Problem 1

Note: Linear maps require additivity $T(u+v) = Tu + Tv : u, v \in V$ and homogeneity $T(\lambda v) = \lambda(Tv) : \lambda \in F, v \in V$

(i) Define $T : \mathcal{P}_4 \to \mathcal{P}_4$ by

$$(Tp)(x) = x^2 p''(x).$$

Let $p, q \in \mathcal{P}_4$. Show that $T(p+q)(x) = (Tp)(x) + (Tq)(x)$ and $T(\lambda p)(x) = x^2(\lambda p)(x) = \lambda(Tp)(x)$

Pf. Show T is a linear map.

$$T(p+q)(x) = x^2(p+q)''(x)$$

by definition of T

$$x^2(p+q)''(x) = x^2(p''+q'')(x) = x^2(p'')(x) + x^2(q'')(x)$$

by definition of differentiation, distribution in $\mathcal{P}_4$.

$$x^2(p'')(x) + x^2(q'')(x) = (Tp)(x) + (Tq)(x)$$

by definition of T.

So $T(p+q)(x) = (Tp)(x) + (Tq)(x)$ and thus T has additivity

$$T(\lambda p)(x) = x^2(\lambda p)''(x)$$

by definition of T

$$x^2(\lambda p)''(x) = \lambda x^2 p''(x) = \lambda(Tp)(x)$$

by definition of T, differentiation

So $T(\lambda p)(x) = \lambda(Tp)(x)$ and thus T has homogeneity.

Since T has additivity and homogeneity, T is linear $\square$

(ii) Define $S : \mathcal{P}_4 \to \mathcal{P}_4$ by

$$(Sp)(x) = p''(x) + x^2$$

Pf. Show that S is not a linear map by example. Let $p(x) = x$ and $q(x) = x$

$$(Sp)(x) = x'' + x^2 = 0 + x^2 = x^2$$

By definition of S, differentiation, additive identity in $\mathcal{P}_4$

$$(Sq)(x) = x'' + x^2 = 0 + x^2 = x^2$$

By definition of S, differentiation, additive identity in $\mathcal{P}_4$

$$(Sp)(x) + (Sq)(x) = x^2 + x^2 = 2x^2$$

However

$$(S(p+q))(x) = (x+x)'' + x^2 = 0 + x^2 = x^2$$

By definition of S, differentiation, additive identity in $\mathcal{P}_4$, and

$$x^2 \neq 2x^2$$

So $(S(p+q))(x) \neq (Sp)(x) + (Sq)(x)$, and S does not have additivity, so S is not linear. $\square$

(iii) $H\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ x_1 \\ 5x_3 + x_4 \end{pmatrix}$ Pf. Show H does not have additivity and is thus not linear through example.

Let $\vec{p} \in \mathbb{R}^4 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ and $\vec{q} \in \mathbb{R}^4 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

$$H\vec{p} = H\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1*1\\1\\5*1+1\end{pmatrix} = \begin{pmatrix}1\\1\\6\end{pmatrix}$$

$$H\vec{q} = H\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1*1\\1\\5*1+1\end{pmatrix} = \begin{pmatrix}1\\1\\6\end{pmatrix}$$

$$H(\vec{p}) + H(\vec{q}) = \begin{pmatrix}1\\1\\6\end{pmatrix} + \begin{pmatrix}1\\1\\6\end{pmatrix} = \begin{pmatrix}1+1\\1+1\\6+6\end{pmatrix} = \begin{pmatrix}2\\2\\12\end{pmatrix}$$

$$H(\vec{p}+\vec{q}) = H\begin{pmatrix}1+1\\1+1\\1+1\\1+1\end{pmatrix} = H\begin{pmatrix}2\\2\\2\\2\end{pmatrix} = \begin{pmatrix}2*2\\2\\5(2)+2\end{pmatrix} = \begin{pmatrix}4\\2\\12\end{pmatrix}$$

However $\begin{pmatrix}4\\2\\12\end{pmatrix} \neq \begin{pmatrix}2\\2\\12\end{pmatrix}$, so $H\vec{p} + H\vec{q} \neq H(\vec{p}+\vec{q})$, so H does not have additivity and is not linear. □

(iv) Pf. Show I is not linear since I does not have homogeneity.

Let $\vec{o} \in \mathbb{R}^4$ be the zero vector $\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ and $\lambda \in \mathbb{R} = 0$

$$H\vec{o} = \lambda H\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0-3*0\\0\\0+7\\5*0+0\end{pmatrix} = \begin{pmatrix}0\\0\\7\\0\end{pmatrix}$$

Note that

$$H(\lambda\vec{o}) = H(0\vec{o}) = H\vec{o}$$

Since any vector multiplied by zero is zero

$$\lambda H\vec{o} = \lambda H\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \lambda\begin{pmatrix}0-3*0\\0\\0+7\\5*0+0\end{pmatrix} = 0*\begin{pmatrix}0\\0\\7\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Since $\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \neq \begin{pmatrix}0\\0\\7\\0\end{pmatrix}$, $\lambda H\vec{o} \neq H(\lambda\vec{o})$, H is not homogeneous and thus not linear. □

(v) Pf. Show that J has both additivity and homogeneity, and is thus linear.

Problem 1 continued on next page...

Let $\vec{v} \in \mathbb{R}^4 = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$ Additivity.

$$J(\vec{v} + \vec{w}) = J \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \\ v_4 + w_4 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 - 3(v_2 + w_2) \\ v_1 + w_1 \\ 5(v_3 + w_3) + v_4 + w_4 \end{pmatrix}$$

$$J\vec{v} = J \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} v_1 - 3v_2 \\ v_1 \\ 5v_3 + v_4 \end{pmatrix}$$

$$J\vec{w} = J \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} w_1 - 3w_2 \\ w_1 \\ 5w_3 + w_4 \end{pmatrix}$$

$$J\vec{w} + J\vec{v} = \begin{pmatrix} w_1 - 3w_2 \\ w_1 \\ 5w_3 + w_4 \end{pmatrix} + \begin{pmatrix} v_1 - 3v_2 \\ v_1 \\ 5v_3 + v_4 \end{pmatrix} = \begin{pmatrix} w_1 - 3w_2 + v_1 - 3v_2 \\ w_1 + w2 \\ 5w_3 + w_4 + 5v_3 + w_4 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 - 3(v_2 + w_2) \\ v_1 + w_1 \\ 5(v_3 + w_3) + v_4 + w_4 \end{pmatrix}$$

So $J\vec{w} + J\vec{v} = J\vec{w}\vec{v}$ and J has additivity.
Homogeneity.

$$\lambda J\vec{v} = \lambda \begin{pmatrix} v_1 - 3v_2 \\ v_1 \\ 5v_3 + v_4 \end{pmatrix} = \begin{pmatrix} \lambda v_1 - \lambda 3v_2 \\ \lambda v_1 \\ \lambda 5v_3 + \lambda v_4 \end{pmatrix}$$

$$J(\lambda\vec{v}) = J(\lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}) = J \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \\ \lambda v_4 \end{pmatrix} = \begin{pmatrix} \lambda v_1 - 3\lambda v_2 \\ \lambda v_1 \\ 5\lambda v_3 + \lambda v_4 \end{pmatrix} = \begin{pmatrix} \lambda v_1 - \lambda 3v_2 \\ \lambda v_1 \\ \lambda 5v_3 + \lambda v_4 \end{pmatrix}$$