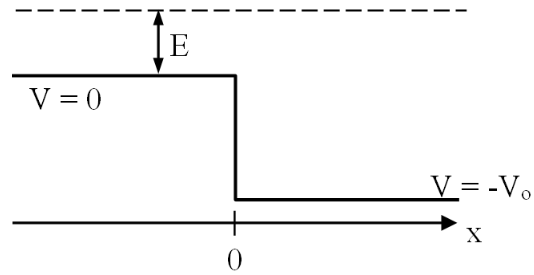
Physics 471 – Fall 2023

Homework #13 (the last one!) – due Wednesday, December 6

1. [6] Consider the "downstep" potential shown in the figure. A particle of mass m and energy E, incident from the left, strikes a potential energy drop-off of depth V_0:



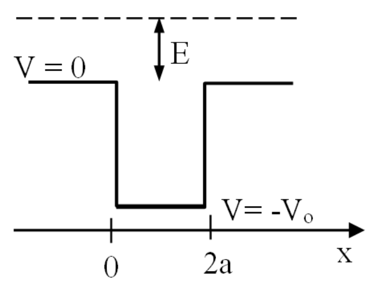
a) [1] In classical physics, consider a particle of mass m coming in from the left with given initial velocity v_i. Find the final velocity of the particle, v_f, after it passes the potential step, i.e. for $x > 0$.

b) [2] Now write down and solve the energy eigenvalue equation (also called the time-independent Schrodinger equation) for energy $E > 0$ in both regions and impose appropriate boundary conditions at $x = 0$. (Write down the boundary condition equations before you plug in the solutions so we can see what you are doing.)

c) [2] Interpret your solutions as a steady flux of particles, and assume there are no particles coming in from the right. Calculate the reflection coefficient R. How does this compare to the classical result? What happens when E is just above 0? When V_0 gets small? When $E >> V_0$? In which of these limits is the quantum result most different from the classical physics result?

d) [1] Plot (or sketch) R vs (E/V_0), like McIntyre does in Fig 6.15 or 6.20. Briefly, comment.

2. [5] A particle of mass m and kinetic energy $E > 0$ approaches (from the left) an abrupt potential drop of magnitude V_0, but then the well rises back up again, after a distance $2a$.



a) [3] Suppose $E = \frac{V_0}{3}$. What is the probability that the particle will reflect back in the 3 cases:

$$i)\ \frac{2mV_0}{\hbar^2}a^2 \ll 1, \quad ii)\ \frac{2mV_0}{\hbar^2}a^2 = \frac{\pi^2}{48}, \quad iii)\ \frac{2mV_0}{\hbar^2}a^2 = \frac{3\pi^2}{16}\ ?$$

Hint: McIntyre does *all the nasty algebra for you – feel free to use results from the book!*

b) [2] Consider two more cases: What if V_0 is fixed but E approaches 0? How about $E \gg V_0$? Briefly, discuss the physics of all these cases – can you make any physical sense of the results?

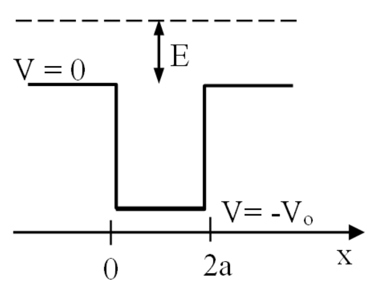
3. [4] **Tunneling**

a) [3] Use the result from the previous problem (McIntyre's equation (6.93)) but flip the sign of V_0 in that formula to change the dip to a bump. Now assume that $E < V_0$, so that you are solving for the "tunneling" case. Derive a simple relation connecting $\sin(ix)$ to $\sinh(x)$ to quickly derive T (and thus also $R = 1 - T$), verifying McIntyre Eq 6.105 and 6.106. (I will do this in class.)

b) [1] Consider an electron approaching a square barrier. Its initial energy is 0.5 eV, the barrier height is 1.0 eV, while the full width of the barrier is 5 Å. (Classically, this is a brick wall - the electron would just turn around. *NOTE: "full width" of 5 Å means a = 2.5 Å.)* What is the numerical quantum probability for the particle to make it to the other side of the barrier?

4. [5] In Figure 6.16, McIntyre shows a cartoon depicting transmission through a square-well potential as a coherent sum over an infinite number of trajectories that reflect and/or transmit from the front and back "walls" of the potential. He claims that the $T = 1$ "resonances" occur due to constructive interference between the transmitting trajectories. Let's find out if this just a cartoon, or if this approach can be used as a real tool for calculating the transmission.

a) [1] In part (c) of problem 1, you derived a formula for the reflection probability R from a down-step potential. Assuming you used similar notation to McIntyre (6.88), the reflection amplitude in that problem was $r_1 = B/A$, so that $R = |r_1|^2$. Calculate the transmission amplitude, $t_1 = C/A$. Now consider an up-step potential that is the mirror-image of the potential in problem 1. You can immediately calculate r_2 and t_2 for this potential just by switching k_1 and k_2 in your formulas.



b) [2] Assume that the system is "on resonance", i.e. equation (6.96) is satisfied. In that case, the particle acquires a phase shift of $2n\pi$ each time it bounces back and forth inside the well, which we can ignore. Using Figure 6.16 as a guide, calculate the full transmission amplitude, *t,* through the square-well potential as a sum over an infinite number of trajectories. Use t_1 or r_1, as appropriate, whenever a trajectory approaches a boundary of the well from the outside, and use t_2 or r_2 whenever a trajectory approaches either boundary from the inside. *Hint: the first term in your infinite sum should be* t_1t_2*, because the first trajectory transmits through both interfaces without any reflections.*

c) [1] If you did part (b) correctly, you should be able to evaluate the sum using the formula for a geometric series: $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$. You should obtain the on-resonance result $t = 1$.

d) [1] To check your answer, do the same thing as you did in parts (b) and (c) for the reflection coefficient, *r*. The first term in your series should be simply r_1. If you do this correctly, you will find that the sum of all the other terms in the series exactly cancels the first term, so you get the result $r = 0$.

Extra Credit: [2] The calculations above were valid only for the special case when $2k_2a = n\pi$, where k_2 is the wavevector inside the well region. For arbitrary values of k_2a, the wave function acquires a phase factor e^{2ik_2a} every time it traverses the well, so it acquires a phase factor e^{4ik_2a} every time it makes a round trip across the well and back. Put that phase factor into your calculation of the full transmission amplitude and evaluate the sum as before. Show that your answer is equivalent to the first line in McIntyre's equation (6.91) up to an overall phase factor.