Deep Dive #4

Intro

(1) Choose an x_0 and write the solution y as a power series expansion centered at a point x_0,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

(2) Introduce the power series expansion above into the differential equation and find a *recurrence relation*—an equation where the coefficient a_n is related to a_{n-1} (and possibly a_{n-2}).

(3) Solve the recurrence relation—find a_n in terms of a_0 (and possibly a_1).

(4) If possible, add up the resulting power series for the solution $y(x)$.

Question 1

First Order Equations with Constant Coefficients

The problem below involves a first order, constant coefficient, equation. We use this simple equation to practice the Power Series Method. But recall, this method is useful to solve variable coefficient equations.

Question 1: Use a power series around the point $x_0 = 0$ to find all solutions y of the equation

$$y' + c\,y = 0, \qquad c \in \mathbb{R}.$$

(1a) *(10 points)* Find the recurrence relation relating the coefficient a_n with a_{n-1}.

(1b) *(10 points)* Solve the recurrence relation, that is, find a_n in terms of a_0.

(1c) *(10 points)* Write the solution y as a power series one multiplied by a_0. Then add the power series expression.

Note: Using the integrating factor method we know that the solution is $y(x) = a_0\, e^{-cx}$, with $a_0 \in \mathbb{R}$. We want to recover this solution using the Power Series Method.

With $x_0 = 0$ the power series expansion is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{(n-1)}$$

(1a) Using the relationship between y and y' we get

$$y' + cy = 0$$

$$c\sum_{n=0}^{\infty} a_n x^n = c\sum_{n=1}^{\infty} a_{n-1}x^{n-1}$$

$$\sum_{n=0}^{\infty} na_n x^{n-1} = \sum_{n=1}^{\infty} na_n x^{n-1}$$

$$0 = \sum_{n=0}^{\infty} na_n x^{n-1} + c\sum_{n=0}^{\infty} a_n x^n$$

$$0 = \sum_{n=1}^{\infty} na_n x^{n-1} + c\sum_{n=1}^{\infty} a_{n-1}x^{n-1}$$

$$0 = \sum_{n=1}^{\infty} (na_n + ca_{n-1})x^{n-1}$$

Since each x^{n-1} doesn't interact, each coefficient must also equal zero

$$na_n+ca_{n-1}=0 \ a_n = -\frac{c}{n}a_{n-1}

(1b) Using the above formula, we'll derive any a_n in terms of a_0. Ideally we'd do a proof by induction but this'll do for me

$$a_1 = -ca_0$$

$$a_2 = -\frac{c}{2}a_1 = \frac{c^2}{2}a_0$$

$$a_3 = -\frac{c}{3}a_2 = -\frac{c^3}{6}a_0$$

$$a_n = (-1)^n \frac{c^n}{n!}a_0$$

(1c)

$$y(x) = \sum_{n=0}^{\infty} (-1)^n \frac{c^n}{n!} a_0 x^n = a_0 \sum_{n=0}^{\infty} \frac{(-cx)^n}{n!}$$

$$y(x) = a_0 e^{-cx}$$

Question 2

Second Order Equations with Constant Coefficients

The problem below involves a second order, constant coefficient, equation. As above, we use this simple equation to practice the Power Series Method.

Question 2: Use a power series around the point $x_0 = 0$ to find all solutions y of the equation

$$y'' + y = 0.$$

(2a) *(10 points)* Find the recurrence relation relating the coefficient a_n with a_{n-2}.

(2b) *(10 points)* Solve the recurrence relation, which in this case means to find a_n in terms of a_0 for n even, and a_n in terms of a_1 for n odd.

(2c) *(10 points)* Write the solution y as a combination of two power series, one multiplied by a_1, the other multiplied by a_0. Then add both power series expressions.

Note: Guessing the fundamental solutions we know that the solution is $y(x) = a_0 \cos(x) + a_1 \sin(x)$, with $a_0, a_1 \in \mathbb{R}$. We want to recover these solutions using the Power Series Method.

(2a)

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

We'll shift down to get the same x^n powers

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} ((n+2)(n+1) a_{n+2} + a_n) x^n = 0$$

Again, each coefficient of the power must also be zero

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$$

$$a_n = -\frac{1}{n(n-1)} a_{n-2}$$

(2b) We'll work through it using the same methods

$$a_2 = \frac{-a_0}{(2)(1)}$$
$$a_4 = \frac{a_0}{(4)(3)(2)(1)}$$
$$a_{2n} = \frac{(-1)^n a_0}{(2n)!}$$
$$a_3 = \frac{-a_1}{(3)(2)}$$
$$a_5 = \frac{a_1}{(5)(4)(3)(2)}$$
$$a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}$$

(2c) We need a_0 and a_1 before we can get to the a_n stuff

$$\begin{aligned} y(x) = \sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x + \sum_{n=2}^{\infty} \frac{(-1)^n a_0}{(2n)!} x^{2n} + \frac{(-1)^n a_1}{(2n+1)!} x^{2n+1} \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^n \\ &= a_0 \cos(x) + a_1 \sin(x) \end{aligned}$$

Question 3

Second Order Equations with Variable Coefficients

In the previous two cases we used power series to solve a differential equation we already knew how to solve with other, simpler, methods. Now, however, we use the power series method to solve a differential equation that cannot be solved with the methods we studied in the previous chapters. The equation is called the Legendre equation. It appears as part of a more complicated set of equations that need to be solved when we try to find the static electric field in situations with spherical symmetry.

Question 3: Find all solutions of the Legendre equation

$$(1 - x^2)\, y'' - 2x\, y' + l(l+1)\, y = 0,$$

where l is any real constant, using power series centered at $x_0 = 0$.

(3a) *(10 points)* Find the recurrence relation relating the coefficient a_n with a_{n-2}.

(3b) *(10 points)* Solve the recurrence relation to find a_4, a_2 in terms of a_0 and to find a_5, a_3 in terms of a_1.

(3c) *(10 points)* If we write the solution y as

$$y(x) = a_0\, y_0(x) + a_1\, y_1(x),$$

then find the first three terms of the series expansions of y_0 and y_1. These functions y_0 and y_1 are called Legendre functions.

(3d) *(10 points)* The powers series that define the Legendre functions have infinitely many terms when the constant l is not an integer. But when l is an integer, either y_0 or y_1 have a finite number of nonzero terms—they terminate—and they are just polynomials. The collection of all polynomials—appropriately normalized—are called the Legendre polynomials. Find the first four Legendre polynomials, P_i, for $i = 0, 1, 2, 3$, which are defined as follows,

$$\begin{aligned} P_0(x) &= y_0(x) && \text{for } l = 0 \\ P_1(x) &= y_1(x) && \text{for } l = 1 \\ P_2(x) &= -\frac{1}{2}\, y_0(x) && \text{for } l = 2 \\ P_3(x) &= -\frac{3}{2}\, y_1(x) && \text{for } l = 3. \end{aligned}$$

Hey I know this one from E&M

(3a) Multiply the power series for y' and y'' by their variable coefficients

$$2xy' = \sum_{n=1} \infty 2na_n x^n = \sum_{n=0} \infty 2na_n x^n$$
$$(1-x^2)y'' = \sum_{n=2} \infty (n)(n-1)a_n x^{n-2} - \sum_{n=2} \infty n(n-1)a_n x^n$$

Do the shift

$$= \sum_{n=0} \infty((n+2)(n+1)a_{n+2} - n(n-1)a_n)x^n$$

y(x) is trivially $l(l+1)a_n x^n$. With that we've got enough to join all the e^x terms and chew down to the a_n coefficients.

$$0 = (n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + l(l+1)a_n$$
$$a_{n+2} = -\frac{(l-n)(l+n+1)}{(n+1)(n+2)}$$

(3b) Now for the part that really isn't super fun

$$a_2 = -\frac{l(l+1)}{12} a_0$$
$$a_4 = -\frac{(l-2)(l+3)}{12} a_2 = \frac{l(l-2)(l+1)(l+3)}{24} a_0$$
$$a_3 = -(l-1)(l_2)/6 * a_1$$
$$a_6 = -(l-3)(l+4)/20 * a_3 = (l-1)(l-3)(l+2)(l+4)/120 * a_1$$

(3c)

$$y_0 = 1 - \frac{l(l+1)}{12} x^2 + \frac{l(l-2)(l+1)(l+3)}{24} x^4$$
$$y_1 = x - (l-1)(l+2)/6 * x^3 + (l-1)(l-3)(l+2)(l+4)/120 * x^5$$

(3d)

- $P_0(x) = 1 - 0 + 0 = 1$
- $P_1(x) = x - 0 + 0 = x$
- $P_2(x) = -1/2(1 - \frac{2(3)}{12} x^2 + 0) = -1/2 + \frac{1}{4} x^2$

- $P_3(x) = -3/2(x - (2)(5)/6 * x^3 + 0) = -3/2 * x + 15/6x^3$

Have a great day!