Variable Transformations

We transform equations by changing the unknown function

Objectives

In this dive we show how to solve certain differential equations by changing the unknown function in the equation, that is, by changing the dependent variable in the equation. We study three cases.

In the first case we introduce *scale invariant* equations. A large class of scale invariant equations cannot be transformed into separable equations by simple algebraic manipulations. However, we will show that there is a change in the unknown function so that the equation for the new unknown function is separable.

In the second case we solve *linear* differential equations having variable coefficients, which are not separable, using the *variation of parameters method*, which is different from the integrating factor method used in class. In the variation of parameters method we transform the non-separable linear equation into a separable linear equation by, once again, a change in the unknown function. Then we solve the separable equation for the transformed function, and we use this function to compute the solution of the non-separable equation.

In the third case we show how to solve a particular type of nonlinear equations, called the *Bernoulli equation* (this is not the Bernoulli equation from fluid dynamics). These nonlinear equations can be transformed into a linear equation by, one more time, a change in the unknown function. Then we solve the linear equation, for example using the variation of parameters method or the integrating factor method, and the we use this function to compute the solution of the Bernoulli equation.

Further Reading

If the summaries in this Dive are not enough to fully understand any of the transformations in this dive, then students may find useful to read from our textbook the appropriate parts of the following sections or subsections:

- Section 1.3, Scale Invariant Equations, for help with scale invariant equations.
- Subsection 1.4.4, Variation of Parameters, for help on variation of parameters on linear equations.
- Section 1.5, Bernoulli Equation, for help on the Bernoulli equation.

Scale Invariant Equations

In section 1.3 of our textbook we have a fairly long discussion on scale invariant functions and scale invariant differential equations. The summary of that discussion is Proposition 1.3.4 in the textbook, which we will use as our definition of a scale invariant differential equation.

Definition 1 (Scale Invariant Equations)**.** *A differential equation for $y(t)$ is scale invariant iff holds,*

$$y' = F(y/t).$$

Remark: That is, the right-hand side of the differential equation written in normal form depends only on the quotient y/t.

Question 1.*(10 points)* Use simple algebraic transformations to show that the differential equation

$$2ty\, y' - 5\, t^2 - 3\, y^2 = 0, \qquad t > 0,$$

is a scale invariant equation and write it in the normal form

$$y' = F(y/t),$$

that is, find the function F.

Now we show our main result. We can transform a scale invariant equation on a function $y(t)$ into a separable equation for a function

$$v(t) = \frac{y(t)}{t}.$$

Theorem 2 (Scale Invariant into Separable)**.** *The scale invariant equation for the function $y(t)$ given by*

$$y' = F\Big(\frac{y}{t}\Big)$$

determines a separable equation for the function $v(t) = y(t)/t$, given by

$$\frac{v'}{\big(F(v) - v\big)} = \frac{1}{t}.$$

Question 2. *(12 points)* Prove Theorem 2 above.

Hint: The relation $v(t) = y(t)/t$ implies a relation between the derivatives v' and y'.

Question 3. *(12 points)* Use the result in Theorem 2 to find all solutions of the scale invariant (and non-separable) differential equation in Question 1.

The Variation of Parameters for Linear Equations

In Theorem 1.4.3 in the textbook we say that all solutions to the linear $y(t)$ of the linear equation

$$y' = a(t)\, y + b(t) \tag{1}$$

are given by the formula

$$y(t) = c\, e^{A(t)} + e^{A(t)} \int e^{-A(t)}\, b(t)\, dt,$$

where c is an arbitrary constant. In the first proof in the textbook of this result we use the integrating factor method. Now we are going to reobtain this result using the variation of parameters method.

In the variation of parameters method we write the solution $y(t)$ of the linear equation (1) in the form

$$y(t) = v(t)\, y_h(t), \tag{2}$$

where $y_h(t)$ is a solution of the linear, homogeneous, and also separable equation

$$y_h' = a(t)\, y_h. \tag{3}$$

The function $v(t)$ is the parameter we try to find, while we know how to compute the function $y_h(t)$, because the linear homogeneous equation in (3) is also separable.

Question 4.

(4a) *(5 points)* Find one solution $y_h(t)$ of the separable equation

$$y_h' = a(t)\, y_h.$$

Denote by $A(t)$ any antiderivative of $a(t)$, that is, $A(t) = \int a(t)\, dt$.

(4b) *(10 points)* Write any solution $y(t)$ of the linear equation (1) as

$$y(t) = v(t)\, y_h(t),$$

where $y_h(t)$ is a solution found in part (1a). Then show that the differential equation for $v(t)$ is

$$v' = e^{-A(t)}\, b(t),$$

where $A(t)$ is the function defined in part (1a).

(4c) *(5 points)* Solve this differential equation for $v(t)$ obtained in part (1b) and show that all solutions $y(t)$ of the linear equation (1) are given by the formula given in the textbook, Section 1.4, Theorem 1.4.3, Equation (1.4.7), that is,

$$y(t) = c\, e^{A(t)} + e^{A(t)} \int e^{-A(t)}\, b(t)\, dt,$$

where c is an arbitrary constant.

Question 5. *(12 points)* Use the Variation of Parameters Method to find all solutions, y, of the equation

$$y' = \frac{3}{t}\,y + t^5, \qquad t > 0.$$

Hint: First find y_h, solution of the homogeneous linear equation, as in Question (4a), then find $v(t)$ as defined in Question (4b), and then get the solution $y(t) = v(t)\,y_h(t)$.

The Bernoulli Equation

Jacob Bernoulli.

The equation we are interested here is named after Jacob Bernoulli (1655-1705), who first wrote it in 1694. This equation is not the Bernoulli equation from fluid dynamics, which was found by Daniel Bernoulli (1700-1782), nephew of Jacob, and published it in his book Hydrodynamics, in 1738. Unlike Daniel's equation, Jacob's equation does not have any known physical application, except for a particular case when it reduces to the logistic equation. We are interested in Jacob's equation because it is a famous example of an equation that can be solved by a variable transformation. We already used a change of variable to transform an Euler Homogeneous equation into a separable equation. Now we show that another change of variable can transform Jacob Bernoulli's equation (from now on called Bernoulli equation), which is nonlinear, into a linear equation.

Definition 3. The ***Bernoulli equation*** with coefficients functions p, q, and index $n \in \mathbb{R}$ is given by

$$y' = p(t)\, y + q(t)\, y^n. \tag{4}$$

Remarks:

- For $n \neq 0, 1$ the equation is nonlinear.
- If $n = 2$ and p, q are constants, we get the *logistic equation*. Just rename $p = r$ and $q = -r/K$, then
$$y' = ry\left(1 - \frac{y}{K}\right).$$
- The Bernoulli equation is special because *it is a nonlinear equation that can be transformed into a linear equation*. This was first shown (without many details) by Gottfried Leibniz in 1696, and one year later, probably independently with all details by Johann Bernoulli (younger brother of Jacob).

Theorem 4. *The function y is a solution of the Bernoulli equation*

$$y' = p(t)\, y + q(t)\, y^n, \qquad n \neq 1,$$

iff the function $v = 1/y^{(n-1)}$ *is solution of the linear differential equation*

$$v' = -(n-1)p(t)\, v - (n-1)q(t).$$

Question 6. *(12 points)* Prove Theorem 4.

Question 7. *(10 points)* Find every nonzero solution of the differential equation

$$y' = y + 3\,y^4.$$

Question 8. *(12 points)* Find every nonzero solution of the constant coefficients Bernoulli equation

$$y' = p\,y + q\,y^n, \qquad n \neq 0, 1, \qquad p \neq 0,$$

where p, q are constants. Write the implicit form of the solution as

$$\frac{1}{y^{n-1}} = f(t, n, p, q, c)$$

where c is an integration constant. Find the right-hand side above, $f(t, n, p, q, c)$.