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Spinning test-particles in general relativity. II

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The equations of motion for spinning test particles are discussed for particles characterized by the condition $S^{i4} = 0$.

1. Introduction

In the present paper we shall discuss some applications of the equations of motion for spinning test particles derived in part I. The Schwarzschild field produced by a large massive body will be assumed as the basic field. As for the test particles, we shall make the assumption that they are of the macroscopic type, that is to say, that the centre of mass always lies inside the (three-dimensional) space occupied by the particle at a given time, the spin being the result of an ordinary rotation. The motion of such particles should be determined entirely by the equations of motion expressed by (5·3) and (5·4) of part I.

These, however, are not sufficient for the determination of all unknowns. In fact, the equations for the spin components are three, while the independent components of $S^{\alpha\beta}$ are six. Therefore, it is necessary to reduce the number of independent components by imposing suitable conditions to the $S^{\alpha\beta}$.

The condition
$$S^{i4} = 0 \tag{1·1}$$
suggests itself because of its simplicity. It must be pointed out, however, that an equation like (1·1) has no meaning, unless we specify the frame of reference in which it holds. In special relativity, where all frames are fully equivalent, one would be inclined to demand (1·1) in the rest-frame of the test particle. Then (1·1) could be generalized unambiguously to any frame of reference,
$$u_\beta S^{\alpha\beta} = 0. \tag{1·1a}$$

In general relativity, however, the choice of the frame in which (1·1) will be demanded is rather ambiguous. Let us consider, for example, a test particle which moves in the Schwarzschild field of an incomparably larger body. In the rest-frame of the test particle the central body is in motion, and it is not at all evident that this frame will be fully equivalent to the frame in which the central body is at rest. On the contrary, it has been suggested that this second frame—which we may call the rest-frame of the Schwarzschild field—would be of primary importance (Rosen 1940; Papapetrou 1948). Thus, we arrive at the alternative possibility, which consists in demanding that the condition (1·1) be fulfilled in the rest-frame of the Schwarzschild field. The first consequence of this formulation of the condition (1·1) is a considerable simplification in the calculations, since three of the spin components vanish identically just in the frame in which the calculations will be performed. Furthermore, this assumption, far from being arbitrary and abstract, can be justified by the following physical argument. Let us suppose for a moment that the reduction of the independent components $S^{\alpha\beta}$ has been obtained by using the condition (1·1 a). Then, in the frame of the Schwarzschild field we will have $S^{i4} \neq 0$. The physical meaning of this result is that the point X^α chosen to 'represent' the particle would not coincide with the centre of mass of the test particle in that frame (Papapetrou 1939). If so, however, the most simple procedure would be to replace the point X^α by the mass-centre X'^α of the particle. Then, automatically, the equations $S'^{i4} = 0$ would hold for the new components of the spin. This can be done for all values of t. Therefore, it all amounts to abandoning the representation of the test particle by the world line L, locus of X^α, and adopting instead that provided by the world line L' of the centre of mass. This argument proves that the condition (1·1 a) corresponds to a fictitious description of the test particle.

Therefore, we shall assume that *the condition* (1·1) *holds in the rest-frame of the Schwarzschild field.*

As for the calculations presented in this paper, their purpose will be twofold. In the first place, the study of the motion in the Schwarzschild field will have a theoretical interest in itself. The integrals of motion representing the conservation of energy and angular momentum of spinning test particles will be derived. By comparing them with the corresponding expressions deduced from the geodesic equations, the implications of the general theory presented in part I will be greatly clarified. On the other hand, we shall give an estimate of the influence of the spin on the planetary motion and on the deflexion of light rays in a gravitational field.

2. Equations of motion in polar co-ordinates

In polar co-ordinates the Schwarzschild field can be described by the line element*

$$ds^2 = e^{\mu}\,dt^2 - e^{-\mu}\,dr^2 - r^2(d\theta^2 + \cos^2\theta\,d\phi^2) \tag{2·1}$$

with

$$e^{\mu} = 1 - \frac{2r_0}{r}, \tag{2·2}$$

where r_0 is the gravitational radius of the central body. The non-vanishing Christoffel symbols are:

$$\left.\begin{aligned} &\Gamma^1_{11} = -\frac{\mu'}{2}, \quad \Gamma^1_{22} = -r\,e^{\mu}, \quad \Gamma^1_{33} = -r\,e^{\mu}\cos^2\theta, \quad \Gamma^1_{44} = \frac{\mu'}{2}\,e^{2\mu},\\ &\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^2_{33} = \cos\theta\sin\theta,\\ &\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = -\tan\theta, \quad \Gamma^4_{14} = \Gamma^4_{41} = \frac{\mu'}{2}, \end{aligned}\right\} \tag{2·3}$$

where $\mu' = d\mu/dr$. As usual, the indices 1, 2, 3, 4 will refer to the co-ordinates r, θ, ϕ and t, respectively.

The definition of $DS^{\alpha\beta}/Ds$ given by (5·1) of part I, together with (1·1), gives†

$$\frac{DS^{i4}}{Ds} = \Gamma^4_{k\nu}S^{ik}u^{\nu}. \tag{2·4}$$

* See, for instance, Bergmann (1947, p. 212). The angle θ is chosen in such a way that the value $\theta = 0$ corresponds to the plane $z = 0$.

† As in part I, Latin indices will take the values 1, 2, 3, only, while Greek indices will run from 1 to 4.

With the help of this relation, the equations for the spin components

$$\frac{DS^{\alpha\beta}}{Ds}+\frac{u^\alpha}{u^4}\frac{DS^{\beta 4}}{Ds}-\frac{u^\beta}{u^4}\frac{DS^{\alpha 4}}{Ds}=0 \tag{2·5}$$

reduce to

$$\frac{DS^{ik}}{Ds}+\Gamma^4_{l\lambda}\frac{u^\lambda}{u^4}(u^iS^{kl}-u^kS^{il})=0. \tag{2·6}$$

Using the expressions (2·3) for the Christoffel symbols, these equations, after a simple calculation, go over into the form

$$\left.\begin{aligned}&\dot{S}^{12}+\left(\frac{1}{r}-\mu'\right)\dot{r}S^{12}+r\,e^{\mu}\cos^2\theta\dot{\phi}S^{23}-\cos\theta\,\sin\theta\dot{\phi}S^{31}=0,\\&\dot{S}^{23}+\left(\frac{\mu'}{2}-\frac{1}{r}\right)\dot{\phi}S^{12}+\left(\frac{2}{r}\dot{r}-\tan\theta\dot{\theta}\right)S^{23}+\left(\frac{\mu'}{2}-\frac{1}{r}\right)\dot{\theta}S^{31}=0,\\&\dot{S}^{31}+\tan\theta\dot{\phi}S^{12}+r\,e^{\mu}\,\dot{\theta}S^{23}+\left[\left(\frac{1}{r}-\mu'\right)\dot{r}-\tan\theta\dot{\theta}\right]S^{31}=0;\end{aligned}\right\} \tag{2·7}$$

the dot indicates the differentiation with respect to s.

The equations of motion

$$\frac{d}{ds}\left(mu^\alpha+u_\beta\frac{DS^{\alpha\beta}}{Ds}\right)+\Gamma^\alpha_{\mu\nu}u^\nu\left(mu^\mu+u_\beta\frac{DS^{\mu\beta}}{Ds}\right)+S^{\mu\nu}u^\sigma(\Gamma^\alpha_{\nu\sigma,\mu}+\Gamma^\alpha_{\mu\rho}\Gamma^\rho_{\nu\sigma})=0 \tag{2·8}$$

must now be written for the Schwarzschild field. Inserting the values (2·3) of the Christoffel symbols into (2·8), one finds after some elementary calculations

$$\left.\begin{aligned}&\frac{d}{ds}[\dot{t}(m+m_s)]+\lambda^4(m+m_s)=0,\\&\frac{d}{ds}[\dot{r}(m+m_s)]+\lambda^1(m+m_s)=0,\\&\frac{d}{ds}[\dot{\theta}(m+m_s)]+\lambda^2(m+m_s)+\frac{3\mu'}{2r}\dot{r}S^{12}+\frac{3\mu'}{2}r\,e^{\mu}\cos^2\theta\dot{\phi}S^{23}=0,\\&\frac{d}{ds}[\dot{\phi}(m+m_s)]+\lambda^3(m+m_s)-\frac{3\mu'}{2}r\,e^{\mu}\,\dot{\theta}S^{23}-\frac{3\mu'}{2r}\dot{r}S^{31}=0.\end{aligned}\right\} \tag{2·9}$$

Here

$$\lambda^\alpha=\Gamma^\alpha_{\mu\nu}u^\mu u^\nu \tag{2·10}$$

(which is *not* a vector) has the following components:

$$\left.\begin{aligned}&\lambda^4=\mu'\dot{r}\dot{t},\\&\lambda^1=\frac{\mu'}{2}e^{2\mu}\,\dot{t}^2-\frac{\mu'}{2}\dot{r}^2-r\,e^{\mu}\,\dot{\theta}^2-r\,e^{\mu}\cos^2\theta\dot{\phi}^2,\\&\lambda^2=\frac{2}{r}\dot{r}\dot{\theta}+\cos\theta\,\sin\theta\dot{\phi}^2,\\&\lambda^3=\frac{2}{r}\dot{r}\dot{\phi}-2\tan\theta\dot{\theta}\dot{\phi}.\end{aligned}\right\} \tag{2·10a}$$

The quantity m_s is defined by the formula

$$m_s=\frac{r^2\mu'}{2}(\cos^2\theta\dot{\phi}S^{31}-\dot{\theta}S^{12}). \tag{2·11}$$

It represents a sort of interaction energy describing a spin-orbit coupling, and will be the object of a more complete discussion in the following sections. In the meantime, we want only to point out that $m+m_s$ will play the role of an 'effective mass' throughout.

Multiplying the equations (2·9) by u_α, and taking into account the relation

$$u_\alpha \dot{u}^\alpha + u_\alpha \lambda^\alpha = 0,$$

we obtain
$$\frac{d}{ds}(m+m_s) = \frac{3\mu'}{2} r\dot{r}(\dot{\theta}S^{12} - \cos^2\theta\dot{\phi}S^{31}). \tag{2·12}$$

Using (2·12) the equations (2·9) go over into the form

$$\left.\begin{aligned}
&(m+m_s)(\ddot{t}+\lambda^4) - \frac{3}{r}\dot{r}\dot{t}m_s = 0,\\
&(m+m_s)(\ddot{r}+\lambda^1) - \frac{3}{r}\dot{r}^2 m_s = 0,\\
&(m+m_s)(\ddot{\theta}+\lambda^2) - \frac{3}{r}\dot{r}\dot{\theta}m_s + \frac{3\mu'}{2r}\dot{r}S^{12} + \frac{3r\mu'}{2}\,e^{\mu}\cos^2\theta\dot{\phi}S^{23} = 0,\\
&(m+m_s)(\ddot{\phi}+\lambda^3) - \frac{3}{r}\dot{r}\dot{\phi}m_s - \frac{3\mu'}{2r}\dot{r}S^{31} - \frac{3r\mu'}{2}\,e^{\mu}\dot{\theta}S^{23} = 0,
\end{aligned}\right\} \tag{2·13}$$

which is more suitable for the applications.

Using (2·9) or (2·13) it is possible to derive first integrals expressing the conservation of energy and angular momentum. The procedure is analogous to that followed in the study of spinless particles. The first of (2·9) combined with the first of (2·10a) gives

$$e^{\mu}\dot{t}(m+m_s) = E = \text{const.}, \tag{2·14}$$

which is the integral of energy. On the other hand, the last of (2·9), combined with the equations (2·7), gives

$$r^2\cos^2\theta\left[(m+m_s)\dot{\phi} + \frac{\mu'}{2}S^{31}\right] + S_z = I_z = \text{const.} \tag{2·15}$$

Here
$$S_z = -r^2\cos\theta\sin\theta S^{23} - r\cos^2\theta S^{31}$$

is the component of the spin in the z-direction of a system of Cartesian co-ordinates,* and I_z represents the z-component of the total angular momentum.

There are two other relations, of a structure similar to (2·15), which express the conservation of the x- and y-components of the angular momentum. These are usually disregarded in the discussion of the motion of spinless particles, the reason being that such particles are always moving on plane orbits; one can assume that the motion takes place on the plane $\theta = 0$, in which case I_x and I_y vanish identically. On the contrary, the motion of spinning particles is, in general, not a plane one, and consequently the full expression of the conservation of angular momentum will be needed. We shall give it in § 4 in Cartesian co-ordinates.

* Cf. the formulae (4·1) and (4·3).

3. Plane motion

We shall now discuss the question whether the equations (2·7) and (2·9) admit any solution representing a motion in a plane passing through the central body. Without any loss of generality, this plane can be taken as the equatorial plane $\theta = 0$. Apart from some special cases of doubtful physical significance, the general condition for a motion in the plane $\theta = 0$ is found to be

$$S^{23} = S^{12} = 0, \tag{3·1}$$

with only $S^{31} \neq 0$. The meaning of this condition can be clarified by noticing that the Cartesian components of the spin in this case are (cf. (4·3))

$$S_x = S_y = 0, \quad S_z = rS^{13}.$$

Thus the spin must be perpendicular to the plane of motion $z = 0$.

On the assumption (3·1), the equations (2·7) reduce to

$$\dot{S}^{31} + \left(\frac{1}{r} - \mu'\right)\dot{r}S^{31} = 0. \tag{3·2}$$

This can be easily integrated,

$$r\,\mathrm{e}^{-\mu} S^{31} \equiv -\mathrm{e}^{-\mu} S_z = \text{const.} \tag{3·3}$$

Therefore, in the present case (3·3) stands besides (2·14) and (2·15), as a first integral for the spin only.

The formula (3·3) has a remarkable, though qualitative analogy with the gravitational red shift of spectral lines. In fact, let us assume that S_z be proportional to the angular velocity of the internal rotation of the test particle; the period of such a rotation is then proportional to $\mathrm{e}^{-\mu} = \left(1 - \frac{2r_0}{r}\right)^{-1}$, i.e. it increases with decreasing r according to a law similar to that describing the influence of the gravitational potential on the wave-length of light.

Besides the strict plane motion based on the condition (3·1), the motion will be very nearly plane in all cases when the spin is much smaller than the orbital angular momentum. This condition is fulfilled in the planetary motions. An interesting question is whether the introduction of the spin in the planetary motion will lead to any appreciable effects. To answer this question we shall have to make an estimate of the order of magnitude of the spin terms for the planetary motion.

By using the integral (2·14), and noticing that

$$\frac{d^2}{ds^2}(\ldots) = \dot{t}^2 \frac{d^2}{dt^2}(\ldots) + \ddot{t}\frac{d}{dt}(\ldots)$$

the variable s can be eliminated from (2·13). Thus we obtain

$$\left.\begin{aligned}
&\frac{d^2r}{dt^2} - \frac{3\mu'}{2}\left(\frac{dr}{dt}\right)^2 - r\,\mathrm{e}^{\mu}\left(\frac{d\theta}{dt}\right)^2 - r\,\mathrm{e}^{\mu}\cos^2\theta\left(\frac{d\phi}{dt}\right)^2 + \frac{\mu'}{2}\mathrm{e}^{2\mu} = 0,\\
&\frac{d^2\theta}{dt^2} + \left(\frac{2}{r} - \mu'\right)\frac{dr}{dt}\frac{d\theta}{dt} + \cos\theta\sin\theta\left(\frac{d\phi}{dt}\right)^2 + \frac{3\mu'}{2E}\mathrm{e}^{\mu}\left[r\,\mathrm{e}^{\mu}\cos^2\theta\frac{d\phi}{dt}S^{23} + \frac{1}{r}\frac{dr}{dt}S^{12}\right] = 0,\\
&\frac{d^2\phi}{dt^2} + \left(\frac{2}{r} - \mu'\right)\frac{dr}{dt}\frac{d\phi}{dt} - 2\tan\theta\frac{d\theta}{dt}\frac{d\phi}{dt} - \frac{3\mu'}{2E}\mathrm{e}^{\mu}\left[r\,\mathrm{e}^{\mu}\frac{d\theta}{dt}S^{23} + \frac{1}{r}\frac{dr}{dt}S^{31}\right] = 0.
\end{aligned}\right\} \tag{3·4}$$

For comparison, the Newtonian equations

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{r_0\mathbf{r}}{r^3}$$

will be written in polar co-ordinates:

$$\left.\begin{aligned} &\frac{d^2r}{dt^2}+\frac{r_0}{r^2}-r\left(\frac{d\theta}{dt}\right)^2-r\cos^2\theta\left(\frac{d\phi}{dt}\right)^2 = 0,\\ &\frac{d^2\theta}{dt^2}+\frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt}+\cos\theta\sin\theta\left(\frac{d\phi}{dt}\right)^2 = 0,\\ &\frac{d^2\phi}{dt^2}+\frac{2}{r}\frac{dr}{dt}\frac{d\phi}{dt}-2\tan\theta\frac{d\theta}{dt}\frac{d\phi}{dt} = 0. \end{aligned}\right\} \tag{3·4a}$$

For vanishing spin, the difference between corresponding equations of the two systems is a perturbation of the order of magnitude r_0/r compared with the Newtonian terms. It is responsible, for example, for the rotation of the perihelion of the planet. We shall refer to such a perturbation as the 'ordinary relativistic terms'. Let us now consider one typical spin-dependent term, e.g. the last term in the third of the (3·4). Its order of magnitude, compared with the Newtonian terms, is

$$\frac{S_z}{E}\frac{r_0}{r^2}\left(r\frac{d\phi}{dt}\right)^{-1}.$$

Since
$$S_z \approx \frac{ER^2}{\tau}, \quad \left(r\frac{d\phi}{dt}\right) \approx \frac{r}{T},$$

R being the radius of the planet, τ the period of the rotation of the planet around its axis, and T the period of the revolution around the sun, the term in question is seen to be of the order of magnitude $\frac{r_0}{r}\frac{R^2}{r^2}\frac{T}{\tau}$ compared with the Newtonian terms, and therefore of the order $\frac{R^2}{r^2}\frac{T}{\tau}$ compared with the ordinary relativistic terms. The ordinary relativistic perturbation, however, is itself very small, and is observable for Mercury alone. In the case of this planet, $T = \tau$ and $R^2/r^2 \approx 10^{-8}$. Similar numerical results would be obtained for the other spin-dependent terms in (3·4). Therefore, the effect of the spin is exceedingly small in the case of the planetary motion.

4. Equations of motion in Cartesian co-ordinates

In this section the equations (2·6) and (2·8) will be written in Cartesian co-ordinates. The use of these has the advantage of making possible a better understanding of the meaning of m_s; it also simplifies the discussion of deflexion problems.

Therefore, let us introduce a spatial frame of reference x, y, z, related to the system of polar co-ordinates used in § 2 by the equations

$$x^1 \equiv x = r\cos\theta\cos\phi, \quad x^2 \equiv y = r\cos\theta\sin\phi, \quad x^3 \equiv z = r\sin\theta. \tag{4·1}$$

For the new system of co-ordinates the non-vanishing Christoffel symbols are*

$$\left.\begin{aligned} \Gamma^l_{ik} &= \mu' \frac{x^l}{r}\left[e^{\mu}\,\delta_{ik} - \frac{x^i x^k}{2r^2}(1+2e^{\mu})\right], \quad \Gamma^l_{44} = \frac{\mu'}{2r} e^{2\mu} x^l, \\ \Gamma^4_{i4} &= \Gamma^4_{4i} = \frac{\mu'}{2r} x^i, \quad \Gamma^4_{44} = 0. \end{aligned}\right\} \tag{4·2}$$

Let us introduce a three-dimensional vector $\mathbf{S}$, whose components S_x, S_y, S_z are the values of S^{23}, S^{31}, S^{12} in the Cartesian frame. Remembering that $S^{\alpha\beta}$ is a tensor, we find that the relation between S_x, S_y, S_z and the components S^{23}, S^{31}, S^{12} in polar co-ordinates, reads

$$\left.\begin{aligned} r^2 \cos\theta S^{23} &= -\cos\theta\,\cos\phi S_x - \cos\theta\,\sin\phi S_y - \sin\theta S_z, \\ r\cos\theta S^{31} &= \sin\theta\,\cos\phi S_x + \sin\theta\,\sin\phi S_y - \cos\theta S_z, \\ rS^{12} &= \sin\phi S_x - \cos\phi S_y. \end{aligned}\right\} \tag{4·3}$$

We can now write the equations (2·6) and (2·8) in Cartesian co-ordinates either directly by using the expressions (4·2) instead of (2·3), or by transforming the equations (2·7) and (2·13) with the help of (4·1) and (4·3). The equations for the spin in Cartesian co-ordinates are found to be:

$$\dot{\mathbf{S}} = \mu'\left[\dot{r}\mathbf{S} + \frac{1}{r}e^{\mu}(\mathbf{rS})\,\dot{\mathbf{r}} - \frac{1}{2r}(\dot{\mathbf{r}}\mathbf{S})\,\mathbf{r} - \frac{1}{2r^2}(1+2e^{\mu})\,\dot{r}(\mathbf{rS})\,\mathbf{r}\right], \tag{4·4}$$

where $\mathbf{r} \equiv (x, y, z)$.

By the equation
$$r^2 \dot{t}\boldsymbol{\omega} = (\mathbf{r} \times \dot{\mathbf{r}}), \tag{4·5}$$

we define a quantity $\boldsymbol{\omega}$, which can be interpreted as the angular velocity, referring to the motion of the test particle on its orbit. Then, after some elementary calculations, we find that the co-ordinates x, y, z of the test particle must satisfy the equation

$$(m+m_s)(\ddot{\mathbf{r}} + \boldsymbol{\lambda}) - \frac{3m_s}{r}\dot{r}\dot{\mathbf{r}} - \frac{3\mu'}{2r}\left(e^{\mu}\,\dot{t}(\mathbf{rS})\,\boldsymbol{\omega} + \frac{\dot{r}}{r}(\mathbf{r}\times\mathbf{S})\right) = 0. \tag{4·6}$$

Here the vector $\boldsymbol{\lambda}$ is defined by the equation

$$\boldsymbol{\lambda} = \frac{\mu'}{2r}(e^{2\mu}\,\dot{t}^2 + 2e^{\mu}\,\dot{\mathbf{r}}^2 - (1+2e^{\mu})\,\dot{r}^2)\,\mathbf{r}. \tag{4·7}$$

The equation for t is not altered by a purely spatial transformation, and is the same as the first of (2·13).

Finally, equation (2·11) for the definition of the quantity m_s, which also appears in (4·6), takes the form

$$m_s = -\frac{\mu' r}{2}\,\dot{t}(\boldsymbol{\omega}\mathbf{S}). \tag{4·8}$$

Here m_s has assumed the characteristic form of a spin-orbit interaction energy. In fact, it is proportional to the scalar product of the spin with the vector $\boldsymbol{\omega}$, which is connected with the orbital angular momentum.

The energy integral retains its form (2·14). The integral expressing the conservation of the angular momentum, on the other hand, can be derived either from (4·6) or by transforming (2·15) to Cartesian co-ordinates: once the equation for the

* See, for instance, Weyl (1922, p. 253).

component I_z has been written in these co-ordinates, the other two components can be derived very easily by making use of symmetry considerations. Thus we obtain the vector equation

$$\mathbf{S}+E\,\mathrm{e}^{-\mu}r^2\boldsymbol{\omega}+\frac{\mu'}{2r}[(\mathbf{rS})\,\mathbf{r}-r^2\mathbf{S}] = \mathbf{I} = \text{const.} \tag{4·9}$$

It is remarkable that in the right-hand side of (4·9), besides the first and second term, which are, respectively, the spin and what can be interpreted as the orbital angular momentum, there is an additional term, which causes a drastic departure from the ordinary Newtonian mechanics.

The existence of the spin-orbit coupling expressed by (4·8) seems to suggest that a precession of the orbital angular momentum and of the spin around **I** might take place.

Here we shall confine ourselves to the case of the planetary motion. In this case an approximate treatment is possible, based on the two assumptions: (i) $|\mathbf{S}|\ll|\mathbf{I}|$, (ii) $r\gg r_0$. Because of (i) the motion is very nearly plane (the plane of motion changing slowly with the time, and always passing through the central body). Because of (ii) the orbit is very nearly an ellipse (with a superimposed slow rotation of the perihelion). Let $\mathbf{S}_0$ be the value of the spin at a certain time, say $t=0$. At a later time, the spin will have assumed the value

$$\mathbf{S} = \mathbf{S}_0+\delta\mathbf{S}. \tag{4·10}$$

If t is not very large compared with the period T of the orbital motion, then $|\delta\mathbf{S}|\ll|\mathbf{S}_0|$, and the equation (4·4) can be approximated as follows:

$$\dot{\mathbf{S}}\approx\mu'\left[\dot{r}\mathbf{S}_0-\frac{3\dot{r}}{2r^2}(\mathbf{rS}_0)\,\mathbf{r}+\frac{1}{r}\mathbf{S}_0\times(\dot{\mathbf{r}}\times\mathbf{r})\right]. \tag{4·11}$$

Averaging (4·11) over one period T (of the orbital motion), we obtain

$$\left(\frac{d\mathbf{S}}{dt}\right)_{\text{av.}} = \left(\frac{2r_0}{r}\right)_{\text{av.}}(\boldsymbol{\omega}\times\mathbf{S}_0), \tag{4·12}$$

since the average value of the first two terms vanishes. Being $|\boldsymbol{\omega}|\approx 2\pi/T$, it is evident from (4·12) that **S** describes a precession around $\boldsymbol{\omega}$ with the period

$$T_{\text{prec.}}\approx\frac{T}{(2r_0/r)_{\text{av.}}}. \tag{4·13}$$

As the orbits of the planets are very nearly circles, $(1/r)_{\text{av.}}$ will not be very sensitive to the way in which the averaging process is carried out. For the earth we find

$$T_{\text{prec.}}\approx 5\times10^7\,T = 5\times10^7 \text{ years}. \tag{4·14}$$

For comparison, we might mention that the non-spherical shape of the earth has the consequence of inducing a precession with a period of 26000 years.

We would like to end this section with a general remark. If the spin precession is connected with any energy-consuming secondary effects, e.g. because of gravitational radiation, one should expect the motion to tend towards a final state characterized by a minimum of the spin-orbit interaction. According to (4·8), m_s will have a minimum when the spin is orientated in the same direction as the orbital angular momentum (in which case the motion will be plane, cf. § 3). This remark might be of interest for the theories of the origin of the planetary system.

5. The hyperbolic motion

Besides the elliptic motion, the Newtonian equations admit solutions representing the motion of particles on hyperbolic orbits. Such orbits, on the other hand, will be practically rectilinear, with a very small deflexion, in all cases when the particle is very energetic and the 'impact parameter', b, is much larger than the gravitational radius r_0. (By 'impact parameter' we mean the distance between the particle and the central body at the position of closest approach.)

The 'hyperbolic motion' has been already treated in general relativity by using the geodesic equations, this corresponding to the motion of spinless particles. In the present section we shall study the problem of the deflexion of very energetic spinning particles in the Schwarzschild field. Thus we shall be able to see whether the influence of the spin could alter appreciably the numerical results deduced by using the equations of the geodesics. In particular, we shall give an estimate of the influence that the spin of the photon might have on the deflexion of light.

In the literature, the discussion of deflexion problems is based on the use of polar co-ordinates. Here, on the contrary, Cartesian co-ordinates will be introduced. Their use makes the calculations much shorter and more compact, as it will be shown.

Using the energy integral (2·14), the variable s can be eliminated from the equation (4·6). Thus we obtain

$$\frac{d^2\mathbf{r}}{dt^2}+\frac{\mu'}{2r}\left[e^{2\mu}+2e^{\mu}\left(\frac{d\mathbf{r}}{dt}\right)^2-(1+2e^{\mu})\left(\frac{dr}{dt}\right)^2\right]\mathbf{r} \\ -\mu'\frac{dr}{dt}\frac{d\mathbf{r}}{dt}-\frac{3\mu' e^{\mu}}{2rE}\left[\frac{1}{r}\frac{dr}{dt}(\mathbf{r}\times\mathbf{S})+e^{\mu}(\mathbf{r}\mathbf{S})\,\boldsymbol{\omega}\right]=0. \qquad (5\cdot1)$$

Let the equations $\qquad x = b, \quad y = \beta t, \quad z = 0 \qquad (5\cdot2)$

with $b \gg r_0$ and $\beta \to 1$, represent the 'unperturbed' motion of a spinning test particle in the Schwarzschild field. As for the spin, arbitrary constant values (S_x, S_y, S_z) can be attributed to its components in the unperturbed motion, with the only restriction that they should not be too large, so that the following perturbation method be permissible.

In all small terms of the equation (5·1) we shall replace $\mathbf{r}$ and $\mathbf{S}$ by the unperturbed values. Thus we can write the first-order equations

$$\left.\begin{aligned}\frac{d^2x}{dt^2}+\frac{3r_0}{(b^2+t^2)^{\frac{5}{2}}}\left(b^3-\frac{S_z t^2}{E}\right)&=0,\\ \frac{d^2y}{dt^2}+\frac{r_0 t}{(b^2+t^2)^{\frac{5}{2}}}\left(b^2-2t^2+\frac{3bS_z}{E}\right)&=0,\\ \frac{d^2z}{dt^2}-\frac{3r_0}{E(b^2+t^2)^{\frac{5}{2}}}[(b^2-t^2)S_x+2bS_y t]&=0.\end{aligned}\right\} \qquad (5\cdot3)$$

Similarly, from (4·4) we could write the first order equations for the spin components.

We shall restrict ourselves to the discussion of the following cases:

(i) $\mathbf{S} = (0, 0, S)$. On this assumption the motion will be strictly in the plane $z = 0$ (cf. (3·1) et seq.). Integrating the first two equations (5·3) we obtain

$$\left.\begin{aligned} \frac{dx}{dt} &= -\frac{br_0 t}{\sqrt{(b^2+t^2)}}\left(\frac{2}{b^2}+\frac{1}{b^2+t^2}\right)+\frac{r_0 S t^3}{Eb^2(b^2+t^2)^{\frac{3}{2}}} \\ \frac{dy}{dt} &= 1-\frac{r_0}{\sqrt{(b^2+t^2)}}\left(2-\frac{b^2}{b^2+t^2}\right)+\frac{r_0 bS}{E(b^2+t^2)^{\frac{3}{2}}}. \end{aligned}\right\} \tag{5·4}$$

The values of the integration constants have been fixed by assuming that $dx/dt = 0$ and $dy/dt = 1$ for $t = 0$. The angle of deflexion $\delta\phi$ can now be obtained very easily from the asymptotic value of the ratio $(dx/dt):(dy/dt)$ for $t \to \pm\infty$. We find

$$\delta\phi \approx \frac{2r_0}{b}\left(1-\frac{S}{2bE}\right). \tag{5·5}$$

The spin-independent part of this deflexion agrees with the value given in the literature for the bending of light rays in the Schwarzschild field. (As for the spin-dependent term, we notice that it vanishes when E tends to infinity.)

(ii) If the unperturbed values of the spin are $(0, S, 0)$ (spin parallel to the velocity), or $(S, 0, 0)$ (spin perpendicular to the velocity and in the xy-plane), the perturbed motion will not be plane. In both these cases, however, the first two of (5·3) are independent of S, and, consequently, the deflexion will be identical with that which is found for spinless particles. Also the corrections to the spin can be evaluated very easily. It does not, however, seem worthwhile to give the details of the calculation.

These results could be applied tentatively to the problem of the deflexion of light. One finds then that the spin of the photon has practically no influence on the value of the deflexion. Assuming, namely, the spin to be orientated as in (i), and taking $S \approx \hbar$, $E = h\nu = h\frac{c}{\lambda}$, one finds for the second term in the bracket of (5·5) the extremely small value λ/b. Actually, one should expect the spin of the photon to be orientated as in (ii) (first case); then, however, it would not give any contribution to the deflexion.

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