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RESEARCH ARTICLE

The Ergo Region of the Kerr Black Hole in the Isotropic Coordinate

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ABSTRACT

The isotropic coordinate is the more physically meaningful coordinate in the Schwarzschild black hole. Then we apply this isotropic coordinate to the Kerr black hole, and we have found the ergo region does not appear and all metrices $g_{\mu\nu}$ become regular even at the "event horizon" in this coordinate. But the determinant $\det g_{\mu\nu}$ becomes zero at the "event horizon", which means that $g^{\mu\nu}$ becomes singular at the "event horizon".

Keywords: Equivalence principle, Ergo region, Isotropic coordinate, Kerr black hole.

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1. Introduction

The Kerr black hole solution [1] is quite important from the theoretical and observational point of view. The Penrose process [2] is one of examples. Because the Kerr black hole is the rotating black hole, the Lense-Thirring effect [3] of frame dragging occurs. The important spacelike region, where the metric is rotating and the region is outside of the event horizon, is called ergo region.

Penrose proposed the interesting process to extract the rotational kinetic energy from the black hole through the rotating metric. This is the prototype of extracting the energy from the black hole. From the Penrose process, we can classically extract the energy from the black hole by using Lense-Thirring effect. While, from the Hawking process [4], we can quantum theoretically extract the energy from the black hole by using the Unruh effect [5].

In the theory of general relativity, the general coordinate transformations are allowed, which means that the physical meaning of coordinates of space-time becomes ambiguous. Then we must take the physically more meaningful coordinate. In this paper, we apply the isotropic coordinate [6], [7], which is the more physically meaningful coordinate in the Schwarzschild and Kerr black holes. Applying this isotropic coordinate to the Kerr black hole, we have found that the ergo region does not appear in this coordinate.

2. Special Relativity vs. General Relativity

In the theory of special relativity, the Lorentz transformation

$$x^{\mu\prime} = a^{\mu}_{\nu} x^{\nu}, \quad \eta_{\mu\nu} = \eta_{\rho\sigma} a^{\rho}_{\mu} a^{\sigma}_{\nu}, \tag{1}$$

is the physically meaningful transformation of one physical inertial frame to the other physical inertial frame. Coordinates $(\mathbf{x}, t)$ of one inertial frame and coordinates $(\mathbf{x}', t')$ of the other inertial frame both have the physical meaning of actual space and time coordinates.

While, in the theory of general relativity, the general coordinate transformation, $x^{\mu\prime} = f^{\mu}(x)$ for arbitrary functional form of x^{ν}, is the formal transformation. The metric transforms in the form

$$g_{\mu\nu'}(x') = \frac{\partial x^{\rho}}{\partial x^{\mu'}} \frac{\partial x^{\sigma}}{\partial x^{\nu'}} g_{\rho\sigma}(x). \tag{2}$$

We can start from any coordinate $(\mathbf{x}, t)$ where $\mathbf{x}$ and t are not necessary to have the meaning of space and time. The transformed coordinate $(\mathbf{x}', t')$ are also not necessary to be physically meaningful.

These $\mathbf{x}$ and $\mathbf{x}'$ are similar to the space coordinates but there is no criterion whether these are really the space coordinates. For t and t', the situation is the same. Thus, we consider that the general coordinate transformation is the formal transformation.

The Einstein equation is the differential equation, and we must put the boundary condition by ourselves to obtain the solution. The natural boundary condition of $g_{\mu\nu}(\mathbf{x}, t) \to \eta_{\mu\nu}$ as $|\mathbf{x}| \to \infty$ is not sufficient to uniquely determine the solution. There is no criterion of what kind of metric is the best physically meaningful metric, Thus, among various solutions, we must further make the coordinate transformation in such a way as $g_{\mu\nu}(\mathbf{x}, t)$ give the more physically meaningful metric. In section 2, as the more meaningful coordinate, we adopt the isotropic coordinate in the Schwarzschild black hole. Further, we apply that coordinate to the Kerr black hole.

In connection to the general coordinate transformation in the general relativity, we here make comments on the equivalence principle.

Suppose that one particle is on the surface of earth. Though the gravitational force acts on this particle, this particle cannot radiate the gravitational wave because of the energy conservation law. If this particle can radiate the gravitational wave, we eventually obtain infinite amount of energy through the energy of the gravitational wave without losing any energy somewhere else.

While, suppose that the other particle falls freely by the gravitational force of the earth, which is realized by the formal general coordinate transformation from the frame on the surface of earth in general relativity. Because the Christoffel symbol does not transform as the real tensor under the general coordinate transformation, by making the proper general coordinate transformation, we can always make the transformed Christoffel symbol to be locally zero. More explicitly, in order to make the transformed Christoffel symbol ${\Gamma^{\rho}_{\mu\nu}}'(x')$ to be zero near $x^{\mu} = 0$, it is necessary and sufficient to make the following infinitesimal transformation

$$x^{\mu} = x^{\mu\prime} - \frac{1}{2}\Gamma^{\rho}_{\mu\nu}(x = 0)x^{\rho\prime}x^{\sigma\prime}. \tag{3}$$

This procedure is called to take the local inertial frame. In this situation, because of the energy conservation, the free-falling particle is possible to lose the velocity by radiating the gravitational wave. This is nothing but the phenomena of the particle falls freely into the star, which is the actual process to radiate the gravitational wave. The key point of this argument is that the freely falling particle is possible to lose the velocity, which gives the possibility to receive the back reaction because of the energy conservation.

This is similar to the reason why the light is not radiated from the ground state of the hydrogen atom. Because the electron is accelerating in the hydrogen atom, we may imagine that the light is radiated even from the ground state. In this example, the key point is that the ground state of the hydrogen atom cannot lose the energy, so that the light is not radiated from the ground state of the hydrogen atom because of the energy conservation.

The terminology of the local inertial frame is misleading. From the relation of the inertial and gravitational masses m_I and m_G respectively being equivalent, we obtain that the inertial and gravitational forces $m_I a$ and $m_G g$ respectively are equivalent in the form $F = m_I a = m_G g$. But if the particle actually begins to fall, the back reaction by the gravitational wave comes in. Therefore, the particle on the surface of the earth, where the gravitational wave is not radiated, and the freely falling particle, where the gravitational wave is radiated, are not equivalent.

Therefore, under the formal general coordinate transformation, we can choose the local inertial frame, that is, we can make the inertial force and the gravitational force to be equivalent, but the back reaction of losing the velocity of the particle cannot be realized by the general coordinate transformation, which means that the general coordinate transformation is the formal transformation. While the physical transformation from the frame on the surface of the earth to the freely falling frame with the back reaction gives the different result with respect to the radiation of the gravitational wave.

Whether the energy and momentum are conserved or not depends on the translational symmetry of the theory under the physical time and space transformation. While, we construct the theory in such a way as the theory is always invariant under the general but not always physical coordinate transformation, where the physical energy-momentum conservation is not guaranteed.

3. Isotropic Coordinate in the Kerr Metric

3.1. Schwarzschild Coordinate of the Schwarzschild Black Hole

In the unit of $G = 1$, $c = 1$, the Schwarzschild coordinate of the Schwarzschild black hole is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + \sin^2\theta d\phi^2. \quad (4)$$

The determinant of this metric is given by $\det g_{\mu\nu} = -r^4 \sin^2\theta$. In this coordinate, we obtain $g_{00} = 0$ and $|g_{rr}| = \infty$ at $r = 2\,M$, so that the radius of the event horizon is given by $r_H = 2\,M$.

3.2. *Isotropic Coordinate of the Schwarzschild Black Hole*

In the Schwarzschild coordinate of the Schwarzschild black hole, the role of timelike and spacelike direction of dt and dr is exchanged when we consider metric outside or inside of the event horizon. While, $rd\theta$ and $r\sin\theta d\phi$ directions remain to be spacelike. Then it is more desirable to use the more physically meaningful coordinate, the isotropic coordinate [6], [7] of the Schwarzschild black hole, to analyze the free fall motion of the particle into the black hole. The difference between the Schwarzschild coordinate and the isotropic coordinate is only for the radial coordinate. We denote the radial coordinate of the Schwarzschild and isotropic coordinates by r and R respectively, and the relation between two radial coordinates is given by $r = R(1 + M/2R)^2$. By using this relation, we rewrite (4) in the form:

$$ds^2 = -\frac{(1 - M/2R)^2}{(1 + M/2R)^2} dt^2 + \left(1 + \frac{M}{2R}\right)^4 (dX^2 + dY^2 + dZ^2) \quad (5)$$

$$= -\frac{(1 - M/2R)^2}{(1 + M/2R)^2} dt^2 + \left(1 + \frac{M}{2R}\right)^4 (dR^2 + R^2\, d\theta^2 + R^2\, \sin^2\theta\, d\phi^2), \quad (6)$$

$$R = \sqrt{X^2 + Y^2 + Z^2},\ R_H = \frac{M}{2} (= \frac{1}{4} r_H). \quad (7)$$

The determinant of the metric of Cartesian coordinate (5) is given by $\det g_{\mu\nu} = -(1 - M/2R)^2 (1 + M/2R)^{10}$ and the determinant of the metric of radial coordinate (6) is given by

$$\det g_{\mu\nu} = -\left(1 - \frac{M}{2R}\right)^2 \left(1 + \frac{M}{2R}\right)^{10} R^4 \sin^2\theta. \quad (8)$$

It is evident that this isotropic coordinate is more physically meaningful coordinate. This metric is isotropic, so that X, Y and Z comes in the symmetric way, and $g_{00} \geq 0$ and $g_{RR} > 0$. In this isotropic coordinate of the Schwarzschild black hole, $g_{\mu\nu}$ have no singularity except $R = 0$, but there comes the new type of the singularity. For $R = M/2$, we obtain $\det g_{\mu\nu} = 0$ from (8), which comes from the zero point of g_{00}. Then $g^{\mu\nu}$ becomes singular at $R = M/2$. Thus $R_H = M/2$ is the new type of the event horizon, which we call the "event horizon". We must notice that the radius of the "event horizon" $R_H = M/2$ of the isotropic coordinate is different from the radius of the event horizon $r_H = 2M$ of the Schwarzschild coordinate, so that we can determine which is the physically meaningful coordinate by observing the radius of the event horizon.

3.3. *Boyer-Lindquist Coordinate of the Kerr Black Hole*

The Boyer-Lindquist coordinate [8] of the Kerr black hole [1] is given by

$$ds^2 = -\left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right) dt^2 - \frac{4aMr\sin^2\theta}{r^2 + a^2\cos^2\theta} dt d\phi + \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2} dr^2$$
$$+ (r^2 + a^2\cos^2\theta) d\theta^2 + \left(r^2 + a^2 + \frac{2a^2 Mr\sin^2\theta}{r^2 + a^2\cos^2\theta}\right) \sin^2\theta d\phi^2. \quad (9)$$

The determinant of the Kerr metric in Boyer-Lindquist coordinate is given by

$$\det g_{\mu\nu} = -(r^2 + a^2\cos^2\theta)^2 \sin^2\theta. \quad (10)$$

In this metric, the ergo region of $g_{00} < 0$ and $g_{rr} > 0$, that is,

$$M + \sqrt{M^2 - a^2} < r < M + \sqrt{M^2 - a^2\cos^2\theta}, \quad (11)$$

appears. Further, we obtain $|g_{rr}| = \infty$ at $r_\pm = M \pm \sqrt{M^2 - a^2}$, so that $r = r_\pm$ becomes the event horizon.

3.4. Isotropic Coordinate of the Kerr Black Hole

By using $r = R(1 + M/2R)^2$, we obtain isotropic coordinate of the Kerr black hole in the form

$$
\begin{aligned}
ds^2 &= g_{00}dt^2 + g_{RR}dR^2 + g_{t\phi}dtd\phi + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \\
&= -\left(\frac{(R^2 - M^2/4)^2 + a^2R^2\cos^2\theta}{(R+M/2)^4 + a^2R^2\cos^2\theta}\right) dt^2 \\
&\quad + \left(\frac{(R^2 - M^2/4)^2\left\{(R+M/2)^4 + a^2R^2\cos^2\theta\right\}}{R^4\left\{(R^2 - M^2/4)^2 + a^2R^2\right\}}\right) dR^2 \\
&\quad - \left(\frac{4aMR\sin^2\theta\,(R+M/2)^2}{(R+M/2)^4 + a^2R^2\cos^2\theta}\right) dtd\phi \\
&\quad + \left(\frac{(R+M/2)^4 + a^2R^2\cos^2\theta}{R^2}\right) d\theta^2 \\
&\quad + \left(\frac{(R+M/2)^4 + a^2R^2}{R^2} + \frac{2a^2MR\sin^2\theta\,(R+M/2)^2}{(R+M/2)^4 + a^2R^2\cos^2\theta}\right)\sin^2\theta\, d\phi^2.
\end{aligned}
\tag{12}
$$

The determinant of the Kerr metric in this coordinate is given by

$$
\det g_{\mu\nu} = -\frac{((R+M/2)^4 + a^2R^2\cos^2\theta)^2(R^2 - M^2/4)^2\sin^2\theta}{R^8}. \tag{13}
$$

We have $g_{tt} \le 0$, $g_{RR} \ge 0$, $g_{\theta\theta} > 0$, $g_{\phi\phi} \ge 0$ and $g_{t\phi}$ takes $\mp$ values for $a = \pm$ value. In this coordinate, the ergo region $g_{tt} > 0$ does not appear. There is no singularity for all metrices $g_{\mu\nu}$, but we obtain $\det g_{\mu\nu} = 0$ at the surface of $R_H = M/2$, which comes from the zero point of g_{RR}. Thus $g^{\mu\nu}$ becomes singular and we cannot define $g^{\mu\nu}$, so that the new type of singularity at $R = M/2$ appears again, which we call the “event horizon”.

4. Geodesic Equation

Let’s consider the idealistic black hole, that is, the radius of the star of the black hole is zero and the accretion disk does not exit around the black hole. In such situation, we consider whether the mass of the black hole continues to increase or not as time passes. Intuitively, by changing the direction of time $t \to -t$, the particle which enter into the event horizon can come out.

In order to examine the trajectory of the particle falling into the black hole, we use the following geodesic equation

$$
\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{14}
$$

$$
\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \tag{15}
$$

where $\tau = \sqrt{1 - v^2/c^2}dt$.

For the Schwarzschild coordinate, the determinant of the metric is given by $\det g_{\mu\nu} = -r^4\sin^2\theta \le 0$ and $\det g_{\mu\nu}$ becomes zero for $\theta = 0$. But this is the artifact of taking the radial coordinate, because we obtain the same expression of $\det g_{\mu\nu}$ for the flat Minkowski space-time. Thus we physically obtain $\det g_{\mu\nu} \ne 0$, and $g^{\mu\nu}$ becomes regular. In order to take the local inertial frame and/or to follow the trajectory of the particle across the event horizon, $\Gamma^\rho_{\mu\nu}$ must be regular at the event horizon. In this case, though $g^{\mu\nu}$ is regular, but g_{rr} becomes singular at the event horizon. Then the Christoffel symbol becomes singular. Thus we cannot take the local inertial frame and/or we cannot follow the trajectory of the particle across the event horizon.

For the isotropic coordinate of the Schwarzschild black hole, though all metrices $g_{\mu\nu}$ is regular even at the “event horizon”, the determinant of the metric $\det g_{\mu\nu}$ given by (8) becomes zero at $R = M/2$, so that $g^{\mu\nu}$ becomes singular at $R = M/2$. Then the Christoffel symbol becomes singular. Thus we cannot take the local inertial frame and/or we cannot follow the trajectory of the particle across the “event horizon”.

For the Boyer-Lindquist coordinate of the Kerr black hole, the determinant of the metric $\det g_{\mu\nu}$ given by (11) becomes physically non-zero, which means that $g^{\mu\nu}$ is regular and we can well define $g^{\mu\nu}$.

But g_{rr} becomes singular at the event horizon. Then the Christoffel symbol becomes singular. Thus we cannot take the local inertial frame and/or we cannot follow the trajectory of the particle across the "event horizon".

For the isotropic coordinate of the Kerr black hole, though all metrices $g_{\mu\nu}$ becomes regular even at the "event horizon", the determinant of the metric (11) becomes zero at the "event horizon" $R = M/2$, which means that $g^{\mu\nu}$ becomes singular. Then the Christoffel symbol becomes singular. Thus we cannot take the local inertial frame and/or we cannot follow the trajectory of the particle across the "event horizon".

Therefore, by adopting the isotropic coordinate in the Schwarzschild black hole and in the Kerr black hole, we can remove the singularity of metrices $g_{\mu\nu}$. But the determinant of the metric $\det g_{\mu\nu}$ becomes zero and $g^{\mu\nu}$ becomes singular at the "event horizon", which gives the new type of singularity. Thus we cannot examine the starting problem to follow the trajectory of the particle falling into the black hole and to see what happen if time is reversed even if we adopt the isotropic coordinate.

5. Summary and Discussions

The isotropic coordinate is the more physically meaning coordinate in the Schwarzschild black hole metric. Then, in this paper, we apply this isotropic coordinate to the Kerr black hole metric and we have found the ergo region does not appear in this coordinate, and there is no singularity for all $g_{\mu\nu}$ even at the "event horizon". But the determinant of the metric $\det g_{\mu\nu}$ becomes zero at the "event horizon". Then $g^{\mu\nu}$ becomes singular, so that new type of singularity appears. Because of $\det g_{\mu\nu} = 0$ at the "event horizon", the Christoffel symbol becomes singular at the "event horizon", Thus we cannot take the local inertial frame and/or we cannot follow the trajectory of the particle across the "event horizon". Therefore we cannot examine the problem to follow the trajectory of the particle falling into the black hole and to see what happen if time is reversed even if we adopt the isotropic coordinate.

Conflict of Interest

Author declares that there is no conflict of interest.

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