Dedekind cuts and binary representation of $\mathbb{R}$:

Each $X \in \mathbb{R}$: (restrict, for simplicity, on $X \in (0,1]$: strictly positive).

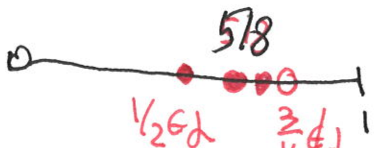
$$0.1 = \frac{1}{2} \;;\; 0.01 = \frac{1}{4} \;;\; 0.10 = \frac{1}{2} \;;\; 0.11 = \frac{3}{4}$$

$$0.001 = \frac{1}{8},\; 0.010 = \frac{1}{4},\; 0.011 = \frac{3}{8},\; 0.100 = \frac{1}{2},\; 0.101 = \frac{5}{8},$$

$$0.110 = \frac{3}{4},\; 0.111 = \frac{7}{8} \text{ etc.}$$

all $X = 0.101\cdots10000$ are rational.

We then ask whether $X \in \alpha$ OR not.



(we say $1 \notin \alpha$ otherwise $1 = \max \alpha$.)

$0.1011\cdots0$, so we obtain an infinite sequence of "0" and "1" such that it is never end up with all "0":

first place

$0.101100\cdots0\cdots$ since then 0.1011 is <u>the largest number</u>

We then have a <u>bijection</u>:

$$\alpha : \{0 \in \alpha \text{ and } 1 \notin \alpha\} \leftrightarrow 0.1011001101\ldots$$

where we have ∞ number of 1's.

it is such that if we replace ANY of 0 by 1 then the corresponding numbers $0.11\underbrace{\quad}_{\text{ANYTHING}} \notin \alpha$.

<u>QUE</u>: what corresponds to "1/2"?

Chapter II. Basic Topology of sets

DEF A function OR MAP BETWEEN TWO (nonempty) sets A & B (denoted $A \to B$, $A \xrightarrow{f} B$, $f(A)$, etc) is a CORRESPONDENCE that set in correspondence to every element $a \in A$ a unique $b \in B$: $b = f(a)$.

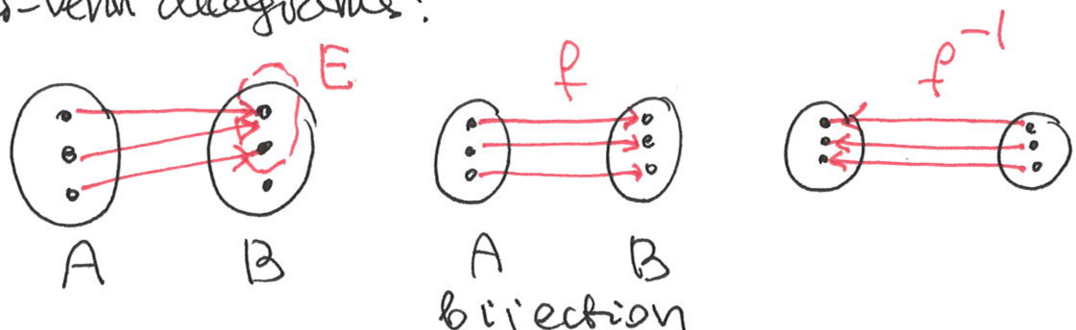
$f(A) = E \subseteq B$, A is the domain of f and E is the range of f.

DEF IF $E = B$ then the map is called surjective (OR mapping ONTO)

IF $f(a_1) \neq f(a_2)$ for $a_1 \neq a_2$, the map is injective.

DEF If a map is surjective and injective it is called a bijection, OR 1–1 correspondence

Euler-Venn diagrams:



If f is a bijection, $\forall y \in B$ $\exists!$ $x \in A$: $f(x) = y$, then we have an inverse function $f^{-1}(y) = x$.

unique

SEQUENCE $f: \mathbb{N} \to \mathbb{R}$. $(x_1, x_2, x_3, \dots)$

Cardinality: "size" of a set: $|A|$

$|A| = |B|$ $\underbrace{\text{iff}}$ $\exists f$ a bijection $A \leftrightarrow B$ $(A \sim B)$

"if and only if"

$|A| \leq |B|$ if $\exists$ an injection $A \to B$ $(A \leftrightarrow E \subseteq B)$

Theorem (hard) $|A| \leq |B| \wedge |B| \leq |A| \Rightarrow |A| = |B|$

[see Appendix]

Classification: let $I_n = 1, 2, \ldots, n \subset \mathbb{N}$ then

(a) A is finite if $A \leftrightarrow I_n$ for some n.
(then $|A| = n$)

(b) A is infinite if not finite

(c) A is countable if $A \sim \mathbb{N}$

(d) A is uncountable if it is not finite nor countable

(e) A is at most countable if A is finite or countable.

Theorem $|A| = |B|$ is an equivalence relation : $A \sim B$

(i) . $A \sim A$

(ii) if $A \sim B$ then $B \sim A$. $B = f(A)$: $\exists f^{-1}$ so $A = f^{-1}(B)$

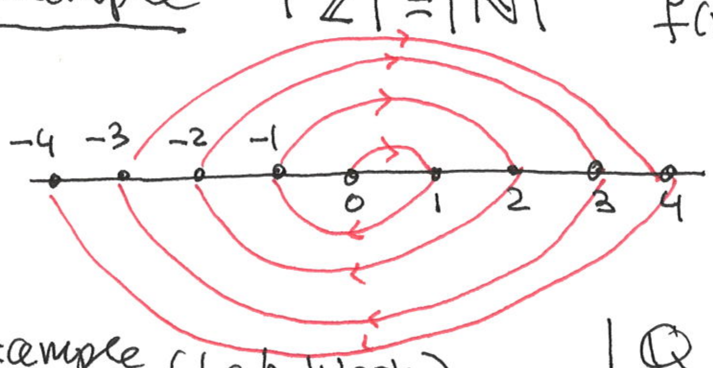
by injection

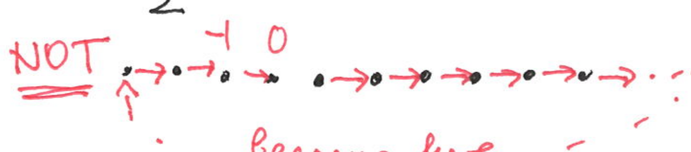
(iii) if $A \sim B$ and $B \sim C$ then $A \sim C$

(composite function) $B = f(A)$, $C = g(B)$ then

$C = g(f(A)) = g \circ f(A)$

Example $|\mathbb{Z}| = |\mathbb{N}|$ $\quad f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ -\frac{n-1}{2} & n \text{ odd} \end{cases}$





because here we cannot assign a specific finite n to, say -1, or 0.

Example (Lab Work) $|\mathbb{Q}| = |\mathbb{N}|$.

X - XVIII

The set of $\alpha \in (0,1]$ is <u>uncountable</u>

Assume α is countable; so $\exists\, f(n) \Leftrightarrow (0,1]$:

Arrange $x \in \alpha$ is an order of increasing n:

n				
1	$0.10110110\ldots$	x_1	0.00110110	
2	$0.01110010\ldots$	x_2	0.00110010	and take
3	$0.10100101\ldots$	x_3 $\rightarrow$	0.10000101	
4	$0.01100100\ldots$	x_4	0.01110100	
		$\vdots$		

diagonal

$x_0 = 0.0001011\ldots$ such that each n'th element is the changed (swapped) n^{th} digit of x_n

Then $x_0 \neq x_n \;\forall n \in \mathbb{N}$ (since they differ by n^{th} digit)

So x_0 is NOT in the list of Dedekind cuts $\{x_1, \ldots x_n \ldots\}$

so this list is incomplete – contradiction.

[This is called Cantor's "diagonal process"]

$\boxed{\text{DEF}}$ <u>Union</u> of sets $\bigcup_{\alpha \in A} E_\alpha$: all x : $x \in E_\alpha$ for <u>SOME</u> $\alpha \in A$

<u>intersection</u> of sets $\bigcap_{\alpha \in A} E_\alpha$: all x: $x \in E_\alpha$ for <u>ALL</u> $\alpha \in A$.

A can be finite or infinite.

<u>interesting</u> EXAMPLES: $\{ E_n = (0, \frac{1}{n}], n = 1, 2, \dots \}$

$$\bigcup_{n=1}^{\infty} E_n = E_1 = (0,1] \qquad E_1 \supseteq E_2 \supset E_3 \supset E_4 \cdots$$

$$\bigcap_{n=1}^{\infty} E_n = \{ \text{all } x : \quad 0 < x \leq \frac{1}{n} \text{ for all } n \in \mathbb{N} \}$$

but $\forall x > 0 \quad \exists n : \quad \frac{1}{n} < x \Rightarrow$ no such x exist

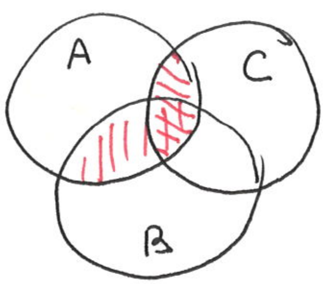
$$\bigcap_{n=1}^{\infty} E_n = \varnothing$$

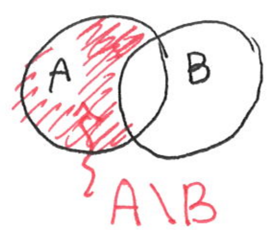
<u>properties</u> $A \cup B = B \cup A \qquad A \cap B = B \cap A$

distributive law:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

what about $A \cup (B \cap C)$?



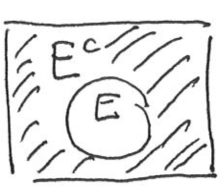


$A \setminus B = ?$

$$\underbrace{A \setminus B = A \cap B^c}$$

<u>Complement</u> E^c: implies we HAVE some ambient space

[... let E be a nonempty subset of $\underline{\mathbb{R}}$ ...]



$$E^c = \{ x \in \mathbb{R} : \ x \notin E \}.$$

$$[A \cup B]^c = A^c \cap B^c.$$