# Power Series Solutions

*We use general power series to solve differential equations*

## Objectives

We generalize the Taylor power series method to general power series expansions to find solutions of differential equations. We learn this method solving both first and second order linear differential equations having constant coefficients. Then, we use this method to solve the Legendre equation, which is a second order linear equation having *variable* coefficients.

## Introduction

In Section 2.3 we solved second order, linear, homogeneous, *constant coefficients* equations

$$y'' + a_1\, y' + a_0\, y = 0,$$

by guessing that the solutions have the form $y(x) = e^{rx}$ and finding the appropriate values for the exponent $r$. Notice that we are using $x$ for the independent variable instead of $t$, in this way we agree with most of the literature on this subject. Once we have two fundamental solutions, we then have all the solutions to the homogenous equation. However, this guessing method is not useful with *variable coefficient* equations,

$$y'' + a_1(x)\, y' + a_0(x)\, y = 0,$$

because the fundamental solutions are too difficult to guess. In the case that the solution $y(x)$ is smooth and has a Taylor series expansion around a point $x_0$ where $y(x_0)$ is defined, then we can write $y(x)$ as

$$y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^n.$$

The idea of the power series method is to put the expression above into the differential equation and then find the values of the coefficients $a_n$. The Power Series method can be summarized as follows:

(1) Choose an $x_0$ and write the solution $y$ as a power series expansion centered at a point $x_0$,

$$y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^n.$$

(2) Introduce the power series expansion above into the differential equation and find a *recurrence relation*—an equation where the coefficient $a_n$ is related to $a_{n-1}$ (and possibly $a_{n-2}$).

(3) Solve the recurrence relation—find $a_n$ in terms of $a_0$ (and possibly $a_1$).

(4) If possible, add up the resulting power series for the solution $y(x)$.

## Further Reading

Students may need to read Section 4.1, Solutions Near Regular Points, in the textbook.

## First Order Equations with Constant Coefficients

The problem below involves a first order, constant coefficient, equation. We use this simple equation to practice the Power Series Method. But recall, this method is useful to solve variable coefficient equations.

**Question 1:** Use a power series around the point $x_0 = 0$ to find all solutions $y$ of the equation

$$y' + c\,y = 0, \qquad c \in \mathbb{R}.$$

**(1a)** *(10 points)* Find the recurrence relation relating the coefficient $a_n$ with $a_{n-1}$.

**(1b)** *(10 points)* Solve the recurrence relation, that is, find $a_n$ in terms of $a_0$.

**(1c)** *(10 points)* Write the solution $y$ as a power series one multiplied by $a_0$. Then add the power series expression.

**Note:** Using the integrating factor method we know that the solution is $y(x) = a_0\, e^{-cx}$, with $a_0 \in \mathbb{R}$. We want to recover this solution using the Power Series Method.

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## Second Order Equations with Constant Coefficients

The problem below involves a second order, constant coefficient, equation. As above, we use this simple equation to practice the Power Series Method.

**Question 2:** Use a power series around the point $x_0 = 0$ to find all solutions $y$ of the equation

$$y'' + y = 0.$$

**(2a)** *(10 points)* Find the recurrence relation relating the coefficient $a_n$ with $a_{n-2}$.

**(2b)** *(10 points)* Solve the recurrence relation, which in this case means to find $a_n$ in terms of $a_0$ for $n$ even, and $a_n$ in terms of $a_1$ for $n$ odd.

**(2c)** *(10 points)* Write the solution $y$ as a combination of two power series, one multiplied by $a_1$, the other multiplied by $a_0$. Then add both power series expressions.

**Note:** Guessing the fundamental solutions we know that the solution is $y(x) = a_0 \cos(x) + a_1 \sin(x)$, with $a_0, a_1 \in \mathbb{R}$. We want to recover these solutions using the Power Series Method.

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## Second Order Equations with Variable Coefficients

In the previous two cases we used power series to solve a differential equation we already knew how to solve with other, simpler, methods. Now, however, we use the power series method to solve a differential equation that cannot be solved with the methods we studied in the previous chapters. The equation is called the Legendre equation. It appears as part of a more complicated set of equations that need to be solved when we try to find the static electric field in situations with spherical symmetry.

**Question 3:** Find all solutions of the Legendre equation

$$(1-x^2)\,y'' - 2x\,y' + l(l+1)\,y = 0,$$

where $l$ is any real constant, using power series centered at $x_0 = 0$.

**(3a)** *(10 points)* Find the recurrence relation relating the coefficient $a_n$ with $a_{n-2}$.

**(3b)** *(10 points)* Solve the recurrence relation to find $a_4$, $a_2$ in terms of $a_0$ and to find $a_5$, $a_3$ in terms of $a_1$.

**(3c)** *(10 points)* If we write the solution $y$ as

$$y(x) = a_0\,y_0(x) + a_1\,y_1(x),$$

then find the first three terms of the series expansions of $y_0$ and $y_1$. These functions $y_0$ and $y_1$ are called Legendre functions.

**(3d)** *(10 points)* The powers series that define the Legendre functions have infinitely many terms when the constant $l$ is not an integer. But when $l$ is an integer, either $y_0$ or $y_1$ have a finite number of nonzero terms—they terminate—and they are just polynomials. The collection of all polynomials—appropriately normalized—are called the Legendre polynomials. Find the first four Legendre polynomials, $P_i$, for $i = 0, 1, 2, 3$, which are defined as follows,

$$\begin{aligned} P_0(x) &= y_0(x) && \text{for} \quad l = 0 \\ P_1(x) &= y_1(x) && \text{for} \quad l = 1 \\ P_2(x) &= -\frac{1}{2}\,y_0(x) && \text{for} \quad l = 2 \\ P_3(x) &= -\frac{3}{2}\,y_1(x) && \text{for} \quad l = 3. \end{aligned}$$

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