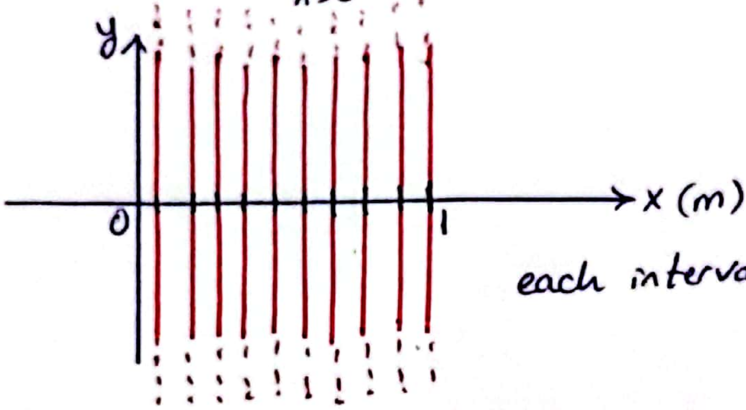
Describing charge distributions with delta functions

① In 2D, a single delta function represents a line that "goes on forever". For example, $\delta(x-\frac{1}{10})$ is a line at $x=\frac{1}{10}$ that extends parallel to y-axis.

For $\lambda(x) = \sum_{n=0}^{10} q_0 n^2 \delta(x-\frac{1}{10})$, we have



each interval is separated by $\frac{1}{10}$ m.

② $$Q = \int_{-\infty}^{\infty} \lambda(x)\,dx = \int_{-\infty}^{\infty} \sum_{n=0}^{10} q_0 n^2 \delta(x-\tfrac{1}{10})\,dx = \sum_{n=0}^{10} q_0 n^2 \underbrace{\int_{-\infty}^{\infty} \delta(x-\tfrac{1}{10})\,dx}_{1}$$

$$= q_0 \sum_{n=0}^{N=10} n^2 = q_0 \frac{N(N+1)(2N+1)}{6}$$

For $N=10$, $Q = 385\,q_0$

Spherical charge distributions are special

① $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \rho = \epsilon_0 \vec{\nabla}\cdot\vec{E}$

In spherical coordinates:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, E_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(E_\phi)$$

For the $\vec{E}$-field in this problem: $E_\theta = 0$ and $E_\phi = 0$.

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot c r^n) = \frac{c}{r^2}\frac{\partial}{\partial r}(r^{n+2}) = \frac{c(n+2)r^{n+1}}{r^2}$$

$$= c(n+2)r^{n-1}$$

Hence, $\rho = \epsilon_0 \vec{\nabla}\cdot\vec{E} = \epsilon_0 c(n+2)r^{n-1}$ for $n \neq -2$

② For $n=-2$, $\rho = 0$ according to the equation above.

If $\rho = 0$, then we would not get $\vec{E} = cr^{-2}\hat{r}$

For $n=-2$: $\vec{\nabla}\cdot\vec{E} = c\,\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = c\,4\pi\,\delta^3(\vec{r})$

Hence, $\rho = 4\pi\epsilon_0 c\,\delta^3(\vec{r})$ for $n=-2$

Extra info: Actually not all values of n is physically possible. We need to have $\int_{\text{volume}} \rho(r)\,d\tau = \text{finite}$.

For $n \geq -2$, we have a finite total charge $\int \rho\, d\tau$.

However for $n < -2$, $\int \rho\, d\tau$ runs to infinity.

So only $n \geq -2$ cases are physically possible.

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③ $E = \text{const.} \rightarrow n = 0$.

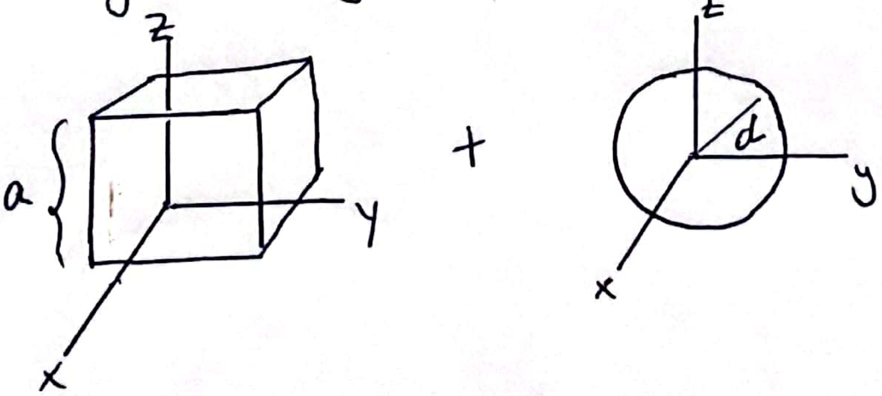
So $\rho = \epsilon_0 c(n+2)r^{n-1}$ becomes

$\rho(r) = \frac{2\epsilon_0 c}{r}$ for $n=0$.

Cube with a hole

①
- Gauss' law is always true.
- In this problem, Gauss' law is partially useful. It can be used to find E-field due to the spherical hole for which we can use a spherical Gaussian surface.
- A cube does not have spherical, cylindirical, or planar symmetry. So we cannot find a gaussian surface with such symmetries.

② We can use superposition.
A cube with uniform charge density $+\rho$ with a spherical hole at its center is equivalent to a solid cube with charge density $+\rho$ plus a sphere with charge density $-\rho$

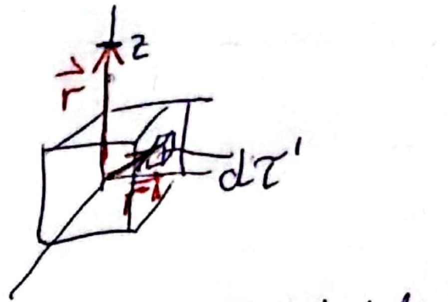


Let's use Gauss' law for the sphere of radius d. by choosing a spherical Gaussian surface of radius z.

$$\oint_S \vec{E}\cdot d\vec{a} = E\oint_S da = E\cdot 4\pi z^2 = \frac{Q_{encl}}{\epsilon_0} = \frac{-\rho\frac{4}{3}\pi d^3}{\epsilon_0}$$

So $\vec{E}_{sphere} = -\dfrac{\rho d^3}{3\epsilon_0 z^2}\hat{z}$

For $\vec{E}_{cube}$ use direct integration



$\vec{r} = z\hat{z}$

$\vec{r}' = x'\hat{x}' + y'\hat{y}' + z'\hat{z}'$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = -x'\hat{x}' - y'\hat{y}' + (z-z')\hat{z}'$

$d\tau = dx'dy'dz'$

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{-x'\hat{x}' - y'\hat{y}' + (z-z')\hat{z}'}{\left[x'^2 + y'^2 + (z-z')^2\right]^{3/2}}$$

$$\vec{E}_{cube} = \frac{1}{4\pi\epsilon_0}\int \rho \frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\, d\tau' = \frac{\rho}{4\pi\epsilon_0}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{-x'\hat{x} - y'\hat{y} + (z-z')\hat{z}}{\left[x'^2 + y'^2 + (z-z')^2\right]^{3/2}}\, dx'dy'dz'$$

Actually by symmetry we can see that x and y components of $\vec{E}_{cube}$ should vanish; i.e.

$$E_{x\,cube} = \frac{\rho}{4\pi\epsilon_0}\underbrace{\iiint}_{\text{symmetric limits}} \underbrace{\left(\frac{-x'}{\mathscr{r}^3}\right)}_{\text{odd function}} dx'dy'dz' = 0$$ and similarly $E_{y\,cube} = 0$.

Hence $$\vec{E}_{cube} = \frac{\rho}{4\pi\epsilon_0}\int_{-\frac{a}{2}}^{a/2}\int_{-a/2}^{a/2}\int_{-\frac{a}{2}}^{a/2} \frac{(z-z')\,dx'dy'dz'}{\left[x'^2 + y'^2 + (z-z')^2\right]^{3/2}}\;\hat{z}$$

and $\vec{E}_{total} = \vec{E}_{cube} + \vec{E}_{sphere}$.