PHY 481 - Fall 2023
Homework 06

Due Wednesday November 1, 2023

Preface

Homework 06 focuses on Laplace's equation and solving it using infinite series solution in Cartesian, spherical, and cylindrical coordinates. You should become comfortable with setting boundary conditions for PDE problems like this and develop a sense of the process for solving these problems analytically.

1 Rectangular Pipe: Separation of Variables-Cartesian-2D

A square rectangular pipe (sides of length a) runs parallel to the z-axis (from $-\infty$ to ∞) The 4 sides are maintained with boundary conditions given in the figure. (Each of the 4 sides is insulated from the others at the corners).

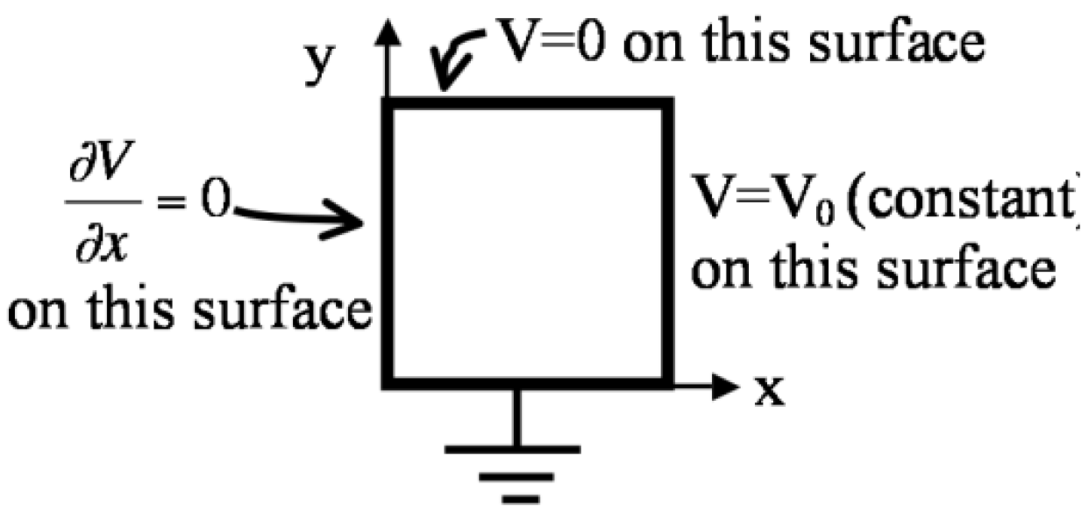

Figure 1: Tube

1. Find the potential $V(x, y, z)$ at all points in this pipe.

2. Find the charge density $\sigma(x, y = 0, z)$ everywhere on the bottom conducting wall ($y = 0$). Check the units for your charge density (show us!).

2 Cubical Box: Separation of Variables-Cartesian-3D

You have a cubical box (sides all of length a) made of 6 metal plates that are insulated from each other. The left wall is located at $x = -a/2$ and the right wall is at $x = +a/2$. Both left and right walls are held at constant potential $V = V_0$. All four other walls are grounded ($V = 0$ for these walls).

(Note that we've set up the geometry so the cube runs from $y = 0$ to $y = a$, and from $z = 0$ to $z = a$, but from $x = -a/2$ to $x = +a/2$ This should actually make the math work out a little easier!)

1. Find the potential $V(x, y, z)$ everywhere inside the box.

2. The method of relaxation (a numerical method) relies on the fact that the value of the potential at some point is equal to the average of the potential at the surrounding points. Based on this argument, what do you think the potential is at the center of the cube assuming that $V_0 = 3$ volts? *(Hint: Use the six walls of the box as the surrounding points!)* Check your answer by calculating the double infinite sum using only the first term of each sum.

3. Is $\vec{E} = 0$ at the center of the cube? Why or why not? *(Hint: think about the how the charge distributions on the six walls of the box are different, if at all.)*

3 Sphere with a known potential

We have a sphere (radius, R) where we have glued charges to the outside such that the electric potential at the surface of the sphere is given by:

$$V_0 = k \cos 3\theta$$

where k is some constant.

You are going to find the potential inside and outside the sphere (there are no charges other than those at the surface of the sphere) as well as the charge density $\sigma(\theta)$ on the surface of the sphere. Each part of this problem is meant to walk you through the process for solving these kinds of boundary-value problems.

1. Rewrite the potential at the surface using Legendre polynomials. *You will need to dust off some trigonometric identities to do this.* You can find a listing of Legendre polynomials online.

2. Using this boundary condition and the knowledge that V should be finite inside the sphere, find the electric potential, $V(r, \theta)$, inside this sphere. You do not have to re-derive the general solution to Laplace's equation, just use the result:

$$V(r, \theta) = \sum_{\ell} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

3. Using the same boundary condition and the knowledge that V should vanish far from the sphere, find the electric potential, $V(r,\theta)$, outside this sphere.

4. Show explicitly that your solutions to parts 2 and 3 match at the surface of the sphere.

5. Take the "normal" derivative of each of your solutions ($\partial V/\partial r$) and use their difference at the surface to find the charge on the surface:

$$\left(\frac{\partial V_{\text{outside}}}{\partial r} - \frac{\partial V_{\text{inside}}}{\partial r}\right) = -\frac{\sigma}{\varepsilon_0}$$

6. What is the total charge on the sphere? *Hint: Note that $P_0(\cos(\theta)) = 1$ and use orthogonality: Eqn. 3.68 of Griffiths!*

4 Separation of Variables in Cylindrical Coordinates

We have gone through how to solve Laplace's equation in Cartesian and spherical coordinates. In both cases, finding a separable and general solution was possible. In fact, there are a number of possible coordinate systems where we can do this, but the most relevant to this class (besides Cartesian and spherical) is cylindrical coordinates.

In this problem, you will develop the general solution to Laplace's equation in cylindrical coordinates where there is no dependence on the z coordinate (i.e., where we have cylindrical symmetry).

1. Starting from Laplace's equation in Cylindrical coordinates, use the ansatz $V(s,\phi) = S(s)\Phi(\phi)$ to convert the problem from one partial differential equation to two 2nd order ordinary differential equations – one for $S(s)$ and one for $\Phi(\phi)$.

2. As we have argued twice, each of those differential equations is equal to a constant. Which constant is positive and which is negative? Explain your choice. *Think about what happens when you rotate your problem by 2π in the ϕ direction, should the physics care that you've done that?* Going forward, choose the positive constant to be $+k^2$ and the negative one to be $-k^2$.

3. Solve the differential equation for $\Phi(\phi)$ to obtain the general solution for $\Phi(\phi)$. Hint: $\Phi(\phi) = \Phi(\phi + 2\pi)$ so this puts an additional condition on k that it must be an integer with $k \geq 0$.

4. Armed with this information about k, solve the differential equation for $S(s)$ to obtain the general solution for $S(s)$. *Be careful to treat $k = 0$ separately as that generates an additional and completely physical solution!*

5. Combine your solutions to Parts 3 and 4 to generate the complete general solution $V(s,\phi) = S(s)\Phi(\phi)$.

6. The potential at a distance s away from an infinite line charge (which should be captured by this solution) is: $V(s) = \dfrac{2\lambda}{4\pi\varepsilon_0}\log(s)$ + constant, which terms in general solution vanish to capture this solution?

This problem is tough. But here's a little help. The general solution for the electric potential in cylindrical coordinates (with cylindrical symmetry) is:

$$V(s,\phi) = a_0 + b_0 \log s + \sum_{k=1}^{\infty} \left[s^k (a_k \cos k\phi + b_k \sin k\phi) + s^{-k}(c_k \cos k\phi + d_k \sin k\phi) \right]$$

You will not get full credit for this problem unless your work clearly shows how you this solution is developed.