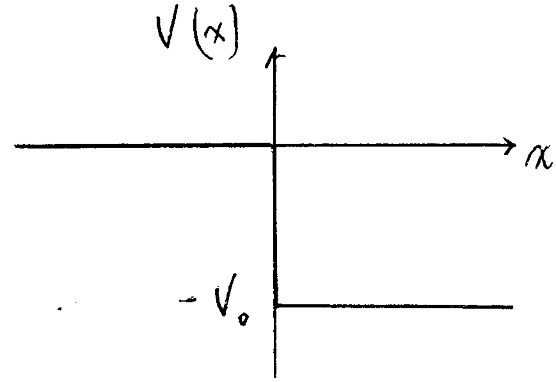
# Physics 471: Homework 13 Solutions

1. $$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ -V_0 & \text{for } x > 0 \end{cases}$$




a) Classical physics:

Energy is conserved: $\frac{1}{2} m v_i^2 = \frac{1}{2} m v_f^2 - V_0$

$$v_F^2 = v_i^2 + \frac{2V_0}{m}$$

$$\underline{\underline{v_F = \sqrt{v_i^2 + \frac{2V_0}{m}}}}$$

b) Energy eigenvalue eqn is $\hat{H}|E\rangle = E|E\rangle$

In position representation: $\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\varphi_E(x) = E\varphi_E(x)$

$x < 0$: $$\frac{d^2\varphi_E}{dx^2} = -\frac{2mE}{\hbar^2}\varphi_E \equiv -k_1^2\varphi_E \qquad k_1^2 = \frac{2mE}{\hbar^2}$$

$$\varphi_E^I(x) = Ae^{ik_1x} + Be^{-ik_1x}$$

$x > 0$: $$\frac{d^2\varphi_E}{dx^2} = -\frac{2m(E+V_0)}{\hbar^2}\varphi_E \equiv -k_2^2\varphi_E \qquad k_2^2 = \frac{2m(E+V_0)}{\hbar^2}$$

$$\varphi_E^{II}(x) = Ce^{ik_2x} \qquad (D=0)$$

Boundary conditions at $x = 0$:

$\varphi_E$ is continuous: $A + B = C$

$\frac{d\varphi_E}{dx}$ is continuous: $ik_1(A - B) = ik_2 C$

c) Substitute $C$ from (1) into (2): $ik_1(A - B) = ik_2(A + B)$

$$A(k_1 - k_2) = B(k_1 + k_2)$$

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2} = \frac{\sqrt{E} - \sqrt{E + V_0}}{\sqrt{E} + \sqrt{E + V_0}}$$

$$R = \left|\frac{B}{A}\right|^2 = \underline{\underline{\left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2}} = \left(\frac{\sqrt{E} - \sqrt{E + V_0}}{\sqrt{E} + \sqrt{E + V_0}}\right)^2$$

Classically, we expect $R = 0$, so QM is different.

If $E$ is small, then $k_1$ is small but $k_2$ is not small, so $k_1 \ll k_2$ and $R \to 1$ total reflection!

If $V_0$ is small, then $k_2 \approx k_1$ and $R \to 0$.

If $E \gg V_0$, then $k_2 \approx k_1$ and $R \to 0$.

So the peculiarity of QM only appears for small $E$.

d) For plotting $R$ vs. $E/V_0$, re-write $R$:

$$R = \left( \frac{\sqrt{E} - \sqrt{E+V_0}}{\sqrt{E} + \sqrt{E+V_0}} \right)^2 = \left( \frac{\sqrt{\frac{E}{V_0}} - \sqrt{\frac{E}{V_0}+1}}{\sqrt{\frac{E}{V_0}} + \sqrt{\frac{E}{V_0}+1}} \right)^2 = \left( \frac{\sqrt{x} - \sqrt{x+1}}{\sqrt{x} + \sqrt{x+1}} \right)^2$$

where $x = E/V_0$

In Mathematica I wrote $R$ as

$$R = \left( \frac{1 - \sqrt{1 + \frac{1}{x}}}{1 + \sqrt{1+\frac{1}{x}}} \right)^2$$ which is equivalent.

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Plot[((1 - Sqrt[1 + 1 / x]) / (1 + Sqrt[1 + 1 / x]))^2, {x, 0, 1}, PlotRange → {0, 1}]
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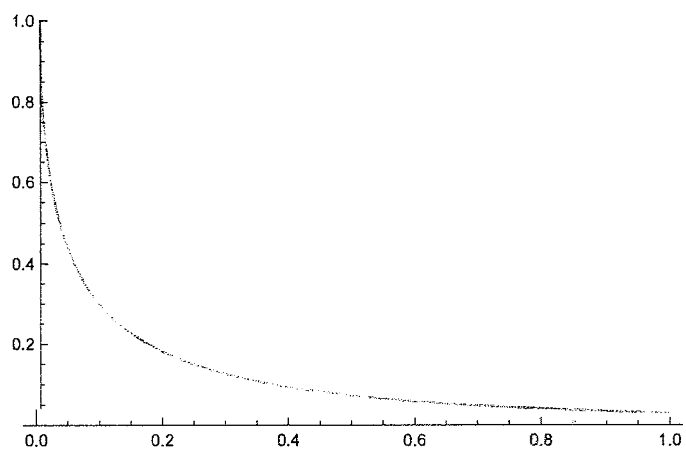
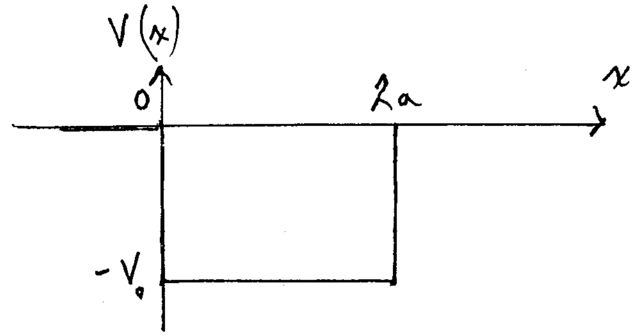



All my comments are in part (c).

2. $V(x) = \begin{cases} 0 & \text{for } x < 0 \\ -V_0 & \text{for } 0 \le x \le 2a \\ 0 & \text{for } x > 2a \end{cases}$




a) This is physically equivalent to the well potential discussed in McIntyre section 6.4. So we can use all the formulas from there.

I will re-write equation (6.94) because I don't like the $\sin^2$ term in the denominator of the denominator!

$$R = \frac{\frac{(k_2^2 - k_1^2)^2}{4k_1^2 k_2^2}\sin^2(2k_2 a)}{1 + \frac{(k_2^2 - k_1^2)^2}{4k_1^2 k_2^2}\sin^2(2k_2 a)} \qquad k_1^2 = \frac{2mE}{\hbar^2} \qquad k_2^2 = \frac{2m(E+V_0)}{\hbar^2}$$

$$\frac{k_2^2 - k_1^2}{4k_1^2 k_2^2} = \frac{V_0^2}{4E(E+V_0)}$$

$$R = \frac{\frac{V_0^2}{4E(E+V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)}{1 + \frac{V_0^2}{4E(E+V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)}$$

With $E = \frac{V_0}{3}$, $\frac{V_0^2}{4E(E+V_0)} = \frac{V_0^2}{\frac{4}{3}V_0 \cdot \frac{4}{3}V_0} = \frac{9}{16}$

$$\frac{2a}{\hbar}\sqrt{2m(E+V_0)} = \frac{2a}{\hbar}\sqrt{2m\cdot\frac{4}{3}V_0} = \frac{2a}{\hbar}\sqrt{\frac{8}{3}mV_0}$$

2. a) continued

Case i) $\frac{2mV_0}{\hbar^2} a^2 \ll 1$ sin term is very small $\Rightarrow R \to 0$

Case ii) $\frac{2mV_0}{\hbar^2} a^2 = \frac{\pi^2}{48}$ Argument of sin is

$$\frac{2a}{\hbar^2}\sqrt{\frac{8}{3}mV_0} = \left(\frac{32}{3}\,\frac{mV_0 a^2}{\hbar^2}\right)^{\frac{1}{2}}$$

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\sin^2\left(\frac{\pi}{3}\right) = \frac{3}{4}$

$$= \left(\frac{16}{3}\cdot\frac{\pi^2}{48}\right)^{\frac{1}{2}} = \frac{\pi}{3}$$

$$R = \frac{\frac{9}{16}\cdot\frac{3}{4}}{1+\frac{9}{16}\cdot\frac{3}{4}} = \frac{27/64}{1+27/64} = \frac{27/64}{91/64} = \frac{27}{91} = 0.297 \approx \underline{\underline{0.30}}$$

Case iii) $\frac{2mV_0}{\hbar^2} a^2 = \frac{3\pi^2}{16}$ Argument of sin (from above) is

$$\left(\frac{16}{3}\cdot\frac{3\pi^2}{16}\right)^{\frac{1}{2}} = \pi$$

$\sin(\pi) = 0$

$\Rightarrow R = 0$, so $T = 1$ This corresponds to a "transmission resonance" discussed in the book and on problem 4 of this homework.

b) $V_0$ fixed, $E \to 0$ ; $R \to 1$ Similar to single step in problem 1

$E \gg V_0$ : $R \to 0$ from numerator

At very high energies, the small step is hardly noticeable.

3. Tunneling

a) Start with McIntyre (6.93) for a potential <u>well</u>:

$$T = \frac{1}{1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)}$$

Change well to bump: $V_0 \rightarrow -V_0$

$$T = \frac{1}{1 + \frac{V_0^2}{4E(E-V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E-V_0)}\right)}$$

Now let $E < V_0$, so $E - V_0 < 0$

$$\sqrt{2m(E-V_0)} = i\sqrt{2m(V_0-E)}$$

We need

$$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} = i \cdot \frac{e^{x} - e^{-x}}{2}$$

$$= i \sinh x$$

$$T = \frac{1}{1 + \frac{V_0^2}{-4E(V_0-E)} \cdot i^2 \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right)}$$

$$T = \frac{1}{1 + \frac{V_0^2}{4E(V_0-E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right)}$$

matches (6.105)

$$R = 1 - T = \frac{\frac{V_0^2}{4E(V_0-E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right)}{1 + \frac{V_0^2}{4E(V_0-E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right)}$$

matches my version of (6.106)

b) Electron $m = 9.1 \cdot 10^{-31}$ kg

$E = 0.5\,\text{eV}$, $V_0 = 1.0\,\text{eV}$, $V_0 - E = 0.5\,\text{eV}$

$$\frac{V_0^2}{4E(V_0-E)} = \frac{(1.0\,\text{eV})^2}{4 \cdot 0.5\,\text{eV} \cdot 0.5\,\text{eV}} = 1$$

Argument of sinh: Use SI units $\quad 2a = 5\,\text{Å} = 5 \cdot 10^{-10}\,\text{m}$

$$V_0 - E = 0.5\,\text{eV} \cdot 1.6 \cdot 10^{-19}\,\tfrac{\text{J}}{\text{eV}} = 0.8 \cdot 10^{-19}\,\text{J}$$

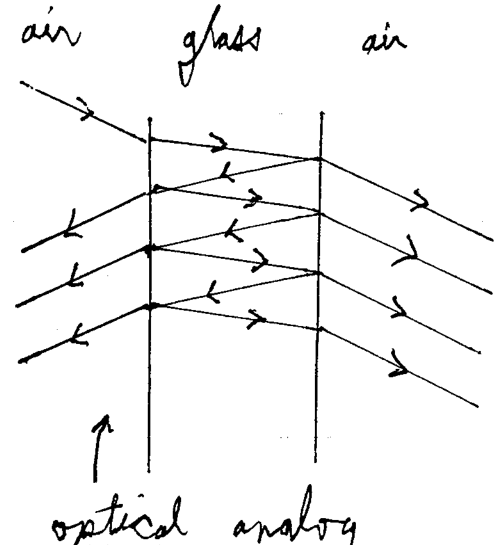
$$\sqrt{2m(V_0-E)} = \sqrt{2 \cdot 9.1 \cdot 10^{-31}\,\text{kg} \cdot 0.8 \cdot 10^{-19}\,\text{J}} = 3.8 \cdot 10^{-25}\,\text{kg}\,\tfrac{\text{m}}{\text{s}}$$

$$\frac{2a}{\hbar}\sqrt{2m(V_0-E)} = \frac{5 \cdot 10^{-10}\,\text{m}}{1.05 \cdot 10^{-34}\,\text{J-s}} \cdot 3.8 \cdot 10^{-25}\,\text{kg}\,\tfrac{\text{m}}{\text{s}} = 1.82$$

$$\sinh^2(1.82) = 9.0$$

$$T = \frac{1}{1 + \sinh^2(1.82)} = \frac{1}{1 + 9.0} = \underline{\underline{0.10}} = 10\,\%$$

(There may be some round-off error in my answer.)




4. Interference approach to understanding transmission resonances.

a) In problem 16), I derived

$$r_1 \quad \frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$ for a single interface

where $k_1 = \sqrt{\frac{2mE}{\hbar^2}}$ is the wave vector in free space

and $k_2 = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$ is the wave vector inside the "medium", i.e. the potential well between $x=0$ and $x=2a$.

$$C = A + B = A + A\frac{k_1 - k_2}{k_1 + k_2} = A\left[1 + \frac{k_1 - k_2}{k_1 + k_2}\right] = A\frac{2k_1}{k_1 + k_2}$$

$$t_1 = \frac{C}{A} = \frac{2k_1}{k_1 + k_2}$$

When we approach the same interface from the "medium" side, reverse the roles of $k_1$ and $k_2$:

$$r_2 = \frac{k_2 - k_1}{k_2 + k_1} \qquad t_2 = \frac{2k_2}{k_2 + k_1}$$

4. b) From the pictures, sum over trajectories:

$$t = t_1 t_2 + t_1 r_2^2 t_2 + t_1 r_2^4 t_2 + \dots$$
$$= t_1 t_2 \left(1 + r_2^2 + r_2^4 + \dots\right)$$

c) $$t = t_1 t_2 \cdot \frac{1}{1 - r_2^2} = \left(\frac{2k_1}{k_1 + k_2}\right)\left(\frac{2k_2}{k_2 + k_1}\right) \cdot \frac{1}{1 - \left(\frac{k_2 - k_1}{k_2 + k_1}\right)^2}$$

$$t = \frac{4k_1 k_2}{(k_1 + k_2)^2} \cdot \frac{(k_2 + k_1)^2}{\left[(k_2 + k_1)^2 - (k_2 - k_1)^2\right]} = \frac{4k_1 k_2}{4k_1 k_2} = \underline{1}$$

d) $$r = r_1 + t_1 r_2 t_2 + t_1 r_2^3 t_2 + \dots$$
$$= r_1 + t_1 r_2 t_2 \left(1 + r_2^2 + r_2^4 + \dots\right)$$
$$= r_1 + t_1 r_2 t_2 \frac{1}{1 - r_2^2}$$
$$= \frac{k_1 - k_2}{k_1 + k_2} + \frac{2k_1 \cdot (k_2 - k_1) \cdot 2k_2}{(k_1 + k_2)^3} \cdot \underbrace{\frac{(k_1 + k_2)^2}{4k_1 k_2}}_{\text{from part (c)}}$$
$$= \frac{k_1 - k_2}{k_1 + k_2} + \frac{k_2 - k_1}{k_1 + k_2} = \underline{0}$$

Extra credit:

$$t = t_1 t_2 \left(1 + r_2^2 e^{4ik_2a} + r_2^4 e^{8ik_2a} + \dots\right)$$

$$= \frac{t_1 t_2}{1 - r_2^2 e^{4ik_2a}} = \frac{4k_1k_2}{(k_1+k_2)^2} \cdot \frac{1}{1 - \left(\frac{k_2-k_1}{k_2+k_1}\right)^2 e^{4ik_2a}}$$

$$= \frac{4k_1k_2}{(k_1+k_2)^2 - (k_2-k_1)^2 e^{4ik_2a}}$$

Multiply numerator & denominator by $e^{-2ik_2a}$:

$$t = \frac{4k_1k_2\, e^{-2ik_2a}}{(k_1^2 + 2k_1k_2 + k_2^2)e^{-2ik_2a} - (k_1^2 - 2k_1k_2 + k_2^2)e^{2ik_2a}}$$

$$= \frac{4k_1k_2\, e^{-2ik_2a}}{(k_1^2+k_2^2)(e^{-2ik_2a} - e^{2ik_2a}) + 2k_1k_2(e^{-2ik_2a} + e^{2ik_2a})}$$

$$= \frac{4k_1k_2\, e^{-2ik_2a}}{(k_1^2+k_2^2)\cdot(-2i\sin(2k_2a)) + 2k_1k_2 \cdot 2\cos(2k_2a)}$$

$$t = \frac{e^{-2ik_2a}}{\cos(2k_2a) - i\,\frac{k_1^2+k_2^2}{2k_1k_2}\sin(2k_2a)}$$

Agrees w/ (6.91) up to an overall phase factor!