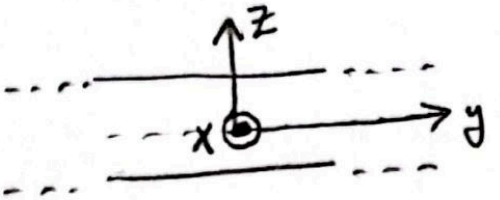
Ampere's Law - Volume Current

Rotated view:

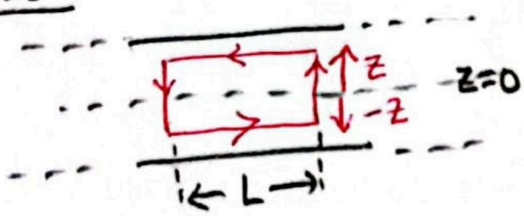


$\vec{J} = J_0 |z| \hat{x}$

Right-hand rule → For $z > 0$, $B = B(z)(-\hat{y})$

For $z < 0$, $B = B(z)\,\hat{y}$

Inside:



Form an Amperian loop symmetric about $z=0$ plane.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

Only contributions to $\oint \vec{B} \cdot d\vec{l}$ is from top and bottom

$$\int_{top} \vec{B} \cdot d\vec{l} + \int_{bottom} \vec{B} \cdot d\vec{l} = \int B(z)(-\hat{y}) \cdot dl(-\hat{y}) + \int B(z)\hat{y} \cdot dl(\hat{y}) = 2BL$$

$$2BL = \mu_0 \int (J_0 |z'| \hat{x}) \cdot (dy' dz' \hat{x}) = \mu_0 \left\{ \int_{-z}^{0} J_0(-z')\,dz' \underbrace{\int dy'}_{L} + \int_{0}^{z} J_0 z'\,dz' \underbrace{\int dy'}_{L} \right\}$$

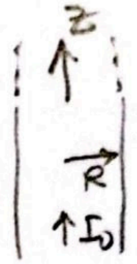
$$2BL = \mu_0 J_0 L \left\{ \frac{z^2}{2} + \frac{z^2}{2} \right\} = \mu_0 J_0 L z^2$$

$$\Rightarrow \vec{B} = \begin{cases} -\dfrac{\mu_0 J_0 z^2}{2} \hat{y}, & z > 0 \text{ inside} \\ \dfrac{\mu_0 J_0 z^2}{2} \hat{y}, & z < 0 \text{ inside} \end{cases}$$

Outside: Loop is similar but we need to integrate from $z = -h$ to $z = +h$ to enclose all current. We get

$$\vec{B} = \begin{cases} -\dfrac{\mu_0 J_0}{2} h^2 \hat{y}, & z > h \\ \dfrac{\mu_0 J_0}{2} h^2 \hat{y}, & z < -h \end{cases}$$

Magnetic Vector Potential



① Uniform current I_0 $\rightarrow$ $\vec{J} = \frac{I_0}{\pi R^2}\hat{z}$

Let's show that $\vec{A}$ could be $cI_0\left(1-\frac{s^2}{R^2}\right)$
given $\vec{\nabla}\cdot\vec{A} = 0$ and $\vec{A}(s=R) = 0$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{A})}_{0 \text{ given}} - \nabla^2 A = -\nabla^2\vec{A} = \mu_0\vec{J}$

We expect $\vec{A}$ points in the same direction as the current.
So $A_s = 0$, $A_\phi = 0$, and $A_z = cI_0\left(1-\frac{s^2}{R^2}\right)$

In cylindrical coordinates : $\nabla^2 A_z = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_z}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 A_z}{\partial\phi^2} + \frac{\partial^2 A_z}{\partial z^2}$

In our case, the only non-vanishing term is

$\frac{\partial A_z}{\partial s} = cI_0\left(-\frac{2s}{R^2}\right)$.

Therefore, $\nabla^2 A_z = \frac{1}{s}\frac{\partial}{\partial s}\left(s\, cI_0\left(\frac{-2s}{R^2}\right)\right) = -\frac{4cI_0}{R^2}$.

Using $\nabla^2 A_z = -\mu_0 J_z$; $-\frac{4cI_0}{R^2} = -\mu_0\frac{I_0}{\pi R^2}$ $\Rightarrow$ $\boxed{c = \frac{\mu_0}{4\pi}}$

To decide if the solution is unique, let's start with checking if $\vec{\nabla}\cdot\vec{A} = 0$;

$\vec{\nabla}\cdot\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} = 0$ ✓

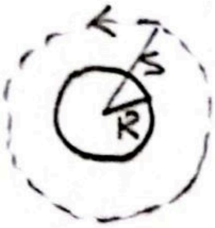
because $A_s = 0$, $A_\phi = 0$ and $\frac{\partial A_z}{\partial z} = 0$ (A_z is not a function of z here)

So. $\vec{A}_{inside} = \frac{\mu_0}{4\pi}I_0\left(1-\frac{s^2}{R^2}\right)\hat{z}$ and it satisfies $\vec{A}(s=R) = 0$

The solution inside is <u>unique</u>.

(2) Outside of the wire, use Ampere's law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl.}$$



$$B \cdot 2\pi s = \mu_0 I_0$$

So $\vec{B} = \vec{\nabla} \times \vec{A} = \dfrac{\mu_0 I_0}{2\pi s} \hat{\phi}$

Because $\vec{A} = A(s)\,\hat{z}$, the only non-vanishing partial derivative in

$$\vec{\nabla} \times \vec{A} = \frac{1}{s}\left[\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial \phi}\right]\hat{z}$$

is $-\dfrac{\partial A_z}{\partial s}$.

So $\vec{B} = \dfrac{\mu_0 I_0}{2\pi s}\hat{\phi} = -\dfrac{\partial A_z}{\partial s}\hat{\phi}$

$$\Rightarrow A_z = \frac{-\mu_0 I_0}{2\pi} \ln s + \underbrace{c_0}_{\text{const.}}$$

If $A_z(s=R) = 0$, then $-\dfrac{\mu_0 I_0}{2\pi}\ln R + c_0 = 0 \Rightarrow c_0 = \dfrac{\mu_0 I_0}{2\pi}\ln R$

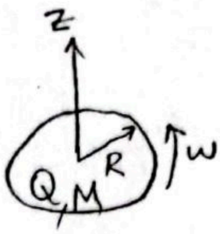
Now, $A_z = -\dfrac{\mu_0 I_0}{2\pi}\ln s + \dfrac{\mu_0 I_0}{2\pi}\ln R$.

So, $\vec{A} = -\dfrac{\mu_0 I_0}{2\pi}\ln\left(\dfrac{s}{R}\right)\hat{z}$ outside.

Satisfies $\vec{\nabla} \cdot \vec{A} = 0$ and $\vec{A}(s=R) = 0$. It is unique.

Semi-classical electron magnetic dipole moment

① $\vec{m} = I \int d\vec{a}$



$$I = \lambda v = \frac{Q}{2\pi R} \omega R = \frac{\omega Q}{2\pi} \qquad \int d\vec{a} = \pi R^2 \hat{z}$$

$$\boxed{\vec{m} = \frac{\omega Q R^2}{2} \hat{z}} \qquad [m] = [I][a] = A m^2 = \frac{C}{s} m^2$$

② Angular momentum = moment of inertia · ω

$$L = I_{ring} \cdot \omega = MR^2 \omega$$

$$\frac{m}{L} = \frac{\frac{\omega Q R^2}{2}}{MR^2 \omega} = \frac{Q}{2M}$$

③ If the gyromagnetic ratio of a single ring depends only on Q and M, each ring that makes up the sphere contributes to $\frac{m}{L}$ by $\frac{Q_{ring}}{2M_{ring}}$.

so that $\left(\frac{m}{L}\right)_{sphere} = \frac{Q_{total}}{2M_{total}}$ where Q_{total} and M_{total} are total charge and mass of the sphere, srespectively.

④ $$\frac{m}{L} = \frac{Q_{total}}{2M_{total}} \rightarrow m = \frac{Q_{total}}{2M_{total}} \cdot L = \frac{-e}{2m_e} \frac{\hbar}{2} = -\frac{1}{2}\mu_B$$

$$m = \frac{-1.6 \times 10^{-19} C}{2 \times (9.11 \times 10^{-31} kg)} \frac{1.0546 \times 10^{-34} kg\, m^2/s}{2} \qquad \left(\frac{e\hbar}{2m_e} = \mu_B \text{ Bohr magneton}\right)$$

$$= -4.64 \times 10^{-24} A \cdot m^2$$ "semi-classical" result

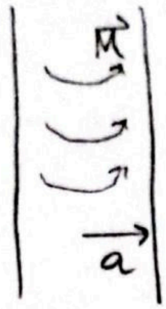
The magnetic dipole moment of the electron is written as $\mu_e = -g\frac{\mu_B}{2}$ where $\mu_B = \frac{e\hbar}{2m_e}$ and g is called the "g-factor".

Compared to what we found "semi-classically" $\left(\mu_e = -\frac{\mu_B}{2}\right)$ which implies $g=1$, the current measured value of g-factor is

$$g = 2 \times 1.001\,159\,652\,180\,73\underbrace{(28)}_{\text{uncertainty in the last two digits}}$$ [PRA83, 052122 (2011)]

There is also a more recent measurement published in 2023
PRL 130, 071801 (2023) $\quad \frac{g}{2} = 1.001\,159\,652\,180\,59(13)$

Bound Currents

1 

$$\vec{M} = cs\,\hat{\phi}$$

$$\vec{J}_b = \vec{\nabla}\times\vec{M} = \frac{1}{s}\frac{\partial}{\partial s}(sM_\phi)\,\hat{z}$$ → only non-vanishing term of $\vec{\nabla}\times\vec{M}$ here

$$= \frac{1}{s}\frac{\partial}{\partial s}(s\,cs)\,\hat{z}$$

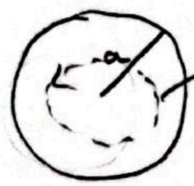
$$= 2c\,\hat{z}$$

$$\vec{K}_b = \vec{M}\times\underbrace{\hat{n}}_{\hat{s}\text{ here}} = cs\,\hat{\phi}\times\hat{s}\Big|_{s=a} = -cs\,\hat{z}\Big|_{s=a} = -ca\,\hat{z}$$

2 $$[J_b] = \left[\frac{dI}{da_\perp}\right] = \frac{A}{m^2} = [c]$$

Double check $$[K_b] = \left[\frac{dI}{dl_\perp}\right] = \frac{A}{m} = [c]\underbrace{[a]}_{m} \rightarrow [c] = \frac{A}{m^2}.$$

3

$s<a$:  Amperian loop of radius s

$\hat{z}$ is out of the page

$$\oint \vec{B}\cdot d\vec{l} = \mu_0\int \vec{J}\cdot d\vec{a} \rightarrow B\cdot 2\pi s = \mu_0\cdot 2c\,\pi s^2$$

$$\Rightarrow \boxed{\vec{B}_{in} = \mu_0 cs\,\hat{\phi}} \quad \text{and} \quad \vec{H}_{in} = \frac{\vec{B}_{in}}{\mu_0} - \vec{M}_{in} = \left(\frac{\mu_0 cs}{\mu_0} - cs\right)\hat{\phi} = \boxed{0}$$

$$s>a: \quad I_{encl} = \int \vec{J}_b\cdot d\vec{a} + \int \vec{K}_b\cdot d\vec{l} = (2c)(\pi a^2) + (-ca)(2\pi a) = 0$$

$$\boxed{\vec{B}_{out} = 0} \qquad \vec{M}_{out} = 0 \Rightarrow \boxed{\vec{H}_{out} = 0}$$