Integrating Factors for Linear Systems

Exponential solution formula for linear systems of differential equations

Objectives

To generalize the integrating factor method from one linear differential equation to systems of linear differential equations. We will need to use several properties of the exponential of a function discussed in a previous Dive.

Introduction

The integrating factor method is a way to find solutions to a linear differential equation

$$y' = a\,y + b,$$

where a, b are constants. One multiplies the equation above by the integrating factor

$$\mu(t) = e^{-at},$$

then we get

$$e^{-at}\,y' - a\,e^{-at}\,y = e^{-at}\,b.$$

But the left-side is a total derivative,

$$\big(e^{-at}\,y\big)' = e^{-at}\,b,$$

which leads us to the final formula

$$e^{-at}y(t) - y(0) = \int_0^t e^{-a\tau}\,b\,d\tau \quad \Rightarrow \quad e^{-at}y(t) - y(0) = -\frac{b}{a}\,e^{-at} + \frac{b}{a}.$$

Then we get that

$$y(t) = \Big(y(0) + \frac{b}{a}\Big)\,e^{at} - \frac{b}{a}.$$

The idea of this project is to *generalize* this solution formula to *systems* of linear differential equations.

Further Reading

Students may need to read Section 6.3, "General Linear Systems" and and Section 6.4, "Solutions Formulas" in our textbook. Make sure you download the last version of our textbook.

Homogeneous Systems

Question 1: *(20 points)* Generalize the integrating factor method used to solve linear scalar equations to prove the following statement: If A is an $n \times n$ matrix and $\boldsymbol{x}_0$ is an n-vector, then the initial value problem

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}^{\circ}$$

has a unique solution given by

$$\boldsymbol{x}(t) = e^{At}\,\boldsymbol{x}^{\circ}.$$

Note: Highlight every property of the matrix exponential you use in your proof.

Question 2: *(20 points)* In the case that an $n \times n$ matrix A is diagonalizable, with eigenpairs given by λ_i, $\boldsymbol{v}_i$, for $i = 1, \cdots, n$, we know that the general solution of the linear system

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t)$$

is given by

$$\boldsymbol{x}(t) = c_1\, e^{\lambda_1 t}\, \boldsymbol{v}_1 + \cdots + c_n\, e^{\lambda_n t}\, \boldsymbol{v}_n.$$

Use this formula for the general solution to show that the unique solution of the initial value problem

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}^0$$

can actually be written in the way given in the question above, that is,

$$\boldsymbol{x}(t) = e^{At}\, \boldsymbol{x}^0.$$

Question 3: *(20 points)* Compute the exponential function e^{At} and use it to express the vector-valued function $\boldsymbol{x}(t)$ solution to the initial value problem

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}.$$

Non-Homogeneous Systems

Question 4: *(20 points)* Prove that the integrating factor method can be generalized to non-homogeneous linear differential systems, that is, prove the following: If A is an $n \times n$ *invertible* matrix, $\boldsymbol{x}^0$ is an n-vector, and $\boldsymbol{b}$ is a constant n-vector, then the initial value problem

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t) + \boldsymbol{b}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}^0$$

has a unique solution given by

$$\boldsymbol{x}(t) = e^{At}\,(\boldsymbol{x}^0 + A^{-1}\boldsymbol{b}) - A^{-1}\boldsymbol{b}.$$

Note: Mention very carefully every property of the matrix exponential you use on each step of your proof.

Question 5: *(20 points)* Find the vector-valued solution $\boldsymbol{x}(t)$ to the differential system

$$\boldsymbol{x}' = A\,\boldsymbol{x} + \boldsymbol{b}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$