Review of Linear Algebra

We review orthogonal vectors, matrices, inverses, and eigenvectors

Objectives

To review the main definitions and properties of vectors in $\mathbb{R}^n$, with $n = 1, 2, 3, \cdots$, of $n \times n$ matrices and their eigenvalues and eigenvectors.

Further Reading

Students may need to review Chapter 5, "Overview of Linear Algebra" in our textbook.

Vectors and Linear Dependence-Independence

A ***vector*** in $\mathbb{R}^n$, with $n = 1, 2, 3, \cdots$, is a collection of n numbers $\boldsymbol{v} = \langle v_1, \cdots, v_n \rangle$ together with the operation ***linear combination*** given by

$$a\,\boldsymbol{u} + b\,\boldsymbol{v} = a\langle u_1 \cdots, u_n\rangle + b\langle v_1 \cdots, v_n\rangle = \langle (au_1 + bu_2), \cdots, (au_n + bv_n)\rangle \qquad \forall a, b \in \mathbb{R}.$$

The numbers a, b above are called ***scalars***, to tell them apart from the numbers u_i, for $i = 1, \cdots, n$, which are vector components. We will also use the column vector notation for vectors,

$$\boldsymbol{u} = \langle u_1, \cdots, u_n\rangle = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.$$

A finite set of vectors $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k\}$, with $k \geq 1$, is ***linearly dependent*** iff there exists a set of scalars $\{c_1, \cdots, c_k\}$, not all of them zero, such that,

$$c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}.$$

The set $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k\}$ is called ***linearly independent*** iff the equation above implies that every scalar vanishes, $c_1 = \cdots = c_k = 0$. The ***dimension*** of a vector space $\mathbb{R}^n$ is the maximum number of vectors that are linearly independent. It is not difficult to see that the dimension of the vector space $\mathbb{R}^n$ is indeed n. A linearly independent set of n vectors in $\mathbb{R}^n$ is called a ***base*** of $\mathbb{R}^n$. Again, it is not difficult to see that any vector in the space $\mathbb{R}^n$ is a linear combination of the vectors in a base.

Question 1: *(10 points)* Determine whether the set of vectors below are linearly dependent or independent.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Orthogonal Vectors

The ***dot product*** of vectors $\boldsymbol{u} = \langle u_1, \cdots, u_n\rangle$ $\boldsymbol{v} = \langle v_1, \cdots, v_n\rangle$ in $\mathbb{R}^n$ is

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_1 v_1 + \cdots + u_n v_n.$$

The length of a vector $\boldsymbol{u} = \langle u_1, \cdots, u_n\rangle$ is

$$\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u}\cdot\boldsymbol{u}} = \sqrt{(u_1)^2 + \cdots (u_n)^2}.$$

And any vector $\boldsymbol{v}$ can be rescaled into a unit vector by dividing by its magnitude. So, the vector $\boldsymbol{u}$ below is a unit vector in the direction of the vector $\boldsymbol{v}$,

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}.$$

The dot product of two vectors can also be written in the alternative form

$$\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\cos(\theta),$$

where $\|\boldsymbol{u}\|$, $\|\boldsymbol{v}\|$ are the length of the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\theta \in [0, \pi]$ is the angle between the vectors. A set of vectors is an ***orthogonal set*** if all the vectors in the set are mutually perpendicular. An ***orthonormal set*** is an orthogonal set where all the vectors are unit vectors.

Theorem 1. Given an orthogonal set $\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_n\}$ in $\mathbb{R}^n$, every vector $\boldsymbol{v} \in \mathbb{R}^3$ can be decomposed as

$$\boldsymbol{v} = v_1\,\boldsymbol{u}_1 + \cdots + v_n\,\boldsymbol{u}_n.$$

Furthermore, there is a formula for the vector components,

$$v_1 = \frac{(\boldsymbol{v}\cdot\boldsymbol{u}_1)}{(\boldsymbol{u}_1\cdot\boldsymbol{u}_1)}, \quad \cdots, \quad v_n = \frac{(\boldsymbol{v}\cdot\boldsymbol{u}_n)}{(\boldsymbol{u}_n\cdot\boldsymbol{u}_n)}.$$

If the vectors are orthonormal, that is orthogonal and unit vectors, then the formula for the components reduces to

$$v_1 = \boldsymbol{v}\cdot\boldsymbol{u}_1, \quad \cdots, \quad v_n = \boldsymbol{v}\cdot\boldsymbol{u}_n.$$

Proof of Theorem 1: Since the vectors $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_n$ are mutually perpendicular, that means they are linearly independent, so the set of all possible linear combinations of these vectors is the whole space $\mathbb{R}^n$. Therefore, given any vector $\boldsymbol{v} \in \mathbb{R}^n$, there exists constants v_1, $\cdots$, v_n such that

$$\boldsymbol{v} = v_1\,\boldsymbol{u}_1 + \cdots + v_n\,\boldsymbol{u}_n.$$

Since the vectors $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_n$ are mutually orthogonal, we can compute the dot product of the equation above with $\boldsymbol{u}_1$, and we get

$$\boldsymbol{u}_1\cdot\boldsymbol{v} = v_1\,\boldsymbol{u}_1\cdot\boldsymbol{u}_1 + \cdots + v_n\,\boldsymbol{u}_1\cdot\boldsymbol{u}_n = v_1\,\boldsymbol{u}_1\cdot\boldsymbol{u}_1 + 0 + \cdots + 0,$$

therefore, we get a formula for the component v_1,

$$v_1 = \frac{(\boldsymbol{v}\cdot\boldsymbol{u}_1)}{(\boldsymbol{u}_1\cdot\boldsymbol{u}_1)}.$$

If the vector $\boldsymbol{u}_1$ is an unit vector, then $\boldsymbol{u}_1\cdot\boldsymbol{u}_1 = 1$. A similar calculation provides the formulas for v_i, with $i = 2, \cdots n$. This establishes the Theorem. $\square$

Question 2: *(10 points)* Find the expansion of the vector

$$\boldsymbol{v} = \langle 3, 2, 1 \rangle$$

on the orthonormal set

$$\{\boldsymbol{u}_1 = \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle,\ \boldsymbol{u}_2 = \frac{1}{\sqrt{6}}\langle -2, 1, 1 \rangle,\ \boldsymbol{u}_3 = \frac{1}{\sqrt{2}}\langle 0, -1, 1 \rangle\}.$$

Overview of Matrices

An $m \times n$ ***matrix***, A, is an ordered array of numbers,

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}.$$

The numbers $A_{i,j} \in \mathbb{R}$, for $i = 1, \cdots, m$ and $j = 1, \cdots n$, are called the matrix components. The space of all $m \times n$ matrices with components in $\mathbb{R}$ is called $\mathbb{R}^{m,n}$. The $n \times n$ matrices are called ***square*** matrices.

The ***matrix-vector product*** of an $m \times n$ matrix $A = [A_{ij}]$ and an n-vector $\boldsymbol{u} = [u_j]$ is the m-vector $A\boldsymbol{u}$ given by

$$A\boldsymbol{u} = \begin{bmatrix} A_{11}u_1 + \cdots + A_{1n}u_n \\ \vdots \\ A_{m1}u_1 + \cdots + A_{mn}u_n \end{bmatrix}.$$

A matrix together with the matrix-vector product imply that a matrix is a function on the space of vectors.

The ***linear combination*** ofthe $m \times n$ matrices $A = [A_{ij}]$ and $B = [B_{ij}]$ with the scalars a, b is also an $m \times n$ matrix denoted as $(a\,A + b\,B)$ given by

$$a\,A + b\,B = [a\,A_{ij} + b\,B_{ij}].$$

That is, we compute the linear combination of matrices in a similar way as the linear combination of vectors, component wise. The ***matrix multiplication*** of an $m \times n$ matrix A and an $n \times \ell$ matrix $B = \big[\boldsymbol{b}_1, \cdots, \boldsymbol{b}_\ell\big]$ is given by

$$AB = \big[A\boldsymbol{b}_1, \cdots, A\boldsymbol{b}_\ell\big], \tag{1}$$

where $A\boldsymbol{b}_i$ are the matrix-vector product of matrix A and the n-vectors $\boldsymbol{b}_i$ for $i = 1, \cdots, \ell$. There is an equivalent expression for the matrix multiplication using the components of bot matrices,

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}, \qquad i = 1, \cdots, m \;\text{ and }\; j = 1, \cdots, \ell.$$

In the case of square matrices we use the standard power notation $A^2 = AA$, $A^3 = AAA$, and so on.

Remarks:

(a) Notice that the matrix multiplication is not commutative, that is, in general we have $AB \neq BA$.

(b) Also notice we may have matrices A, B such that $BA = 0$ but $A \neq 0$ and $B \neq 0$.

(c) Therefore, the product $AB = 0$ *does not imply* that either $A = 0$ or $B = 0$.

An $n \times n$ matrix A is called ***invertible*** iff there exists another $n \times n$ matrix, denoted as A^{-1}, such that

$$\big(A^{-1}\big)A = I \quad \text{and} \quad A\big(A^{-1}\big) = I.$$

where I is the $n \times n$ identity matrix. There is a simple formula for the inverse of a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which is given by

$$(A^{-1}) = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad \text{in the case that} \quad \det(A) \neq 0,$$

where $\det(A) = ad - bc$ is the ***determinant*** of a matrix A. It is simple to verify that this matrix (A^{-1}) is the inverse of matrix A because

$$(A^{-1})\, A = I, \qquad A\,(A^{-1}) = I, \quad \text{where} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Also recall that the ***trace*** of a $n \times n$ matrix is the sum of its diagonal elements, so the trace of the 2×2 matrix A above is

$$\operatorname{tr}(A) = a + d.$$

Question 3: *(10 points)* Prove the Cayley-Hamilton Theorem in the case of 2×2 matrices, that is, show that every 2×2 matrix A satisfies the following *matrix equation*,

$$A^2 - \operatorname{tr}(A)\, A + \det(A)\, I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Properties of Determinants

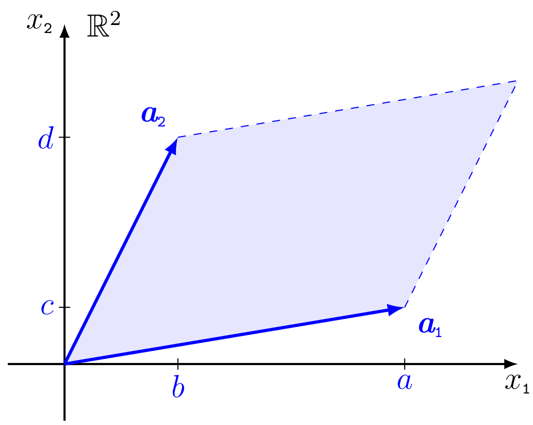
Question 4: *(10 points)* Prove that $\det(AB) = \det(A)\det(B)$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

Question 5: *(10 points)* Determine whether the equation $\det(A + B) = \det(A) + \det(B)$ is true or not. If it is true, prove it for all 2×2 matrices A and B; if it is not true, give an example.

Question 6: *(10 points)* Denote a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in terms of its column vectors as $A = \begin{bmatrix} \boldsymbol{a}_1, \boldsymbol{a}_2 \end{bmatrix}$. Suppose that the vectors $\boldsymbol{a}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\boldsymbol{a}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$ are given in the figure below. Use that picture to prove

$$\text{Area of the shaded parallelogram} = |\det(A)|.$$

Hint: Relate the parallelogram area with areas you can easily compute, such as triangle and rectangle areas.



Properties of Inverse Matrices

Question 7: *(10 points)* Prove that for every invertible 2×2 matrix holds that $\big((A^{-1})^{-1}\big) = A$.

Question 8: *(10 points)* Prove that every invertible 2×2 matrix satisfy $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Question 9: *(10 points)* Prove that every invertible 2×2 matrices A, B, satisfy $(AB)^{-1} = (B^{-1})(A^{-1})$.

Eigenvalues and Eigenvectors

A number $\lambda \in \mathbb{R}$ and a nonzero n-vector $\boldsymbol{v} \in \mathbb{R}^n$ are an ***eigenvalue*** with corresponding ***eigenvector*** (eigenpair) of an $n \times n$ matrix A iff they satisfy the equation

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

Theorem 2 (Eigenvalues-Eigenvectors)**.**

(a) All the eigenvalues λ of an $n \times n$ matrix A are the solutions of the scalar equation

$$\det(A - \lambda I) = 0. \tag{2}$$

(b) Given an eigenvalue λ of an $n \times n$ matrix A, the corresponding eigenvectors $\boldsymbol{v}$ are the nonzero solutions to the homogeneous linear system

$$(A - \lambda I)\boldsymbol{v} = \mathbf{0}. \tag{3}$$

Question 10: *(10 points)* Compute the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.