Applications of First order Equations

Radioactive dating, Newton's cooling law, and mixing problems

Objectives

In this dive we use differential equations to describe a few physical systems. We start using the radioactive decay equation on Carbon-14, which is radioactive, to date old remains. Then we introduce Newton's cooling law to determine the temperature of an object cooling down or warming up in a room at a fixed temperature. The last application is describe salt concentration in a water tank with salty water coming in and going out of the tank at possible different rates.

Further Reading

If the summaries in this Dive are not enough to fully understand any of the transformations in this dive, then students may find useful to read from our textbook the appropriate parts of the following sections or subsections:

- In Section 1.2, Separable Equations, see subsection 1.2.4 or help with Carbon-14 dating
- In Section 1.2, Separable Equations, see subsection 1.2.5 for help with Newton's cooling law.
- In Section 1.4, Linear Equations, see subsection 1.4.5 for help on mixing problems.

Carbon-14 Dating

A material is radioactive when the nucleus spontaneously changes. One possible change is that the nucleus breaks into two main pieces plus radiation and many smaller pieces. Another possible change is that a neutron in the nucleus spontaneously changes into a proton plus radiation and other smaller pieces.

The radioactive decay of a single nucleus cannot be predicted, but the decay of a large number can. It turns out the amount of a radioactive material, N, as function of time, t, is described by the exponential decay equation,

$$N' = -r\, N$$

where r is the radioactive decay constant of the material. This constant describes how fast the material decays. Radioactive materials are often characterized not by their decay constant k but by their half-life τ—the time it takes for half the material to decay.

Definition 1. The ***half-life*** of a radioactive substance is the time τ such that

$$N(\tau) = \frac{N(0)}{2}.$$

Question 1.

(1a) *(10 points)* Show that the decay constant k and half-life τ of a radioactive material satisfy

$$k\tau = \ln(2).$$

(1b) *(10 points)* Use the half-life constant τ to write all the solutions of the initial value problem for the radioactive decay equation

$$N = k\, N, \qquad N(0) = N_{\circ}.$$

Simplify your answer.

Carbon-14 is a radioactive isotope of Carbon-12. An atom is an *isotope* of another atom if their nuclei have the same number of protons but different number of neutrons. The Carbon atom has 6 protons. The stable Carbon atom has also 6 neutrons, so it is called Carbon-12. Carbon-13 is another stable isotope of Carbon having 7 neutrons. The Carbon-14 has 8 neutrons and it happens to be radioactive with half-life $\tau = 5730$ years.

The radioactive decay of the Carbon-14 is the following: one of the neutrons in the Carbon-14 changes into a proton plus other stuff, changing the Carbon-14 into Nitrogen-14, which is the stable isotope of Nitrogen, responsible for 99% of the Nitrogen on Earth. The Carbon on Earth is made up of 99% of Carbon-12 and almost 1% of Carbon-13. The Carbon-14 is very rare, it can be found in the atmosphere, where there is 1 Carbon-14 atom per 10^{12} Carbon-12 atoms.

Carbon-14 is being constantly created in the upper atmosphere by collisions of the Carbon-12 with outer space radiation. These collisions create Carbon-14 in such a way that the proportion of Carbon-14 and Carbon-12 in the atmosphere is constant in time. The Carbon atoms are accumulated by living organisms in that same proportion. When the organism dies, the amount of Carbon-14 in the dead body decays while the amount of Carbon-12 remains constant. The proportion between radioactive over normal Carbon isotopes in the dead body decays in time. Therefore, one can measure this proportion in old remains and then find out how old are such remains—this is called Carbon-14 dating.

Question 2. *(10 points)* Bone remains in an ancient excavation site contain only 14% of the Carbon-14 found in living animals today. Estimate how old are the bone remains. Use that the half-life of the Carbon-14 is $\tau = 5730$ years.

Newton's Cooling Law

Newton's cooling law describes how the temperature of an object changes in time (cools down or heat up) when the object is paced in a surrounding which is held at a constant temperature.

Definition 2. *The* ***Newton cooling law*** *says that the temperature $T(t)$ at a time t of an object placed in a surrounding medium kept at a constant temperature T_s satisfies the differential equation*

$$(\Delta T)' = -k\,(\Delta T),$$

with $\Delta T(t) = T(t) - T_s$, and $k > 0$ is a constant that characterizes the thermal properties of the object.

Remark: Newton's cooling law for ΔT is the same as the radioactive decay equation. But, unlike radioactive materials, the initial temperature difference, $(\Delta T)(0) = T(0) - T_s$, can be either positive or negative.

Question 3. *(10 points)* Prove that the solution of the initial value problem

$$(\Delta T)' = -k\,(\Delta T), \qquad T(0) = T_0$$

is given by

$$T(t) = (T_0 - T_s)\,e^{-kt} + T_s.$$

Question 4. *(10 points)* A cup of coffee at 85 C is placed in a cold room held at 5 C. After 5 minutes the water temperature is 25 C. When will the water temperature be 15 C?

Mixing Problems

Consider a tank containing salty water as pictured in Fig. 1, where salty water comes in and goes out of the tank. The amount of water in the tank at a time t is proportional to the water volume, $V(t)$, while the amount of salt dissolved in the water at that time is given by $Q(t)$. Water is pouring into the tank at a rate $r_i(t)$ with a salt concentration $q_i(t)$. Water is also leaving the tank at a rate $r_o(t)$ with a salt concentration $q_o(t)$. Recall that a water rate, r, means water volume per unit time, and a salt concentration, q, means salt mass per unit volume. If we denote by $[r_i]$ the units of the quantity r_i, then we have

$$[V] = \text{Volume}, \quad [Q] = \text{Mass}, \quad [r_i] = [r_o] = \frac{\text{Volume}}{\text{Time}}, \quad [q_i] = [q_o] = \frac{\text{Mass}}{\text{Volume}}.$$

We want to write a mathematical model to describe how the water volume and salt mass change in time. We make one important assumption that will simplify such mathematical model. We assume that the salt inside the tank gets *instantaneously mixed*, which means that—at every time—the salt concentration in one part of the tank is the same as in any other part of the tank. When that happens the salt concentration inside the tank is constant in space and changes only in time.

The physical system described above is called a *mixing problem*. Now we want to introduce a mathematical description of this mixing problem. This mathematical description is based in the conservation of mass: the change in time of the amount of water in the tank has to be equal to the difference between the water rates in and out of the tank. The same must happen to the salt in the tank. Below we write these relations using differential equations.

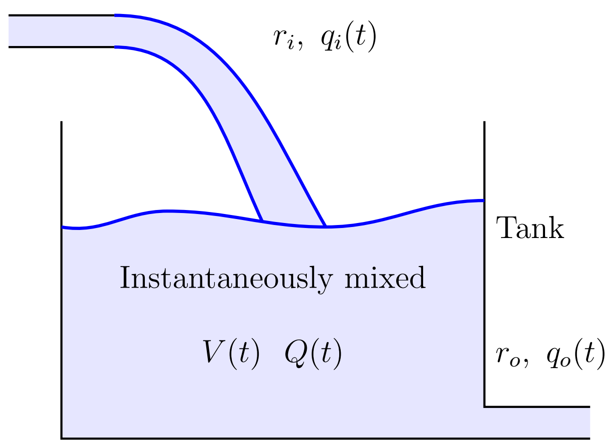

Figure 1: Description of a mixing problem in a water tank.

Definition 3. *A **Mixing Problem** consists of water coming into a tank at a rate r_i with salt concentration q_i, and going out of the tank at a rate r_o with salt concentration q_o, so that the water volume V and the total amount of salt Q, which is instantaneously mixed, in the tank satisfy the following equations,*

$$V'(t) = r_i(t) - r_o(t), \tag{1}$$
$$Q'(t) = r_i(t)\, q_i(t) - r_o(t)\, q_o(t), \tag{2}$$
$$q_o(t) = \frac{Q(t)}{V(t)}, \tag{3}$$
$$r_i'(t) = r_o'(t) = 0. \tag{4}$$

Remarks:

(1) The first equation says that the variation in time of the water volume inside the tank is the difference of volume rates coming in and going out of the tank. In other words, the water volume cannot be created from nothing. Any variation in the water volume inside the tank is produced by the difference between the water volume rates.

(2) The second equation above says that the variation in time of the amount of salt in the tank is the difference of the amount of salt rates coming in and going out of the tank. These salt rates are the product of a water rate r times a salt concentration q. Notice that this product has units of mass per time, which are the units of salt rates. This equation states the conservation of the salt mass.

(3) Eq. (3) is the consequence of the instantaneous mixing mechanism in the tank. Since the salt in the tank is well-mixed, the salt concentration is homogeneous in the tank, with value $Q(t)/V(t)$.

(4) Finally the two equations in (4) say that both rates, in and out, are time independent, hence constants. We include this restriction to get simple coefficients in the resulting differential equation for the salt mass Q.

From the definition above it is not hard to prove the following result.

Theorem 4 (Mixing Problem)**.** The amount of salt in the mixing problem above satisfies the equation

$$Q'(t) = a(t)\,Q(t) + b(t), \tag{5}$$

where the coefficients in the equation are given by

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0}, \qquad b(t) = r_i\,q_i(t). \tag{6}$$

Remark: The equation in (5) is a linear differential equation, which variable coefficients in the case that the rates $r_i \neq r_o$ and/or the salt concentration q_i is non-constant.

Question 5.*(20 points)* Prove Theorem 4.

Question 6.*(15 points)* Consider a mixing problem with *equal* constant water rates $r_i = r_o = r$, with *constant* incoming concentration q_i, and with a given initial water volume in the tank V_0. Use Theorem 4 proven above to find the amount of salt $Q(t)$ in the tank given an arbitrary initial condition $Q(0) = Q_0$.

Question 7. *(15 points)* Consider a tank with a maximum capacity for V_M liters that at time $t = 0$ contains V_0 liters and $Q_0 > 0$ grams of salt, with

$$0 < V_0 < V_M.$$

Denote $\Delta V = V_M - V_0$. Fresh water (that is, $q_i = 0$), is poured in the tank at a constant rate of r_i liters per minute. The well-stirred water pours out of the tank at a constant rate of r_o liters per minute, where

$$0 < r_o < r_i$$

so the tank is slowly filling up with water. Denote $\Delta r = r_i - r_o$. Find the amount of salt in the tank at the time t_c when the tank starts to overflow.