PHY 471 Homework 8 Solutions

1. $|\psi_1\rangle = \frac{1}{3}|\varphi_1\rangle + \frac{\sqrt{8}}{3}|\varphi_2\rangle \qquad |\psi_2\rangle = \frac{\sqrt{8}}{3}|\varphi_1\rangle - \frac{1}{3}|\varphi_2\rangle$

a) $$\langle\psi_1|\psi_2\rangle = \left(\frac{1}{3}\langle\varphi_1| + \frac{\sqrt{8}}{3}\langle\varphi_2|\right)\left(\frac{\sqrt{8}}{3}|\varphi_1\rangle - \frac{1}{3}|\varphi_2\rangle\right)$$

$$= \frac{\sqrt{8}}{9}\underbrace{\langle\varphi_1|\varphi_1\rangle}_{1} - \frac{1}{9}\underbrace{\langle\varphi_1|\varphi_2\rangle}_{0} + \frac{8}{9}\underbrace{\langle\varphi_2|\varphi_1\rangle}_{0} - \frac{\sqrt{8}}{9}\underbrace{\langle\varphi_2|\varphi_2\rangle}_{1}$$

$$= \frac{\sqrt{8}}{9} - \frac{\sqrt{8}}{9} = 0$$ So they are orthogonal. ✓

$$\left.\begin{aligned} \langle\psi_1|\psi_1\rangle &= \frac{1}{9} + \frac{8}{9} = 1 \\ \langle\psi_2|\psi_2\rangle &= \frac{8}{9} + \frac{1}{9} = 1 \end{aligned}\right\}$$ So they are normalized. ✓

b) Measure A, find a_1. System is in state $|\psi_1\rangle$, which is the eigenstate of $\hat{A}$ with eigenvalue a_1.

c) Possible results of B measurement are b_1 and b_2.

$$P(b_1) = |\langle\varphi_1|\psi_1\rangle|^2 = \left|\frac{1}{3}\right|^2 = \frac{1}{9}$$

$$P(b_2) = |\langle\varphi_2|\psi_1\rangle|^2 = \left|\frac{\sqrt{8}}{3}\right|^2 = \frac{8}{9}$$

1\. d) i) After measuring B, system is in state $|\varphi_1\rangle$.

Now measure A: $P(a_1) = |\langle\psi_1|\varphi_1\rangle|^2 = \left|\frac{1}{3}\right|^2 = \underline{\underline{\frac{1}{9}}}$

ii) If the B measurement produced b_1, then $P(a_1) = \frac{1}{9}$

If B produced b_2, then $P(a_1) = |\langle\psi_1|\varphi_2\rangle|^2 = \left|\frac{\sqrt{8}}{3}\right|^2 = \frac{8}{9}$.

These should be multiplied by their probabilities to occur from part c:

$$P(a_1) = \underbrace{\frac{1}{9}\cdot\frac{1}{9}}_{\text{If B gave } b_1} + \underbrace{\frac{8}{9}\cdot\frac{8}{9}}_{\text{If B gave } b_2} = \frac{1}{81} + \frac{64}{81} = \underline{\underline{\frac{65}{81}}}$$

iii) B is not measured. $P(a_1) = 1$ because the system stayed in state $|\psi_1\rangle$ after the initial measurement of A.

iv) $\hat{A}$ and $\hat{B}$ do <u>not</u> commute. There are several ways to tell this:

- They do <u>not</u> share common eigenstates
- Measuring B affects the outcome of measuring A, and vice versa.

2. $$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right)$$

a) $$\langle\psi|\psi\rangle = \frac{1}{2}\left({}_1\langle+| \, {}_2\langle-| - {}_1\langle-| \, {}_2\langle+| \right)\left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right)$$

$$= \frac{1}{2}\Big({}_1\langle+|+\rangle_1 \, {}_2\langle-|-\rangle_2 - {}_1\langle+|-\rangle_1 \, {}_2\langle-|+\rangle_2$$
$$- {}_1\langle-|+\rangle_1 \, {}_2\langle+|-\rangle_2 + {}_1\langle-|-\rangle_1 \, {}_2\langle+|+\rangle_2 \Big)$$

$$= \frac{1}{2}\left(1 - 0 - 0 + 1 \right) = 1$$

b) $$\hat{S}_{1z} |\psi\rangle = \frac{1}{\sqrt{2}}\left(\hat{S}_{1z} |+\rangle_1 |-\rangle_2 - \hat{S}_{1z} |-\rangle_1 |+\rangle_2 \right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\hbar}{2} |+\rangle_1 |-\rangle_2 - \left(-\frac{\hbar}{2}\right) |-\rangle_1 |+\rangle_2 \right)$$

$$= \frac{\hbar}{2\sqrt{2}}\left(|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2 \right)$$ <u>No</u>

$|\psi\rangle$ has a minus sign here.

c) $$\left(\hat{S}_{1z} + \hat{S}_{2z} \right)|\psi\rangle = \frac{1}{\sqrt{2}}\Big[\left(\hat{S}_{1z} + \hat{S}_{2z} \right)|+\rangle_1 |-\rangle_2$$
$$- \left(\hat{S}_{1z} + \hat{S}_{2z} \right)|-\rangle_1 |+\rangle_2 \Big]$$

$$= \frac{1}{\sqrt{2}}\left[\left(\frac{\hbar}{2} - \frac{\hbar}{2} \right)|+\rangle_1 |-\rangle_2 - \left(-\frac{\hbar}{2} + \frac{\hbar}{2} \right)|-\rangle_1 |+\rangle_2 \right]$$

$= 0$ <u>Yes</u>, it is an eigenstate with eigenvalue $= 0$.

2. d) It's easier to do this problem backward: start with $|\varphi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_{1n}|-\rangle_{2n} - |-\rangle_{1n}|+\rangle_{2n}\right)$ and show that $|\varphi\rangle$ is equivalent to $|\psi\rangle$ up to a phase factor.

Use
$$\left.\begin{aligned} |+\rangle_n &= \cos\tfrac{\theta}{2}|+\rangle + \sin\tfrac{\theta}{2}e^{i\phi}|-\rangle \\ |-\rangle_n &= \sin\tfrac{\theta}{2}|+\rangle - \cos\tfrac{\theta}{2}e^{i\phi}|-\rangle \end{aligned}\right\} \begin{array}{c}\text{McIntyre}\\(2.42)\end{array}$$

$$|\varphi\rangle = \frac{1}{\sqrt{2}}\Big[\left(\cos\tfrac{\theta}{2}|+\rangle_1 + \sin\tfrac{\theta}{2}e^{i\phi}|-\rangle_1\right)\left(\sin\tfrac{\theta}{2}|+\rangle_2 - \cos\tfrac{\theta}{2}e^{i\phi}|-\rangle_2\right)$$
$$- \left(\sin\tfrac{\theta}{2}|+\rangle_1 - \cos\tfrac{\theta}{2}e^{i\phi}|-\rangle_1\right)\left(\cos\tfrac{\theta}{2}|+\rangle_2 + \sin\tfrac{\theta}{2}e^{i\phi}|-\rangle_2\right)\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}|+\rangle_1|+\rangle_2 - \cos^2\tfrac{\theta}{2}e^{i\phi}|+\rangle_1|-\rangle_2 + \sin^2\tfrac{\theta}{2}e^{i\phi}|-\rangle_1|+\rangle_2 - \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}e^{2i\phi}|-\rangle_1|-\rangle_2$$
$$- \left(\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}|+\rangle_1|+\rangle_2 + \sin^2\tfrac{\theta}{2}e^{i\phi}|+\rangle_1|-\rangle_2 - \cos^2\tfrac{\theta}{2}e^{i\phi}|-\rangle_1|+\rangle_2 - \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}e^{2i\phi}|-\rangle_1|-\rangle_2\right)\Big]$$

The first terms on both lines cancel each other. So do the last terms. So we are left with all the terms that contain $e^{i\phi}$:

$$|\varphi\rangle = \frac{1}{\sqrt{2}}e^{i\phi}\Big[-\cos^2\tfrac{\theta}{2}|+\rangle_1|-\rangle_2 + \sin^2\tfrac{\theta}{2}|-\rangle_1|+\rangle_2$$
$$- \sin^2\tfrac{\theta}{2}|+\rangle_1|-\rangle_2 + \cos^2\tfrac{\theta}{2}|-\rangle_1|+\rangle_2\Big]$$

Using $\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$ and factoring out an overall $-$ sign:

$$|\varphi\rangle = -\frac{1}{\sqrt{2}}e^{i\phi}\Big[|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\Big] = -e^{i\phi}|\psi\rangle$$

So $|\varphi\rangle$ and $|\psi\rangle$ are the same state up to a phase factor.

2. e) Calculate total probability for Observer A to measure particle 1 with spin up along $\hat{n}$ axis, regardless of what Observer B measures for particle 2.

$$\text{Prob}(+\hat{n}\,1, +\hat{z}\,2) = \left| {}_1\langle +\hat{n}| \, {}_2\langle +\hat{z}| \psi\rangle \right|^2$$

$$= \left| \left(\cos\tfrac{\theta}{2}\, {}_1\langle +| + \sin\tfrac{\theta}{2}\, {}_1\langle -|\right) {}_2\langle +| \cdot \frac{1}{\sqrt{2}}\left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2\right) \right|^2$$

$$= \Big| \cos\tfrac{\theta}{2}\, {}_1\langle +|+\rangle_1 \underbrace{{}_2\langle +|-\rangle_2}_{0} \cdot \frac{1}{\sqrt{2}} - \cos\tfrac{\theta}{2} \underbrace{{}_1\langle +|-\rangle_1}_{0} {}_2\langle +|+\rangle_2 \frac{1}{\sqrt{2}}$$

$$+ \sin\tfrac{\theta}{2} \underbrace{{}_1\langle -|+\rangle_1}_{0} \underbrace{{}_2\langle +|-\rangle_2}_{0} \frac{1}{\sqrt{2}} - \sin\tfrac{\theta}{2}\, {}_1\langle -|-\rangle_1 \, {}_2\langle +|+\rangle_2 \cdot \frac{1}{\sqrt{2}} \Big|^2$$

only the last term survives

$$= \frac{1}{2}\sin^2\frac{\theta}{2}$$

$$\text{Prob}(+\hat{n}\,1, -\hat{z}\,2) = \left| {}_1\langle +\hat{n}| \, {}_2\langle -\hat{z}| \psi\rangle \right|^2$$

$$= \left| \left(\cos\tfrac{\theta}{2}\, {}_1\langle +| + \sin\tfrac{\theta}{2}\, {}_1\langle -|\right) {}_2\langle -| \cdot \frac{1}{\sqrt{2}}\left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2\right) \right|^2$$

This is the only term that survives

$$= \frac{1}{2}\cos^2\frac{\theta}{2}$$

$$\text{Total prob.} = \frac{1}{2}\sin^2\frac{\theta}{2} + \frac{1}{2}\cos^2\frac{\theta}{2} = \frac{1}{2} \quad \checkmark$$

3. a) $|\alpha\rangle = \frac{1}{2}\left(|+\rangle_1|+\rangle_2 - |+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2 - |-\rangle_1|-\rangle_2\right)$

Let's try to factor it into separate states of systems 1 and 2:

$|\alpha\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_1 + |-\rangle_1\right) \cdot \frac{1}{\sqrt{2}}\left(|+\rangle_2 - |-\rangle_2\right)$

$= |+\rangle_{1x}\,|-\rangle_{2x}$

$|\alpha\rangle$ is <u>not</u> entangled because it is a product of system 1 in state $|+\rangle_x$ and system 2 in state $|-\rangle_x$.

b) $|\beta\rangle = \frac{1}{\sqrt{6}}|+\rangle_1|+\rangle_2 + \frac{1}{\sqrt{6}}|+\rangle_1|-\rangle_2 + \frac{2}{\sqrt{6}}|-\rangle_1|+\rangle_2$

$P\left(S_{1z} = +\frac{\hbar}{2}\right) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$ (first 2 terms)

$P\left(S_{1z} = S_{2z} = +\frac{\hbar}{2}\right) = \frac{1}{6}$ (first term only)

$P\left(\text{opposite}\right) = \frac{1}{6} + \frac{4}{6} = \frac{5}{6}$ (last 2 terms)