Atomic hydrogen and the polarization model

① From Griffiths Table 4.1 : $\frac{\alpha}{4\pi\epsilon_0} = 0.667 \times 10^{-30}\, m^3$

The result of Griffiths Example 4.1 says $\alpha = 4\pi\epsilon_0 a^3$

So, $a^3 = \frac{\alpha}{4\pi\epsilon_0} = 0.667 \times 10^{-30}\, m^3 \Rightarrow \boxed{a \simeq 8.7 \times 10^{-11}\, m}$

This compares well to Bohr radius, $5.29 \times 10^{-11}\, m = a_0$

② $E_{external} = \frac{100\, V}{1\, mm} = 10^5\, V/m$

From example 4.1 : $p = qd = 4\pi\epsilon_0 a^3 E_{ext}$

$$d = \frac{4\pi\epsilon_0 a^3}{q} E_{ext}$$

$$= \frac{4\pi \times 8.85 \times 10^{-12} \times (8.7 \times 10^{-11})^3}{1.6 \times 10^{-19}} \cdot 10^5$$

$$\simeq \boxed{4.6 \times 10^{-17}\, m} \sim 0.05\, fm \quad \text{very small !}$$

$$\frac{d}{a} = \frac{4.6 \times 10^{-17}\, m}{8.7 \times 10^{-11}\, m} \simeq 5.3 \times 10^{-7}$$

③ To ionize, we expect $d \approx a \Rightarrow E_{ext} = \frac{q\,a}{4\pi\epsilon_0 a^3} = \frac{q}{4\pi\epsilon_0 a^2}$

$$E_{ext} = \frac{1.6 \times 10^{-19}}{4\pi(8.85 \times 10^{-12})(8.7 \times 10^{-11})^2} = \boxed{1.9 \times 10^{11}\, V/m}$$

This is much higher than break down $\vec{E}$ of air $\sim 3 \times 10^6 \frac{V}{m}$
So, our model is missing some aspects.

Polarized sphere of charge

(1) $\vec{P} = P_0 \vec{r} = P_0 r \hat{r}$

$$\sigma_b = \vec{P} \cdot \hat{n}\Big|_{\text{at surface}} = P_0 r \hat{r} \cdot \hat{r}\Big|_{r=a} = \boxed{P_0 a}$$

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r) = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_0 r) = \boxed{-3P_0}$$

(2) Polarization is spherically symmetric. We can use Gauss' law.

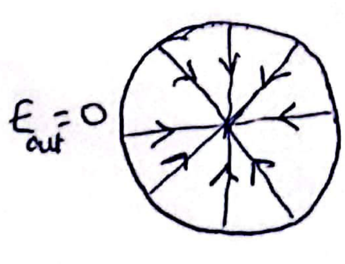
$\underline{r<a:}$ $$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enclosed}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_V \rho_b \, d\tau$$

$$E \cdot 4\pi r^2 = \frac{1}{\epsilon_0}\int_V (-3P_0)\, d\tau = \frac{1}{\epsilon_0}(-3P_0)\frac{4}{3}\pi r^3$$

$$\boxed{\vec{E}_{in} = -\frac{P_0 r}{\epsilon_0}\hat{r}}$$

$r>a:$ $$Q_{encl.} = \int_V \rho_b \, d\tau + \int_S \sigma_b \, da = -3P_0 \cdot \frac{4}{3}\pi a^3 + P_0 a \cdot 4\pi a^2 = 0$$

$$\Rightarrow \boxed{E = 0 \text{ outside.}}$$



Bound charges and the D-field I

① $\vec{P} = ks\hat{s}$

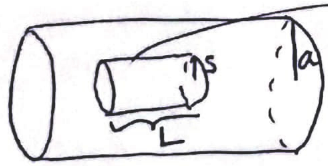
$$\sigma_b = \vec{P}\cdot\hat{n}\Big|_{\text{at surface}} = ks\,\hat{s}\cdot\hat{s}\Big|_{s=a} = \boxed{ka}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{1}{s}\frac{\partial}{\partial s}(sP_s) = -\frac{1}{s}\frac{\partial}{\partial s}(s\,ks) = \boxed{-2k}$$

$$[P] = \frac{C}{m^2} = [k][s] = [k]\cdot m \Rightarrow \boxed{[k] = \frac{C}{m^3}}$$

$$[\sigma_b] = [k][a] = \frac{C}{m^3}\cdot m = \frac{C}{m^2} \checkmark$$

$$[\rho_b] = [k] = \frac{C}{m^3} \checkmark$$

② 

Gaussian cylinder of radius s and height L

$\underline{s<a}$: $\oint \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\int_V \rho_b\, d\tau$

$$E\cdot 2\pi s L = \frac{1}{\epsilon_0}(-2k)\cdot \pi s^2 L \Rightarrow \boxed{\vec{E}_{in} = -\frac{ks}{\epsilon_0}\hat{s}}$$

$s>a$: $\oint \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\left[\int \rho_b\, d\tau + \int \sigma_b\, da\right]$

$$E\cdot 2\pi s L = \frac{1}{\epsilon_0}\left[(-2k)\cdot \pi a^2 L + (ka)\cdot 2\pi a L\right] = 0$$

$$\boxed{\vec{E}_{out} = 0}$$

3 $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

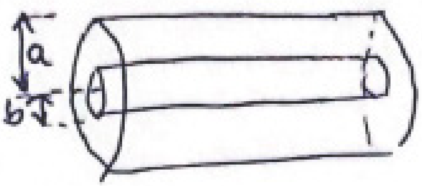
$$\vec{D}_{inside} = \epsilon_0 \vec{E}_{in} + \vec{P} = \epsilon_0 \left(-\frac{ks}{\epsilon_0}\hat{s}\right) + ks\hat{s} = 0$$

Makes sense because $\oint \vec{D}_{in} \cdot d\vec{a} = \underbrace{Q_{free}}_{0} = 0 \rightarrow \vec{D}_{in} = 0$ ✓

$$\vec{D}_{outside} = \epsilon_0 \vec{E}_{out} + \underbrace{\vec{P}}_{0 \text{ b/c no medium}} = 0 + 0 = 0$$

Also makes sense because $\oint \vec{D}_{out} \cdot d\vec{a} = Q_{free} = 0 \rightarrow \vec{D}_{out} = 0$ ✓

Bound charges and the D-field II

① 

$\vec{P} = k\hat{s}$ for $b < s < a$

For $s<b$ and $s>a$, we have vacuum. So, $\vec{P} = 0$ in those regions.

$\sigma_b = \vec{P}\cdot\hat{n}\Big|_{\text{at surface}}$: For outer surface at $s=a$: $\sigma_{b_{out}} = k\hat{s}\cdot\hat{s}\Big|_{s=a} = \underline{\underline{k}}$

For inner surface at $s=b$: $\sigma_{b_{in}} = k\hat{s}\cdot\underbrace{(-\hat{s})}_{\hat{n}}\Big|_{s=b} = \underline{\underline{-k}}$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{1}{s}\frac{\partial}{\partial s}(s P_s) = -\frac{1}{s}\frac{\partial}{\partial s}(s\cdot k) = \underline{\underline{-\frac{k}{s}}}$$

② $\underline{s<b}$: $\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \rightarrow 0 \Rightarrow \boxed{\vec{E} = 0 \text{ for } s<b}$

$\underline{b<s<a}$ $\oint \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\left[\int_V \rho_b \, d\tau + \int_S \sigma_b \, da\right]$

$$E\cdot 2\pi s L = \frac{1}{\epsilon_0}\left[2\pi L \int_b^s -\frac{k}{s'}\cdot s' ds' + (-k)\, 2\pi b L\right]$$

$$E \cdot 2\pi s L = \frac{1}{\epsilon_0}\left[-2\pi L k(s-b) - 2\pi L k b\right] = -2\pi k L s$$

$$\boxed{\vec{E} = -\frac{k}{\epsilon_0}\hat{s} \text{ for } b<a<a}$$

$s>b$: $\oint \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \left[\underbrace{\int \sigma_{b_{in}} da} + \int \rho_b d\tau + \underbrace{\int \sigma_{b_{out}} da} \right]$

$$= \frac{1}{\epsilon_0} \left[-k\, 2\pi b L + k\, 2\pi a L + 2\pi L \int_b^a \left(-\frac{k}{s'}\right) s' ds' \right]$$

$$= \frac{1}{\epsilon_0} \left[2\pi k L (a-b) + 2\pi L (-k)(a-b) \right] = 0$$

$\Rightarrow \boxed{\vec{E} = 0 \text{ for } s>b}$

③ $\oint \vec{D} \cdot d\vec{a} = Q_{free} = 0 \Rightarrow \vec{D} = 0$ everywhere

$s<b$: $\vec{P} = 0 \Rightarrow \vec{E} = \dfrac{\vec{D} - \vec{P}}{\epsilon_0} = 0$ ✓

$b<s<a$: $\vec{P} = k\hat{s} \Rightarrow \vec{E} = \dfrac{\vec{D} - \vec{P}}{\epsilon_0} = -\dfrac{k}{\epsilon_0} \hat{s}$ ✓

$s>a$: $P = 0 \Rightarrow E = \dfrac{\vec{D} - \vec{P}}{\epsilon_0} = 0$ ✓