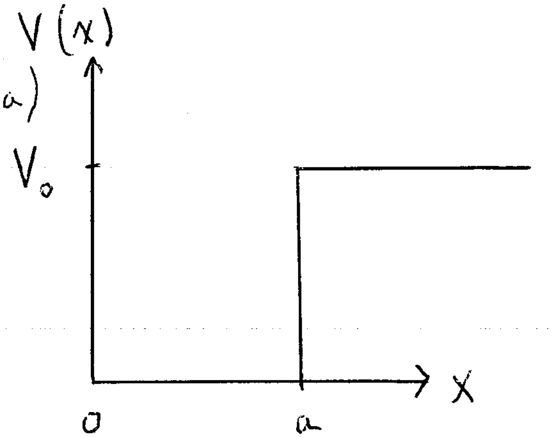
PHY 471 : Homework 10 Solutions

1. a)



Since $V(x) \to \infty$ for $x < 0$, we must have $\varphi_E(x=0) = 0$. This is true for the odd solutions to the symmetric square well.

So we have
$$\varphi_{odd}(x) = \begin{cases} C \sin(kx) & 0 \le x \le a \\ A e^{-qx} & x > a \end{cases} \qquad (5.81)$$

(Note that I removed the minus sign that McIntyre put here.)

I'll show the steps that lead to (5.85):

φ is continuous at $x = a$: $C \sin(ka) = A e^{-qa}$

$\frac{d\varphi}{dx}$ is continuous at $x = a$: $C k \cos(ka) = -A q e^{-qa}$

Divide the 2nd eqn by the 1st : $\boxed{k \cot(ka) = -q}$

The normalization is different because our wavefunction is nonzero only for $x > 0$, so we'll have

$$\int_0^\infty |\varphi_E(x)|^2 dx = 1$$

instead of $\int_{-\infty}^{\infty} |\varphi_E(x)|^2 dx = 1$. So the normalization constant will be $\sqrt{2}$ larger.

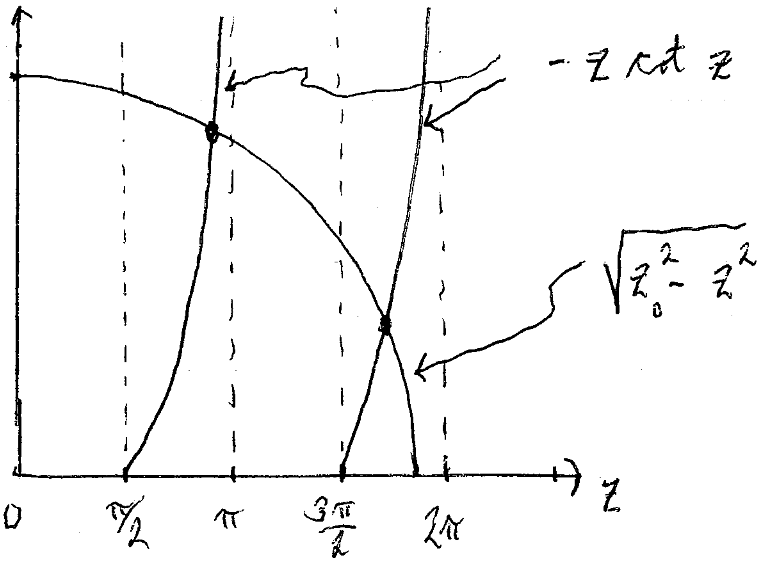
1. b) Following McIntyre (5.86) - (5.88):

Define $z = ka = \sqrt{\frac{2mE}{\hbar^2} a^2}$, $z_0 = \sqrt{\frac{2mV_0}{\hbar^2} a^2}$, $qa = \sqrt{\frac{2m(V_0 - E)}{\hbar^2} a^2}$

The boxed equation on the previous page becomes:

$-ka \cot(ka) = qa \rightarrow -z \cot z = qa = \sqrt{z_0^2 - z^2}$

Plot both the LHS and RHS as a function of z:



The RHS is the equation for a circle of radius z_0.

The LHS goes to zero at odd multiples of $\pi/2$ and diverges at even multiples of $\pi/2$

See Figure 5.16 in McIntyre. As the well gets wider or deeper, z_0 increases so the circle gets bigger and the number of bound state solutions increases.

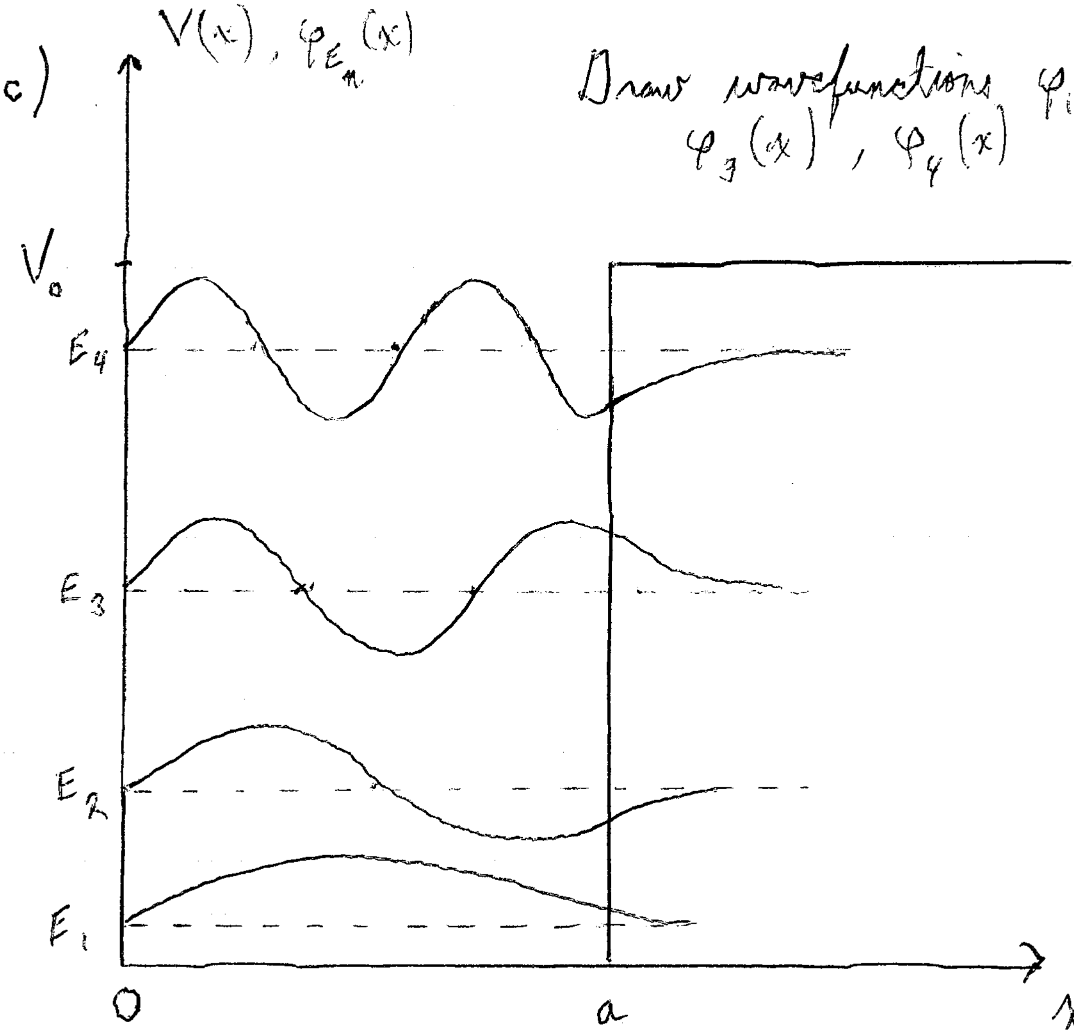
When z_0 is very large, the solutions approach

$z = \pi, 2\pi, 3\pi$, etc., i.e. $z = n\pi$

$$z = \sqrt{\frac{2mE}{\hbar^2} a^2} = n\pi \Rightarrow \frac{2mE}{\hbar^2} a^2 = n^2\pi^2 \Rightarrow E = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

Those are the energies of the infinite square well. From the graph above, it's clear that the differences between the finite well and infinite well energies <u>increase</u> with E.

1. c) Draw wavefunctions $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$, $\varphi_4(x)$



Features:

i) $\varphi_n(x)$ has $n-1$ nodes, not counting the node at $x=0$.

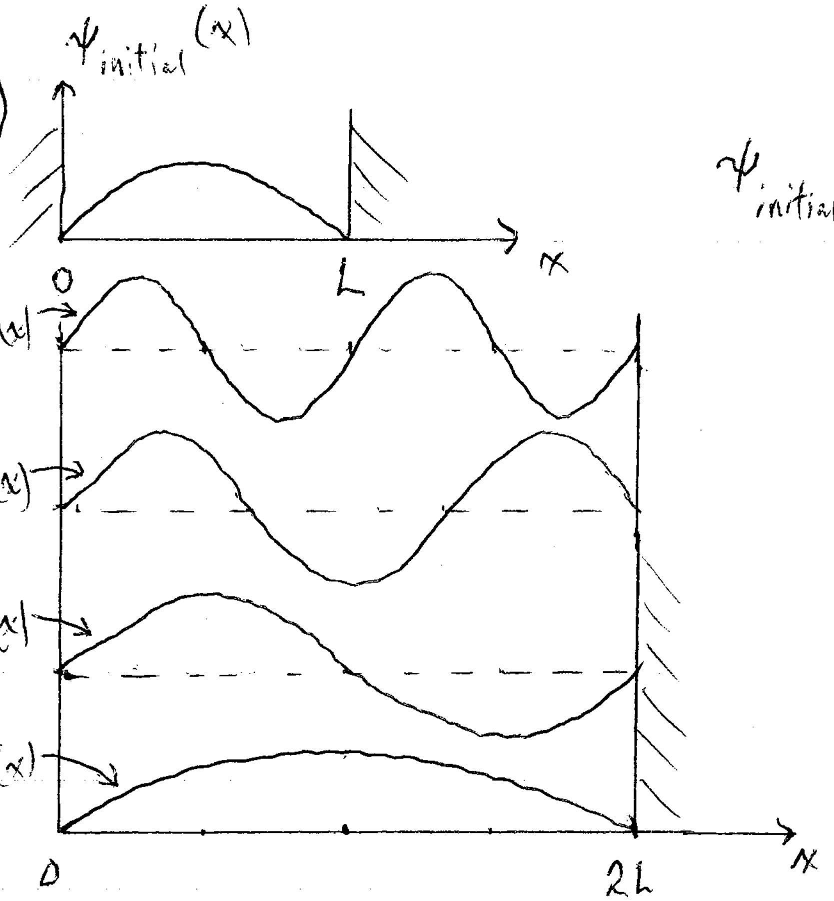
ii) All 4 wavefunctions have approximately equal amplitudes, since they are all normalized

iii) The wavefunctions oscillate in the region $x<a$ and decay expontially in the region $x>a$.

iv) The higher energy wavefunctions extend further into the barrier region as the energy increases, because

$$q = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$ gets smaller as E increases.

2. a)



$$\psi_{initial}(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} & \text{for } x<L \\ 0 & \text{for } x>L \end{cases}$$

In the new, wider well, we can write

$$\psi_{initial}(x) = \sum_n c_n \varphi_{E_n}(x)$$

where $c_n = \langle E_n | \psi_{initial} \rangle = \int_0^{2L} \varphi^*_{E_n}(x)\, \psi_{initial}(x)\, dx$

From the pictures, my guesses are:

$$\boxed{c_2 > c_1 > c_3 \;, \text{ and } c_4 = 0}$$

We can tell that $c_4 = 0$ because in the interval $0 < x < L$, $\psi_{initial}$ is even around $L/2$ while $\varphi_4(x)$ is odd around $L/2$, so their inner product is zero.

2. b) $$c_n = \int_0^L \underbrace{\sqrt{\frac{2}{2L}}\sin\left(\frac{n\pi x}{2L}\right)}_{\varphi^*_{E_n}(x)} \cdot \underbrace{\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)}_{\psi_{initial}(x)} \, dx$$

So $n = 2$ first:

$$c_2 = \frac{1}{\sqrt{2}} \cdot \frac{2}{L} \int_0^L \sin^2 \frac{\pi x}{L} \, dx = \frac{1}{\sqrt{2}} \cdot \frac{2}{L} \cdot \frac{L}{2} = \underline{\underline{\frac{1}{\sqrt{2}}}}$$

For $n \neq 2$:

$$c_n = \frac{1}{\sqrt{2}} \cdot \frac{2}{L} \int_0^L \sin\left(\frac{n}{2}\frac{\pi x}{L}\right) \sin\left(\frac{\pi x}{L}\right) dx$$

Use $\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$

or ask the computer to do the integral for you.

$$c_n = \frac{1}{\sqrt{2}} \frac{1}{L} \int_0^L \left[\cos\left(\left(1 - \frac{n}{2}\right)\frac{\pi x}{L}\right) - \cos\left(\left(1 + \frac{n}{2}\right)\frac{\pi x}{L}\right)\right] dx$$

$$= \frac{1}{\sqrt{2}L}\left[\frac{\sin\left(\left(1 - n/2\right)\frac{\pi x}{L}\right)}{1 - n/2} - \frac{\sin\left(\left(1 + n/2\right)\frac{\pi x}{L}\right)}{1 + n/2}\right]_0^L \cdot \frac{L}{\pi}$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{\sin\left(\left(1 - n/2\right)\pi\right)}{1 - n/2} - \frac{\sin\left(\left(1 + n/2\right)\pi\right)}{1 + n/2}\right]$$

$$\sin\left(\left(1 - \frac{n}{2}\right)\pi\right) = \sin\left[(2 - n)\frac{\pi}{2}\right] = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

$$\sin\left(\left(1 + \frac{n}{2}\right)\pi\right) = \sin\left[(2 + n)\frac{\pi}{2}\right] = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n+1}{2}} & n \text{ odd} \end{cases}$$

$$c_n = \frac{1}{\sqrt{2}\,\pi}\left[\frac{2}{2-n}(-1)^{\frac{n-1}{2}} - \frac{2}{2+n}(-1)^{\frac{n+1}{2}}\right]$$

$$= (-1)^{n-1/2}\frac{1}{\sqrt{2}\,\pi}\left[\frac{2}{2-n} + \frac{2}{2+n}\right] = (-1)^{\frac{n-1}{2}}\frac{1}{\sqrt{2}\,\pi}\frac{(4+2n)+(4-2n)}{4-n^2}$$

$$c_n = (-1)^{\frac{n-1}{2}}\frac{\sqrt{2}}{\pi}\frac{4}{4-n^2} \quad \text{for } n \text{ odd}$$

$$c_1 = \frac{4}{3}\frac{\sqrt{2}}{\pi} = 0.600 \qquad P(E_1) = |c_1|^2 = \underline{\underline{0.360}}$$

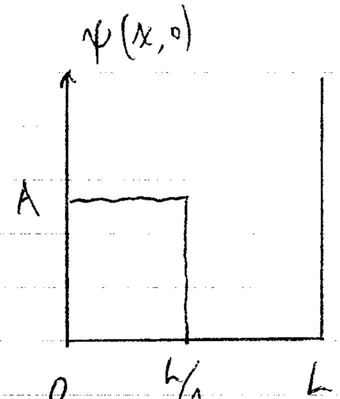
$$c_2 = \frac{1}{\sqrt{2}} = 0.707 \text{ from previous page} \qquad P(E_2) = |c_2|^2 = \underline{\underline{0.500}}$$

$$c_3 = \frac{4}{5}\frac{\sqrt{2}}{\pi} = 0.360 \qquad P(E_3) = |c_3|^2 = \underline{\underline{0.130}}$$

$$c_4 = 0 \text{ explained in part (a)} \qquad P(E_4) = \underline{\underline{0}}$$

As I guessed, $c_2 > c_1 > c_3$

3. a)



$$\psi(x,0) = \begin{cases} A & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases}$$

Normalize:

$$\int_0^L |\psi(x,0)|^2 dx = |A|^2 \cdot \frac{L}{2} = 1$$

$$\Rightarrow \underline{\underline{A = \sqrt{\frac{2}{L}}}}$$

b) $\psi(x,0) = \sum_n c_n \varphi_{E_n}(x)$ with $c_n = \int_0^L \varphi^*_{E_n}(x)\,\psi(x,0)\,dx$

$$c_n = \int_0^{L/2} \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \cdot \sqrt{\frac{2}{L}}\, dx = \frac{-2}{L} \cdot \frac{L}{n\pi} \cos\frac{n\pi x}{L}\Big|_0^{L/2}$$

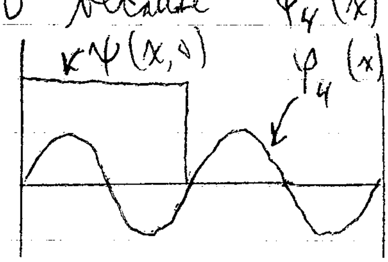
$$= \frac{-2}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right)$$

$$c_1 = \frac{2}{\pi}, \quad c_2 = \frac{1}{\pi}(-1-1) = \frac{-2}{\pi}, \quad c_3 = \frac{2}{3\pi}, \quad c_4 = 0$$

The probability that an energy measurement will yield the result E_n is $|\langle E_n|\psi\rangle|^2 = |c_n|^2$

$$\underline{\underline{P(E_n) = \frac{4}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)^2}} = \begin{cases} 4/\pi^2 & n=1 \\ 4/\pi^2 & n=2 \\ 4/9\pi^2 & n=3 \\ 0 & n=4 \end{cases}$$

$c_4 = 0$ because $\varphi_4(x)$ is orthogonal to $\psi(x,0)$, i.e. $\langle E_4|\psi\rangle = 0$



or $\int_0^L \varphi_4(x)\,\psi(x,0) = 0$

4. $[\hat{x}, \hat{p}]$ in position representation $\rightarrow \left[x, -i\hbar \frac{\partial}{\partial x}\right]$

$$\left[x, -i\hbar \frac{\partial}{\partial x}\right] \psi(x) = x\left(-i\hbar \frac{\partial \psi}{\partial x}\right) - \left(-i\hbar \frac{\partial}{\partial x}\right)(x\psi)$$

Use product rule:

$$= -i\hbar x \frac{\partial \psi}{\partial x} + i\hbar\left(\psi + x \frac{\partial \psi}{\partial x}\right) = i\hbar \psi(x)$$

This holds for any $\psi(x)$, so $\underline{\underline{[\hat{x}, \hat{p}] = i\hbar}}$