Deep Dive #6

Review of Linear Algebra

We review orthogonal vectors, matrices, inverses, and eigenvectors

Question 1: *(10 points)* Determine whether the set of vectors below are linearly dependent or independent.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

We'll check if the third matrix is a linear combination of the first two

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & -7 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$a = \frac{3}{2},\ b = \frac{1}{2}$$

As such these are linearly dependent. This can be verified by

$$\begin{cases} 1 * \frac{3}{2} + 3 * \frac{1}{2} = 3 \\ 2 * \frac{3}{2} + 2 * \frac{1}{2} = 2 \\ 3 * \frac{3}{2} - 7 * \frac{1}{2} = 1 \end{cases}$$

Question 2: *(10 points)* Find the expansion of the vector

$$\boldsymbol{v} = \langle 3, 2, 1 \rangle$$

on the orthonormal set

$$\{\boldsymbol{u}_1 = \frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle,\ \boldsymbol{u}_2 = \frac{1}{\sqrt{6}}\langle -2, 1, 1\rangle,\ \boldsymbol{u}_3 = \frac{1}{\sqrt{2}}\langle 0, -1, 1\rangle\}.$$

In other words, we'll write v in the new basis

Furthermore, there is a formula for the vector components,

$$v_1 = \frac{(\boldsymbol{v} \cdot \boldsymbol{u}_1)}{(\boldsymbol{u}_1 \cdot \boldsymbol{u}_1)}, \quad \cdots, \quad v_n = \frac{(\boldsymbol{v} \cdot \boldsymbol{u}_n)}{(\boldsymbol{u}_n \cdot \boldsymbol{u}_n)}.$$

$$v \cdot u_1 = \frac{3+2+1}{\sqrt{3}} = \frac{6}{\sqrt{3}}$$
$$v \cdot u_2 = \frac{-6+2+1}{\sqrt{6}} = \frac{-3}{\sqrt{6}}$$
$$v \cdot u_3 = \frac{0-2+1}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

This is an orthonormal set, so we'll just use the dot products to make the expansion

$$v = u_1\frac{6}{\sqrt{3}} + u_2\frac{-3}{\sqrt{6}} + u_3\frac{-1}{\sqrt{2}}$$

Question 3: *(10 points)* Prove the Cayley-Hamilton Theorem in the case of 2×2 matrices, that is, show that every 2×2 matrix A satisfies the following *matrix equation*,

$$A^2 - \operatorname{tr}(A)\, A + \det(A)\, I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Lets convert to the values of a generic matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Thus we have

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+a_1b \\ ac+dc & d^2+bc \end{bmatrix}$$
$$-\operatorname{tr}(A)A = -(a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\begin{bmatrix} a(a+d) & (a+d)b \\ (a+d)c & d(a+d) \end{bmatrix} = -\begin{bmatrix} a^2+ad & ab+db \\ ac+dc & da+d^2 \end{bmatrix}$$
$$\det(A)I = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

It is clear that summing these returns the empty 2x2 matrix. Thus this equality shows the Cayley-Hamilton Theorem is true

Properties of Determinants

Question 4: *(10 points)* Prove that $\det(AB) = \det(A)\det(B)$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

Again, we simply convert these to their matrix components

$$\det(A)\det(B) = (a_{11}a_{22} - a_{12}a_{21}) * (b_{11}b_{22} - b_{12}b_{21}) = a_{12}a_{21}b_{12}b_{21} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{11}a_{22}b_{11}b_{22}$$
$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$
$$\det(AB) = a_{12}a_{21}b_{12}b_{21} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{11}a_{22}b_{11}b_{22}$$

Thus $\det(AB) = \det(A)\det(B)$

Question 5: *(10 points)* Determine whether the equation $\det(A+B) = \det(A) + \det(B)$ is true or not. If it is true, prove it for all 2×2 matrices A and B; if it is not true, give an example.

I will show an example that this is false

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} B = A \det(A) = \det(B) = 1 * 1 = 1$$

$$A + B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

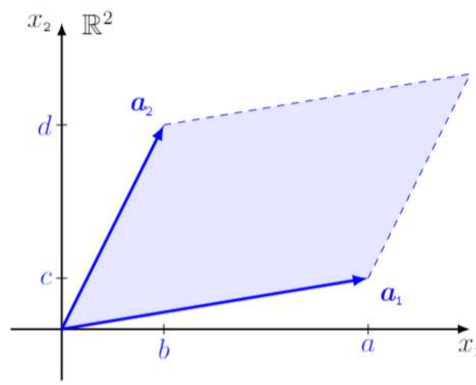
$$\det A + B = 2 * 2 = 4$$

Since $\det(A) + \det(B) = 2 \neq 4$, this property is not true.

Question 6: *(10 points)* Denote a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in terms of its column vectors as $A = \begin{bmatrix} \boldsymbol{a}_1, \boldsymbol{a}_2 \end{bmatrix}$. Suppose that the vectors $\boldsymbol{a}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\boldsymbol{a}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$ are given in the figure below. Use that picture to prove

$$\text{Area of the shaded parallelogram} = |\det(A)|.$$

Hint: Relate the parallelogram area with areas you can easily compute, such as triangle and rectangle areas.



Remember $\det(A) = ad - bc$

Let us consider half the selected paralelogram. It is a triangle with points (0,0), (a,c), and (b,d). The area of an arbitrary triangle is $\frac{1}{2}|x1(y2 - y3) + x2(y3 - y1) + x3(y1 - y2)|$. Thus this we assign our three coordinates to 1, 2, and 3 respectively, and the area is

$$\frac{1}{2}|0 * (c - d) + a(d - 0) + b(0 - c)|$$

and double that is the parallelogram area, which comes out to

$$|0 * (c - d) + a(d - 0) + b(0 - c)| = |ad - bc| = |\det(A)|$$

Thus the equality is proven true

Question 7: *(10 points)* Prove that for every invertible 2×2 matrix holds that $((A^{-1})^{-1}) = A$.

$$(A^{-1}) = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad \text{in the case that} \quad \det(A) \neq 0,$$

Lets take the inverse of A^{-1}

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$$\det(A^{-1}) = \frac{d}{ad-bc}\frac{a}{ad-bc} - \frac{-b}{ad-bc}\frac{-c}{ad-bc} = \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc}$$

$$(A^{-1})^{-1} = (ad-bc)\begin{bmatrix} \frac{a}{ad-bc} & \frac{b}{ad-bc} \\ \frac{c}{ad-bc} & \frac{d}{ad-bc} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Question 9: *(10 points)* Prove that every invertible 2×2 matrices A, B, satisfy $(AB)^{-1} = (B^{-1})(A^{-1})$.

Again, we'll simply show each value to correctly correspond

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$(AB)^{-1}$$

$$= \frac{1}{a_{12}a_{21}b_{12}b_{21} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{11}a_{22}b_{11}b_{22}}\begin{pmatrix} a_{21}b_{12} + a_{22}b_{22} & -a_{11}b_{12} - a_{12}b_{22} \\ -a_{21}b_{11} - a_{22}b_{21} & a_{11}b_{11} + a_{12}b_{21} \end{pmatrix}$$

$$B^{-1}A^{-1} = \left(\frac{1}{(a_{11}a_{22} - a_{12}a_{21})}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\right)\left(\frac{1}{(b_{11}b_{22} - b_{12}b_{21})}\begin{pmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{pmatrix}\right)$$

$$= \frac{1}{(a_{11}a_{22} - a_{12}a_{21})}\frac{1}{(b_{11}b_{22} - b_{12}b_{21})}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{pmatrix}$$

$$= \frac{1}{a_{12}a_{21}b_{12}b_{21} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{11}a_{22}b_{11}b_{22}}\begin{pmatrix} a_{21}b_{12} + a_{22}b_{22} & -a_{11}b_{12} - a_{12}b_{22} \\ -a_{21}b_{11} - a_{22}b_{21} & a_{11}b_{11} + a_{12}b_{21} \end{pmatrix}$$

Thus these are equivalent

Question 10: *(10 points)* Compute the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

This isn't too bad.

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda - 3 = 0$$

$$\lambda = 3, -1$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{cases} a + 2b = 3a \\ 2a + b = 3b \end{cases} \implies a = b$$

$$v_1 = (1, 1)$$

$$\begin{cases} a + 2b = -a \\ 2a + b = -b \end{cases} \implies a = -b$$

$$v_2 = (1, -1)$$