Physics 471 – Fall 2023

Homework #11 – due Wednesday, November 22

1. [16] Free particles are often modeled as "wave packets". For example, we might start with an initial Gaussian wave function, $\psi(x, t = 0) = Ae^{-\alpha x^2}$, with A and α both real and positive constants.

a) [3] Normalize $\psi(x, t = 0)$, and then calculate its momentum-space distribution:

$\phi(p, t = 0) = \int_{-\infty}^{\infty} \psi(x, t = 0) \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} dx$ for this wave function.

Hints: If you are feeling ambitious, there's a trick to do integrals of the form $\int_{-\infty}^{\infty} e^{-(ax^2+bx)} dx$, *called "completing the square": Let* $y = \sqrt{a}(x + b/2a)$ *so that* $(ax^2 + bx) = y^2 - \frac{b^2}{4a}$. *(This is valid even for complex b! What is "b" for your problem at hand?) Also note the handy Gaussian integral:* $\int_{-\infty}^{\infty} e^{-cz^2} dz = \sqrt{\pi/c}$. *If you don't feel up to this, just use Eqn. (F.23) in Appendix F.*

b) [3] Now calculate the time dependent wave function $\psi(x, t)$.

Hint: See McIntyre 6.40. The answer here is $\psi(x, t) = \left(\frac{2\alpha}{?}\right)^{1/4} \frac{e^{-\alpha x^2/(1+i\Omega t)}}{\sqrt{(1+i\Omega t)}}$, with $\Omega \equiv 2\alpha\hbar/m$

(You will need to work out what is the dimensionless constant indicated by "?" above)

c) [3] Find the probability density $|\psi(x, t)|^2$. Simplify your result by expressing it in terms of a new quantity defined as $\Gamma = 1 + (\Omega t)^2$. Then, find $\langle x \rangle$ and $\langle p \rangle$ as functions of time, and briefly discuss.

d) [3] Find $\langle x^2 \rangle$ and $\langle p^2 \rangle$, and then Δx and Δp (all as functions of time). (If you need a computer to help, that's fine, just let the grader know.) Discuss the Heisenberg uncertainty principle for this problem. When is the system closest to the lower limit of $\Delta x \Delta p$?

Helpful math hint: $\int_{-\infty}^{\infty} z^2 e^{-cz^2} dz = \frac{1}{2}\sqrt{\pi/c^3}$. *Another hint: For* $\langle p^2 \rangle$ *, you may find using the result from part a) makes it easier – work in momentum space. See McIntyre equation (6.58).*

e) [4] Sketch $|\psi(x,t)|^2$ you found in part c) as a function of *x*, first at *t* = 0, and then again for a rather large time *t*.

Then, address the following questions (with words and formulas):

i) What happens qualitatively to $|\psi(x,t)|^2$ as time goes on? (Can you interpret this physically?)

ii) What do the quantities Ω and Γ defined above represent or tell you?

iii) I said, "at a rather large time". What sets this timescale?

iv) How does the characteristic time $\tau = \Omega^{-1}$ depend on the spatial width of the initial wave packet?

2. [4] You should have found that the momentum distribution $\phi(p)$ for the wave packet in the previous problem is a Gaussian function centered at zero; the particle in that problem is stationary. In Section 6.2.2 of the textbook, McIntyre discusses the more general and interesting situation where the particle has a nonzero value of $\langle p \rangle$. (The integrals are trickier, unfortunately, so I didn't assign them as homework.) Let's compare the two problems qualitatively.

a) [1] Sketch the $\phi(p)$ you calculated in problem 1, and sketch McIntyre's Equation (6.45) on the same plot. Assume that they both have the same width, i.e. the same Δp.

b) [1] On different plots, sketch the real part of $\psi(x, t = 0)$ both for your wavefunction in problem 1 and for McIntyre's wavefunction given in Equation (6.48) evaluated at $t = 0$.

Note: You are not expected to calculate the real part of $\psi(x, t = 0)$. Just use what you know about free-particle wave packets from the figures in sections 6.2.2 and 6.3 of the textbook.

Hint: Since the momentum distributions have the same width, the envelopes of the initial wavefunctions also have the same widths. What is the difference between his wave function and your wavefunction from problem 1?

c) [2] A short time later, McIntyre's wave packet will have moved to the right, assuming that $p_0 > 0$. Your wave packet, on the other hand, hasn't moved in space. But both wave packets have spread out. Sketch the real part of $\psi(x, t)$ for both wave functions at some time $t > 0$, so that your sketches illustrate both effects.