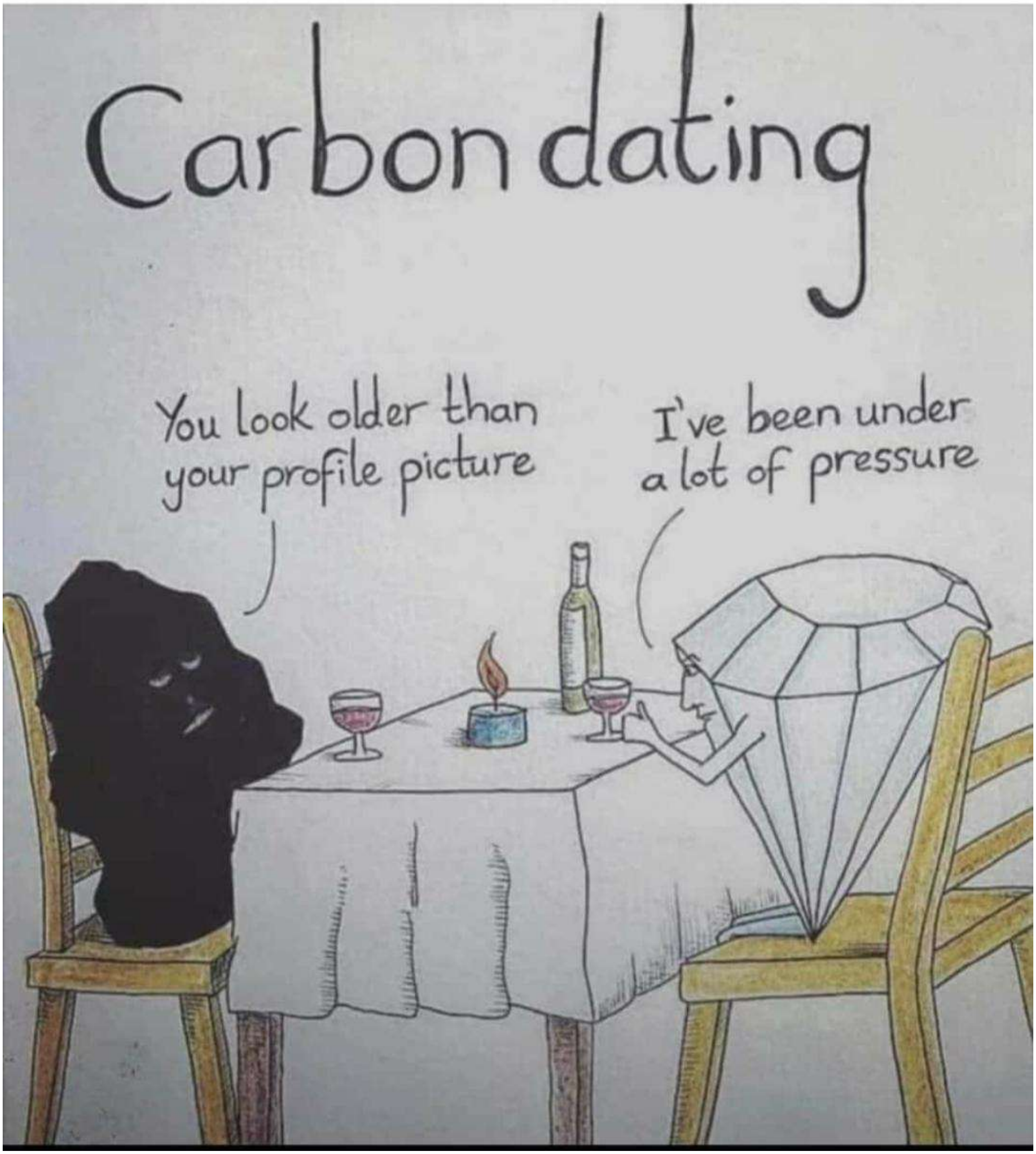
Deep Dive #2

Part 1: Carbon-14 Dating



Radioactive decay is described by the exponential decay equation $N' = -rN$

Half-life is defined as $N(\tau) = \dfrac{N(0)}{2}$

(1a)

(1a) *(10 points)* Show that the decay constant k and half-life τ of a radioactive material satisfy

$$k\tau = \ln(2).$$

$r = k$

$$\int \frac{dN}{-kN} = \int dt + c$$
$$\frac{-1}{k}\ln(N) = t + c$$
$$\ln(N) = -kt + c$$
$$N(t) = ce^{-kt}$$

where c encorperates e^c

$$N(\tau) = \frac{N(0)}{2} = \frac{c}{2}$$
$$N(\tau) = ce^{-k\tau}$$

$$ce^{-k\tau} = \frac{c}{2}e^{-k\tau} = \frac{1}{2}$$

inverting these we have

$$e^{k\tau} = 2$$

and taking the log of each gives us our final result

$$\ln(e^{k\tau}) = \ln(2) \to k\tau = \ln(2)$$

(1b)

(1b) *(10 points)* Use the half-life constant τ to write all the solutions of the initial value problem for the radioactive decay equation

$$N = k\,N, \qquad N(0) = N_0.$$

Simplify your answer.

$k = \frac{\ln(2)}{\tau}$

$N(0) = N_0 = ce^{-0} = c$

$$N(t) = N_0 e^{-\frac{t\ln(2)}{\tau}} = N_0 * 2^{-\frac{t}{\tau}} = \frac{N_0}{2^{\frac{t}{\tau}}}$$

(2)

Question 2. *(10 points)* Bone remains in an ancient excavation site contain only 14% of the Carbon-14 found in living animals today. Estimate how old are the bone remains. Use that the half-life of the Carbon-14 is $\tau = 5730$ years.

$$.14N_0 = \frac{N_0}{2^{\frac{t}{5730}}}$$

$$\ln(.14) = -\frac{t}{5730}\ln(2)$$

$$t = -5730\frac{\ln(.14)}{\ln(2)} = 16253 \text{ years}$$

Part 2: Newton's Cooling Law

Definition 2. *The* **Newton cooling law** *says that the temperature $T(t)$ at a time t of an object placed in a surrounding medium kept at a constant temperature T_s satisfies the differential equation*

$$(\Delta T)' = -k\,(\Delta T),$$

with $\Delta T(t) = T(t) - T_s$, and $k > 0$ is a constant that characterizes the thermal properties of the object.

(3)

Question 3. *(10 points)* Prove that the solution of the initial value problem

$$(\Delta T)' = -k\,(\Delta T), \qquad T(0) = T_0$$

is given by

$$T(t) = (T_0 - T_s)\,e^{-kt} + T_s.$$

Considering the fact that the solution has no ΔT's in it, lets start there

$$(\Delta T)' = (T(t) - T_s)' = T'$$
$$T' = -kT(t) - kT_s$$

Theorem 1.4.2 (Constant Coefficients). *The linear non-homogeneous equation*

$$y' = a\,y + b \tag{1.4.2}$$

with $a \neq 0$, b constants, has infinitely many solutions,

$$y(t) = c\,e^{at} - \frac{b}{a}, \tag{1.4.3}$$

$$T(t) = ce^{-kt} + T_s$$

Applying the initial conditions

$$T_0 = c + T_s \implies c = T_0 - T_s$$
$$T(t) = (T_0 - T_s)e^{-kt} + T_s$$

(4)

Question 4. *(10 points)* A cup of coffee at 85 C is placed in a cold room held at 5 C. After 5 minutes the water temperature is 25 C. When will the water temperature be 15 C?

$T_0 = 85, T_s = 5, t = 5, T(t) = 25$ Find k_t

$$25 = (85-5)e^{-5k} + 5$$
$$\ln(\frac{20}{80}) = \ln(e^{-5k})$$
$$k = -\ln(\frac{1}{4})/5 = \ln(4)/5$$

From this we plug in the new resultant value $T(t) = 15$ to find the new time value t

$$15 = (80)2^{-\frac{2t}{5}} + 5$$
$$\log_2(\frac{1}{8}) = \log_2(2^{-\frac{2t}{5}})$$
$$-\frac{5}{2}\log_2(\frac{1}{8}) = t = 7.5 \text{ minutes}$$

Part 3: Mixing

I think we did these in CMSE 201

$$V'(t) = r_i(t) - r_o(t),$$
$$Q'(t) = r_i(t)\,q_i(t) - r_o(t)\,q_o(t),$$
$$q_o(t) = \frac{Q(t)}{V(t)},$$
$$r_i'(t) = r_o'(t) = 0.$$

Thank you for equation #4. That means that flow rates are constant

Theorem 4 (Mixing Problem). The amount of salt in the mixing problem above satisfies the equation

$$Q'(t) = a(t)\,Q(t) + b(t), \tag{5}$$

where the coefficients in the equation are given by

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0}, \qquad b(t) = r_i\,q_i(t). \tag{6}$$

(4)

Question 5. *(20 points)* Prove Theorem 4.

This is not hard to prove [1]

Pf. We will show that $Q'(t) = r_i(t)q_i(t) - r_0(t)q_0(t)$ can be linearized. I'll refer to the equations in Definiton 3 as (1-4)

We already have $Q'(t) = b(t) - r_0(t)q_0(t)$, so we will show $-r_0(t)q_0(t) = a(t)Q(t)$

From (3) we have

$$-r_0 q_0 = \frac{-r_0 Q}{V}$$

Since (4) implies r_i and r_0 are independent of t, the integral of V' is simply

$$V = \int V' dt = (r_i - r_0)t + c$$

We choose $c = V_0$ and insert what we've found into the above

$$\frac{-r_0 Q}{V} = \frac{-r_0}{(r_i - r_0)t + V_0} Q = a(t)Q(t)$$

Thus our proof is complete.

1

From the definition above it is not hard to prove the following result.

(6)

Question 6. *(15 points)* Consider a mixing problem with *equal* constant water rates $r_i = r_o = r$, with *constant* incoming concentration q_i, and with a given initial water volume in the tank V_o. Use Theorem 4 proven above to find the amount of salt $Q(t)$ in the tank given an arbitrary initial condition $Q(0) = Q_o$.

With this setup a and b is independent of t, $Q' = aQ(t) + b$

$a = \frac{r}{V_0}$
$b = rq_i$

Using Theorem 1.4.2 of the book this becomes

$$Q(t) = ce^{\frac{r}{V_0}t} - \frac{rq_i V_0}{r}$$

The initial value $Q(0) = Q_i$ yeilds

$$Q_i = c - q_i V_0$$
$$c = Q_i + q_i V_0$$

Giving us the final equation

$$Q(t) = (Q_i + q_i V_0)e^{\frac{r}{V_0}t} - q_i V_0$$

(7)

Question 7. *(15 points)* Consider a tank with a maximum capacity for V_M liters that at time $t = 0$ contains V_o liters and $Q_o > 0$ grams of salt, with

$$0 < V_o < V_M.$$

Denote $\Delta V = V_M - V_o$. Fresh water (that is, $q_i = 0$), is poured in the tank at a constant rate of r_i liters per minute. The well-stirred water pours out of the tank at a constant rate of r_o liters per minute, where

$$0 < r_o < r_i$$

so the tank is slowly filling up with water. Denote $\Delta r = r_i - r_o$. Find the amount of salt in the tank at the time t_c when the tank starts to overflow.

ΔV is from the top, not bottom, and is the sum of two constants.

Objective: find $Q(t_c)$

1. find t_c.

$$V(t) = (r_i - r_o)t + c$$

Since $V(t_c) = V_m$

$$V_m = (r_i - r_o)t_c + V_0$$

$$t_c = \frac{\Delta V}{\Delta r}$$

2. Find Q(t). Since r_i does not necessarily equal r_o, we have to start at theorem #4

$$a(t) = \frac{-r_o}{\Delta r t + V_0}$$
$$b(t) = r_i q_i = r_i * 0 = 0$$

As such, this becomes a seperable equation

$$\frac{Q'}{Q(t)} = \frac{-r_o}{\Delta r t + V_0}$$
$$\int \frac{dQ}{Q(t)} = -r_0 \int \frac{dt}{\Delta r t + V_0} + c_1$$
$$\ln(Q(t)) = \frac{-r_0}{\Delta r} \ln(\Delta r t + V_0) + c_1 = \ln((\Delta r t + V_0)^{\frac{-r_0}{\Delta r}}) + c_1$$
$$Q(t) = (\Delta r t + V_0)^{\frac{-r_0}{\Delta r}} * e^c = c_1(\Delta r t + V_0)^{\frac{-r_0}{\Delta r}}$$

Applying $Q(0) = Q_0$ we have

$$Q_0 = c_1(V_0)^{\frac{-r_0}{\Delta r}}$$
$$c_1 = Q_0 V_0^{\frac{r_0}{\Delta r}}$$

From here we encorperate t_c

$$Q(t_c) = Q_0 V_0^{\frac{r_0}{\Delta r}} (\Delta r \frac{\Delta V}{\Delta r} + V_0)^{\frac{-r_0}{\Delta r}}$$
$$= Q_0 V_0^{\frac{r_0}{\Delta r}} (\Delta V + V_0)^{\frac{-r_0}{\Delta r}}$$

Which becomes our final equation

$$Q(t_c) = Q_0 \left(\frac{V_0}{\Delta V + V_0} \right)^{\frac{r_0}{\Delta r}}$$