Physics 471 Homework 4 Solutions

1. a) Find matrix representation of $\hat{S}_y$: Let $S_y \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

We know the eigenvectors & eigenvalues:

$$\hat{S}_y |+\rangle_y = \frac{\hbar}{2} |+\rangle_y \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$a + ib = \frac{\hbar}{2} \qquad c + id = i\frac{\hbar}{2}$$

$$\hat{S}_y |-\rangle_y = -\frac{\hbar}{2} |-\rangle_y \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = -\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$a - ib = -\frac{\hbar}{2} \qquad c - id = i\frac{\hbar}{2}$$

Add: $2a = 0 \Rightarrow a = 0 \qquad 2c = i\hbar \Rightarrow c = i\frac{\hbar}{2}$

Subtract: $2ib = \hbar \Rightarrow b = \frac{\hbar}{2i} = -i\frac{\hbar}{2} \qquad 2id = 0 \Rightarrow d = 0$

$$\hat{S}_y \doteq \begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

b) If we switch to the S_y basis, then by definition

$$|+\rangle_y \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |-\rangle_y \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

S_y is diagonal in its own basis:

$$S_y \doteq \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(It looks just like S_z expressed in its own basis)

1. c) Express $|+\rangle_z$ and $|-\rangle_z$ in the S_y basis

The easiest method is to invert the transformations we know:

$$|+\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle_z + i\,|-\rangle_z\right)$$
$$|-\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle_z - i\,|-\rangle_z\right)$$

Add these: $|+\rangle_y + |-\rangle_y = \frac{2}{\sqrt{2}}|+\rangle_z = \sqrt{2}\,|+\rangle_z$

$$\Rightarrow |+\rangle_z = \frac{1}{\sqrt{2}}\left(|+\rangle_y + |-\rangle_y\right) \quad \text{or} \quad \underline{\underline{|+\rangle_z \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_y}}$$

Subtract them: $|+\rangle_y - |-\rangle_y = \sqrt{2}\,i\,|-\rangle_z$

$$\Rightarrow |-\rangle_z = \frac{1}{\sqrt{2}\,i}\left(|+\rangle_y - |-\rangle_y\right) = \frac{-i}{\sqrt{2}}\left(|+\rangle_y - |-\rangle_y\right) \doteq \underline{\underline{\frac{1}{\sqrt{2}}\begin{pmatrix}-i\\i\end{pmatrix}_y}}$$

See next page for alternate method.

1. c) Alternate method

In the z-basis, we defined column vector notation this way:

$$|\psi\rangle \doteq \begin{pmatrix} \langle + | \psi \rangle \\ \langle - | \psi \rangle \end{pmatrix} \qquad \text{McIntyre } (1.50)$$

So we can do the same with the y-basis:

$$|\psi\rangle \doteq \begin{pmatrix} {}_y\langle + | \psi \rangle \\ {}_y\langle - | \psi \rangle \end{pmatrix}_y$$

y indicates what basis I am using.

Let $|\psi\rangle = |+\rangle_z$: so $|+\rangle_z = \begin{pmatrix} {}_y\langle + | + \rangle_z \\ {}_y\langle - | + \rangle_z \end{pmatrix}_y$

Evaluate all the brackets (inner products) in the z basis:

$${}_y\langle + | + \rangle_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$${}_y\langle - | + \rangle_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\underline{\underline{|+\rangle_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_y}}$$

Similarly for $|-\rangle_z$:

$${}_y\langle + | - \rangle_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-i}{\sqrt{2}}$$

$${}_y\langle - | - \rangle_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{i}{\sqrt{2}}$$

$$\underline{\underline{|-\rangle_z = \frac{-i}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}_y}}$$

2. From class or McIntyre (2.40)

a) $\hat{S}_n = \hat{S}_x \sin\theta\cos\phi + \hat{S}_y \sin\theta\sin\phi + \hat{S}_z \cos\theta$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\phi + \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta\sin\phi + \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & (\sin\theta\cos\phi - i\sin\theta\sin\phi) \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

b) $e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta}$

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$

Equate real parts: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Equate imaginary parts: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Set $\alpha = \beta$ to get double-angle formulas:

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha \qquad \sin 2\alpha = 2\sin\alpha\cos\alpha$$

Now set $\alpha \to \alpha/2$:

$$\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} \qquad \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

2. a) Diagonalize $\hat{S}_n$: Set $\det(\hat{S}_n - \lambda \hat{I}) = 0$

$$\det \begin{vmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2}\sin\theta\, e^{-i\phi} \\ \frac{\hbar}{2}\sin\theta\, e^{i\phi} & -\frac{\hbar}{2}\cos\theta - \lambda \end{vmatrix} = 0$$

$$\left(\frac{\hbar}{2}\cos\theta - \lambda\right)\left(-\frac{\hbar}{2}\cos\theta - \lambda\right) - \left(\frac{\hbar}{2}\right)^2 \sin^2\theta\, e^{i\phi} e^{-i\phi} = 0$$

$$-\left(\frac{\hbar}{2}\right)^2 \cos^2\theta + \lambda^2 - \left(\frac{\hbar}{2}\right)^2 \sin^2\theta = 0$$

$$\lambda^2 = \left(\frac{\hbar}{2}\right)^2 \left(\cos^2\theta + \sin^2\theta\right) = \left(\frac{\hbar}{2}\right)^2$$

$$\underline{\underline{\lambda = \pm \frac{\hbar}{2}}}$$

This must be true, because we could measure S_n by orienting a Stern-Gerlach apparatus along the $\hat{n}$ direction.

Find first ("up" along $\hat{n}$) eigenvector:

$$\hat{S}_n |+\rangle_n = \frac{\hbar}{2} |+\rangle_n \qquad \text{Represent } |+\rangle_n \doteq \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\frac{\hbar}{2}\left(a\cos\theta + b\sin\theta\, e^{-i\phi}\right) = \frac{\hbar}{2} a$$

$$\frac{\hbar}{2}\left(a\sin\theta\, e^{i\phi} - b\cos\theta\right) = \frac{\hbar}{2} b$$

It turns out that these two equations are equivalent. I'll use the first one:

$$a(\cos\theta - 1) + b\sin\theta\, e^{-i\phi} = 0$$

$$b = a\,\frac{1-\cos\theta}{\sin\theta\, e^{-i\phi}} = a\,\frac{1-\cos\theta}{\sin\theta}\, e^{i\phi}$$

Express $\cos\theta$ and $\sin\theta$ in terms of $\theta/2$ from part b.

$$b = a\,\frac{1-\cos^2\theta/2 + \sin^2\theta/2}{2\sin\theta/2\,\cos\theta/2}\, e^{i\phi} = a\,\frac{2\sin^2\theta/2}{2\sin\theta/2\,\cos\theta/2}\, e^{i\phi}$$

$$b = a\,\frac{\sin\theta/2}{\cos\theta/2}\, e^{i\phi} \quad (*)$$

We need $|+\rangle_n$ to be normalized: $|a|^2 + |b|^2 = 1$

$$|a|^2 + \left|a\,\frac{\sin\theta/2}{\cos\theta/2}\, e^{i\phi}\right|^2 = 1$$

Let a be real and positive: $a^2\left(1 + \frac{\sin^2\theta/2}{\cos^2\theta/2}\right) = 1$

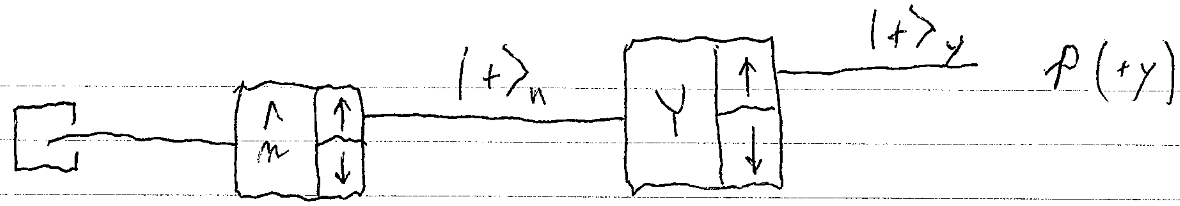
$$a^2\left(\frac{\cos^2\theta/2 + \sin^2\theta/2}{\cos^2\theta/2}\right) = 1$$

$$a^2 = \cos^2\theta/2 \Rightarrow a = \cos\theta/2$$

then $b = \sin\theta/2\, e^{i\phi}$ from $(*)$ above

$$\Rightarrow |+\rangle_n \doteq \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\phi} \end{pmatrix} = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$$

2. d)



where $\hat{n}$ is in the direction $\theta = \pi/4$, $\phi = \pi/3$

We are calculating the probability of measuring up-y given that the system was prepared in the state $|+\rangle_n$.

so $P(+y) = \left| {}_y\langle + | + \rangle_n \right|^2$ $\qquad {}_y\langle + | = \frac{1}{\sqrt{2}}\left(\langle + | - i \langle - | \right)$

note the - sign

$${}_y\langle + | + \rangle_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \, e^{i\phi} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \left(\cos\theta/2 - i \sin\theta/2 \, e^{i\phi} \right)$$

$$\left| {}_y\langle + | + \rangle_n \right|^2 = \frac{1}{2} \left(\cos\theta/2 + i \sin\theta/2 \, e^{-i\phi} \right)\left(\cos\theta/2 - i \sin\theta/2 \, e^{i\phi} \right)$$

$$= \frac{1}{2} \left[\cos^2\theta/2 + \sin^2\theta/2 + i \sin\theta/2 \cos\theta/2 \underbrace{\left(e^{-i\phi} - e^{i\phi} \right)}_{-2i\sin\phi} \right]$$

$$= \frac{1}{2} \left[1 + 2\sin\theta/2 \cos\theta/2 \sin\phi \right]$$

$$= \frac{1}{2} \left[1 + \sin\theta \sin\phi \right]$$

$$= \frac{1}{2} \left[1 + \sin\pi/4 \sin\pi/3 \right] = \frac{1}{2}\left[1 + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} \right]$$

$$= \frac{1}{2} \left(1 + 0.612 \right) = \underline{\underline{0.806}}$$

3. a) $|\psi\rangle = \frac{1}{\sqrt{5}}\left(2|+\rangle - i|-\rangle\right)$

Convert $-i$ to $e^{-i\pi/2}$: $|\psi\rangle = \frac{2}{\sqrt{5}}|+\rangle + \frac{1}{\sqrt{5}}e^{-i\pi/2}|-\rangle$

Standard form $|+\rangle_n = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$

$$\left.\begin{aligned}\cos\tfrac{\theta}{2} &= \tfrac{2}{\sqrt{5}} \\ \sin\tfrac{\theta}{2} &= \tfrac{1}{\sqrt{5}}\end{aligned}\right\} \Rightarrow \theta = 53.1^\circ = 0.295\pi \text{ (radians)} = 0.927 \text{ rad.}$$

$$\phi = -\frac{\pi}{2} \text{ or } +\frac{3\pi}{2}$$

b) Measure S_z, obtain result $+\hbar/2$

Postulate 5 says the new state is

$$|\psi'\rangle = \frac{P_{+z}|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$$

$$P_{+z}|\psi\rangle = |+\rangle\langle+|\psi\rangle = \frac{2}{\sqrt{5}}|+\rangle$$

$$\langle\psi|P_n|\psi\rangle = \langle\psi|\frac{2}{\sqrt{5}}|+\rangle = \left(\frac{2}{\sqrt{5}}\right)^2$$

$$|\psi'\rangle = \frac{2/\sqrt{5}\,|+\rangle}{2/\sqrt{5}} = |+\rangle \quad \text{as it must be!}$$