PHY 481 - Fall 2023
Homework 09

Due Friday, December 1, 2023

Preface

Homework 09 introduce foundational ideas for the magnetic field. You will start with magnetic forces and motion, and then get some practice with current densities.

1 Magnetic Forces and Motion

Griffiths works out (Example 5.2) the general solution to motion of a particle in "crossed E and B fields" ($\mathbf{E}$ points in the z-direction, $\mathbf{B}$ in the x-direction) Work out his solution carefully, make sure you follow it. **Specifically, fill in the missing steps between Equations (5.5) and (5.6).** Then, use it to answer the following questions.

1. Suppose the particle starts at the origin at $t = 0$, with a given velocity $\mathbf{v}(t = 0) = v_0\hat{y}$. Use Griffiths' formal results (Eq. 5.6) to find the "special initial speed", v_0, whose subsequent motion is simple straight-line constant-speed motion. Verify that this answer makes sense by using elementary Phys 184-style right-hand rule arguments and the Lorentz force law, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

2. Assuming that the electrons are moving at $1/3$ the speed of light in a vacuum and that the electric field is produced by two large flat plates with a potential difference of 2 kV and a plate separation of 2 cm, what magnetic field is needed to obtain a single straight-line motion through the apparatus? How does this magnetic field compare to that of the Earth's magnetic field? In other words, was it reasonable for Thomson to neglect the effect of the Earth's magnetic field in his experiment?

3. The discovery of the electron (J.J. Thomson, 1897) used an apparatus with crossed $\mathbf{E}$ and $\mathbf{B}$ just like the above. Thomson adjusted $\mathbf{E}$ until he observed "straight-line, constant-speed" motion of the particle beam. Then, he turned off the E-field, and measured the radius of curvature (R) of the electron beam (deflected purely by the remaining B-field). Given E, B, and R (all measured), deduce Thomson's formula to find the charge to mass ratio (e/m) of the electrons.

4. Go back to Griffiths' Ex 5.2 again, but this time suppose your particle starts at the origin with $\mathbf{v}(t = 0) = v_1\hat{y}$, with starting speed v_1 exactly half the "special value" velocity you found in part 1. Find and sketch the resulting trajectory, $z(t)$ and $y(t)$,

of the particle. Calculate the kinetic energy of the particle $K = \frac{1}{2}m(\dot{y}^2 + \dot{z}^2)$, where $\dot{y}$ and $\dot{z}$ are time derivatives of positions (velocities). Is the kinetic energy of the particle constant with time? Briefly, comment (Is this consistent with energy conservation?!)

2 Current Densities

We are going to be working with "current densities" for the rest of the term. Let's practice writing down some current densities. These are important for understanding books and papers where folks often use these forms as shorthand to describe what they are talking about.

1. A sphere (radius R, total charge Q uniformly distributed throughout the volume) is spinning at angular velocity $\omega\hat{z}$ about its center, which is at the origin. What is the volume current density $\mathbf{J}(r, \theta, \phi)$ at any point (r, θ, ϕ) in the sphere? (Don't forget direction too!)

2. A very thin DVD has been rubbed so that it has a fixed, constant, uniform surface electric charge density σ everywhere on its top surface. It is spinning at angular velocity ω about its center (which is at the origin). What is the magnitude of surface current density $\mathbf{K}$ at a distance r from the center? What is the **volume** current density $\mathbf{J}$ in cylindrical coordinates? (This may be a little tricky, since the disk is "very thin," there will be a δ function. *Hint: write down a formula for $\rho(s, \phi, z)$ first. And, remember that $\mathbf{J}$ should be a vector!)*

3. A very thin plastic ring (radius R) has a constant linear charge density, and total charge Q. The ring spins at angular velocity ω about its center, which is the origin. What is the current I, in terms of given quantities? What is the volume current density $\mathbf{J}$ in cylindrical coordinates? (This may be a little tricky, since the ring is "very thin," there will be some δ functions. *Hint: write down a formula for $\rho(s, \phi, z)$ first. And, remember that $\mathbf{J}$ should be a vector!)*

3 Magnetic field of distributed currents

In the previous problem, we had a DVD (radius R) with a fixed, constant, uniform surface electric charge density σ everywhere on its top surface (figure below). It was spinning at angular velocity ω about its center (the origin). You should find the current density $\mathbf{K}$ at a distance s from the center as $\mathbf{K} = \sigma\omega\mathbf{s}\hat{\phi}$

1. Write $\vec{r}$, $\vec{r'}$, $\vec{\mathscr{r}}$, $\hat{\mathscr{r}}$, and da' using the cylindrical coordinates s, ϕ, and z. Write $\mathbf{K} \times \vec{\mathscr{r}}$ in cylindrical coordinates using the fact that $\hat{\phi} \times \hat{z} = \hat{s}$ and $\hat{\phi} \times \hat{s} = -\hat{z}$

2. To calculate the magnetic field, we will use the equation 5.42 in Griffiths:
$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r'}) \times \vec{\mathscr{r}}}{\mathscr{r}^3} da'$
Using the result from the previous part, construct the integral. What happens to the

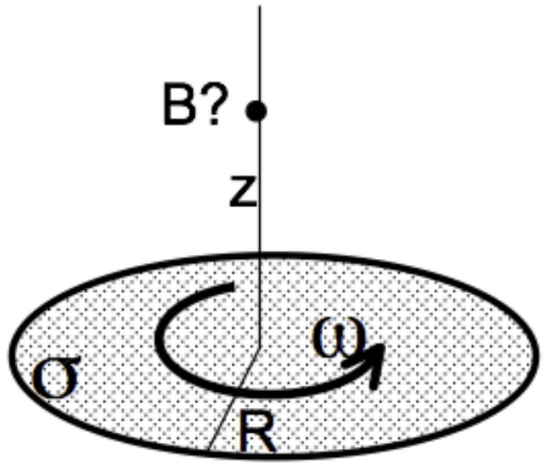

Figure 1: Spinning disk

part of the integral with $\hat{s} = cos\phi\,\hat{x} + sin\phi\,\hat{y}$? Solve the integral with $\hat{z}$ to express $\mathbf{B}(z)$.

3. Does your answer to part 2 seem reasonable? Please check it, with units, and some limiting behaviors (e.g. what do you expect if $R \to 0$? $\omega \to 0$? $z \to \infty$?) For this last one, don't just say "it goes to zero. This is a dipole, so B should go to 0 like $1/z^3$. (Right?) Show that it does!

4 Python : Modeling the motion of a charged particle in a magnetic field

We have shown that the motion of a charged particle in a constant magnetic is a circular when the initial velocity is perpendicular to the field. In this problem, you will complete the code in a Jupyter notebook to model the motion of a proton in a magnetic field. Download the template HW09_MotionOfChargeInMagneticField.ipynb from D2L.

1. Your first task is to read through the code and complete the integration loop to compute the trajectory of the proton and plot it in 3D. (You might need to look up how to construct a 3D plot.) For this first case, you should expect a simple circular orbit because the proton starts it's motion moving perpendicular to the magnetic field.

2. Once you have your code working for part 1, change it to give the proton a component of velocity along the direction of the magnetic field. What does the resulting motion look like? Explain qualitatively why it should look like that.