PHY 471: Homework 7 Solutions

1. a) We know the answers from Chapter 1, but let's calculate them anyway:

$$P\left(S_y = +\frac{\hbar}{2}\right) = \left|{}_y\langle + | + \rangle\right|^2 = \left|\frac{1}{\sqrt{2}}(1 \ \ -i)\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$$

$$P\left(S_y = -\frac{\hbar}{2}\right) = \left|{}_y\langle - | + \rangle\right|^2 = \left|\frac{1}{\sqrt{2}}(1 \ \ i)\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$$

b) $\vec{B} = B_0 \hat{x}$ $\quad \hat{H} = \frac{e}{m}\vec{S}\cdot\vec{B} = \frac{eB_0}{m}\hat{S}_x \equiv \omega_0 \hat{S}_x$

The eigenstates of $\hat{H}$ are $|+\rangle_x$ with energy $+\frac{\hbar\omega_0}{2}$ and $|-\rangle_x$ with energy $-\frac{\hbar\omega_0}{2}$.

We need to express the initial state $|\psi(t=0)\rangle = |+\rangle$ as a linear superposition of energy eigenstates:

$$|+\rangle = c_1 |+\rangle_x + c_2 |-\rangle_x$$

There are several ways to find c_1 and c_2. The most direct way is this:

Take inner product with bra ${}_x\langle +|$:

$${}_x\langle + | + \rangle = c_1 \, {}_x\langle + | + \rangle_x + c_2 \, {}_x\langle + | - \rangle_x = c_1$$

so $c_1 = {}_x\langle + | + \rangle = \frac{1}{\sqrt{2}}(1 \ \ 1)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$

Similarly $c_2 = {}_x\langle - | + \rangle = \frac{1}{\sqrt{2}}(1 \ -1)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$

$$|\psi(t=0)\rangle = |+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_x + |-\rangle_x\right)$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_+t/\hbar}|+\rangle_x + e^{-iE_-t/\hbar}|-\rangle_x\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}|+\rangle_x + e^{+i\omega_0 t/2}|-\rangle_x\right)$$

c) Measure S_y after time t

$$P\left(S_y = +\frac{\hbar}{2}\right) = \left|{}_y\langle + | \psi(t)\rangle\right|^2$$

$$= \frac{1}{2}\left| e^{-i\omega_0 t/2}\,{}_y\langle + | + \rangle_x + e^{+i\omega_0 t/2}\,{}_y\langle + | - \rangle_x \right|^2$$

$${}_y\langle + | + \rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}(1-i)$$

$${}_y\langle + | - \rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}(1+i)$$

$$P\left(S_y = +\frac{\hbar}{2}\right) = \frac{1}{2}\left| e^{-i\omega_0 t/2}\frac{(1-i)}{2} + e^{+i\omega_0 t/2}\frac{(1+i)}{2}\right|^2$$

$$= \frac{1}{8}\left(e^{+i\omega_0 t/2}(1+i) + e^{-i\omega_0 t/2}(1-i)\right)\left(e^{-i\omega_0 t/2}(1-i) + e^{i\omega_0 t/2}(1+i)\right)$$

$$= \frac{1}{8}\left[(1+i)(1-i)\cdot 2 + e^{i\omega_0 t}(1+i)^2 + e^{-i\omega_0 t}(1-i)^2\right]$$

$$= \frac{1}{8}\left[4 + e^{i\omega_0 t}\cdot 2i + e^{-i\omega_0 t}\cdot(-2i)\right]$$

$$= \frac{1}{8}\left[4 - 4\sin\omega_0 t\right] = \underline{\underline{\frac{1}{2}\left(1 - \sin\omega_0 t\right)}} \qquad \omega_0 = \frac{eB_0}{m}$$

Alternative solutions to problems 1.b) and 1.c)

If you chose to write $|\psi(t)\rangle$ as a column vector in the usual z-basis, you would get

b) $$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + e^{+i\omega_0 t/2}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix} e^{-i\omega_0 t/2} + e^{+i\omega_0 t/2} \\ e^{-i\omega_0 t/2} - e^{+i\omega_0 t/2}\end{pmatrix} = \begin{pmatrix}\cos \omega_0 t/2 \\ -i\sin \omega_0 t/2\end{pmatrix}$$

c) $$P\left(S_y = +\frac{\hbar}{2}\right) = \left|{}_y\langle + | \psi(t)\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}\cos \omega_0 t/2 \\ -i\sin \omega_0 t/2\end{pmatrix}\right|^2$$

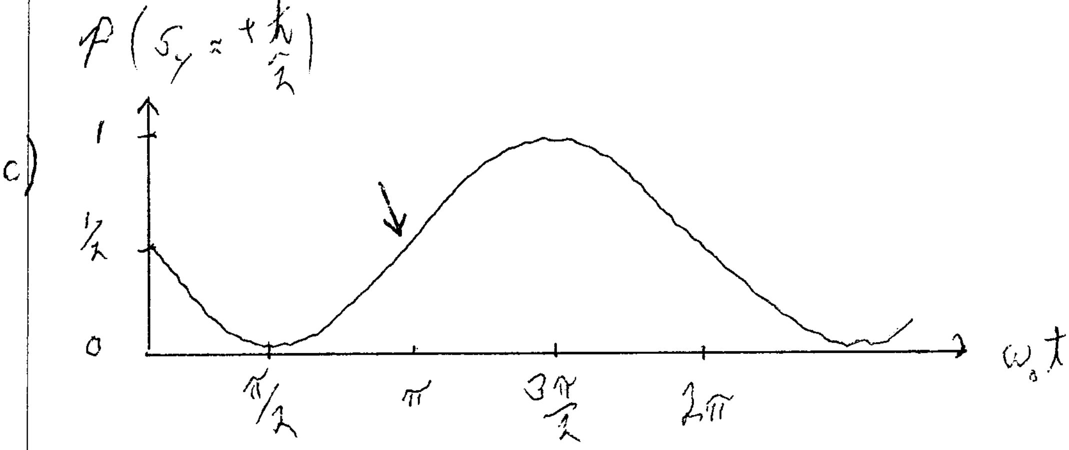
$$= \frac{1}{2}\left|\cos\frac{\omega_0 t}{2} - \sin\frac{\omega_0 t}{2}\right|^2$$

$$= \frac{1}{2}\left(\cos^2\frac{\omega_0 t}{2} + \sin^2\frac{\omega_0 t}{2} - 2\sin\frac{\omega_0 t}{2}\cos\frac{\omega_0 t}{2}\right)$$

$$= \underline{\underline{\frac{1}{2}\left(1 - \sin\omega_0 t\right)}}$$ same answer as before

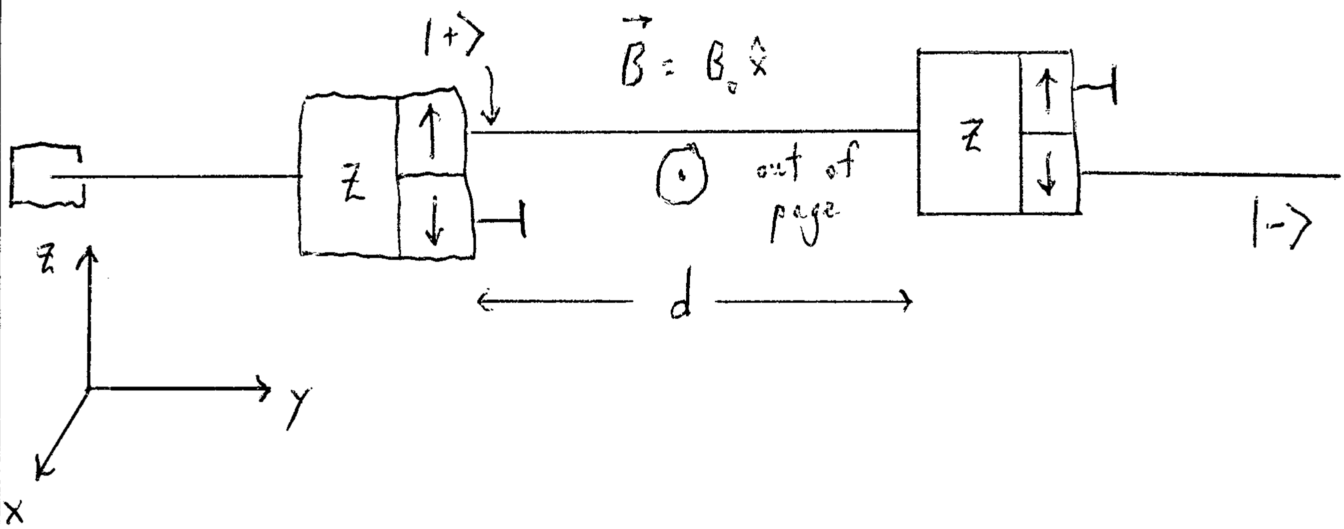
If you leave your answer as $\frac{1}{2}\left(\cos\frac{\omega_0 t}{2} - \sin\frac{\omega_0 t}{2}\right)^2$, that is correct, but it's difficult to see what's going on.

1.c)



The earliest time when $P = \frac{1}{2}$ is $\omega_0 t = \pi$ or $\underline{\underline{t = \frac{\pi}{\omega_0}}}$

2.



a) If $B_0 = 0$, <u>no</u> particles pass through, because

$$P_- = |\langle - | + \rangle|^2 = 0$$

final state   initial state

2. b) Since $\vec{B}$ points along $\hat{x}$, the spins will precess in the y-z plane at angular frequency $\omega_0 = \frac{eB_0}{m}$. The probability of getting through the second Stern-Gerlach will be 50% when $\langle \vec{S} \rangle$ points in the $+\hat{y}$ or $-\hat{y}$ direction, which occurs after $\frac{1}{4}$ of a cycle: $\omega_0 t = \frac{2\pi}{4} = \frac{\pi}{2}$.

Let's check that answer using results from the previous problem: $|\psi(t=0)\rangle = |+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_x + |-\rangle_x\right)$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}|+\rangle_x + e^{+i\omega_0 t/2}|-\rangle_x\right)$$

$$P\left(S_z = -\frac{\hbar}{2}\right) = \left|\langle -|\psi(t)\rangle\right|^2$$

$$= \frac{1}{2}\left|e^{-i\omega_0 t/2}\langle -|+\rangle_x + e^{i\omega_0 t/2}\langle -|-\rangle_x\right|$$

$$\langle -|+\rangle_x = \begin{pmatrix} 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \qquad \langle -|-\rangle_x = \begin{pmatrix} 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{1}{\sqrt{2}}$$

$$P\left(S_z = -\frac{\hbar}{2}\right) = \frac{1}{4}\left|e^{-i\omega_0 t/2} - e^{i\omega_0 t/2}\right|^2 = \frac{1}{4}\left|-2i\sin\frac{\omega_0 t}{2}\right|^2$$

$$= \sin^2\left(\frac{\omega_0 t}{2}\right) \qquad \text{Alternate form: } \frac{1}{2}\left(1 - \cos\omega_0 t\right)$$

So $P = \frac{1}{2}$ when $\sin^2\left(\frac{\omega_0 t}{2}\right) = \frac{1}{2}$

$$\Rightarrow \sin\frac{\omega_0 t}{2} = \frac{1}{\sqrt{2}} \Rightarrow \omega_0 t/2 = \frac{\pi}{4} \qquad \underline{\underline{\omega_0 t = \frac{\pi}{2}}}$$

as we said at the beginning.

Particles are moving at speed v, so $d = vt = v\,\frac{\pi}{2\omega_0}$

Plug in $\omega_0 = \frac{eB_0}{m_e}$: $\quad \underline{\underline{d = \frac{\pi}{2}\frac{m_e v}{eB_0}}}$

c) In part b, the spins precess around $\vec{B} = B_0\hat{x}$, so particles with spins up along z have their spins rotated. In part a, the spins stay in the $|+\rangle$ state.

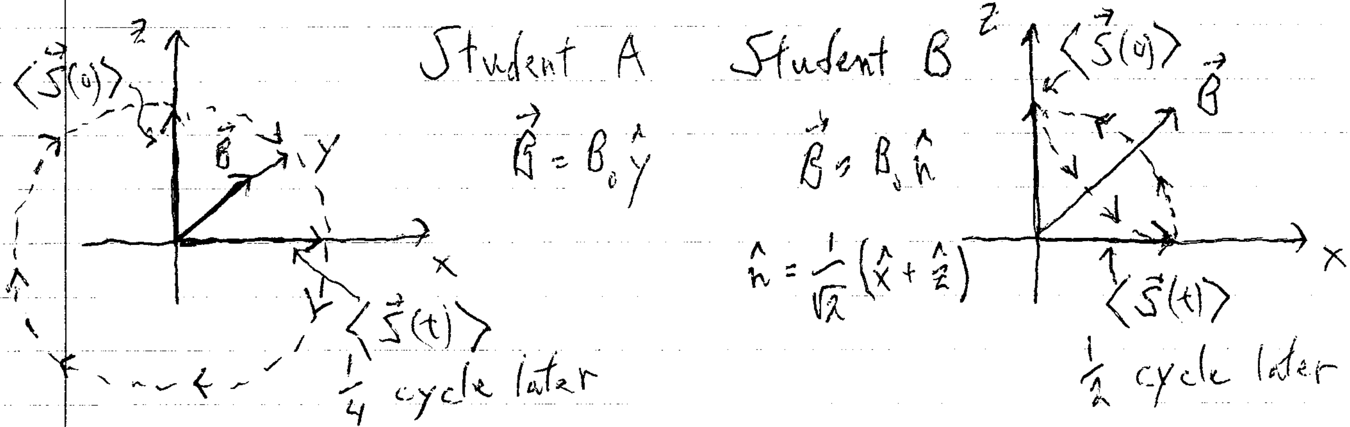
d) $P(S_z = -\frac{\hbar}{2}) = \sin^2\left(\frac{\omega_0 t}{2}\right)$ from part b).

$P = 1$ when $\frac{\omega_0 t}{2} = \frac{\pi}{2}$ or $\omega_0 t = \pi$

This is twice as long as in part b), so

$$\underline{\underline{d = \pi\,\frac{m_e v}{eB_0}}}.$$

3. I'll draw the x-z plane in the plane of the paper.



a) $\langle\vec{S}(t)\rangle$ precesses around $\vec{B} = B_0\hat{y}$ at angular frequency ω_0.
$\langle\vec{S}(t)\rangle$ points along $+\hat{x}$ after $\frac{1}{4}$ cycle, or $\omega_0 t = \frac{\pi}{2} \Rightarrow \underline{\underline{t = \frac{\pi}{2\omega_0}}}$

b) $\langle\vec{S}(t)\rangle$ precesses around $\vec{B} = B_0\hat{n}$, with $\hat{n} = \frac{1}{\sqrt{2}}(\hat{x}+\hat{z})$,
at the same angular frequency ω_0. It points along $+\hat{x}$ after $\frac{1}{2}$ cycle $\Rightarrow \underline{\underline{t = \frac{\pi}{\omega_0}}}$

c) Now $\langle\vec{S}(t)\rangle$ precesses in the opposite direction, so after $\frac{1}{4}$ cycle it points along $-\hat{x}$, which means $|\psi\rangle = |-\rangle_x$.
Since $|{}_x\langle+|-\rangle_x|^2 = 0$, $\underline{\underline{\text{no}}}$ particles get through the x-analyzer.
If they wait $\frac{3}{4}$ cycle, or $\underline{\underline{t = \frac{3\pi}{2\omega_0}}}$, then 100% will be transmitted.

d) 100%. After $\frac{1}{2}$ cycle, you get to the same place if you go around clockwise or counter-clockwise.