PHY 481 HW6
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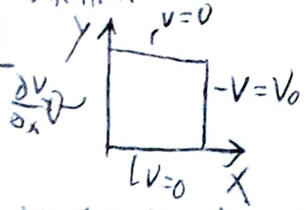
1.1



I'll use separation of variables. This problem doesn't depend on z so I'll assume $V(x,y) = X(x)Y(y)$ which gives equations $\frac{d^2X}{dx^2} = k^2 X$; $\frac{d^2Y}{dy^2} = -k^2 Y$. I've chosen these signs for k^2 to get periodic solutions in the y direction to satisfy boundary conditions. The Y ODE gives: $Y(y) = A\cos(ky) + B\sin(ky)$, but $Y(0)=Y(a)=0$ so $A=0$ and $ka = n\pi \Rightarrow k = \frac{n\pi}{a}$, $n \in \mathbb{Z}$.

In the x direction we need $\frac{\partial V}{\partial x} = 0$ when $x=0$, so our exponential solution will look like $X(x) = C\cosh(kx)$. So $V(x,y) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi}{a}y\right)\cosh\left(\frac{n\pi}{a}x\right)$ for some $F_n \in \mathbb{R}$.

Now we impose the boundary condition $V(a,y) = V_0$:

$V(a,y) = \sum F_n \sin\left(\frac{n\pi}{a}y\right)\cosh(n\pi) = V_0$ Now we employ Fourier's trick:

$$\sum F_n \int_0^a \sin\left(\frac{m\pi}{a}y\right)\sin\left(\frac{n\pi}{a}y\right)\cosh(n\pi)\,dy = V_0\int_0^a \sin\left(\frac{m\pi}{a}y\right)dy$$

$$F_n\left(\frac{a}{2}\right)\cosh(n\pi) = V_0\int_0^a \sin\left(\frac{n\pi}{a}y\right)dy \rightarrow F_n\left(\frac{a}{2}\right)\cosh(n\pi) = V_0\left(\frac{a}{\pi n}\left(-\cos(n\pi)+1\right)\right)$$

$$\Rightarrow F_n = \frac{2V_0}{\pi n\cosh(n\pi)}\left(1-\cos(\pi n)\right)$$

$$\Rightarrow V(x,y) = \sum_{n=1}^{\infty} \frac{2V_0}{n\pi\cosh(n\pi)}\left(1-\cos(\pi n)\right)\sin\left(\frac{n\pi}{a}y\right)\cosh\left(\frac{n\pi}{a}x\right)$$

$$= \sum_{n=1,3,5,\dots}^{\infty} \frac{4V_0}{n\pi\cosh(n\pi)}\sin\left(\frac{n\pi}{a}y\right)\cosh\left(\frac{n\pi}{a}x\right)$$

1.2

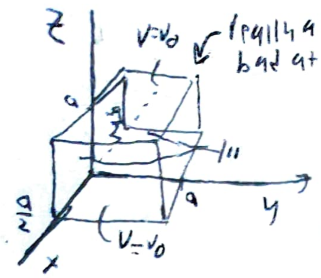
$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n} = -\varepsilon_0 \frac{\partial V}{\partial y}$ along $y=0$:

$$\sigma = -\varepsilon_0\left[\frac{\partial}{\partial y}V(x,y,z)\right] = -\varepsilon_0\left[\sum_{n=1,3,5,\dots}^{\infty} \frac{4V_0}{n\pi\cosh(n\pi)}\left(\frac{n\pi}{a}\right)\underbrace{\cos\left(\frac{n\pi}{a}y\right)}_{=1 \text{ because } y=0}\cosh\left(\frac{n\pi}{a}x\right)\right]$$

$$= -\varepsilon_0\left[\sum_{n=1,3,5,\dots}^{\infty} \frac{4V_0}{a\cosh(n\pi)}\cosh\left(\frac{n\pi}{a}x\right)\right]$$ now lets check units: $\cosh$ ← unitless

$[\varepsilon_0] = \frac{C}{Vm}$ $\quad [V_0] = V$ and $\frac{C}{Vm}V = \frac{C}{m} = [\sigma]$ as expected.

2.1



Now I'll use separation of variables in 3D: $V(x,y,z) = X(x)\,Y(y)\,Z(z)$

Need to choose signs of constants. Since $V(x,y,0)=V(x,y,a)=V(x,0,z)=V(x,a,z)=0$ it would be nice to have sines in z and y directions so lets choose

$$\frac{d^2Y}{dy^2} = -k^2 Y, \quad \frac{d^2Z}{dz^2} = -l^2 Z, \quad \frac{d^2X}{dx^2} = (k^2+l^2)X.$$

Because of B.C.s mentioned above its clear that in Y and z we get

$Y(y) = A\sin(ky)$; $Z(z) = B\sin(lz)$ with $ka = n\pi$; $la = m\pi$.

In the x direction we want $V(\frac{a}{2},y,z) = V(-\frac{a}{2},y,z) = V_0$ so we'll choose the solution

$X(x) = C\cosh(\sqrt{k^2+l^2}\,x)$, so we have: $V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{n,m}\sin(k_n y)\sin(l_m z)\cosh(\sqrt{k_n^2+l_m^2}\,x)$

Fourier's Trick: set $x = \frac{a}{2}$, multiply by $\sin(k_{n'}y)\sin(l_{m'}z)$, integrate over y,z from 0 to a; only need to do this once because cosh is even.

$$\int_0^a\int_0^a V_0 \sin(k_{n'}y)\sin(l_{m'}z)\,dy\,dz = \sum_n\sum_m D_{n,m}\cosh\left(\sqrt{k_n^2+l_m^2}\,\frac{a}{2}\right)\int_0^a\int_0^a \sin(k_n y)\sin(l_m z)\sin(k_{n'}y)\sin(l_{m'}z)\,dy\,dz$$

$$\Rightarrow \int_0^a\int_0^a V_0 \sin(k_n y)\sin(l_m z)\,dy\,dz = D_{n,m}\cosh\left(\sqrt{k_n^2+l_m^2}\,\frac{a}{2}\right)\frac{a^2}{4}$$

$$\sqrt{k_n^2+l_m^2} = \sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{a}\right)^2} = \frac{\pi}{a}\sqrt{n^2+m^2}$$

$$\Rightarrow \frac{a^2 V_0}{nm\pi^2}\left[1-\cos(n\pi)\right]\left[1-\cos(m\pi)\right] = D_{n,m}\cosh\left(\sqrt{k_n^2+l_m^2}\,\frac{a}{2}\right)\frac{a^2}{4}$$

$$\Rightarrow D_{n,m} = \frac{16V_0}{nm\pi^2\cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)} \quad n,m = 1,3,5,\ldots$$

$$\Rightarrow V(x,y,z) = \sum_{n=1,3,5\ldots}^{\infty}\sum_{m=1,3,5\ldots}^{\infty}\frac{16V_0}{nm\pi^2}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{a}z\right)\left[\frac{\cosh\left(\frac{\pi}{a}\sqrt{n^2+m^2}\,x\right)}{\cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}\right]$$

2.2

We would expect $V(0,\frac{a}{2},\frac{a}{2})$ to be the average of the potential of the sides: $V(0,\frac{a}{2},\frac{a}{2}) = \frac{3+3+0+0+0+0}{6} = 1\,V$. I wrote a short Python script to test this - I'll include a picture - it is consistent with the formula for $V(x,y,z)$ found in 2.1.

2.3

It does turn out that $\vec{E}=\vec{0}$ at the center of the cube. There are a couple ways to see this - the boundary conditions give it away since they are totally symmetric about each axis, or we can see it directly from $\vec{E} = -\nabla V$. Ignoring pre-factors, the y and z components look like $\cos(\frac{n\pi}{a}y)$, $\cos(\frac{m\pi}{a}z)$, but if $y=z=\frac{a}{2}$, we have $\cos(\frac{n\pi}{a}\frac{a}{2}) = \cos(\frac{n\pi}{2}) = 0$. Similarly the x term of V looks like $\cosh(\alpha x)$ so its partial derivative with respect to x looks like $\sinh(\alpha x)$ which is zero when $x=0$.

So $\vec{E}(0,\frac{a}{2},\frac{a}{2}) = \vec{0}$.

3.1 $V_0 = k\cos(3\theta)$, $k \in \mathbb{R}$. Use $e^{i\theta} = \cos\theta + i\sin\theta$!

$$\cos(3\theta) + i\sin(3\theta) = e^{i(3\theta)} = (\cos\theta + i\sin\theta)^3$$

$$= \cos^3\theta - \sin^2\theta\cos\theta + 2i\sin\theta\cos\theta + i\cos^2\sin\theta - i\sin^3\theta - 2\sin^2\theta\cos\theta$$

Taking just real parts gives: $\cos(3\theta) = \cos^3\theta - 3\sin^2\cos\theta = \cos^3\theta - 3\cos(1-\cos^2\theta)$

$$= 4\cos^3\theta - 3\cos\theta = \frac{8}{5}P_3(\cos\theta) - \frac{3}{5}P_1(\cos\theta)$$

$$\Rightarrow V_0 = k\left(\frac{8}{5}P_3(\cos\theta) - \frac{3}{5}P_1(\cos\theta)\right)$$

3.2 Finite in sphere $\to B_\ell$'s $= 0$. $V(r,\theta) = \sum_\ell A_\ell r^\ell P_\ell \cos\theta$

Fit "by eye"

$$\left.\begin{array}{l} \ell = 0 \to A_0 = 0 \\ \ell = 1 \to -\frac{3k}{5} = A_1 R \to A_1 = -\frac{3k}{5R} \\ \ell = 2 \to A_2 = 0 \\ \ell = 3 \to \frac{8k}{5} = A_3 R^3 \to A_3 = \frac{8k}{5R^3} \\ \ell = 4+ \to A_{4+} = 0 \end{array}\right\} \Rightarrow V_{in} = -\frac{3k}{5R} r P_1(\cos\theta) + \frac{8k}{5R} r^3 P_3(\cos\theta)$$

3.3 Vanish far away $\to A_\ell$'s $= 0$. $V(R,\theta) = \sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$

$$\left.\begin{array}{l} \ell = 0 \to B_0 = 0 \\ \ell = 1 \to \frac{B_1}{R^2} = -\frac{3k}{5} \to B_1 = -\frac{3}{5}kR^2 \\ \ell = 2 \to B_2 = 0 \\ \ell = 3 \to \frac{B_3}{R^4} = -\frac{8k}{5} \to B_3 = \frac{8}{5}kR^4 \\ \ell\, 4+ \to B_{4+} = 0 \end{array}\right\} \Rightarrow V_{out} = -\frac{3}{5}kR^2\frac{1}{r^2}P_1(\cos\theta) + \frac{8}{5}kR^4\frac{1}{r^4}P_3(\cos\theta)$$

3.4 $-\frac{3k}{5R} R P_1(\cos\theta) + \frac{8k}{5R^3} P_3 = -\frac{3}{5}kR^2\frac{1}{R^2}P_1(\cos\theta) + \frac{8}{5}kR^4\frac{1}{R^4}P_3(\cos\theta)$ ← obviously equal, the Rs cancel.

3.5
$$\left.\begin{array}{l} \frac{\partial V_{out}}{\partial r} = \frac{6}{5}kR^2\frac{1}{r^3}P_1(\cos\theta) - \frac{32}{5}kR^4\frac{1}{r^5}P_3(\cos\theta) \\ \frac{\partial V_{in}}{\partial r} = -\frac{3k}{5r}P_1(\cos\theta) + \frac{24k}{5R^3}r^2 P_3(\cos\theta) \end{array}\right\} \frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r} = \frac{6}{5}kR^2\frac{1}{r^3} + \frac{3k}{5R}P_1(\cos\theta) + \left(-\frac{32}{5}kR^4 - \frac{24k}{5R^3}r^2\right)P_3(\cos\theta)$$

$$= -\frac{\sigma}{\varepsilon_0}$$

$$\Rightarrow 3.6 \quad \sigma = -\frac{\varepsilon_0 k}{R}\left[\frac{9}{5}P_1(\cos\theta) - \frac{56}{5}P_3(\cos\theta)\right]$$

integrate: $q = -\frac{\varepsilon_0 k}{R}\int_0^{2\pi} d\phi \int_0^{\pi} R^2 \sin\theta \, d\theta \left[\quad " \quad \right] = 0$ bc $\sin\theta$ is odd.

4.1 Laplace's eqn in cylindrical (ignoring z): $0 = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}$

Ansatz: $V(s,\phi) = S(s)\Phi(\phi)$; $\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial(S(s)\Phi(\phi))}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2(S(s)\Phi(\phi))}{\partial\phi^2} = 0$

$\Rightarrow \frac{1}{s}\Phi\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{1}{s^2}S\frac{d^2\Phi}{d\phi^2} = 0$ multiply by s^2:

$s\Phi\frac{d}{ds}\left(s\frac{dS}{ds}\right) + S\frac{d^2\Phi}{d\phi^2} = 0$ divide by $S\Phi = V$:

$0 = \frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$

$\Rightarrow \frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = C_1$; $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = C_2$ with $C_1 + C_2 = 0$.

4.2 we need Φ to be 2π periodic so we must choose $C_2 < 0$, otherwise we'd get exponentials which would be totally unphysical. $C_2 = -k^2$, $C_1 = k^2$

4.3 $\frac{d^2\Phi}{d\phi^2} = -k^2\Phi \Rightarrow \Phi(\phi) = A\cos k\phi + B\sin k\phi$ and $k \in \mathbb{Z}^+$

4.4 suppose $k \neq 0$ then we can see that $S = s^n$ with $n = \pm k$ solves the S equation:

$s\frac{d}{ds}\left(s\frac{d}{ds}s^n\right) = s\frac{d}{ds}(s n s^{n-1}) = ns\frac{d}{ds}s^n = n^2 s^n = k^2 S$

so the general solution for $k \neq 0$ is: $S(s) = Cs^k + Ds^{-k}$

If $k = 0$: $s\frac{d}{ds}\left(s\frac{dS}{ds}\right) = 0 \rightarrow s\frac{dS}{ds} = \text{const.} = C \Rightarrow C = \frac{dS}{s} \Rightarrow S = C\log(s) + D$

4.5 $V(s,\phi) = a_0 + b_0\log(s) + \sum_{k=1}^{\infty}\left[s^k(a_k\cos(k\phi) + b_k\sin(k\phi)) + s^{-k}(c_k\cos(k\phi) + d_k\sin(k\phi))\right]$

4.6 Everything but a_0 and b_0 vanishes.