Deep Dive #8 - Integrating Factor Method

Remember, matricies aren't actually scary. They just say that the rate of change of a variable is dependent on the current value of other variables.

Question 1: *(20 points)* Generalize the integrating factor method used to solve linear scalar equations to prove the following statement: If A is an $n \times n$ matrix and $\boldsymbol{x}_0$ is an n-vector, then the initial value problem

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}^0$$

has a unique solution given by

$$\boldsymbol{x}(t) = e^{At}\,\boldsymbol{x}^0.$$

Note: Highlight every property of the matrix exponential you use in your proof.

The integrating factor method introduced in section 1.4 for constant coefficients is $y' = ay \implies y = ce^{at}$ which, with intial value $y_0 = y(0)$ this becomes $y = y_0 e^{at}$

How do we do this when A is a matrix? We'll use the property of matrix exponentials $\left(e^{At}\right)' = e^{At}A$ as follows

$$x' = Ax$$
$$x' - Ax = 0$$
$$e^{-At}x' - e^{-At}Ax = e^{-At}0$$
$$e^{-At}x' + e^{-At}(-A)x = 0$$
$$e^{-At}x' + \left(e^{-At}\right)'x = 0$$
$$\left(e^{-At}x\right)' = 0$$
$$e^{-At}x = c$$
$$x(t) = e^{At}c$$
$$x(0) = e^0 c = x^0 \implies c = x^0$$
$$x(t) = e^{At}x^0$$

Question 2: *(20 points)* In the case that an $n \times n$ matrix A is diagonalizable, with eigenpairs given by λ_i, $\boldsymbol{v}_i$, for $i = 1, \cdots, n$, we know that the general solution of the linear system

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t)$$

is given by

$$\boldsymbol{x}(t) = c_1\,e^{\lambda_1 t}\,\boldsymbol{v}_1 + \cdots + c_n\,e^{\lambda_n t}\,\boldsymbol{v}_n.$$

Use this formula for the general solution to show that the unique solution of the initial value problem

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}^0$$

can actually be written in the way given in the question above, that is,

$$\boldsymbol{x}(t) = e^{At}\,\boldsymbol{x}^0.$$

Show $x(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n = e^{At} x^0$

Note that $x(0) = x^0 = c_1 v_1 + \cdot + c_n v_n$. Thus we have

$$e^{At} x^0 = e^{At}\left(c_1 v_1 + \cdot + c_n v_n\right)$$
$$= e^{At} c_1 v_1 + \cdots + e^{At} c_n v_n = c_1 e^{At} v_1 + \cdots + c_n e^{At} v_n$$

In deep dive #7 Question 6 we showed that $e^{A} v = e^{\lambda} v$. I'll copy-and-paste the proof in here, now with the addition of t.

Each λ and v correspond to a pair of λ_i and v_i.

$$e^{At} v = Iv + Atv + \frac{A^2 t^2 v}{2} + \cdots + \frac{A^n t^n v}{n!}$$
$$= Iv + Avt + \frac{A^2 vt^2}{2} + \cdots + \frac{A^n vt^n}{n!} = Ivt + \lambda vt + \frac{\lambda^2 vt^2}{2} + \cdots + \frac{\lambda^n vt^n}{n!}$$
$$= Itv + \lambda tv + \frac{\lambda^2 t^2 v}{2} + \cdots + \frac{\lambda^n t^n v}{n!}$$
$$= e^{\lambda t} v$$

Returning to question 2, this gives us

$$c_1 e^{At} v_1 + \cdots + c_n e^{At} v_n = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n = x(t)$$

Question 3: *(20 points)* Compute the exponential function e^{At} and use it to express the vector-valued function $\boldsymbol{x}(t)$ solution to the initial value problem

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}.$$

Compute the exponential function e^{At}? Doesn't that have like, infinite terms?

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \begin{pmatrix} e^{-t} & e^{4t} \\ e^{4t} & e^{-t} \end{pmatrix}$$

As such $x(t) = e^{At}x^0$

$$x(t) = \begin{pmatrix} x_1^0 e^{-t} + x_2^0 e^{4t} \\ x_1^0 e^{4t} + x_2^0 e^{-t} \end{pmatrix}$$

Question 4: *(20 points)* Prove that the integrating factor method can be generalized to non-homogeneous linear differential systems, that is, prove the following: If A is an $n \times n$ *invertible* matrix, $\boldsymbol{x}^0$ is an n-vector, and $\boldsymbol{b}$ is a constant n-vector, then the initial value problem

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t) + \boldsymbol{b}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}^0$$

has a unique solution given by

$$\boldsymbol{x}(t) = e^{At}\left(\boldsymbol{x}^0 + A^{-1}\boldsymbol{b}\right) - A^{-1}\boldsymbol{b}.$$

Note: Mention very carefully every property of the matrix exponential you use on each step of your proof.

That looks like slightly more work than question 1.

We'll start by following our solution to question 1.

$$\begin{gathered} x' - Ax = b \\ e^{-At}x' - e^{-At}Ax = e^{-At}b \\ \left(e^{-At}x\right)' = e^{-At}b \end{gathered}$$

Can we go and integrate $e^{-At}b$? Since $\left(e^{-At}\right)' = -Ae^{-At}$ and b is constant, we have

$$\begin{gathered} \left(-A^{-1}e^{-At}b\right)' = A^{-1}Ae^{-At}b = e^{-At}b \\ \implies \int e^{-At}b\,dt = -A^{-1}e^{-At}b + c \end{gathered}$$

$$\begin{gathered} \left(e^{-At}x\right)' = e^{-At}b \implies e^{-At}x = -A^{-1}e^{-At}b + c \\ x = e^{At}c - A^{-1}e^{-At}b \\ x(0) = x^0 = c - A^{-1}b \implies c = x^0 + A^{-1}b \\ x(t) = e^{At}\left(x^0 + A^{-1}b\right) - A^{-1}b \end{gathered}$$

Question 5: *(20 points)* Find the vector-valued solution $\boldsymbol{x}(t)$ to the differential system

$$\boldsymbol{x}' = A\,\boldsymbol{x} + \boldsymbol{b}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Well we can use the formula from above.

$$A = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{15}\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$

$$x(t) = e^{\begin{bmatrix} -t & 4t \\ 4t & -t \end{bmatrix}} \left(\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \frac{1}{15}\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) - \frac{1}{15}\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & e^{4t} \\ e^{4t} & e^{-t} \end{pmatrix} \left(\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \frac{1}{15}\begin{pmatrix} 6 \\ 9 \end{pmatrix} \right) - \frac{1}{15}\begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & e^{4t} \\ e^{4t} & e^{-t} \end{pmatrix} \begin{pmatrix} 2.4 \\ 3.6 \end{pmatrix} - \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} 2.4e^{-t} + 3.6e^{4t} - 0.4 \\ 2.4e^{4t} + 3.6e^{-t} - 0.6 \end{pmatrix}$$