Physics 471 - Homework 3 Solutions

1. $|\psi\rangle = \frac{3}{\sqrt{34}}|+\rangle + i\frac{5}{\sqrt{34}}|-\rangle$

a) Results are $\pm\frac{\hbar}{2}$. $\quad P(\frac{\hbar}{2}) = |\langle +|\psi\rangle|^2 = \frac{9}{34} \approx 0.26$

$P(-\frac{\hbar}{2}) = |\langle -|\psi\rangle|^2 = \frac{25}{34} \approx 0.74$

b) We need $|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $|-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

Results are still $\pm\frac{\hbar}{2}$ (along any axis)

$$P_x(\tfrac{\hbar}{2}) = |{}_x\langle +|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\langle +| + \langle -|\right)\left(\frac{3}{\sqrt{34}}|+\rangle + i\frac{5}{\sqrt{34}}|-\rangle\right)\right|^2$$

$$= \left|\frac{3}{\sqrt{68}} + i\frac{5}{\sqrt{68}}\right|^2 = \frac{9}{68} + \frac{25}{68} = \frac{34}{68} = \frac{1}{2} \quad \text{surprise!}$$

Then we must also have $P_x(-\frac{\hbar}{2}) = \frac{1}{2}$ since probabilities add to 1.

c) We need $|+\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle)$ and $|-\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle - i|-\rangle)$

Results are $\pm\frac{\hbar}{2}$ again. notice complex conjugate

$$P_y(\tfrac{\hbar}{2}) = |{}_y\langle +|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\langle +| - i\langle -|\right)\left(\frac{3}{\sqrt{34}}|+\rangle + i\frac{5}{\sqrt{34}}|-\rangle\right)\right|^2$$

$$= \left|\frac{3}{\sqrt{68}} + \frac{5}{\sqrt{68}}\right|^2 = \left|\frac{8}{\sqrt{68}}\right|^2 = \frac{64}{68} = \frac{16}{17} \approx 0.94$$

We know that $P(-\hbar/2)$ must be $\frac{1}{17}$, but let's calculate it to be sure:

$$P_y(-\tfrac{\hbar}{2}) = |{}_y\langle -|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\langle +| + i\langle -|\right)\left(\frac{3}{\sqrt{34}}|+\rangle + i\frac{5}{\sqrt{34}}|-\rangle\right)\right|^2$$

$$= \left|\frac{3}{\sqrt{68}} - \frac{5}{\sqrt{68}}\right|^2 = \left|\frac{-2}{\sqrt{68}}\right|^2 = \frac{4}{68} = \frac{1}{17} \approx 0.06 \quad \text{as expected.}$$

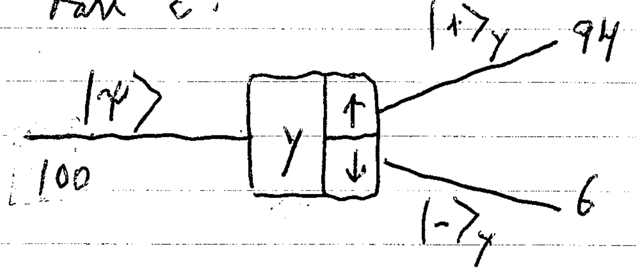
d) After going through the y-S.G., the "spin-up" component is in the state

$$|+\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right)$$

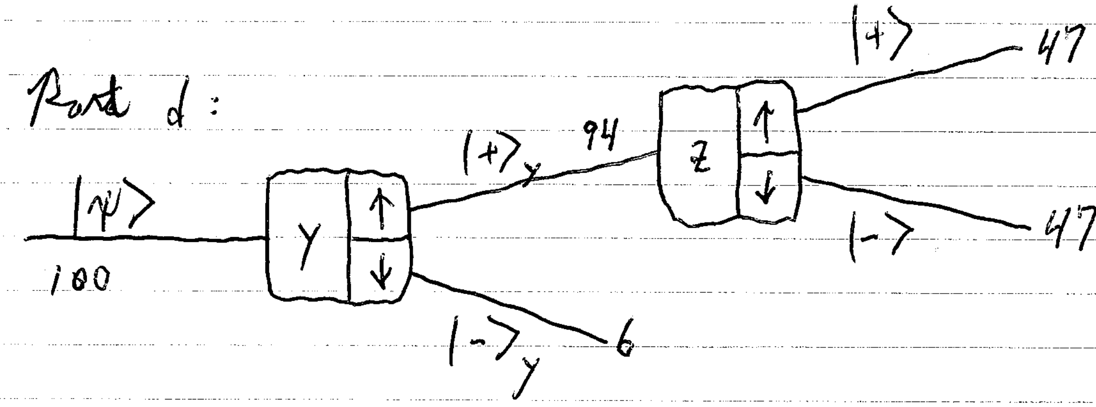
So we already know that a measurement of the z component will give:

$$P\left(+\frac{\hbar}{2}\right) = \frac{1}{2} \quad , \quad P\left(-\frac{\hbar}{2}\right) = \frac{1}{2}$$

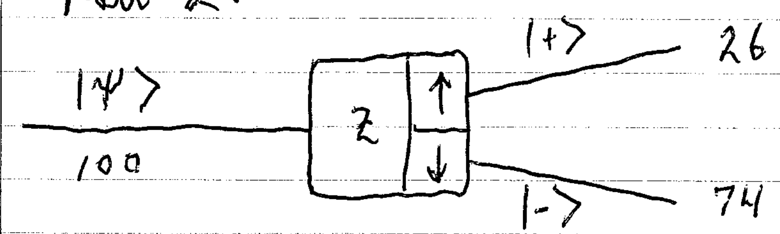
e) Part c:



Part d:



Part a:



In part d), the measurement of S_y made the system "forget" its initial state. The state that is passed on to the next analyzer is $|+\rangle_y$, which contains more $|+\rangle$ than $|\psi\rangle$ does.

2. $|\psi\rangle = \frac{3}{5}|+\rangle_x - \frac{4}{5}|-\rangle_x$

a) Results are $\pm \hbar/2$

$$P\left(+\frac{\hbar}{2}\right) = \left|{}_x\langle+|\psi\rangle\right|^2 = \left|\frac{3}{5}\right|^2 = \frac{9}{25} = 0.36$$

$$P\left(-\frac{\hbar}{2}\right) = \left|{}_x\langle-|\psi\rangle\right|^2 = \left|-\frac{4}{5}\right|^2 = \frac{16}{25} = 0.64$$

b) Results are $\pm \hbar/2$ as usual. I'll show two methods to calculate the measurement probabilities.

<u>Method 1</u>:

$$P\left(+\frac{\hbar}{2}\right) = \left|\langle+|\psi\rangle\right|^2 = \left|\langle+|\left(\frac{3}{5}|+\rangle_x - \frac{4}{5}|-\rangle_x\right)\right|^2$$

$$= \left|\frac{3}{5}\langle+|+\rangle_x - \frac{4}{5}\langle+|-\rangle_x\right|^2$$

Use $\langle+|+\rangle_x = \frac{1}{\sqrt{2}}$, $\langle+|-\rangle_x = \frac{1}{\sqrt{2}}$

$$P\left(+\frac{\hbar}{2}\right) = \left|\frac{3}{5\sqrt{2}} - \frac{4}{5\sqrt{2}}\right|^2 = \left|\frac{-1}{5\sqrt{2}}\right|^2 = \frac{1}{50} = 0.02$$

$$P\left(-\frac{\hbar}{2}\right) = \left|\langle-|\psi\rangle\right|^2 = \left|\frac{3}{5}\langle-|+\rangle_x - \frac{4}{5}\langle-|-\rangle_x\right|^2$$

Use $\langle-|+\rangle_x = \frac{1}{\sqrt{2}}$, $\langle-|-\rangle_x = -\frac{1}{\sqrt{2}}$

$$P\left(-\frac{\hbar}{2}\right) = \left|\frac{3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}}\right|^2 = \left|\frac{7}{5\sqrt{2}}\right|^2 = \frac{49}{50} = 0.98$$

Method 2: Express $|\psi\rangle$ in the z-spin basis.

I'll use "matrix notation" to shorten the calculation:

$$|+\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad |-\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$|\psi\rangle = \frac{3}{5}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} - \frac{4}{5}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \frac{1}{5\sqrt{2}}\begin{pmatrix}3-4\\3+4\end{pmatrix} = \frac{1}{5\sqrt{2}}\begin{pmatrix}-1\\7\end{pmatrix}$$

$$= -\frac{1}{5\sqrt{2}}|+\rangle + \frac{7}{5\sqrt{2}}|-\rangle$$

From this it's easy to see that

$$\mathcal{P}\left(+\frac{\hbar}{2}\right) = \left|\langle +|\psi\rangle\right|^2 = \left|-\frac{1}{5\sqrt{2}}\right|^2 = \frac{1}{50} = 0.02$$

$$\mathcal{P}\left(-\frac{\hbar}{2}\right) = \left|\langle -|\psi\rangle\right|^2 = \left|\frac{7}{5\sqrt{2}}\right|^2 = \frac{49}{50} = 0.98$$

c) After measuring $\hat{S}_x$ and getting the result $-\frac{\hbar}{2}$, the system is in the quantum state $|-\rangle_x$.

We already know the answers for the $\hat{S}_z$ measurement:

$$\mathcal{P}\left(+\frac{\hbar}{2}\right) = \mathcal{P}\left(-\frac{\hbar}{2}\right) = \frac{1}{2}$$

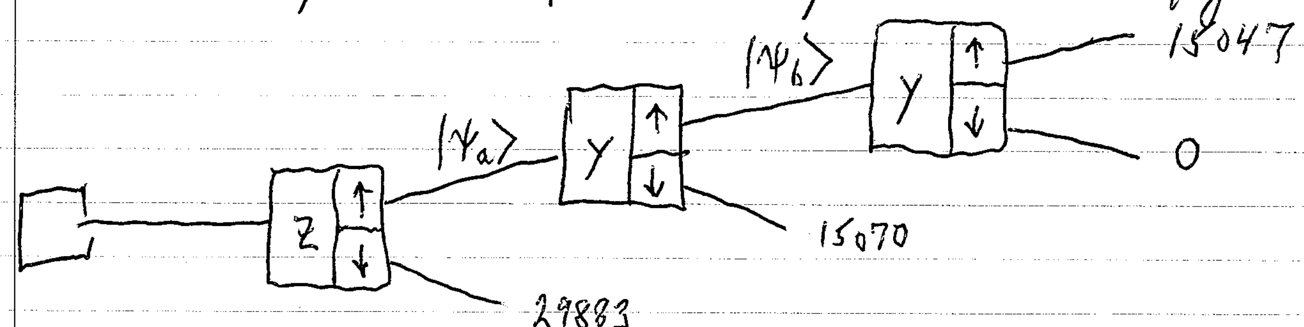
3. a) $29883 + 15070 + 7473 + 7574 \approx 60{,}000$

b) $|\psi_a\rangle = |+\rangle$, $|\psi_b\rangle = |+\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right)$

$|\psi_c\rangle = |-\rangle$, $|\psi_d\rangle = |-\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right)$

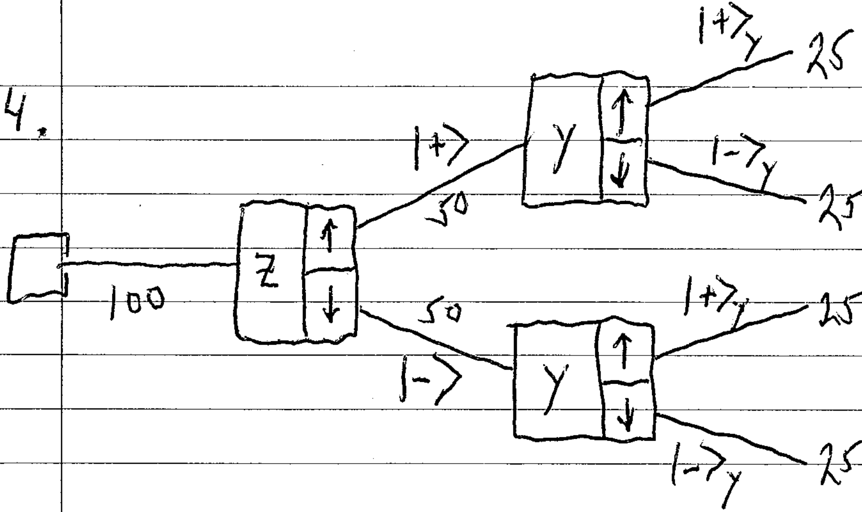
c) Student A is confusing a mixture with a superposition. The particles coming out of the second analyzer are in state $|\psi_b\rangle = |+\rangle_y$, which is a superposition of $|+\rangle$ and $|-\rangle$, not a mixture of $|+\rangle$ and $|-\rangle$.

Here is a way to tell the difference: place another y-SG after the y-SG in the figure:



You will get 100% spin up results. If $|\psi_b\rangle$ were a mixture of $|+\rangle$ and $|-\rangle$, then we would get 50% spin up and 50% spin down from the S_y measurement.

4.



$$P = \left| {}_y\langle + | + \rangle \right|^2 = \frac{1}{2}$$

$$P = \left| {}_y\langle - | + \rangle \right|^2 = \frac{1}{2}$$

$$P = \left| {}_y\langle + | - \rangle \right|^2 = \frac{1}{2}$$

$$P = \left| {}_y\langle - | - \rangle \right|^2 = \frac{1}{2}$$

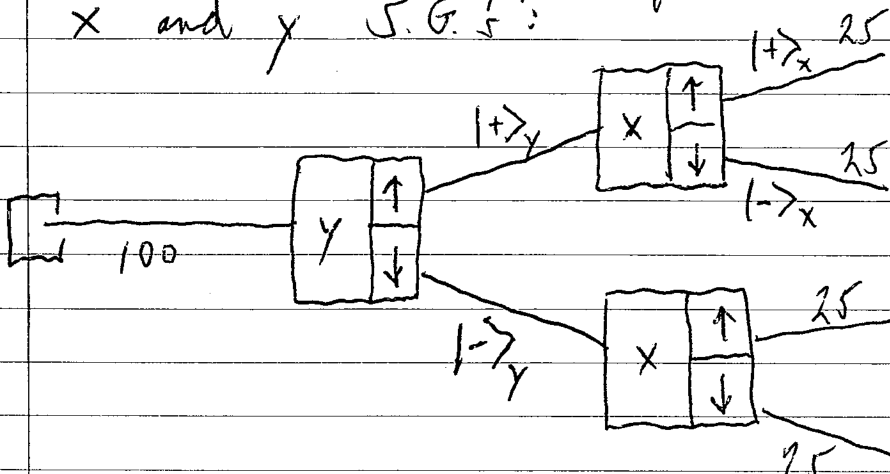
If we write $|+\rangle_y = a|+\rangle + b|-\rangle$, the first condition above implies

$$|a|^2 = \frac{1}{2}$$

and the third condition implies $|b|^2 = \frac{1}{2}$.

Choosing $\underline{a}$ real gives us $|+\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\alpha}|-\rangle\right)$

But now we <u>don't</u> have the freedom to choose the phase α, because we already made that choice for the x-basis states. And we <u>do</u> know what would happen if we chain together x and y S.G.s:



$$P = \left| {}_x\langle + | + \rangle_y \right|^2 = \frac{1}{2}$$

$$P = \left| {}_x\langle - | + \rangle_y \right|^2 = \frac{1}{2}$$

$$P = \left| {}_x\langle + | - \rangle_y \right|^2 = \frac{1}{2}$$

$$P = \left| {}_x\langle - | - \rangle_y \right|^2 = \frac{1}{2}$$

The first condition above:

$$\left| {}_x\langle + | + \rangle_y \right|^2 = \frac{1}{2}$$

$$|+\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \Rightarrow {}_x\langle +| = \frac{1}{\sqrt{2}}\left(\langle +| + \langle -|\right)$$

$$\frac{1}{2} = \left|{}_x\langle + | + \rangle_y\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\langle +| + \langle -|\right)\cdot\frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\alpha}|-\rangle\right)\right|^2$$

$$= \left|\frac{1}{2}\left(\underbrace{\langle +|+\rangle}_{1} + e^{i\alpha}\underbrace{\langle +|-\rangle}_{0} + \underbrace{\langle -|+\rangle}_{0} + e^{i\alpha}\underbrace{\langle -|-\rangle}_{1}\right)\right|^2$$

$$\frac{1}{2} = \left|\frac{1}{2}\left(1 + e^{i\alpha}\right)\right|^2 = \frac{1}{4}\left(1 + e^{-i\alpha}\right)\left(1 + e^{i\alpha}\right)$$

$$= \frac{1}{4}\left(1 + e^{i\alpha} + e^{-i\alpha} + 1\right) = \frac{1}{4}\left(2 + 2\cos\alpha\right)$$

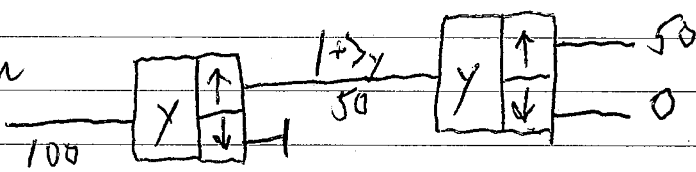
$$\frac{1}{2} = \frac{1}{2}\left(1 + \cos\alpha\right) \Rightarrow \cos\alpha = 0 \Rightarrow \alpha = \pm\frac{\pi}{2}$$

We don't know which one to choose, but McIntyre tells us that $\alpha = +\pi/2$ corresponds to a right-handed coordinate system.

So $$|+\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\pi/2}|-\rangle\right)$$

$$|+\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right)$$

Finally, we know from



that ${}_y\langle - | + \rangle_y = 0$. Given everything else we know, you can easily show that $|-\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right)$