functions. When a function is defined as a solution of a differential equation—instead of defined by the function values—we use that differential equation to characterize the solution.

We can do this is because of the theorem of existence and uniqueness of solutions for second order linear initial value problems which states the following (see **Theorem 2.1.3** in the online Textbook):

**Theorem** (Existence and Uniqueness). *Consider the initial value problem*

$$y'' + a_1(t)\,y' + a_0(t)\,y = b(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_1. \tag{1}$$

*If the functions $a_1$, $a_0$, $b$ are continuous on an open interval $(t_1, t_2)$, then there exists a unique solution $y(t)$ of Eq. (1) defined on that interval $(t_1, t_2)$ for every choice of the initial data $t_0 \in (t_1, t_2)$, and $y_0$, $y_1 \in \mathbb{R}$.*

This theorem enables us to talk about *the solution* of the IVP, just as we did with first order equations.

In this project we want to learn how to use a differential equation to characterize its solutions. One of the first examples are the Bessel's functions, which are solutions of the differential equation

$$x^2\,y'' + x\,y' + (x^2 - \alpha^2)\,y = 0,$$

where $\alpha$ is an arbitrary real or complex number. This equation describes, for example, the electrostatic field around a cylindrical conductor. Solutions of this equation for particular cases were studied by Daniel Bernoulli, Leonhard Euler, Joseph-Louis Lagrange, and a few others. A systematical study of the solutions to this equations was carried out by Friedrich Bessel in 1816.

Unfortunately, Bessel's functions are a bit complicated for our class. So, instead of studying a differential equation having new functions as solutions, such as the Bessel functions, we will study the differential equation

$$y'' + y = 0,$$

with well-known functions as solutions—the Sine and Cosine functions.

The functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$ are usually defined as a ratio of one side and the hypotenuse in a right triangle. A right triangle has two sides and a hypotenuse, so the ratio of each side with the hypotenuse defines each function, the Sine and the Cosine functions. From properties of similar triangles one finds that these ratios are independent of the side sizes. These rations depend only on the angle between the hypotenuse and one of the sides—angle we called $x$ above. This geometric definition implies many properties on the Sine and Cosine functions; below we mention a few of them.

- Parity: Sine is an odd function, $\sin(-x) = -\sin(x)$; Cosine is an even function, $\cos(-x) = \cos(x)$.
- Pythagoras Theorem: $\sin^2(x) + \cos^2(x) = 1$.
- Differentiability Property: $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$. (Hard to prove, but it is done in Calculus 1.)
- Power series expansions: $\sin(x) = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ and $\cos(x) = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$.

The goal of these notes is to ***forget*** *the geometric definition* above of the Sine and Cosine functions, and ***instead define*** *the Sine and Cosine functions as solutions of a particular differential equation* with appropriate initial conditions. Then we use ***only the differential equation*** and the *existence and uniqueness* results about solutions of initial value problems, to prove all the properties highlighted above.

## Further Reading

Students need to review **Section 2.1 "General Properties"** in the course textbook MTH 347H - Differential Equations. In particular students should focus on the **Existence and Uniqueness Theorem**, which is **Theorem 2.1.3**, and the subsection 2.1.5, **The Wronskian function**.

## The Characterization of functions $\gamma$ and $\sigma$

We are now going to find the properties of two functions, $\gamma$ and $\sigma$, solutions of two initial values problems.

**Defintion.** Let the function $\gamma(x)$ be the *unique solution* of the initial value problem

$$\gamma'' + \gamma = 0, \qquad \gamma(0) = 1, \quad \gamma'(0) = 0,$$

and let the function $\sigma(x)$ be the *unique solution* of the initial value problem

$$\sigma'' + \sigma = 0, \qquad \sigma(0) = 0, \quad \sigma'(0) = 1.$$

**Question 1.** *(2 points)* Show that the functions $\gamma$ and $\sigma$ are linearly independent — not proportional to each other.

**Hints:**

**(a)** We cannot use that $\gamma$ is the Cosine function and $\sigma$ is the Sine function, because this is what we want to prove at the end of this Dive.

**(b)** Recall the properties of the Wronskian of two functions which are solutions of linear differential equations.

**Question 2.** *(2 points)* Show that the function $\gamma$ is even and the function $\sigma$ is odd.

**Hints:**

**(a)** Recall that a function $f$ is even iff $f(-x) = f(x)$, while a function $g$ is odd iff $g(-x) = -g(x)$.

**(b)** We cannot use that $\gamma$ is the Cosine function and $\sigma$ is the Sine function, because this is what we want to prove at the end of this Dive.

**(c)** Find what initial value problem satisfy the functions $\hat{\gamma}(x) = \gamma(-x)$ and $\hat{\sigma}(x) = \sigma(-x)$. And recall the uniqueness results for initial value problems.

**Question 3.** *(2 points)* Prove the following relations between the functions $\gamma$ and $\sigma$,

$$\gamma'(x) = -\sigma(x), \qquad \sigma'(x) = \gamma(x).$$

**Hints:**

**(a)** We cannot use that $\gamma$ is the Cosine function and $\sigma$ is the Sine function, because this is what we want to prove at the end of this Dive.

**(b)** Again, recall the uniqueness results for initial value problems.

**Question 4.** *(2 points)* Show that the functions $\gamma$ and $\sigma$ satisfy the Pythagoras' theorem,

$$\gamma^2(x) + \sigma^2(x) = 1 \qquad \text{for all} \quad x.$$

**Hints:**

**(a)** We cannot use that $\gamma$ is the Cosine function and $\sigma$ is the Sine function, because this is what we want to prove at the end of this Dive.

**(b)** Use the results in Question 3 to compute the Wronskian of $\gamma$ and $\sigma$.

**(c)** Recall Abel's Theorem, which is about the differential equation satisfied by the Wronskian of two solutions to a second order differential equation.

**Question 5.** *(2 points)* Show that the power series expansion of the functions $\gamma$ centered at $x = 0$ is

$$\gamma(x) = \sum_{k=0}^{\infty} (-1)^k \, \frac{x^{2k}}{(2k)!}.$$

**Note:** A similar calculation can be done for the function $\sigma$, the result is

$$\sigma(x) = \sum_{k=0}^{\infty} (-1)^k \, \frac{x^{2k+1}}{(2k+1)!},$$

but you do not need to compute it.

**Hints:**

**(a)** We cannot use that $\gamma$ is the Cosine function and $\sigma$ is the Sine function, because this is what we want to prove at the end of this Dive.

**(b)** Use the results from the previous questions.

**Question 6:** *(0 points)* What is the well known name for the functions $\gamma$ and $\sigma$?