Line (or path) integrals

$\vec{F} = y^3 \hat{x} - 2x^2 \hat{y}$ path: $y = x^2 + 1$ from $(0,1)$ to $(2,5)$

$d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$ (since path is only in xy-plane)

$\vec{F} \cdot d\vec{l} = (y^3 \hat{x} - 2x^2 \hat{y}) \cdot (dx\,\hat{x} + dy\,\hat{y}) = y^3\,dx - 2x^2\,dy$

We can choose x or y as our integration variable.
Let's see which case is easier

$y = x^2 + 1 \Rightarrow dy = 2x\,dx$ easy

$x = \sqrt{y-1} \Rightarrow dx = \frac{1}{2}(y-1)^{-1/2}\,dy$ looks less easy

Express all in terms of x.

$\vec{F} \cdot d\vec{l} = y^3\,dx - 2x^2\,dy = (x^2+1)^3\,dx - 2x^2\,2x\,dx$

$= (x^6 + 3x^4 - 4x^3 + 3x^2 + 1)\,dx$

$$\int_{x=0}^{x=2} \vec{F} \cdot d\vec{l} = \left[\frac{x^7}{7} + 3\frac{x^5}{5} - 4\frac{x^4}{4} + 3\frac{x^3}{3} + x \right] \Big|_0^2 = \frac{1102}{35}$$

Now, is $\vec{F}$ a conservative vector field?

Method 1: Choose another path such as the line from $(0,1)$ to $(2,5)$; i.e. $y = 2x + 1$ and see if the result of the integral is different to call it path-dependent $\rightarrow$ not conservative. (If done correctly, it will be accepted as correct answer).

Method 2 : (more elegant)

If $\vec{F}$ is path-independent, then $\vec{F}$ must be a gradient of some scalar function

① We can try to guess what that scalar function might be. (But that may not be always easy)

or

② Use $\vec{\nabla} \times (\vec{\nabla} T) = 0$

So if $\vec{F} = \vec{\nabla} T$, then $\vec{\nabla} \times \vec{F} = 0$ for all points

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^3 & -2x^2 & 0 \end{vmatrix} = \hat{x}(0) - \hat{y}(0) + \hat{z}(-4x - 3y^2)$$

$$= -(3y^2 + 4x)\,\hat{z} \neq 0 \text{ in general.}$$

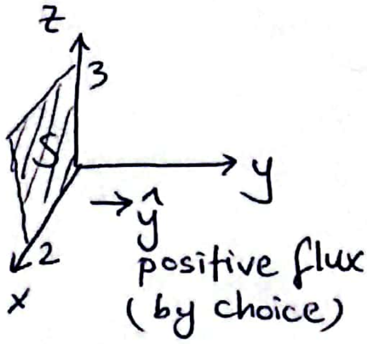
Since $\vec{\nabla} \times \vec{F} \neq 0$ in general, $\vec{F}$ cannot be represented as $\vec{F} = \vec{\nabla} T$. Therefore, the line integral is path dependent

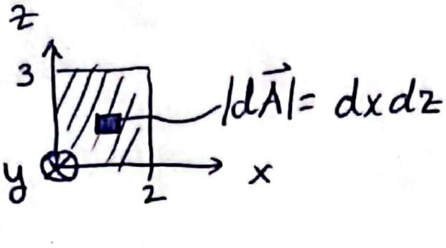
Surface integrals

$\vec{V} = 3zx\,\hat{x} + 5x\,\hat{y} + 2y\,\hat{z}$

$d\vec{A} = dx\,dz\,\hat{y}$ (because area here is on xz-plane)

↑ by choice





integration limits:

$0 \le z \le 3$

$0 \le x \le 2$

$y = 0$

$$\int_S \vec{V}\cdot d\vec{A} = \int_S (3zx\,\hat{x} + 5x\,\hat{y} + 2y\,\hat{z})\cdot(dx\,dz\,\hat{y}) = \int_S 5x\,dx\,dz$$

$$= \int_0^3 \left(\int_0^2 5x\,dx\right) dz = \int_0^3 \left(\frac{5x^2}{2}\Big|_0^2\right) dz = \int_0^3 10\,dz = 10z\Big|_0^3 = 30$$

The result turned out to be positive.

Because $d\vec{A}$ here is in $\hat{y}$ direction, only component of $\vec{V}$ that contributes the flux is its y-component; $5x$.

In the range $0 \le x \le 2$, its value is always positive (pointing in the same direction as $\hat{y}$; not opposite).

So we expect the flux through the surface to be positive.

Volume integrals

$$M = \int_V \rho \, d\tau$$

It is best to use spherical coordinates here :

$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$

<u>Uniform sphere</u> : $\rho = \rho_0 \rightarrow$ constant

$$M_{uni.} = \int_{sphere} \rho \, d\tau = \int_{sph.} \rho_0 \, d\tau = \rho_0 \underbrace{\left[\int_{sph} d\tau \right]}_{\text{This is just the volume of the sphere}} = \rho_0 \frac{4}{3} \pi R^3$$

<u>Non-uniform sphere</u> : $\rho = \frac{4\rho_0}{5R} r$

$$M_{non\text{-}uni.} = \int_{sph.} \frac{4\rho_0}{5R} r \left(r^2 \sin\theta \, dr \, d\theta \, d\phi \right)$$

This can actually be separated in a product of three integrals

$$M_{non\text{-}uni.} = \frac{4\rho_0}{5R} \left[\int_0^R r^3 dr \right] \left[\int_0^\pi \sin\theta \, d\theta \right] \left[\int_0^{2\pi} d\phi \right]$$

$$= \frac{4\rho_0}{5R} \left[\frac{r^4}{4} \Big|_0^R \right] \underbrace{\left[(-\cos\theta) \Big|_0^\pi \right]}_{2} \underbrace{\left[\phi \Big|_0^{2\pi} \right]}_{2\pi} = \frac{4\rho_0}{5R} \frac{R^4}{4} \cdot 4\pi$$

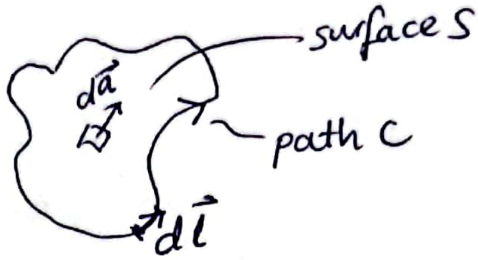
$$= \rho_0 \frac{4}{5} \pi R^3$$

$$M_{uniform} > M_{non\text{-}uniform}.$$

$\boxed{\text{Some vector proofs}}$

① Given $\oint \vec{\nabla} T \cdot dl = 0$ and $\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint_C \vec{v} \cdot d\vec{l}$,

show $\vec{\nabla} \times \vec{\nabla} T = 0$

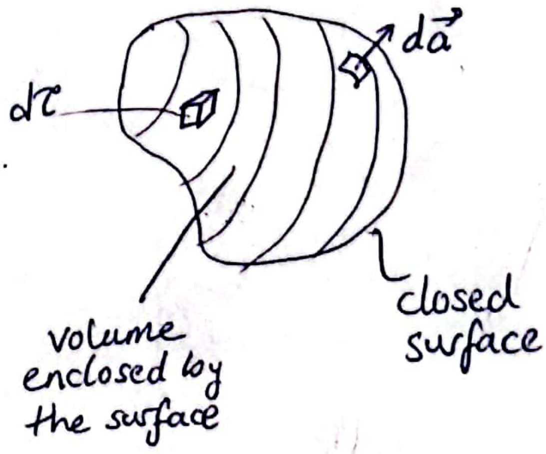


$$0 = \oint_C \vec{\nabla} T \cdot d\vec{l} \underset{\text{Stokes'}}{=} \int_S \underbrace{(\vec{\nabla} \times \vec{\nabla} T)} \cdot d\vec{a}$$

If this is true for any surface bounded by a closed loop, then $\rightarrow \vec{\nabla} \times \vec{\nabla} T = 0$ everywhere

② Given $\oint_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = 0$ and $\int \vec{\nabla} \cdot \vec{v} \, d\tau = \oint_S \vec{v} \cdot d\vec{a}$,

show $\nabla \cdot (\vec{\nabla} \times \vec{v}) = 0$



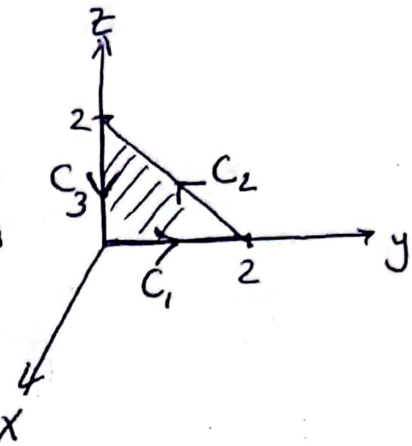
$$0 = \oint_S (\nabla \times \vec{v}) \cdot d\vec{a} = \int \underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v})} \, d\tau$$

If the volume integral is zero for any volume bounded by a closed surface, then $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$ everywhere

Test Stokes' theorem

$$\vec{v} = xy\,\hat{x} + 2yz\,\hat{y} + 3zx\,\hat{z}$$

$$\oint \vec{v}\cdot d\vec{l} = \int_{C_1} \vec{v}\cdot d\vec{l}_1 + \int_{C_2} \vec{v}\cdot d\vec{l}_2 + \int_{C_3} \vec{v}\cdot d\vec{l}_3$$



where
$$d\vec{l}_1 = dy\,\hat{y}$$
$$d\vec{l}_2 = dy\,\hat{y} + dz\,\hat{z}$$
$$d\vec{l}_3 = dz\,\hat{z}$$

$$\int_{C_1} \vec{v}\cdot d\vec{l}_1 = \int_0^2 2yz\,dy = 0$$

$z=0$ on C_1

$$\int_{C_2} \vec{v}\cdot d\vec{l}_2 = \int_{C_2} (2yz\,dy + 3zx\,dz)$$

On C_2, we have $z = 2-y$. Hence $dz = -dy$

Substituting in the integral above:

$$\int_{C_2} \vec{v}\cdot d\vec{l}_2 = \int \left(2y(2-y)\,dy + 3(2-y)\,x(-dy)\right)$$

$x=0$ on C_2

$$= \int_2^0 (4y-2y^2)\,dy = \left(2y^2 - \frac{2y^3}{3}\right)\Big|_2^0 = -\frac{8}{3}$$

$$\int_{C_3} \vec{v}\cdot d\vec{l}_3 = \int_2^0 3zx\,dz = 0$$

$x=0$ on C_3

Hence $\oint \vec{v}\cdot d\vec{l} = -8/3$

Let's now evaluate the surface integral.

$$\int_S (\vec{\nabla} \times \vec{V}) \cdot d\vec{a}$$

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2yz & 3zx \end{vmatrix} = -2y\,\hat{x} - 3z\,\hat{y} - x\,\hat{z}$$

$$d\vec{a} = dy\,dz\,\hat{x}$$

$$\int_S (-2y\hat{x} - 3z\,\hat{y} - x\hat{z}) \cdot (dy\,dz\,\hat{x}) = \int_S -2y\,dy\,dz$$

$$= -2\int_0^2 \left[\int_0^{2-z} y\,dy\right] dz = -2\int_0^2 \left(\frac{y^2}{2}\Big|_0^{2-z}\right) dz = -\cancel{2}\int_0^2 \frac{(2-z)^2}{\cancel{2}}\,dz.$$

$$= -\int_0^2 (4-4z+z^2)\,dz = \left(-4z+2z^2-\frac{z^3}{3}\right)\Big|_0^2 = -\frac{8}{3}$$

So we have $\quad \oint \vec{V} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{V}) \cdot d\vec{a}$