Physics 471: Homework 11 Solutions

1. $\psi(x, t=0) = A e^{-\alpha x^2}$

a) Normalize: $\int_{-\infty}^{\infty} |\psi(x,t=0)|^2 dx = |A|^2 \int_{-\infty}^{\infty} e^{-2\alpha x^2} dx$

$= |A|^2 \cdot \sqrt{\frac{\pi}{2\alpha}} = 1 \Rightarrow \underline{\underline{A = \left(\frac{2\alpha}{\pi}\right)^{1/4}}}$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x,t=0)\, e^{-ipx/\hbar}\, dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \cdot \left(\frac{2\alpha}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-\alpha x^2 - ipx/\hbar}\, dx$$

We could derive McIntyre equation (F.23) by "completing the square", but here I'll just use F.23:

$$\int_{-\infty}^{\infty} e^{-a^2 x^2 + bx}\, dx = \frac{\sqrt{\pi}}{a}\, e^{b^2/4a^2}$$

Compare our integral with his $\Rightarrow a^2 = \alpha$, $b = \frac{-ip}{\hbar}$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \cdot \left(\frac{2\alpha}{\pi}\right)^{1/4} \cdot \frac{\sqrt{\pi}}{\sqrt{\alpha}}\, e^{-p^2/4\alpha\hbar^2}$$

Simplify:

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \cdot \left(\frac{2\pi}{\alpha}\right)^{1/4} e^{-\frac{p^2}{4\alpha\hbar^2}} = \left(\frac{1}{2\alpha\pi\hbar^2}\right)^{1/4} e^{-\frac{p^2}{4\alpha\hbar^2}}$$

I'll use this form in the next step.

b) $$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p)\, e^{ipx/\hbar}\, \underbrace{e^{-\frac{ip^2 t}{2m\hbar}}}\, dp$$

This term is $e^{-iEt/\hbar}$ with $E = p^2/2m$

$$\psi(x,t) = \frac{1}{2\pi\hbar}\left(\frac{2\pi}{\alpha}\right)^{1/4} \int_{-\infty}^{\infty} e^{-\frac{p^2}{4\alpha\hbar^2} - \frac{ip^2 t}{2m\hbar} + ipx/\hbar}\, dp$$

Exponent equals $-p^2\left(\frac{1}{4\alpha\hbar^2} + \frac{it}{2m\hbar}\right) + p \cdot \frac{ix}{\hbar}$

Use McIntyre F.23 again with $a^2 = \left(\frac{1}{4\alpha\hbar^2} + \frac{it}{2m\hbar}\right)$, $b = \frac{ix}{\hbar}$

Simplify $a^2 = \frac{1}{4\alpha\hbar^2}\left(1 + i\,\frac{2\alpha\hbar}{m}\,t\right) = \frac{1}{4\alpha\hbar^2}\left(1 + i\Omega t\right)$

where $\Omega = \frac{2\alpha\hbar}{m}$

From F.23, the integral equals:

$$\frac{\sqrt{\pi}}{a}\, e^{b^2/4a^2} = \sqrt{\pi}\sqrt{\frac{4\alpha\hbar^2}{1+i\Omega t}}\; e^{\frac{-x^2}{4\hbar^2}\cdot\frac{4\alpha\hbar^2}{1+i\Omega t}}$$

Put everything together to get

$$\psi(x,t) = \frac{1}{2\pi\hbar}\left(\frac{2\pi}{\alpha}\right)^{1/4} \cdot \sqrt{\frac{4\pi\alpha\hbar^2}{1+i\Omega t}}\; e^{-\frac{\alpha x^2}{1+i\Omega t}}$$

$$\underline{\underline{\psi(x,t) = \left(\frac{2\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{1+i\Omega t}}\; e^{\frac{-\alpha x^2}{1+i\Omega t}}}}$$

This looks a lot like our initial wave function $\psi(x, t=0)$, except for the factor $(1+i\Omega t)$ in both the prefactor and the exponent.

c) $$|\psi(x,t)|^2 = \psi^*(x,t) \cdot \psi(x,t)$$

$$= \sqrt{\frac{2\alpha}{\pi}}\; \frac{1}{\sqrt{(1-i\Omega t)(1+i\Omega t)}}\; e^{\frac{-\alpha x^2}{1-i\Omega t}}\; e^{\frac{-\alpha x^2}{1+i\Omega t}}$$

Simplify the exponentials:

$$-\alpha x^2\left(\frac{1}{1-i\Omega t} + \frac{1}{1+i\Omega t}\right) = -\alpha x^2\left(\frac{1+i\Omega t + 1-i\Omega t}{1+\Omega^2 t^2}\right) = \frac{-2\alpha x^2}{1+\Omega^2 t^2}$$

$$\underline{\underline{|\psi(x,t)|^2 = \sqrt{\frac{2\alpha/\pi}{1+\Omega^2 t^2}}\; e^{-\frac{2\alpha x^2}{1+\Omega^2 t^2}}}} = \underline{\underline{\sqrt{\frac{2\alpha}{\pi\Gamma}}\; e^{-2\alpha x^2/\Gamma}}}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \, |\psi(x,t)|^2 \, dx = \sqrt{\frac{2\alpha}{\pi\Gamma}} \int_{-\infty}^{\infty} x \, e^{-\frac{2\alpha x^2}{\Gamma}} \, dx = 0$$

because the integrand is an odd function of x.

The wave packet has zero average momentum, because $\phi(p)$ is centered at $p=0$ and is symmetric (even) in p. So the wave packet in position space stays centered at $x=0$ and just spreads out over time.

We can see that $\langle p \rangle = 0$ two ways:

i) $$\langle p \rangle = \int_{-\infty}^{\infty} p \, |\phi(p)|^2 \, dp \propto \int_{-\infty}^{\infty} p \, e^{\frac{-2p^2}{4\alpha\hbar}} \, dp = 0$$

integrand is odd

ii) Ehrenfest's Thm:

$$\langle p \rangle = m \frac{d\langle x \rangle}{dt} = 0$$

1. d) $$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \left| \psi(x,t) \right|^2 dx = \sqrt{\frac{2\alpha}{\pi\Gamma}} \int_{-\infty}^{\infty} x^2 e^{-\frac{2\alpha x^2}{\Gamma}} dx$$

Use result from homework: $$\int_{-\infty}^{\infty} z^2 e^{-cz^2} dz = \frac{1}{2}\sqrt{\frac{\pi}{c^3}}$$

with $c = \frac{2\alpha}{\Gamma}$

$$\langle x^2 \rangle = \sqrt{\frac{2\alpha}{\pi\Gamma}} \cdot \frac{1}{2}\sqrt{\frac{\pi\Gamma^3}{8\alpha^3}} = \frac{\Gamma}{4\alpha} = \frac{1+\Omega^2 t^2}{4\alpha}$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 \left| \phi(p) \right|^2 dp = \frac{1}{(2\alpha\pi\hbar^2)^{1/2}} \int_{-\infty}^{\infty} p^2 e^{-\frac{2p^2}{4\alpha\hbar^2}} dp$$

This time $c = \frac{1}{2\alpha\hbar^2}$

$$\langle p^2 \rangle = \frac{1}{\sqrt{2\alpha\pi}\,\hbar} \cdot \frac{1}{2}\sqrt{\pi(2\alpha\hbar^2)^3} = \alpha\hbar^2$$

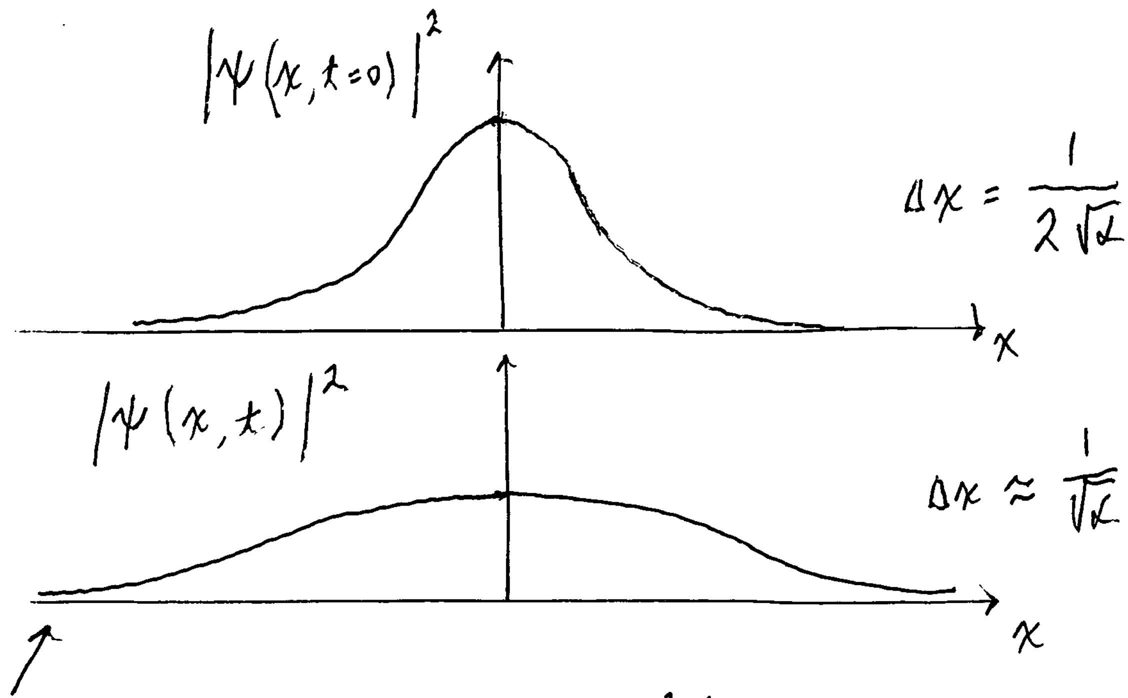
$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\Gamma}{4\alpha}} = \frac{1}{2}\sqrt{\frac{\Gamma}{\alpha}}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\alpha}\,\hbar$$

$$\Delta x\, \Delta p = \sqrt{\Gamma}\,\hbar/2 = \sqrt{1+\Omega^2 t^2}\,\hbar/2$$

The packet starts with minimum uncertainty at $t=0$, then it spreads out in position space as time progresses.

1. e)



I chose t so that $\Gamma = 1 + \Omega^2 t^2 = 4$. At that time Δx has doubled and the peak value of $|\psi(x)|^2$ has decreased to half its initial value.

i) The wave packet spreads with time, so the position of the particle becomes less certain.

ii) Ω determines how quickly the packet spreads. $\sqrt{\Gamma}$ tells you how much it has spread.

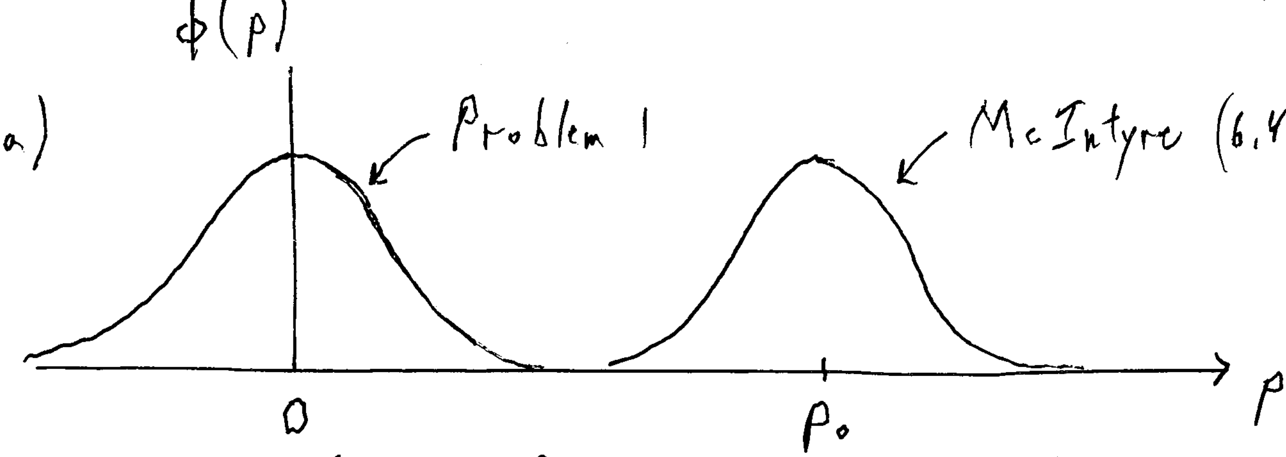
iii) The time scale for spreading is $\tau = \frac{1}{\Omega} = \frac{m}{2\alpha\hbar}$

iv) Since $\Delta x(t=0) = \frac{1}{2\sqrt{\alpha}}$, we can express τ in terms of the width of the initial packet:

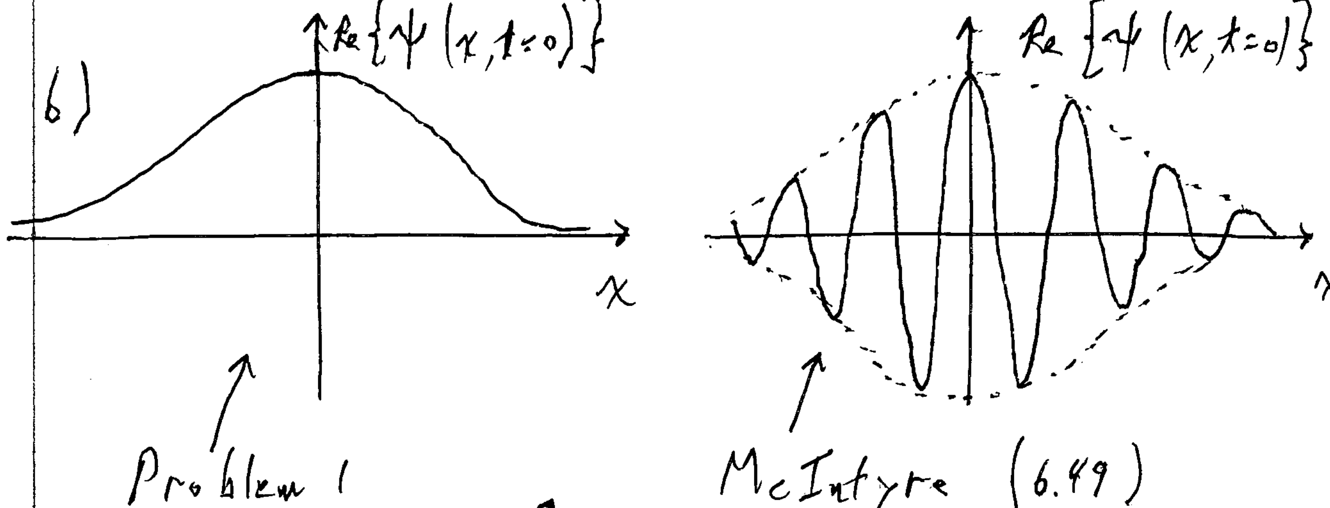
$$\tau = \frac{m}{2\hbar}\frac{1}{\alpha} = \frac{m}{2\hbar}\left(2\,\Delta x(t=0)\right)^2 = \frac{2m}{\hbar}\left(\Delta x(t=0)\right)^2$$

$\Rightarrow$ Narrow packets spread faster than wider packets because they have a broader momentum distribution $\phi(p)$.

2\. a)

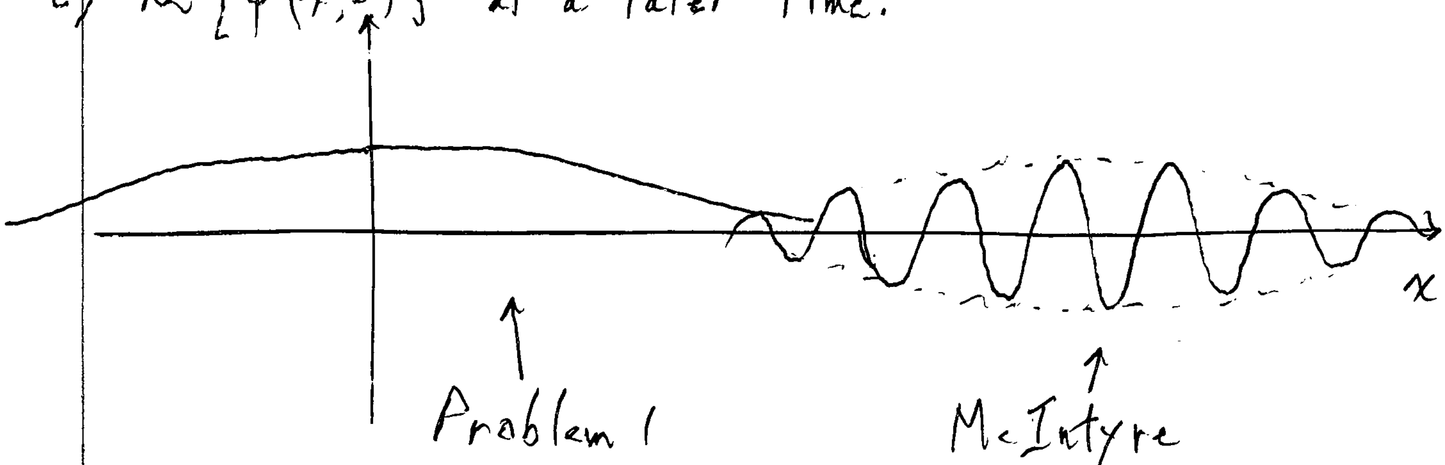


b)



Both packets are centered at $x=0$ when $t=0$.

c) $Re\{\psi(x,t)\}$ at a later time.



Both packets have spread the same amount.

The McIntyre packet has also moved to the right.