MTH 327h Honors Analysis I, Fall 2023 Homework 2

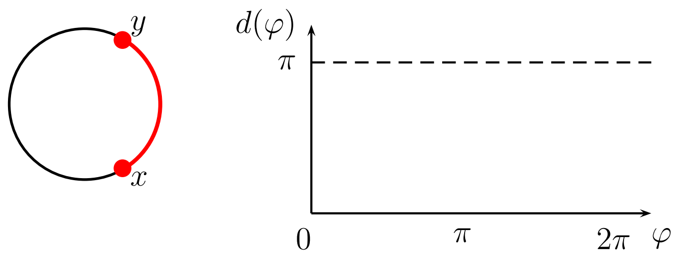
Problem 1 Which of the following formulas define the **distance** on $\mathbb{R}^1$:

(a) $d(x, y) = (x - y)^2$;

(b) $d(x, y) = |x - y|^{1/2}$;

(c) $d(x, y) = e^{|x-y|} - 1$.

Problem 2 Consider the set C^1—the unit circle. We parameterize points $x(\varphi)$ on the circle by the angle $\varphi \in [0, 2\pi]$ with $x(\varphi+2\pi) = x(\varphi)$. Define the distance $d(x, y)$ between two points x and y on the circle C^1 as the length of the shortest path along the circle between these two points. Draw the graph of the distance depending on the angle φ between points on the circle



Problem 3 Construct a bounded set having exactly three limit points.

Problem 4 Prove that any point of an open subset $e \subset \mathbb{R}$ is a limit point of E.

Problem 5 Let the **boundary** ∂E of a subset $E \subset \mathbb{R}$ be $\partial E = \overline{E} \backslash E^o$.

(a) Prove that ∂E is closed. [You may want to use that $A \backslash B = A \cap B^c$.]

(b) Construct a subset E with $\partial E = \{1/n, n \in \mathbb{N}\} \cup \{0\}$.

(c) Can a boundary of a subset E be $\partial E = \{1/n, n \in \mathbb{N}\}$? Explain.

Problem 6 Is it true that $\overline{\left(\overline{E}\right)^c} = (E^o)^c$? If so, prove it, if not, provide a counterexample.

Problem 7

(a) Prove that, for any set I of indices α, if $\bigcup_{\alpha \in I} A_\alpha$ is bounded above, then

$$\sup\Big(\bigcup_{\alpha \in I} A_\alpha\Big) = \sup\big\{\sup A_\alpha,\ \alpha \in I\big\}$$

(b) Is it true that if A_n are bounded, $\ldots A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_2 \subseteq A_1$, and $\bigcap_{n=1}^{\infty} A_n$ is nonempty then $\sup\Big(\bigcap_{n=1}^{\infty} A_\alpha\Big) = \inf\big\{\sup A_n,\ n \in \mathbb{N}\big\}$? If true, prove it, if not, provide a counterexample.

Problem 8* Prove that **any** subset of $\mathbb{R}$ containing only isolated points is at most countable.

[Hint: use the Lindelöf theorem on countable covering and construct a “clever” covering.]