Dynamics of Extended Bodies in General Relativity. I. Momentum and Angular Momentum

Author(s): W. G. Dixon

Source: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Jan. 27, 1970, Vol. 314, No. 1519 (Jan. 27, 1970), pp. 499-527

Published by: Royal Society

Stable URL: https://www.jstor.org/stable/2416466

Proc. Roy. Soc. Lond. A. **314**, 499–527 (1970)
Printed in Great Britain

Dynamics of extended bodies in general relativity
I. Momentum and angular momentum

BY W. G. DIXON
Churchill College, University of Cambridge

(*Communicated by H. Bondi, F.R.S.—Received* 25 *February* 1969)

Definitions are proposed for the total momentum vector p^α and spin tensor $S^{\alpha\beta}$ of an extended body in arbitrary gravitational and electromagnetic fields. These are based on the requirement that a symmetry of the external fields should imply conservation of a corresponding component of momentum and spin. The particular case of a test body in a de Sitter universe is considered in detail, and used to support the definition $p_\beta S^{\alpha\beta} = 0$ for the centre of mass. The total rest energy M is defined as the length of the momentum vector. Using equations of motion to be derived in subsequent papers on the basis of these definitions, the time dependence of M is studied, and shown to be expressible as the sum of two contributions, the change in a potential energy function Φ and a term representing energy inductively absorbed, as in Bondi's illustration of Tweedledum and Tweedledee. For a body satisfying certain conditions described as 'dynamical rigidity', there exists, for motion in arbitrary external fields, a mass constant m such that $M = m + \frac{1}{2}S^\kappa \Omega_\kappa + \Phi$, where Ω_κ is the angular velocity of the body and S^κ its spin vector.

1. INTRODUCTION

This is the first of a series of papers which will develop a theory of the dynamics of extended bodies in the general theory of relativity. The type of body envisaged is a macroscopic body such as a planet orbiting the Sun. The results are not applicable to bodies of atomic dimensions, since general relativity itself breaks down where quantum phenomena come into prominence, nor are they applicable directly to bodies which are too large on a cosmological scale. This latter restriction is made explicit below, but it is possible that it could be removed by suitable modifications to the proofs. Within these limits, imposed by our basic assumptions, the theory will be free of any approximations.

A more specific description of the systems to be treated is as follows. Motions will be considered under the influence of both gravitational and electromagnetic fields. The bodies are taken as described by a symmetric energy-momentum tensor $T^{\alpha\beta}$ and a charge-current vector J^α which satisfy, as consequences of the Einstein–Maxwell equations, the 'generalized conservation equations'

$$\nabla_\beta T^{\alpha\beta} = -F^{\alpha\beta} J_\beta \tag{1.1}$$

and

$$\nabla_\alpha J^\alpha = 0, \tag{1.2}$$

where $F^{\alpha\beta}$ is the electromagnetic field tensor. In addition, we must state our assumptions about differentiability, and conditions which make specific the restriction mentioned above to bodies which are 'not too large'. We shall take the spacetime manifold $\mathfrak{M}$ to be of class C^∞, and both $F^{\alpha\beta}$ and the metric tensor $g_{\alpha\beta}$ also of class

C^∞, although probably this condition could be eased. The Einstein–Maxwell equations then imply that $T^{\alpha\beta}$ and J^α must also be of class C^∞. However, most of the work in this and the following papers uses only the equations (1.1) and (1.2), and not the full field equations. For these purposes it is possible to consider $T^{\alpha\beta}$ and J^α as being only of class C^1. We shall only need to assume this less restrictive condition. We also require them to have supports whose union is a world tube W satisfying the following condition: the intersection of W with an arbitrary spacelike hypersurface Σ lies in some open set N of $\mathfrak{M}$ which is a normal neighbourhood of each of its points, and whose inverse image under the exponential map at any point $p \in N$ is bounded. The existence of such open sets in an arbitrary C^∞ manifold with an affine connexion is proved by Helgason (1962, ch. I, §6), where the definition of normal neighbourhood may also be found. This apparently complicated condition is merely to ensure that the body is spatially finite and that within it, spacelike geodesics are well behaved.

In the present paper we propose definitions of the total momentum vector and spin tensor for such a body, and we discuss their use in defining the centre of mass. For this purpose we rely on some results of Beiglböck (1967), which require the body to satisfy somewhat more restrictive conditions than those expressed above. (The derivation of the equations of motion to be given in a later paper is independent of the concept of the mass centre, and requires only these conditions given above.) To assist in justifying these definitions as opposed to other possible choices of similar form which all agree in the special case of flat spacetime, we develop in some detail the special case of motion in a de Sitter universe. In common with flat spacetime, such a universe admits ten linearly independent Killing vector fields. This maintains the simplification that the equations of motion involve only the momentum vector and spin tensor, terms involving higher moments being absent, but it is a sufficient generalization of flat spacetime to reveal many of the advantages of our choice of definitions. The guiding principle behind our choice is that a constant of the motion should exist corresponding to every group of motions admitted by the spacetime which also preserves the electromagnetic field.

In order also to give some of the consequences of these definitions in a general spacetime, we state without proof in §7 the equations of motion which will be derived in subsequent papers. For simplicity, these equations are here given only up to terms of quadrupole order for the gravitational field and dipole order for the electromagnetic field, as the essential features of the results are all present at this order. The exact form of the results will be given in a later paper. In this section, definitions are proposed for the total force and couple acting on a body, and the rest energy M of the body is defined as the length of the momentum vector. Using the quoted results for the gravitational and electromagnetic force and couple, we then show that a potential energy function Φ exists which agrees with the usual expressions for potential energy in static electric and magnetic fields and in Newtonian gravitational theory. It is such that the difference $M - \Phi$ has the natural interpretation of the internal energy content of the body. The meaning of this will become clear in § 7, but esesentially it is that any change in $M - \Phi$ is due to an exchange of energy with the applied fields by gravitational and electromagnetic induction. I consider this natural appearance of the energy exchange terms of induction to be one of the strongest features in support of the proposed definitions.

Under certain conditions, which can be interpreted as a rigidity criterion, we find that this change in $M - \Phi$ is an exact differential, of a quantity having a natural interpretation as the rotational kinetic energy of the body. Describing these conditions as 'dynamical rigidity', we then have that a dynamically rigid body possesses a well defined mass constant m such that the total rest energy M is the sum of the constant m, the potential energy Φ, and the rotational kinetic energy.

Section 2 investigates constants of the motion in stationary external fields. To present its results in the form most useful to us, we need some results about the equation of geodesic deviation which are obtained in § 3. Section 4 summarizes some well known results of special relativistic dynamics for comparison with their generalizations to a de Sitter universe derived in § 5 from the results of §§ 2 and 3. Section 6 discusses the centre of mass in a general spacetime. Section 7 has been discussed in the preceding paragraph. A brief summary and discussion forms § 8. The appendix contains a summary of standard notation and conventions used throughout the paper, and a discussion of geodesic orbits of Killing vector fields in a de Sitter space which is needed in § 5.

2. Constants of the motion

Suppose $\mathfrak{M}$ admits a Killing vector field ξ^α whose corresponding group of motions preserves $F_{\alpha\beta}$, so that

$$L_\xi g_{\alpha\beta} = 0 \quad \text{and} \quad L_\xi F_{\alpha\beta} = 0. \tag{2.1}$$

Here L_ξ denotes Lie differentiation with respect to ξ^α, for the definition and properties of which see, for example, Schouten (1954, Ch. II, § 10). In addition, $F_{\alpha\beta}$ satisfies the Maxwell equation

$$\nabla_{[\alpha} F_{\beta\gamma]} = 0. \tag{2.2}$$

We first show that (2.1) and (2.2) together imply the existence of a vector potential A_α satisfying both

$$F_{\alpha\beta} = 2\nabla_{[\beta} A_{\alpha]} \tag{2.3}$$

and

$$L_\xi A_\alpha = 0. \tag{2.4}$$

Equation (2.2) implies the existence of a vector field $'A_\alpha$ such that

$$F_{\alpha\beta} = 2\nabla_{[\beta}{}' A_{\alpha]}, \tag{2.5}$$

and that the general A_α satisfying (2.3) is given by

$$A_\alpha = {}'A_\alpha + \partial_\alpha \chi \tag{2.6}$$

for an arbitrary scalar field χ. Condition (2.4) is then equivalent to requiring χ to satisfy

$$\partial_\alpha L_\xi \chi = -L_\xi{}' A_\alpha. \tag{2.7}$$

But by (2.1) and (2.5),

$$\nabla_{[\alpha}(L_\xi{}'A_{\beta]}) = L_\xi(\nabla_{[\alpha}{}'A_{\beta]}) = L_\xi F_{\beta\alpha} = 0. \tag{2.8}$$

Hence (2.7), considered as an equation for the scalar field $L_\xi \chi$, is integrable and its solution is unique up to an arbitrary additive constant. Another integration then yields χ itself, satisfying (2.7) and thus completing the proof. Note for future reference that (2.6) implies

$$\xi^\alpha A_\alpha = \xi^{\alpha\prime} A_\alpha + L_\xi \chi, \tag{2.9}$$

so that $\xi^\alpha A_\alpha$ is also unique up to an arbitrary additive constant.

Equations (2.1) and (2.4) give

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \tag{2.10}$$

and

$$\xi^\beta \nabla_\beta A_\alpha + A_\beta \nabla_\alpha \xi^\beta = 0. \tag{2.11}$$

Using these and (1.2), multiplication of (1.1) by ξ_α yields

$$\nabla_\beta(T^{\alpha\beta}\xi_\alpha + A^\alpha J^\beta \xi_\alpha) = 0. \tag{2.12}$$

Now let Σ be a spacelike cross-section of the world tube W, whose normal is taken as future directed. Define scalar functions $q(\Sigma)$ and $\mathfrak{E}(\Sigma)$ of Σ by

$$q(\Sigma) := \int_\Sigma \mathfrak{J}^\alpha \mathrm{d}\Sigma_\alpha \tag{2.13}$$

and

$$\mathfrak{E}(\Sigma) := \int_\Sigma (\mathfrak{T}^{\alpha\beta} + A^\alpha \mathfrak{J}^\beta)\, \xi_\alpha \mathrm{d}\Sigma_\beta, \tag{2.14}$$

where

$$\mathfrak{J}^\alpha := \surd(-g)\, J^\alpha \quad \text{and} \quad \mathfrak{T}^{\alpha\beta} := \surd(-g)\, T^{\alpha\beta}. \tag{2.15}$$

Then an application of Stokes's theorem, together with equations (1.2) and (2.12), shows that $q(\Sigma)$ and $\mathfrak{E}(\Sigma)$ are independent of the particular cross-section Σ chosen, that is, they are constants of the motion, and the argument Σ may be omitted.

The constant q is the total charge of the body, whose value depends only on $\mathfrak{J}^\alpha$. The value of $\mathfrak{E}$, however, depends not only on $\mathfrak{T}^{\alpha\beta}$ and $\mathfrak{J}^\alpha$, but also on the particular choices made for ξ^α and A^α. The gauge dependence of $\mathfrak{E}$, that is, its dependence on the choice of A^α, is very simple. We remarked above that $A^\alpha \xi_\alpha$ is determined up to an additive constant, and it is only this combination which occurs in $\mathfrak{E}$. Thus we see from (2.13) and (2.14) that if, under a change of gauge,

$$A^\alpha \xi_\alpha \to A^\alpha \xi_\alpha + b, \quad b \text{ constant}, \tag{2.16}$$

then

$$\mathfrak{E} \to \mathfrak{E} + bq. \tag{2.17}$$

The dependence on ξ_α, however, is more complicated, and it is this that leads us to our definitions of momentum and spin.

3. Solution of the equation of geodesic deviation

If the spacetime admits only a one parameter group of motions preserving $F_{\alpha\beta}$, there is only the freedom to multiply ξ_α, and thus $\mathfrak{E}$, by a constant factor, and ξ_α is determined completely by its value at one point. But if the group of motions is of higher dimension, we need to know more than this to so determine ξ_α everywhere.

To see what is required, note that since a group of motions preserves geodesics, its generating Killing vector field must satisfy the equation of geodesic deviation

$$\delta^2\xi_\alpha/\mathrm{d}u^2 + R_{\alpha\beta\gamma\delta}\dot{x}^\beta\dot{x}^\gamma\xi^\delta = 0 \tag{3.1}$$

along every geodesic $x^\alpha(u)$, where u is an affine parameter and $\dot{x}^\alpha(u) := \mathrm{d}x^\alpha/\mathrm{d}u$. This result may also easily be proved directly. A solution of this equation is determined by giving the value of ξ_α and $\delta\xi_\alpha/\mathrm{d}u$ at some fixed value of u. Now choose any fixed point z and suppose the values of ξ_α and $\nabla_{[\alpha}\xi_{\beta]}$ are given at z. On using (2.10), we have

$$\delta\xi_\alpha/\mathrm{d}u = \dot{x}^\beta\nabla_{[\beta}\xi_{\alpha]}. \tag{3.2}$$

Then by constructing all the geodesics through z and integrating (3.1) along them using these initial values, we determine ξ_α throughout any normal neighbourhood of z. Hence in the general case we only need to know, at one point, both ξ_α and $\nabla_{[\alpha}\xi_{\beta]}$ to determine ξ_α everywhere.

We next use the world function biscalar $\sigma(x,z)$, introduced into general relativity by Synge (1960) and DeWitt & Brehme (1960), to formally solve equation (3.1). A summary of the definition and basic properties of σ, together with the conventions we shall use concerning indices on two-point tensors, is given in the appendix. Consider a general spacetime $\mathfrak{M}$ and let $x(u,v)$ be a one parameter family of geodesics in it, where v labels the geodesics and u is an affine parameter along each geodesic. Put

$$\dot{x}^\alpha := \partial x^\alpha/\partial u \quad \text{and} \quad \eta^\alpha := \partial x^\alpha/\partial v. \tag{3.3}$$

Then, as is well known (see, for example, Synge 1960, ch. I, §6), $\eta^\alpha(u,v)$ satisfies the equation of geodesic deviation (3.1) for each fixed v. But by equation (A 1.12) of the appendix,

$$\sigma^\kappa(z(v), x(u,v)) = -u\dot{x}^\kappa(0,v), \tag{3.4}$$

where $z(v) := x(0,v)$ and in accordance with our conventions, σ^κ denotes the derivative of σ at the point $z(v)$. Covariant differentiation of (3.4) with respect to v gives

$$\sigma^\kappa_{\cdot\lambda}\eta^\lambda + \sigma^\kappa_{\cdot\alpha}\eta^\alpha = -u\,\delta\dot{x}^\kappa/\mathrm{d}v. \tag{3.5}$$

But from (3.3),

$$\delta\dot{x}^\kappa/\mathrm{d}v = \delta\eta^\kappa/\mathrm{d}u. \tag{3.6}$$

Define $\overset{-1}{\sigma}{}^\alpha_{\cdot\kappa}$ as the inverse of the matrix $\sigma^\kappa_{\cdot\alpha}$, so that

$$\overset{-1}{\sigma}{}^\alpha_{\cdot\kappa}\sigma^\kappa_{\cdot\beta} = A^\alpha_\beta, \tag{3.7}$$

where A^α_β is the unit tensor. Then (3.5) with (3.6) gives

$$\eta^\alpha = -\overset{-1}{\sigma}{}^\alpha_{\cdot\lambda}\sigma^\lambda_{\cdot\kappa}\eta^\kappa - u\overset{-1}{\sigma}{}^\alpha_{\cdot\kappa}\frac{\delta\eta^\kappa}{\mathrm{d}u}. \tag{3.8}$$

Consider this equation with $v = 0$. We have that $x(u,0)$ represents a geodesic γ along which the vector field $\eta^\alpha(u,0)$ satisfies equation (3.1). Equation (3.8) expresses this in terms of the values η^κ and $\delta\eta^\kappa/\mathrm{d}u$ of it and its derivative at $u = 0$. But by suitably choosing the family of geodesics, γ may be taken as an arbitrary geodesic and η^κ and $\delta\eta^\kappa/\mathrm{d}u$ as arbitrary initial values for the solution of (3.1). Hence (3.8) is the desired formal solution of the equation of geodesic deviation. Since it will play an important role throughout this and the subsequent papers, we find it convenient to define bitensors

$$K^{\alpha}_{\cdot\kappa} = -\overset{-1}{\sigma}{}^{\alpha}_{\cdot\lambda}\sigma^{\lambda}_{\cdot\kappa} \quad \text{and} \quad H^{\alpha}_{\cdot\kappa} = -\overset{-1}{\sigma}{}^{\alpha}_{\cdot\kappa}, \tag{3.9}$$

so that

$$\eta^{\alpha} = K^{\alpha}_{\cdot\kappa}\eta^{\kappa} + uH^{\alpha}_{\cdot\kappa}\,\delta\eta^{\kappa}/\mathrm{d}u. \tag{3.10}$$

If we apply this to the case where ξ^{α} is a Killing vector field, using (3.2) and (3.4), we get

$$\xi_{\alpha} = K_{\alpha}^{\cdot\kappa}\xi_{\kappa} + H_{\alpha}^{\cdot\kappa}\sigma^{\lambda}\nabla_{[\kappa}\xi_{\lambda]}. \tag{3.11}$$

This is valid for all x in a normal neighbourhood of z, and makes explicit the determination of ξ_{α} by the values of ξ_{κ} and $\nabla_{[\kappa}\xi_{\lambda]}$ at one point.

The electromagnetic potential A^{α} occurs in (2.14) in the combination $A^{\alpha}\xi_{\alpha}$, the choice of A^{α} depending on that of ξ^{α} as (2.4) must be satisfied. The allowed gauge transformations are those that preserve the condition (2.4), under which $A^{\alpha}\xi_{\alpha}$ transforms as in (2.16). Hence $A^{\alpha}\xi_{\alpha} - A^{\kappa}\xi_{\kappa}$ is gauge independent, for any pair of points x and z. If x lies in a normal neighbourhood of z, and γ is the geodesic segment joining them, we may express $A^{\alpha}\xi_{\alpha} - A^{\kappa}\xi_{\kappa}$ directly in terms of $F_{\alpha\beta}$ in a form analogous to (3.11). First note that along γ, equation (2.11) implies

$$\mathrm{d}\,(A^{\alpha}\xi_{\alpha})/\mathrm{d}u = \xi^{\alpha}\dot{x}^{\beta}F_{\alpha\beta}. \tag{3.12}$$

With the aid of (3.11) and (A 1.12) of the appendix, this becomes

$$\mathrm{d}(A^{\alpha}\xi_{\alpha})/\mathrm{d}u = u^{-1}\sigma^{\beta}F^{\alpha}_{\cdot\beta}(K_{\alpha}^{\cdot\kappa}\xi_{\kappa} + H_{\alpha}^{\cdot\kappa}\sigma^{\lambda}\nabla_{[\kappa}\xi_{\lambda]}). \tag{3.13}$$

Now define

$$\Psi_{\kappa}(z,y) := \int_0^1 u^{-1}\sigma^{\beta}(z,x)\,K^{\alpha}_{\cdot\kappa}F_{\alpha\beta}\,\mathrm{d}u \tag{3.14}$$

and

$$\Phi_{\kappa}(z,y) := \int_0^1 \sigma^{\beta}(z,x)\,H^{\alpha}_{\cdot\kappa}F_{\alpha\beta}\,\mathrm{d}u, \tag{3.15}$$

where $z := x(0)$ and $y := x(1)$, and note in (3.13) that $u^{-1}\sigma^{\lambda}(z,x) = \sigma^{\lambda}(z,y)$. Then if we integrate (3.13) from $u = 0$ to $u = 1$ and replace y by x in the result, we get

$$A^{\alpha}\xi_{\alpha} - A^{\kappa}\xi_{\kappa} = \Psi^{\kappa}\xi_{\kappa} + \Phi^{\kappa}\sigma^{\lambda}\nabla_{[\kappa}\xi_{\lambda]} \tag{3.16}$$

as required, where the arguments of Ψ^{κ}, Φ^{κ} and σ^{λ} are (z,x).

4. Momentum, spin and the mass centre in special relativity

We now have all the results needed to express $\mathfrak{E}$ in terms of the values of ξ_{κ}, $\nabla_{[\kappa}\xi_{\lambda]}$ and A_{κ} at a single point z. But before continuing, we give now for later comparison a summary of some results obtained by Pryce (1948) and Møller (1949) in their detailed studies of the centre of mass of an extended body in free motion in the special theory of relativity. They define the momentum $p^{\alpha}(\Sigma)$, and the angular momentum $S^{\alpha\beta}(z,\Sigma)$ about an event z, of the matter on a spacelike hypersurface Σ by

$$p^{\alpha}(\Sigma) = \int_{\Sigma} T^{\alpha\beta}\,\mathrm{d}\Sigma_{\beta} \tag{4.1}$$

and
$$S^{\alpha\beta}(z,\Sigma) = 2\int_{\Sigma}(x^{[\alpha}-z^{[\alpha})\,T^{\beta]\gamma}\,\mathrm{d}\Sigma_\gamma. \tag{4.2}$$

Here we are using Minkowskian coordinates and assume Σ to extend to infinity in all directions. Then as a result of (1.1), which now reads
$$\partial_\beta T^{\alpha\beta} = 0, \tag{4.3}$$
it follows that p^α and $S^{\alpha\beta}$ are independent of the particular hypersurface Σ chosen. The momentum p^α is thus a constant of the motion. Further, the dependence of $S^{\alpha\beta}$ on z is given by
$$S^{\alpha\beta}(z_2) = S^{\alpha\beta}(z_1) + 2(z_1^{[\alpha} - z_2^{[\alpha})\,p^{\beta]}, \tag{4.4}$$
which has a differential form
$$\partial_\alpha S_{\beta\gamma} = -2g_{\alpha[\beta}p_{\gamma]}, \tag{4.5}$$
so that along any world-line L with parametric form $z^\alpha(s)$,
$$\mathrm{d}S^{\alpha\beta}/\mathrm{d}s = 2p^{[\alpha}v^{\beta]}, \tag{4.6}$$
where $v^\alpha := \mathrm{d}z^\alpha/\mathrm{d}s$. Hence if L is chosen to be a straight line parallel to the momentum vector p^α, then
$$\mathrm{d}S^{\alpha\beta}/\mathrm{d}s = 0. \tag{4.7}$$
One particular such world line, L_0 say, can be picked out by requiring
$$p_\beta S^{\alpha\beta} = 0; \tag{4.8}$$
if this holds at one point of L then by (4.7) it holds at all points of L. If we choose a Lorentz frame in which $p^\alpha = M\delta^\alpha_0$, M constant, and let Latin indices run from 1 to 3, condition (4.8) reads
$$z^a\int_{x^0=\mathrm{const}} T^{00}\,\mathrm{d}^3x = \int_{x^0=\mathrm{const}} x^a T^{00}\,\mathrm{d}^3x, \tag{4.9}$$
showing that L_0 is uniquely determined and that it is a natural generalization of the Newtonian definition of the mass centre applied in the zero 3-momentum frame. We thus adopt (4.8) as defining the world line of the centre of mass in special relativity.

The superficially similar condition
$$v_\beta S^{\alpha\beta} = 0 \tag{4.10}$$
does not determine an L uniquely, although this also states that (4.9) holds, but now in the zero 3-velocity frame. The trouble with this condition is that this zero 3-velocity frame itself depends on L, the world line we are seeking, and so solving the equation is a somewhat cyclic process. This does not happen in (4.8) as p^α is given in advance by (4.1). Condition (4.10) is actually satisfied by a set of helical world lines, and the straight one L_0, which fill a spacetime cylinder with L_0 as axis. For further details see Møller (1949).

Synge (1956) has shown that we may ensure that p^α will be timelike and that L_0 will lie within the body, i.e. within the convex hull of the world tube W, by requiring
$$n_\alpha n_\beta T^{\alpha\beta} > 0 \quad \text{and} \quad n_\alpha n_\beta T^{\alpha\gamma}T^{\beta}_{\cdot\gamma} > 0 \tag{4.11}$$
for all timelike unit vectors n^α, the signature of $g_{\alpha\beta}$ being -2. These conditions state that the energy density is positive and the 4-momentum density timelike in all Lorentz frames.

5. Motion in a spacetime of constant curvature

Return now to equation (2.14). Let z be any point of $W \cap \Sigma$. Then by hypothesis, $W \cap \Sigma$ lies in some normal neighbourhood $\mathfrak{N}$ of z, so that (3.11) and (3.16) are valid for all x on $W \cap \Sigma$. So if we define

$$p^{\kappa}(z,\Sigma) := \int_{\Sigma} (K_{\alpha}^{\cdot\,\kappa} \mathfrak{T}^{\alpha\beta} + \Psi^{\kappa} \mathfrak{J}^{\beta})\, \mathrm{d}\Sigma_{\beta} \tag{5.1}$$

and

$$S^{\kappa\lambda}(z,\Sigma) := 2\int_{\Sigma} \sigma^{[\lambda}(H_{\alpha}^{\cdot\,\kappa]}\, \mathfrak{T}^{\alpha\beta} + \Phi^{\kappa]} \mathfrak{J}^{\beta})\, \mathrm{d}\Sigma_{\beta}, \tag{5.2}$$

we may express (2.14) in the form

$$\mathfrak{E} = (p^{\kappa} + qA^{\kappa})\, \xi_{\kappa} + \tfrac{1}{2} S^{\kappa\lambda} \nabla_{[\kappa} \xi_{\lambda]}, \tag{5.3}$$

valid for any spacelike cross-section Σ of W and any z obeying the above restriction. In general, the tensors p^{κ} and $S^{\kappa\lambda}$ depend on both z and Σ, but they are independent of the particular choice of either ξ^{α} or A^{α} used. Noting that when $F_{\alpha\beta} = 0$ and $R_{\alpha\beta\gamma\delta} = 0$, the definitions (5.1) and (5.2) agree with (4.1) and (4.2), we tentatively call p^{κ} and $S^{\kappa\lambda}$ the momentum vector and spin tensor relative to z of the matter on Σ.

Using the independence of $\mathfrak{E}$ from z and Σ, we may derive a restriction on the position dependence of p^{κ} and $S^{\kappa\lambda}$. Let $\Sigma(s)$ be a one parameter family of spacelike cross-sections of W, and let $z(s)$ be the parametric form of a world line L such that for each s, $z(s)$ is related to $\Sigma(s)$ by the above condition. For the moment we require neither that L should be timelike nor that s be the proper interval measured along it. Now $p^{\kappa}(z(s), \Sigma(s))$ and $S^{\kappa\lambda}(z(s), \Sigma(s))$ are well defined tensor fields on L, and so (5.3) may be differentiated with respect to s. Putting $v^{\kappa} := \mathrm{d}z^{\kappa}/\mathrm{d}s$ and using (2.11), (2.3) and the well-known result that

$$\nabla_{\alpha\beta} \xi_{\gamma} = R_{\beta\gamma\alpha\delta} \xi^{\delta} \tag{5.4}$$

for any Killing vector field ξ^{α}, we get

$$\xi_{\kappa}[\delta p^{\kappa}/\mathrm{d}s + \tfrac{1}{2} S^{\lambda\mu} v^{\nu} R_{\lambda\mu\nu}^{\cdot\cdot\cdot\,\kappa} + qF^{\kappa}_{\cdot\lambda} v^{\lambda}] + \tfrac{1}{2} \nabla_{[\kappa} \xi_{\lambda]}\, [\delta S^{\kappa\lambda}/\mathrm{d}s - 2p^{[\kappa} v^{\lambda]}] = 0. \tag{5.5}$$

The equation gives a restriction on the manner in which p^{κ} and $S^{\kappa\lambda}$ can vary along an arbitrary world line L, valid for any allowable choice of cross-sections $\Sigma(s)$. It thus also gives information about the dependence on the choice of $\Sigma(s)$. For the special case when ξ_{κ} and $\nabla_{[\kappa} \xi_{\lambda]}$ may be given arbitrary values at a point, it indeed gives complete information. This is the case of a spacetime of constant curvature in which $F_{\alpha\beta} = 0$. Equation (5.5) can then hold for all Killing vector fields if and only if

$$\delta p^{\kappa}/\mathrm{d}s = -\tfrac{1}{2} S^{\lambda\mu} v^{\nu} R_{\lambda\mu\nu}^{\cdot\cdot\cdot\,\kappa} \tag{5.6}$$

and

$$\delta S^{\kappa\lambda}/\mathrm{d}s = 2p^{[\kappa} v^{\lambda]}. \tag{5.7}$$

But in a space of constant curvature,

$$R_{\alpha\beta\gamma\delta} = k(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}), \tag{5.8}$$

where $k := -\frac{1}{12}R$ is its scalar curvature. Equation (5.6) thus becomes

$$\delta p^\kappa/\mathrm{d}s = kS^{\kappa\lambda}v_\lambda. \tag{5.9}$$

The pair of equations (5.7) and (5.9) can now be integrated along L given the values of p^κ and $S^{\kappa\lambda}$ at one point of it. This shows that p^κ and $S^{\kappa\lambda}$ must be independent of the particular choice of Σ, depending only on the point z at which they are evaluated. They are thus well defined tensor fields on $\mathfrak{M}$, a result which also follows from (5.3). Moreover, they satisfy (5.7) and (5.9) along an arbitrary world line L, and hence at a point v^κ is arbitrary. We must thus have

$$\nabla_\alpha p_\beta = -kS_{\alpha\beta} \tag{5.10}$$

and

$$\nabla_\alpha S_{\beta\gamma} = -2g_{\alpha[\beta}p_{\gamma]}. \tag{5.11}$$

For the particular case of flat spacetime, $k = 0$, (5.10) shows that p_α is covariant constant and (5.11) agrees with (4.5), thus rederiving results of the previous section.

The case of a spacetime of constant non-zero curvature, i.e. a de Sitter universe of positive or negative curvature, deserves further attention. It gives natural extensions of many of the results of flat spacetime, e.g. the independence of p^κ and $S^{\kappa\lambda}$ from the choice of Σ, while having $R_{\alpha\beta\gamma\delta} \neq 0$, and so distinguishing between many possible curved-space formulae which reduce to the same formula in flat spacetime. Moreover, being homogeneous and isotropic, we intuitively expect no resultant gravitational force to act on an extended body, and thus would expect of a satisfactory definition of the centre of mass of such a body that its world line be a geodesic in such a spacetime.

We first note that equation (5.10) implies $\nabla_{(\alpha}p_{\beta)} = 0$, showing that p^α is a Killing vector field. It thus satisfies (5.4). But if $k \neq 0$, this then shows that (5.10) implies (5.11). The momentum vector field p^α is thus subject only to the restriction that it be a Killing vector field. Equation (5.10) then serves to determine $S_{\alpha\beta}$, which identically satisfies (5.11), and thus makes $\mathfrak{C}$ of (5.3) identically a constant, for all Killing vector fields ξ^α.

Our next remark is that (5.10) shows that, at a point z_0, the conditions

$$p_\kappa S^{\kappa\lambda} = 0 \tag{5.12}$$

and

$$p^\kappa \nabla_\kappa p_\lambda = 0 \tag{5.13}$$

are equivalent. Let L_0 be the orbit of the vector field p^α through a point z_0 at which these hold. Then, since p^α is a Killing vector field, (5.13) will hold at all points of L_0, so that L_0 will be a geodesic and p^α will have constant magnitude along it. The points at which (5.13), and thus (5.12), hold are thus the points of geodesic orbits of p^α.

We may impose on $T^{\alpha\beta}$ conditions analogous to (4.11) which ensure the existence, within the convex hull of the support W of $T^{\alpha\beta}$, of points at which (5.12) holds, and that at them p^α is timelike. One such set of conditions is to require that for an arbitrary future-directed timelike vector n^α at $x \in W$, $T^{\alpha\beta}n_\beta$ is timelike and future-

33-2

directed, and remains so when propagated by $K_{\alpha}^{;\kappa}$ and $H_{\alpha}^{;\kappa}$ along any spacelike curve lying within the convex hull of W. A proof of the sufficiency of these conditions would follow the lines of a similar result given by Kundt (1966). Physically, the conditions can be thought of as requiring not only that the energy density be positive and the momentum density timelike at every point x, as was needed in flat spacetime, but also that this is still true if we include the gravitational potential energy and momentum of the matter at x relative to any other point z of the body. However, a weaker set of conditions which ensures the existence of such a point as z_0 could probably be found.

Under such conditions we show in §A2 of the appendix that except for one exceptional case, there is a unique timelike geodesic orbit L_0 of p^{α}. Together with our above comments, this then shows that L_0 is the unique world line within the body along which (5.12) holds. Hence, as in flat spacetime, (5.12) characterizes a unique world line which may suitably be called the world line of the centre of mass of the body. Using the parallelism of v^{κ} and p^{κ}, equations (5.7) and (5.9) reduce to

$$\delta p^{\kappa}/\mathrm{d}s = 0 \quad \text{and} \quad \delta S^{\kappa\lambda}/\mathrm{d}s = 0 \quad \text{along } L_0. \tag{5.14}$$

The exceptional case mentioned above only occurs when the curvature k is negative, which is known for other reasons to be a physically unsatisfactory world model. Moreover, it would require the body to have a size comparable with the radius of the universe and to be rotating with a velocity comparable with the speed of light. We thus need not worry if such a case has to be excluded from our results, as it is a physically unreasonable matter distribution. Further properties of this case are given in the appendix, but we will not consider it further here.

Let us now investigate under what conditions the vector field p^{α} is hypersurface orthogonal. The condition for this is

$$p_{[\alpha}\nabla_{\beta}p_{\gamma]} = 0. \tag{5.15}$$

Define a vector field W_{α} by $$W_{\alpha} := \tfrac{1}{2}\eta_{\alpha\beta\gamma\delta}p^{\beta}S^{\gamma\delta}. \tag{5.16}$$

Here, $\eta_{\alpha\beta\gamma\delta} := \sqrt{(-g)}\,\epsilon_{\alpha\beta\gamma\delta}$ and $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita alternating symbol, so that $\eta_{\alpha\beta\gamma\delta}$ is a tensor whose indices may be raised and lowered in the usual manner. Define also a scalar field W by

$$W := -\tfrac{1}{8}\eta_{\alpha\beta\gamma\delta}S^{\alpha\beta}S^{\gamma\delta}. \tag{5.17}$$

By (5.10) the condition (5.15) is equivalent to the vanishing of W_{α}. But from (5.10), (5.11) and (5.16),

$$\nabla_{\alpha}W_{\beta} = -\tfrac{1}{2}k\eta_{\beta}{}^{\gamma\delta\epsilon}S_{\alpha\gamma}S_{\delta\epsilon}, \tag{5.18}$$

which we can put in the form $$\nabla_{\alpha}W_{\beta} = kWg_{\alpha\beta} \tag{5.19}$$

on noticing that $$S_{\alpha[\beta}S_{\gamma\delta]} = S_{[\alpha\beta}S_{\gamma\delta]}. \tag{5.20}$$

Again using (5.10) and (5.11), we obtain from (5.17) that

$$\nabla_{\alpha}W = W_{\alpha}. \tag{5.21}$$

It then follows by integration of (5.19) and (5.21) that if, at one point, both $W = 0$ and $W_\alpha = 0$, then everywhere $W = 0$ and $W_\alpha = 0$. Now note that (5.19) and (5.21) imply $\nabla_\alpha I_2 = 0$, where

$$I_2 := -W_\alpha W^\alpha + kW^2, \tag{5.22}$$

so that I_2 is a scalar constant. But at the centre of mass, condition (5.12) implies that $S_{\alpha\beta}$ is a simple bivector, and hence

$$W = 0 \quad \text{on} \quad L_0. \tag{5.23}$$

Moreover, if p^α is timelike there, then from (5.16) W_α either vanishes or is spacelike, so that $I_2 \geqslant 0$, with equality if and only if $W_\alpha = 0$ at the centre of mass. Combining this with the above remarks, we see that p_α is hypersurface orthogonal if and only if $I_2 = 0$.

Another scalar constant may be formed by taking $\xi_\kappa = p_\kappa$ in (5.3), giving with (5.10) the constancy of

$$I_1 := p^\alpha p_\alpha - \tfrac{1}{2}kS^{\alpha\beta}S_{\alpha\beta}. \tag{5.24}$$

Now define constants M and S, the mass and spin constants of the body, by evaluating

$$M^2 := p_\kappa p^\kappa \quad \text{and} \quad S^2 := \tfrac{1}{2}S_{\kappa\lambda}S^{\kappa\lambda} \tag{5.25}$$

at any point of L_0. By (5.14), these are independent of the point on L_0 at which they are evaluated, but if we used these equations to define scalar fields M and S everywhere, they would not be constant. However, taking into account that (5.12) and (5.23) also hold along L_0, we see that

$$I_1 = M^2 - kS^2 \quad \text{and} \quad I_2 = M^2S^2, \tag{5.26}$$

which may be solved to give

$$M^2 = \tfrac{1}{2}[I_1 + \surd(I_1^2 + 4kI_2)] \tag{5.27}$$

and

$$S^2 = (2k)^{-1}[\surd(I_1^2 + 4kI_2) - I_1]. \tag{5.28}$$

We have here made a choice of one of the two possible sets of roots for M^2 and S^2, corresponding to assuming M and S real and $M \leqslant SR_0$, where $R_0 := |k|^{-\frac{1}{2}}$ is the radius of the universe. Except when $M = SR_0$, i.e. when $k < 0$ and

$$I_1^2 + 4kI_2 = 0, \tag{5.29}$$

there is another set of roots, with M^2 and S^2 negative if $k > 0$ and with $M > SR_0$ if $k < 0$. However, we show in the appendix that the case $k < 0$, $M = SR_0$ is the exceptional case discussed above, which we dismissed as physically unreasonable. The roots with $M > SR_0$ may similarly be excluded; if we assume on physical grounds that $M < SR_0$ then we simultaneously exclude the exceptional case as well as making M and S uniquely determined by I_1 and I_2. We thus have that the magnitudes M and S of the mass and spin at the centre of mass are equivalent to the constants I_1 and I_2 which may be evaluated from a knowledge of p^α and $S^{\alpha\beta}$ at *any* point. The condition for p^α to be hypersurface orthogonal is now simply $S = 0$. A geometrical interpretation of M and S is given in the appendix.

In passing, it is interesting to note the great similarity between equations (5.10) and (5.11), and the commutation relations for the infinitesimal operators of the de Sitter group as given by Gürsey (1964). With the use of these operators as the momentum and spin operators of a quantum theory in a de Sitter universe, our equations then become the classical limit of the operator equations which result from these commutation relations, and the constants I_1 and I_2 become the eigenvalues of the Casimir operators of the group.

6. Momentum, spin and the mass centre in an arbitrary spacetime

The discussion of the previous section has shown that equations (5.1) and (5.2) are suitable definitions of the momentum vector and spin tensor in a space of constant curvature in which $F_{\alpha\beta} = 0$, and that (5.12) is then a suitable characterization of the mass centre. Since, in such a spacetime, the three bitensors $K_{\dot{\alpha}}{}^{\kappa}$, $H_{\dot{\alpha}}{}^{\kappa}$ and the bitensor of parallel propagation $\bar{g}_{\dot{\alpha}}{}^{\kappa}$ are all distinct, we note that such simple results would not have been obtained had we used $\bar{g}_{\dot{\alpha}}{}^{\kappa}$ in (5.1) and (5.2) in place of $K_{\dot{\alpha}}{}^{\kappa}$ and $H_{\dot{\alpha}}{}^{\kappa}$, although this seems, without further investigation, to be a natural generalization of the flat spacetime definitions (4.1) and (4.2). The choice of $\bar{g}_{\dot{\alpha}}{}^{\kappa}$ was made in a previous investigation (Dixon 1964) by the author, and used also in a detailed investigation of the existence and uniqueness of the centre of mass in a general curved spacetime by Beiglböck (1967). I now feel that this choice was made without sufficient study of its implications, and that the results of the present paper and those to appear in the other papers of this series are overwhelmingly in favour of the definitions (5.1) and (5.2). Fortunately, however, Beiglböck's proofs of existence and uniqueness hold with only trivial modifications if we adopt these new definitions. We shall call on his results in the discussion below, understanding that they are meant in this slightly modified form.

Since we needed to take $F_{\alpha\beta} = 0$ in the discussion of the de Sitter universe, we have not yet produced any evidence in support of the electromagnetic contributions in (5.1) and (5.2). The simplest case to consider is a flat spacetime containing an electromagnetic field which appears static in some Lorentz frame K. This requires the existence of a timelike vector field ξ_α satisfying

$$\nabla_\alpha \xi_\beta = 0 \quad \text{and} \quad L_\xi F_{\alpha\beta} = 0. \tag{6.1}$$

Using Minkowskian coordinates adapted to K, we may take $\xi_\alpha = \delta^0_\alpha$. Then as (F^{01}, F^{02}, F^{03}) are the components of the static electric field as seen in K, equation (3.14) shows that $\xi_\kappa \Psi^\kappa(z,x) = \Psi^0(z,x)$ is the electrostatic potential in K of the point x^a relative to z^a, where $a = 1, 2, 3$. From (5.1), the electromagnetic contribution to $\xi_\kappa p^\kappa(z)$ is

$$\int_\Sigma \xi_\kappa \Psi^\kappa \mathfrak{J}^\alpha \mathrm{d}\Sigma_\alpha. \tag{6.2}$$

If we choose Σ as the hypersurface $x^0 =$ constant, this is the electrostatic potential energy in K at time x^0 of the charge distribution of the body relative to the point z^a. Thus, looked at in the frame K at time x^0, $\xi_\kappa p^\kappa(z)$ can be considered as the sum of the body's mechanical energy

$$\int_{x^0=\text{const}} T^{0\alpha}\,\mathrm{d}\Sigma_\alpha \tag{6.3}$$

and its electrostatic potential energy (6.2) relative to z^a. But since (6.1) implies (2.1), equation (5.3) is valid and shows with (6.1) that $\xi_\kappa p^\kappa(z)$ is independent of the hypersurface Σ used in its evaluation. The total energy just considered is thus independent of the time x^0, so expressing the conservation of total mechanical and potential energy in a static electromagnetic field.

If equations (6.1) hold for a spacelike unit vector ξ_α, we may similarly interpret $\xi_\kappa p^\kappa(z)$ as the total mechanical and potential momentum in the ξ direction relative to z^a, which will again be independent of the time x^0. However, potential momentum is less familiar than potential energy, so that this interpretation is not as immediately clear as the previous one.

For the special case of a constant homogeneous electromagnetic field, we can go further, as then $\nabla_\alpha F_{\beta\gamma} = 0$ and so there exist four linearly independent vector fields ξ^α satisfying (6.1). Since equations (5.3) and (5.5) hold for each such ξ_κ, we deduce that $p^\kappa(z, \Sigma)$ itself is independent of Σ, and that along an arbitrary world line,

$$\delta p^\kappa/\mathrm{d}s = -qv^\lambda F^{\kappa}_{\cdot\lambda}. \tag{6.4}$$

In this case, nothing can be deduced about the dependence of $S^{\kappa\lambda}(z, \Sigma)$ on either z or Σ. Equation (6.4) shows that with our definition of momentum, the Lorentz force law holds for an arbitrary extended distribution of charge in a constant homogeneous electromagnetic field, which is again in accordance with what one might expect of a satisfactory definition of momentum.

In the case of a general external electromagnetic field, we may still interpret $p^\kappa(z, \Sigma)$ as the sum of the mechanical and potential energy and momentum of the matter on Σ relative to z. However, it will no longer be independent of Σ, and so the potential terms will lose their immediate physical significance. They agree with the potential terms introduced by Dixon (1967) in deriving equations of motion in an arbitrary electromagnetic field in flat spacetime. Also, it will be seen in the next section that a simple physical meaning may still be attached to the electromagnetic potential contribution to the rest mass of the body.

We can, if we like, make a similar interpretation of the gravitational case. However, while such an interpretation is worth bearing in mind, and should be meaningful in the Newtonian limit, it should not be taken too literally in the exact gravitational theory, as the terms in (5.1) and (5.2) involving $T^{\alpha\beta}$ do not naturally separate into mechanical and potential parts. It will be seen in the next section, though, that a quantitative meaning can be given to the gravitational potential contribution to the rest mass of the body.

Let us now consider generalizing our characterization (5.12) of the world line of the mass centre to the case of general electromagnetic and gravitational fields. In this case, $p^\kappa(z, \Sigma)$ and $S^{\kappa\lambda}(z, \Sigma)$ depend on both z and Σ. But in order to be able to carry

(5.12) over into this general case, we must obtain tensor fields p^κ and $S^{\kappa\lambda}$ which are functions only of position. The natural choice is to evaluate them in the zero 3-momentum frame at the appropriate point, with such a frame defined as a generalization of that used in special relativity in § 4. To make this precise, first define a particular family $\Sigma(z, n)$ of hypersurfaces as functions of a point z and a future-pointing timelike unit vector n^κ at z, by taking $\Sigma(z, n)$ as the hypersurface generated by all geodesics through z orthogonal to n^κ. If $z \in W$, where W is as defined in § 1, by our hypotheses $\Sigma(z, n)$ provides a spacelike cross-section of W provided n^κ lies sufficiently far from the null cone for $\Sigma \cap W$ to be everywhere spacelike. We may interpret $\Sigma(z, n)$ as the instantaneous rest-space of an observer at z with 4-velocity n^κ. Then

$$p^\kappa(z, n) := p^\kappa(z, \Sigma(z, n)) \tag{6.5}$$

is the momentum vector of the body relative to such an observer.

Under rather weak conditions which have the interpretation that the electromagnetic and gravitational fields are not too strong, but which are sufficient to exclude the exceptional case which caused trouble in § 5, Beiglböck (1967) has proved the existence of a unique timelike unit vector n^κ at each point $z \in W$ satisfying

$$n^{[\kappa} p^{\lambda]}(z, n) = 0. \tag{6.6}$$

This n^κ is the unique 4-velocity an observer at z can have if he is to be travelling parallel to the 4-momentum of the body relative to himself. The vectors n^κ together form a vector field, and $\Sigma(z, n(z))$ is the desired zero 3-momentum frame at z. If we now put

$$p^\kappa(z) := p^\kappa(z, \Sigma(z, n(z))) \quad \text{and} \quad S^{\kappa\lambda}(z) := S^{\kappa\lambda}(z, \Sigma(z, n(z))), \tag{6.7}$$

we have a momentum vector field and spin tensor field depending only on position, as required. We may now invoke another result of Beiglböck (1967) which states that, under these same conditions, the points z within W at which

$$p_\lambda(z)\, S^{\kappa\lambda}(z) = 0 \tag{6.8}$$

form a unique timelike world line which, as before, we call the world line L_0 of the centre of mass of the body. Our terminology actually differs slightly from that of Beiglböck, who calls this the centre-of-motion line, reserving the term centre-of-mass from a somewhat different line.

7. Force, couple and induction of energy

Subsequent papers of this series will derive the analogues of (5.6) and (5.7) for the general case. These will have the form

$$\delta p^\kappa / \mathrm{d}s = F^\kappa \tag{7.1}$$

and

$$\delta S^{\kappa\lambda} / \mathrm{d}s = 2 p^{[\kappa} v^{\lambda]} + G^{\kappa\lambda} \tag{7.2}$$

along an arbitrary world line L with parametric equation $z(s)$, where $v^\kappa := \mathrm{d}z^\kappa / \mathrm{d}s$. We here assume a field $n^\kappa(s)$ of timelike unit vectors have been given along L, and that $p^\kappa(s)$ and $S^{\kappa\lambda}(s)$ are evaluated by using the hypersurface $\Sigma(z(s), n(s))$, in the notation of the previous section. Exact expressions for F^κ and $G^{\kappa\lambda}$ will be given in terms of multipole moments of $T^{\alpha\beta}$ and J^α defined relative to L as explicit integrals over $\Sigma(z, n)$, generalizing the moments defined by Dixon (1967) in flat spacetime. The results will be obtained using only the restrictions given in §1. For present purposes, however, we shall assume that the slightly stronger conditions of Beiglböck (1967) are satisfied, so that L may be taken as the centre-of-mass line L_0 defined by (6.8), with s the proper time along it, and $n^\kappa(s)$ as given by (6.6). Then F^κ and $G^{\kappa\lambda}$ may be called the (four-)force and tensor couple acting on the body. We also define $M(s)$, $u^\kappa(s)$ and $\chi(s)$ along L_0 by

$$p^\kappa = Mu^\kappa \quad \text{and} \quad \chi = u^\kappa v_\kappa, \quad \text{where} \quad u^\kappa u_\kappa = 1 \quad \text{and} \quad M > 0. \tag{7.3}$$

Then M may be called the mass of the body, and u^κ and v^κ its dynamical and kinematical velocities. The meaning of the factor χ, which enters into a number of equations below, will be considered later.

If the body is small compared with a typical length scale for the external fields, we may to a good approximation retain the electromagnetic and gravitational contributions to F^κ and $G^{\kappa\lambda}$ only as far as the dipole and quadrupole terms respectively. Then to this order,

$$F^\kappa = -qv^\lambda F^\kappa{}_{.\lambda} + \tfrac{1}{2}v^\lambda S^{\mu\nu} R^\kappa{}_{.\lambda\mu\nu} + \chi\{\tfrac{1}{6}J^{\lambda\mu\nu\rho}\nabla^\kappa R_{\lambda\mu\nu\rho} - \tfrac{1}{2}Q^{\lambda\mu}\nabla^\kappa F_{\lambda\mu}\} \tag{7.4}$$

and

$$G^{\kappa\lambda} = \chi\{\tfrac{4}{3}J^{\mu\nu\rho[\kappa} R_{\mu\nu\rho}{}^{\cdot\cdot\cdot\lambda]} - 2Q^{\mu[\kappa} F_\mu{}^{\cdot\lambda]}\}, \tag{7.5}$$

where $J^{\kappa\lambda\mu\nu}$ is the mass quadrupole moment tensor, having the same symmetries as $R_{\kappa\lambda\mu\nu}$, and $Q^{\kappa\lambda} = Q^{[\kappa\lambda]}$ is the electromagnetic dipole moment tensor. Exact definitions of $J^{\kappa\lambda\mu\nu}$ and $Q^{\kappa\lambda}$ will be given in a later paper, but they can be considered to a good approximation as given by the definitions used in Dixon (1967).

We have separated off on the right-hand side of (7.2) the only term which does not explicitly involve the external fields, and which is present even in the flat spacetime case. Whenever a Killing vector field satisfying (2.1) exists, the contributions to F^κ and $G^{\kappa\lambda}$ from the higher moments, i.e. all except the monopole electromagnetic and dipole gravitational contributions, are restricted by (5.5), which expresses the vanishing of a component of these contributions for each such symmetry of the applied fields. This is in accordance with one's intuitive notions of the effects of such a symmetry, and supports this identification of the force and couple.

Along L_0, (6.8) is satisfied, and hence $S_{\kappa\lambda}$ has only three linearly independent components there. The spin may thus be alternatively described by a vector S_κ, defined by

$$S_\kappa := \tfrac{1}{2}\eta_{\kappa\lambda\mu\nu} u^\lambda S^{\mu\nu}, \tag{7.6}$$

related to W_α of (5.16) by $W_\kappa = MS_\kappa$. Then using (6.8) we see that $S^{\kappa\lambda}$ is itself determined by S^κ thus:

$$S^{\kappa\lambda} = \eta^{\kappa\lambda\mu\nu} u_\mu S_\nu, \tag{7.7}$$

and we also note that

$$u^\kappa S_\kappa = 0, \tag{7.8}$$

so that S_κ has only three linearly independent components. If we now also define the vector couple G_κ by

$$G_\kappa := \tfrac{1}{2}\eta_{\kappa\lambda\mu\nu} u^\lambda G^{\mu\nu}, \tag{7.9}$$

so that also

$$u^\kappa G_\kappa = 0, \tag{7.10}$$

we find that (7.2) implies
$$\delta S^\kappa/\mathrm{d}s = -u^\kappa \dot{u}_\lambda S^\lambda + G^\kappa, \tag{7.11}$$
where $\dot{u}^\kappa = \delta u^\kappa/\mathrm{d}s$. Note that S^κ is not parallelly transported along L_0 even if $G^\kappa = 0$; this deviation from parallel propagation results physically in the Thomas precession.

This equation is not directly equivalent to (7.2). First note that (7.9) implies
$$G^{\kappa\lambda} = \eta^{\kappa\lambda\mu\nu} u_\mu G_\nu - 2u^{[\kappa} H^{\lambda]}, \tag{7.12}$$
where
$$H^\kappa := G^{\kappa\lambda} u_\lambda. \tag{7.13}$$
Then if we evaluate $\delta S^{\kappa\lambda}/\mathrm{d}s$ by using (7.7) with (7.11) and compare the result with (7.2) and (7.12), we find that we must also have
$$p^{[\kappa} v^{\lambda]} = M^{-1}(\tfrac{1}{2}\eta^{\kappa\lambda\mu\nu} P_\mu S_\nu + p^{[\kappa} H^{\lambda]}), \tag{7.14}$$
where
$$P^\kappa := F^\kappa - u^\kappa u_\lambda F^\lambda. \tag{7.15}$$
Note that
$$P^\kappa u_\kappa = H^\kappa u_\kappa = 0. \tag{7.16}$$
Together with $v^\kappa v_\kappa = 1$, equation (7.14) determines v^κ in terms of p^κ, S^κ and the applied force and couple F^κ and $G^{\kappa\lambda}$. In particular, p^κ and v^κ are parallel whenever F^κ and $G^{\kappa\lambda}$ both vanish, and hence by (7.1), L_0 will then be a geodesic, as we should expect.

Recalling equation (7.8), we may use the free propagation of the spin vector, i.e. that corresponding to $G^\kappa = 0$, to define a standard non-rotating orthonormal reference tetrad along L_0, one of whose vectors is u^κ. We choose one vector to be u^κ rather than v^κ as our moments are defined as integrals over $\Sigma\,(z, u)$, and hence the triad of spacelike vectors will lie at $z(s)$ in this hypersurface of integration. If we require the three spacelike vectors to obey (7.11) with $G^\kappa = 0$, then along L_0, each vector of the tetrad will satisfy
$$\delta A^\kappa/\mathrm{d}s = (\dot{u}^\kappa u_\lambda - u^\kappa \dot{u}_\lambda)\, A^\lambda, \tag{7.17}$$
and so will any vector field along L_0 whose components relative to this tetrad are independent of s. This law of propagation, which we call M-transport as it preserves orthogonality to momentum, preserves the scalar product of an arbitrary pair of vectors. If $u^\kappa = v^\kappa$, it reduces to Fermi–Walker transport, which is well known to correspond physically to a standard of non-rotation if orthogonality to v^κ is to be preserved, as is shown, for example, by Synge (1960) and Trautman (1965).

M-transport may be used to define a derivative operator $\overset{M}{\delta}/\mathrm{d}s$ along L_0 analogous to the Fermi derivative $\overset{F}{\delta}/\mathrm{d}s$ introduced by Tulczyjew & Tulczyjew (1962). For a vector field B^κ along L_0 we define
$$\frac{\overset{M}{\delta} B^\kappa}{\mathrm{d}s} = \frac{\delta B^\kappa}{\mathrm{d}s} - 2\dot{u}^{[\kappa} u^{\lambda]} B_\lambda, \tag{7.18}$$
with an extension to tensors of arbitrary rank by requiring that $\overset{M}{\delta} g_{\kappa\lambda}/\mathrm{d}s = 0$ and that the Leibnitz rule should hold for products. This derivative operator represents the covariant rate of change relative to the non-rotating tetrad, and vanishes when applied to any M-transported tensor field. Using it, (7.11) may be written as
$$\overset{M}{\delta} S^\kappa/\mathrm{d}s = G^\kappa. \tag{7.19}$$

Just as we have decomposed (7.2) into the two equations (7.11) and (7.14), so we may also decompose (7.1), by resolving it parallel and orthogonal to u^κ. If we define Θ by

$$\Theta := u_\kappa F^\kappa, \tag{7.20}$$

then, with (7.3) and (7.15), we get

$$M\,\delta u^\kappa/\mathrm{d}s = P^\kappa \tag{7.21}$$

and

$$\mathrm{d}M/\mathrm{d}s = \Theta. \tag{7.22}$$

If we interpret M as giving the mass, and thus also the energy, of the body relative to a comoving non-rotating reference frame, the scalar Θ must thus represent the rate at which the body absorbs energy from the external fields. We shall show next that to the order of accuracy of equations (7.4) and (7.5), this energy absorbed can be separated into two parts. First, there will be a change in the potential energy of the body relative to its centre of mass, and secondly there will be a real change in the internal energy content of the body. This latter part is the energy inductively absorbed from the external fields. In the example of gravitational inductive transfer of energy proposed by Bondi, see for example, Bondi (1965), and now well known as the experiment of Tweedledum and Tweedledee, a decrease in this internal energy content represents the work performed by the batteries in driving the motors which change the shape of the bodies from oblate to prolate spheroids. We shall see that this part also includes any change in the rotational kinetic energy of the body. Furthermore, although these results are only obtained here to the approximation given in equations (7.4) and (7.5), we shall show in a later paper that they hold also when the exact expressions are used for F^κ and $G^{\kappa\lambda}$.

Let us first obtain a new expression for Θ involving $v_\kappa F^\kappa$ rather than $u_\kappa F^\kappa$. Multiply (7.1) by v_κ using (7.3) and (7.22) to give

$$\chi\Theta + M v_\kappa \dot{u}^\kappa = v_\kappa F^\kappa. \tag{7.23}$$

But also, by differentiating (6.8), we have

$$u_\kappa \dot{u}_\lambda \,\delta S^{\kappa\lambda}/\mathrm{d}s = -\dot{u}_\kappa \dot{u}_\lambda S^{\kappa\lambda} = 0. \tag{7.24}$$

On substituting (7.2) into this, we get

$$M v_\kappa \dot{u}^\kappa = \dot{u}_\kappa u_\lambda G^{\kappa\lambda}, \tag{7.25}$$

which when used in (7.23) gives the desired alternative expression for Θ,

$$\Theta = \chi^{-1}(v_\kappa F^\kappa - \dot{u}_\kappa u_\lambda G^{\kappa\lambda}). \tag{7.26}$$

Now use (7.4) to obtain, approximately,

$$\chi^{-1} v_\kappa F^\kappa = \frac{\mathrm{d}\Phi}{\mathrm{d}s} - \tfrac{1}{6} R_{\kappa\lambda\mu\nu} \frac{\delta J^{\kappa\lambda\mu\nu}}{\mathrm{d}s} + \tfrac{1}{2} F_{\kappa\lambda} \frac{\delta}{\mathrm{d}s} Q^{\kappa\lambda}, \tag{7.27}$$

where

$$\Phi := \tfrac{1}{6} J^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} - \tfrac{1}{2} Q^{\kappa\lambda} F_{\kappa\lambda}. \tag{7.28}$$

Putting this into (7.26), and using (7.5) and (7.18) yields the approximate expression for Θ,

$$\Theta = \frac{\mathrm{d}\Phi}{\mathrm{d}s} - \tfrac{1}{6}R_{\kappa\lambda\mu\nu}\frac{\overset{M}{\delta} J^{\kappa\lambda\mu\nu}}{\mathrm{d}s} + \tfrac{1}{2}F_{\kappa\lambda}\frac{\overset{M}{\delta} Q^{\kappa\lambda}}{\mathrm{d}s}. \tag{7.29}$$

This verifies our claims about the form of Θ. The term $\mathrm{d}\Phi/\mathrm{d}s$ can be considered as the rate of change of a potential energy function Φ, while the other two terms in (7.29) express the energy absorbed by induction, due to a change in the mass quadrupole and electromagnetic dipole moments relative to a non-rotating reference tetrad. The exact result will be similar; Φ will contain terms of all orders of multipole moments, while the inductive terms will all involve M-derivatives of these moments.

Our interpretation of Φ is supported by the contribution $-\frac{1}{2}Q^{\kappa\lambda}F_{\kappa\lambda}$ agreeing with the familiar terms $-(\boldsymbol{p}.\boldsymbol{E}+\boldsymbol{m}.\boldsymbol{H})$ known from electrostatics and magnetostatics, where $\boldsymbol{p}$, $\boldsymbol{m}$ are the static electric and magnetic dipole moments and $\boldsymbol{E}$, $\boldsymbol{H}$ are the electric and magnetic field vectors. This may be seen using the expression for $Q^{\kappa\lambda}$ in flat spacetime given in Dixon (1967). Similarly, in a static gravitational field, in the Newtonian approximation to general relativity we may take

$$g_{\alpha\beta} = \mathrm{diag}\,(1+2\phi,\ -1+2\phi,\ -1+2\phi,\ -1+2\phi), \tag{7.30}$$

where ϕ is the Newtonian gravitational potential and units are such that the velocity of light is unity. In this approximation also, for a slowly moving body the only non-negligible components of $J^{\kappa\lambda\mu\nu}$ are given by

$$J^{a0b0} = \tfrac{3}{8}(I_{cc}\delta_{ab} - 2I_{ab}) \quad (a, b = 1, 2, 3) \tag{7.31}$$

where I_{ab} is the usual moment of inertia tensor of the body relative to its mass centre. Hence to this order of accuracy,

$$\tfrac{1}{6}J^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu} = \tfrac{1}{4}(I_{cc}\delta_{ab} - 2I_{ab})\,\partial_{ab}\phi. \tag{7.32}$$

But Newtonian gravitational theory gives the potential energy of a quadrupole body relative to its centre of mass as

$$\begin{aligned}\Phi_{\text{Newtonian}} &= \int \rho\phi(z+r)\,\mathrm{d}\tau \\ &= \tfrac{1}{2}\partial_{ab}\phi \int \rho r^a r^b\,\mathrm{d}\tau \\ &= \tfrac{1}{4}(I_{cc}\delta_{ab} - 2I_{ab})\,\partial_{ab}\phi,\end{aligned} \tag{7.33}$$

where ρ is its mass density, r^a the position vector relative to the centre of mass z, and we have used the centre-of-mass condition

$$\int \rho r^a\,\mathrm{d}\tau = 0.$$

Thus to this order our gravitational potential energy also agrees with the Newtonian value. Note that the absence of a dipole gravitational energy in the Newtonian theory is due to z having been chosen as the centre of mass of the body, and consequently the absence of such a dipole term in the relativistic Φ is good evidence in favour of the suitability of our mass centre condition (6.8) used in deriving (7.29).

Let us also verify our statement above that the inductive absorption of energy includes any change in the rotational kinetic energy of the body. We may interpret a pure rotational motion of the body as meaning that the moments $J^{\kappa\lambda\mu\nu}$ and $Q^{\kappa\lambda}$, and all higher moments also, have numerically constant components relative to some system of orthonormal tetrads along L_0, each having u^κ as one vector, but which are not necessarily related by M-transport along L_0. The basis vectors of these new tetrads will then each satisfy along L_0 an equation of the form

$$\overset{M}{\delta} B^\kappa/\mathrm{d}s = \chi \Omega^{\kappa}_{\cdot\lambda} B^\lambda, \tag{7.34}$$

where
$$\Omega_{\kappa\lambda} = \Omega_{[\kappa\lambda]} \quad \text{and} \quad \Omega^{\kappa}_{\cdot\lambda} u^\lambda = 0. \tag{7.35}$$

The skew tensor $\Omega_{\kappa\lambda}$ describes the angular velocity of the tetrads relative to an M-transported one; the factor χ in (7.34) has been put in for later convenience and its physical significance will be discussed later.

The corresponding propagation law for $J^{\kappa\lambda\mu\nu}$ and $Q^{\kappa\lambda}$ along L_0 will then be

$$\overset{M}{\delta} J^{\kappa\lambda\mu\nu}/\mathrm{d}s = -2\chi \Omega^{[\kappa}_{\cdot\rho} J^{\lambda]\rho\mu\nu} - 2\chi J^{\kappa\lambda\rho[\mu} \Omega^{\nu]}_{\cdot\rho} \tag{7.36}$$

and
$$\overset{M}{\delta} Q^{\kappa\lambda}/\mathrm{d}s = -2\chi \Omega^{[\kappa}_{\cdot\rho} Q^{\lambda]\rho}. \tag{7.37}$$

On putting these into (7.29) and using (7.5), we get

$$\Theta = \mathrm{d}\Phi/\mathrm{d}s + \tfrac{1}{2}\Omega_{\kappa\lambda} G^{\kappa\lambda}, \tag{7.38}$$

a result which holds also in the exact theory. Defining the angular velocity vector Ω_κ by

$$\Omega_\kappa := \tfrac{1}{2}\eta_{\kappa\lambda\mu\nu} u^\lambda \Omega^{\mu\nu} \tag{7.39}$$

and using (7.9) and the analogue of (7.7) for $\Omega^{\kappa\lambda}$, this may also be written as

$$\Theta = \mathrm{d}\Phi/\mathrm{d}s + \Omega_\kappa G^\kappa. \tag{7.40}$$

The term $\Omega_\kappa G^\kappa$ thus represents the inductive absorption of energy in a purely rotational motion. But in Newtonian dynamics, if a rigid body has angular velocity vector $\boldsymbol{\omega}$ and experiences a couple $\boldsymbol{G}$, its kinetic energy of rotation increases at a rate $\boldsymbol{\omega}.\boldsymbol{G}$. This agrees with the Newtonian limit of the $\Omega_\kappa G^\kappa$ of (7.40), which may thus also be interpreted as the rate of increase of rotational kinetic energy, as stated above.

Now consider the meaning of the factor χ in (7.34). Let us reparametrize L_0 with a new parameter τ, whose change $\delta\tau$ between two neighbouring points $z(s)$ and $z(s+\delta s)$ of L_0 is the time between these two events as measured in the instantaneous zero 3-momentum frame at $z(s)$. This definition of τ is simply obtained from that of the proper time s by replacing the zero 3-velocity frame by the zero 3-momentum frame. Then

$$\mathrm{d}\tau/\mathrm{d}s = \chi, \tag{7.41}$$

and hence if we define $\overset{M}{\delta}/\mathrm{d}\tau$ by replacing s by τ in (7.18) (including in the definition of $\dot{u}^\kappa$), we see that

$$\overset{M}{\delta}B^\kappa/\mathrm{d}s = \chi\overset{M}{\delta}B^\kappa/\mathrm{d}\tau. \tag{7.42}$$

So (7.34) may be written as

$$\overset{M}{\delta}B^\kappa/\mathrm{d}\tau = \Omega^\kappa{}_{.\lambda}B^\lambda, \tag{7.43}$$

showing that $\Omega^\kappa{}_{.\lambda}$ represents the rotation rate *as measured in the zero 3-momentum frame,* and that the factor χ expresses the time dilation effect occurring between the frames of zero 3-momentum and zero 3-velocity.

Finally, let us consider the possible relation between the spin S^κ and the angular velocity Ω^κ for a body whose internal motion is purely rotational. It appears that there is no theoretical necessity for any such relationship unless we impose some further physical restriction such as rigidity. As a simple example, consider a toroidal tube containing a gas in which a compression wave is travelling, so that viewed from the centre of the torus, the angular velocity of the wave is Ω^κ. Then one would expect that referred to a tetrad whose angular velocity was also Ω^κ, the multipole moments would have numerically constant components, as required. The tube, however, could also be rotating, not affecting Ω^κ but contributing to the angular momentum S^κ. Hence there would be no necessary relation between S^κ and Ω^κ. Of course, the above argument is not rigorous, but it is certainly suggestive.

In accordance with our Newtonian notions of rigidity, for a body which could in any sense be called rigid, we should expect such a relation. It is interesting to investigate the consequences of directly adopting the Newtonian relationship

$$S^\kappa = I^{\kappa\lambda}\Omega_\lambda, \tag{7.44}$$

where

$$I^{\kappa\lambda} = I^{(\kappa\lambda)} \quad \text{and} \quad I^{\kappa\lambda}u_\lambda = 0. \tag{7.45}$$

We also require that $I^{\kappa\lambda}$ has constant components with respect to the rotating tetrad, so that

$$\overset{M}{\delta}I^{\kappa\lambda}/\mathrm{d}s = 2\chi\Omega^{(\kappa}{}_{.\mu}I^{\lambda)\mu}. \tag{7.46}$$

Then, as $\Omega_\lambda\Omega^{\kappa\lambda} = 0$ from the analogue of (7.7) for $\Omega^{\kappa\lambda}$, we have

$$\Omega_\kappa\Omega_\lambda\overset{M}{\delta}I^{\kappa\lambda}/\mathrm{d}s = 0. \tag{7.47}$$

Combining this with (7.19) and (7.44) gives

$$\Omega_\kappa G^\kappa = \tfrac{1}{2}\mathrm{d}(\Omega_\kappa S^\kappa)/\mathrm{d}s, \tag{7.48}$$

and hence with (7.40) and (7.22) we get

$$\mathrm{d}(M - \Phi - \tfrac{1}{2}\Omega_\kappa S^\kappa)/\mathrm{d}s = 0. \tag{7.49}$$

There thus exists a constant m, which we may call the mass constant of the body, such that

$$M = m + \tfrac{1}{2}\Omega_\kappa S^\kappa + \Phi. \tag{7.50}$$

The term $\frac{1}{2}\Omega_\kappa S^\kappa = \frac{1}{2}I^{\kappa\lambda}\Omega_\kappa\Omega_\lambda$ can be interpreted as the kinetic energy of rotation, and hence we have the total rest mass M expressed as the sum of a constant part, the kinetic energy of rotation, and the potential energy Φ. Once again, (7.50) holds also in the exact theory. A body whose multipole moments have constant components with respect to a rotating tetrad and which satisfies (7.44) to (7.46) may be called *dynamically rigid*. We have thus proved the existence of a mass constant m satisfying (7.50) for any dynamically rigid body.

8. Summary and discussion

Using the definitions proposed here for the momentum and spin of an extended body in gravitational and electromagnetic fields, we have shown the following, which are sufficiently in accordance with our intuitive physical requirements for these quantities that we feel that they are the most suitable definitions yet suggested for them.

In a spacetime of constant curvature k, the momentum $p^\kappa(z, \Sigma)$ and spin $S^{\kappa\lambda}(z, \Sigma)$ relative to the point z of the matter on the spacelike hypersurface Σ is, for fixed z, independent of Σ. This expresses the laws of conservation of momentum and angular momentum in such a universe. They are thus well defined tensor functions of position, whose dependence on position is given by equations (5.6) and (5.7). These equations agree with those obtained for a pole-dipole particle in an arbitrary external field by Papapetrou (1951) and Dixon (1964). They appear in (5.6) and (5.7) in the form given by Dixon (1964), but this is equivalent to the formulation of Papapetrou, although this equivalence was only claimed as approximate in Dixon (1964). This error was pointed out in Dixon (1965). It is important to realize, however, that equations (5.6) and (5.7) hold exactly for an extended body, whereas if the pole–dipole equations are treated as equations for an extended body, then they are only approximate. Indeed, it is open to doubt whether it is valid to treat them as even approximately valid for an extended body, as explained by Dixon (1967).

If $k \neq 0$, equations (5.6) and (5.7) merely reduce to the statement that p^α is a Killing vector field, with $S^{\alpha\beta}$ related to p^α by $S_{\alpha\beta} = k^{-1}\nabla_\beta p_\alpha$. This Killing vector field is hypersurface orthogonal if and only if the body has zero internal angular momentum, $S = 0$. The condition $p_\lambda S^{\kappa\lambda} = 0$ is a suitable characterization of the world line of the centre of mass of the body. A point satisfies this condition if and only if the orbit of p^α through this point is a geodesic. As a result, this condition picks out a unique world line within the body provided $S \neq MR_0$, and this world line is a timelike geodesic.

With a suitable interpretation, given in § 6, this condition can be taken as defining the mass centre of a body in arbitrary external fields. The mass M of the body, defined as the length of the momentum vector, can be interpreted as the energy of the body as seen in a comoving non-rotating reference frame. By studying the time dependence of M, we are led to define a potential energy function Φ such that any change in $M - \Phi$ represents energy extracted from the external fields by induction, due to a change in the multipole moments of the body relative to this frame. Φ agrees with the usual expressions for potential energy in electrostatics and magnetostatics, and in Newtonian gravitational theory. No gravitational dipole term appears in Φ, which strongly supports our mass centre condition, as in Newtonian gravitational theory this dipole potential energy vanishes if and only if the origin is taken as the centre of mass.

If the only internal motion of the body is purely rotational, the only inductively absorbed energy is the work $G^\kappa \Omega_\kappa$ done on the body by the external couple G^κ. If we further impose that the spin vector S^κ and angular velocity vector Ω^κ are related by $S^\kappa = I^{\kappa\lambda}\Omega_\lambda$, where $I^{\kappa\lambda}$ generalizes the moment of inertia tensor of Newtonian dynamics and satisfies (7.45) and (7.46), we call the body *dynamically rigid*. Then $G^\kappa \Omega_\kappa$ is the rate of increase of the kinetic energy $\frac{1}{2}S^\kappa\Omega_\kappa$, so that a mass constant m exists for the body satisfying $M = m + \frac{1}{2}S^\kappa \Omega_\kappa + \Phi$.

Finally, let us comment on the concept of 'test bodies'. In our considerations of a body in a de Sitter universe, we have neglected the effect of the body's own gravitational field on the metric, and thus we are dealing with what are generally known as 'test bodies'. However, this is in order to verify that our intuitive expectations in such a universe are satisfied by our theory. In the full theory to be developed in subsequent papers, the external fields will be completely general, and so can be taken to include the effect of the body under consideration. As we are dealing with extended bodies, we do not run into problems of singularities in the metric. Although for practical computation, bodies such as the planets are frequently treated as test bodies for simplicity, our full theory will not be restricted to this, and so, for example, the result that a mass constant exists for a dynamically rigid body is exactly true, even if the body's own field is taken into account.

References

Beiglböck, W. 1967 *Commun. Math. Phys.* **5**, 106–130.

Bondi, H. 1965 Article in *Lectures on general relativity*, Brandeis Summer Institute 1964. Englewood Cliffs, N. J.: Prentice-Hall.

DeWitt, B. S. & Brehme, R. W. 1960 *Ann. Phys.* (*N.Y.*) **9**, 220–259.

Dixon, W. G. 1964 *Nuovo Cim.* **34**, 317–339.

Dixon, W. G. 1965 Ph.D. Thesis, Cambridge University.

Dixon, W. G. 1967 *J. Math. Phys.* (*N.Y.*) **8**, 1591–1605.

Gürsey, F. 1964 Article in *Relativity, groups and topology* (ed. B. S. DeWitt, and C. M. DeWitt). London: Blackie and Son.

Helgason, S. 1962 *Differential geometry and symmetric spaces*, New York: Academic Press.

Kundt, W. 1966 *Centre lines*, preprint dated July 1966, Hamburg.

Møller, C. 1949 *Commun. Dublin Inst. Adv. Stud.* A **5**.

Papapetrou, A. 1951 *Proc. Roy. Soc. Lond.* A **209**, 248–258.

Pryce, M. H. L. 1948 *Proc. Roy. Soc. Lond.* A **195**, 62–81.

Schouten, J. A. 1954 *Ricci Calculus*, 2nd ed. Berlin: Springer-Verlag.

Schrödinger, E. 1956 *Expanding universes*. Cambridge University Press.

Synge, J. L. 1956 *Relativity: the special theory*. Amsterdam: North-Holland Publ. Co.

Synge, J. L. 1960 *Relativity: the general theory*. Amsterdam: North-Holland Publ. Co.

Trautman, A. 1965 Article in *Lectures on general relativity*, Brandeis Summer Institute 1964. Englewood Cliffs, N. J.: Prentice-Hall.

Tulczyjew, B. & Tulczyjew, W. 1962 Article in *Recent developments in general relativity*. London: Pergamon Press.

Appendix

A1. *Summary of notation and conventions*

Spacetime is considered as a four-dimensional pseudo-Riemannian manifold of class C^∞, with metric tensor $g_{\alpha\beta}$ of signature -2, with respect to which covariant differentiation is denoted by ∇_α, partial differentiation being denoted by ∂_α. In repeated differentiations the kernel ∇ or ∂ is only written once, e.g. $\nabla_{\alpha\beta}A_\gamma = \nabla_\alpha\nabla_\beta A_\gamma$. Absolute differentiation along a curve with respect to a parameter u is denoted by

$$\frac{\delta}{\mathrm{d}u} := \frac{\mathrm{d}x^\alpha}{\mathrm{d}u}\nabla_\alpha,$$

where, as elsewhere, a colon placed before an equals sign indicates that the equation is to be considered as defining the quantity to the left of the colon.

Symmetrization and antisymmetrization of indices is denoted by ( ) and [ ] respectively, indices to be omitted from these operations being enclosed between vertical lines, e.g.

$$A_{[\alpha|\beta\gamma|\delta]} = \tfrac{1}{2}(A_{\alpha\beta\gamma\delta} - A_{\delta\beta\gamma\alpha}). \tag{A 1.1}$$

This, as also much else of our notation, follows that used by Schouten (1954).

The sign of the curvature tensor is such that the Ricci identity for a covariant vector A_α is

$$\nabla_{[\alpha\beta]}A_\gamma = -\tfrac{1}{2}R_{\dot\alpha\dot\beta\dot\gamma}{}^{\delta}A_\delta. \tag{A 1.2}$$

The Ricci tensor and the curvature scalar are given by

$$R_{\beta\gamma} := R_{\dot\alpha\dot\beta\dot\gamma}{}^{\alpha} \quad \text{and} \quad R := g^{\alpha\beta}R_{\alpha\beta}. \tag{A 1.3}$$

The electromagnetic field tensor $F_{\alpha\beta}$ is taken such that in flat spacetime, with Minkowskian coordinates with metric tensor

$$g_{\alpha\beta} = \mathrm{diag}\,(1, -1, -1, -1), \tag{A 1.4}$$

the electric and magnetic field vectors in the 3-spaces $x^0 = \text{const}$ are given by

$$\boldsymbol{E} = (F^{01}, F^{02}, F^{03}) \quad \text{and} \quad \boldsymbol{H} = (F^{23}, F^{31}, F^{12}) \tag{A 1.5}$$

respectively. The electromagnetic 4-vector potential A^α is such that

$$F_{\alpha\beta} = \nabla_\beta A_\alpha - \nabla_\alpha A_\beta. \tag{A 1.6}$$

We make use of the theory of bitensors, or two-point tensors, developed by Synge (1960) and by DeWitt & Brehme (1960), following closely the notation of DeWitt & Brehme. Let $x(u)$ be the parametric form of a geodesic joining $x_1 := x(u_1)$ and $x_2 := x(u_2)$, with u an affine parameter along it. Then the world function biscalar $\sigma(x_1, x_2)$ is defined by

$$\sigma(x_1, x_2) := \tfrac{1}{2}(u_2 - u_1)\int_{u_1}^{u_2} g_{\alpha\beta}(x(u))\,\frac{\mathrm{d}x^\alpha}{\mathrm{d}u}\frac{\mathrm{d}x^\beta}{\mathrm{d}u}\,\mathrm{d}u, \tag{A 1.7}$$

and is independent of the particular affine parametrization chosen. We shall use bitensors only in a region of spacetime where there exists a unique geodesic joining every pair of points, and then $\sigma(x_1, x_2)$ is a well defined single valued function of the point pair (x_1, x_2). If we did not make this restriction, in general it would be a many-valued function.

If we have a bitensor function of the point pair (x, z) it is necessary to distinguish which indices are tensor indices at x and which are at z. Unless otherwise stated, from now on we shall always use x and z as labels for a point pair, and then $\alpha, \beta, \ldots$ will be used as indices at x and $\kappa, \lambda, \ldots$ at z. We denote covariant derivatives of $\sigma(x, z)$ simply by appropriate suffixes, thus

$$\sigma_{\alpha\beta\kappa}(x, z) := \nabla_\kappa \nabla_{\beta\alpha} \sigma(x, z),$$

where $\nabla_{\beta\alpha}$ acts at x and ∇_κ at z. Note that for any bitensor, covariant derivatives at x commute with those at z. Note also that we may now unambiguously write A^α and A^κ to denote the value of a vector field A^α at x and z respectively, without any need for the appropriate argument to be written explicitly.

We now develop the elementary properties of σ used in the present paper; further details may be found in the above references. The integral in (A 1.7) is well known to be stationary, in the sense of the calculus of variations, if $x(u)$ is a geodesic with u an affine parameter along it, and the allowed variations $\delta x(u)$ leave the end-points fixed, i.e. $\delta x(u_1) = \delta x(u_2) = 0$. Hence if a variation $\delta x(u)$ is made in the path of integration which varies the end-points, the induced change in σ is given by

$$\delta\sigma = (u_2 - u_1)\,[g_{\alpha\beta}\dot{x}^\beta \delta x^\alpha]_{u_1}^{u_2}, \tag{A 1.8}$$

where

$$\dot{x}^\alpha := \mathrm{d}x^\alpha/\mathrm{d}u. \tag{A 1.9}$$

But also

$$\delta\sigma = \frac{\partial\sigma}{\partial x_1^\alpha}\,\delta x_1^\alpha + \frac{\partial\sigma}{\partial x_2^\alpha}\,\delta x_2^\alpha, \tag{A 1.10}$$

and hence from this and (A 1.8),

$$\left.\begin{aligned} \partial\sigma/\partial x_1^\alpha &= -(u_2 - u_1)\,g_{\alpha\beta}(x_1)\,\dot{x}^\beta(u_1), \\ \text{and}\quad \partial\sigma/\partial x_2^\alpha &= (u_2 - u_1)\,g_{\alpha\beta}(x_2)\,\dot{x}^\beta(u_2). \end{aligned}\right\} \tag{A 1.11}$$

If we take the particular case of $u_1 = 0$, and write u_2 as u, x_1 as z and x_2 as x, then (A 1.11) can be written in the simpler form

$$\sigma^\kappa = -u\dot{x}^\kappa \quad \text{and} \quad \sigma^\alpha = u\dot{x}^\alpha, \tag{A 1.12}$$

where we are using our convention on indices stated above. Now using the fact that $\delta\dot{x}^\alpha/\mathrm{d}u = 0$, the integrand in (A 1.7) is seen to be actually independent of u, and hence the integration may be performed to give

$$\sigma(x_1, x_2) = \tfrac{1}{2}(u_2 - u_1)^2 g_{\alpha\beta}(x_1)\,\dot{x}^\alpha(u_1)\,\dot{x}^\beta(u_1). \tag{A 1.13}$$

Using (A 1.11) and changing the notation to that of (A 1.12) gives the important relations

$$2\sigma = \sigma_\kappa \sigma^\kappa = \sigma_\alpha \sigma^\alpha. \tag{A 1.14}$$

Equation (A 1.12) shows that $-\sigma^\kappa(x, z)$ is a vector at z tangent to the geodesic joining z to x and whose length is equal to that of this geodesic. Combining this with

(A 1.14) shows that $\sigma = \pm\frac{1}{2}s^2$, where s is the length of this geodesic, the positive/negative sign being taken according as the geodesic is timelike/spacelike respectively. These geometric interpretations of σ and σ^κ and the equivalent formulae (A 1.12) and (A 1.14) will be very important throughout the development of this dynamical theory. We see that $-\sigma^\kappa(x,z)$ is a natural generalization of the position vector of x relative to z, and reduces to this position vector in flat spacetime. Consequently, in flat spacetime and using Minkowskian coordinates,

$$\sigma^\alpha_{\cdot\beta} = \delta^\alpha_\beta, \quad \sigma^\alpha_{\cdot\kappa} = -\delta^\alpha_\kappa \quad \text{and} \quad \sigma^\kappa_{\cdot\lambda} = \delta^\kappa_\lambda. \tag{A 1.15}$$

Hence also, by (3.9), in flat spacetime

$$K^\alpha_{\cdot\kappa} = H^\alpha_{\cdot\kappa} = \delta^\alpha_\kappa. \tag{A 1.16}$$

These results have been used in the comparison between the definitions (4.1) and (4.2) in flat spacetime and (5.1) and (5.2) in curved spacetime. Applying them to (3.11) and taking z as the origin, we recover the well-known result that the Killing vectors of a flat spacetime have the form

$$\xi^\alpha = a^\alpha + \omega^\alpha_{\cdot\beta} x^\beta, \tag{A 1.17}$$

where a^α and $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ are constants.

A 2. *Geodesic orbits of Killing vectors in a de Sitter universe*

The arguments presented in the first part of this section are based on those of Schrödinger (1956). The de Sitter universe of either positive or negative curvature k can be embedded in a 5-dimensional flat manifold of suitable signature, as the hypersurface U with equation

$$g_{AB} y^A y^B = -k^{-1}. \tag{A 2.1}$$

Here, capital Latin indices run from 1 to 5, Minkowskian coordinates are used and the scalar curvature k agrees with that used in § 5. To obtain the required signature -2 for the metric induced in U by g_{AB}, we must take

$$g_{AB} = \left\{\begin{array}{ll} \operatorname{diag}(-1,-1,-1,-1,+1) & (k>0), \\ \operatorname{diag}(-1,-1,-1,+1,+1) & (k<0). \end{array}\right\} \tag{A 2.2}$$

This is most easily seen by noting that any infinitesimal displacement at a point P of U and which lies in U must be orthogonal to the radius vector to P from the origin. But by (A 2.1), this radius vector is spacelike ($g_{AB}y^Ay^B < 0$) or timelike ($g_{AB}y^Ay^B > 0$) according as $k > 0$ or $k < 0$ respectively. Hence the additional dimension added to the required signature for U of $(- - - +)$ must be respectively spacelike or timelike, as given in (A 2.2).

Geodesics in the hypersurface U are given by the intersection of U with hyperplanes through the origin, while isometries of U are induced by those isometries of the 5-dimensional manifold which leave the origin fixed. By (A 1.17), these isometries are generated by Killing vectors of the form

$$\xi^A = \omega^A_{\cdot B} y^B, \tag{A 2.3}$$

where $\omega_{AB} = -\omega_{BA}$. Now we are interested, for its application in § 5, in the geodesic orbits of the induced Killing vector fields of U. They must thus be given by the intersection of U with any plane Π through the origin invariant under (A 2.3), i.e. such that

$$y^A \in \Pi \quad \text{implies} \quad \omega^A_{.B} y^B \in \Pi. \tag{A 2.4}$$

Enumeration of all possibilities is equivalent to determining the canonical forms of the skew tensor ω_{AB} under Lorentz transformations of the indefinite metrics (A 2.2). However, we only need the canonical forms of a restricted class of skew tensors, as the physical considerations of § 5 showed that at least one point z_0 exists in the body satisfying equation (5.12), and hence that the orbit γ of the Killing vector field p^α through z_0 is a timelike geodesic. Hence $\omega^A_{.B}$ leaves invariant a plane Π_0 such that $U \cap \Pi_0 = \gamma$. This restriction on the class of skew tensors simplifies the derivation of the canonical forms. Since Π_0 also contains the radius vector to any point of γ, which vector meets γ orthogonally, and since such radius vectors are timelike or spacelike according as $k < 0$ or $k > 0$, we see that the metric induced in Π_0 by g_{AB} is of signature $(++)$ or $(-+)$ respectively. Thus if, by a Lorentz transformation, we bring the y_4, y_5 axes to lie in the plane Π_0, g_{AB} will maintain the form (A 2.2) and ω_{AB} will take the form

$$\omega_{AB} = \left(\begin{array}{c|cc} \omega_{ab} & \multicolumn{2}{c}{\cdot} \\ \hline \cdot & 0 & \alpha \\ & -\alpha & 0 \end{array}\right) \tag{A 2.5}$$

for some real α. Here ω_{ab} is a 3×3 skew matrix, and the two off-diagonal matrices are zero. As the directions of the y_1, y_2 and y_3 axes are, by (A 2.2), all spacelike, by a 3-dimensional rotation of these axes we may reduce ω_{ab} to the usual canonical form for a 3-dimensional rotation,

$$\omega_{ab} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \beta \\ \cdot & -\beta & \cdot \end{pmatrix}, \tag{A 2.6}$$

where again the dots denote zeros. This thus reduces ω_{AB} to the canonical form

$$\omega_{AB} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \beta & \cdot & \cdot \\ \cdot & -\beta & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & -\alpha & \cdot \end{pmatrix}. \tag{A 2.7}$$

Since the orbit γ is, on physical grounds, non-degenerate, we must have $\alpha \neq 0$. If $\alpha^2 \neq \beta^2 \neq 0$, the only planes left invariant by this ω_{AB} are the (y_4, y_5) plane, which meets U in γ, and the (y_2, y_3) plane, which only meets U if $k > 0$, and then it meets it in a spacelike geodesic. If $\beta = 0$, then again the (y_4, y_5) plane is invariant, but so also is any plane through the origin in the 3-space $y_4 = y_5 = 0$. These again meet U only if $k > 0$, when their intersections consist of all geodesics on the spacelike 2-sphere in which $y_4 = y_5 = 0$ meets U.

In both these cases, the geodesic orbits other than γ are spacelike and 'half way around the universe from γ'. One final case need be considered, when $\alpha^2 = \beta^2 \neq 0$. In this case, if $k > 0$ the situation is exactly as it was for $\alpha^2 \neq \beta^2 \neq 0$, but if $k < 0$, then a 2-parameter family of geodesic orbits exists, consisting of the section of U by planes through the origin and through points

$$(0, a, b, c, d) \quad \text{and} \quad (0, \beta b, -\beta a, -\alpha d, \alpha c),$$

where a, b, c, d are arbitrary. Thus the orbit through every point of the section of U by $y_1 = 0$ is geodesic. This includes γ, and also a two-parameter family of other timelike geodesics in the neighbourhood of γ.

This is the only case in which (5.12) fails to determine a unique timelike curve lying within the world tube W of the body, as the spacelike geodesics admitted in the other cases are far from the body, and are also excluded by our requirement that p^α be timelike throughout W. To see the physical implication of the condition $\alpha^2 = \beta^2$, we need to determine the parameters α and β of (A 2.7) in terms of the invariants M and S defined in § 5. Now letting x^α be any coordinate system in U, the five y^A will be functions of the four x^α, from which, following Schouten (1954, ch. 5) we may define the connecting quantities

$$B^A_\alpha := \partial y^A / \partial x^\alpha. \tag{A 2.8}$$

Then since p_α is to be the vector field induced in U by ξ_A of (A 2.3),

$$p_\alpha = B^A_\alpha \omega_{AB} y^B. \tag{A 2.9}$$

Hence

$$\nabla_\beta p_\alpha = B^{AB}_{\alpha\beta} \omega_{AB}, \tag{A 2.10}$$

where $B^{AB}_{\alpha\beta} := B^A_\alpha B^B_\beta$, so that by (5.10)

$$S_{\alpha\beta} = k^{-1} B^{AB}_{\alpha\beta} \omega_{AB}. \tag{A 2.11}$$

Now let n^A be the unit outward normal to U, and put $k^{-1} = \epsilon R_0^2$, where $\epsilon = \pm 1$. Then from (A 2.1),

$$n^A = y^A R_0^{-1}. \tag{A 2.12}$$

Equations (A 2.9) and (A 2.11) now become

$$p_\alpha = R_0 B^A_\alpha n^B \omega_{AB} \quad \text{and} \quad S_{\alpha\beta} = \epsilon R_0^2 B^{AB}_{\alpha\beta} \omega_{AB}, \tag{A 2.13}$$

showing that p_α and $S_{\alpha\beta}$ together give all the components of ω_{AB}. Moreover,

$$B^{AB}_{\alpha\beta} g^{\alpha\beta} = g^{AB} + \epsilon n^A n^B, \tag{A 2.14}$$

as $n^A n_A = -\epsilon$, and hence

$$p_\alpha p^\alpha = R_0^2 \omega^{AB} \omega_{AC} n_B n^C \tag{A 2.15}$$

and

$$S_{\alpha\beta} S^{\alpha\beta} = R_0^4 \omega^{AB} \omega_{AB} - 2\epsilon R_0^2 p_\alpha p^\alpha. \tag{A 2.16}$$

If the coordinates y^A are chosen so that ω_{AB} takes the canonical form (A 2.7), then along γ (the L_0 of § 5) we have $n^1 = n^2 = n^3 = 0$. Hence the definitions (5.25) of M^2 and S^2 yield

$$M^2 = R_0^2 \alpha^2 \quad \text{and} \quad S^2 = R_0^4 \beta^2. \tag{A 2.17}$$

By choosing the directions of the coordinate axes suitably, we may ensure that α and β are both positive. Then

$$\alpha = M/R_0 \quad \text{and} \quad \beta = S/R_0^2, \tag{A 2.18}$$

giving the required physical interpretations of α and β.

The condition $\alpha^2 = \beta^2$ is now seen to be equivalent to $S = MR_0$. Physically, this could only be satisfied by a body whose size was comparable with the radius of the universe and which was rotating with a velocity comparable with that of light. Such a body comes outside the range of applicability claimed for our results, and we shall not consider this case further. Nevertheless, it is interesting to note that the uniqueness of our definition of the mass centre does break down under extreme conditions such as this.

We may now also understand the ambiguity in the solution of equations (5.26) for M and S in terms of I_1 and I_2. For on using (A 2.17) in (5.26) we get

$$I_1 = R_0^2(\alpha^2 - \epsilon\beta^2) \quad \text{and} \quad I_2 = R_0^6\alpha^2\beta^2. \tag{A 2.19}$$

If $k > 0$, i.e. $\epsilon = 1$, the solution for real and non-negative α^2 and β^2 is unique, and hence the corresponding Killing vector field is unique up to rotations and reflexions. The possible solutions for M^2 and S^2 are $M^2 = R_0^2\alpha^2$, $S^2 = R_0^4\beta^2$, giving the correct (real) values of M and S obtained by applying (5.25) at a point of γ, and

$$M^2 = -R_0^2\beta^2, \quad S^2 = -R_0^4\alpha^2,$$

corresponding to a spacelike p^κ and timelike S^κ, giving the values which would be obtained by applying the definitions (5.25) at any point of the spacelike geodesic (if $\beta \neq 0$) or 2-sphere (if $\beta = 0$) which consists of all points other than those of γ at which (5.12) holds.

If $k < 0$, i.e. $\epsilon = -1$, the values of I_1 and I_2 do not determine unique real and non-negative values of α^2 and β^2 unless the solution has $\alpha^2 = \beta^2$, as the solutions for α^2 and β^2 may be interchanged. The physical solution may be picked out by requiring $\beta^2 \leqslant \alpha^2$, i.e. $S \leqslant MR_0$, in accordance with the discussion above. This gives the values (5.27) and (5.28) for M and S. In contrast with the case $k > 0$ the other root for M and S does not correspond to points at which (5.12) is satisfied but which do not lie on γ; instead it corresponds to the essentially different Killing vector field with $\beta^2 > \alpha^2$ which yields the same values for the constants I_1 and I_2. The case of equal roots, $I_1^2 = -4kI_2$, is the exceptional case $\alpha^2 = \beta^2$ discussed above.

It is of interest to note that I_1 and I_2 may be expressed in terms of ω_{AB} in a covariant manner, avoiding the special coordinate system used in the canonical form (A 2.7). It may be shown (cf. Gürsey (1964)) that

$$I_1 = -\tfrac{1}{2}k^{-1}\omega^{AB}\omega_{AB} \quad \text{and} \quad I_2 = -\epsilon k^{-3}W_A W^A, \tag{A 2.20}$$

where

$$W_A := \tfrac{1}{8}\epsilon_{ABCDE}\,\omega^{BC}\omega^{DE} \tag{A 2.21}$$

and ϵ_{ABCDE} is the 5-dimensional Levi-Civita alternating symbol.

In summary, in this appendix we have shown that the problem of the uniqueness of solutions of the centre-of-mass relation

$$p_\kappa S^{\kappa\lambda} = 0 \tag{A 2.22}$$

in a de Sitter universe falls into several possible cases. Provided the physical assumption is made that there exists one point within the world tube of the body at which p^κ is timelike and (A 2.22) holds, then there exists a timelike geodesic γ through this point along which this condition holds, and scalars M and S may be defined by applying (5.25) at any point of γ. The points at which (A 2.22) holds which do not lie on γ form:

(i) a spacelike geodesic if $k > 0$ and $S > 0$;

(ii) a spacelike 2-sphere if $k > 0$ and $S = 0$, in which case the momentum Killing vector field p^α is hypersurface orthogonal;

(iii) a 2-surface containing γ if $k < 0$ and $S = MR_0$; and

(iv) there are no such points if $k < 0$ and $S \neq MR_0$.

Only in case (iii) is the centre of mass line ambiguously defined by requiring it to be timelike and satisfy (A 2.22), and this exceptional case comes outside the scope claimed for the present theory.