MTH 327h Honors Analysis I, Fall 2023 Homework 1

Problem 1 Find all rational solutions to the algebraic equation $x^3 - 7x^2 - 289x + 2023 = 0$

Problem 2 Prove by induction that $2^n > n$ for all $n \in \mathbb{N}$.

Problem 3 Prove by induction that any finite subset of $\mathbb{R}$ contains a maximum element. (Similarly, any finite subset of $\mathbb{R}$ contains a minimum element.)

Problem 4 Prove by induction that the triangle inequality $|x + y| \leq |x| + |y|$ implies

$$|x_1 + x_2 + \cdots + x_{n-1} + x_n| \leq |x_1| + |x_2| + \cdots + |x_{n-1}| + |x_n| \quad \forall n \geq 2.$$

Problem 5 Consider the sequence $(r_n) \subset \mathbb{Q}$ defined recursively:

$$r_1 = 1, \qquad r_{n+1} = \frac{3}{4} r_n + \frac{1}{2r_n}, \quad n \geq 1.$$

(a) Prove that $1 \leq r_{n+1} < \sqrt{2}$ provided $1 \leq r_n < \sqrt{2}$;

(b) Prove, using (a), that $r_n < r_{n+1}$ $\forall n \in \mathbb{N}$.

Problem 6 Prove that for nonempty subset $A \subseteq B$, where B is bounded above, $\sup(A) \leq \sup(B)$.

Problem 7 Prove that for two nonempty sets A and B bounded above,

$$\sup(A + B) = \sup(A) + \sup(B).$$

Here the set $A + B = \{s + t : \ s \in A, \ t \in B\}$—the set of all possible sums of elements of A and B.

[Hint: Start with a simpler statement that for any bounded set A and a number t, $\sup\{s + t : \ s \in A\} = \sup(A) + t$.]

Problem 8 Prove that a finite field (e.g. $\mathbb{F}_p$) does not admit **any** order relation satisfying an axiom of addition.

[Hint: use the result of Problem 2 that any finite set has maximum and minimum, so it exists $x \in \mathbb{F}_p$ such that $x < y$ for any $y \neq x$, $y \in \mathbb{F}_p$ and find a proper $c \in \mathbb{F}_p$ such that $x + c < y + c$ will be a clear contradiction.]

Problem 9* Consider the sequence in Problem 5 as a set $A = \{r_n, \ n \in \mathbb{N}\}$. What do you think is $\sup(A)$? For this, show that

$$|r_{n+1} - \sqrt{2}| \leq \frac{1}{2}|r_n - \sqrt{2}| \quad \forall n \geq 1,$$

then use the Archimedean property of the set of natural numbers and the result of Problem 2 to show that for any positive numbers $\varepsilon > 0$ and $\beta > 0$ there exists a natural number n such that $\beta/2^n < \varepsilon$.