PHY 471: Homework 9 Solutions

1. $|\psi(t=0)\rangle = |E_2\rangle$ $\qquad E_n = \dfrac{n^2\pi^2\hbar^2}{2mL^2}$

a) $\langle \hat{H} \rangle = E_2 = \dfrac{2\pi^2\hbar^2}{mL^2}$ at <u>all</u> times.

$$|\psi(t)\rangle = e^{-iE_2t/\hbar}|E_2\rangle$$

An energy measurement will always yield E_2.

b) $\psi(x,t) = e^{-iE_2t/\hbar}\,\varphi_{E_2}(x) = e^{-iE_2t/\hbar}\sqrt{\dfrac{2}{L}}\sin\left(\dfrac{2\pi x}{L}\right)$

McIntyre (5.66)

$$|\psi(x,t)|^2 = \frac{2}{L}\sin^2\left(\frac{2\pi x}{L}\right)$$

$$\langle x(t)\rangle = \int_0^L x\,|\psi(x,t)|^2\,dx = \frac{2}{L}\int_0^L x\sin^2\left(\frac{2\pi x}{L}\right)dx$$

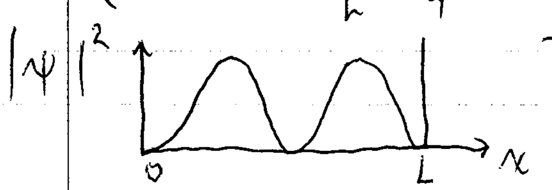
<u>Method 1</u>: Change to dimensionless variables $u = \frac{x}{L}$, $x = Lu$, $dx = L\,du$

$$\langle x(t)\rangle = \frac{2}{L}\int_0^1 (Lu)\sin^2(2\pi u)\cdot(L\,du) = 2L\underbrace{\int_0^1 u\sin^2(2\pi u)\,du}$$

Mathematica evaluates dimensionless integral $= \frac{1}{4}$

$$\langle x(t)\rangle = \frac{2}{L}\cdot\frac{1}{4} = \underline{\underline{\frac{L}{2}}}$$

This makes sense because $|\psi|^2$ is symmetric about the middle of the well.



1.b) Method 2 : Evaluate the integral yourself:

$$\int_0^1 u \sin^2(2\pi u)\,du \qquad\qquad \sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$= \frac{1}{2}\int_0^1 u\,(1-\cos 4\pi u)\,du = \frac{1}{2}\left[\frac{1}{2}u^2\Big|_0^1 - \int_0^1 u\cos(4\pi u)\,du\right]$$

Integrate by parts : $dv = \cos(4\pi u)\,du \qquad v = \frac{1}{4\pi}\sin(4\pi u)$

$$= \frac{1}{4} - \frac{1}{2}\left[\underbrace{\frac{u}{4\pi}\sin(4\pi u)\Big|_0^1}_{0} - \underbrace{\frac{1}{4\pi}\int_0^1 \sin(4\pi u)\,du}_{0}\right]$$

$$= \frac{1}{4}$$

Method 3 : Use Symbolab or Mathematica to give you an algebraic expression that contains L.

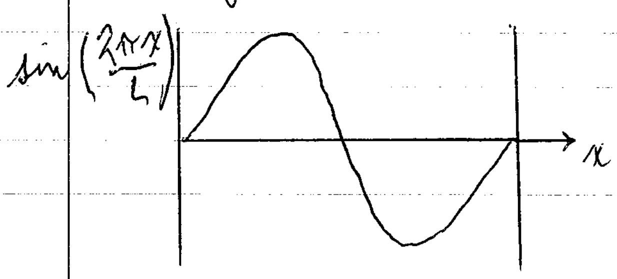
Then you don't have to change to dimensionless variables in the first place. (But it's a good trick to know!)

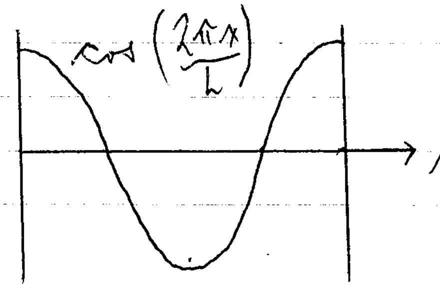
c) Calculate $\langle \hat{p}(t) \rangle$

Method 1 : Brute force

$$\langle \hat{p}(t)\rangle = \langle \psi(t)|\hat{p}|\psi(t)\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\left(-i\hbar\frac{d}{dx}\right)\psi(x,t)\,dx$$

$$= \frac{2}{L}\int_0^L e^{+iE_2 t/\hbar}\sin\left(\frac{2\pi x}{L}\right)\cdot(-i\hbar)\cdot\frac{2\pi}{L}\cos\left(\frac{2\pi x}{L}\right)e^{-iE_2 t/\hbar}\,dx$$





1.c) $\sin\frac{2\pi x}{L}$ is odd around $x = \frac{L}{2}$ } Their product is odd, so the integral is 0.
$\cos\frac{2\pi x}{L}$ is even around $x = L/2$ }

If you aren't confident about that result, then continue

$$\langle \hat{p}(t)\rangle = -i\hbar \frac{4\pi}{L^2}\int_0^L \sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right)dx$$

$$= -i\hbar \frac{4\pi}{L^2}\cdot\frac{L}{2\pi}\sin^2\left(\frac{2\pi x}{L}\right)\Big|_0^L = \underline{\underline{0}}$$

Method 2: Ehrenfest's Theorem: $\langle \hat{p}(t)\rangle = m\frac{d}{dt}\langle \hat{x}(t)\rangle$

But $\langle \hat{x}(t)\rangle = L/2$ is independent of time, so $\langle \hat{p}\rangle = 0$.

This makes sense because the system is in a stationary state!

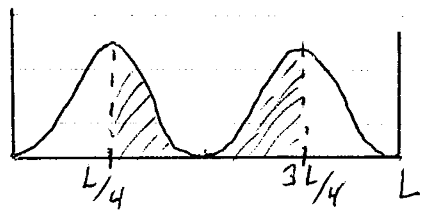
d) $$P\left[L/4 < x < \tfrac{3}{4}L\right] = \int_{L/4}^{3L/4} |\psi(x,t)|^2\, dx$$

$$= \frac{2}{L}\int_{L/4}^{3L/4}\sin^2\left(\frac{2\pi x}{L}\right)dx = \frac{1}{L}\int_{L/4}^{3L/4}\left(1-\cos\left(\frac{4\pi x}{L}\right)\right)dx$$

$$= \frac{1}{L}\cdot\left[x - \frac{L}{4\pi}\sin\left(\frac{4\pi x}{L}\right)\right]_{L/4}^{3L/4}$$

$$= \frac{1}{L}\left[\left(\frac{3L}{4}-\frac{L}{4}\right) - \frac{L}{4\pi}\left(\sin 3\pi - \sin\pi\right)\right] = \frac{1}{L}\cdot\frac{L}{2} = \underline{\underline{\frac{1}{2}}}$$

We could have guessed this from $|\psi|^2$:



1. e) We already have $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$, so we need $\langle \hat{x}^2 \rangle$ and $\langle \hat{p}^2 \rangle$

$$\langle x^2 \rangle = \int_0^L x^2 \left| \psi(x,t) \right|^2 dx = \frac{2}{L} \int_0^L x^2 \sin^2\left(\frac{2\pi x}{L}\right) dx$$

Again I'll use $u = \frac{x}{L}$:

$$\langle x^2 \rangle = 2L^2 \underbrace{\int_0^1 u^2 \sin^2(2\pi u)\, du} = L^2\left(\frac{1}{3} - \frac{1}{8\pi^2}\right)$$

For the integral, Symbolab gives $\frac{32\pi^3 - 12\pi}{192\pi^3} = \frac{1}{6} - \frac{1}{16\pi^2}$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{L^2\left(\frac{1}{3} - \frac{1}{8\pi^2}\right) - \frac{L^2}{4}} = \sqrt{L^2\left(\frac{1}{12} - \frac{1}{8\pi^2}\right)}$$

$$= L\sqrt{0.0707\ldots} \approx \underline{\underline{0.266\, L}}$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi^* \left(-\hbar^2 \frac{d^2}{dx^2}\psi\right) dx = -\hbar^2 \frac{2}{L} \int_0^L \sin\left(\frac{2\pi x}{L}\right)\left(\frac{2\pi}{L}\right)^2\left(-\sin\frac{2\pi x}{L}\right) dx$$

$$= \frac{8\pi^2}{L^3}\hbar^2 \underbrace{\int_0^L \sin^2\left(\frac{2\pi x}{L}\right) dx}_{= L/2} = 4\pi^2 \frac{\hbar^2}{L^2}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{4\pi^2 \frac{\hbar^2}{L^2}} = \underline{\underline{\frac{2\pi\hbar}{L}}}$$

$$\Delta x\, \Delta p = 0.266\, L \cdot \frac{2\pi\hbar}{L} = \underline{\underline{1.67\, \hbar}}$$

Uncertainty Principle says $\Delta x \Delta p \geq \frac{1}{2}\hbar$, so this is consistent.

2. $|\psi(t=0)\rangle = A\left(|E_1\rangle + 3i|E_2\rangle\right)$

a) $1 = \langle\psi|\psi\rangle = |A|^2(1+9) = 10|A|^2 \rightarrow A = \frac{1}{\sqrt{10}}$

Measure energy: results are E_1 or E_2

$$P(E_1) = 1/10 \quad P(E_2) = 9/10$$

$$\langle \hat{H}\rangle = \frac{1}{10}E_1 + \frac{9}{10}E_2 = \frac{1}{10}\cdot\frac{\pi^2\hbar^2}{2mL^2} + \frac{9}{10}\cdot\frac{4\pi^2\hbar^2}{2mL^2}$$

$$= (0.1 + 3.6)\frac{\pi^2\hbar^2}{2mL^2} = \underline{\underline{3.7\cdot\frac{\pi^2\hbar^2}{2mL^2}}} = 3.7\,E_1$$

b) $|\psi(t)\rangle = \frac{1}{\sqrt{10}}e^{-iE_1t/\hbar}|E_1\rangle + \frac{3i}{\sqrt{10}}e^{-iE_2t/\hbar}|E_2\rangle$

Energy measurements are the same as in part (a)!

c) $\psi(x,t) = \frac{1}{\sqrt{10}}e^{-iE_1t/\hbar}\cdot\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L} + \frac{3i}{\sqrt{10}}e^{-iE_2t/\hbar}\sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$

$$|\psi(x,t)|^2 = \frac{1}{10}\cdot\frac{2}{L}\cdot\left|e^{-iE_1t/\hbar}\sin\frac{\pi x}{L} + 3i\,e^{-iE_2t/\hbar}\sin\frac{2\pi x}{L}\right|^2$$

$$= \frac{1}{5L}\left(e^{iE_1t/\hbar}\sin\frac{\pi x}{L} - 3i\,e^{iE_2t/\hbar}\sin\frac{2\pi x}{L}\right)\left(e^{-iE_1t/\hbar}\sin\frac{\pi x}{L} + 3i\,e^{-iE_2t/\hbar}\sin\frac{2\pi x}{L}\right)$$

$$= \frac{1}{5L}\left[\sin^2\frac{\pi x}{L} + 9\sin^2\frac{2\pi x}{L} + 3i\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\underbrace{\left(e^{i(E_1-E_2)t/\hbar} - e^{i(E_2-E_1)t/\hbar}\right)}_{2i\sin\frac{(E_1-E_2)t}{\hbar}}\right]$$

$$E_2 - E_1 = \frac{4\pi^2\hbar^2}{2mL^2} - \frac{\pi^2\hbar^2}{2mL^2} = \frac{3\pi^2\hbar^2}{2mL^2} = \hbar\omega$$

2. c) $$\left|\psi(x,t)\right|^2 = \frac{1}{5L}\left[\sin^2\frac{\pi x}{L} + 9\sin^2\frac{2\pi x}{L} + 6\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\sin\omega t\right]$$

$t_0 = 0 \rightarrow \sin\omega t = 0$, $t_1 = \frac{\pi}{2\omega} \rightarrow \sin\omega t = \sin\frac{\pi}{2} = 1$

$t_2 = \frac{\pi}{\omega} \rightarrow \sin\omega t = \sin\pi = 0$, $t_3 = \frac{3\pi}{2\omega} \rightarrow \sin\omega t = \sin\frac{3\pi}{2} = 1$

$t_4 = \frac{2\pi}{\omega} \rightarrow \sin\omega t = 0$

$\Rightarrow$ The plots for $t = t_0, t_2$, and t_4 will be identical. See next page for plots made in Mathematica. The probability density sloshes back and forth in the well.

d) $$\langle x(t)\rangle = \int_0^L x\left|\psi(x,t)\right|^2 dx$$

Let's separate the integral into a time-independent part (first two terms) plus a time-dependent part (last term):

$$\frac{1}{5L}\int_0^L dx\left(x\sin^2\frac{\pi x}{L} + 9x\sin^2\frac{2\pi x}{L}\right) = \frac{1}{5L}\left(\frac{L^2}{4} + 9\frac{L^2}{4}\right) = \frac{L}{2}$$

$$\frac{1}{5L}\cdot\int_0^L 6x\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\,dx = \frac{6}{5L}\left(\frac{-8L^2}{9\pi^2}\right) = \frac{-16}{15\pi^2}L$$

from Mathematica

$$\langle x(t)\rangle = \frac{L}{2} - \frac{16}{15\pi^2}L\sin(\omega t)$$

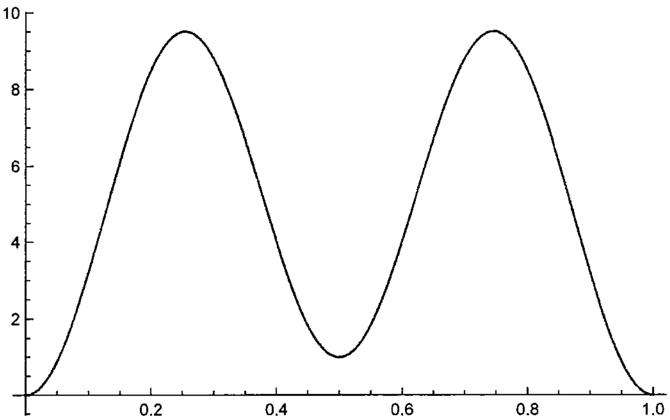
$$x_{max} = L\left(\frac{1}{2} + \frac{16}{15\pi^2}\right) = 0.608L$$

$$x_{min} = L\left(\frac{1}{2} - \frac{16}{15\pi^2}\right) = 0.392L$$

In problem 1, $\langle x(t)\rangle = L/2$ always.

In[6]:= `Plot[Sin[π * x]^2 + 9 * Sin[2 * π * x]^2, {x, 0, 1}]`

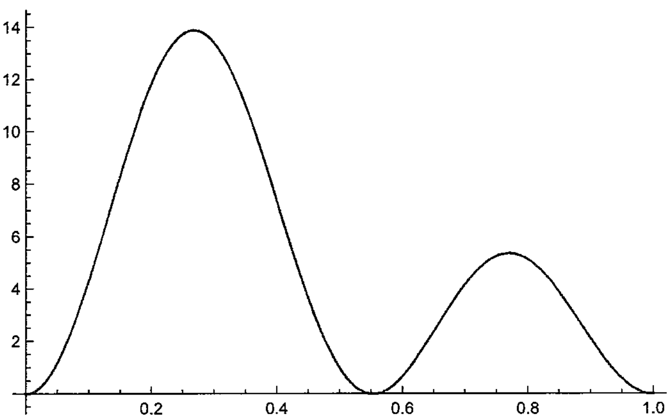
Out[6]=



$t = t_0 , t_2 , \text{or } t_4$

In[7]:= `Plot[Sin[π * x]^2 + 9 * Sin[2 * π * x]^2 + 6 * Sin[π * x] * Sin[2 * π * x], {x, 0, 1}]`

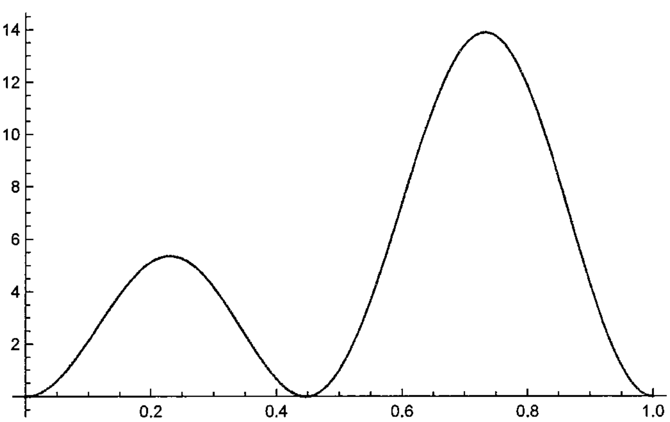
Out[7]=



$t = t_1$

In[8]:= `Plot[Sin[π * x]^2 + 9 * Sin[2 * π * x]^2 - 6 * Sin[π * x] * Sin[2 * π * x], {x, 0, 1}]`

Out[8]=



$t = t_3$

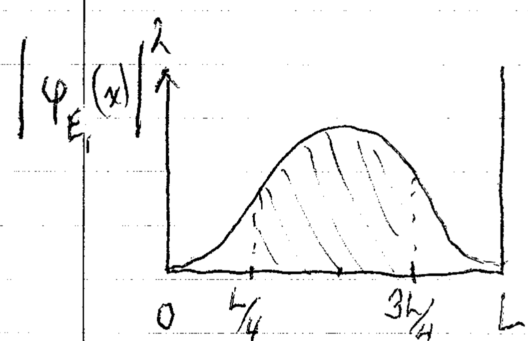
2. a) Measure energy, obtain result E_1. The system is now in the state

$$|E_1\rangle \longrightarrow \varphi_{E_1}(x) = \sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}$$

$$P\left[\tfrac{L}{4},\tfrac{3L}{4}\right] = \int_{L/4}^{3L/4} \frac{2}{L}\sin^2\frac{\pi x}{L}\,dx = \frac{2}{L}\cdot\frac{1}{2}\int_{L/4}^{3L/4}\left(1-\cos\frac{2\pi x}{L}\right)dx$$

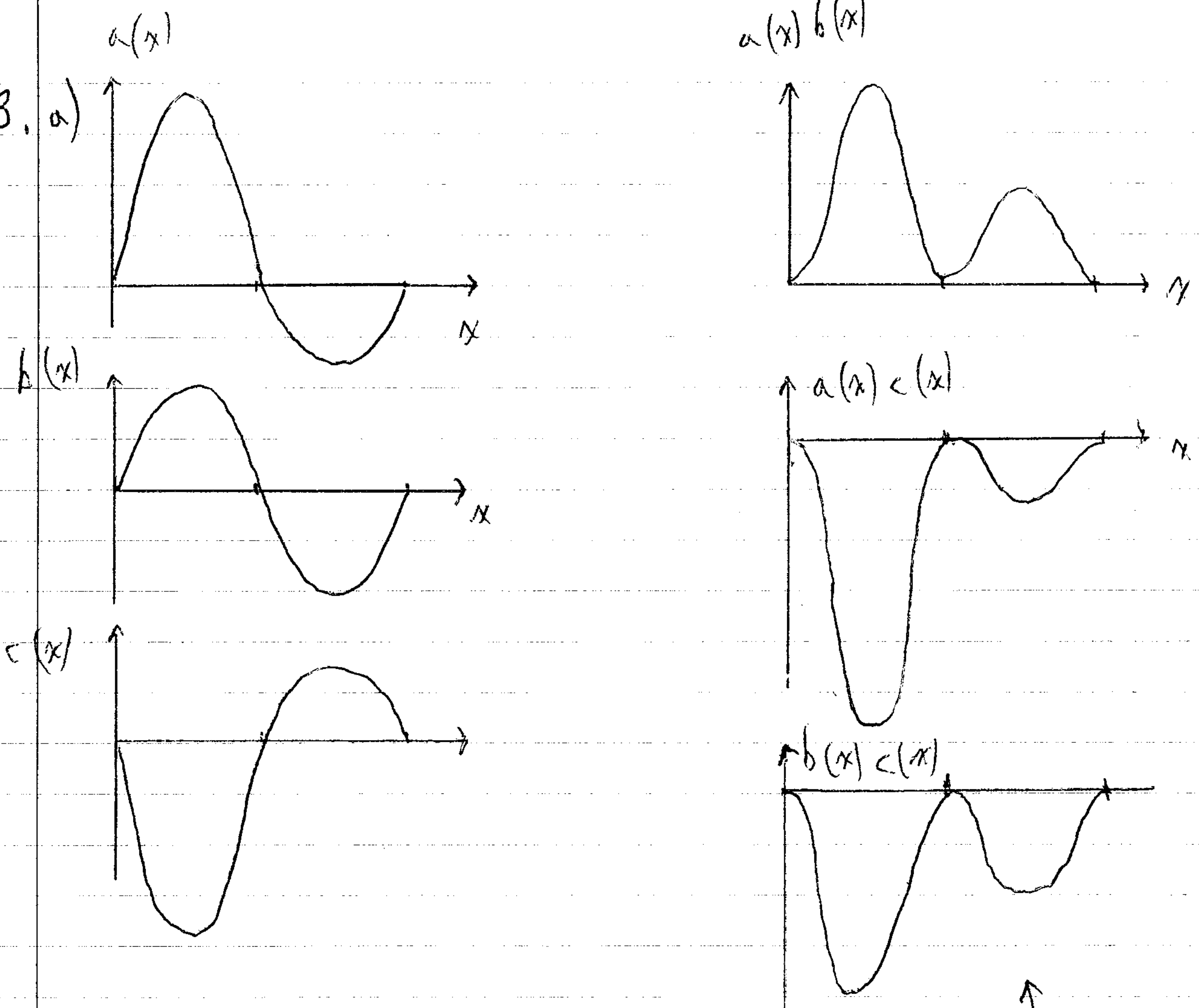
$$= \frac{1}{L}\cdot x\Big|_{L/4}^{3L/4} - \frac{1}{L}\cdot\frac{L}{2\pi}\sin\frac{2\pi x}{L}\Big|_{L/4}^{3L/4}$$

$$= \frac{1}{L}\cdot\frac{L}{2} - \frac{1}{2\pi}\left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right) = \frac{1}{2} + \frac{1}{\pi} = \underline{\underline{0.818}}$$



In the ground state the probability of finding the particle in the middle half of the well is large.

3. a)



The inner products are the integrals of these functions

i) $\langle a | b \rangle > 0$ ii) $\langle a | c \rangle < 0$ iii) $\langle b | c \rangle < 0$

b) $|\langle a | c \rangle| > |\langle a | b \rangle| = |\langle b | c \rangle|$

$\langle a | c \rangle$ is largest in magnitude because $a(x)c(x)$ has the largest bump

$\langle a | b \rangle = -\langle b | c \rangle$ because $c(x) = -a(x)$

so $|\langle a | b \rangle| = |\langle b | c \rangle|$