Graviton physics

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The interactions of gravitons with matter are calculated in parallel with the familiar photon case. It is shown that graviton scattering amplitudes can be factorized into a product of familiar electromagnetic forms. The cross sections for various reactions are evaluated straightforwardly using helicity methods.
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I. INTRODUCTION

The evaluation of the Compton scattering cross section is a standard exercise in relativistic quantum mechanics, because gauge invariance together with the zero mass of the photon allows the results to be presented in terms of simple analytic forms.[1]

Superficially, a similar analysis should be applicable to the interactions of gravitons. Like photons, such particles are massless and subject to a gauge invariance, so that similar analytic results for graviton cross sections can be expected. Also, just as virtual photon exchange leads to a detailed understanding of electromagnetic interactions between charged systems, a careful treatment of virtual graviton exchange allows an understanding not just of Newtonian gravity, but also of spin-dependent phenomena associated with general relativity that will be tested in the recently launched gravity probe B.[2] However, despite this obvious parallel, quantum mechanics texts (with one exception[3]) do not discuss graviton interactions in any detail. There are at least three reasons for this situation: (1) the graviton is a spin-two particle, in contrast to the spin-one photon, so that the interaction forms are more complex, involving symmetric second rank tensors rather than simple Lorentz four-vectors; (2) there exist few experimental results with which to compare the theoretical calculations; (3) to guarantee gauge invariance in some processes we must include, in addition to the usual Born and seagull diagrams, the contribution from a graviton pole term, which involves a triple-graviton coupling. This vertex is a sixth rank tensor and contains a multitude of kinematic forms.

Recently, using powerful (string-based) techniques, which simplify conventional quantum field theory calculations, it has been demonstrated that the elastic scattering of gravitons from an elementary target of arbitrary spin must factorize,[4] a feature that had been noted earlier based on gauge theory arguments.[5] This factorization permits a relatively painless evaluation of the various graviton amplitudes. In the following we show how this factorization comes about and evaluate some relevant cross sections. These calculations can be used as an interesting auxiliary topic in an advanced quantum mechanics course.

In Sec. II we review the electromagnetic case and develop the corresponding gravitational formalism. In Sec. III we give the factorization results and calculate the relevant cross sections. Our results are summarized in Sec. IV. Two appendices contain some of the formalism and calculational details.

II. PHOTON INTERACTIONS: A LIGHTNING REVIEW

Before treating the case of gravitons it is useful to review photon interactions, because this familiar formalism can be used as a bridge to our understanding of the gravitational case. We begin by generating the photon interaction Lagrangian, which is accomplished by giving the free matter Lagrangian together with the minimal substitution[6]

$$i\partial_\mu \to iD_\mu \equiv i\partial_\mu - eA_\mu, \tag{1}$$

where e is the particle charge and A_μ is the photon field. We will discuss the case of a scalar field and a spin 1/2 field, because these are familiar to most readers. For example, the Lagrangian for a free charged Klein-Gordon field is

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + m^2 \phi^\dagger \phi, \tag{2}$$

which becomes

$$\mathcal{L} = (\partial_\mu - ieA_\mu)\phi^\dagger(\partial^\mu + ieA^\mu)\phi + m^2\phi^\dagger\phi \tag{3}$$

after the minimal substitution. The corresponding interaction Lagrangian can then be identified:

$$\mathcal{L}_{\text{int}} = -ieA_\mu(\partial^\mu\phi^\dagger\phi - \phi^\dagger\partial^\mu\phi) + e^2A^\mu A_\mu\phi^\dagger\phi. \tag{4}$$

Similarly, for spin 1/2 the free Dirac Lagrangian

$$\mathcal{L} = \bar\psi(i\nabla\!\!\!/ - m)\psi, \tag{5}$$

becomes

$$\mathcal{L} = \bar\psi(i\nabla\!\!\!/ - eA\!\!\!/ - m)\psi, \tag{6}$$

where for a four-vector V_μ, $V\!\!\!/ \equiv \gamma_\mu V^\mu$, and the interaction Lagrangian is found to be

$$\mathcal{L}_{\text{int}} = -e\bar\psi A\!\!\!/\psi. \tag{7}$$

The single-photon vertices are then

$$\langle p_f|V^\mu_{\text{em}}|p_i\rangle_{S=0} = e(p_f + p_i)^\mu \tag{8}$$

for spin zero and

$$\langle p_f|V^\mu_{\text{em}}|p_i\rangle_{S=1/2} = e\bar u(p_f)\gamma^\mu u(p_i) \tag{9}$$

for spin 1/2, and the amplitudes for photon (Compton) scattering can be calculated (see Fig. 1). For spin zero, three diagrams are involved—two Born terms and a seagull—and the total amplitude is

$$A_{\text{Compton}}(S=0) = 2e^2\left[\frac{2\epsilon_i\cdot p_i\epsilon_f^*\cdot p_f}{p_i\cdot k_i} - \frac{\epsilon_i\cdot p_f\epsilon_f^*\cdot p_i}{p_i\cdot k_f} - \epsilon_f^*\cdot\epsilon_i\right]. \tag{10}$$

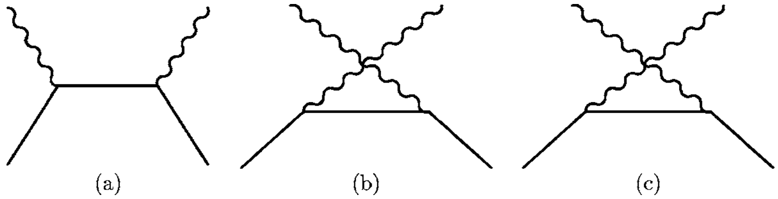

Fig. 1. Diagrams relevant to Compton scattering.

Note that all three diagrams must be included to satisfy the stricture of gauge invariance, which requires that the amplitude be unchanged under a gauge change $\epsilon_\mu \to \epsilon_\mu + \lambda k_\mu$. We easily verify that under such a change for the incident photon

$$\delta A_{\text{Compton}}(S=0) = \lambda 2e^2\left[\frac{p_i\cdot k_i \epsilon_f^* \cdot p_f}{k_i\cdot p_i} - \frac{k_i\cdot p_f \epsilon_f\cdot p_i}{p_f\cdot k_i} - k_i\cdot\epsilon_f^*\right] \tag{11a}$$

$$= \lambda 2e^2 \epsilon_f^* \cdot (p_f - p_i - k_i) = \lambda 2e^2 \epsilon_f^* \cdot k_f = 0. \tag{11b}$$

For spin 1/2 there exists no seagull diagram and only the two Born diagrams exist, yielding[7]

$$A_{\text{Compton}}\left(S=\frac{1}{2}\right) = e^2\bar{u}(p_f)\left[\frac{\not{\epsilon}_f^*(\not{p}_i + \not{k}_i + m)\not{\epsilon}_i}{2p_i\cdot k_i} - \frac{\not{\epsilon}_i(\not{p}_f - \not{k}_i + m)\not{\epsilon}_f^*}{2p_f\cdot k_i}\right]u(p_i). \tag{12}$$

It can be easily verified that this amplitude is gauge-invariant:

$$\delta A_{\text{Compton}}\left(S=\frac{1}{2}\right) = \lambda e^2\bar{u}(p_f)\left[\frac{\not{\epsilon}_f^*(\not{p}_i + \not{k}_i + m)\not{k}_i}{2p_i\cdot k_i} - \frac{\not{k}_i(\not{p}_f - \not{k}_i + m)\not{\epsilon}_f^*}{2p_f\cdot k_i}\right]u(p_i) \tag{13a}$$

$$= \bar{u}(p_f)[\not{\epsilon}_f^* - \not{\epsilon}_f^*]u(p_i) = 0. \tag{13b}$$

The corresponding cross sections can then be found via standard methods, as shown in many texts.[7] The results are usually presented in the laboratory frame for which $p_i = (m, \vec{0})$ and the incident and final photon energies are related by

$$\omega_f = \frac{\omega_i}{1 + 2(\omega_i/m)\sin^2(\theta/2)}, \tag{14}$$

where θ is the scattering angle. For unpolarized scattering, we sum (average) over final (initial) spins via

$$\sum_\lambda \epsilon_\lambda^{*\mu}\epsilon_\lambda^\nu = -\eta^{\mu\nu}, \quad \sum_s u(p,s)_i\bar{u}(p,s)_j = \frac{(\not{p}+m)_{ij}}{2m}. \tag{15}$$

The resulting cross sections are well known:

$$\frac{d\sigma_{\text{lab}}(S=0)}{d\Omega} = \frac{\alpha^2}{m^2}\left(\frac{\omega_f}{\omega_i}\right)^2\frac{1}{2}(1+\cos^2\theta), \tag{16}$$

$$\frac{d\sigma_{\text{lab}}(S=1/2)}{d\Omega} = \frac{\alpha}{2m^2}\left(\frac{\omega_f}{\omega_i}\right)^2\left[\frac{\omega_f}{\omega_i} + \frac{\omega_i}{\omega_f} - \ 1 + \cos^2\theta)\right] \tag{17}$$

$$= \frac{\alpha^2}{m^2}\frac{\omega_f^2}{\omega_i^2}\left[\frac{1}{2}(1+\cos^2\theta)\left(1 + 2\frac{\omega_i}{m}\sin^2\frac{\theta}{2}\right) + 2\frac{\omega_i^2}{m^2}\sin^4\frac{\theta}{2}\right]. \tag{18}$$

Helicity methods. In the gravitational case considered below it is useful to derive these results in an alternative fashion, using the helicity formalism, wherein the amplitude is expressed in terms of components of definite helicity.[8] Helicity is defined by the projection of the particle spin along its momentum direction. For a photon moving along the z direction, we choose states

$$\epsilon_i^{\lambda_i} = -\frac{\lambda_i}{\sqrt{2}}(\hat{x} + i\lambda_i\hat{y}) \quad (\lambda_i = \pm), \tag{19}$$

and for a photon moving in the direction $\hat{k}_f = \sin\theta\hat{x} + \cos\theta\hat{z}$, we use the states

$$\epsilon_f^{\lambda_f} = -\frac{\lambda_f}{\sqrt{2}}(\cos\theta\hat{x} + i\lambda_f\hat{y} - \sin\theta\hat{z}) \quad (\lambda_f = \pm). \tag{20}$$

In the center of mass frame we can calculate the amplitude for transitions between states of definite helicity. If we use

$$\epsilon_i^\pm \cdot p_f = -\epsilon_f^{*\pm}\cdot p_i = \mp\frac{p}{\sqrt{2}}\sin\theta, \tag{21}$$

$$\epsilon_f^{*\pm}\cdot\epsilon_i^\pm = -\tfrac{1}{2}(1+\cos\theta), \tag{22a}$$

$$\epsilon_f^{*\pm}\cdot\epsilon_i^\mp = -\tfrac{1}{2}(1-\cos\theta), \tag{22b}$$

we find for spin zero Compton scattering

$$A_{++} = A_{--} = -e^2\left(1 + \cos\theta + \frac{p^2\sin^2\theta}{p_i\cdot k_f}\right), \tag{23a}$$

$$A_{+-} = A_{-+} = -e^2\left(1 - \cos\theta - \frac{p^2\sin^2\theta}{p_i\cdot k_f}\right). \tag{23b}$$

Although these results can be found by direct calculation, the calculation can be simplified by realizing that under a parity transformation, the momentum reverses but the spin stays the same. Thus the helicity reverses, so parity conservation assures the equality of $A_{a,b}$ and $A_{-a,-b}$, while under time reversal both spin and momentum change sign, as do the initial and final states, guaranteeing that the helicity amplitudes are symmetric and hence $A_{a,b}=A_{b,a}$.

By using the standard definitions

$$s=(p_i+k_i)^2, \quad t=(k_i-k_f)^2, \quad u=(p_i-k_f)^2, \tag{24}$$

it is easy to see from simple kinematical considerations that

$$p^2=\frac{(s-m^2)^2}{4s}, \tag{25a}$$

$$\cos\frac{\theta}{2}=\frac{[(s-m^2)^2+st]^{1/2}}{s-m^2}=\frac{(m^4-su)^{1/2}}{s-m^2}, \tag{25b}$$

$$\sin\frac{\theta}{2}=\frac{(-st)^{1/2}}{s-m^2}. \tag{25c}$$

We can then write[9]

$$A_{++}=A_{--}=2e^2\frac{(s-m^2)^2+st}{(s-m^2)(u-m^2)}, \tag{26a}$$

$$A_{+-}=A_{-+}=2e^2\frac{-m^2t}{(s-m^2)(u-m^2)}. \tag{26b}$$

(The general form of these amplitudes follows from simple kinematical constraints, as shown in Ref. 10.) The cross section can now be written in terms of Lorentz invariants as

$$\frac{d\sigma}{dt}=\frac{1}{16\pi(s-m^2)^2}\frac{1}{2}\sum_{i,j=\pm}|A_{ij}|^2 \tag{27a}$$

$$=4e^4\frac{(m^4-su)^2+m^4t^2}{16\pi(s-m^2)^4(u-m^2)^2}, \tag{27b}$$

and can be evaluated in any desired frame. In the laboratory frame we have $s-m^2=2m\omega_i$, $u-m^2=-2m\omega_f$, $m^4-su=4m^2\omega_i\omega_f\cos^2(\theta/2)$, and $m^2t=-4m^2\omega_i\omega_f\sin^2(\theta/2)$. Because

$$\frac{dt}{d\Omega}=\frac{d}{2\pi d\cos\theta}\left(-\frac{2\omega_i^2(1-\cos\theta)}{1+\frac{\omega_i}{m}(1-\cos\theta)}\right)=\frac{\omega_f^2}{\pi}, \tag{28}$$

the laboratory cross section is found to have the form

$$\frac{d\sigma_{\text{lab}}(S=0)}{d\Omega}=\frac{d\sigma(S=0)}{dt}\frac{dt}{d\Omega}=\frac{\alpha^2}{m^2}\frac{\omega_f^2}{\omega_i^2}\left(\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}\right). \tag{29}$$

If we use the identity

$$\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}=\frac{1}{2}(1+\cos^2\theta), \tag{30}$$

Eq. (29) is seen to be identical to Eq. (16) derived by conventional means.

A corresponding analysis can be performed for spin 1/2. Working again in the center of mass frame and using helicity states for both photons and spinors, we can calculate the various amplitudes. In this case it is convenient to define the photon as the target particle, so that the corresponding polarization vectors are

$$\epsilon_i^{\lambda_i}=-\frac{\lambda_i}{\sqrt{2}}(-\hat{x}+i\lambda_i\hat{y}), \tag{31a}$$

$$\epsilon_f^{\lambda_f *}=-\frac{\lambda_f}{\sqrt{2}}(\sin\theta\hat{x}+\cos\theta\hat{z}-i\lambda_f\hat{y}) \tag{31b}$$

for initial (final) state helicity λ_i (λ_f). If we work in the center of mass, the corresponding helicity amplitudes can then be evaluated via

$$B_{s_f\lambda_f;s_i\lambda_i}=\bar{u}(p_f,s_f)\left[\frac{2\epsilon_i\cdot p_i\not{\epsilon}_f^*}{p_i\cdot k_i}-\frac{2\epsilon_i\cdot p_f\not{\epsilon}_f^*}{p_i\cdot k_f}+\frac{\not{\epsilon}_f^*\not{k}_i\not{\epsilon}_i}{p_i\cdot k_i}-\frac{\not{\epsilon}_i\not{k}_i\not{\epsilon}_f^*}{p_i\cdot k_f}\right]u(p_i,s_i). \tag{32}$$

Useful identities in this evaluation are

$$\mathcal{O}_1\equiv\not{\epsilon}_f^*\not{k}_i\not{\epsilon}_i=\frac{p\lambda_i\lambda_f}{2}\begin{pmatrix}A+\Sigma & \lambda_i(A+\Sigma)\\ -\lambda_i(A+\Sigma) & -(A+\Sigma)\end{pmatrix}, \tag{33a}$$

$$\mathcal{O}_2\equiv\not{\epsilon}_i\not{k}_i\not{\epsilon}_f^*=\frac{p\lambda_i\lambda_f}{2}\begin{pmatrix}A-\Sigma & -\lambda_i(A-\Sigma)\\ \lambda_i(A-\Sigma) & -(A-\Sigma)\end{pmatrix}, \tag{33b}$$

$$\mathcal{O}_3\equiv\epsilon_i\cdot p_f\not{\epsilon}_f^*=\frac{p\sin\theta\lambda_i\lambda_f}{2}\begin{pmatrix}0 & \Upsilon\\ -\Upsilon & 0\end{pmatrix}, \tag{33c}$$

where

$$A=\lambda_i\lambda_f+\cos\theta, \tag{34a}$$

$$\Sigma=(\cos\theta\lambda_i+\lambda_f)\sigma_z+\lambda_i\sin\theta\sigma_x-i\sin\theta\sigma_y, \tag{34b}$$

$$\Upsilon=-\cos\theta\sigma_x+\sin\theta\sigma_z-i\lambda_f\sigma_y. \tag{34c}$$

We have then

$$B_{s_f\lambda_f;s_i\lambda_i}=\frac{E+m}{2m}\begin{pmatrix}\chi_f^\dagger\\ \frac{-2s_fp}{E+m}\chi_f^\dagger\end{pmatrix}\times\left[\frac{1}{s-m^2}\mathcal{O}_1-\frac{1}{u-m^2}(\mathcal{O}_2+\mathcal{O}_3)\right]\begin{pmatrix}\chi_i\\ \frac{2s_ip}{E+m}\chi_i\end{pmatrix}. \tag{35}$$

After a straightforward (but tedious) exercise, we find the amplitudes[11]

$$B_{(1/2)1;(1/2)1}=B_{-(1/2)-1;-(1/2)-1}=\frac{2\sqrt{s}e^2p\cos\frac{\theta}{2}}{m^2-u}\left(-\frac{1}{m}+\frac{m}{s}\sin^2\frac{\theta}{2}\right), \tag{36a}$$

$$B_{(1/2)1;(1/2)-1}=B_{-(1/2)1;-(1/2)-1}=B_{(1/2)-1;(1/2)1}=B_{-(1/2)-1;-(1/2)1}=-\frac{2e^2mp}{(m^2-u)\sqrt{s}}\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}, \tag{36b}$$

$$B_{(1/2)-1;(1/2)-1}=B_{-(1/2)1;-(1/2)1}=\frac{-2\sqrt{s}e^2p}{m(m^2-u)}\cos^3\frac{\theta}{2}, \tag{36c}$$

$$B_{(1/2)1;-(1/2)1} = -B_{-(1/2)-1;(1/2)-1} = B_{-(1/2)1;(1/2)1} = -B_{(1/2)-1;-(1/2)-1} = -\frac{2e^2 p}{(m^2-u)} \sin\frac{\theta}{2}\cos^2\frac{\theta}{2}, \quad (36d)$$

$$B_{(1/2)-1;-(1/2)1} = -B_{-(1/2)1;(1/2)-1} = \frac{-2e^2 p}{m^2-u}\sin^3\frac{\theta}{2}, \quad (36e)$$

$$B_{(1/2)1;-(1/2)-1} = -B_{-(1/2)-1;(1/2)1} = -\frac{2e^2 m^2 p}{s(m^2-u)}\sin^3\frac{\theta}{2}. \quad (36f)$$

By summing (averaging) over final (initial) spin 1/2 states, we define spin-averaged photon helicity quantities

$$|B_{++}|^2_{\text{av}} = |B_{--}|^2_{\text{av}} = \frac{2p^2 e^4 s \sin^2(\theta/2)}{m^2(m^2-u)^2} \times\left[1+\cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2}\frac{m^4}{s^2}\left(1-2\frac{s}{m^2}\right)\right], \quad (37a)$$

$$|B_{+-}|^2_{\text{av}} = |B_{-+}|^2_{\text{av}} = \frac{2p^2 e^4 \sin^4(\theta/2)}{(m^2-u)^2} \times\left[2\frac{m^2}{s}\cos^2\frac{\theta}{2} + \left(1+\frac{m^4}{s^2}\right)\sin^2\frac{\theta}{2}\right], \quad (37b)$$

which, in terms of invariants, have the form

$$|B_{++}|^2_{\text{av}} = |B_{--}|^2_{\text{av}} = \frac{e^4}{2m^2(u-m^2)^2(s-m^2)^2}(m^4-su) \times(2(m^4-su)+t^2), \quad (38a)$$

$$|B_{+-}|^2_{\text{av}} = |B_{-+}|^2_{\text{av}} = \frac{e^4 t^2(2m^2-t)}{2(s-m^2)^2(u-m^2)^2}. \quad (38b)$$

The laboratory frame cross section can be determined as before [note that this expression differs from Eq. (29) by the factor $4m^2$, which is due to the different normalizations for fermion and boson states $-m/E$ for fermions and $1/2E$ for bosons]

$$\frac{d\sigma(S=1/2)}{d\Omega} = \frac{4m^2\omega_f^2}{16\pi^2(s-m^2)^2}[|B_{++}|^2_{\text{av}} + |B_{+-}|^2_{\text{av}}] \quad (39a)$$

$$= \frac{\alpha^2}{m^2}\frac{\omega_f^2}{\omega_i^2}\left[\frac{1}{2}(1+\cos^2\theta)\left(1+2\frac{\omega_i}{m}\sin^2\frac{\theta}{2}\right) + 2\frac{\omega_i^2}{m^2}\sin^4\frac{\theta}{2}\right], \quad (39b)$$

which is seen to be identical to Eq. (18).

So far, all we have done is to derive the usual forms for Compton cross sections by nontraditional means. In Sec. III we shall see how the use of helicity methods allows the derivation of the corresponding graviton cross sections in an equally straightforward fashion.

III. GRAVITATION

The theory of graviton interactions can be developed in direct analogy to that of electromagnetism. Some of the details are given in Appendix A. Here we shall be content with a brief outline.

Just as the electromagnetic interaction can be written in terms of the coupling of a vector current j_μ to the vector potential A^μ with a coupling constant given by the charge e, $\mathcal{L}_{\text{int}} = -e j_\mu a^\mu$, the gravitational interaction can be described in terms of the coupling of the energy-momentum tensor $T_{\mu\nu}$ to the gravitational field $h^{\mu\nu}$ with a coupling constant κ,

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{2}\kappa T_{\mu\nu}h^{\mu\nu}. \quad (40)$$

Here the field tensor is defined in terms of the metric via

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \quad (41)$$

where κ is defined in terms of Newton's constant as $\kappa^2 = 32\pi G$. The energy-momentum tensor is defined in terms of the free matter Lagrangian via

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_{\text{int}}}{\delta g^{\mu\nu}}, \quad (42)$$

where

$$\sqrt{-g} = \sqrt{-\det g} = \exp\left(\tfrac{1}{2}\text{Tr}\log g\right) \quad (43)$$

is the square root of the determinant of the metric. This prescription yields the forms

$$T_{\mu\nu} = \partial_\mu\phi^\dagger\partial_\nu\phi + \partial_\nu\phi^\dagger\partial_\mu\phi - g_{\mu\nu}(\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi) \quad (44)$$

for a scalar field and

$$T_{\mu\nu} = \bar\psi\left[\frac{1}{4}\gamma_\mu i\overleftrightarrow{\nabla}_\nu + \frac{1}{4}\gamma_\nu i\overleftrightarrow{\nabla}_\mu - g_{\mu\nu}\left(\frac{i}{2}\overleftrightarrow{\nabla\!\!\!/} - m\right)\right]\psi \quad (45)$$

for spin 1/2, where we have defined

$$\bar\psi i\overleftrightarrow{\nabla}_\mu\psi \equiv \bar\psi i\nabla_\mu\psi - (i\nabla_\mu\bar\psi)\psi. \quad (46)$$

The matrix elements of $T_{\mu\nu}$ can be read off as

$$\langle p_f|T_{\mu\nu}|p_i\rangle_{S=0} = p_{f\mu}p_{i\nu} + p_{f\nu}p_{i\mu} - \eta_{\mu\nu}(p_f\cdot p_i - m^2), \quad (47a)$$

$$\langle p_f|T_{\mu\nu}|p_i\rangle_{S=1/2} = \bar u(p_f)\left[\tfrac{1}{4}\gamma_\mu(p_f+p_i)_\mu + \tfrac{1}{4}\gamma_\nu(p_f+p_i)_\mu\right]u(p_i). \quad (47b)$$

We shall work in the harmonic (deDonder) gauge which satisfies, in lowest order,

$$\partial^\mu h_{\mu\nu} = \tfrac{1}{2}\partial_\nu h, \quad (48)$$

where

$$h = \text{Tr}\, h_{\mu\nu}, \quad (49)$$

and in which the graviton propagator has the form

$$D_{\alpha\beta;\gamma\delta}(q) = \frac{i}{2q^2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\beta}\eta_{\gamma\delta}). \quad (50)$$

Just as the (massless) photon is described in terms of a spin-one polarization vector ϵ_μ which can have projection (helicity) plus or minus one along the momentum direction, the (massless) graviton is a spin two particle which can have the projection (helicity) plus or minus two along the momentum

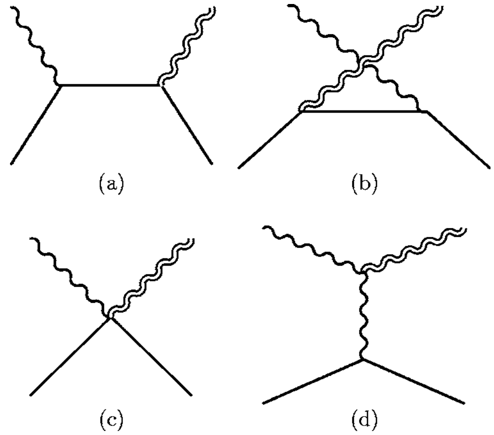

Fig. 2. Diagrams relevant to graviton photoproduction.

direction. Because $h_{\mu\nu}$ is a symmetric tensor, it can be described in terms of a simple product of unit spin polarization vectors

$$h^{(2)}_{\mu\nu} = \epsilon^+_\mu \epsilon^+_\nu \quad (\text{helicity} = +2), \tag{51a}$$

$$h^{(-2)}_{\mu\nu} = \epsilon^-_\mu \epsilon^-_\nu \quad (\text{helicity} = -2). \tag{51b}$$

Just as in electromagnetism, there is a gauge condition—in this case Eq. (48), which must be satisfied. Note that the helicity states given in Eq. (51) are consistent with the gauge requirement, because $\eta^{\mu\nu}\epsilon_\mu\epsilon_\nu=0$ and $k^\mu\epsilon_\mu=0$. With this background we can now examine various interesting reactions involving gravitons, as we detail in the following.

A. Graviton photoproduction

Before dealing with our ultimate goal, which is the treatment of graviton Compton scattering, we first warm up with a simpler process—that of graviton photoproduction

$$\gamma + s \to g + s, \quad \gamma + f \to g + f. \tag{52}$$

The relevant diagrams are shown in Fig. 2 and include the Born diagrams accompanied by a seagull and by the photon pole. The existence of a seagull is required by the feature that the energy-momentum tensor is momentum-dependent and therefore yields contact interactions when the minimal substitution is made, yielding the amplitudes

$$\langle p_f;k_f,\epsilon_f\epsilon_f|T|p_i;k_i,\epsilon_i\rangle_{\text{seagull}} = \kappa e \begin{cases} -2\epsilon^*_f\cdot\epsilon_i\epsilon^*_f\cdot(p_f+p_i) & (S=0) \\ \bar{u}(p_f)\not{\epsilon}^*_f\epsilon^*_f\cdot\epsilon_i u(p_i) & \left(S=\tfrac{1}{2}\right). \end{cases} \tag{53}$$

For the photon pole diagram we require a new ingredient, the graviton-photon coupling, which can be found from the expression for the photon energy-momentum tensor[6]

$$T_{\mu\nu} = -F_{\mu\alpha}F^\alpha_\nu + \tfrac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \tag{54}$$

and generates the photon pole term

$$\begin{aligned}\langle p_f;k_f,\epsilon_f\epsilon_f|T||p_i;k_i\epsilon_i\rangle_{\gamma\text{-pole}} &= e\langle p_f|j_\alpha|p_i\rangle\frac{1}{(p_f-p_i)^2}\times\frac{\kappa}{2}[2\epsilon^{*\alpha}_f(k_f\cdot k_i\epsilon^*_f\cdot\epsilon_i \\ &\quad -\epsilon^*_f\cdot k_i\epsilon_i\cdot k_f)+2\epsilon_i\cdot k_f(\epsilon^*_f\cdot\epsilon_i k^\alpha_f-\epsilon_i\cdot k_f\epsilon^{*\alpha}_f)].\end{aligned} \tag{55}$$

If we add the four diagrams together, we find [after considerable but simple algebra (see Appendix B)] a remarkably simple result

$$\langle p_f;k_f,\epsilon_f\epsilon_f|T|p_i;k_i,\epsilon_i\rangle = H(\epsilon^*_{f\alpha}\epsilon_{i\beta}T^{\alpha\beta}_{\text{Compton}}(S)), \tag{56}$$

where the factor H is

$$H = \frac{\kappa}{4e}\frac{\epsilon^*_f\cdot p_f k_f\cdot p_i - \epsilon^*_f\cdot p_i k_f\cdot p_f}{k_i\cdot k_f}, \tag{57}$$

and $\epsilon^*_{f\alpha}\epsilon_{i\beta}T^{\alpha\beta}_{\text{Compton}}(S)$ is the Compton scattering amplitude for particles of spin S calculated in Sec. II. The gauge invariance of Eq. (56) is obvious, because it follows directly from the gauge invariance already shown for the corresponding photon amplitudes together with that of the factor H under $\epsilon_f\to\epsilon_f+\lambda k_f$. Also we note that in Eq. (56) the factorization condition mentioned in Sec. I is made manifest, and consequently the corresponding cross sections can be obtained simply. In principle, conventional techniques can be used, but this use is somewhat challenging in view of the tensor structure of the graviton polarization vector. However, factorization means that the helicity amplitudes for graviton photoproduction are simple products of the corresponding photon amplitudes times the universal factor H, and the cross sections are given by the simple photon forms times the universal factor H^2. In the center of mass frame we have $\epsilon^*_f\cdot p_i=-\epsilon^*_f\cdot k_i$, and the factor H assumes the form

$$|H| = \frac{\kappa}{4e}\left|\frac{\epsilon^*_f\cdot k_i k_f\cdot p_f}{k_i\cdot k_f}\right| = \frac{\kappa}{4e}\frac{p\sin\theta}{\sqrt{2}}\frac{s-m^2}{-t} = \frac{\kappa}{2e}\left[\frac{m^4-st}{-2t}\right]^{1/2}. \tag{58}$$

In the lab frame H becomes

$$|H_{\text{lab}}|^2 = \frac{\kappa^2 m^2}{8e^2}\frac{\cos^2(\theta/2)}{\sin^2(\theta/2)}, \tag{59}$$

and the graviton photoproduction cross sections are found to be

$$\frac{d\sigma}{d\Omega} = G\alpha\cos^2\frac{\theta}{2}\begin{cases}\left(\dfrac{\omega_f}{\omega_i}\right)^2\left[\cot^2\dfrac{\theta}{2}\cos^2\dfrac{\theta}{2}+\sin^2\dfrac{\theta}{2}\right] & (S=0) \\ \left(\dfrac{\omega_f}{\omega_i}\right)^3\left[\left(\cot^2\dfrac{\theta}{2}\cos^2\dfrac{\theta}{2}+\sin^2\dfrac{\theta}{2}\right)+\dfrac{2\omega_i}{m}\left(\cos^4\dfrac{\theta}{2}+\sin^4\dfrac{\theta}{2}\right)+2\dfrac{\omega_i^2}{m^2}\sin^2\dfrac{\theta}{2}\right] & (S=1/2).\end{cases} \tag{60}$$

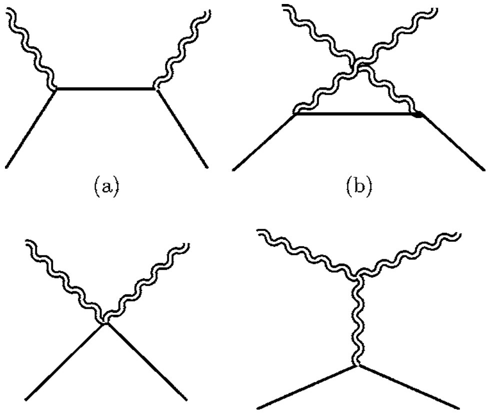

Fig. 3. Diagrams relevant for gravitational Compton scattering.

The form of the latter has been given by Voronov.[12]

B. Graviton Compton scattering

Finally, we can proceed to our primary goal, which is the calculation of graviton Compton scattering. To produce a gauge invariant scattering amplitude we require four separate contributions as shown in Fig. 3. Two of these diagrams are Born terms and can be written down straightforwardly. There are also seagull terms for spin 0 and for spin 1/2 whose forms can be found in Appendix A. For the scalar case we have

$$\begin{aligned}\langle p_f;k_f,\epsilon_f\epsilon_f|T|p_i;k_i\epsilon_i\epsilon_i\rangle_{\text{seagull}} &= \left(\frac{\kappa}{2}\right)^2\Bigg[-2\epsilon_f^*\cdot\epsilon_i(\epsilon_i\cdot p_i\epsilon_f^*\cdot p_f+\epsilon_i\cdot p_f\epsilon_f^*\cdot p_i) \\ &\quad -\frac{1}{2}(\epsilon_f^*\cdot\epsilon_i)^2k_i\cdot k_f\Bigg]. \qquad (61)\end{aligned}$$

In the spin 1/2 case we have

$$\begin{aligned}\langle p_f;k_f,\epsilon_f\epsilon_f|T|p_i;k_i\epsilon_i\epsilon_i\rangle_{\text{seagull}} &= \left(\frac{\kappa}{2}\right)^2\bar{u}(p_f)\times\Bigg[\frac{3}{16}\epsilon_f^*\cdot\epsilon_i(\not{\epsilon}_i\epsilon_f^*\cdot(p_i+p_f) \\ &\quad +\not{\epsilon}_f^*\epsilon_i\cdot(p_i+p_f))+\frac{i}{16}\epsilon_f^*\cdot\epsilon_i\epsilon^{\rho\sigma\eta\lambda}\gamma_\lambda\gamma_5 \\ &\quad \times(\epsilon_{i\eta}\epsilon_{f\sigma}^*k_{f\rho}-\epsilon_{f\eta}\epsilon_{i\sigma}k_{i\rho})\Bigg]u(p_i). \qquad (62)\end{aligned}$$

Despite the complex form of the various contributions, the final form, which results [after considerable algebra (see Appendix B)], is remarkably simple after the summation of the various components:

$$\begin{aligned}\epsilon_{f\alpha}\epsilon_{f\beta}A_{\text{grav}}^{\alpha\beta;\gamma\delta}\epsilon_{i\gamma}\epsilon_{i\delta} &= F(\epsilon_{f\mu}^*\epsilon_{i\nu}T_{\text{Compton}}^{\mu\nu}(S=0)) \\ &\quad \times(\epsilon_{f\alpha}^*\epsilon_{i\beta}T_{\text{Compton}}^{\alpha\beta}(S)), \qquad (63)\end{aligned}$$

where F is the universal factor

$$F=\frac{\kappa^2}{8e^4}\frac{p_i\cdot k_ip_i\cdot k_f}{k_i\cdot k_f}, \qquad (64)$$

and the Compton amplitudes are those calculated in Sec. II. The gauge invariance of this form is obvious from the already-demonstrated gauge invariance of the photon amplitudes; Eq. (63) is the factorized form guaranteed by general arguments.[5]

In principle it is possible though very challenging to evaluate the cross section by standard means, but the result follows directly by the use of helicity methods. From the form of Eq. (63) it is clear that the helicity amplitudes for graviton scattering have the simple form of a product of corresponding helicity amplitudes for spinless and spin S Compton scattering. For graviton scattering from a spinless target we have

$$|C_{++}|^2=|C_{--}|^2=F^2|A_{++}|^4, \qquad (65a)$$

$$|C_{+-}|^2=|C_{-+}|^2=F^2|A_{+-}|^4, \qquad (65b)$$

and for scattering from a spin 1/2 target, we find for the target-spin averaged helicity amplitudes

$$|D_{++}|_{\text{av}}^2=|D_{--}|_{\text{av}}^2=F^2|A_{++}|^2|B_{++}|_{\text{av}}^2, \qquad (66a)$$

$$|D_{+-}|_{\text{av}}^2=|D_{-+}|_{\text{av}}^2=F^2|A_{+-}|^2|B_{+-}|_{\text{av}}^2. \qquad (66b)$$

Here the factor F has the form

$$F=\frac{\kappa^2}{8e^4}\frac{(s-m^2)(u-m^2)}{t}, \qquad (67)$$

whose laboratory frame value is

$$F_{\text{lab}}=\frac{\kappa^2m^2}{8e^4}\frac{1}{\sin^2(\theta/2)}. \qquad (68)$$

The corresponding laboratory cross sections are then found to be

$$\begin{aligned}\frac{d\sigma_{\text{lab}}(S=0)}{d\Omega} &= \frac{\omega_f^2}{\pi}F^2\frac{1}{16\pi(s-m^2)^2}(|C_{++}|^2+|C_{+-}|^2) \qquad (69a) \\ &= G^2m^2\left(\frac{\omega_f}{\omega_i}\right)^2\left[\cot^4\frac{\theta}{2}\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}\right] \qquad (69b)\end{aligned}$$

for a spinless target. For a spin 1/2 target we have

$$\begin{aligned}\frac{d\sigma_{\text{lab}}\left(S=\frac{1}{2}\right)}{d\Omega} &= \frac{\omega_f^2}{\pi}F^2\frac{1}{16\pi(s-m^2)^2}(|D_{++}|_{\text{av}}^2+|D_{+-}|_{\text{av}}^2) \qquad (70a) \\ &= G^2m^2\left(\frac{\omega_f}{\omega_i}\right)^3\Bigg[\left(\cot^4\frac{\theta}{2}\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}\right) \\ &\quad +2\frac{\omega_i}{m}\left(\cot^2\frac{\theta}{2}\cos^6\frac{\theta}{2}+\sin^6\frac{\theta}{2}\right) \\ &\quad +2\frac{\omega_i^2}{m^2}\left(\cos^6\frac{\theta}{2}+\sin^6\frac{\theta}{2}\right)\Bigg]. \qquad (70b)\end{aligned}$$

The latter form agrees with that given by Voronov.[12]

We have given the results for unpolarized scattering from an unpolarized target, but having the form of the helicity amplitudes means that we can also obtain the cross sections for polarized photons or gravitons. However, that is a subject for another paper.

IV. SUMMARY

In other work we showed how the parallel between the exchange of virtual gravitons and photons could be used to understand the phenomena of geodetic and Lense–Thirring precession in terms of the related spin-orbit and spin-spin interactions in quantum electrodynamics.[2] In this paper we have shown how the treatment of graviton scattering processes can benefit from use of this analogy. Of course, such amplitudes are inherently more complex and involve tensor polarization vectors and the addition of somewhat complex photon or graviton pole diagrams. It is remarkable that when all the effects are summed, the resulting amplitudes factorize into simple products of photon amplitudes times kinematic factors. Using helicity methods, this factorization property allows the relatively elementary calculation of cross sections. It is hoped that this remarkable result will allow the introduction of graviton reactions into the quantum mechanics curriculum as at least a special topic. In any case, the simplicity associated with this result means that graviton interactions can be considered a subject that is no longer only associated with advanced research papers.

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APPENDIX A: GRAVITATIONAL FORMALISM

Here we present some of the basics of gravitational field theory. The details can be found in various references.[13,14] The full gravitational action is given by

$$S_g = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G}R + \mathcal{L}_m\right), \tag{A1}$$

where $\mathcal{L}_m$ is the Lagrange density for matter and R is the scalar curvature. Variation of Eq. (A1) via

$$g_{\mu\nu} \to \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{A2}$$

yields the Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}, \tag{A3}$$

where the energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial}{\partial g^{\mu\nu}}(\sqrt{-g}\mathcal{L}_m). \tag{A4}$$

We work in the weak field limit, with an expansion in powers of the gravitational coupling G,

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h^{(1)}_{\mu\nu} + \cdots, \tag{A5a}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{(1)\mu\nu} + \kappa^2 h^{(1)\mu\lambda}h^{(1)^\nu}_\lambda + \cdots, \tag{A5b}$$

where the superscript indicates the number of powers of G which appear, and the indices are understood to be raised or lowered by $\eta_{\mu\nu}$. We shall also need the determinant, which is given by

$$\sqrt{-g} = \exp\tfrac{1}{2}\mathrm{Tr}\log g = 1 + \tfrac{1}{2}\kappa h^{(1)} + \cdots . \tag{A6}$$

The corresponding curvatures are given by

$$R^{(1)}_{\mu\nu} = \frac{\kappa}{2}[\partial_\mu\partial_\nu h^{(1)} + \partial_\lambda\partial^\lambda h^{(1)}_{\mu\nu} - \partial_\mu\partial_\lambda h^{(1)^\lambda}_\nu - \partial_\nu\partial_\lambda h^{(1)^\lambda}_\mu], \tag{A7a}$$

$$R^{(1)} = \eta^{\mu\nu}R^{(1)}_{\mu\nu} = \kappa[\Box h^{(1)} - \partial_\mu\partial_\nu h^{(1)\mu\nu}]. \tag{A7b}$$

To define the graviton propagator, we must make a gauge choice. We shall work in the harmonic (or deDonder) gauge, $g^{\mu\nu}\Gamma^\lambda_{\mu\nu}=0$, which requires, to first order in the field expansion,

$$0 = \partial^\beta h^{(1)}_{\beta\alpha} - \tfrac{1}{2}\partial_\alpha h^{(1)}. \tag{A8}$$

If we use Eq. (A7), the Einstein equation reads, in lowest order,

$$\Box h^{(1)}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\Box h^{(1)} - \partial_\mu\left(\partial^\beta h^{(1)}_{\beta\nu} - \tfrac{1}{2}\partial_\nu h^{(1)}\right) - \partial_\nu\left(\partial^\beta h^{(1)}_{\beta\mu} - \tfrac{1}{2}\partial_\mu h^{(1)}\right) = -16\pi G T^{\mathrm{matter}}_{\mu\nu}, \tag{A9}$$

which, using the gauge condition Eq. (A8), can be written as

$$\Box\left(h^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{(1)}\right) = -16\pi G T^{\mathrm{matter}}_{\mu\nu}, \tag{A10}$$

or in the equivalent form

$$\Box h^{(1)}_{\mu\nu} = -16\pi G\left(T^{\mathrm{matter}}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}T^{\mathrm{matter}}\right). \tag{A11}$$

APPENDIX B: GRAVITATIONAL INTERACTIONS: SPIN 0

The coupling to matter via one-graviton and two-graviton vertices can be found by expanding the spin zero matter Lagrangian

$$\sqrt{-g}\mathcal{L}_m = \sqrt{-g}\left(\tfrac{1}{2}D_\mu\phi g^{\mu\nu}D_\nu\phi - \tfrac{1}{2}m^2\phi^2\right) \tag{B1}$$

as

$$\sqrt{-g}\mathcal{L}^{(0)}_m = \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2), \tag{B2a}$$

$$\sqrt{-g}\mathcal{L}^{(1)}_m = -\frac{\kappa}{2}h^{(1)\mu\nu}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial_\alpha\phi\partial^\alpha\phi - m^2\phi^2)\right), \tag{B2b}$$

$$\sqrt{-g}\mathcal{L}_m^{(2)} = \frac{\kappa^2}{2}\left(h^{(1)\mu\lambda}h_\lambda^{(1)\nu} - \frac{1}{2}h^{(1)}h^{(1)\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi - \frac{\kappa^2}{8}\left(h^{(1)\alpha\beta}h_{\alpha\beta}^{(1)} - \frac{1}{2}h^{(1)2}\right)(\partial^\alpha\phi\partial_\alpha\phi - m^2\phi^2). \tag{B2c}$$

The one- and two-graviton vertices are then, respectively,

$$\tau_{\alpha\beta}(p,p') = \frac{-i\kappa}{2}[p_\alpha p'_\beta + p'_\alpha p_\beta - \eta_{\alpha\beta}(p\cdot p' - m^2)], \tag{B3a}$$

$$\tau_{\alpha\beta,\gamma\delta}(p,p') = i\kappa^2\Big[I_{\alpha\beta,\rho\xi}I^{\xi}_{\sigma,\gamma\delta}(p^\rho p'^\sigma + p'^\rho p^\sigma) - \tfrac{1}{2}(\eta_{\alpha\beta}I_{\rho\sigma,\gamma\delta} + \eta_{\gamma\delta}I_{\rho\sigma,\alpha\beta})p'^\rho p^\sigma - \tfrac{1}{2}\Big(I_{\alpha\beta,\gamma\delta} - \tfrac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta}\Big)(p\cdot p' - m^2)\Big], \tag{B3b}$$

where we have defined

$$I_{\alpha\beta;\gamma\delta} = \tfrac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma}). \tag{B4}$$

We also require the triple graviton vertex $\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}(k,q)$ whose form is

$$\begin{aligned}\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}(k,q) = \frac{i\kappa}{2}\Bigg\{&\left(I_{\alpha\beta,\gamma\delta} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta}\right)\left[k^\mu k^\nu + (k-q)^\mu(k-q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2\right] + 2q_\lambda q_\sigma\big[I^{\lambda\sigma,}_{\alpha\beta}I^{\mu\nu,}_{\gamma\delta} + I^{\lambda\sigma,}_{\gamma\delta}I^{\mu\nu,}_{\alpha\beta} - I^{\lambda\mu,}_{\alpha\beta}I^{\sigma\nu,}_{\gamma\delta}\\ &- I^{\sigma\nu,}_{\alpha\beta}I^{\lambda\mu,}_{\gamma\delta}\big] + \big[q_\lambda q^\mu(\eta_{\alpha\beta}I^{\lambda\nu,}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\nu,}_{\alpha\beta}) + q_\lambda q^\nu(\eta_{\alpha\beta}I^{\lambda\mu,}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\mu,}_{\alpha\beta} \quad - q^2(\eta_{\alpha\beta}I^{\mu\nu,}_{\gamma\delta} + \eta_{\gamma\delta}I^{\mu\nu,}_{\alpha\beta}) - \eta^{\mu\nu}q^\lambda q^\sigma(\eta_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}\\ &+ \eta_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}) + \big[2q^\lambda(I^{\sigma\nu,}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\mu + I^{\sigma\mu,}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\nu - I^{\sigma\nu,}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\mu - I^{\sigma\mu,}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\nu) + q^2(I^{\sigma\mu,}_{\alpha\beta}I^{\nu}_{\gamma\delta,\sigma} + I^{\nu}_{\alpha\beta,\sigma}I^{\sigma\mu,}_{\gamma\delta})\\ &+ \eta^{\mu\nu}q^\lambda q_\sigma(I_{\alpha\beta,\lambda\rho}I^{\rho\sigma,}_{\gamma\delta} + I_{\gamma\delta,\lambda\rho}I^{\rho\sigma,}_{\alpha\beta})\big] + \left[(k^2 + (k-q)^2)\left(I^{\sigma\mu,}_{\alpha\beta}I^{\nu}_{\gamma\delta,\sigma} + I^{\sigma\nu,}_{\alpha\beta}I^{\mu}_{\gamma\delta,\sigma} - \frac{1}{2}\eta^{\mu\nu}P_{\alpha\beta,\gamma\delta}\right) - (k^2\eta_{\gamma\delta}I^{\mu\nu,}_{\alpha\beta} + (k\right.\\ &\left.\left. - q)^2\eta_{\alpha\beta}I^{\mu\nu,}_{\gamma\delta})\right]\Bigg\}.\end{aligned} \tag{B5}$$

APPENDIX C: GRAVITATIONAL INTERACTIONS: SPIN 1/2

For spin 1/2 we require some additional formalism in order to extract the gravitational couplings. In this case the matter Lagrangian reads

$$\sqrt{e}\mathcal{L}_m = \sqrt{e}\bar{\psi}(i\gamma^a e_a^\mu D_\mu - m)\psi, \tag{C1}$$

and involves the vierbein e_a^μ which links global coordinates with those in a locally flat space.[15,16] The vierbein is in some sense the "square root" of the metric tensor $g_{\mu\nu}$ and satisfies the relations

$$e_\mu^a e_\nu^b \eta_{ab} = g_{\mu\nu}, \quad e_\mu^a e_{a\nu} = g_{\mu\nu} \tag{C2a}$$

$$e^{a\mu}e_{b\mu} = \delta_b^a, \quad e^{a\mu}e_a^\nu = g^{\mu\nu}. \tag{C2b}$$

The covariant derivative is defined via

$$D_\mu\psi = \partial_\mu\psi + \frac{i}{4}\sigma^{ab}\omega_{\mu ab}, \tag{C3}$$

where

$$\omega_{\mu ab} = \tfrac{1}{2}e_a^\nu(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}) - \tfrac{1}{2}e_b^\nu(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}) + \tfrac{1}{2}e_a^\rho e_b^\sigma(\partial_\sigma e_{c\rho} - \partial_\rho e_{c\sigma})e_\mu^c. \tag{C4}$$

The connection with the metric tensor can be made via the expansion

$$e_\mu^a = \delta_\mu^a + \kappa c_\mu^{(1)a} + \cdots. \tag{C5}$$

The inverse of this matrix is

$$e_a^\mu = \delta_a^\mu - \kappa c_a^{(1)\mu} + \kappa^2 c_b^{(1)\mu}c_a^{(1)b} + \cdots, \tag{C6}$$

and we find

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa c_{\mu\nu}^{(1)} + \kappa c_{\nu\mu}^{(1)} + \cdots. \tag{C7}$$

For our purposes we shall use only the symmetric component of the c matrices, because these are physical and can be connected to the metric tensor. We find

$$c_{\mu\nu}^{(1)} \to \tfrac{1}{2}(c_{\mu\nu}^{(1)} + c_{\nu\mu}^{(1)}) = \tfrac{1}{2}h_{\mu\nu}^{(1)}. \tag{C8}$$

We have

$$\det e = 1 + \kappa c + \cdots = 1 + \frac{\kappa}{2}h + \cdots. \tag{C9}$$

By using Eq. (C9), we see that the matter Lagrangian has the expansion

$$\sqrt{e}\mathcal{L}_m^{(0)} = \bar{\psi}\left(\frac{i}{2}\gamma^\alpha\delta_\alpha^\mu\overleftrightarrow{\nabla}_\mu - m\right), \tag{C10a}$$

$$\sqrt{e}\mathcal{L}_m^{(1)} = -\frac{\kappa}{2}h^{(1)\alpha\beta}\bar{\psi}i\gamma_\alpha\overleftrightarrow{\nabla}_\beta\psi - \frac{\kappa}{2}h^{(1)}\bar{\psi}\left(\frac{i}{2}\overleftrightarrow{\nabla} - m\right)\psi, \tag{C10b}$$

$$\begin{aligned}\sqrt{e}\mathcal{L}_m^{(2)} = {}&\frac{\kappa^2}{8}h_{\alpha\beta}^{(1)}h^{(1)\alpha\beta}\bar{\psi}i\gamma^\gamma\overleftrightarrow{\nabla}_\lambda\psi + \frac{\kappa^2}{16}(h^{(1)})^2\bar{\psi}i\gamma^\gamma\overleftrightarrow{\nabla}_\gamma\psi\\ &- \frac{\kappa^2}{8}h^{(1)}\bar{\psi}i\gamma^\alpha h_\alpha^\lambda\overleftrightarrow{\nabla}_\lambda\psi + \frac{3\kappa^2}{16}h_{\delta\alpha}^{(1)}h^{(1)\alpha\mu}\bar{\psi}i\gamma^\delta\overleftrightarrow{\nabla}_\mu\psi\\ &+ \frac{\kappa^2}{4}h_{\alpha\beta}^{(1)}h^{(1)\alpha\beta}\bar{\psi}m\psi - \frac{\kappa^2}{8}(h^{(1)})^2\bar{\psi}m\psi + \frac{i\kappa^2}{16}h_{\delta\nu}^{(1)}\\ &\times(\partial_\beta h_\alpha^{(1)\nu} - \partial_\alpha h_\beta^{(1)\nu})\epsilon^{\alpha\beta\delta\epsilon}\bar{\psi}\gamma_\epsilon\gamma_5\psi.\end{aligned} \tag{C10c}$$

The corresponding one- and two-graviton vertices are then found to be

$$\tau_{\alpha\beta}(p,p') = \frac{-i\kappa}{2}\left[\frac{1}{4}(\gamma_\alpha(p+p')_\beta + \gamma_\beta(p+p')_\alpha) - \frac{1}{2}\eta_{\alpha\beta}\left(\frac{1}{2}(\not{p}+\not{p}') - m\right)\right] \tag{C11a}$$

$$\begin{aligned}\tau_{\alpha\beta,\gamma\delta}(p,p') = i\kappa^2\Bigg\{ & -\frac{1}{2}\left(\frac{1}{2}(\not{p}+\not{p}') - m\right)P_{\alpha\beta,\gamma\delta} \\ & - \frac{1}{16}[\eta_{\alpha\beta}(\gamma_\gamma(p+p')_\delta + \gamma_\delta(p+p')_\gamma) \\ & + \eta_{\gamma\delta}(\gamma_\alpha(p+p')_\beta + \gamma_\beta(p+p')_\alpha)] \\ & + \frac{3}{16}(p+p')^\epsilon\gamma^\xi(I_{\xi\phi,\alpha\beta}I^\phi_{\epsilon,\gamma\delta} + I_{\xi\phi,\gamma\delta}I^\phi_{\epsilon,\alpha\beta}) \\ & + \frac{i}{16}\epsilon^{\rho\sigma\eta\lambda}\gamma_\lambda\gamma_5(I^\nu_{\alpha\beta,\eta}I_{\gamma\delta,\sigma\nu}k'_\rho \\ & - I^\nu_{\gamma\delta,\eta}I_{\alpha\beta,\sigma\nu}k_\rho)\Bigg\}.\end{aligned} \tag{C11b}$$

APPENDIX D: GRAVITON SCATTERING AMPLITUDES

In this appendix we summarize the independent contributions to the various graviton scattering amplitudes that must be added to produce the complete amplitudes given in the text. We leave it to the (perspicacious) reader to perform the appropriate additions to verify the factorized forms shown earlier.

1. Graviton photoproduction: Spin 0

Born a: $$A_a = 4e\kappa\frac{(\epsilon_f^*\cdot p_f)^2\epsilon_i\cdot p_i}{2p_i\cdot k_i}. \tag{D1a}$$

Born b: $$A_b = -4e\kappa\frac{(\epsilon_f^*\cdot p_i)^2\epsilon_i\cdot p_f}{2p_i\cdot k_f}. \tag{D1b}$$

Seagull: $$A_c = -2e\kappa\epsilon_f^*\cdot\epsilon_i\epsilon_f^*\cdot(p_i+p_f). \tag{D1c}$$

γ-pole: $$\begin{aligned}A_d = \frac{e\kappa}{k_i\cdot k_f}[&\epsilon_f^*\cdot(p_i+p_f)(k_i\cdot k_f\epsilon_f^*\cdot\epsilon_i \\ &- \epsilon_f^*\cdot k_i\epsilon_i\cdot k_f) + \epsilon_f^*\cdot k_i(\epsilon_f^*\cdot\epsilon_i k_i\cdot(p_i+p_f) \\ &- \epsilon_f^*\cdot k_i\epsilon_i\cdot(p_i+p_f)\ ].\end{aligned} \tag{D1d}$$

2. Graviton photoproduction: Spin 1/2

Born a: $$A_a = e\kappa\frac{\epsilon_f^*\cdot p_f}{2p_i\cdot k_i}\bar{u}(p_f)[\not{\epsilon}_f^*(\not{p}_i+\not{k}_i+m)\not{\epsilon}_i]u(p_i). \tag{D2a}$$

Born b: $$A_b = -e\kappa\frac{\epsilon_f^*\cdot p_i}{2p_i\cdot k_f}\bar{u}(p_f)[\not{\epsilon}_i(\not{p}_i-\not{k}_f+m)\not{\epsilon}_f^*]u(p_i). \tag{D2b}$$

Seagull: $$A_c = -e\kappa\bar{u}(p_f)\not{\epsilon}_f^* u(p_i). \tag{D2c}$$

γ pole: $$\begin{aligned}A_d = e\kappa\frac{1}{k_i\cdot k_f}\bar{u}(p_f)[&\not{\epsilon}_f^*(k_i\cdot k_f\epsilon_f^*\cdot\epsilon_i - \epsilon_f^*\cdot k_i\epsilon_i\cdot k_f) \\ &+ \not{k}_f\epsilon_f^*\cdot\epsilon_i\epsilon_f^*\cdot k_i - \not{\epsilon}_i(\epsilon_f^*\cdot k_i)^2]u(p_i).\end{aligned} \tag{D2d}$$

3. Graviton scattering: Spin 0

Born a: $$A_a = 2\kappa^2\frac{(\epsilon_i\cdot p_i)^2(\epsilon_f^*\cdot p_f)^2}{p_i\cdot k_i}. \tag{D3a}$$

Born b: $$A_b = -2\kappa^2\frac{(\epsilon_f^*\cdot p_i)^2(\epsilon_i\cdot p_f)^2}{p_i\cdot k_f}. \tag{D3b}$$

Seagull: $$A_c = \kappa^2\left[\epsilon_f^*\cdot\epsilon_i(\epsilon_i\cdot p_i\epsilon_f^*\cdot p_f + \epsilon_i\cdot p_f\epsilon_f^*\cdot p_i) - \frac{1}{2}k_i\cdot k_f(\epsilon_f^*\cdot\epsilon_i)^2\right]. \tag{D3c}$$

g pole: $$\begin{aligned}A_d = \frac{4\kappa^2}{k_i\cdot k_f}\Bigg[&\epsilon_f^*\cdot p_f\epsilon_f^*\cdot p_i(\epsilon_i\cdot(p_i-p_f))^2 + \epsilon_i\cdot p_i\epsilon_i\cdot p_f(\epsilon_f^*\cdot(p_i+p_f))^2 + \epsilon_i\cdot(p_i-p_f)\epsilon_f^*\cdot(p_f-p_i)(\epsilon_f^*\cdot p_f\epsilon_i\cdot p_i \\ &+ \epsilon_f^*\cdot p_i\epsilon_i\cdot p_f) - \epsilon_f^*\cdot\epsilon_i(\epsilon_i\cdot(p_i-p_f)\epsilon_f^*\cdot(p_f-p_i)(p_i\cdot p_f-m^2) + k_i\cdot k_f(\epsilon_f^*\cdot p_f\epsilon_i\cdot p_i + \epsilon_f^*\cdot p_i\epsilon_i\cdot p_f) + \epsilon_i\cdot(p_i-p_f) \\ &\times(\epsilon_f^*\cdot p_f p_i\cdot k_f + \epsilon_f^*\cdot p_i p_f\cdot k_f) + \epsilon_f^*\cdot(p_f-p_i)(\epsilon_i\cdot p_i p_f\cdot k_i + \epsilon_i\cdot p_f p_i\cdot k_i)) + (\epsilon_f^*\cdot\epsilon_i)^2\Big(p_i\cdot k_i p_f\cdot k_i + p_i\cdot k_f p_f\cdot k_f \\ &- \frac{1}{2}(p_i\cdot k_i p_f\cdot k_f + p_i\cdot k_f p_f\cdot k_i) + \frac{3}{2}k_i\cdot k_f(p_i\cdot p_f-m^2)^2\Big)\Bigg].\end{aligned} \tag{D3d}$$

4. Graviton scattering: Spin 1/2

Born a: $$A_a = \kappa^2 \frac{\epsilon_f^* \cdot p_f \epsilon_i \cdot p_i}{8 p_i \cdot k_i} \bar{u}(p_f)[\not{\epsilon}_f^* (\not{p}_i + \not{k}_i + m) \not{\epsilon}_i] u(p_i). \tag{D4}$$

Born b: $$A_b = -\kappa^2 \frac{\epsilon_f^* \cdot p_i \epsilon_i \cdot p_f}{8 p_i \cdot k_f} \bar{u}(p_f)[\not{\epsilon}_i (\not{p}_i - \not{k}_f + m) \not{\epsilon}_f^*] u(p_i). \tag{D5}$$

Seagull: $$A_c = \kappa^2 \bar{u}(p_f) \Bigg[\frac{3}{16} \epsilon_f^* \cdot \epsilon_i (\not{\epsilon}_i \epsilon_f^* \cdot (p_i + p_f) + \not{\epsilon}_f^* \epsilon_i \cdot (p_i + p_f)) + \frac{i}{8} \epsilon_f^* \cdot \epsilon_i \epsilon^{\rho\sigma\eta\lambda} \gamma_\lambda \gamma_5 (\epsilon_{i\eta} \epsilon_{f\sigma}^* k_{f\rho} - \epsilon_{f\eta}^* \epsilon_{i\sigma} k_{i\rho}) \Bigg] u(p_i).$$

g pole: $$A_d = \frac{\kappa^2}{k_i \cdot k_f} \bar{u}(p_f) \Bigg[(\not{\epsilon}_i \epsilon_f^* \cdot k_i + \not{\epsilon}_f^* \epsilon_i \cdot k_f) \times (\epsilon_i \cdot p_i \epsilon_f^* \cdot p_f - \epsilon_i \cdot p_f \epsilon_f^* \cdot p_i) - (\epsilon_f^* \cdot \epsilon_i) \times (k_i \cdot k_f (\not{\epsilon}_f^* \epsilon_i \cdot k_f + \not{\epsilon}_i \epsilon_f^* \cdot p_i) + \not{k}_i (\epsilon_f^* \cdot p_f \epsilon_i \cdot p_i - \epsilon_f^* \cdot p_i \epsilon_i \cdot p_f) + p_i \cdot k_i (\not{\epsilon}_i \epsilon_f^* \cdot k_i + \not{\epsilon}_f^* \epsilon_i \cdot k_f)) + (\epsilon_f^* \cdot \epsilon_i)^2 \not{k}_i \left(p_i \cdot k_i - \frac{1}{2} k_i \cdot k_f \right) \Bigg] u(p_i). \tag{D6}$$

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