Physics 471 - Homework 5 Solutions

1. $S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$S_x S_y \doteq \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$S_y S_x = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$[S_x, S_y] \doteq S_x S_y - S_y S_x \doteq \frac{\hbar^2}{4}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right] \doteq \frac{\hbar^2}{4}\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

$$= \frac{\hbar^2}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \doteq i\hbar \cdot \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \doteq i\hbar S_z \qquad \checkmark$$

2. $|\psi\rangle = a|+\rangle + b|-\rangle$

a) $\langle \hat{S}_z \rangle \equiv \langle \psi | \hat{S}_z | \psi \rangle = \begin{pmatrix} a^* & b^* \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

$$= \frac{\hbar}{2} \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} a \\ -b \end{pmatrix} = \underline{\underline{\frac{\hbar}{2}\left(|a|^2 - |b|^2\right)}}$$

$$\Delta S_z = \sqrt{\langle \hat{S}_z^2 \rangle - \langle \hat{S}_z \rangle^2}$$

$$\hat{S}_z^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} \cdot \hat{I} \leftarrow \text{identity operator}$$

$$\langle S_z^2 \rangle = \frac{\hbar^2}{4} \langle \psi | \hat{I} | \psi \rangle = \frac{\hbar^2}{4} \langle \psi | \psi \rangle = \frac{\hbar^2}{4}$$

$$\Delta S_z = \sqrt{\frac{\hbar^2}{4} - \left[\frac{\hbar}{2}\left(|a|^2 - |b|^2\right)\right]^2} = \underline{\underline{\frac{\hbar}{2}\sqrt{1 - \left(|a|^2 - |b|^2\right)^2}}}$$

This answer is fine, but it can be simplified further:

$$\left(|a|^2 - |b|^2\right)^2 = |a|^4 - 2|a|^2|b|^2 + |b|^4$$

$$= |a|^4 + 2|a|^2|b|^2 + |b|^4 - 4|a|^2|b|^2$$

$$= \left(|a|^2 + |b|^2\right)^2 - 4|a|^2|b|^2 = 1 - 4|a|^2|b|^2$$

So $\Delta S_z = \frac{\hbar}{2}\sqrt{1 - \left(1 - 4|a|^2|b|^2\right)} = \frac{\hbar}{2}\sqrt{4|a|^2|b|^2}$

$$\Delta S_z = \underline{\underline{\hbar\,|a|\,|b|}}$$

2. b) Using either form of ΔS_z from part a), you can see that $\Delta S_z = 0$ if either:

i) $|a| = 1$, $|b| = 0 \Rightarrow |\psi\rangle = |+\rangle$, $\langle \hat{S}_z \rangle = \frac{\hbar}{2}$

or ii) $|a| = 0$, $|b| = 1 \Rightarrow |\psi\rangle = |-\rangle$, $\langle \hat{S}_z \rangle = -\frac{\hbar}{2}$

Those two states have the minimum uncertainty in $\hat{S}_z$.

The maximum uncertainty of $\hat{S}_z$ occurs if $\left(|a|^2 - |b|^2\right)^2 = 0$, so $|a|^2 = |b|^2 = \frac{1}{2}$

That is the case for $|\psi\rangle = |+\rangle_x, |-\rangle_x, |+\rangle_y, |-\rangle_y$, or any other state that "points" somewhere in the x-y plane.

Let's use $|\psi\rangle = |+\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$, so $a = b = \frac{1}{\sqrt{2}}$

Then $\langle \hat{S}_z \rangle = \frac{\hbar}{2}\left(|a|^2 - |b|^2\right) = 0$

and $\Delta S_z = \frac{\hbar}{2}\sqrt{1 - \left(|a|^2 - |b|^2\right)^2} = \frac{\hbar}{2}$

3. a) $|\psi\rangle = |-\rangle$

From previous problem, with $a=0$, $b=1$, we have

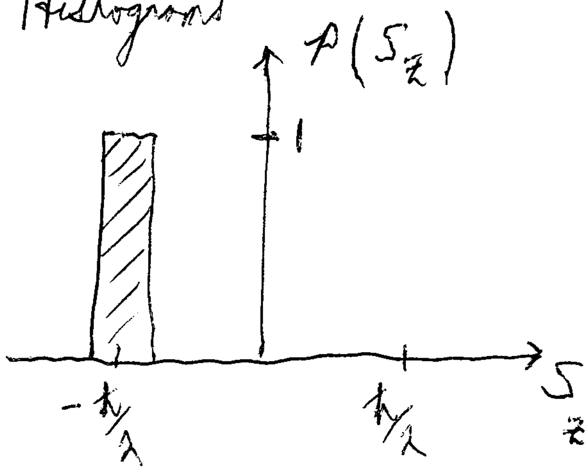
$$\langle \hat{S}_z \rangle = -\frac{\hbar}{2} \qquad \Delta S_z = 0$$

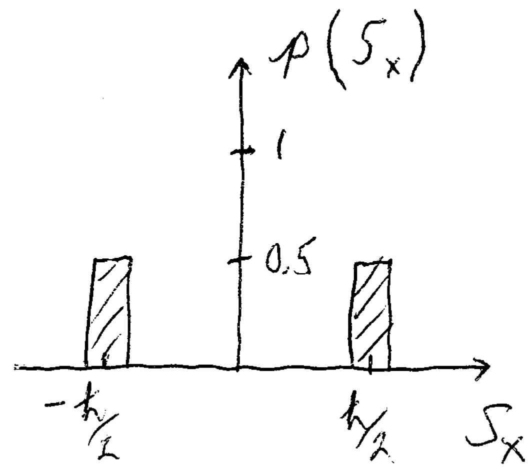
$$\langle \hat{S}_x \rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$\hat{S}_x^2 = \frac{\hbar^2}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4} \hat{I} \quad \Rightarrow \langle \hat{S}_x^2 \rangle = \frac{\hbar^2}{4}$$

$$\Delta S_x = \sqrt{\langle \hat{S}_x^2 \rangle - \langle \hat{S}_x \rangle^2} = \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2}$$

Histograms

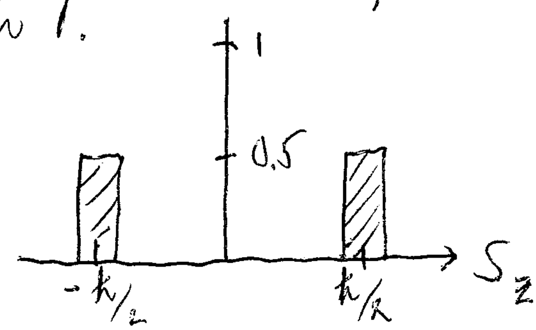




b) $|\psi\rangle = |-\rangle_y$ Instead of doing the calculation, think about the Stern-Gerlach experiments from Chapter 1.

$$\langle \hat{S}_z \rangle = 0$$

$$\Delta S_z = \frac{\hbar}{2}$$



3. c) $|\psi\rangle = \frac{1}{\sqrt{5}}\left(2|+\rangle - i|-\rangle\right)$

$$\langle \hat{S}_z \rangle = \langle \psi | \hat{S}_z | \psi \rangle = \frac{1}{5}\begin{pmatrix} 2 & +i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -i \end{pmatrix}\cdot\frac{\hbar}{2}$$

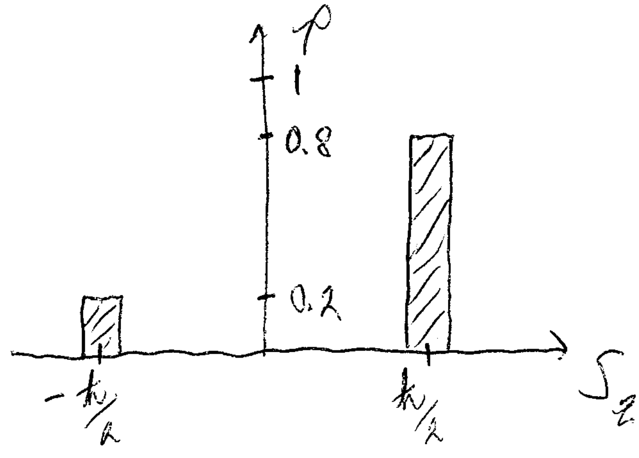
$$= \frac{\hbar}{2}\frac{1}{5}\begin{pmatrix} 2 & +i \end{pmatrix}\begin{pmatrix} 2 \\ +i \end{pmatrix} = \frac{\hbar}{2}\frac{1}{5}(4-1) = \frac{3}{5}\cdot\frac{\hbar}{2} = \underline{\underline{\frac{3}{10}\hbar}}$$

By now we know that $\hat{S}_z^2 = \frac{\hbar^2}{4}\hat{I}$ so $\langle \hat{S}_z^2 \rangle = \frac{\hbar^2}{4}$

$$\Delta S_z = \sqrt{\frac{\hbar^2}{4} - \left(\frac{3}{10}\hbar\right)^2} = \hbar\sqrt{\frac{1}{4} - \frac{9}{100}} = \frac{\hbar}{2}\sqrt{1 - \frac{9}{25}}$$

$$= \frac{\hbar}{2}\sqrt{\frac{16}{25}} = \frac{4}{5}\cdot\frac{\hbar}{2} = \underline{\underline{\frac{4}{10}\hbar}} = \frac{2}{5}\hbar$$

$$P\left(+\frac{\hbar}{2}\right) = |a|^2 = \frac{4}{5} \quad , \quad P\left(-\frac{\hbar}{2}\right) = |b|^2 = \frac{1}{5}$$



$|\psi\rangle$ points closer to "up" than "down"

4. a) $\hat{S}_x$ and $\hat{S}_z$ do <u>not</u> commute: $[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$ so they are incompatible observers. A measurement of $\hat{S}_x$ will destroy any previous knowledge of $\hat{S}_z$ and vice versa. So we <u>cannot</u> know both $\hat{S}_x$ and $\hat{S}_z$ simultaneously.

b) We have seen that $\hat{S}_z^2 = \frac{\hbar^2}{4}\hat{I}$ for spin-$\frac{1}{2}$ systems

The identity operator commutes with all other operators, so

$$[\hat{S}_x, \hat{S}_z^2] = 0$$

This doesn't contradict part (a) because a measurement of $\hat{S}_z^2$ does <u>not</u> give us any information about which way $|\psi\rangle$ is pointing.

$\langle \hat{S}_z^2 \rangle = \frac{\hbar^2}{4}\langle\psi|\hat{I}|\psi\rangle = \frac{\hbar^2}{4}$ for <u>any</u> state $|\psi\rangle$.