Deep Dive #5

Functions Defined by Differential Equations

A differential equation can be used to define new functions

The Characterization of functions γ and σ

We are now going to find the properties of two functions, γ and σ, solutions of two initial values problems.

Defintion. Let the function $\gamma(x)$ be the *unique solution* of the initial value problem

$$\gamma'' + \gamma = 0, \qquad \gamma(0) = 1, \quad \gamma'(0) = 0,$$

and let the function $\sigma(x)$ be the *unique solution* of the initial value problem

$$\sigma'' + \sigma = 0, \qquad \sigma(0) = 0, \quad \sigma'(0) = 1.$$

Question 1. *(2 points)* Show that the functions γ and σ are linearly independent — not proportional to each other.

Hints:

(a) We cannot use that γ is the Cosine function and σ is the Sine function, because this is what we want to prove at the end of this Dive.

(b) Recall the properties of the Wronskian of two functions which are solutions of linear differential equations.

Properties of σ and γ:

- Homogenous
- Constant coefficients
- Linear
- Defined for all initial data ($f(x) = 1$ and $f(x) = 0$ are continuous)
- $W_{\gamma\sigma}(0) = 1$

By Theorem 2.1.12, σ and γ aren't fundamental solutions of the of a SOLDE $L(y) = 0$, so they are linearly independent

Question 2. *(2 points)* Show that the function γ is even and the function σ is odd.

Hints:

(a) Recall that a function f is even iff $f(-x) = f(x)$, while a function g is odd iff $g(-x) = -g(x)$.

(b) We cannot use that γ is the Cosine function and σ is the Sine function, because this is what we want to prove at the end of this Dive.

(c) Find what initial value problem satisfy the functions $\hat{\gamma}(x) = \gamma(-x)$ and $\hat{\sigma}(x) = \sigma(-x)$. And recall the uniqueness results for initial value problems.

How do we do this? Define a function $f(x)$ that (1) has the same initial values as γ or σ, and (2) satisfies the orignal differential equation $f'' + f = 0$.

γ : Choose f s.t. $f(t) = \gamma(-t)$

(1) Show initial values are `equivalent

$$f(0) = \gamma(-0) = \gamma(0) = 1 \checkmark$$
$$f'(t) = \frac{d}{dt}\gamma(-t) = -\gamma'(-t)$$
$$f'(0) = -\gamma'(-0) = -\gamma'(0) = -0 \checkmark$$

(2) Show $f(t)$ satisfies $\gamma'' + \gamma = 0$

$$f''(t) = \frac{d}{dt} - \gamma'(-t) = \gamma''(-t)$$
$$\gamma''(0) + \gamma(0) = 0 \implies \gamma''(0) = -1$$
$$f''(0) = \gamma''(-0) = \gamma''(0) = -1$$
$$f''(0) + f(0) = -1 + 1 = 0 \checkmark$$

Since f has the same initial conditions as γ and agrees with the differential equation which γ is a solution of, f and γ are equivalent unique solutions by the existence uniqueness theorem

σ : Choose g s.t. $g(t) = -\sigma(-t)$

(1) Show initial values are equivalent

$$g(0) = \sigma(-0) = \sigma(0) = 0 \checkmark$$
$$g'(t) = \frac{d}{dt} - \sigma(-t) = \sigma'(-t)$$
$$g'(0) = \sigma'(-0) = \sigma'(0) = 1 \checkmark$$

(2) Show $g(t)$ satisfies $\sigma'' + \sigma = 0$

$$g''(t) = \frac{d}{dt} - \sigma'(-t) = \sigma''(-t)$$
$$\sigma''(0) + \sigma(0) = 0 \implies \sigma''(0) = 0$$
$$g''(0) = \sigma''(-0) = \sigma''(0) = 0$$
$$g''(0) + f(0) = 0 + 0 = 0 \checkmark$$

Since g has the same initial conditions as σ and agrees with the differential equation which σ is a solution of, f and σ are equivalent unique solutions by the existence uniqueness theorem

Question 3. *(2 points)* Prove the following relations between the functions γ and σ,

$$\gamma'(x) = -\sigma(x), \qquad \sigma'(x) = \gamma(x).$$

Hints:

(a) We cannot use that γ is the Cosine function and σ is the Sine function, because this is what we want to prove at the end of this Dive.

(b) Again, recall the uniqueness results for initial value problems.

Let us consider the initial values of $-\gamma'(x)$, hopefully they'll be the same as $\sigma(x)$

$$-\gamma'(0) = -0 = 0$$
$$\frac{d}{dx} - \gamma'(x) = -\gamma''(x) = -(-1) = 1$$

How fortunate! Now all we need is for it to satisfy the original differential eqation and we'll know it is the same as the unique solution $\sigma(x)$

$$\gamma = -\gamma'' \implies (\gamma)' = (-\gamma'')'$$
$$\frac{d^2}{dx^2}(-\gamma') + (-\gamma') = 0 \checkmark$$

Thus these equations are the same unique solution to the original differential equation

From this the next proof is quite easy. Take the derivative of our previous result

$\gamma''(x) = -\sigma'(x) = -\gamma(x) \implies \sigma'(x) = \gamma(x)$

Question 4. *(2 points)* Show that the functions γ and σ satisfy the Pythagoras' theorem,

$$\gamma^2(x) + \sigma^2(x) = 1 \qquad \text{for all} \quad x.$$

Hints:

(a) We cannot use that γ is the Cosine function and σ is the Sine function, because this is what we want to prove at the end of this Dive.

(b) Use the results in Question 3 to compute the Wronskian of γ and σ.

(c) Recall Abel's Theorem, which is about the differential equation satisfied by the Wronskian of two solutions to a second order differential equation.

Lets start by computing the Wronskian of γ and σ

$W_{\gamma\sigma} = \gamma\sigma' - \sigma\gamma' = \gamma\gamma + \sigma\sigma$

Since we computed in step 1 that $W(0) = 1$ and according to Abel's theorem W'=0 for SOLDEs with no f' component, it is a constant function $W(x) = 1$ and thus $\gamma^2(x) + \sigma^2(x) = 1\ \forall x$

Question 5. *(2 points)* Show that the power series expansion of the functions γ centered at $x = 0$ is

$$\gamma(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

Note: A similar calculation can be done for the function σ, the result is

$$\sigma(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!},$$

but you do not need to compute it.

Hints:

(a) We cannot use that γ is the Cosine function and σ is the Sine function, because this is what we want to prove at the end of this Dive.

(b) Use the results from the previous questions.

Okay lets do it.

We know $\gamma(0) = 1, \gamma'(0) = 0$, and $\gamma''(0) = -1$. We can take the derivative continuously.

$$\gamma(0) = -\gamma''(0) \implies \gamma'(0) = -\gamma'''(0)$$

Thus the sign of $\gamma^{(2n)}$ flips continuously, while $\gamma^{2n+1} = 0$. The taylor expansion of γ becomes

$$\gamma(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k!)}$$

done! yaya!!!