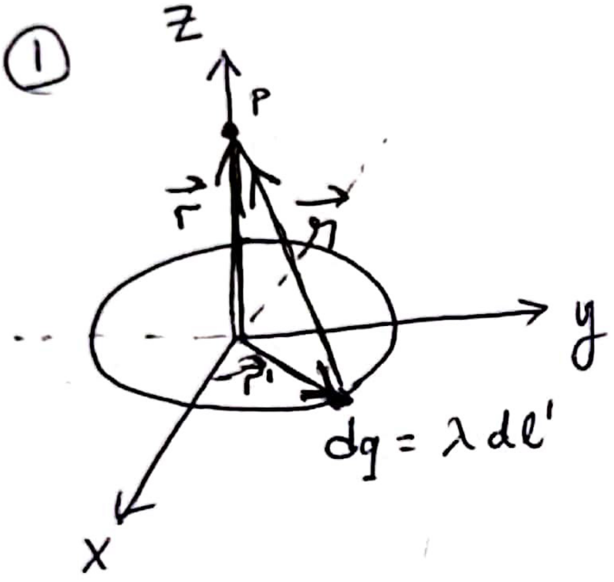
Electric Field of a ring of charge

① 

② $\vec{r} = z\,\hat{z}$

$\vec{r}\,' = R\,\hat{s}$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = z\,\hat{z} - R\,\hat{s}$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{|\vec{\mathscr{r}}|} = \frac{z\,\hat{z} - R\,\hat{s}}{\sqrt{z^2 + R^2}}$$

$dl' = R\,d\phi'$

③ $$\hat{\mathscr{r}} = \frac{z\,\hat{z} - R\left(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right)}{\sqrt{z^2 + R^2}}$$

$$= -\frac{R\cos\phi}{\sqrt{z^2+R^2}}\,\hat{x} - \frac{R\sin\phi}{\sqrt{z^2+R^2}}\,\hat{y} + \frac{z}{\sqrt{z^2+R^2}}\,\hat{z}$$

④ $$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda}{\mathscr{r}^2} \hat{\mathscr{r}} \, dl'$$

$$= \frac{\lambda}{4\pi\epsilon_0} \int \frac{-R\cos\phi \, \hat{x} - R\sin\phi \, \hat{y} + z \, \hat{z}}{\left(\sqrt{z^2+R^2}\right)^3} R \, d\phi$$

⑤ $$E_x = -\frac{1}{4\pi\epsilon_0} \frac{\lambda R^2}{(z^2+R^2)^{3/2}} \underbrace{\int_0^{2\pi} \cos\phi}_{0} = 0$$

$$E_y = -\frac{1}{4\pi\epsilon_0} \frac{\lambda R^2}{(z^2+R^2)^{3/2}} \underbrace{\int_0^{2\pi} \sin\phi}_{0} = 0$$

$$E_z = \frac{1}{4\pi\epsilon_0} \frac{\lambda z R}{(z^2+R^2)^{3/2}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} = \frac{1}{4\pi\epsilon_0} \frac{\overbrace{\lambda(2\pi R)}^{Q} \, z}{(z^2+R^2)^{3/2}}$$

There is a symmetry in x and y. So it makes sense that $E_x = 0$ and $E_y = 0$. Now we have

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{z}{(z^2+R^2)^{3/2}} \hat{z}$$

Checking your result

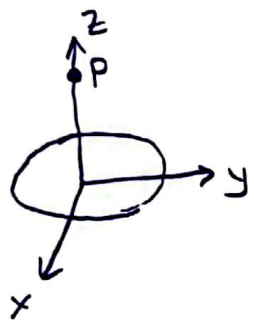
① $$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\lambda\, 2\pi R z}{(z^2+R^2)^{3/2}}\, \hat{z}$$

$$\left[\frac{1}{4\pi\epsilon_0}\right] = \left[\frac{N m^2}{C^2}\right], \quad [\lambda] = \left[\frac{C}{m}\right], \quad [R \cdot z] = [m^2]$$

$$\left[(z^2+R^2)^{3/2}\right] = [m^3]$$

$$\frac{\left[\frac{N m^2}{C^2}\right]\left[\frac{C}{m}\right][m^2]}{[m^3]} = \left[\frac{N}{C}\right] \quad ✓$$

②



Check 1: At $z=0$, we expect $E=0$ by symmetry.

Check 2: At $z \gg R$, we expect the charge on the ring looks like a point charge.

③ • At $z=0$: $$E(z{=}0) = \frac{1}{4\pi\epsilon_0} \frac{\lambda\, 2\pi R z}{(z^2+R^2)^{3/2}} = 0 \quad ✓$$

0

0

• At $z \gg R$: $$(z^2+R^2)^{-3/2} = z^{-3}\left(1+\frac{R^2}{z^2}\right)^{-3/2} \approx z^{-3}\left(1-\frac{3}{2}\frac{R^2}{z^2}\right)$$

Using $(1\pm\varepsilon) \approx 1 \pm n\varepsilon$ if $\varepsilon \ll 1$

So $$E \approx \frac{1}{4\pi\epsilon_0} \lambda 2\pi R z\, z^{-3}\left(1-\frac{3}{2}\frac{R^2}{z^2}\right) = \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}\left(1-\frac{3}{2}\frac{R^2}{z^2}\right)$$

point charge small correction

So: units ✓

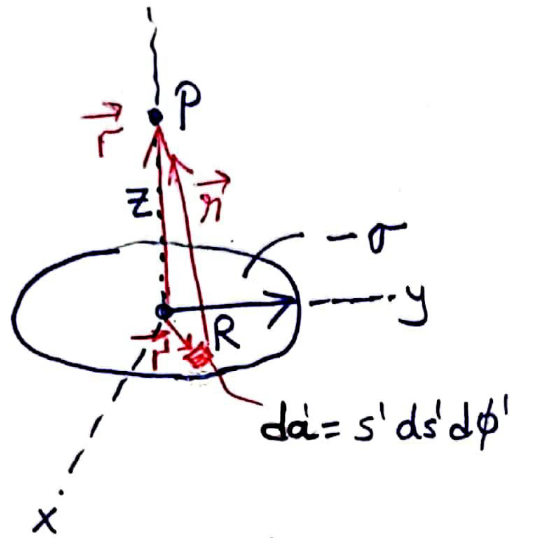
$E \to 0$ as $z \to 0$ ✓

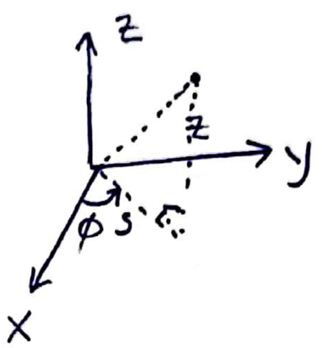
$E \to$ field due to point charge as $z \gg R$ ✓

Based on those checks, I believe this result is correct.

$\boxed{\text{Electric field of a disk of charge}}$

①





$$\vec r = z\hat z$$

$$\vec r\,' = s'\hat s$$

$$\vec{\mathscr{r}} = \vec r - \vec r\,' = z\hat z - \underbrace{s'\hat s'} = z\hat z - \underbrace{s'\cos\phi'\hat x - s'\sin\phi'\hat y}$$

$$|\vec{\mathscr{r}}| = \sqrt{z^2+s'^2}$$

$$dq = -\sigma\, da' = -\sigma\, s'ds'd\phi'$$

$$\vec E = \frac{1}{4\pi\epsilon_0}\int_S \left(-\sigma\, da'\,\frac{\vec{\mathscr{r}}}{|\vec{\mathscr{r}}|^3}\right) = \frac{-\sigma}{4\pi\epsilon_0}\int \frac{-s'\cos\phi\hat x - s'\sin\phi'\hat y + z\hat z}{(z^2+s'^2)^{3/2}} \times s'ds'd\phi'$$

$$E_x = \frac{-\sigma}{4\pi\epsilon_0}\int_0^R \frac{-s'^2 ds}{(z^2+s'^2)^{3/2}} \underbrace{\int_0^{2\pi}\cos\phi'\,d\phi'}_{0} = 0$$

Similarly $E_y = 0$ because $\int_0^{2\pi}\sin\phi'\,d\phi' = 0$

$$E_z = \frac{-\sigma}{4\pi\epsilon_0} \underbrace{\int_0^R \frac{z\, s'\, ds'}{(z^2+s'^2)^{3/2}}}_{\left.\frac{-z}{\sqrt{z^2+s'^2}}\right|_{s'=0}^{s'=R}} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} = \frac{-\sigma \cdot 2\pi}{4\pi\epsilon_0}\left(\frac{-z}{\sqrt{R^2+z^2}} + \frac{z}{|z|}\right)$$

Hence $\vec{E} = \frac{-\sigma}{2\epsilon_0}\left(\frac{z}{|z|} - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{z}$

(2) If $z \gg R$, then $\frac{R}{z} \ll 1$ (Assume $z>0$)

$$\frac{z}{\sqrt{R^2+z^2}} = \frac{z}{z\sqrt{\frac{R^2}{z^2}+1}} = \left(1+\frac{R^2}{z^2}\right)^{-1/2} \approx 1 - \frac{R^2}{2z^2}$$

using $(1 \pm \varepsilon)^n \approx 1 \pm n\varepsilon$ where $\varepsilon \ll 1$.

So $\frac{z}{|z|} - \frac{z}{\sqrt{R^2+z^2}} \approx 1 - \left(1 - \frac{R^2}{2z^2}\right) = \frac{R^2}{2z^2}$ (for $z>0$)

$$|\vec{E}|_{far} \approx -\frac{\sigma}{2\epsilon_0}\frac{R^2}{2z^2} = \frac{-\sigma \pi R^2}{4\pi\epsilon_0 z^2} = \frac{-|Q|}{4\pi\epsilon_0 z^2} \quad \text{(for } z>0\text{)}$$

As expected, $|\vec{E}|$ behaves like E-field of a point charge

③ If $z \ll R$, then $\frac{z}{R} \ll 1$ (Assume $z > 0$)

$$\frac{z}{\sqrt{R^2+z^2}} = \frac{z}{R\sqrt{1+\frac{z^2}{R^2}}} = \frac{z}{R}\left(1+\frac{z^2}{R^2}\right)^{-1/2} \approx \frac{z}{R}\left(1-\frac{z^2}{2R^2}\right)$$

$$= \frac{z}{R} - \underbrace{\frac{z^3}{2R^3}}_{\text{very small}} \approx \frac{z}{R}$$

So $\frac{z}{|z|} - \frac{z}{\sqrt{R^2+z^2}} \approx 1 - \frac{z}{R}$ (for $z > 0$)

Therefore, $|\vec{E}|_{near} \approx -\frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{R}\right]$

field from infinite sheet of charge + small correction

④ $$\vec{E} = -\frac{\sigma}{2\epsilon_0}\left[\frac{z}{|z|} - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{z}$$

At $z = 0$ → discontinuity because of $\frac{z}{|z|}$

For $z > 0$ → $\frac{z}{|z|} = 1$

For z 0 → $z/|z| = -1$