

DIE GRUNDELHREN DER MATHEMATISCHEN  
WISSENSCHAFTEN IN EINZELDARSTELLUNGEN

BAND X

J. A. SCHOUTEN

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SECOND EDITION

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AN INTRODUCTION TO TENSOR ANALYSIS  
AND ITS GEOMETRICAL APPLICATIONS

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ALLE RECHTE,  
INSBESONDERE DAS DER ÜBERSETZUNG IN FREMDE SPRACHEN  
VORBEHALTEN

OHNE AUSDRÜCKLICHE GENEHMIGUNG DES VERLAGES  
IST ES AUCH NICHT GESTATTET, DIESES BUCH ODER TEILE DARAUS  
AUF PHOTOMECHANISCHEM WEGE (PHOTOKOPIE, MIKROKOPIE) ZU VERVIELFÄLTIGEN

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*This book is dedicated to the memory of*

**DR. GREGORIO RICCI CURBASTRO**

*in life Professor of Mathematics in the  
University of Padua,  
who laid the foundations  
of tensor calculus.*

## Preface to the second edition.

This is an entirely new book. The first edition appeared in 1923 and at that time it was up to date. But in 1935 and 1938 the author and Prof. D. J. STRUIK published a new book, their *Einführung I* and *II*, and this book not only gave the first systematic introduction to the kernel-index method but also contained many notions that had come into prominence since 1923. For instance densities, quantities of the second kind, pseudo-quantities, normal coordinates, the symbolism of exterior forms, the LIE derivative, the theory of variation and deformation and the theory of subprojective connexions were included. Now since 1938 there have been many new developments and so a book on RICCI calculus and its applications has to cover quite different ground from the book of 1923. Though the purpose remains to make the reader acquainted with RICCI's famous instrument in its modern form, the book must have quite a different methodical structure and quite different applications have to be chosen.

The first chapter contains algebraical preliminaries but the whole text is modernized and there is a section on hybrid quantities (quantities with indices of the first and of the second kind) and one on the many abridged notations that have been developed by several authors.

In the second chapter the most important analytical notions that come before the introduction of a connexion are dealt with in full. The theory of integrability and PFAFF's problem are treated here and do not get a chapter of their own this time as they did in the first edition because SCHOUTEN and v. d. KULK's book on PFAFF's problem of 1949 can be referred to. A special section pays attention to CARTAN's symbolism.

In the first edition the chapters IV–VI contained the theory of affine, riemannian and WEYL connexions and in these chapters problems of projective and conformal transformation and of imbedding were also dealt with. In the light of modern developments this mixing up of quite different topics could not be tolerated. So there is now a chapter III dealing with linear connexions, curvature and normal coordinates, also with respect to anholonomic systems of reference and in different notations. Then there is a chapter V on imbedding and curvature, in the last section of which the theory of higher curvatures

is developed. Chapter VI is dedicated to projective and conformal transformations of connexions and in this chapter subprojective connexions and concircular transformations of a  $V_n$  find their natural place.

Between chapter III and these latter chapters it has been necessary to insert a chapter IV on the theory of LIE groups. This theory not only gives a very beautiful example of special linear connexions but it also forms the necessary foundation for the following chapters, especially for chapter VII.

This chapter VII on the theory of variations, deformations and movements leads in §§ 4, 5 to the theory of the holonomy groups that is used in § 6 to give an application of CARTAN's method.

The last chapter in the first edition dealt with the invariant decomposition of tensors and in 1923 this was a new and interesting topic. But now we can refer to D. E. LITTLEWOOD's book of 1951 and it does not seem any longer justifiable to treat such a special purely algebraic problem in a book on RICCI calculus and differential geometry.

Instead a chapter VIII has been added containing miscellaneous examples, mostly taken from recent literature. Here the harmonic  $V_n$  (§ 1), the many different connexions for hybrid quantities and their properties with respect to imbedding and transformation (§§ 2–8) and the spaces of recurrent curvature (§ 9) are considered. The choice of these subjects was entirely free. Of course this chapter could have been made many times longer. But there must be a limit to the size of a book of this kind and this is also the reason why all those topics contained in the following long list, however interesting they are, could at most be mentioned very shortly:

1. properties in the large as for instance properties of compact manifolds;
2. theory of geometric objects; fibre bundles etc.;
3. theory of VITALI;
4. geometries of FINSLER, CARTAN, A. KAWAGUCHI;
5. extensors of CRAIG and generalizations of M. KAWAGUCHI;
6. linear elements and surface elements of higher order;
7. connexions of higher order;
8. projective and conformal geometry with supernumerary coordinates either in the tangent spaces or in the whole space;
9. path spaces;  $K$ -spreads;
10. natural families of curves;
11. non-linear connexions;
12. special geometries for small values of the dimensions;

13. contact transformations and their connexions;
14. treatment of differential equations by means of linear connexions belonging to them;
15. spaces with a fundamental tensor of rank  $< n$ ;
16. G. KRON's applications of tensor analysis to electric circuits.

In the first edition there was an extensive literature list. With its 196 titles it covered nearly all the literature on many dimensional differential geometry. The literature list of the *Einführung* of 1935–1938 with its 488 titles represents only a selection of the literature at that time. This is even more true of the literature list presented here which is by no means exhaustive though it contains about 1400 titles and refers to about 350 authors. By selecting the titles in such a way that the reader interested in some topic will always find at least a few titles that can lead him to more references, the author has tried to retain something of the encyclopedic character of the first edition.

The author owes many thanks to the Mathematical Institute of Amsterdam University and the Mathematical Centre at Amsterdam for the valuable help of many of their collaborators. In the first place I mention Dr. NIJENHUIS (M. I. and M. C.) who has given much time to the study of literature and the collection of exercises and who proposed valuable improvements to the text in nearly every chapter. Mr. BARNING (M. I.) and Mrs. v. ROOTSELAAR (M. I.) did important work by studying literature and checking formulae. For the correction of proofs I had the valuable help of Mr. BARNING (M. I.) and Mr. VERHOEFF (M. C.). Mr. WOOD of the Clarendon Press, Oxford, has kindly given advice on the language. This is the third book with which he has helped me and I am very grateful to him for all the English idiom he has taught me. Last but not least I have to thank my wife Mrs. SCHOUTEN-BYLSMA who spent much time on the administration of all literature references.

I wish to express my best thanks to the publishers for their most agreeable collaboration.

Epe (Holland), im Juli 1954.

J. A. SCHOUTEN.

# Contents.

## I. Algebraic preliminaries (1).

### § 1. The $E_n$ (1).

affine space (1) —  $E_n$  (1) — affine group (1) —  $G_a$  (1) — minor (1) — summation convention (1) —  $A_{\lambda}^x$ ,  $A_{\lambda}^y$  (1) — point space (1) — allowable coordinate transformations (1) — rectilinear coordinates (1) — homogeneous linear group (2) —  $G_h$  (2) — centred  $E_n$  (2) — point transformations (2) —  $\delta_{\lambda}^{\mu}$  (2) — dragging along of coordinate system (2) —  $\delta_{\lambda}^x$  (2) — KRONECKER symbol (2, 7) —  $\delta_{\lambda}^{*\mu}$  (2) — generalized KRONECKER symbol (2) — kernel letter (2) — running indices (2) — fixed indices (2) — kernel-index method (3) — flat submanifold (3) — linear submanifold (3) —  $C_{\lambda}^x$  (3) — rank (3) —  $B_b^x$  (3) — point (3) — straight line (3) — plane (3) — hyperplane (3) — coordinate axes (4) — coordinate planes (4) — coordinate  $E_p$ 's (4) — net (4) — lie in (4) — contain (4) — translation (4) — parallel (4) —  $p$ -direction (4) — improper  $E_{p-1}$  (4) —  $E_{p-1}$  "at infinity" (4) — direction (4) — reduction (5) —  $t/p$ -parallel (4) — section (4, 5) — spanning (4) — join (4) — projection (4, 5) — screwsense (5) — sense (5) — sense of rotation (5) — opposite (5) — inner orientation (5) — outer orientation (5).

### § 2. Quantities in $E_n$ (6).

quantity (6) — components (6) — kind (6) — sum of quantities (5) — manner of transformation (6) — scalar (6) — contravariant vector (6) — contragredient (6, 7) — contravariant basis (7) — dead indices (7) — living indices (7) — covariant vector (7) — covariant basis (8) — transvection (8) — reciprocal sets (9) — strangling (10) — tensor (10) — affinor (10) — valence (10) — co- and contravariant tensor (10) — cocontravariant (10) — mixed (10) — unity tensor (10) —  $A_{\lambda}^x$  (10) — intermediate components (11) — equiform (11) — pseudo-tensor (11) — pseudo-scalar (11) — pseudo-vector (11) — class of pseudo-quantity (11) — tensor density (12) — tensor  $\Delta$ -density (12) — weight (12) —  $W$ -tensor (12) —  $\tilde{\omega}$  (12) —  $G_{sa}$  (12) — connecting quantities (13).

### § 3. Invariant processes and relations (13).

addition of quantities (13) — isomer (13) — general multiplication (13) — contraction (13) — transvection (14) — saturated indices (14) — dummy indices (14) — free indices (14) — mixing (14) — round brackets (14) — symmetric (14) — symmetric part (14) — alternation (14) — square brackets (14) — alternating (14) — alternating part (14) — concomitant (15) — invariant (15) — rational integral (15).

### § 4. Section and reduction with respect to an $E_m$ in $E_n$ (16).

decomposition with respect to a rigged  $E_m$  (16) — section (16) — reduction (17) — rigging (17) — projection (18) —  $E_m$ -part (19).

### § 5. Rank, domain and support of domain with respect to one or more indices (20). $\mu$ -rank (20) — $\mu$ -domain (20) — support of domain (20) — double brackets $[\square]$ (20).

### § 6. Symmetric tensors (21).

symmetric tensor (21) — symmetric multiplication (21) — divisor of a symmetric tensor (21) — irreducible symmetric tensors (21).

### § 7. Multivectors (22).

multivector (22) —  $p$ -vector (22) — bivector, trivector, quadrivector (23) — alternating multiplication (23) — divisor of a multivector (23) — simple multivector (23) —  $n$ -vector (23) —  $E_{\lambda_1 \dots \lambda_n}^{x_1 \dots x_n}$  (25) —  $\tilde{e}_{\lambda_1 \dots \lambda_n}^{(x)}$  (25) —  $\infty$  (25) —  $\tilde{\epsilon}_{\lambda_1 \dots \lambda_n}^{x_1 \dots x_n}$  (26) —  $\tilde{e}_{\lambda_1 \dots \lambda_n}$  (26) — identifications (27) — pseudo-scalar (27) — tables of alternating quantities in  $E_3$  (28).

### § 8. Tensors of valence 2 (28).

Cocontravariant tensor of valence 2 (28) — trace (29) — spur (29) — eigenvector (29) — eigenvalue (29) — characteristic equation (29) — first canonical form of the matrix (33) — elementary divisors (33) — co- and contravariant tensors (34) —  $s$ -vector tensor (34) — adjoint (34, 38) — symmetric co- and contravariant tensors (34) — index (34) — signature (34) — rank tensors (34) — definite (35) — indefinite (35) — semi-definite (35) — co- and contravariant bivectors (35) — blades (36) — classification of trivectors (36) — reduction number (37).

### § 9. Introduction of a metric in an $E_n$ (40).

length (40) — distance (40) — indicatrix (40) — fundamental tensor (41) — euclidean  $R_n$  (41) — minkowskian  $R_n$  (41, 42) — time-like (41) — space-like (41) —  $\pm$ -region (41) — nullcone (41) — unitvector (41) — angle (41) — duration (41) — isotropic  $E_p$  (42) — full isotropic  $E_p$  (42) — null  $E_p$  (42) — perpendicular (42) — cartesian coordinate system (42) — minkowskian proper rotations (43) — LORENTZ transformations (43) — minkowskian reflexions (43) — improper minkowskian rotations (44) — vector (44) — raising and lowering of indices (44) —  $I^{x_1 \dots x_n}$  (45) —  $i_{\lambda_1 \dots \lambda_n}^{(h)}$  (45) —  $i, \tilde{i}$  (45) —  $g$  (45) — principal multidirection (46) — theorem of principal axes (46) — theorem of principal blades (46) — orthogonal coordinate transformations (47 ff.) — rotations (47) — reflexotations (47) — principal angles (48) — reflexional orthogonal transformations (50) — proper and improper rotations (50) —  $t/p$ -parallel (51) —  $t/p$ -perpendicular (51).

### § 10. Hybrid quantities (51).

quantities of the first and second kind (51) — hybrid quantities (52) — weight and antiweight (52) — hermitian (symmetric) tensor (52) — hermitian alternating tensor (52) — hybrid bivector (52) —  $\pm$ -invertible (53) — index (53) — signature (53) —  $\pm$ -definite (53) — indefinite (53) — semi-definite (53) — the auxiliary  $E_{2n}$  (53) — principal  $E_n$  (53) — first and second set of invariant  $E_n$ 's (54) — hermitian fundamental tensor (54) —  $\tilde{R}_n$  (54) — ordinary  $\tilde{R}_n$  (54) — kets and bras (54) — unitary perpendicular vectors (54) — norm of vector (54) — unitvector (54) — unitary cartesian coordinate systems (54) — unitary orthogonal group (55) — fundamental figure in  $\tilde{R}_n$  (55) — nullcone in  $\tilde{R}_n$  (55) — the auxiliary  $R_{2n}$  (55) — principal multidirections of a hermitian tensor (56) — theorem of principal axes (56).

### § 11. Abridged notations (57).

collecting indices (57) — representative indices (57) — the notation of Grvens (57) — VITALI's method (57) — CRAIG's extensors (58) — ideal vectors (58) — systems of "direct" calculus (58) — skeleton of a formula (59) — the method of the radius-vector in  $E_n$  and  $R_n$  (60) — CARTAN's method (61) — abbreviations ad hoc (61).

## II. Analytic preliminaries (61).

### § 1. The arithmetic $n$ -dimensional manifold $\mathfrak{A}_n$ (61).

arithmetic point (61) — arithmetic manifold  $\mathfrak{A}_n$  (61) — components of the arithmetic point (61) — polycylinder (61) — box (61) — region of  $\mathfrak{A}_n$  (62) — neighbourhood (62).

### § 2. The geometric $n$ -dimensional manifold $X_n$ (62).

coordinate system (62) — analytic functions (62) — class of a function (62) — coordinate transformation (63) — pseudo-group (64) — groupoid (64) — supernumerary coordinates (64) — allowable coordinate systems (64) — geometric manifold (64) —  $X_n$  (64) — geometric point (64) — KLEIN's principle (65) — cell (65) — region of  $X_n$  (65) — imbedding (66) — neighbourhood (66) — analyticity in  $X_n$  (66) — geometry in the large or global geometry (66) — class in  $X_n$  (66).

### § 3. Geometric objects and quantities in $X_n$ (67).

geometric object at a point (67) —  $\{\}$  (67) — components of geometric object (68) — kind of an object (68) — quantities (68) — connecting objects (68) — pseudo-quantities (69) — pseudo-scalar (69) — class of pseudo-quantities (69) — tangent  $E_n$  (69) — local  $E_n$  (69) — object field (70) — analyticity of field (70) — class of field (70) — natural derivative (70) — gradient (70) — Grad (70) —  $D$  (70) — convention about operators (70) — covariant and contravariant basis vectors in  $X_n$  (70f.) —  $e_\lambda^\alpha, \tilde{e}_\lambda^\alpha$  (71) —  $A_\lambda^\alpha$  (71) —  $\delta_\lambda^\alpha$  (71) — region of constant rank (71) — contra- and covariant measuring vectors (71).

### § 4. The $X_m$ in $X_n$ (72).

functional matrix (72) — rank of a set of functions (72) — theorem of adaptation (73) — rank of a system of equations (73) — null point of a system of equations (73) — nullmanifold (73) — null form (73) — equivalent systems (73) — minimal regular (73) — dimension of nullmanifold (73) — imbedding (74f.) —  $m$ -dimensional surface (74) — curve, surface, hypersurface (74) — covariant connecting quantity (74) — base (74) — first and second base theorem (74f.) — base transformation (75) — parametric form or representation (75) — minimal regular (75) — contravariant connecting quantity (75) — coordinate  $X_m$ 's (76) — normal system of  $X_m$ 's (76) — net of a coordinate system (76) — reduction (Zusammenlegung) of  $X_n$  (76) — tangent  $E_m$  (77) — section with  $X_m$  (77) — reduction with respect to  $X_m$  (77) — rigged (77) — parts of a tensor (77) — prolongation of a field of  $X_m$  over  $X_n$  (78).

### § 5. The $E_m$ -field or $X_m^m$ in $X_n$ . Systems of linear partial differential equations (78).

section with  $X_n^m$  (78) — reduction with respect to  $X_n^m$  (78) — rigged  $X_n^m$  (79) — parts of a tensor (79) — enveloping (79) — outer and inner problem (79) — PFAFF's problem (79) — solution of a system of partial differential equations (79) — nullpoint of a system (80) — totally integrable systems (80) — conditions of integrability (80) — homogeneous linear equations (80) — adjoint systems (80) — integral function (80) — integral (80) — derived systems (81) — complete systems (81) —  $X_m$ -forming  $E_m$ -field (81) — non homogeneous linear equations (82) — reduced system (82) — complete non homogeneous systems (82) — gradient equation (83).

### § 6. The invariant operators Rot and Div (83).

alternated derivative (83) — natural derivative (83, 84) — rotation (83) — Rot (83) —  $D$  (83) — divergence (84) — Div (84) — gradient product (84) — theorem of rotation and divergence (85) — other conditions of integrability (86 ff.) — section of a rotation (89).

### § 7. PFAFF's problem (89).

class  $K$  (89) — rotation class  $2\varrho$  (89) — similarity class  $k$  (89) — supports of rotation (90) — supports (90) — characteristics (90) —  $S_1, S'_1, S_4, S'_4$  (90) — similarity transformations (91) — gradient transformations (91) — table of results (92) — canonical form (93) —  $w_\lambda$  is  $X_\nu$ -enveloping,  $\nu = n - \varrho - \varepsilon$  (94) — special canonical forms (94).

### § 8. Theorem of STOKES (95).

generalized theorem of STOKES (97) — differential forms (97 f.) — non invariant forms (99).

### § 9. Anholonomic coordinates (99).

anholonomic coordinate system ( $h$ ) (99) — net (100) —  $A_\lambda^h$  (100) —  $=^h$  (100) —  $(d\xi)^h$  (100) — object of anholonomy  $\Omega_{ji}^h$  (100) —  $X_m$  in  $X_n$  with anholonomic coordinates (100) —  $X_n^m$  with anholonomic coordinates (101) — intermediate components  $\Omega_{\mu\lambda}^h$  of  $\Omega_{ji}^h$  (101).

### § 10. The LIE derivative (102).

dragging along of a coordinate system (102) — dragging along of an object field (102) — dragging along over  $v^\kappa dt$  (103) — operator  $e^{tX}$  (104) — displacement over  $\tau v^\kappa$  (104) — infinitesimal point transformation (104) — group generated by an infinitesimal transformation (104) — LIE derivative, LIE differential (104) — LIE derivative of sums, products, transvections, contractions (106) — LIE derivative of scalars, vectors, tensors,  $p$ -vectors, densities (106) —  $\Delta$ -densities (107) — geometric objects (107) — absolutely invariant fields (107) — absolute invariance with respect to streamlines (107) — LIE derivative of  $\Pi_{\mu\lambda}^\kappa$  (108) — operator  $e^{t\xi}$  working on linear geometric objects (108) —  $\xi$  and  $\xi^\kappa$  (109) — commutativity of  $\xi$  and  $D$  (110).

### § 11. The LAGRANGE derivative (111).

LAGRANGE derivative (112) — LAGRANGE equation (112) — relations for LAGRANGE derivative (114).

### § 12. CARTAN's symbolical method (117).

alternating differential form (118) — symbolical method of CARTAN and KÄHLER (119) — symbolical method applied to geometric objects (119) — scheme of translation (120) — examples (120 f.).

## III. Linear connexions (121).

### § 1. Parallel displacement in an $E_n$ (121).

covariant differential and derivative (122) — parameters of parallel displacement (122) —  $\delta, V_\mu$  (122).

### § 2. Parallel displacement in $X_n$ (123).

parallel displacement of quantities (123) — conditions for the covariant differential (124) —  $I_{\mu\lambda}^\kappa$  (124) — linear displacement (125) — linear connexion (125) — geometric interpretation (125) — covariant stationary (125 ff.) — covariant constant (125 f.) — recurrent or parallel field (126) —  $S_{\mu j}^\kappa$  (126) — symmetric and semi-symmetric connexion (126) — geometric interpretation (127) — displacement of points (127) — CARTAN displacement (128 f.) — covariant differentiation of densities (129) — covariant differentiation of pseudo-quantities (130) — parallel displaced field (130).

**§ 3. A linear connexion expressed in terms of  $S_{\mu\lambda}^{\alpha\beta}$ , an auxiliary symmetric tensor field  $\mathbf{g}_{\lambda\kappa}$  of rank  $n$  and  $V_\mu \mathbf{g}_{\lambda\kappa}$  (131).**

CHRISTOFFEL symbol (132) —  $\{\nu_\lambda^\alpha\}$  (132) —  $\{\mu\lambda\kappa\}$  (132) — metric connexion (132) —  $U_n$  (133) — fundamental tensor (133) —  $V_n$  (133) — riemannian geometry (133) — ordinary  $U_n$  and  $V_n$  (133) — semi-metric connexion (133) —  $W_n$  (133) — WEYL's geometry (133) — conformal transformation of  $g_{\lambda\kappa}$  (134) — “Umeichung” (134) —  $W_n$  dealt with by means of pseudo-tensor (135) — necessity of fundamental tensor of valence 2 (135 ff.) — rings of quantities (137).

**§ 4. Curvature (138).**

$R_{\nu\mu\lambda}^{\alpha\beta}$  (138) — RIEMANN-CHRISTOFFEL tensor (139) — curvature tensor (139) — integrable connexion (139) —  $V_{[\nu} V_{\mu]}$  and the rule of LEIBNIZ (139f.) — application to densities and pseudo-quantities (140f.) —  $V_{,\mu}$  (140) —  $R_{\mu\lambda}$ ,  $K_{\mu\lambda}$  Ricci tensor (141) — teleparallelism or absolute parallelism (142) —  $R_{\nu\mu\lambda}^{\alpha\beta}$  and integrability conditions (142) — the ideal factors  $a_\lambda$ ,  $b_\lambda$  (143) — special semi-symmetric and metric connexions (143) — volume preserving connexions (144).

**§ 5. The identities for the curvature tensor (144).**

first and second identity (144) — third and fourth identity (145) — principal directions of  $V_n$  (146) — BIANCHI's identity (146) —  $K$  (148) —  $G_{\kappa\lambda}$  (148) — scalar curvature  $\kappa$  (148) — Gaussian curvature (148) — Ricci space, special (148) — EINSTEIN space, special (148) — theorem of HERGLOTZ (148) — space of constant curvature (148) —  $S_n$  (148) —  $\dot{V}_{[\mu\dots]}^{\dot{\nu}}$  (149) —  $\dot{V}_{[\nu}^{\dot{\mu}} \dot{V}_{\mu\dots]}^{\dot{\nu}}$  and the rule of LEIBNIZ (149) — geometric interpretation of  $\dot{V}_{[\mu\dots]}^{\dot{\nu}}$  (150) — field in equilibrium (150) — geometric interpretation by means of CARTAN displacements (151f.) — LIE derivative of  $I_{\mu\lambda}^{\alpha\beta}$  (152) —  $\frac{d}{v}$  expressed in terms of  $V$  instead of  $\partial$  (152) —  $\frac{d}{v} V_\mu - V_\mu \frac{d}{v}$  (152) — riemannian curvature with respect to a 2-direction (153) — theorem of SCHUR (153).

**§ 6. Integrability conditions in  $L_n$  (153).**

integrability conditions with  $V$  (154) — possibility of covariant constant  $n$ -vector fields (155).

**§ 7. Geodesics and normal coordinates (155).**

geodesic (155) — affine parameter (155) — symmetric connexion determined by geodesics and their affine parameters (156) — coordinate system geodesic at a point (157) —  $I_{\mu_p\dots\mu_1\lambda}^{\alpha\beta}$  (157) — normal coordinates in  $A_n$  (158) — normal tensors (159f.) — extension (160) — relations between the  $R$ 's and the  $N$ 's (162) —  $A_n$  symmetric with respect to a point (163) —  $R_{\nu\mu\lambda}^{\alpha\beta}$  covariant constant in symmetric  $A_n$  (163) — differential concomitants (164) — absolute and relative differential invariants (164) — first and second reduction theorem (164).

**§ 8. FERMI coordinates (166).**

FERMI coordinates (169) — generalization for  $X_m$  (169).

**§ 9. Linear connexions expressed in anholonomic coordinates (169).**

$I_{ji}^h$  (169) —  $I_j$  (169) —  $\Omega_{jih}$  (170) —  $\chi_{jhi}$  (170) —  $I_{jih}$  (170) —  $\gamma_{ihj}$  (171) — coefficients of rotation (171) —  $R_{kji}^{h\dots h}$  (172) —  $V_{kj}$  (172).

**§ 10. CARTAN's symbolical method used for connexions (172).**

$A^h, \Omega^h, I_i^h, S^h, R_i^h$  (172) — structural equations of  $L_n$  (173) — théorème de conservation de la courbure et de la torsion (173) —  $\overset{\rho}{[V\dots]}$  (174) — mapping of two sets of  $n$  vector fields on each other (174f.) — mapping of two  $L_n$ 's on each other (176ff.).

**§ 11. Linear connexions depending on a non-symmetric fundamental tensor (179).**

non-symmetric fundamental tensor (179) —  $s_{\lambda\kappa}, S^{\kappa\lambda}, \overset{-1}{s}{}^{\lambda\kappa}, \tilde{s}, \tilde{S}^{\kappa\lambda}, h_{\lambda\kappa}, h_{\lambda\kappa}$  (180) — the connexion  $I_{\mu\lambda}^\alpha$  (180) —  $\tilde{G}_{\mu\lambda}^\alpha$  (180) — the  $(---+)$ -differentiation (180) —  $\tilde{G}_{[\mu\lambda]}$  and its divergence (181) —  $*I_{\mu\lambda}^\alpha$  (181ff.) — the condition for  $*S_\lambda$  (181) — the four necessary relations derived by means of the LIE derivative (183) — the connexion  $*I_{\mu\lambda}^\alpha$  in special cases (184) — first geometrical interpretation of the condition (184) — EINSTEIN's interpretation (184) — discussion of results (184f.).

**IV. LIE groups and linear connexions (185).**

**§ 1. Finite continuous groups (185).**

$r$ -parameter finite continuous group (185) — unity element (185) — group germ (186) — group space (186) —  $(\pm)$ -equipollent (186) —  $(\pm)$ -constant (187) —  $\overset{\pm}{I}_{\gamma\beta}^\alpha$  (187) —  $(\pm)$ -connexion (187) —  $(\pm)$ -parallel (187) —  $R_{\delta\gamma\beta}^\alpha = 0$  (187) — anholonomic coordinate systems (*a*) and (*A*) (187ff.) — geodesic (188) —  $I_{\gamma\beta}^\alpha$  (188) —  $(0)$ -connexion (188) —  $V, \overset{+}{V}, \overset{-}{V}$  (189) —  $\overset{\pm}{S}_{\gamma\beta}^\alpha$  (189) —  $c_{\gamma\beta}^\alpha, c_{cb}^\alpha, c_{CB}^A$  (190) —  $c_{cb}^\alpha = \text{const.}, c_{CB}^A = \text{const.}$  (190) —  $R_{\delta\gamma\beta}^\alpha$  (191) —  $\nabla_e R_{\delta\gamma\beta}^\alpha = 0$  (191).

**§ 2. The parameter-groups and the adjoint group of a finite continuous group (191).** continuous group (191) — parameter-groups (191f.) — special anholonomic coordinate systems in general  $X_r$  (195ff.) — application to group space (196) — structural formulae of the parameter-groups after LIE and after MAURER-CARTAN (196) — structural constants (196) —  $\xi, \dot{\xi}$  (197) — finite transformations of the parameter-groups (197) — adjoint group (198) — centre (198) — homologous elements and subgroups (198) — invariant or normal subgroup (198) — linear adjoint group (199) — normal coordinates  $\eta^a$  (199) — series for  $A_B^a$  (200) — linear transformations of the  $\eta^a$  (200).

**§ 3. Finite continuous transformation groups (201).**

essential parameters (201) — connected and dependent infinitesimal transformations (203) — theorem I. 1 (203) — structural formulae of a transformation group after LIE and their generalization by means of the LIE derivative (205) — structural constants (205) — theorems II. 1, III. 1, I. 2 (206) — theorem II. 2 (207) — generators of a group germ (208) — theorem III. 2 (208).

**§ 4. The geometry of group space (209).**

relations between general coordinates and the normal coordinates  $\eta^a$  (209ff.) — series for  $A_b^a, A_b^a, A_b^A$  and  $A_B^a$  (211) — series for  $A_B^a, I_{cb}^a, \overset{+}{I}_{cb}^a$  and  $c_{cb}^{;a}$  (212f.) — reflexions at the point  $\eta^a$  (213).

**§ 5. Invariants of a transformation group in the  $X_n$  of the  $\xi^a$  (214).**

invariant functions (214) — invariant subspace (215) — relative invariant sets of functions (215) — smallest invariant subspace through a point (215) — group induced in invariant subspace (216) — transitive (216) — simply transitive (217f.) — invariant contravariant vector fields (217ff.) — primitive (218).

**§ 6. Invariants of a group in group space (220).**

$g_b, g_{ba}, g_{cba}$  (220) —  $X_p$ 's representing subgroups (221 ff.) — the group of  $g_b$ , the centre (224) — the derived groups (224 f.) — the group of  $g_{ba}$  (225) — the group of  $c_c^b d g_{da}$  (225).

**§ 7. Properties of integrable groups (225).**

integrable groups (225) — necessary and sufficient conditions (225) — canonical form for an integrable group (226).

**§ 8. Simple and semi-simple groups (227).**

the group space of a semi-simple group is a  $V_r$  (227).

**V. Imbedding and Curvature (227).**

**§ 1. The  $X_1$  in  $V_n$  (227).**

linear element (227) — first curvature (228) — curvatura geodetica (228) — curvature vector (228) — first normal (228) — osculating  $R_2$  (228) — stationary point (228) — second curvature (229) — second normal (229) — osculating  $R_3$  (229) — stationary osculating  $R_2$  (229) — higher curvatures (229) — osculating  $R_m$  (229) — FRENET formulae (229 f.) — natural equations (230) — invariance of the proportions of the curvatures (230) — contact of order  $u$  of a curve and a hypersurface (231).

**§ 2. The  $X_1$  in  $W_n$  and  $L_n$  (231).**

$X_1$  in  $W_n$  (231) —  $X_1$  in  $L_n$  (231) — osculating  $p$ -direction (232) — affine parameter (232) — affine length (233).

**§ 3. The  $X_{n-1}$  in  $A_n$  (235).**

$X_{n-1}$  in  $A_n$  (235) — rigged  $X_{n-1}$  (235) — pseudo-normal vector (235) — tangent vector (235) — first normalizing condition (235) — induced connexion (236) —  $h_{cb}$  (236) — principal tangents (237) — asymptotic lines (237) — directions of principal curvature (237) — equation of GAUSS for  $A_{n-1}$  in  $A_n$  (237) — second condition of normalization (237) — equations of CODAZZI for  $A_{n-1}$  in  $A_n$  (238) — different cases of normalization in a volume preserving  $A_n$  (238 ff.) — integrability conditions for an  $A_{n-1}$  in  $E_n$  (240).

**§ 4. The  $V_{n-1}$  in  $V_n$  (242).**

$V_{n-1}$  in  $V_n$  (242) — equations of GAUSS and CODAZZI for  $V_{n-1}$  in  $V_n$  and  $R_n$  (242) — second fundamental tensor (243) — integrability conditions for  $V_{n-1}$  in  $R_n$  (243) — bending (244).

**§ 5. Congruences in  $V_n$  (244).**

real congruences (244) — canonical congruences (245) — normal congruences (245) — geodesic congruences (245) — orthogonal net (246) — orthogonal system (246) — congruences belonging to an orthogonal system (246 ff.) — conditions for the principal directions of a symmetric tensor to be  $V_{n-1}$ -normal (247 f.) — conditions for the eigenvectors of a co-contravariant tensor to be  $X_{n-1}$ -forming (248 f.).

**§ 6. Properties of curvature of a  $V_{n-1}$  in  $V_n$  (250).**

absolute, relative and enforced curvatures and curvature vectors (250) — theorem of MEUSNIER (250) — directions of principal curvature (250) — principal curvatures (250) — mean curvature (250) — theorem of DUPIN (250) — principal tangents (250) — asymptotic lines (250) — geodesic  $V_{n-1}$  in  $V_n$  (251) — umbilical point (251) — umbilical  $V_{n-1}$  (251) — SCHUR space (252) —  $V_{n-1}$  in  $S_n$  (252) —  $S_{n-1}$  in  $S_n$  (252) — indicatrix of DUPIN (252) — theorem of EULER (253).

**§ 7. The rigged  $X_n^m$  in  $L_n$  and  $X_n$  (253).**

$X_n^m$  rigged in  $L_n$  (253 ff.) —  $D$ -symbolism,  $D_\mu$ ,  $D_c$ ,  $D_z$  (254) — full sets of components (255) — skeleton of a formula (255) — first and second curvature tensors  $H_{cb}^{m'}, L_{c,y}^{m,a}, H_{z,y}^{m',a}, L_{z,b}^{m,x}$  (256) —  $L_n^m(A_n^m)$  in  $L_n(A_n)$  (257) —  $Z_{cb}^{m'}, Z_{z,y}^{m',a}$  (257) —  $\Omega_{cb}^a, \Omega_{cy}^a, \Omega_{zy}^x, \Omega_{zy}^x$  (258) —  $r_{dcb}^{m,a}, r_{uzy}^{m,x}$  (260) —  $\mathfrak{D}_c, \mathfrak{D}_z$  (260 f.) —  $R_{dcy}^{m,m}, R_{uzb}^{m,m}$  (261) — equation of GAUSS for  $L_n^m$  in  $L_n$  (261) —  $R_{dcy}^{m,x}, R_{uzb}^{m,a}$  (261 f.) — equations of Ricci for  $L_n^m$  in  $L_n$  (261 f.) — equations of CODAZZI for  $L_n^m$  in  $L_n$  (262 f.) — geodesic  $L_n^m$  (263) —  $L_n^m(A_n^m)$  rigged in  $X_n$  (263 f.) — generalization of  $\mathfrak{D}_c$  and  $\mathfrak{D}_z$  (264).

**§ 8. The rigged  $X_m$  in  $A_n$  (265).**

$X_n^m$  rigged in  $A_n$  (265) — equations of GAUSS, CODAZZI and RICCI for  $A_m$  in  $A_n$  (266 f.) — equations for  $A_m$  in  $E_n$  (267) — superposition of SCHLÄFLI (268) — class of a  $V_m$  (268).

**§ 9. The  $V_n^m$  and  $V_m$  in  $V_n$  and  $X_n$  (269).**

$V_n^m$  in  $V_n$  (269) —  $V_n^m$  in  $X_n$  (270) —  $V_m$  in  $V_n$  (271) — absolute, relative and enforced curvature vectors (271) — principal tangent (271) — asymptotic lines (271) — first curvature region (271) — curvature figure (Krümmungsgebilde) (271) — axial, planar and spatial points (272) — umbilical point (272) — umbilical  $V_m$  (272) — mean curvature vector (272) — minimal  $V_m$  (272) — transposition of a problem of variation (273).

**§ 10. Higher curvatures of a  $V_m$  in  $V_n$  (275).**

$p$ -th curvature region (275) — asymptotic  $V_m$  of order  $p$  (275) — asymptotic lines of order  $p$  on  $V_m$  (275) — higher curvature tensors of valence 3 (276) — generalization of the FRENET formulae for  $V_m$  in  $V_n$  (277) — equations of GAUSS, CODAZZI and RICCI involving higher curvatures (278 ff.) — imbedding theorems (280 ff.) —  $W_m$  in  $W_n$  and  $A_m$  in  $A_n$  (285).

**§ 11. Product spaces (285).**

product spaces (285) — decomposable spaces (285) — breakable and decomposable objects (285) — covariant constant symmetric tensor fields (286).

**VI. Projective and conformal transformations of connexions (287).**

**§ 1. Projective transformation of a symmetric connexion (287).**

vector  $p_\lambda$  of projective transformation (287) — preservation of parallelism (287) — restricted projective transformation (288) — volume preserving connexion (288) — projectively euclidean connexion (288 ff.) —  $D_n$  (288) — projective curvature tensor (289) — necessary projective transformation of an  $E_n$  into an  $E_n$  (290).

**§ 2. Projective transformation of the connexion in a  $V_n$  (292).**

every  $S_n$  a  $D_n$  (292) — projective transformation of a  $V_n$  into a  $V_n$  (292) — equations of LAMÉ (296).

**§ 3. Imbedded spaces in  $A_n$  under projective transformations of the connexion (296).**

induced connexion transformed projectively (297) —  $X_{n-1}$  in  $A_n$  (297 f.) — curve in  $A_n$  (298) — quasi-plane curve (298) — affine parameter (299) — affine normal (299) — affine conic (299).

**§ 4. Projective connexions (300).**

projective parameters of T. Y. THOMAS (301f.) of WHITEHEAD and of BORTOLOTTI (302) — connexion of CARTAN (302) — connexion of KÖNIG (303) — geometry of paths (303) — homogeneous coordinates of VAN DANTZIG (303) —  $K$ -spreads of DOUGLAS (303).

**§ 5. Conformal transformation of a connexion in  $V_n$  (304).**

geodesic and conformal properties fix a  $V_n$  (304) — conformally euclidean  $V_n$  (305) —  $C_n$  (305) — every  $V_2$  an  $S_2$  (306) — conformal curvature tensor (306f.) — orthogonal systems in  $C_n$  (307) — umbilical  $V_{n-1}$ 's (308f.) — conformally geodesic  $V_m$ 's (309ff.) — theorem of THOMSON and TAIT (310) — null geodesic invariant (310) — umbilical points invariant (310) — theorem of LIOUVILLE and LIE (312).

**§ 6. Conformal transformations of the connexion in an EINSTEIN space (312).**  
integrability conditions (313) — conformally invariant tensor density  $\mathfrak{C}_{\alpha\lambda}$  (313) — theorem of BRINKMAN (314).

**§ 7. Conformal connexions (315).**

conformal parameters of J. M. THOMAS (315) — connexions of CARTAN, KÖNIG, T. Y. THOMAS and of SCHOUTEN-HAANJTES (316).

**§ 8. Subprojective connexions (317).**

$k$ -fold projective  $A_n$  (317) — subprojective  $A_n$  (317) — pole direction (317) — theorem of RACHEVSKI (317) — every subprojective connexion projectively invariant (319) — auxiliary algebraic theorem (320) — subprojective  $D_n$  (321).

**§ 9. ADATI's problem (321).**

torse forming fields (322) — concircular, special concircular, concurrent, recurrent, parallel and covariant constant fields (322f.) — chief theorem of the subprojective connexions (324) — special solutions (325f.).

**§ 10. Subprojective transformations of a connexion in  $A_n$  (327).**

subprojective transformation of a connexion belonging to a direction field (327) — subgeodesics belonging to a direction field (327).

**§ 11. The subprojective  $V_n$  (328).**

every subprojective  $V_n$ ,  $n > 2$ , a special  $C_n$  (329) special cases (329).

**§ 12. Concircular transformations of a  $V_n$  (330).**

geodesic circles (330) — concircular transformations (330ff.) — relations with the principal directions of  $V_n$  (321f.) — umbilical  $V_m$ 's (321) — necessary and sufficient conditions for a  $V_n$  to admit a concircular transformation (322f.) — concircular curvature tensor (334) — concircular euclidean  $V_n$  (334).

**VII. Variations and deformations (335).**

**§ 1. General deformation problems (335).**

natural value of a field (335) — value after dragging along (335) — value after pseudoparallel displacement (335) — operators invariant with respect to certain objects (336) — covariant differential (336) — LIE differential  $\oint_v dt$  (336) — apparent differential  $\overset{s}{D} dt$  (336) — identities (337) — natural variation  $\overset{n}{D} dt$  (335) — absolute variation  $\overset{a}{D} dt$  (338) — geodesic variation  $\overset{g}{D} dt$  (338) — relations between these operators (338) — a deformation problem (339) — product integrals (340ff.) —

transposition of a variation problem (341) —  $'D = \overset{n}{D}$  (341) — BERTRAND curves and their generalizations (342ff.).

### § 2. Groups of “motions” in $V_n$ and $L_n$ (346).

affine, conformal, homothetic and projective motions (346) — details on affine motions (346f.) — number of parameters of groups of motions in  $V_n$  and  $A_n$  (348) — KILLING’s equation (348) — translation (349) — translations in  $V_n$  (349) — groups of motions in  $V_n$  and projective mapping (349) — number of parameters of the group of all motions in a  $V_n$  (350).

### § 3. Deformation of subspaces (352).

details on  $V_m$  in  $V_n$  under infinitesimal transformations (352ff.) — deformation of  $V_m^m$  in  $V_n$ ,  $L_m$  in  $L_n$  and  $L_m^m$  in  $L_n$  (352ff.) — method of transposition impossible for  $X_m$  in  $X_n$  (354) — method of NIJENHUIS for the rigged  $X_n^m$  in  $X_n$  (354ff.) — the rigged  $L_n^m$  in  $X_n$  (358ff.).

### § 4. The holonomy group of an $L_n$ (361).

the holonomy group (361) — non homogeneous holonomy group (361) — the infinitesimal transformations of the holonomy group deduced by means of product integrals (362ff.) — generators of the holonomy group (363f.) — introduction of an auxiliary tangent  $E_r$  (364) — extension of covariant differentiation (364ff.) — covariant constant fields and the holonomy groups (367) — special results for  $L_n = V_n$  (367f.) — the  $V_n$  with a recurrent curvature tensor (368) — product spaces (370).

### § 5. Affine motions and the holonomy group in a symmetric $A_n$ (370).

the holonomy group in a symmetric  $A_n$  (370) — equations for the affine motions (371) — extension of the LIE operator (372) — group of isotropy (373) — the group of affine motions is transitive (374) — number of parameters (374).

### § 6. CARTAN’s method applied to the holonomy group and the symmetric $A_n$ (375).

introduction of local coordinate systems (375) — the allowable anholonomic coordinate system in a region (375) — the infinitesimal transformations of the holonomy group (376) — the case of the symmetric  $A_n$  (377) — the case of the general  $A_n$  (377ff.) — group of point transformations in a symmetric  $A_n$  leaving the connexion invariant (381).

## VIII. Miscellaneous Examples (381).

### § 1. The harmonic $V_n$ (381).

$V_n$  harmonic at a point (381) — characteristic function (382) — completely harmonic (382) — centrally harmonic (382) — RUSE’s invariant (382) — simply harmonic (383) — equation of LICHNEROWICZ (384) — equations of COPSON and RUSE (385) — condition for a completely harmonic  $V_n$  to be an EINSTEIN space, an  $S_n$  or an  $R_n$  (385ff.) — SCHUR space and centrally harmonic space (385) — a completely harmonic  $V_n$  that is not an  $S_n$  (386) — the inequality of LICHNEROWICZ (388) — imbedding of an EINSTEIN  $V_n$  and a completely harmonic  $V_n$  in  $S_{n+1}$  (388).

### § 2. Connexions for hybrid quantities (388ff.).

analytic and semi-analytic fields (389) — the new conditions for the connexion of hybrid quantities (389) — most general form of connexion (389) — the auxiliary  $X_{2n}$  (390) — the two invariant sets of  $\infty^n X_n$ ’s in  $X_{2n}$  (390) — equipollent figures (390) — the principal  $X_n$  in  $X_{2n}$  (390) — most general connexion in  $X_{2n}$  (391) — first invariant conditions: the tangent  $n$ -directions are parallel (391) — second invariant condition: the principal  $X_n$  is geodesic (391) — third invariant condition about equipollence and parallelism (392) — the  $\tilde{L}_n$  (393) — the auxiliary  $X_{2n}$

is an  $L_{2n}$  (393) — curvature tensors in  $\tilde{L}_n$  (393 ff.) — first and second identity (393) — identity of BIANCHI (393) —  $V_{\bar{\nu}\mu}$  and  $V'_{\bar{\nu}\mu}$  (394) — identities found by contraction (394 f.) — the  $\tilde{A}_n$  (394) — necessary and sufficient conditions for a semi-analytic scalar field to be analytic (395) — necessary and sufficient conditions for a semi-analytic transformation of  $X_n$  to be analytic (395).

### § 3. Unitary connexion (395 ff.).

hermitian tensor field  $a_{\lambda\bar{\lambda}}$  (395) — unitary connexion (396) — the  $\tilde{U}_n$  (396) — fundamental tensor of a  $\tilde{U}_n$  (396) — connexion in a  $\tilde{U}_n$  derived from a connexion in an  $X_{2n}$  with a symmetric fundamental tensor (397 f.) — the auxiliary  $X_{2n}$  is a  $U_{2n}$  (397) — the  $\tilde{V}_n$  and its auxiliary  $V_{2n}$  (397) — the  $a_{\lambda\bar{\lambda}}$  in a  $\tilde{V}_n$  derived from a scalar (397 f.) — KÄHLER space (397) — the curvature tensors of a  $\tilde{U}_n$  (398 ff.) —  $V_{\bar{\nu}\mu}$  vanishes and  $V'_{\bar{\nu}\mu}$  is hermetian (398) — the four identities in  $\tilde{U}_n$  (398 f.) — BIANCHI's identity in  $\tilde{U}_n$  (399) —  $S_\lambda$  a gradient in a semi-symmetric  $\tilde{U}_n$  (399) —  $R_{\bar{\nu}\lambda}$  hermitian in a  $\tilde{V}_n$  (399) — the identities in  $\tilde{V}_n$  (399) — list of formulae in  $\tilde{L}_n$ ,  $\tilde{A}_n$ ,  $\tilde{U}_n$  and  $\tilde{V}_n$  (400 f.) — mutually perpendicular analytic fields of unitvectors in  $\tilde{U}_n$  and the integrability and analyticity of the connexion (402 f.) — a  $\tilde{V}_n$  with an analytic connexion is a  $\tilde{R}_n$  (403).

### § 4. The $\tilde{V}_n$ of constant curvature (404 ff.).

$\tilde{V}_n$  of constant curvature (404) —  $\tilde{S}_n$  (405) — projective transformation of an  $\tilde{S}_n$  into an  $\tilde{R}_n$  (405) — the linear element of FUBINI and STUDY in  $\tilde{S}_n$  (406) — hermitian non euclidean geometry (406).

### § 5. Imbedding in an $\tilde{L}_n$ (407 ff.).

a rigged  $X_m$  in  $\tilde{L}_n$  (407) —  $B_b^\lambda$ ,  $C_\lambda^\xi$  (407) — the  $X_m$  is an  $L_m$  (408) — induced connexion (408) — a rigged  $X_m$  in  $\tilde{A}_n$  is an  $\tilde{A}_m$  (408) —  $H_{\dot{c}b}^{\dot{a}\lambda}$ ;  $L_{\dot{c}}^{\dot{a}\lambda}$ ;  $l_{\dot{c}}^{\dot{a}\lambda}$  (408) — geodesic  $L_m$  in  $\tilde{L}_n$  (409) — two GAUSS equations (410) — five CODAZZI equations (410) — two RICCI equations (410) — the GAUSS, CODAZZI and RICCI equations in  $\tilde{U}_n$  (411 f.) — imbedding in  $\tilde{V}_n$  (412).

### § 6. Curves in a $\tilde{U}_n$ with a positive definite fundamental tensor (412 ff.).

the  $\tilde{U}_1$  in  $\tilde{U}_n$  (412) — the  $dz$  of COBURN (412) — the  $\tilde{R}_1$  in  $\tilde{U}_n$  (414) — curves with a real parameter (415).

### § 7. Conformal transformation of a connexion in $\tilde{U}_n$ (413 ff.).

a conformal transformation is also restricted projective transformation (415) — a  $\tilde{V}_n$  determined by its conformal properties only (416) — conformal euclidean  $\tilde{V}_n$  (416) —  $\tilde{C}_n$  (416) — conformal curvature tensor (417) —  $C_{\bar{\nu}\mu\lambda}^{\dot{\nu}\dot{\mu}\dot{\lambda}}$  (417) — conformally symmetric  $\tilde{U}_n$  (417).

### § 8. Conformal unitary connexion (418 ff.).

$\mathfrak{A}_{\lambda\bar{\lambda}}$  (418) — conformal connexion  $\tilde{H}_{\mu\lambda}^\kappa$  (419) —  $\tilde{V}$  (419) —  $\tilde{R}_{\nu\mu\lambda}^{\dot{\nu}\dot{\mu}\dot{\lambda}}$  (419) —  $\tilde{R}_{\bar{\nu}\mu\lambda}^{\dot{\nu}\dot{\mu}\dot{\lambda}}$  (419).

### § 9. Spaces of recurrent curvature (421 ff.).

recurrent curvature tensor (421) —  $K_n$ ,  $K_n^*$  (421) —  $k_\omega$  always a gradient (421) — the decomposable  $K_n^*$  (421 f.) — flat extension (422) — null-extension (422) — the three cases for the simple  $K_n^*$  (423) — classification of simple  $K_n^*$ 's (423 f.) — linear elements of WALKER (424).

Bibliography . . . . . 425

Index . . . . . 512

# I. Algebraic preliminaries.

## § 1. The $E_n$ .

In this chapter we consider the “ordinary”  $n$ -dimensional *affine space*  $E_n$ , i.e. the space in which the coordinates of a point are subject to transformations of the *affine group*  $G_a$ .<sup>1)</sup> If the  $\xi^\kappa$ ;  $\kappa=1, \dots, n^2)$  are the “old” coordinates of a point, we can get “new” coordinates  $\xi'^\kappa$ ;  $\kappa'=1', \dots, n'$  by means of the coordinate transformation

$$(1.1) \quad \xi'^\kappa = A_\kappa^{\kappa'} \xi^\kappa + a^\kappa; \quad \Delta = \text{Det}(A_\kappa^{\kappa'}) \neq 0^3)$$

where the  $A_\kappa^{\kappa'}$  and  $a^\kappa$  are constants. The coordinate systems are usually denoted by  $(\kappa)$  and  $(\kappa')$  respectively. From (1.1) it follows that there exists a transformation of  $\xi'^\kappa$  into  $\xi^\kappa$ , called the inverse transformation

$$(1.2) \quad \xi^\kappa = A_{\kappa'}^{\kappa} \xi'^{\kappa'} + a^{\kappa}; \quad \text{Det}(A_{\kappa'}^{\kappa}) = \Delta^{-1}$$

with constant coefficients. For every value of  $\varrho$  and  $\sigma'$  the coefficient  $A_\varrho^{\sigma'}$  equals the minor<sup>4)</sup> of the element  $A_\varrho^{\sigma'}$  in the matrix of the  $A_\kappa^{\kappa'}$ , divided by  $\Delta$  and  $a^\kappa$  and  $a^{\kappa'}$  are related by the equation

$$(1.3) \quad a^{\kappa'} = - A_\kappa^{\kappa'} a^\kappa.$$

No mention has been made so far of the meaning of the words “point” and “space”. We leave this to Ch. II where the  $E_n$  will be defined as a special case of more general manifolds. For the moment it is sufficient to know that each point has certain coordinates, i.e. a set of  $n$  real or complex numbers assigned to it and that the linear transformations (1.1, 2) play an important role. The transformations (1.1, 2) will be called *allowable coordinate transformations* and the coordinates obtainable by them *rectilinear coordinates*.

<sup>1)</sup> A set of transformations is said to form a *group* if 1° any two transformations of the set performed after each other yield a transformation belonging to the set, 2° the inverse of every transformation of the set belongs to the set, 3° the set contains the identical transformation.

<sup>2)</sup> Unless otherwise specified the indices  $\kappa, \lambda, \mu, \nu, \varrho, \sigma, \tau$  and sometimes  $\omega$  always take the values  $1, \dots, n$  (in italics);  $\kappa', \lambda', \mu', \nu', \varrho', \sigma', \tau'$  and sometimes  $\omega'$  take the values  $1', \dots, n'$  (in italics) etc.

<sup>3)</sup> We adopt the summation convention: if an index appears twice in the same term, *once as a subscript and once as a superscript*, the sign  $\sum$  will be omitted.

<sup>4)</sup> The minor of  $A_\varrho^{\sigma'}$  is the determinant remaining after dropping the row and the column containing  $A_\varrho^{\sigma'}$  in the matrix of  $A_\kappa^{\kappa'}$  and multiplying by  $(-1)^{\varrho+\sigma'}$ .

If in (1.1) we take  $a'' = 0$  we get the transformations of the *homogeneous linear group*  $G_{ho}$ . The  $E_n$  is then called a *centred*  $E_n$ . In a centred  $E_n$  the origin has the coordinates zero with respect to all allowable coordinate systems.

There are also other transformations called *point transformations*, for instance the linear transformation

$$(1.4) \quad \eta'' = P''_{\lambda} \xi^{\lambda} + p''$$

with constant coefficients  $P''_{\lambda}$  and  $p''$ . Here the point  $\xi''$  is transformed into the point  $\eta''$  but the coordinates are not changed.

To every coordinate transformation  $(\varkappa) \rightarrow (\varkappa')$  there belongs a point transformation such that the *old* coordinates of every point are numerically equal to the *new* coordinates of the transformed point. We may express this by the equation

$$(1.5) \quad \eta'' * \delta''_{\varkappa} \xi''^{-1})$$

valid for the coordinate systems  $(\varkappa)$  and  $(\varkappa')$  only. The sign  $*$  (speak: star equals) will always be used if we wish to emphasize the fact that an equation is only valid or that its validity is only asserted for the coordinate system or coordinate systems occurring explicitly in the formula itself. From (1.4, 5) we get

$$(1.6) \quad A''_{\mu} P''_{\lambda} \xi^{\lambda} + A''_{\lambda} p^{\lambda} + a'' * \delta''_{\varkappa} \xi''$$

valid for every choice of  $\xi''$ , and this is only possible if

$$(1.7) \quad a'' * A''_{\lambda} \delta''_{\lambda}; \quad b) \quad p'' * -A''_{\lambda} a''.$$

If a point transformation belongs to the coordinate transformation  $(\varkappa) \rightarrow (\varkappa')$  we say that  $(\varkappa)$  is *dragged along* by this point transformation. Hence the numerical values of the coordinates of a point remain invariant if any point transformation is applied and if at the same time the coordinate system is dragged along by the same point transformation.

In all formulae there are *kernel letters* like  $A$ ,  $\xi$ ,  $P$ <sup>2)</sup>, *running indices* like  $\varkappa$ ,  $\varkappa'$  and *fixed indices* like  $1, \dots, n$ ;  $1', \dots, n'$ . With coordinate transformations the kernel letters do not change but we get a new set of running indices and fixed indices. But with point transformations the kernel letters change and the running and fixed indices remain the same.

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<sup>1)</sup> The symbol  $\delta''_{\lambda}$  used hereafter is the KRONECKER symbol. Its value is  $+1$  if  $\varkappa = \lambda$  (not to summarize) and zero if  $\varkappa \neq \lambda$ . It occurs also in the forms  $\delta''^{\lambda}_{\varkappa}$  and  $\delta''_{\varkappa\lambda}$ . The symbol  $\delta''^{\varkappa'}$  is the generalized KRONECKER symbol. Its value is  $+1$  if  $\varkappa$  and  $\varkappa'$  take corresponding values from  $1, \dots, n$  and  $1', \dots, n'$  and zero if they do not.

<sup>2)</sup> A kernel can also be symbolized by a more complicated symbol, cf. for instance (VII 1, 24).

This is the principle of the *kernel-index method* used in this book and in many other publications.<sup>1)</sup>

A kernel letter can be changed in the following ways:

1. By changing the letter itself, e.g.  $\xi \rightarrow \eta$ ;
2. by adjoining an accent or another sign, e.g. \* preferably to the left in order to avoid collision with upper running or fixed indices to the right;
3. by adjoining an index *over* or *under* the kernel<sup>2)</sup>. The upper and lower places to the right are reserved for running or fixed indices;
4. by putting upper or lower indices, without moving them, between brackets as will be shown later.

A *flat* or *linear  $m$ -dimensional submanifold* or  $E_m$  in  $E_n$  can be given by a set of  $n - m$  linear equations

$$(1.8) \quad C_{\lambda}^x \xi^{\lambda} + C^x = 0; \quad x = m + 1, \dots, n \text{ } ^3)$$

with the conditions that the  $C_{\lambda}^x$  and the  $C^x$  are constants and that the matrix of the  $C_{\lambda}^x$  has the rank<sup>4)</sup>  $n - m$ . But it can also be given by a set of  $m$  parametric equations

$$(1.9) \quad \xi^x = B_b^x \eta^b + B^x; \quad b = 1, \dots, m$$

with  $m$  parameters  $\eta^b$  and the condition that the  $B_b^x$  and the  $B^x$  are constants and that the matrix of  $B_b^x$  has rank  $m$ . From (1.9) we see that the  $\eta^b$  can be used as rectilinear coordinates in the submanifold and that accordingly this submanifold is really an  $E_m$ , because every other set of parameters that could be used in (1.9) must be linearly connected with the  $\eta^b$ . From (1.8, 9) it follows that

$$(1.10) \quad C_{\lambda}^x B_b^{\lambda} = 0; \quad b = 1, \dots, m; \quad x = m + 1, \dots, n$$

and that

$$(1.11) \quad C^x = -C_{\lambda}^x B^{\lambda}.$$

An  $E_0$  is a *point*, an  $E_1$  a *straight line*, an  $E_2$  a *plane* and an  $E_{n-1}$  a *hyperplane*. To every rectilinear coordinate system there belong  $n E_1$ 's

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<sup>1)</sup> The kernel-index method was sketched for the first time in the authors lectures on continuous groups at Leiden University 1926/27 (opalographic syllabus 1927, 1). The first printed exposition of the method appeared in SCHOUTEN and VAN KAMPEN 1930, 2 and GOLAB (thesis) 1930, 1.

<sup>2)</sup> If possible it is advisable to avoid using upper and lower indices of this kind simultaneously.

<sup>3)</sup> In reasonings like this we always suppose that  $m$  and  $m$  and also  $n$  and  $n$  stand for the same integer written in italics and vertical.

<sup>4)</sup> A matrix has rank  $p$  if it contains at least one non zero subdeterminant with  $p$  rows but none with more rows.

the coordinate axes,  $\binom{n}{2}$  coordinate planes and  $\binom{n}{p}$  coordinate  $E_p$ 's,  $p=1, \dots, n-1$ . These flat manifolds form the *net* of the coordinate system. If a coordinate system is dragged along by a point transformation, this means that the point transformation is applied to its net.

$E_p$  is said to *lie in*  $E_q$  and  $E_q$  is said to *contain*  $E_p$  if every point of  $E_p$  is a point of  $E_q$ . In order to discuss the common points of an  $E_p$  and an  $E_q$  we first define parallelism in  $E_n$ . A *translation* is a point transformation of the form

$$(1.12) \quad 'x^{\alpha} = x^{\alpha} + c^{\alpha}; \quad c^{\alpha} = \text{constant}$$

with respect to some rectilinear coordinate system and thus with respect to all these systems. Two  $E_p$ 's are said to be *parallel* if they can be transformed into each other by a translation. The set of all  $E_p$ 's parallel to a given  $E_p$  is called a *p-direction* or an *improper*  $E_{p-1}$ .<sup>1)</sup> In a centred  $E_n$  there is a one to one correspondence between all improper  $E_{p-1}$ 's and all  $E_p$ 's containing the origin. A *1-direction* is called a *direction* for short.<sup>2)</sup> An  $E_p$  is said to *contain* an improper  $E_q$  and the  $E_q$  is said to *lie in* the  $E_p$  if the  $E_p$  contains one of the  $E_{q+1}$ 's of the  $E_q$ . The equation

$$(1.13) \quad C_x^{\alpha} \xi^{\lambda} = \zeta^{\alpha}; \quad x = m+1, \dots, n$$

with variable  $\zeta^{\alpha}$  represents all  $E_m$ 's parallel to the  $E_m$  fixed by  $C_x^{\alpha}$  through the origin. In the manifold of these  $E_m$ 's the  $\zeta^{\alpha}$  can be used as rectilinear coordinates. Accordingly this manifold is an  $E_{n-m}$  each point of which represents an  $E_m$ . We call this  $E_{n-m}$  the *reduction* of  $E_n$  with respect to the  $E_m$  through the origin or with respect to its *m-direction*.

The following statements are valid both for proper and for improper flat submanifolds.

1. An  $E_p$  and an  $E_q$ ;  $0 < p \leq q$ , are called *t/p-parallel*,  $t > 0$ , and for  $t=p$  *parallel* if they contain the same *t-direction* but no common  $(t+1)$ -direction:

2. The *section*<sup>3)</sup> of an  $E_p$  and an  $E_q$  consists of all common  $E_0$ 's. This section is an  $E_s$ ;  $p \geq s \geq p+q-n$ ,  $q \geq s$ .

3. An  $E_p$  and an  $E_q$  *span* an  $E_r$  if they are both contained in  $E_r$  and if there is no  $E_{r-1}$  containing both  $E_p$  and  $E_q$ .  $E_r$  is called the *join* ) of  $E_p$  and  $E_q$ .  $r$  and  $s$  are related by the equation  $r=p+q-s$ .

<sup>1)</sup> Formerly often called an  $E_{p-1}$  "at infinity".

<sup>2)</sup> A direction in every day language is according to our definition a direction with a sense (arrow).

<sup>3)</sup> Some authors use the terms *intersection* and *meet*.

<sup>4)</sup> In P. P. 1949, 1 we used the term *junction*.

If in  $E_n$  a proper  $E_p$  and a  $q$ -direction,  $q = n - p$ , which have no direction in common, are given, then every geometric figure in  $E_n$  can be subjected to the following processes:

1. *Section with  $E_p$ :* All points of the figure not lying in  $E_p$  are dropped. Only the  $E_p$  is used.

2. *Reduction with respect to the  $q$ -direction:* All points of the figure lying in an  $E_q$  with this  $q$ -direction are identified. The resulting figure lies in the  $E_p$  that arises from reduction of the  $E_n$  with respect to the  $q$ -direction. Only the  $q$ -direction is used.

3. *Projection on  $E_p$  in the  $q$ -direction:* Through every point of the figure an  $E_q$  is laid with the given  $q$ -direction. The section of this  $E_q$  with  $E_p$ , is the projection of this point. Both the  $E_p$  and the  $q$ -direction are used.

An  $n$ -dimensional screwsense in  $E_n$  is determined by  $n$  directions, with a sense (arrow), which are not contained in the same  $(n-1)$ -direction and are given in a definite order. For  $n=2$  the screwsense is often called *sense of rotation* and for  $n=1$  *sense*. Two screwsenses are *equal* if the defining figures can be transformed into each other by a point transformation (1.4) with  $\text{Det}(P_{\cdot\lambda}^*) > 0$  and *opposite* if they can not. Hence there exist only two screwsenses in an  $E_n$ . The screwsense can also be given by a part of a general curve, not lying in an  $E_{n-1}$ , which is provided with a sense (arrow). If  $\frac{1}{2}n(n+1)$  is even, e.g. for  $n=3$ , the sense of the curve can be omitted. In this case it makes no difference if the senses of all directions are changed and their order is inverted at the same time. A screwsense is invariant for all linear point transformations (1.4) with  $\text{Det}(P_{\cdot\lambda}^*) > 0$ . To every coordinate system ( $\kappa$ ) there belongs a definite screwsense fixed by the directions of the axes, their +-senses and the order  $1, \dots, n$ . This screwsense is invariant for rectilinear coordinate transformations if and only if  $\Delta > 0$ .

An  $E_p$  with a  $p$ -dimensional screwsense fixes a screwsense in every  $E_p$  parallel to it. Hence a  $p$ -direction may have a screwsense. An  $E_p$  or a  $p$ -direction with screwsense are said to be *oriented* with *inner orientation*. If to a given  $E_p$  or  $p$ -direction an  $(n-p)$ -dimensional screwsense is given in some  $E_{n-p}$  which has no direction in common with  $E_p$  or the  $p$ -direction, this screwsense fixes a screwsense in every  $E_{n-p}$  with this property. Then the  $E_p$  or the  $p$ -direction are said to be *oriented* with *outer orientation*. For instance an  $E_1$  in  $E_3$  has inner orientation if there is an arrow *in* it and it has outer orientation if there is an arrow *around* it.<sup>1)</sup>

<sup>1)</sup> The idea of orientation has been dealt with from a very general and highly interesting point of view by E. CARTAN 1941, 1.

### Exercises.

I 1,1. Prove that (cf. Exerc. II 2,1)

$$\text{I 1,1 } \alpha) \quad A_{\sigma'}^{\theta} = \frac{\partial \log \varLambda}{\partial A_{\sigma'}^{\theta}}.$$

I 1,2. An  $E_p$  in  $E_n$  can be fixed by  $(p+1)(n-p)$  numbers. An  $E_p$  through a given  $E_q$ ,  $q < p$ , can be fixed by  $(p-q)(n-p)$  numbers.

I 1,3. If an  $E_p$  and an  $E_q$ , having no direction in common, are given in a definite order, and if in each of them a screwsense is given, a screwsense in the join  $E_{p+q}$  is determined. If the order of  $E_p$  and  $E_q$  is changed the screwsense in  $E_{p+q}$  changes if and only if  $pq$  is odd.

## § 2. Quantities in $E_n$ .

A quantity in  $E_n$  is a correspondence between the rectilinear coordinate systems and the ordered sets of  $N$  numbers, satisfying the conditions:

1. To every coordinate system  $(\varkappa)$  there corresponds one and only one set of  $N$  numbers;

2. if  $\Phi_A$ ;  $A = I, \dots, N$  corresponds to  $(\varkappa)$  and  $\Phi_{A'}$ ;  $A' = I', \dots, N'$  to  $(\varkappa')$  the  $\Phi_{A'}$  are functions of the  $\Phi_A$  and the  $A_{\varkappa'}^{\varkappa}$ , homogeneous linear in  $\Phi_A$  and homogeneous algebraic in  $A_{\varkappa'}^{\varkappa}$ . The  $\Phi_A$  are called the components of the quantity with respect to  $(\varkappa)$ .<sup>1)</sup> Quantities with the same number of components and the same manner of transformation are said to be of the same kind. According to the definition, by addition of corresponding components of two quantities of the same kind we get components of another quantity of the same kind called the sum of both.

Quantities are distinguished by the manner of transformation of their components. We give here a list of some of the most important quantities in  $E_n$ . Other quantities will be defined later.

1. A scalar with one component, invariant for the transformations (1.1).

2. A contravariant vector<sup>2)</sup> with  $n$  components  $v^{\varkappa}$  and the transformation

$$(2.1) \quad v^{\varkappa'} = A_{\varkappa'}^{\varkappa} v^{\varkappa}.$$

<sup>1)</sup> It is often convenient to denote a quantity in the text by its kernel letter only. This must not of course be done if the same kernel letter is being used for two or more different quantities.

<sup>2)</sup> In the classical theory of invariants two transformations of the kind (2.1) and (2.4) were called contragredient with respect to each other. Then the two transformations were distinguished by the words covariant and contravariant. But the reasons for calling (2.1) contravariant and (2.4) covariant (and not the other way round) are purely historical.

A contravariant vector  $v^*$  is represented by two points  $\xi^*$  and  $\xi^* + v^*$  in definite order fixed to within a common translation. The order can be represented by numbering the points or by a curve or straight line with an arrow joining the points. In a centred  $E_n$  a contravariant vector  $v^*$  can be represented by one point with the coordinates  $v^*$ , the origin playing the role of first point. Addition of two contravariant vectors can be illustrated by the wellknown figure called "parallelogram of forces" in mechanics.  $|v^I|$  is the projection of  $v^*$  on the  $I$ -axis in the  $(n - I)$ -direction of the other axes measured by the unit in the  $I$ -axis.  $v^I$  is positive if the projected sense of  $v^*$  on the  $I$ -axis is the same as the  $+$ -sense of the  $I$ -axis and negative otherwise.

To every coordinate system  $(\kappa)$  there belong  $n$  contravariant vectors

$$(2.2) \quad \begin{matrix} e_\lambda^* \\ \lambda \end{matrix} = \delta_\lambda^* = \begin{cases} 1 & \text{for } \kappa = \lambda \\ 0 & \text{for } \kappa \neq \lambda \end{cases}$$

called the *contravariant basis vectors* of  $(\kappa)$ .<sup>1)</sup> The components of these vectors with respect to another coordinate system  $(\kappa')$  in general no longer have the values 1 or 0 because

$$(2.3) \quad \begin{matrix} e_\lambda^{*\prime} \\ \lambda \end{matrix} = A_{\kappa'}^\kappa \begin{matrix} e_\kappa^* \\ \lambda \end{matrix} = A_{\kappa'}^\kappa$$

and to  $(\kappa')$  there belongs another contravariant basis  $\begin{matrix} e_\lambda^* \\ \lambda \end{matrix}$ . Here we have an example of kernels with indices *under* the letter. Such indices are not subject to the transformation of coordinates and are called *dead indices* in contradistinction to the indices to the right that are called *living*.<sup>2)</sup>

3. A *covariant vector* with  $n$  components  $w_\lambda$  and the transformation

$$(2.4) \quad w_{\lambda'} = A_{\lambda'}^\lambda w_\lambda.$$

The transformations (2.4) and (2.1) are said to be *contragredient*<sup>3)</sup> to each other. A covariant vector  $w_\lambda$  is represented by two parallel hyperplanes

$$(2.5) \quad w_\lambda \xi^\lambda = c; \quad w_\lambda \xi^\lambda = c + 1; \quad c = \text{const.}$$

<sup>1)</sup> In former publications we used the term measuring vectors instead of basis vectors and called a basis a set of measuring vectors.

<sup>2)</sup> The KRONECKER symbol  $\delta_\lambda^*$  stands for  $n^2$  scalars and accordingly its indices are dead. So it ought to be written  $\begin{matrix} \delta \\ \lambda \end{matrix}$  but we shall not do so for historical reasons and also to avoid such towerlike constructions as much as possible. The same holds for  $\delta_\lambda^{*\prime}$ ,  $\delta_{\kappa\lambda}$ ,  $\delta^{\kappa\lambda}$  etc. All these symbols represent  $n^2$  scalars with the values 1 or 0 and not, as might be expected, different components of some geometric object  $\delta$ . This is an exception to the general rule for the use of kernels.

<sup>3)</sup> See footnote 2 on p. 6.

given in this order and fixed to within a common translation, i.e. addition of the same arbitrary constant to  $c$ .<sup>1)</sup> The order can be represented by a (preferably curved)<sup>2)</sup> arrow in a curve through the hyperplanes. Fig. 1 represents a covariant vector in  $E_3$ . In a centred  $E_n$  a covariant vector  $w_\lambda$  can be represented by one hyperplane  $w_\lambda \xi^\lambda = 1$ , the parallel  $E_{n-1}$  through the origin playing the role of first hyperplane. Addition of two covariant vectors is illustrated in Fig. 2, representing a section with an arbitrary  $E_2$ .<sup>3)</sup>

$|w_1|$  is the inverse of the segment on the  $\mathbf{1}$ -axis cut out by the  $E_{n-1}$ 's of  $w_\lambda$ , measured by the unit on this axis.  $w_1$  is positive if the

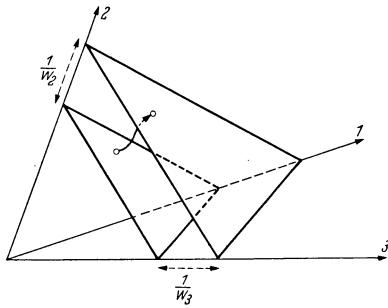


Fig. 1.

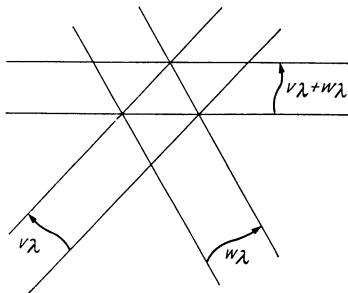


Fig. 2.

sense of  $w_\lambda$  corresponds to the  $+$ -sense of the  $\mathbf{1}$ -axis and negative otherwise.

To every coordinate system  $(\varkappa)$  there belong  $n$  covariant vectors

$$(2.6) \quad \overset{\varkappa}{e}_\lambda \equiv \delta_\lambda^\varkappa$$

called the *covariant basis vectors*<sup>4)</sup> of  $(\varkappa)$ . Their components with respect to  $(\varkappa')$  no longer have in general the values  $\mathbf{1}$  and  $0$ , and to  $(\varkappa')$  there belongs another covariant basis  $\overset{\varkappa'}{e}_\lambda$ .

The combination of a contravariant and a covariant vector  $v^\varkappa w_\varkappa$  is an invariant

$$(2.7) \quad v^\varkappa w_\varkappa = v^\varkappa A_\varkappa^{\varkappa'} A_{\varkappa'}^\lambda w_\lambda = v^\varkappa w_\varkappa$$

and is called the *transvection* of these vectors.  $|v^\varkappa w_\varkappa|$  is the length of  $v^\varkappa$  measured by the segment cut out from the line of  $v^\varkappa$  by the hyperplanes of  $w_\lambda$ .  $v^\varkappa w_\varkappa$  is positive if the senses of  $v^\varkappa$  and  $w_\lambda$  correspond and

<sup>1)</sup> Cf. SCHOUTEN, 1923, 1, p. 164.

<sup>2)</sup> A straight arrow could suggest that a special direction were given.

<sup>3)</sup> If  $v_\lambda$  and  $w_\lambda$  are parallel the construction can be effected by means of an auxiliary vector  $u_\lambda$ :  $v_\lambda + w_\lambda = (v_\lambda + u_\lambda) + (w_\lambda - u_\lambda)$ .

<sup>4)</sup> See footnote 1 on p. 7.

negative otherwise. If  $v^\lambda w_\lambda = \pm 1$  the vector  $v^\lambda$  fits exactly between the hyperplanes of  $w_\lambda$ . If  $v^\lambda w_\lambda = 0$  the direction of  $v^\lambda$  lies in the  $(n-1)$ -direction of  $w_\lambda$ .

In Fig. 3 we see the six basis vectors belonging to a coordinate system in  $E_3$ .

If  $n$  linearly independent contravariant vectors  $v_i^\lambda$  are given, there always exist  $n$  covariant vectors  $\hat{w}_\lambda^h$ ;  $h, i = 1, \dots, n$  such that

$$(2.8) \quad v_i^\lambda \hat{w}_\lambda^h = \delta_i^h.$$

The sets  $v_i^\lambda$  and  $\hat{w}_\lambda^h$  are said to be *reciprocal* to each other. For each value of  $i$ ,  $v_i^\lambda$  fits between the hyperplanes of  $\hat{w}_\lambda^h$  and lies in the  $(n-1)$ -direction of every other covariant vector  $\hat{w}_\lambda^k$ ;  $k \neq i$ . The co- and contravariant basis vectors of a rectilinear coordinate system are reciprocal sets and conversely two reciprocal sets always constitute the basis vectors of some rectilinear coordinate system.

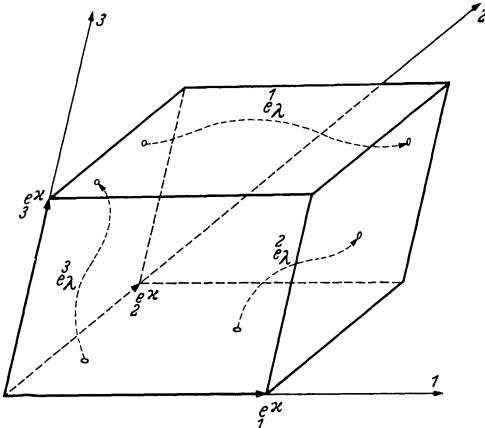


Fig. 3.

If a vector  $v^\lambda$  is transvected by the  $n$  covariant basis vectors  $\hat{e}_\lambda$ , we get  $n$  scalars

$$(2.9) \quad \tilde{v} = v^\lambda \hat{e}_\lambda$$

each representing the result of measuring the projection on an axis by the unit on this axis. The equation

$$(2.10) \quad v^\lambda = \tilde{v} \hat{e}_\lambda$$

gives the construction of  $v^\lambda$  from these scalars and the contravariant basis vectors. The two equations (2.9, 10) are numerically equal if we use the coordinate system  $(\lambda)$ , but by introducing  $(\lambda')$  we get the equations

$$(2.11) \quad \text{a)} \quad \tilde{v} = v^{\lambda'} \hat{e}_{\lambda'} \quad \text{b)} \quad v^{\lambda'} = \tilde{v} \hat{e}_{\lambda'}$$

that are no longer numerically equal. The  $\lambda$  in  $\tilde{v}$  is a dead index and  $\tilde{v}$  stands for a set of  $n$  scalars, but the  $\lambda'$  in  $v^{\lambda'}$  is a living index transforming

with transformations of coordinates. Accordingly we must write

$$(2.12) \quad \overset{\ast}{v} = v^*$$

but  $\overset{\ast}{v} \neq v^*$ . Also  $\overset{\ast}{e}_\lambda = e_\lambda^*$  but  $\overset{\ast}{e}_\lambda \neq e_\lambda^*$ . This illustrates the fact that it is not sufficient to give only the components of a quantity with respect to some coordinate system, and that we must also give the manner of transformation because the components of two quantities of different kind with respect to some special coordinate system may happen to be equal.

The process in (2.9) that leads from a living index to a dead one by means of transvection with the basis vectors is called the *strangling* of an index. It is clear that strangling of the same index or indices at both sides of an equation leaves the equation invariant. But if the strangled indices are not the same on both sides the sign  $=$  must be changed into  $\neq$ . Moreover it is clear that in an equation with  $\underline{*}$  that has the invariant form, both sides have the same living running indices and it is therefore allowable to introduce  $=$  instead of  $\underline{*}$ .

4. A tensor<sup>1)</sup> of *contravariant valence p* and *covariant valence q* with  $n^{p+q}$  components  $P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p} {}_{\lambda_1 \dots \lambda_q}$  and the transformation

$$(2.13) \quad P_{\cdot \cdot \cdot}^{\kappa'_1 \dots \kappa'_p} {}_{\lambda'_1 \dots \lambda'_q} = A_{\kappa_1 \dots \kappa_p}^{\kappa'_1 \dots \kappa'_p} {}^{\lambda_1 \dots \lambda_q} P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p} {}_{\lambda_1 \dots \lambda_q}.^2)$$

The place of the indices is essential, for instance we must not write sometimes  $P_{\cdot \cdot \cdot \mu}^{\kappa \lambda}$  and at another time  $P_{\cdot \cdot \cdot \mu}^{\kappa \lambda}$  for the same quantity. If only lower (upper) indices occur the tensor is called *co-(contra-)variant*. If both occur it is often called *cocontravariant* (or *mixed*).

The tensor of valence 1,1

$$(2.14) \quad A_\lambda^{\kappa *} = \delta_\lambda^\kappa$$

is called the *unity tensor*. Transvection of a quantity over one index with the unity tensor does not change the quantity, for instance

$$(2.15) \quad A_\lambda^\kappa P_{\cdot \cdot \cdot \nu}^{\lambda \mu} = P_{\cdot \cdot \cdot \nu}^{\kappa \mu}.$$

Tensors with more than one index also have components with respect to two or more different coordinate systems, e.g.

$$(2.16) \quad P_{\cdot \cdot \cdot \nu}^{\kappa \lambda} = A_\kappa^\kappa P_{\cdot \cdot \cdot \nu}^{\lambda \lambda} = A_{\lambda' \nu}^{\lambda \nu} P_{\cdot \cdot \cdot \nu}^{\kappa \lambda'}$$

<sup>1)</sup> In former publications we used the word affinor instead of tensor and the word tensor was used in a more restricted sense (cf. footnote 1, p. 21).

<sup>2)</sup>  $A_{\kappa \lambda}^{\kappa' \lambda'}$  stands for  $A_\kappa^{\kappa'} A_\lambda^{\lambda'}$ . Generally we do not put upper and lower indices in the same vertical line so as to avoid ambiguity if the process of raising and lowering of indices (to be defined later on) is to be applied. But there are a few exceptions.

These components are called *intermediate*. From this we see that  $A_{\lambda}^{\kappa}$  and  $A_{\lambda'}^{\kappa'}$  in (2.1) and (2.4) are intermediate components of  $A_{\lambda}^{\kappa}$ . This justifies the use of the same kernel letter  $A$  in all three cases. The equations (2.1) and (2.4) now also express the fact that a vector is invariant for transvection with the unity tensor. The same holds mutatis mutandis for (2.13).

Two quantities of the same kind are said to be *equiform* if there exist two coordinate systems  $(\kappa)$  and  $(\kappa')$  such that the components of the first quantity with respect to  $(\kappa)$  are equal to the corresponding components of the second with respect to  $(\kappa')$ , e.g.

$$(2.17) \quad P_{..,\mu}^{\kappa\lambda} = \delta_{\kappa'\lambda'}^{\kappa\lambda}\mu Q_{..,\mu'}^{\kappa'\lambda'}.$$

5. A tensor may be determined to within a scalar factor. Then it is called a *pseudo-tensor* (*pseudo-scalar*, *pseudo-vector* if the valence is 0 or 1). If more pseudo-tensors occur it may be that the unknown scalar factors in them are not entirely independent. For instance a direction and an  $(n-1)$ -direction fix a contravariant and a covariant pseudo-vector whose scalar factors are entirely free and independent. But if the scalar factors in  $v^{\kappa}$  and  $w_{\lambda}$  are free, the scalar factor in the pseudo-tensor  $P_{..,\lambda}^{\kappa} = v^{\kappa}w_{\lambda}$  depends on the scalar factors in  $v^{\kappa}$  and  $w_{\lambda}$ . In the first case a pseudo-tensor can be distinguished if necessary by the sign  $[ ]$ . For instance  $[P_{..,\lambda}]$  is the pseudo-tensor belonging to the tensor  $P_{..,\lambda}$  and  $[ ]$  only means that there is an entirely unknown scalar factor. In the second case it is most convenient to introduce next to the  $\xi^{\kappa}$  an auxiliary variable  $\xi^{\theta}$  (or if necessary more of these variables)<sup>1)</sup> with the transformation

$$(2.18) \quad \xi^{\theta'} = \sigma \xi^{\theta}.$$

A pseudo-tensor of *class c* is then defined by the transformation formula of the form

$$(2.19) \quad \mathfrak{P}_{..,\mu'}^{\kappa'\lambda'} = \sigma^c A_{\kappa'\lambda'\mu'}^{\kappa\lambda} \mathfrak{P}_{..,\mu}^{\kappa\lambda}.$$

For pseudo-tensors defined in this way we usually prefer gothic kernel letters. Pseudo-scalars have only one component and no indices. It is therefore necessary to introduce an extra notation in order to distinguish between the different values that the pseudo-scalar can take.

If we write  $\overset{(0)}{\mathfrak{P}}$  or  $\overset{(0)}{\mathfrak{P}}[0]$  for the component belonging to  $\xi^{\theta}$  we get the transformation formula

$$(2.20) \quad \overset{(0')}{\mathfrak{P}} = \sigma^c \overset{(0)}{\mathfrak{P}} \quad \text{or} \quad \overset{(0')}{\mathfrak{P}}[0'] = \sigma^c \overset{(0)}{\mathfrak{P}}[0].$$

The second notation is used if there is some reason for avoiding an upper index, for instance if there is already another upper or lower index.

<sup>1)</sup> SCHOUTEN and HLAVATY 1929, 2.

6. If a quantity transforms like a tensor but with an extra factor  $\tau$  dependent on  $\Delta$  it is called

- a) *tensor  $\Delta$ -density of weight  $w$  and antiweight  $w'$*  if  $\tau = \Delta^{-w} \bar{\Delta}^{-w'}$ ;  
 $\bar{\Delta}$  = complex conjugate of  $\Delta$ ;
- b) *tensor density of weight  $w$*  if  $\tau = |\Delta|^{-w}$  (cf. I § 10);
- c)  *$W$ -tensor* if  $\tau = \Delta / |\Delta|$ .

$\Delta$ -densities frequently occur in differential geometry but densities and  $W$ -tensors are more used in physics. For densities we prefer a gothic kernel and for  $\Delta$ -densities a gothic kernel with the sign  $\sim$  (snake) above it, for instance  $\tilde{P}_{\cdot\cdot\mu}^{\kappa\lambda}$ . For non-scalar  $W$ -tensors we prefer roman kernel letters and for  $W$ -scalars a small greek letter, both with a snake. But there are exceptions. For instance the mass or charge per parallelepiped of basis vectors is a scalar density of weight  $+1$  and is generally denoted by a small greek letter.

If we consider real coordinate systems only, there are two kinds of coordinate systems, transforming into themselves by coordinate transformations with  $\Delta > 0$  and into each other by transformations with  $\Delta < 0$ . They can be distinguished by calling them *right* and *left* systems if some geometric object suitable for comparison is introduced in  $E_n$ .<sup>1)</sup> We call this the introduction of a screwsense in  $E_n$ . The screwsense can be fixed algebraically by the  $W$ -scalar  $\tilde{\omega}$ , whose component is  $+1$  ( $-1$ ) with respect to right (left) systems.

After the introduction of  $\tilde{\omega}$  there is a one to one correspondence between  $\Delta$ -densities and densities and between  $W$ -tensors and tensors. For instance

$$(2.21) \quad \tilde{P}_{\cdot\cdot\mu}^{\kappa\lambda} = \tilde{\omega} P_{\cdot\cdot\mu}^{\kappa\lambda}; \quad \tilde{Q}_{\cdot\cdot\lambda}^{\kappa} = \tilde{\omega} Q_{\cdot\cdot\lambda}^{\kappa}.$$

If, instead of  $G_a$ , we use only the group of all real affine transformations with  $\Delta > 0$ , corresponding quantities have the same manner of transformation and can be identified. This is the first example of identification of quantities after introducing a subgroup of  $G_a$ .<sup>2)</sup>

Analogous to the case of pseudo-scalars we denote the value of a scalar density or  $\Delta$ -density or a  $W$ -scalar with respect to the coordinate system  $(\kappa)$  by writing  $(\kappa)$  right over the kernel or  $[\kappa]$  behind it, for instance

$$(2.22) \quad \tilde{p}^{(\kappa')} = \Delta^{-w} \tilde{p}^{(\kappa)} \quad \text{or} \quad \tilde{p}^{[\kappa']} = \Delta^{-w} \tilde{p}^{[\kappa]}$$

$$(2.23) \quad \tilde{p}^{(\kappa')} = \frac{\Delta}{|\Delta|} \tilde{p}^{(\kappa)} \quad \text{or} \quad \tilde{p}^{[\kappa']} = \frac{\Delta}{|\Delta|} \tilde{p}^{[\kappa]}.$$

<sup>1)</sup> This is just what we do in ordinary space if we call a system right or left after comparison with our own right hand.

<sup>2)</sup> We get another example if instead of  $G_a$  we use the so called special affine group  $G_{sa}$  consisting of all transformations of  $G_a$  with  $\Delta = +1$ . Then the differences between tensors, tensor densities and tensor  $\Delta$ -densities vanish.

7. If a quantity has different kinds of indices belonging to different spaces it is called a *connecting quantity*. These quantities occur frequently. E.g.  $B_b^x$  in (1.9) is a connecting quantity,  $x$  belongs to  $E_n$  and  $b$  to  $E_m$ .  $C_\lambda^x$  in (1.8) is another connecting quantity.  $\lambda$  belongs to  $E_n$  and  $x$  belongs to the  $E_{n-m}$  that arises from reducing the  $E_n$  with respect to the  $m$ -direction of  $E_m$ .  $C_\lambda^x$  and  $B_b^x$  are called the *covariant* and the *contravariant connecting quantity* of the  $E_m$  in  $E_n$ .

### Exercises.

I 2,1.  $n^p$  numbers  $P^{x_1 \dots x_p}$  are components of a contravariant tensor of valence  $p$  if and only if for some definitely chosen value of  $q \leq p$

$$(I 2,1 \alpha) \quad P^{x_1 \dots x_p} Q_{x_1 \dots x_q}$$

is a contravariant tensor for every choice of the tensor  $Q_{x_1 \dots x_q}$ .

I 2,2.  $P_{\cdot \lambda}^x = \alpha A_\lambda^x$  if and only if  $P_{\cdot \lambda}^x v^\lambda = \alpha v^x$  for every choice of  $v^x$ .

I 2,3.  $P_{\cdot \lambda}^x = \alpha A_\lambda^x + p_\lambda q^x$  if and only if  $P_{\cdot \lambda}^x v^\lambda = \alpha v^x$  for every  $v^x$  satisfying the equation  $p_\lambda v^\lambda = 0$ .

I 2,4. Let  $P_{\cdot \lambda}^x$  be the tensor of the linear vector transformation transforming the  $e_\lambda^x$  into the  $e_\lambda^x$ .

$$(I 2,4 \alpha) \quad e_\lambda^x = \delta_\lambda^\mu P_{\cdot \mu}^x e_\lambda^\mu.$$

Prove that

$$P_{\cdot \lambda}^x = \delta_\lambda^\mu.$$

I 2,5. Prove that two equiform quantities can always be transformed into each other by a homogeneous linear vector transformation.

### § 3. Invariant processes and relations.

The following processes and relations are invariant for all rectilinear coordinate transformations.

1. The *addition* of two quantities of the same kind (cf. I § 2).
2. The construction of an *isomer* by changing the location of the upper indices and/or of the lower indices, e.g.

$$(3.1) \quad Q_{\lambda x}^{\cdot \cdot \mu} = P_{x \cdot \lambda}^{\cdot \cdot \mu}.$$

3. The *(general) multiplication* of two quantities, e.g.

$$(3.2) \quad P_{\cdot \cdot \mu}^{\cdot \cdot \lambda} Q_{\nu \cdot \sigma}^{\cdot \cdot \rho} = R_{\cdot \cdot \sigma \mu}^{\cdot \cdot \rho \lambda}.$$

4. The *contraction* (Faltung, Verjüngung) with respect to an upper and a lower index or with respect to several such pairs of indices, e.g.

$$(3.3) \quad P_{\cdot \cdot x \sigma \varrho}^{\cdot \cdot \lambda \cdot \cdot \sigma} = Q_{\cdot \varrho}^{\lambda}.$$

5. The *transvection* (Überschiebung) of two quantities by application of the process of contraction to the general product, e.g.

$$(3.4) \quad P_{\cdot\cdot\cdot\mu}^{x\lambda} Q_{\cdot\cdot\cdot\lambda}^{\nu} = R_{\cdot\cdot\cdot\nu}^x.$$

The indices used for summation in contractions and transvections are called *saturated* (also *dummy*), the other *free* indices. The transvection is a kind of multiplication, i.e. it is distributive with respect to addition.

6. The process of *mixing* over  $p$  upper or  $p$  lower indices by constructing all  $p!$  isomers that can be obtained by permutation of these indices, addition of these isomers and division by  $p!$ . The process is denoted by round brackets ( ).<sup>1)</sup> If indices have to be singled out the sign || is used, e.g.

$$(3.5) \quad \left\{ \begin{array}{l} P_{(\alpha\lambda|\mu|\nu)}^{x\cdot\cdot\cdot} = \frac{1}{6}(P_{\alpha\lambda\mu\nu}^{x\cdot\cdot\cdot\omega} + P_{\lambda\nu\mu\alpha}^{x\cdot\cdot\cdot\omega} + P_{\nu\alpha\mu\lambda}^{x\cdot\cdot\cdot\omega} + \\ + P_{\alpha\nu\mu\lambda}^{x\cdot\cdot\cdot\omega} + P_{\nu\lambda\mu\alpha}^{x\cdot\cdot\cdot\omega} + P_{\lambda\alpha\mu\nu}^{x\cdot\cdot\cdot\omega}) \end{array} \right.$$

where  $\mu$  stays put and the three other lower indices permute in the remaining places. The effect of ( ) is not intercepted by any form of ordinary brackets, e.g.

$$(3.6) \quad (P_{(\alpha|\lambda} + Q_{(\alpha|\lambda}) R_{\mu|\nu)} = P_{(\alpha|\lambda} R_{\mu|\nu)} + Q_{(\alpha|\lambda} R_{\mu|\nu)}.$$

A quantity is called *symmetric* in  $p$  upper or  $p$  lower indices if it is invariant for mixing over these indices. It is then also invariant if two of these indices are interchanged.  $P^{(x_1 \dots x_p)}$  is called the *symmetric part* of  $P^{x_1 \dots x_p}$ .

7. The process of *alternation* over  $p$  upper or  $p$  lower indices, effected in the same way as the mixing process with the only difference that all isomers obtained by odd permutations get a negative sign. This process is denoted by square brackets [ ], e.g.

$$(3.7) \quad \left\{ \begin{array}{l} P_{[\alpha\lambda|\mu|\nu]}^{x\cdot\cdot\cdot} = \frac{1}{6}(P_{\alpha\lambda\mu\nu}^{x\cdot\cdot\cdot\omega} + P_{\lambda\nu\mu\alpha}^{x\cdot\cdot\cdot\omega} + P_{\nu\alpha\mu\lambda}^{x\cdot\cdot\cdot\omega} - \\ - P_{\alpha\nu\mu\lambda}^{x\cdot\cdot\cdot\omega} - P_{\nu\lambda\mu\alpha}^{x\cdot\cdot\cdot\omega} - P_{\lambda\alpha\mu\nu}^{x\cdot\cdot\cdot\omega}) \end{array} \right.$$

As far as interception by other brackets is concerned [ ] has the same property as ( ). A quantity is called *alternating* in  $p$  upper or  $p$  lower indices if it is invariant for alternation over these indices. It then changes sign if two of these indices are interchanged. Alternation over more than  $n$  indices always leads to zero.  $P^{[x_1 \dots x_p]}$  is called the *alternating part* of  $P^{x_1 \dots x_p}$ .

Of these seven processes the first four are fundamental, the others being only combinations of them. In order to see what can be done

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<sup>1)</sup> The use of ( ) and [ ] for mixing and alternating was proposed by BACH 1921, 1, p. 113.

by these four fundamental processes we need a theorem concerning concomitants. If a set of quantities is given, every quantity whose components can be expressed as functions of the components of the quantities of the set is called a (simultaneous) *concomitant* of the set. For a scalar the term *invariant* also is used. If all these functions are rational integral the concomitant is called *rational integral*. The theorem in question is:

*All rational integral concomitants of a given set of quantities can be derived from the quantities of the set by means of addition, construction of isomers, multiplication and contraction.*

This theorem is an immediate consequence of the well known first fundamental theorem of the symbolic method used in the theory of invariants.<sup>1)</sup>,<sup>2)</sup>

### Exercises.

I 3,1. Prove that  $P^{\kappa_1 \dots \kappa_p}$  vanishes if it is symmetric in  $\kappa_1 \dots \kappa_s$  and if  $P^{\kappa_1 \dots \kappa_p} w_{\kappa_1} \dots w_{\kappa_s} = 0$  for every choice of the vector  $w_\lambda$ .

I 3,2. If  $u_{\lambda \kappa} v^\lambda v^\kappa = 0$  for every vector  $v^\kappa$  for which  $v^\lambda w_\lambda = 0$ ,  $u_{(\lambda \kappa)}$  has the form

$$\text{I 3,2 } \alpha) \quad u_{(\lambda \kappa)} = w_{(\lambda} p_{\kappa)} {}^3)$$

I 3,3. If  $w_\kappa P_{\cdot \lambda}^\kappa = \alpha w_\lambda$  for every vector  $w_\lambda$  for which  $v^\lambda w_\lambda = 0$ ,  $P_{\cdot \lambda}^\kappa$  has the form

$$\text{I 3,3 } \alpha) \quad P_{\cdot \lambda}^\kappa = \alpha A_\lambda^\kappa + v^\kappa p_\lambda {}^3)$$

and  $\alpha$  is independent of the choice of  $w_\lambda$ .

I 3,4. If  $P_{\cdot \lambda \mu}^\kappa w_\kappa v^\lambda v^\mu = 0$  for every choice of  $v^\kappa$  and  $w_\lambda$  satisfying  $v^\lambda w_\lambda = 0$ ,  $P_{\cdot (\lambda \mu)}^\kappa$  has the form

$$\text{I 3,4 } \alpha) \quad P_{\cdot (\lambda \mu)}^\kappa = A_{(\lambda}^\kappa p_{\mu)} {}^4), {}^3).$$

<sup>1)</sup> Cf. e.g. WEITZENBÖCK, 1923, 1, p. 93.

<sup>2)</sup> There is a large amount of literature on the problem of finding algebraical concomitants of a quantity or a set of quantities. Cf. for instance WEITZENBÖCK 1923, 1; 1939, 1; DROST 1931, 1; OBERTI 1936, 1; WEYL 1939, 1; CISOTTI 1940, 1; 1941, 1; 1943, 1; 1944, 1; WADE 1941, 1; 1943, 1; 1944, 1; 1945, 1; DOYLE 1941, 1; DUBNOV 1941, 1; BRUCK and WADE 1943, 1; BRUCK 1944, 1; LITTLEWOOD 1944, 1; 1950, 1; RUTHERFORD 1948, 1; RACAH 1949, 1.

<sup>3)</sup> Propositions of this kind can always be proved by means of the following well known algebraic theorem. If a polynomial  $f(x^1, \dots, x^n)$  does not contain multiple irreducible factors and if the polynomial  $F(x^1, \dots, x^n)$  is zero for all values for which  $f = 0$ , then  $F$  is divisible by  $f$ .

<sup>4)</sup> From a letter of A. FRIEDMANN.

I 3,5. Verify that  $v^{[\kappa_1 \dots \kappa_p]}$  can be written out in two ways

$$\text{I 3,5 } \alpha) \quad v^{[\kappa_1 \dots \kappa_p]} = \frac{1}{p} \sum_s^{1, \dots, p} (-1)^{s-1} v^{\kappa_s [\kappa_1 \dots \kappa_{s-1} \kappa_{s+1} \dots \kappa_p]}.$$

$$\text{I 3,5 } \beta) \quad v^{[\kappa_1 \dots \kappa_p]} = \frac{1}{p} \sum_s^{1, \dots, p} (-1)^{s-1} v^{[\kappa_2 \dots \kappa_s | \kappa_1 | \kappa_{s+1} \dots \kappa_p]}.$$

I 3,6. If  $P_{\cdot \lambda \mu}^x = P_{[\lambda \mu]}^x$  and if  $P_{\cdot \lambda \mu}^x w_\lambda v^\mu \propto w_\lambda$  for every choice of  $v^x$  and  $w_\lambda$  satisfying  $v^\lambda w_\lambda = 0$ ,  $P_{\cdot \lambda \mu}^x$  has the form

$$\text{I 3,6 } \alpha) \quad P_{\cdot \lambda \mu}^x = A_{[\lambda}^x \rho_{\mu]}.$$

#### § 4. Section and reduction with respect to an $E_m$ in $E_n$ . Decomposition with respect to a rigged $E_m$ .

Let an  $E_m$  in  $E_n$  through the origin be given by the equations

$$(4.1) \quad \begin{cases} \text{a)} & \xi^x = B_b^x \eta^b \quad \text{or} \quad \text{b)} & C_\lambda^x \xi^\lambda = 0 \\ b = 1, \dots, m; & x = m + 1, \dots, n. \end{cases}$$

The  $E_m$ 's parallel to it are given by the parametric equation (1.13) with the parameters  $\zeta^x$ . The  $\eta^b$  are rectilinear coordinates in the  $E_m$  and the  $\zeta^x$  rectilinear coordinates in the  $E_{m'}$ ;  $m' = n - m$ , arising from the reduction of  $E_n$  with respect to the  $m$ -direction of  $E_m$ .

The points of the  $E_m$  which lie in the hyperplanes of a vector  $w_\lambda$  satisfy the equations [cf. (2.5)]

$$(4.2) \quad w_\lambda B_b^\lambda \eta^b = c; \quad w_\lambda B_b^\lambda \eta^b = c + 1.$$

Hence

$$(4.3) \quad 'w_b \stackrel{\text{def}}{=} B_b^\lambda w_\lambda$$

is the *section* of  $w_\lambda$  with the  $E_m$ . Conversely  $w_\lambda$  is not uniquely determined by ' $w_b$ '. It is only determined to within an additive vector whose  $(n-1)$ -direction contains the  $m$ -direction of  $E_m$ . The section of a *covariant* tensor with valence  $>1$  is formed in the same way, e.g.

$$(4.4) \quad 'w_{c b a} \stackrel{\text{def}}{=} B_{c b a}^{\mu \lambda x} w_{\mu \lambda x}.$$

To every *contravariant* vector  $\rho^a$  in  $E_m$  there belongs one and only one contravariant vector in  $E_n$

$$(4.5) \quad \rho^x \stackrel{\text{def}}{=} B_a^x \rho^a$$

$\rho^x$  vanishes if it is transvected with  $C_\lambda^x$ . Conversely to every contravariant vector in  $E_n$  which satisfies this condition there belongs one and only one contravariant vector in  $E_m$ . In fact  $\rho^a$  and  $\rho^x$  have the

same geometric interpretation, viz. an arrow lying in the  $m$ -direction of  $E_m$ . Therefore we consider  $p^x$  and  $p^a$  as two different sets of components of the *same* quantity and accordingly we use the same kernel letter. The  $E_n$ -components of a *contravariant* tensor of  $E_m$  with valence  $>1$  are formed in the same way, e.g.

$$(4.6) \quad p^{x\lambda\mu} \stackrel{\text{def}}{=} B_{abc}^{x\lambda\mu} p^{abc}; \quad a, b, c = 1, \dots, m.$$

The  $E_m$ 's parallel to the given  $E_m$  and passing through the points  $\xi^x, \xi^x + v^x$  have the coordinates  $\zeta^x = C_x^x \xi^x, \zeta^x = C_x^x \xi^x + C_x^x v^x$ . From this we see that to the *contravariant* vector  $v^x$  of  $E_n$  there corresponds the vector

$$(4.7) \quad "v^x \stackrel{\text{def}}{=} C_x^x v^x, \quad x = m+1, \dots, n$$

in the  $E_{m'}$ . We call " $v^x$ " the *reduction* of  $v^x$  with respect to the given  $E_m$ . Conversely  $v^x$  is not uniquely determined by " $v^x$ ". It is only determined to within an additive vector lying in the  $m$ -direction of  $E_m$ . The reduction of a *contravariant* tensor with valence  $>1$  is formed in the same way, e.g.

$$(4.8) \quad "v^{xyz} \stackrel{\text{def}}{=} C_{x\lambda\mu}^{xyz} v^{x\lambda\mu}.$$

To every *covariant* vector  $s_y$  in  $E_{m'}$  there belongs one and only one covariant vector in  $E_n$

$$(4.9) \quad s_\lambda \stackrel{\text{def}}{=} C_\lambda^y s_y.$$

$s_\lambda$  vanishes if it is transvected with  $B_b^\lambda$ . Conversely to every covariant vector in  $E_n$  satisfying this condition there belongs one and only one covariant vector in  $E_{m'}$ . In fact  $s_y$  is represented by the two  $E_{m'-1}$ 's in  $E_{m'}$  resulting from reduction of the two  $E_{n-1}$ 's of  $s_\lambda$ . Therefore we consider  $s_y$  and  $s_\lambda$  as two different sets of components of the *same* quantity and express this by using the same kernel letter. The  $E_n$ -components of a *covariant* tensor of  $E_{m'}$  with valence  $>1$  can be formed in the same way, e.g.

$$(4.10) \quad S_{\mu\lambda\kappa} \stackrel{\text{def}}{=} C_{\mu\lambda\kappa}^{zyx} S_{zyx}.$$

If in  $E_n$  an  $E_{m'}$  through the origin is given which has no direction in common with the  $E_m$ , the  $E_m$  is called *rigged* (eingespannt). Conversely the  $E_{m'}$  is rigged by the  $E_m$ . This  $E_{m'}$  is not identical with the  $E_{m'}$  arising from reduction of the  $E_n$  with respect to the  $E_m$ . But there is a one to one correspondence between the points of these two  $E_{m'}$ 's because every point of the  $E_{m'}$  arising from reduction represents an  $E_m$  and this  $E_m$  has just one point in common with the rigging  $E_{m'}$ . Hence we may now, *after having carried out the rigging, identify* both  $E_{m'}$ 's. From now on we have only one  $E_{m'}$  in which the  $\zeta^x$  in (1.13) can be

used as rectilinear coordinates and this  $E_{m'}$  can be interpreted geometrically in two ways. Firstly it may be interpreted as an  $E_m$  in  $E_n$  through the origin with the equations [cf. (4.1a)]

$$(4.11) \quad \xi^x = C_x^x \zeta^x$$

and the contravariant connecting quantity  $C_x^x$ . Secondly it can be interpreted as the manifold of all  $E_m$ 's parallel to the given  $E_m$  in which every  $E_m$  is considered as a point. Substituting (4.11) in (1.13) we get

$$(4.12) \quad C_y^x C_y^\lambda \zeta^y = \zeta^x$$

and this proves that  $C_y^x \stackrel{\text{def}}{=} C_\lambda^x C_\lambda^\lambda$  is the unity tensor of  $E_{m'}$ .

The  $E_{m'}$  is rigged by the  $E_m$ . Hence the  $E_m$  can be identified with the  $E_m$  arising from reduction of the  $E_n$  with respect to the  $E_{m'}$ . The  $E_m$ 's constituting this latter  $E_m$  are the  $E_m$ 's parallel to the rigging  $E_{m'}$  and are given by the equations [cf. (1.13)]

$$(4.13) \quad B_\lambda^a \xi^\lambda = \eta^a.$$

$B_\lambda^a$  is the covariant connecting quantity of the rigging  $E_{m'}$ . Substituting (4.1a) in (4.13) we see that  $B_b^a \stackrel{\text{def}}{=} B_\lambda^a B_\lambda^b$  is the unity tensor of  $E_m$ .

After the rigging has been performed we are able to form

1. the *sections* of covariant quantities of  $E_n$  with  $E_m$  and  $E_{m'}$ , for instance

$$(4.14) \quad \text{a) } 'w_b = B_b^\lambda w_\lambda; \quad \text{b) } ''w_y = C_y^\lambda w_\lambda.$$

2. the *reductions* of contravariant quantities of  $E_n$  with respect to  $E_m$  and to  $E_{m'}$ , for instance

$$(4.15) \quad \text{a) } ''v^x = C_x^x v^x; \quad \text{b) } 'v^a = B_a^\lambda v^\lambda.$$

These reductions can also be considered as projections on  $E_{m'}(E_m)$  in the  $m$ -( $m'$ )-direction of  $E_m(E_{m'})$ .

3. the  $E_n$ -*components* of covariant and contravariant quantities of  $E_m$  and  $E_{m'}$ , for instance

$$(4.16) \quad \text{a) } p^x = B_a^x p^a; \quad \text{b) } q_\lambda = B_\lambda^b q_b$$

$$(4.17) \quad \text{a) } s_\lambda = C_\lambda^y s_y; \quad \text{b) } r^x = C_x^y r^y.$$

In (4.14–17) the (a) formulae hold for every  $E_m$  in  $E_n$  but the (b) formulae are only valid if the  $E_m$  is rigged. Applying the processes (4.16) and (4.17) to  $B_b^a$  and  $C_y^x$  respectively we see that  $B_b^a$ ,  $B_b^x$  and  $B_\lambda^a$  and in the same way  $C_y^x$ ,  $C_\lambda^x$  and  $C_y^\lambda$  are three different sets of components

of the same quantity and that

$$(4.18) \quad B_{\lambda}^{\kappa} \stackrel{\text{def}}{=} B_b^{\kappa} B_{\lambda}^b \quad \text{and} \quad C_{\lambda}^{\kappa} \stackrel{\text{def}}{=} C_x^{\kappa} C_{\lambda}^x$$

form a fourth set. This justifies the use of the same kernel letter  $B$  and  $C$  respectively in all four cases.

In the following table results are gathered together for the rigged  $E_m$ .

$$(4.19) \quad \left\{ \begin{array}{ccc} E_m & E_n & E_{m'} \\ p^a & \boxed{p^{\kappa} = B_a^{\kappa} p^a} & r^{\kappa} = C_x^{\kappa} r^x \\ \xrightarrow[\text{change of components}]{} & & \xleftarrow[\text{change of components}]{} r^x \\ v^a & \boxed{v^{\kappa} \text{ with } C_x^{\kappa} v^{\kappa} = 0; \quad v^x \text{ with } B_a^a v^x = 0} & \xrightarrow[\text{change of components}]{} v^x \\ & \xleftarrow[\text{reduction (projection)}]{} v^{\kappa} & \xrightarrow[\text{reduction (projection)}]{} \boxed{''v^{\kappa} = C_x^{\kappa} v^x} \\ 'v^a = B_{\kappa}^a v^{\kappa} & & ''v^{\kappa} = C_x^{\kappa} ''v^x = C_{\lambda}^{\kappa} v^{\lambda} \\ \boxed{'w_b = B_b^{\lambda} w_{\lambda}} & \xleftarrow[\text{section}]{} w_{\lambda} & \xrightarrow[\text{section}]{} \boxed{''w_y = C_y^{\lambda} w_{\lambda}} \\ 'w_{\lambda} = B_{\lambda}^b 'w_b = B_{\lambda}^{\kappa} w_{\kappa} & & ''w_{\lambda} = C_{\lambda}^y ''w_y = C_{\lambda}^{\kappa} w_{\kappa} \\ w_b & \xleftarrow[\text{change of components}]{} w_{\lambda} \text{ with } C_y^{\lambda} w_{\lambda} = 0; \quad w_{\lambda} \text{ with } B_b^{\lambda} w_{\lambda} = 0 & \xrightarrow[\text{change of components}]{} w_y \\ q_b & \xrightarrow[\text{change of components}]{} & \boxed{s_{\lambda} = C_{\lambda}^y s_y} \\ & & \xleftarrow[\text{change of components}]{} s_y \end{array} \right.$$

If the  $E_m$  is not rigged the  $E_{m'}$  is the reduced  $E_n$ . Then only the formulae put in a frame remain valid and the reduction of  $v^{\kappa}$  can no longer be interpreted as a projection. From (4.18), (1.10) and the corresponding formulae for  $B_{\lambda}^a$  and  $C_y^{\lambda}$  it follows that for a rigged  $E_m$

$$(4.20) \quad (B_{\lambda}^{\kappa} + C_{\lambda}^{\kappa}) B_b^{\lambda} = B_b^{\kappa}; \quad (B_{\lambda}^{\kappa} + C_{\lambda}^{\kappa}) C_y^{\lambda} = C_y^{\kappa}$$

and

$$(4.21) \quad B_{\lambda}^{\kappa} + C_{\lambda}^{\kappa} = A_{\lambda}^{\kappa}.$$

Hence by means of  $B_{\lambda}^{\kappa}$  and  $C_{\lambda}^{\kappa}$  a vector can be split up into two parts in  $E_m$  and in  $E_{m'}$  called the  $E_m$ -part and the  $E_{m'}$ -part respectively. Besides its components in  $E_n$ , each part has also components in the submanifold to which it belongs. Tensors with a valence  $> 1$  split up into more parts, for instance

$$(4.22) \quad P_{\cdot \lambda}^{\kappa} = B_{\cdot \lambda}^{\kappa \sigma} P_{\cdot \sigma}^{\rho} + B_{\cdot \lambda}^{\kappa \sigma} C_{\lambda}^{\rho} P_{\cdot \sigma}^{\sigma} + C_{\cdot \lambda}^{\kappa \sigma} B_{\lambda}^{\rho} P_{\cdot \sigma}^{\sigma} + C_{\cdot \lambda}^{\kappa \sigma} P_{\cdot \sigma}^{\rho}.$$

The first is the  $E_m$ -part and the last the  $E_{m'}$ -part.

### § 5. Rank, domain and support of domain with respect to one or more indices.

From a co- or contravariant or mixed tensor of valence  $\phi$  we choose one index at some definite place and write  $\mu$ , say, for this index. If the tensor is then transvected over *all other* indices with all basis vectors belonging to some arbitrary coordinate system ( $\varkappa$ ) (that is, if  $1, \dots, n$  are substituted for all other indices with respect to this system) we get  $n^{\phi-1}$  co- or contravariant vectors according as  $\mu$  is an upper or a lower index. The number  $r$  of linearly independent ones among them is an arithmetic invariant and called the  $\mu$ -rank of the tensor. This rank is always  $\geq 0$  and  $\leq n$ . The set of all vectors which are linearly dependent on these  $r$  is called the  $\mu$ -domain. If  $\mu$  is an upper (lower) index the  $r$  vectors span an  $r$ -direction [ $(n-r)$ -direction] called the support of the  $\mu$ -domain. The  $\mu$ -domain and its support are invariants of the given tensor. The same holds mutatis mutandis for every quantity of valence  $\phi$ . Instead of vectors we then have pseudo-vectors or vector densities or  $W$ -vectors.

The rank and the domain with respect to more than one index is defined in the same way.<sup>1)</sup> Obviously the rank with respect to *all* indices is always equal to one.

If the valence is 2 the rank with respect to one index is equal to the rank with respect to the other one. Hence in this case there exists only one non-trivial rank and this rank is equal to the rank of the matrix of the components.

#### Exercises.

I 5,1. If a symmetric tensor with valence  $\phi$  has  $\mu$ -rank  $r$ , it can be written as a sum of  $r$  products of a vector in the  $\mu$ -domain with a symmetric tensor of valence  $\phi-1$ . Hence every symmetric tensor can be written as a sum of products of vectors. (Cf. I § 6.)

I 5,2. Prove that the rank with respect to some indices is always equal to the rank with respect to the other indices.

I 5,3. If a tensor  $v^{\varkappa_1 \dots \varkappa_p}$  is symmetric or alternating in all indices its rank is the same with respect to every index. This rank is  $r$  if and only if the equation  $v^{\varkappa_1 \dots \varkappa_p} w_{\varkappa_1} = 0$  has exactly  $n - r$  linearly independent solutions  $w_1$ .

I 5,4. Prove that the  $v^{\varkappa_1 \dots \varkappa_p}$  from Exerc. I 5,3 has rank  $r$  if and only if<sup>2)</sup>

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<sup>1)</sup> Cf. ALEXANDER 1925, 1; RICE 1928, 1.

<sup>2)</sup> In (I 5,4α) the alternation is only over the last index of each factor. In (I 5,4β) the last indices are alternated, and independently the last but one of each factor are also alternated. The notation  $[[\cdot]]$  is very often used in this sense.

$$\text{I } 5,4\alpha) \quad v^{\kappa_1 \dots [\mu_1} \dots v^{\kappa_s \dots |\mu_s]} \quad \begin{cases} \neq 0 & \text{for } s \leq r \\ = 0 & \text{for } s > r \end{cases}$$

and also if and only if

$$\text{I } 5,4\beta) \quad v^{\kappa_1 \dots [\lambda_1[\mu_1} \dots v^{\kappa_s \dots |\lambda_s]\mu_s]} \quad \begin{cases} \neq 0 & \text{for } s \leq r \\ = 0 & \text{for } s > r. \end{cases}$$

## § 6. Symmetric tensors.

A co- or contravariant tensor that is symmetric in *all* indices is called a *symmetric tensor*.<sup>1)</sup> Hence a symmetric tensor is invariant for interchange of any two indices. Symmetric pseudo-tensors, tensor densities and  $W$ -tensors are defined in the same way. The number of the linearly independent components of a symmetric tensor of valence  $\phi$  is  $\binom{n+\phi-1}{\phi}$ . In a centred  $E_n$ , a covariant symmetric tensor  $P_{\lambda_1 \dots \lambda_p}$  is represented by the hypersurface with the equations

$$(6.1) \quad P_{\lambda_1 \dots \lambda_p} \xi^{\lambda_1} \dots \xi^{\lambda_p} = \pm 1$$

in point coordinates and a contravariant symmetric tensor  $Q^{\kappa_1 \dots \kappa_p}$  is represented by the hypersurface with the equations

$$(6.2) \quad Q^{\kappa_1 \dots \kappa_p} u_{\kappa_1} \dots u_{\kappa_p} = \pm 1$$

in hyperplane coordinates. If we put zero instead of  $\pm 1$  in (6.1, 2) we get the equation of the hypercone belonging to the symmetric tensor. The hypercone determines only the symmetric pseudo-tensor  $[P_{\lambda_1 \dots \lambda_p}]$  or  $[Q^{\kappa_1 \dots \kappa_p}]$  respectively. From two covariant (contravariant) symmetric tensors of valences  $\phi$  and  $q$  a symmetric tensor of valence  $\phi+q$  can be derived by general multiplication and mixing over all indices. This process is called *symmetric multiplication* of symmetric tensors. It can also be applied to a greater number of symmetric tensors. The polynomial at the left hand side of the equation of the hypersurface of the symmetric product is the product of the polynomials on the left hand side of the equations belonging to the factors. Hence the notions *division* and *divisor* known from the theory of homogeneous polynomials can be extended to symmetric tensors. A symmetric tensor  $Q$  is a divisor of a symmetric tensor  $P$  if and only if there exists a symmetric tensor  $R$  such that  $P$  is the symmetric product of  $Q$  and  $R$ . A symmetric tensor without a divisor is called *irreducible*. It follows from the theory of homogeneous polynomials that every symmetric tensor is the

<sup>1)</sup> In former publications we used the term tensor instead of symmetric tensor (cf. footnote 1 on p. 10). The word "tenseur" in CARTAN 1938, I, 130 has quite another meaning.

symmetric product of a finite number of irreducible symmetric tensors and that these factors are uniquely determined to within a scalar factor. If all irreducible divisors have valence 1 the hypercone of a covariant symmetric tensor of valence  $p$  consists of  $p$  hyperplanes and the hypercone of a contravariant symmetric tensor of valence  $p$  consists of  $p$  straight lines.

### Exercises.

I 6,1<sup>1)</sup>. The domain of a symmetric tensor is the same with respect to all its indices. If the rank is  $r$  the tensor can be written as a sum of symmetric products of vectors from this domain. The support of the domain of a symmetric tensor  $w_{\lambda_1 \dots \lambda_p}$  of rank  $r$  is an  $(n-r)$ -direction. Its hypersurface is a cylinder consisting of  $\infty^{r-1} E_{n-r}$ 's with this  $(n-r)$ -direction. The hypercone consists for  $r > 1$  of  $\infty^{r-2} E_{n-r+1}$ 's all of them containing the  $E_{n-r}$ , with the equation

$$\text{I 6,1}\alpha) \quad w_{\lambda_1 \dots \lambda_p} \xi^{\lambda_p} = 0.$$

I 6,2<sup>2)</sup>. The rank of the symmetric tensor  $w_{\lambda_1 \dots \lambda_p}$  is  $< n$  if and only if  
 I 6,2 $\alpha)$   $w_{(\kappa_1 \dots \lambda_1 [\mu_1 \nu_1 \dots \nu_{n-1}] \dots \kappa_n) \mu_n] \nu_n} = 0.$

In this expression the mixing has to be effected over all indices except the last two of every factor. The homogeneous polynomial of degree  $n(p-2)$  belonging to the left hand side of (I 6,2 $\alpha$ ) is the hessian covariant of the polynomial belonging to  $w_{\lambda_1 \dots \lambda_p}$ .

I 6,3<sup>3)</sup>. If the symmetric tensors  $P_{\lambda \kappa}$  and  $Q_{\lambda \kappa}$  satisfy the equation  $P_{[\kappa} (\lambda Q_{\mu]} \nu)} = 0$  they differ only by a scalar factor.

I 6,4. A symmetric tensor of valence 2 is for  $n \leq 2$  always the symmetric product of two vectors.

I 6,5. A symmetric tensor  $w_{\lambda_1 \dots \lambda_p}$  is the symmetric product of  $p$  equal vectors if and only if

$$\text{I 6,5}\alpha) \quad w_{[\lambda_1 [\lambda_2 | \lambda_3 \dots \lambda_p] w_{\kappa_1} \kappa_2] \kappa_3 \dots \kappa_p} = 0.$$

## § 7. Multivectors.

A co- or contravariant tensor that is alternating in *all* indices is called a *multivector* or *alternating tensor*. It is also called a  $p$ -vector if its valence is  $p$ .<sup>4)</sup> The valence can not be  $> n$ . Pseudo-multivectors, multivector densities and  $W$ -multivectors are defined in the same way. A multivector changes its sign if two indices are interchanged. The number of linearly independent components of an alternating quantity

<sup>1)</sup> P. P. 1949, 1, p. 12.

<sup>2)</sup> SINIGALLIA 1905, 1, p. 371.

<sup>3)</sup> SCHOUTEN 1924, 1, p. 205; EISENHART 1926, 1, p. 32.

<sup>4)</sup> A multivector is also called antisymmetric or skew symmetric tensor by some authors.

of valence  $p$  is  $\binom{n}{p}$ . It is usual to call a 2-vector *bivector*, a 3-vector *trivector* and a 4-vector *quadrivector*.

From two covariant (contravariant) multivectors of valences  $p$  and  $q$ ;  $p+q \leq n$  a multivector of valence  $p+q$  can be derived by general multiplication and alternation over all indices. This process is called *alternating multiplication* of multivectors. It can also be applied to a greater number of multivectors, provided that the sum of the valences is  $\leq n$ . For this alternating product we often use a notation without indices. Instead of  $u^{[x_1 \dots x_p} v^{\lambda_1 \dots \lambda_q]}$  we write shortly  $[uv]$ .<sup>1)</sup> Note that

$$(7.1) \quad [uv] = (-1)^{p,q} [vu].$$

A multivector  $Q$  is called a *divisor* of a multivector  $P$  if and only if there exists a multivector  $R$  such that  $P$  is the alternating product of  $Q$  and  $R$ . A vector  $v^\kappa$  is a divisor of  $v^{\kappa_1 \dots \kappa_p}$  if and only if  $v^{[x_1 \dots x_p} v^{\lambda]} = 0$ . To prove this we take  $e^\kappa = v^\kappa$ . Then  $v^{\kappa_1 \dots \kappa_p}$  cannot have non vanishing components with all indices  $\neq 1$ . Hence  $e^\kappa$  is a divisor. If  $v^{\kappa_1 \dots \kappa_p}$  has  $s$  linearly independent vector divisors this  $p$ -vector can be written as the alternating product of these vectors and a  $(p-s)$ -vector. This can be proved by taking the  $s$  vectors as the first  $s$  contravariant basis vectors.

A  $p$ -vector that can be written as the alternating product of  $p$  vectors is called *simple*. The necessary and sufficient condition that  $v^{\kappa_1 \dots \kappa_p}$  is simple is

$$(7.2) \quad v^{[x_1 \dots x_p} v^{\lambda_1]} \dots \lambda_p] = 0. \text{ } ^2)$$

The same holds mutatis mutandis for covariant  $p$ -vectors. The necessity of the condition is obvious. In order to prove the sufficiency we take the coordinate system in such a way that  $v^1 \dots ^p \neq 0$ . Then (7.2) states that the  $p$  vectors  $v^{\kappa_2 \dots \kappa_p}$ ,  $v^{\kappa_1 1 \dots \kappa_p}$ ,  $v^{\kappa_1 2 \dots \kappa_p}$ , ...,  $v^{\kappa_1 \dots (p-1)}$  are divisors of  $v^{\kappa_1 \dots \kappa_p}$ . But these  $p$  vectors can not be linearly dependent because the  $i$ -th component  $\pm v^1 \dots ^p$  of the  $i$ -th vector is  $\neq 0$  and the  $i$ -th components of all other vectors are zero. Hence  $v^{\kappa_1 \dots \kappa_p}$  has  $p$  independent vector factors.<sup>3)</sup>

Every  $n$ -vector is simple. Hence any two  $n$ -vectors differ only by a scalar factor. But every  $(n-1)$ -vector is also simple.<sup>4)</sup> In order

<sup>1)</sup> Cf. P. P. 1949, 1, I § 7,10. This notation is intimately connected with CARTAN's notations for differential forms (cf. II § 12).

<sup>2)</sup> Givens proved 1937, 1 that  $v^{[x_1 \dots x_p} v^{\lambda_1 \dots \lambda_p]} = 0$  is another necessary and sufficient condition and moreover that from  $v^{[x_1 \dots x_p} v^{\lambda_1 \dots \lambda_{s+1}]} \dots \lambda_p] = 0$  it always follows that  $v^{[x_1 \dots x_s} v^{\lambda_1 \dots \lambda_s]} \dots \lambda_p] = 0$  if  $s$  is odd. This proposition was proved by WEITZENBÖCK 1908, 1 for  $p=3$ ,  $n=5$  and formulated 1923, 1, p. 87 for the general case Cf. BOMPIANI 1952, 2.

<sup>3)</sup> If  $v^{\kappa_1 \dots \kappa_p}$  is written as the product of these  $p$  vectors and a scalar, we get the relations of VAHLEN. Cf. WEITZENBÖCK 1923, 1, p. 118.

<sup>4)</sup> GRASSMANN 1862, 1, p. 61.

to prove this we suppose that  $v^{[\kappa_1 \dots \kappa_{n-1} \lambda_1] \dots \lambda_{n-1}} \neq 0$ . From this it would follow that at least one of the vectors obtained by replacing in  $v^{\lambda_1 \dots \lambda_{n-1}}$  the last  $n-2$  indices by  $n-2$  of the indices  $1, \dots, n$  were not zero. By interchanging indices we can always arrange that  $v^{\lambda_2 \dots (\lambda_{n-1})}$  is this vector. But then it would follow that

$$(7.3) \quad v^1 \dots (n-1) v^{n,2 \dots (n-1)} - v^{n,2 \dots (n-1)} v^1 \dots (n-1) \neq 0$$

and this is impossible.

The component  $v^1 \dots p$  of the simple  $p$ -vector

$$(7.4) \quad v^{\kappa_1 \dots \kappa_p} = p! v_1^{\kappa_1} \dots v_p^{\kappa_p}$$

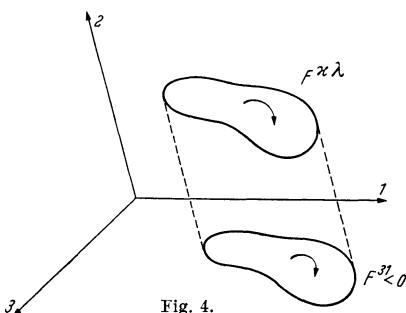


Fig. 4.

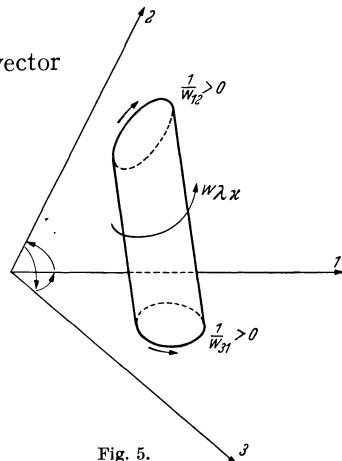


Fig. 5.

is the  $p$ -dimensional volume of the projection of the parallelotope of  $v_1^{\kappa}, \dots, v_p^{\kappa}$  on the  $E_p$  of  $e_1^{\kappa}, \dots, e_p^{\kappa}$  in the  $(n-p)$ -direction of  $e_{p+1}^{\kappa}, \dots, e_n^{\kappa}$ , measured by the parallelotope of  $e_1^{\kappa}, \dots, e_p^{\kappa}$ . Its sign is + if the screwsense of the projections of  $v_1^{\kappa}, \dots, v_p^{\kappa}$  in this order is the same as the screwsense of  $e_1^{\kappa}, \dots, e_p^{\kappa}$  and - if it is not. From this it follows that a simple contravariant  $p$ -vector is represented geometrically by a definite  $p$ -dimensional volume with an arbitrary form but a definite screwsense in a definite  $E_p$ . Fig. 4 shows a simple contravariant bivector in  $E_3$  with its projection on the 31-plane.

A simple covariant  $p$ -vector is represented geometrically by a cylinder of  $\infty^{p-1}$  parallel  $E_{n-p}$ 's with an outer orientation, that is a  $p$ -dimensional screwsense around the cylinder.<sup>1)</sup> The component  $w_{1\dots p}$  of  $w_{\lambda_1 \dots \lambda_p}$  is the inverse of the  $p$ -dimensional volume of the section of this cylinder with the  $E_p$  of  $e_1^{\kappa}, \dots, e_p^{\kappa}$  measured by the parallelotope of these vectors. Its sign is + if the outer orientation of the  $p$ -vector corresponds to the screwsense of  $e_1^{\kappa}, \dots, e_p^{\kappa}$  and - if it does not. Fig. 5 shows a simple covariant bivector in  $E_3$  with its sections with the 31- and 12-plane.<sup>2)</sup> To every rectilinear coordinate system in  $E_n$  there belongs

<sup>1)</sup> SCHOUTEN-STRIJK 1922, 1.

<sup>2)</sup>  $w_{12} > 0$  should be  $w_{12} < 0$ .

for every value of  $p$  from 1 to  $n$ , a set of  $\binom{n}{p}$  simple contravariant  $p$ -vectors and a set of  $\binom{n}{p}$  simple covariant  $p$ -vectors, uniquely determined for  $p=1$  and  $p=n$  and determined except for the sign for  $1 < p < n$ . These  $p$ -vectors are alternating products of  $p$  basis vectors multiplied by  $p!$ . Their  $p$ -directions and  $(n-p)$ -directions are the  $\binom{n}{p}$  coordinate- $E_p$ 's and coordinate- $E_{n-p}$ 's respectively. The  $n$ -vectors

$$(7.5) \quad \begin{cases} E_{(\alpha)}^{\alpha_1 \dots \alpha_n} \stackrel{\text{def}}{=} n! e_1^{[\alpha_1} \dots e_n^{\alpha_n]} \\ e_{\lambda_1 \dots \lambda_n}^{(\alpha)} \stackrel{\text{def}}{=} n! e_{[\lambda_1}^1 \dots e_{\lambda_n]}^n \end{cases}$$

are most important among them. They are both represented by the parallelotop of the basis vectors with the screwsense fixed by these vectors in the order  $1, \dots, n$ . If the choice of the coordinate system is free, they are determined to within a scalar factor. Their components satisfy the identities (not summarize over  $\alpha$ )

$$(7.6) \quad E_{(\alpha)}^{I \dots n} = e_{I \dots n}^{(\alpha)} = +1$$

$$(7.7) \quad E_{(\alpha)}^{\mu_1 \dots \mu_n} e_{\mu_1 \dots \mu_n}^{(\alpha)} = n!$$

$$(7.8) \quad E_{(\alpha)}^{\mu_1 \dots \mu_m \alpha_{m+1} \dots \alpha_n} e_{\mu_1 \dots \mu_m \alpha_{m+1} \dots \alpha_n}^{(\alpha)} = m! (n-m)! A_{[\alpha_{m+1} \dots \alpha_n]}^{[\alpha_1 \dots \alpha_m]}.$$

These can easily be verified.

From (7.5) it follows that

$$(7.9) \quad E_{(\alpha)}^{I' \dots n'} = A_{\lambda_1 \dots \lambda_n}^{I' \dots n'} E_{(\alpha)}^{\lambda_1 \dots \lambda_n} = n! A_{[I \dots n]}^{I' \dots n'} E_{(\alpha)}^{I \dots n} = \Delta E_{(\alpha)}^{I \dots n}.$$

Hence  $E_{(\alpha)}^{I \dots n}$  is the component with respect to  $(\alpha)$  of a scalar  $\Delta$ -density of weight  $-1$ . In the same way  $e_{I \dots n}^{(\alpha)}$  is the component of a scalar  $\Delta$ -density of weight  $+1$ . This is the first example of sets of numbers that can be looked upon in two different ways as components of a quantity.

The  $n$ -vectors  $E_{(\alpha)}$  and  $e_{I \dots n}^{(\alpha)}$  determine a one to one correspondence between contravariant pseudo- $p$ -vectors and covariant pseudo- $(n-p)$ -vectors:

$$(7.10) \quad \begin{cases} [v^{\alpha_1 \dots \alpha_p}] \propto [w_{\lambda_1 \dots \lambda_q}] E_{(\alpha)}^{\lambda_1 \dots \lambda_q \alpha_1 \dots \alpha_p}; & q = n - p \\ [w_{\lambda_1 \dots \lambda_q}] \propto e_{\lambda_1 \dots \lambda_q \alpha_1 \dots \alpha_p}^{(\alpha)} [v^{\alpha_1 \dots \alpha_p}] \cdot 1 \end{cases}$$

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<sup>1)</sup> The sign  $\propto$  means proportional to.

If  $v^{\alpha_1 \dots \alpha_p}$  is simple,  $|v^{\alpha_1 \dots \alpha_p}|$  can be represented in a centred  $E_n$  by an  $E_p$  through the origin.<sup>1)</sup> Now this  $E_p$  can be given by  $p(n-p)$  numbers, hence  $v^{\alpha_1 \dots \alpha_p}$  has only  $p(n-p) + 1$  independent components and among the equations (7.2) there must be  $\binom{n}{p} - p(n-p) - 1$  independent ones. In order to obtain a convenient set of  $p(n-p)$  independent components of  $|v^{\alpha_1 \dots \alpha_p}|$  we choose the coordinate system in such a way that  $v^{1 \dots p} \neq 0$  and put

$$(7.11) \quad B_{\beta}^{\alpha} \stackrel{\text{def}}{=} \frac{v^{1 \dots (\beta-1) \alpha (\beta+1) \dots p}}{v^{1 \dots p}}; \quad \beta = 1, \dots, p. \quad ^2)$$

Then it is easily proved that

$$(7.12) \quad \frac{v^{\alpha_1 \dots \alpha_p}}{v^{1 \dots p}} = p! \quad B_I^{[\alpha_1} \quad B_p^{\alpha_p]}; \quad B_{\beta}^{\alpha} \stackrel{\text{def}}{=} \delta_{\beta}^{\alpha}$$

or

$$(7.13) \quad \left\{ \begin{array}{l} \frac{v^{\xi_1 \dots \xi_s \alpha_{s+1} \dots \alpha_p}}{v^{1 \dots p}} = s! \quad B_{\alpha_1 \dots \alpha_p}^{[\xi_1 \dots \xi_s]}; \quad \xi_1, \dots, \xi_s = p+1, \dots, n, \\ \alpha_1, \dots, \alpha_p = \text{even permutation of } 1, \dots, p. \end{array} \right.$$

The  $B_{\alpha}^{\xi}$  are the  $p(n-p)$  components required. They are very often used in problems concerning systems of partial differential equations.<sup>3)</sup>

The alternating quantities  $\tilde{\mathfrak{E}}^{\alpha_1 \dots \alpha_n}$  and  $\tilde{\epsilon}_{\lambda_1 \dots \lambda_n}$  defined by

$$(7.14) \quad \tilde{\mathfrak{E}}^{1 \dots n} = +1 \quad \tilde{\epsilon}_{1 \dots n} = +1$$

with respect to every rectilinear coordinate system in  $E_n$  are  $n$ -vector  $\Delta$ -densities of weight  $+1$  and  $-1$  respectively. They exist a priori in  $E_n$  just like  $A_{\lambda}^{\alpha}$  and they establish a one to one correspondence between contra-(co-)variant  $p$ -vectors and co- (contra-) variant  $(n-p)$ -vector  $\Delta$ -densities of weight  $-1$  ( $+1$ ):

$$(7.15) \quad \left\{ \begin{array}{l} \text{a)} \quad p! \quad \tilde{\mathfrak{v}}_{\lambda_1 \dots \lambda_q} = (-1)^{\alpha_q} \tilde{\epsilon}_{\lambda_1 \dots \lambda_q \alpha_1 \dots \alpha_p} v^{\alpha_1 \dots \alpha_p} \\ \text{b)} \quad q! \quad v^{\alpha_1 \dots \alpha_p} = (-1)^{-\alpha_q} \tilde{\mathfrak{v}}_{\lambda_1 \dots \lambda_q} \tilde{\mathfrak{E}}^{\lambda_1 \dots \lambda_q \alpha_1 \dots \alpha_p} \\ \text{c)} \quad p! \quad \tilde{\mathfrak{w}}^{\alpha_1 \dots \alpha_p} = (-1)^{\beta_p} w_{\lambda_1 \dots \lambda_p} \tilde{\mathfrak{E}}^{\lambda_1 \dots \lambda_p \alpha_1 \dots \alpha_q} \\ \text{d)} \quad q! \quad w_{\lambda_1 \dots \lambda_p} = (-1)^{-\beta_p} \tilde{\epsilon}_{\lambda_1 \dots \lambda_p \alpha_1 \dots \alpha_q} \tilde{\mathfrak{w}}^{\alpha_1 \dots \alpha_q}; \quad q = n-p \end{array} \right.$$

<sup>1)</sup> A contra-(co-)variant pseudo- $p$ -vector can also be represented by an  $E_{p-1}(E_{n-p-1})$  in an  $E_{n-1}$  if it is simple. The components are then the homogeneous coordinates of this  $E_{p-1}(E_{n-p-1})$ . If it is not simple it can be represented by an  $E_{n-p-1}(E_{p-1})$ -complex in  $E_{n-1}$ . Hence for  $n=4$  a pseudo-bivector can be represented by a line complex in  $E_3$  that degenerates into all straight lines intersecting the same straight line if and only if the pseudo-bivector is simple. Cf. e.g. WEITZEN-BÖCK 1923, 1, p. 69, 83.

<sup>2)</sup> KÄHLER 1934, 1, p. 17.

<sup>3)</sup> Cf. e.g. KÄHLER 1934, 1; P. P. 1949, 1, Ch. X.

or in another form

$$(7.16) \quad \begin{cases} \text{a)} \quad \tilde{v}_{\lambda_1 \dots \lambda_q} = (-1)^{\alpha_q} v^{\kappa_1 \dots \kappa_p}; \\ \text{b)} \quad \tilde{w}^{\kappa_1 \dots \kappa_p} = (-1)^{\beta_p} w_{\lambda_1 \dots \lambda_q}; \\ \lambda_1 \dots \lambda_q \kappa_1 \dots \kappa_p = \text{even permutation of } 1, \dots, n. \end{cases}$$

The choice of the exponents  $\alpha$  and  $\beta$  is free. We may take them all equal to zero. But if subgroups of  $G_a$  are introduced and if we wish to make use of the possibility of identification of quantities with respect to such a group (cf. e.g. I § 2) it is more convenient to give the  $\alpha$ 's and  $\beta$ 's values suitable for this special purpose.<sup>1)</sup>

From (7.16) we see that the components of a contra-(co-)variant  $p$ -vector, if labelled in another way, transform as the components of a co-(contra-) variant  $(n-p)$ -vector  $\Delta$ -density of weight  $-1$  ( $+1$ ). Hence there is no geometric difference between these quantities. As an example we prove explicitly that the components  $v^{23}, v^{31}, v^{12}$  of a contravariant bivector in  $E_3$  transform like the components  $v_1, v_2, v_3$  of a covariant vector  $\Delta$ -density of weight  $(-1)$ . In fact

$$(7.17) \quad \begin{cases} v^{2'3'} = A_{\kappa\lambda}^{2'3'} v^{\kappa\lambda} = (A_{23}^{2'3'} - A_{32}^{2'3'}) v^{23} + \text{cycl. } 1, 2, 3 \\ \qquad \qquad \qquad = \Delta (A_{I'}^1 v^{23} + A_{I'}^2 v^{31} + A_{I'}^3 v^{12}) \end{cases}$$

according to the construction of the  $A_{\lambda}^{\kappa}$  from the  $A_{\lambda}^{\kappa}$  given in I § 1.

The densities most frequently occurring in physics are contra-(co-)variant multivectordensities of weight  $+1$  ( $-1$ ) which transform with  $|\Delta|$  instead of  $\Delta$ . Hence for real coordinate transformations they transform in the same way as  $\Delta$ -densities if  $\Delta > 0$  and change the sign for  $\Delta < 0$ . This implies that their geometric interpretation is the same as that of the corresponding  $\Delta$ -densities to within the orientation. Instead of an inner (outer) orientation they get an outer (inner) one. If transvected over all indices with  $\tilde{\epsilon}$  or  $\tilde{\epsilon}$  they give rise to  $W$ -multivectors, transforming like ordinary multivectors but with a factor<sup>2)</sup>  $\Delta/|\Delta|$ . The value of a  $W$ -scalar<sup>3)</sup> is invariant for  $\Delta > 0$  and changes its sign for  $\Delta < 0$  (transformation of a right hand system into a left hand system).

The alternating quantities in  $E_3$  with their geometric interpretations and their notations are to be seen in the following tables.<sup>4)</sup>

<sup>1)</sup> These identifications are elaborately dealt with in T.P. 1951, 1, Ch. III.

<sup>2)</sup> SCHOUTEN 1938, 1. The two different kinds of  $p$ -vectors were used in DE RHAM and KODAIRA 1950, 1.

<sup>3)</sup> Often called pseudo-scalar in physical publications.

<sup>4)</sup> Cf. SCHOUTEN 1938, 1; SCHOUTEN and VAN DANTZIG 1940, 1 (note that in these papers the sign  $\sim$  is used in another less satisfactory way); T.P. 1951, 1, Ch. II.

| Interpretation | Ordinary notation                                                            | Second notation                                                                                                                   | Weight       | Orientation |
|----------------|------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------|--------------|-------------|
| Number         | $s$ scalar                                                                   | $\tilde{s}_{\mu \lambda \kappa}$ cov. triv. $\Delta$ -dens.<br>$\tilde{\sigma}^{\kappa \lambda \mu}$ contr. triv. $\Delta$ -dens. | $-1$<br>$+1$ | $\pm$ sign  |
|                | $v^\kappa$ contr. vect.                                                      | $\tilde{v}_{\lambda \kappa}$ cov. biv. $\Delta$ -dens.                                                                            | $-1$         | inner       |
|                | $w_\lambda$ cov. vect.                                                       | $\tilde{w}^{\kappa \lambda}$ contr. biv. $\Delta$ -dens.                                                                          | $+1$         | outer       |
|                | $f^{\kappa \lambda}$ contr. biv.                                             | $\tilde{f}_\lambda$ cov. vect. $\Delta$ -dens.                                                                                    | $-1$         | inner       |
|                | $h_{\lambda \kappa}$ cov. biv.                                               | $\tilde{h}^{\kappa}$ contr. vect. $\Delta$ -dens.                                                                                 | $+1$         | outer       |
|                | $p^{\kappa \lambda \mu}$ contr. triv.<br>$q_{\mu \lambda \kappa}$ cov. triv. | $\tilde{p}$ scalar $\Delta$ -dens.<br>$\tilde{q}$ scalar $\Delta$ -dens.                                                          | $-1$<br>$+1$ | screw       |

| Interpretation      | Ordinary notation                                                                                             | Weight       | Second notation                                                                                        | Orientation |
|---------------------|---------------------------------------------------------------------------------------------------------------|--------------|--------------------------------------------------------------------------------------------------------|-------------|
| Number + screwsense | $\tilde{s}_{\mu \lambda \kappa}$ cov. triv. dens.<br>$\tilde{\sigma}^{\kappa \lambda \mu}$ contr. triv. dens. | $-1$<br>$+1$ | $\tilde{s}$ $W$ -scalar                                                                                | screw       |
|                     | $v_{\lambda \kappa}$ cov. biv. dens.                                                                          | $-1$         | $\tilde{v}^\kappa$ contr. $W$ -vect.                                                                   | outer       |
|                     | $\tilde{w}^{\kappa \lambda}$ contr. biv. dens.                                                                | $+1$         | $\tilde{w}_\lambda$ cov. $W$ -vect.                                                                    | inner       |
|                     | $\tilde{f}_\lambda$ cov. vect. dens.                                                                          | $-1$         | $\tilde{f}^{\kappa \lambda}$ contr. $W$ -biv.                                                          | outer       |
|                     | $\tilde{h}^{\kappa}$ contr. vect. dens.                                                                       | $+1$         | $\tilde{h}_{\lambda \kappa}$ cov. $W$ -biv.                                                            | inner       |
|                     | $\tilde{p}$ scalar dens.<br>$\tilde{q}_{\mu \lambda \kappa}$ cov. $W$ -triv.                                  | $-1$<br>$+1$ | $\tilde{p}^{\kappa \lambda \mu}$ contr. $W$ -triv.<br>$\tilde{q}_{\mu \lambda \kappa}$ cov. $W$ -triv. | $\pm$ sign  |

### Exercise.

I 7.1. Prove that for every  $m$ -vector  $v^{x_1 \dots x_m}$ :

$$\text{I 7.1 } \alpha) \quad v^{x_1 \dots x_m} E_{(\kappa)}^{\lambda_1 \dots \lambda_m x_{m+1} \dots x_n} = \binom{n}{m} v^{[\lambda_1 \dots \lambda_m E_{(\kappa)}^{x_{m+1} \dots x_n}] x_1 \dots x_m}.$$

### § 8. Tensors of valence 2.

#### A. Cocontravariant tensors.

A tensor  $P_{\cdot \lambda}^{\kappa}$  is represented by the homogeneous linear vector transformation (which is in a centred  $E_n$  also a point transformation)

$$(8.1.) \quad 'v^\kappa = P_{\cdot \lambda}^{\kappa} v^\lambda.$$

If the rank of  $P_{\cdot \lambda}^*$  is  $n$  this transformation is invertible

$$(8.2) \quad v^* = P_{\cdot \lambda}^{-1} v^\lambda; \quad P_{\cdot \mu}^* P_{\cdot \lambda}^{-1} = P_{\cdot \mu}^{-1} P_{\cdot \lambda}^* = A_\lambda^*; \quad \text{Det}(P_{\cdot \lambda}^*) \neq 0.$$

For every value of  $\varrho$  and  $\sigma$ ,  $P_{\cdot \sigma}^*$  is equal to the minor of the element  $P_{\cdot \sigma}^*$  in the matrix of the  $P_{\cdot \lambda}^*$  divided by  $\text{Det}(P_{\cdot \lambda}^*)$  (cf. Exerc. I 1,1 and I 8,1 ; 2).

The components of

$$(8.3) \quad s! P_{\cdot [\lambda_1}^{\kappa_1} \dots P_{\cdot \lambda_s]}^{\kappa_s}] = s! P_{\cdot [\lambda_1}^{\kappa_1} \dots P_{\cdot \lambda_s]}^{\nu_s}] = s! P_{\cdot [\lambda_1}^{\kappa_1} \dots P_{\cdot \lambda_s]}^{\nu_s}]$$

are the  $s$ -rowed subdeterminants of the matrix of  $P_{\cdot \lambda}^*$ . Hence the rank is  $r$  if and only if

$$(8.4) \quad P_{\cdot [\lambda_1}^{\kappa_1} \dots P_{\cdot \lambda_s]}^{\nu_s}] \begin{cases} \neq 0 & \text{for } s=r \text{ (hence for } s < r) \\ = 0 & \text{for } s > r. \end{cases}$$

The determinant of  $P$  is the scalar

$$(8.5) \quad n! P_{\cdot 1}^{[I} \dots P_{\cdot n]}^{n].}$$

Another concomitant of  $P$  is the trace (Spur, spur)  $P_{\cdot \lambda}^\lambda$ .

A vector  $v^*$  is said to be an *eigenvector* of  $P$  if it satisfies an equation of the form

$$(8.6) \quad P_{\cdot \lambda}^* v^\lambda = \lambda v^*$$

and  $\lambda$  is called the *eigenvalue* of  $P$  belonging to this eigenvector. From (8.6) the rank of  $P_{\cdot \lambda}^* - \lambda A_\lambda^*$  is  $< n$ , hence the possible values of  $\lambda$  are the solutions of the algebraic equation of degree  $n$

$$(8.7) \quad \left\{ \begin{array}{l} \varphi(\lambda) \stackrel{\text{def}}{=} (-1)^n \text{Det}(P_{\cdot \lambda}^* - \lambda A_\lambda^*) \\ = (-1)^n n! (P_{\cdot [1}^{[I} - \lambda A_{[1]}^{[I}] \dots (P_{\cdot n]}^{n]} - \lambda A_{n]}^{n]}) = 0 \end{array} \right.$$

the so called *characteristic equation* of  $P$ . From this we see that if in (8.6) we take the transvection with a covariant vector instead of with a contravariant vector we get the same eigenvalues. Writing (8.7) in the form

$$(8.8) \quad \varphi(\lambda) = \lambda^n - \sum_1^p \lambda^{n-1} + \sum_2 \lambda^{n-2} - \dots + (-1)^n \sum_n = 0$$

we have

$$(8.9) \quad \left\{ \begin{array}{l} \sum_1^p = nn! A_{[1}^{[I} \dots A_{n-1}^{n-1} P_{\cdot n]}^{n]} = P_{\cdot \kappa}^* \\ \vdots \\ \sum_i = \binom{n}{i} n! A_{[1}^{[I} \dots A_{n-i}^{n-i} P_{\cdot n-i+1}^{n-i+1} \dots P_{\cdot n]}^{n]} = P_{\cdot \kappa_i}^* \dots I^{\nu_i} \\ \vdots \\ \sum_n = n! P_{\cdot [1}^{[I} \dots P_{\cdot n]}^{n]} = P_{\cdot \kappa_1}^* \dots P_{\cdot \kappa_n}^* = \text{Det}(P_{\cdot \lambda}^*). \end{array} \right.$$

Using the short notation of matrix calculus

$$(8.10) \quad P Q \stackrel{\text{def}}{=} P_{\cdot, \lambda}^{\kappa} Q_{;\lambda}^o; \quad P^2 = P P, \text{ etc.}$$

we prove now that  $P$  itself satisfies the equation (8.8):

$$(8.11) \quad \varphi(P) \stackrel{\text{def}}{=} \sum_{i=1}^n P_i^{\kappa} P^{n-i} + \sum_{i=2}^n P_i^{\kappa} P^{n-2} - \cdots + (-1)^n P_n^{\kappa} A = 0.$$

Equation (8.11) is easily verified by writing out the identity

$$(8.12) \quad P_{\cdot, [\kappa_1}^{\kappa_n} \cdots P_{\cdot, \kappa_n]}^{\kappa_n} A_{\lambda}^{\kappa_{n+1}]} = 0$$

and making use of (8.9).

If the roots of (8.8) are determined, the eigenvectors belonging to some root  $\lambda$  can be found by solving the equation

$$(8.13) \quad (P_{\cdot, \lambda}^{\kappa} - \lambda A_{\lambda}^{\kappa}) v^{\lambda} = 0.$$

But if we wish to compute eigenvectors for all roots it is more convenient to determine the following auxiliary quantities first.<sup>1)</sup>

$$(8.14) \quad \left\{ \begin{array}{ll} P_1^{\kappa} \stackrel{\text{def}}{=} P_{\cdot, \lambda}^{\kappa} A - P; & P_1^{\kappa} \stackrel{\text{def}}{=} \text{trace } P; \quad P_2^{\kappa} \stackrel{\text{def}}{=} \frac{1}{2} \text{trace } P P \\ P_2^{\kappa} \stackrel{\text{def}}{=} P_{\cdot, \lambda}^{\kappa} A - P P; & P_2^{\kappa} \stackrel{\text{def}}{=} \frac{1}{3} \text{trace } P P \\ \vdots & \vdots \\ P_{n-1}^{\kappa} \stackrel{\text{def}}{=} P_{\cdot, \lambda}^{\kappa} A - P_{\cdot, \lambda}^{\kappa} P_{n-2}; & P_{n-1}^{\kappa} \stackrel{\text{def}}{=} \frac{1}{n} \text{trace } P_{\cdot, \lambda}^{\kappa} P_{n-1} \\ P_n^{\kappa} \stackrel{\text{def}}{=} P_{\cdot, \lambda}^{\kappa} A - P_{\cdot, \lambda}^{\kappa} P_{n-1}. & \end{array} \right.$$

It will be found that  $P_n^{\kappa} = 0$  and this can be used as a check on the correctness of the operations. We now define an auxiliary function of  $\lambda$

$$(8.15) \quad \left\{ \begin{array}{l} (-1)^{n-1} Q(\lambda) \stackrel{\text{def}}{=} -A \lambda^{n-1} + \\ \quad + P_1^{\kappa} \lambda^{n-2} - P_2^{\kappa} \lambda^{n-3} + \cdots + (-1)^{n-1} P_{n-2}^{\kappa} \lambda + (-1)^n P_{n-1}^{\kappa}. \end{array} \right.$$

Then we have

$$(8.16) \quad \lambda Q(\lambda) = A \text{ Det}(P - \lambda A) + P Q(\lambda).$$

Hence

$$(8.17) \quad P Q(\lambda) = \lambda Q(\lambda)$$

and this proves that  $Q_{;\lambda}^o(\lambda) v^{\lambda}$  is an eigenvector of  $P$  belonging to the eigenvalue  $\lambda$  for every choice of  $v^{\kappa}$ . This gives an easy method for the computing of eigenvectors if the auxiliary quantities (8.14) and the eigenvalues are known. But if a root  $\lambda$  has a multiplicity  $> 1$  it may

<sup>1)</sup> FETTIS 1950, 1; Souriau 1950, 1.

<sup>2)</sup> FETTIS uses  $\lambda Q$  instead of  $Q$  but this can only be done for  $\lambda \neq 0$ .

happen that  $Q(\lambda)$  vanishes and in this case the method can not be used<sup>1).</sup>

The matrix of a given quantity  $P^*\lambda$  can be brought into a canonical form by means of transformation of coordinates. We give here briefly the results of the theory of elementary divisors<sup>2).</sup>

a) For every eigenvalue  $\lambda_p$  with frequency  $r_p$  there exist  $r_p$  linear independent vectors from which  $\sigma_{p,i}$  are annihilated by  $(P - \lambda_p A)^i$  but not by  $(P - \lambda_p A)^{i-1}$ . We call them vectors of *order*  $i$  with respect to  $\lambda_p$ .  $i$  takes the values  $1, \dots, s_p$  and the numbers  $\sigma_{p,i}$  satisfy the conditions

$$(8.18) \quad \begin{cases} \sigma_{p,i} \leq \sigma_{p,(i-1)} \\ \sigma_{p,1} + \dots + \sigma_{p,s_p} = r_p. \end{cases}$$

The numbers  $r_p, \sigma_{p,i}$  are arithmetic invariants of  $P$ . Also the  $E_{\sigma_{p,1}}$  spanned by the first  $\sigma_{p,1}$  vectors, the  $E_{\sigma_{p,1} + \sigma_{p,2}}$  spanned by the first  $\sigma_{p,1} + \sigma_{p,2}$  vectors etc. are invariant but the directions in these flat spaces are not individually uniquely determined by  $P$ .

b) The  $r_p + r_q$  vectors belonging to two different eigenvalues  $\lambda_p$  and  $\lambda_q$  are linearly independent.

c) The coordinates in  $E_n$  can be chosen in such a way that the matrix of  $P$  splits up into a number of square matrices arranged along the main diagonal each of which belongs to one eigenvalue. See fig. 6 for  $n=14, r_1=5, r_2=4, r_3=2, r_4=2, r_5=1$ .

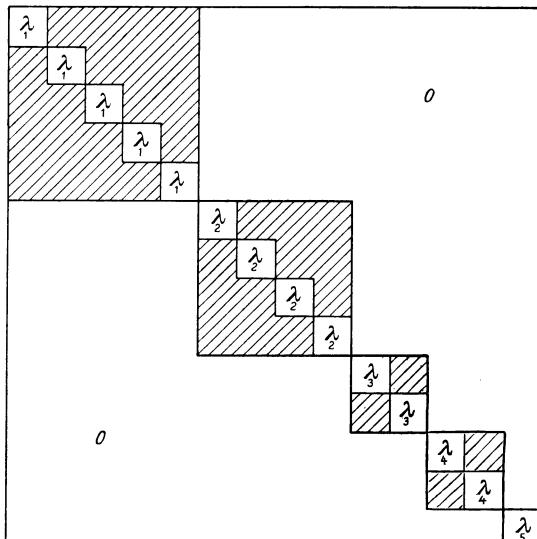


Fig. 6.

If the rank of  $P$  is  $r$ , the last square has  $n-r$  rows and only zero's in the main diagonal. The quantity  $P$  is now split up into a number of

<sup>1)</sup> It can in fact be proved that  $Q(\lambda)$  is always zero if, in the canonical form fig. 6 of the square belonging to  $\lambda_1$ , there is at least one gap in the secondary diagonal.

<sup>2)</sup> MUTH 1899, 1; BÖCHER 1910, 1, p. 282; Pascal Répertorium I, 1, p. 102; WEYL 1922, 1, p. 88; SCHREIER in KLEIN 1926, 1, p. 379ff.; DICKSON 1926, 1; VAN DER WAERDEN 1931, 1, p. 120; E I, 1935, 1, p. 38ff.

quantities in an  $E_{r_1}, E_{r_2}$ , etc. and each of these quantities has only one eigenvalue  $\lambda_1, \lambda_2$ , etc., in its own space. Hence, in order to get a further re-arrangement of the matrix of  $P$  we may suppose that  $P$  has only one eigenvalue  $\lambda$ . Then the result can be applied later to all subspaces  $E_{r_1}, E_{r_2}$  etc.

d) Let  $P$  have only one eigenvalue  $\lambda$ . For  $\sigma_{pi}$  and  $s_p$  we now write  $\sigma_i$  and  $s$  [cf. (8.18)]

$$(8.20) \quad \begin{cases} \sigma_i \leq \sigma_{i-1} \\ \sigma_1 + \cdots + \sigma_s = n. \end{cases}$$

The basis vectors in  $E_n$  can now be chosen as follows.

Let  $e^*$  be a vector of order  $s$  with respect to  $\lambda$ . That means that it is annihilated by  $(P - \lambda A)^s$  but not by  $(P - \lambda A)^{s-1}$ . Then we choose the first  $s$  basis vectors as follows

$$(8.21) \quad \begin{cases} e_1^* \\ e_2^* = (P - \lambda A)_{\lambda} e_1^* \\ \vdots \\ e_s^* = \{(P - \lambda A)^{s-1}\}_{\lambda} e_1^*. \end{cases}$$

The last vector  $e_s^*$  is an eigenvector. For  $e_{s+1}^*$  we choose another vector of order  $s$  which is linearly independent of  $e_1^*, \dots, e_s^*$  and another set of  $s$  vectors  $e_{s+1}^*, \dots, e_{2s}^*$  is constructed. This process can be repeated  $\sigma_s$  times, the first vectors of every set and also the  $\sigma_s$  vectors of all sets being linearly independent. In the  $E_{\sigma_{s-1}}$  of all vectors of order  $s-1$  there are still  $\sigma_{s-1} - \sigma_s$  vectors which are linearly independent of each other and of the  $\sigma_s$  vectors of this order already chosen. From these  $\sigma_{s-1} - \sigma_s$  vectors arise  $\sigma_{s-1} - \sigma_s$  sets of  $s-1$  vectors and among them there are exactly  $\sigma_{s-1} - \sigma_s$  vectors of each order from 1 to  $s-1$ . Proceeding in this way we get at last  $n$  linearly independent vectors and among them  $\sigma_s$  of order  $s$ ,  $\sigma_{s-1}$  of order  $s-1$  and so on down to  $\sigma_1$  vectors of order 1. It may happen that some of the differences  $\sigma_{s-1} - \sigma_s$ ,  $\sigma_{s-2} - \sigma_{s-1}$ , etc. vanish. Now if  $z_1 = s, z_2, \dots, z_{\sigma_1}$  are the numbers of vectors in the sets constructed above, in the order in which they arise we have

$$(8.22) \quad z_1 = s \geq z_2 \geq z_3 \geq \cdots \geq z_{\sigma_1}$$

and the relations between the numbers  $\sigma_1, \dots, \sigma_{\sigma_1}$  and  $z_1, \dots, z_{\sigma_1}$  can be made clear in a diagram, e.g. for  $\sigma_1 = 5, \sigma_2 = 4, \sigma_3 = 3, \sigma_4 = 1, n = 13$ :

$$(8.23) \quad \left\{ \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & \sigma_1 = 5 \\ 0 & 0 & 0 & 0 & & \sigma_2 = 4 \\ 0 & 0 & 0 & & & \sigma_3 = 3 \\ 0 & & & & & \sigma_4 = 1 \\ s = z_1 = 4 & z_2 = 3 & z_3 = 3 & z_4 = 2 & z_5 = 1 \end{array} \right.$$

The numbers  $z$  are of course also invariants of  $P$ . If the basis vectors in  $E_n$  are chosen in this way the matrix of  $P$  takes the form

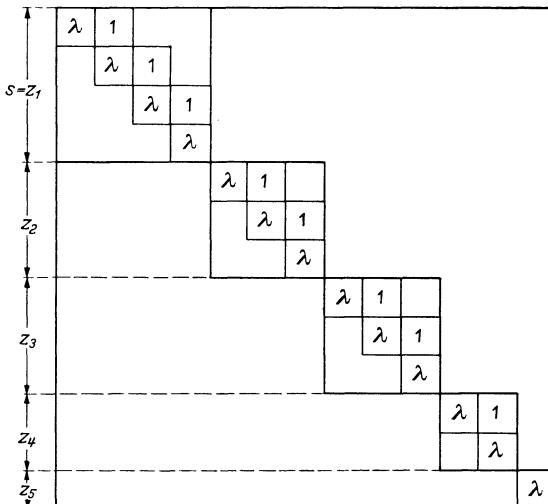


Fig. 7.

If this process is applied to each part of a general quantity  $P$ , which comes from splitting up the quantity into parts in  $E_r$ ,  $E_s$ , etc. then every square matrix in a diagram like fig. 6 can be brought into a form like fig. 7. We then get the *first canonical form* of the matrix of  $P$ .<sup>1)</sup>

The determinant  $\text{Det}(P - \lambda A)$  can be written in the form

$$(8.24) \quad \text{Det}(P - \lambda A) = \prod_p (\lambda - \lambda_p)^{z_{p1}} \dots (\lambda - \lambda_p)^{z_{p\sigma_{p1}}}.$$

The factors on the right hand side, for instance  $(\lambda - \lambda_p)^{z_{p1}}$ , are called the *elementary divisors* of the matrix of  $P$ . Obviously two mixed tensors of valence 2 are equiform (cf. I § 2) if and only if their matrices have the same canonical form, in other words if they have the same invariants  $z$ , that is the same elementary divisors.

<sup>1)</sup> Another canonical form was given by WEYL 1922, 1, p. 99.

*B. Co- and contravariant tensors.*

Every tensor  $P^{\kappa\lambda}$  of rank  $r$  can of course be written in the form

$$(8.25) \quad P^{\kappa\lambda} = \underset{1}{u^\kappa} v^\lambda + \cdots + \underset{r}{u^\kappa} v^\lambda.$$

The vectors  $u^\kappa$  span the  $\kappa$ -domain and the vectors  $v^\lambda$  the  $\lambda$ -domain. The same holds mutatis mutandis for the covariant case.

The components of

$$(8.26) \quad s! P^{[\kappa_1 | \lambda_1] \dots [P^{\kappa_s} | \lambda_s]} = s! P^{[\kappa_1 | \lambda_1] \dots P^{\kappa_s} \lambda_s} = s! P^{\kappa_1 | \lambda_1} \dots P^{[\kappa_s | \lambda_s]}$$

are the  $s$ -rowed subdeterminants of the matrix of  $P^{\kappa\lambda}$ . Hence the rank of  $P^{\kappa\lambda}$  is  $r$  if and only if the quantity (8.26) vanishes for  $s > r$  but not for  $s = r$ .<sup>1)</sup>

If  $P^{\kappa\lambda}$  is symmetric the quantity (8.26) is invariant for the operation of interchanging the set of indices  $\kappa_1, \dots, \kappa_s$  with the set of indices  $\lambda_1, \dots, \lambda_s$ . Such a quantity is called a symmetric  $s$ -vector-tensor. It plays the same role with respect to  $s$ -vectors as a symmetric tensor of valence 2 does with respect to vectors.

It is possible to define to a co- or contravariant tensor of valence 2 an *adjoint* (cf. Exerc. I 8,2). If the determinant of the matrix is not zero this adjoint is equal to the product of the determinant and the inverse. Relations like (Exerc. I 8,2  $\beta$ ) can be established for the adjoint.<sup>2)</sup> Note that  $P^{\kappa\mu} \overset{-1}{P}_{\mu\lambda} = A_\lambda^\kappa$  and that generally  $P^{\kappa\mu} \overset{-1}{P}_{\lambda\mu} \neq A_\lambda^\kappa$ .

*C. Symmetric co- and contravariant tensors.*

If we consider real and complex symmetric tensors and real and complex coordinate transformations the following theorem holds:

*If the symmetric tensor  $P_{\kappa\lambda}(Q^{\kappa\lambda})$  has rank  $r$  the coordinate system ( $\kappa$ ) can always be chosen in such a way that the matrix of  $P_{\kappa\lambda}(Q^{\kappa\lambda})$  has the diagonal form with  $r$  numbers +1 and  $n - r$  numbers 0 in the main diagonal.*

If only *real* symmetric tensors and *real* coordinate transformations are considered we have the theorem:

*If the symmetric tensor  $P_{\kappa\lambda}(Q^{\kappa\lambda})$  is **real** and of rank  $r$ , there always exists a **real** coordinate system such that the matrix of  $P_{\kappa\lambda}(Q^{\kappa\lambda})$  has the diagonal form with  $s$  numbers -1,  $r - s$  numbers +1 and  $n - r$  numbers 0 in the main diagonal.<sup>3)</sup>*

The number  $s$ , called the *index*, is an invariant of the symmetric tensor. The sequence  $-- \dots + + \dots$  with  $s$  - signs and  $(r - s)$  + signs is called the *signature*. The signature is said to be *even* (*odd*) if  $s$

<sup>1)</sup> WADE 1943, 1, calls therefore (8.26) the *rank tensors*.

<sup>2)</sup> P. P. 1949, 1, p. 28.

<sup>3)</sup> Cf. for the proof of these two theorems DICKSON 1926, 1, p. 70.

is even (odd). The symmetric tensor is called *positive (negative) definite* if  $s=0(n)$  and  $r=n$  and *indefinite* in all other cases. It is called *positive (negative) semi-definite* if  $n-r > 0$  and  $s=0(r)$ .

It follows from these theorems that two symmetric tensors are equiform for general transformations if and only if they have the same rank and that two real symmetric tensors are equiform for real transformations if and only if they have the same rank and the same index.

#### D. Co- and contravariant bivectors.<sup>1)</sup>

An antisymmetric matrix always has an even rank. Hence the rank of a bivector is always even. If  $v^{\lambda\mu}$  has the rank  $r$  it can be proved that<sup>2)</sup>

$$(8.27) \quad r! v^{[\lambda_1 \lambda_1} \dots v^{\lambda_r] \lambda_r]} = \left( \frac{r!}{2^{\frac{1}{2}r} (\frac{1}{2}r)!} \right)^2 v^{[\lambda_1 \lambda_2} \dots v^{\lambda_{r-1} \lambda_r]} v^{[\lambda_1 \lambda_2} \dots v^{\lambda_{r-1} \lambda_r]}.$$

Hence, if  $r=n$

$$(8.28) \quad \text{Det}(v^{\lambda\mu}) = \left( \frac{n!}{2^{\frac{1}{2}n} (\frac{1}{2}n)!} v^{[12} \dots v^{n-1} n]} \right)^2.$$

The same holds mutatis mutandis for a covariant bivector.

Let  $F_{\lambda\mu}$  be a bivector of rank  $r$  and let  $u^\lambda$  and  $v^\mu$  be two vectors such that

$$(8.29) \quad F_{\lambda\mu} u^\lambda v^\mu = 1.$$

Then, if

$$(8.30) \quad w_{\lambda\mu} \stackrel{\text{def}}{=} 2F_{[\lambda|\mu|} v^\mu F_{\mu]\nu} u^\nu$$

we will prove that the rank ' $r$ ' of

$$(8.31) \quad 'F_{\lambda\mu} \stackrel{\text{def}}{=} F_{\lambda\mu} - w_{\lambda\mu}$$

is  $r-2$ . Because the rank of  $F_{\lambda\mu}$  is  $r$ , there exist  $n-r$  linearly independent contravariant vectors whose transvections with  $F_{\lambda\mu}$  vanish. Because of (8.29 to 31) their transvections with ' $F_{\lambda\mu}$ ' also vanish. Hence ' $r \leq r$ '. Now, according to (8.30, 31) the transvections of  $u^\lambda$  and  $v^\mu$  with ' $F_{\lambda\mu}$ ' vanish and these vectors are linearly independent of each other and of the  $n-r$  vectors just mentioned. Hence ' $r \leq r-2$ '. Now we have

$$(8.32) \quad \begin{cases} F_{[\lambda_1 [\lambda_1} \dots F_{\lambda_r] \lambda_r]} = 'F_{[\lambda_1 [\lambda_1} \dots 'F_{\lambda_r] \lambda_r]} - r w_{[\lambda_1 [\lambda_1} 'F_{\lambda_2 \lambda_2} \dots 'F_{\lambda_r] \lambda_r]} + \\ \quad + \binom{r}{2} w_{[\lambda_1 [\lambda_1} w_{\lambda_2 \lambda_2} 'F_{\lambda_3 \lambda_3} \dots 'F_{\lambda_r] \lambda_r]} \end{cases}$$

and from this we see that the rank of  $F_{\lambda\mu}$  could not be  $r$  if the rank of ' $F_{\lambda\mu}$ ' were  $< r-2$ . Hence ' $r=r-2$ '. If this process is carried out

<sup>1)</sup> There is a great number of papers on special properties of tensors, especially of symmetric tensors and multivectors. We mention only a few: RUSE 1936, 1; WONG 1940, 1; GUREWITSCH 1950, 2; 3; 4; MOREAU 1950, 1.

<sup>2)</sup> P. P. 1949, 1, p. 28.

$\frac{1}{2}r - 1$  times we get a decomposition of  $F_{\lambda\kappa}$  into  $\frac{1}{2}r$  simple bivectors. This is called the splitting up of  $F_{\lambda\kappa}$  into  $\frac{1}{2}r$  blades.<sup>1)</sup> The same can be done with a contravariant bivector. Note that the blades are by no means uniquely determined by the bivector. If  $F_{\lambda\kappa}$  is real,  $u^\kappa$  and  $v^\kappa$  can be chosen to be real and the process can be carried out in such a way that all blades are real.

If

$$(8.33) \quad F_{\lambda\kappa} = \overset{1}{f}_{\lambda\kappa} + \cdots + \overset{\varrho}{f}_{\lambda\kappa}; \quad \varrho = \frac{1}{2}r$$

is such a decomposition, the  $r$ -dimensional domain of  $F_{\lambda\kappa}$  is spanned by the  $\frac{1}{2}r$  two-dimensional domains of the blades. Hence these domains can have no vector in common and from this it follows that

$$(8.34) \quad F_{[\lambda_1 \lambda_2} \dots F_{\lambda_{r-1} \lambda_r]} \neq 0.$$

Hence for a bivector, besides the necessary and sufficient condition

$$(8.35) \quad F_{[\lambda_1 [\lambda_2 \dots F_{\lambda_s] \kappa_s}]} \begin{cases} \neq 0 & \text{for } s = r \\ = 0 & \text{for } s > r \end{cases}$$

for the rank to be equal to  $r$ , we still have another necessary and sufficient condition<sup>2)</sup>

$$(8.36) \quad F_{[\lambda_1 \lambda_2 \dots F_{\lambda_{s-1} \lambda_s]} \begin{cases} \neq 0 & \text{for } s = r \\ = 0 & \text{for } s > r. \end{cases}$$

This also holds mutatis mutandis for a contravariant bivector.

If we start from (8.33) we can always choose the coordinate system ( $\kappa$ ) in such a way that

$$(8.37) \quad \overset{1}{f}_{\lambda\kappa} = 2 \overset{1}{e}_{[\lambda} \overset{2}{e}_{\kappa]}; \quad \overset{2}{f}_{\lambda\kappa} = 2 \overset{3}{e}_{[\lambda} \overset{4}{e}_{\kappa]}; \quad \text{etc.}$$

That proves the theorem:

*If a **real** bivector has rank  $r$ , there is always a **real** coordinate system such that the matrix of the components with respect to this system consists of exactly  $\frac{1}{2}r$  matrices of the form  $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$ , arranged along the main diagonal and zero's at all other places.<sup>3)</sup>*

<sup>1)</sup> There is a mistake in the proof in E I 1935, 1, p. 46 which is corrected here.

<sup>2)</sup> Cf. (8.27) and GRASSMANN 1844, 1, p. 206; 1862, 1, p. 56; ROTHE 1912, 1, p. 1039.

<sup>3)</sup> The results obtained under  $A$ ,  $C$  and  $D$  of this section give a possibility of classification for cocontravariant and symmetric tensors of valence 2 and for bivectors. The problem of the classification of quantities with a valence  $> 2$  is only solved in very special cases. For trivectors the classification was given for  $n = 6$  by REICHEL 1907, 1, for  $n = 7$  by SCHOUTEN 1931, 1 and for  $n = 8$  by GUREWITSCH 1935, 1. Cf. W. KÄMMERER 1927, 1; GUREWITSCH 1933, 1; 2; 1934, 1; 2; 3; 1935, 2; 1950, 1; WEITZENBÖCK 1937, 1; 2; 1938, 1; PAPY 1946, 1; PAPY and TOURNAY 1946, 2; HUTCHINSON 1948, 1.

If a covariant bivector  $F_{\lambda\kappa}$  and  $p$  linearly independent covariant vectors  $\overset{1}{u}_\lambda, \dots, \overset{p}{u}_\lambda$  are given, we may require the section of  $F_{\lambda\kappa}$  (cf. I § 4) with the  $E_{n-p}$  spanned by these  $p$  vectors. Taking  $(\varkappa)$  in such a way that the vectors are  $\overset{q+1}{e}_\lambda, \dots, \overset{n}{e}_\lambda$ ;  $q = n - p$ , the  $\xi^\alpha$ ;  $\alpha = 1, \dots, q$  can be taken as coordinates in the  $E_q$  (cf. I § 4) and we have for the connecting quantity

$$(8.38) \quad B_\beta^\kappa \stackrel{*}{=} e_\beta^\kappa; \quad \beta = 1, \dots, q$$

and for the section

$$(8.39) \quad 'F_{\beta\alpha} \stackrel{*}{=} F_{\lambda\kappa} \underset{\beta}{e}_\alpha^\lambda \underset{\kappa}{e}^\kappa \stackrel{*}{=} F_{\beta\alpha}.$$

This means that with this special choice of the coordinate system we have only to drop all components of  $F_{\lambda\kappa}$  with indices  $q+1, \dots, n$ . In order to find the rank of ' $F$ ' we have to consider the set of multivectors (8.36)

$$(8.40) \quad \begin{cases} \overset{2}{I}_{\lambda\kappa} \stackrel{\text{def}}{=} F_{\lambda\kappa} \\ \overset{4}{I}_{\lambda_1\kappa_1 \lambda_2\kappa_2} \stackrel{\text{def}}{=} F_{[\lambda_1\kappa_1} F_{\lambda_2\kappa_2]} \\ \vdots \end{cases} \quad \text{in short (cf. I § 7):} \quad \begin{cases} \overset{2}{I} \stackrel{\text{def}}{=} F \\ \overset{4}{I} \stackrel{\text{def}}{=} [FF] \\ \vdots \end{cases}$$

the last non vanishing multivector of this set being  $\overset{r}{I}$ . The multivectors  $\overset{2}{I}, \overset{4}{I}, \dots$  belonging to ' $F$ ' are the sections of  $\overset{2}{I}, \overset{4}{I}, \dots$  with the  $E_{n-p}$  and these sections can be found in the same way as ' $F$ ', i.e. by dropping all components with indices  $q+1, \dots, n$  with respect to the special coordinate system. Now consider for instance the multivector

$$(8.41) \quad [\overset{4}{I} \overset{1}{u} \dots \overset{p}{u}] = [\overset{4}{I} \overset{q+1}{e} \dots \overset{n}{e}].$$

Components of  $\overset{4}{I}$  with indices  $q+1, \dots, n$  can not occur in this quantity. Hence it is zero if and only if ' $I$ ' vanishes. This proves the following theorem:

*The section of the covariant bivector  $F_{\lambda\kappa}$  with the  $E_{n-p}$  spanned by the covariant vectors  $\overset{1}{u}_\lambda, \dots, \overset{p}{u}_\lambda$  has rank ' $r$ ' if and only if*

$$(8.42) \quad [\overset{2s}{I} \overset{1}{u} \dots \overset{p}{u}] \begin{cases} \neq 0 & \text{for } 2s = r \\ = 0 & \text{for } 2s > r. \end{cases}$$

We call  $r - r'$  the *reduction number* of the vectors  $\overset{1}{u}, \dots, \overset{p}{u}$  with respect to the bivector  $F$ .

It can be proved that the reduction number of  $\dot{u}^1, \dots, \dot{u}^p$  with respect to  $F$  is  $\varkappa$  if and only if the matrix

$$(8.43) \quad \begin{array}{c|ccc|cc} & \xleftarrow{n} & & \xleftarrow{p} & & \\ \begin{array}{c} \uparrow \\ n \\ \downarrow \\ \uparrow \\ p \\ \downarrow \end{array} & \left| \begin{array}{cccc} F_{\mu\lambda} & u_\lambda^1 & \dots & u_\lambda^p \\ -u_\mu^1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ -u_\mu^p & 0 & \dots & 0 \end{array} \right| \end{array}$$

has the rank  $R = r + 2p - \varkappa$ .<sup>1)</sup>

### Exercises.

I 8,1. (Cf. Exerc. I 1,1.) Prove that for every tensor  $P_{\cdot\lambda}^\varkappa$  of rank  $n$  (cf. Exerc. II 2,2)

$$\text{I 8,1 } \alpha) \quad \bar{P}_{\cdot\lambda}^1 = \frac{\partial \log \text{Det}(P_{\cdot\sigma}^\varrho)}{\partial P_{\cdot\varkappa}^1}.$$

I 8,2. The quantity

$$\text{I 8,2 } \alpha) \quad \overset{*}{P}_{\cdot\lambda}^\varkappa \stackrel{\text{def}}{=} \frac{\partial \text{Det}(P_{\cdot\sigma}^\varrho)}{\partial P_{\cdot\varkappa}^1}$$

is often called the *adjoint* of  $P$ . It exists always, also if  $\text{Det}(P) = 0$ . The elements of the adjoint are the minors of the elements of  $P$ . The adjoint of the adjoint of  $P$  does in general not equal  $P$ . Prove that for every tensor  $P_{\cdot\lambda}^\varkappa$  of rank  $n$  (cf. Exerc. II 2,2)

$$\text{I 8,2 } \beta) \quad \overset{*}{P}_{\cdot\lambda}^\varkappa = n \cdot n! A_{[1}^\varkappa A_{|\lambda|}^{[1} P_{\cdot 2}^2 \dots P_{\cdot n]}^n.$$

I 8,3. Prove that  $Q(\lambda)$  defined in (8.15) is the adjoint (cf. Exerc. I 8,2) of  $P - \lambda A$ .

I 8,4. If  $p, q$  and  $r$  are the ranks of  $P, Q$  and  $R = P Q$ , prove that  $r \leq p, r \leq q$  and that  $r = p$  if  $q = n$ .

I 8,5. Prove that the trace of  $P Q - Q P$  is always zero.

I 8,6. There exist always two coordinate systems  $(\varkappa)$  and  $(\varkappa')$  such that the matrix of the intermediate components  $P^{\varkappa\varkappa'}$  has the diagonal form. There exist always two tensors  $Q_{\cdot\lambda}^\varkappa, R_{\cdot\lambda}^\varkappa$  such that the matrix of  $Q_{\cdot\varrho}^\varkappa R_{\cdot\sigma}^\lambda P^{\varrho\sigma}$  has the diagonal form.

I 8,7. If  $P_{\lambda\varkappa}$  and  $Q_{\lambda\varkappa}$  satisfy the equation

$$\text{I 8,7 } \alpha) \quad P_{[\varkappa[\lambda} Q_{\mu]\nu]} = 0$$

and if the rank of  $P_{\lambda\varkappa}$  is 1 or 2, then the rank of  $Q_{\lambda\varkappa}$  is 0, 1 or 2. If the rank of  $P_{\lambda\varkappa}$  is  $> 2$ ,  $Q_{\lambda\varkappa}$  is zero.

<sup>1)</sup> Cf. P. P. 1949, 1, p. 27.

I 8,8. If  $P_{\lambda\kappa}$  is a tensor of rank  $n$  and  $D$  the determinant of its matrix, prove that

$$\text{I 8,8 } \alpha) \quad \begin{vmatrix} P_{\lambda\kappa} & u_\kappa \\ v_\lambda & 0 \end{vmatrix} = -P^{\kappa\lambda} u_\kappa v_\lambda D.$$

I 8,9.  $P^{\kappa\lambda}$  is determined to within a factor  $\sqrt[8]{+1}$  by the quantity (8.26) for  $s=1, \dots, r-1$  but not for  $s=r$ . The same holds mutatis mutandis for covariant and mixed tensors.

I 8,10. The reduction number of one vector with respect to a bivector is always 0, 1 or 2. It is zero if and only if the vector belongs to the domain of the bivector.

I 8,11. Prove the inequalities [cf. (8.43)]

$$\text{I 8,11 } \alpha) \quad \begin{cases} 0 \leq \kappa \leq r & 2p \leq R \leq r + 2p \\ 2p \geq \kappa & R \geq r \\ r - \kappa \leq n - p & R \leq n + p. \end{cases}$$

I 8,12. The  $m \leq n$  vectors  $\vec{v}_\lambda^1, \dots, \vec{v}_\lambda^m$  are linearly independent and the  $m$  vectors  $\vec{w}_\lambda^1, \dots, \vec{w}_\lambda^m$  satisfy the equation

$$\text{I 8,12 } \alpha) \quad \vec{v}_{[\lambda}^1 \vec{w}_{\mu]}^1 + \dots + \vec{v}_{[\lambda}^m \vec{w}_{\mu]}^m = 0.$$

Prove that there exist  $m^2$  coefficients  $\alpha_{uv}$  such that  $\vec{w}_\lambda^u = \alpha_{uv} \vec{v}_\lambda^v$  and that  $\alpha_{uv} = \alpha_{vu}$ ;  $u, v = 1, \dots, m$ .<sup>1)</sup>

I 8,13. If  $P_{\cdot\lambda}^\alpha$  has rank  $n$  and if  $P_{\cdot\eta}^\alpha = 0$ ;  $\alpha = 1, \dots, m$ ;  $\eta = m+1, \dots, n$  prove that

$$\text{I 8,13 } \alpha) \quad \overset{-1}{P}_{\cdot\eta}^\alpha = 0;$$

$$\text{I 8,13 } \beta) \quad \overset{-1}{P}_{\cdot\gamma}^\alpha P_{\cdot\beta}^\gamma = A_\beta^\alpha; \quad \alpha, \beta, \gamma = 1, \dots, m$$

$$\text{I 8,13 } \gamma) \quad \overset{-1}{P}_{\cdot\xi}^\xi P_{\cdot\eta}^\zeta = A_\eta^\xi; \quad \xi, \eta, \zeta = m+1, \dots, n.$$

I 8,14. An  $E_m$  is imbedded in  $E_n$ . For the tensor  $P_{\mu\lambda}$  of  $E_n$  the rank of  $B_c^\mu P_{\mu\lambda}$  is  $r$  and the rank of  $B_{cb}^\mu P_{\mu\lambda}$  is  $r'$ . The support of the  $\lambda$ -domain of  $B_c^\mu P_{\mu\lambda}$  intersects  $E_m$  in an  $E_s$ . Prove that,

$$\text{I 8,14 } \alpha) \quad r - (n - m) \leq r' = m - s \leq r.$$

I 8,15<sup>2)</sup>. If  $c_{\lambda\kappa} = a_{\lambda\kappa} + i b_{\lambda\kappa}$ , where  $a_{\lambda\kappa}$  and  $b_{\lambda\kappa}$  are real and symmetric,  $c_{\lambda\kappa}$  has a rank  $< n$  and  $a_{\lambda\kappa}$  is positive definite or positive semi-definite, then there exists a real vector  $s^\kappa$  such that  $c_{\lambda\kappa} s^\kappa = 0$ .

<sup>1)</sup> CARTAN 1949, 1, p. 151.

<sup>2)</sup> PEREMANS, DUPARC and LEKKERKERKER 1952, 1; QUADE 1953, 1.

### § 9. Introduction of a metric in an $E_n$ .

In an  $E_n$  the magnitude of two contravariant vectors can only be compared if they are parallel. If we wish to establish a comparison of magnitudes of non-parallel vectors we have to introduce a metric, i.e. a definition of "length". This definition must satisfy the conditions that the length of a real vector is always positive and that multiplication of a vector by a real number  $k$  alters its length by a factor  $|k|$ . Hence, if  $F(\xi^x - \xi^x)$  is the *length* of  $\xi^x - \xi^x$  or the *distance* of the points  $\xi^x$  and  $\xi^x$ , we have

$$(9.1) \quad F\left(k\left(\begin{smallmatrix} \xi^x & - \\ 2 & 1 \end{smallmatrix}\right)\right) = |k| F\left(\begin{smallmatrix} \xi^x & - \\ 2 & 1 \end{smallmatrix}\right).$$

We assume that the function  $F$  is single valued and continuous and that it has continuous derivatives up to a certain order. The equation

$$(9.2) \quad F\left(\begin{smallmatrix} \xi^x & - \\ 0 & 1 \end{smallmatrix}\right) = 1$$

represents the *indicatrix* of the metric with respect to the point  $\xi^x$ .

In a centred  $E_n$  we have  $\xi^x = 0$  and  $\xi^x$  is the radius vector. According to (9.1) the indicatrix is symmetric with respect to the point  $\xi^x$ . To every direction there belongs one and only one  $(n-1)$ -direction viz. the tangent  $(n-1)$ -direction of the indicatrix at its section with the straight line through  $\xi^x$  in the given direction. If the equation of the indicatrix is algebraic, its degree is even because  $\xi^x$  must be the centre.

The projection of a vector  $u^x$  on a vector  $v^x$  can now be defined as the projection in the  $(n-1)$ -direction that belongs to the direction of  $v^x$ . If  $u$  is the length of  $u^x$  and ' $u$ ' the length of the projection, the angle between  $u^x$  and  $v^x$  (in this order) can be defined by

$$(9.3) \quad \cos(u^x, v^x) \stackrel{\text{def}}{=} \frac{'u}{u} \cdot ^1)$$

But in general  $\cos(u^x, v^x) \neq \cos(v^x, u^x)$ .

If we wish to define a "rotation" in a centred  $E_n$  in a natural way we will have to define it as a linear homogeneous transformation which leaves invariant both an  $n$ -dimensional screwsense and the indicatrix.<sup>2)</sup> That excludes all forms of indicatrix which are not invariant for some subgroup of  $G_{ho}$ . This subgroup must be chosen big enough to guarantee some freedom of "motion". For instance it could be required that

<sup>1)</sup> FINSLER 1918, 1, p. 39.

<sup>2)</sup> Cf. WEYL 1921, 1, p. 125 ff.

an  $E_1$  through the centre, an  $E_2$  through  $E_1$ , an  $E_3$  through  $E_2$ , etc. being given, there always exists a rotation turning  $E_1$  in a given direction,  $E_2$  in a 2-direction containing this direction, etc. Then it can be proved<sup>1)</sup> that the only possible indicatrix is a non-degenerate real quadratic hypersurface. Following another line of thought it would be possible to require that the angle between two directions be independent of their order. This condition also leads to a real quadratic indicatrix.<sup>2)</sup>

Accordingly, let us take the equation of the indicatrix in the form

$$(9.4) \quad g_{\lambda s} \left( \frac{\xi^\lambda - \xi^s}{\partial} \right) \left( \frac{\xi^s - \xi^\lambda}{\partial} \right) = \pm 1$$

where  $g_{\lambda s}$  is a real symmetric tensor of rank  $n$  and index  $s \leq n$ .  $g_{\lambda s}$  and its inverse  $g^{s\lambda}$  are then called the *covariant* and *contravariant fundamental tensor*. An  $E_n$  with a symmetric fundamental tensor is called an  $R_n$  and is said to be *ordinary* if  $g_{\lambda s}$  is positive definite. The geometry in  $R_n$  is called *euclidean* if  $g_{\lambda s}$  is *definite* and *minkowskian* if  $g_{\lambda s}$  is *indefinite*. The *length* of a vector  $v^s$

$$(9.5) \quad \sqrt{|g_{\lambda s} v^\lambda v^s|}$$

is always positive or zero but  $g_{\lambda s} v^\lambda v^s$  can be  $> 0$  or  $< 0$  or zero. In the first case  $v^s$  is called *time-like* and in the second *space-like*.<sup>3)</sup> The time-like vectors through the origin fill the *+ -region* and the space-like vectors the *- -region* of a centred  $R_n$ . The vectors with zero length fill the *nullcone*. A real nullcone only exists in the indefinite case  $0 < s < n$ . A vector with length  $\pm 1$  is called a *unitvector*. The *angle* between two *real* vectors, both of which lie in the *+ -region* or both in the *- -region* is defined by

$$(9.6) \quad \cos(u^s, v^s) = \cos(v^s, u^s) \stackrel{\text{def}}{=} \frac{\pm g_{\lambda s} u^\lambda v^\lambda}{\sqrt{|g_{\theta\sigma} u^\theta u^\sigma g_{\tau\omega} v^\tau v^\omega|}} \quad \text{for the } \pm\text{-region.}$$

The possible values of the cos depend on the form of the section of the plane of the vectors with the nullcone:

$$(9.7) \quad \left\{ \begin{array}{ll} \text{Section} & \cos \\ \begin{array}{l} 2 \text{ real } E_1 \text{'s} \\ 1 \text{ real } E_1 \\ 2 \text{ imaginary } E_1 \text{'s} \end{array} & \left\{ \begin{array}{ll} \geq +1 \\ \leq -1 \\ \pm 1 \\ \leq +1 \\ \text{and} \\ \geq -1. \end{array} \right. \end{array} \right.$$

<sup>1)</sup> HELMHOLZ 1868, 1; LIE 1890, 1; WEYL 1921, 1, p. 26.

<sup>2)</sup> FINSLER 1918, 1, p. 40.

<sup>3)</sup> The space of special relativity is an  $R_4$  with signature  $-- - +$  and a minkowskian metric. For time-like vectors the term *duration* (Dauer) is sometimes used instead of the term length.

If we take a unitvector as first vector we see from this that the length of its projection on the other vector is  $\geq 1$ ,  $= 1$  and  $\leq 1$  in these three cases. In the first case the geometry in the plane is minkowskian and in the third case it is euclidean. In the second case the plane is not an  $R_2$  because it does not contain a symmetric tensor of rank 2. Such an  $E_2$  in  $R_n$  is called *isotropic*.

An  $E_p$  in  $R_n$  is called *isotropic* if the symmetric tensor of its metric has a rank  $< p$  and *full isotropic* or a *null*  $E_p$  if this rank is zero.

Any two vectors  $u^*$  and  $v^*$  for which  $g_{\lambda\mu} u^\lambda v^\mu = 0$  are said to be *mutually perpendicular*. With this exception the angle between a vector in the + -region and a vector in the -- -region is not defined.

An  $E_p$  and an  $E_q$  in  $R_n$  are mutually perpendicular if every direction in  $E_p$  is perpendicular to every direction of  $E_q$ . All directions perpendicular to  $E_p$  span an  $E_{n-p}$ . The section of  $E_p$  and this  $E_{n-p}$  is an  $E_t$  with the property that all its directions are mutually perpendicular. Such an  $E_t$  is a null- $E_t$ . Its  $t$ -direction lies in the nullcone. A null- $E_t$  can never be an  $R_t$ . For  $t > 0$  the  $E_p$  must be isotropic because it contains a direction perpendicular to all its directions.

By using the second theorem of I § 8.C, a real coordinate system  $(h)$ ;  $h = 1, \dots, n$  may be introduced such that

$$(9.8) \quad \begin{cases} \text{a)} g_{11} = -1; \dots; g_{ss} = -1; g_{s+1s+1} = +1; \dots; g_{nn} = +1; g_{ih} = 0 \\ \text{b)} g^{11} = -1; \dots; g^{ss} = -1; g^{s+1s+1} = +1; \dots; g^{nn} = +1; g^{hi} = 0 \end{cases} \text{ for } h \neq i.$$

Such a coordinate system is called *cartesian*. We write  $i^h$ ,  $i_i$  for its basis vectors. These vectors are mutually perpendicular unitvectors. We always use *roman* running indices and *vertical* fixed indices for cartesian coordinate systems in  $R_n$ .

Fig. 8 shows the minkowskian geometry in two dimensions. The signature is  $-+$ . The indicatrix consists of the hyperbolas

$$(9.9) \quad -(x^1)^2 + (x^2)^2 = \pm 1$$

or in parametric form

$$(9.10) \quad x^1 = \cosh \varphi; \quad x^2 = \sinh \varphi; \quad x^1 = \sinh \varphi; \quad x^2 = \cosh \varphi.$$

The fact that, if the figure is looked at from a euclidean point of view, the 1-axis seems to be perpendicular to the 2-axis and that  $i_1$  and  $i_2$  seem to have the same length, has no significance at all. The whole figure could be subjected to an affine transformation and would then represent just the same thing. In a minkowskian metric, length is fixed by the indicatrix, hence  $i_1$ ,  $i_2$ ,  $i_1'$ , and  $i_2'$  all have the same length.

The area of any part of the plane happens to be the same for both metrics. The angle  $\varphi$  is real and  $\frac{1}{2}\varphi$  is the shaded area. According to (9.6) we have

$$(9.11) \quad \cos(i_1, i_1) = \cosh \varphi \geq 1$$

for the angle between 1-axis and 1'-axis. The vectors  $i_1$  and  $i_2$  are perpendicular to each other and so are  $i_1'$  and  $i_2'$ . But the angle between arbitrary directions that do not belong to the same region is not defined.

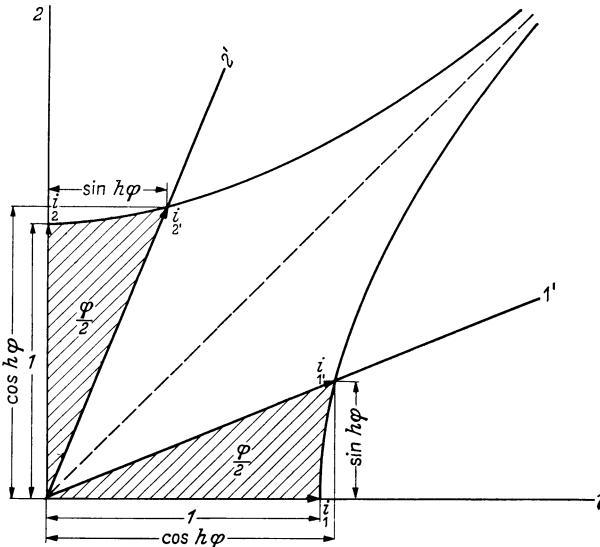


Fig. 8.

The + -region and the -- -region each consist of two parts. Every orthogonal transformation leaves each region invariant as a whole, but it may happen that it interchanges its parts. So we have four different kinds of orthogonal transformations in a minkowskian  $R_2$

|        | + -region           | -- -region       | $\Delta$ | example                                        |
|--------|---------------------|------------------|----------|------------------------------------------------|
| (9.12) | A) not interchanged | not interchanged | +1       | $i_1 \rightarrow i_1'; i_2 \rightarrow i_2'$   |
|        | B) not interchanged | interchanged     | -1       | $i_1 \rightarrow -i_1'; i_2 \rightarrow i_2'$  |
|        | C) interchanged     | not interchanged | -1       | $i_1 \rightarrow i_1'; i_2 \rightarrow -i_2'$  |
|        | D) interchanged     | interchanged     | +1       | $i_1 \rightarrow -i_1'; i_2 \rightarrow -i_2'$ |

The transformations (A) form a group and are called *proper minkowskian rotations* and in the theory of relativity, *LORENTZ transformations*. (B) and (C) are *minkowskian reflexions* and (D) are *improper*

*minkowskian rotations.* They do not form a group neither separately nor together. But (A), (B), (C) and (D) together form the group of all minkowskian orthogonal transformations.

For all values of  $n$ , the fundamental tensor establishes a one to one correspondence between co- and contravariant vectors

$$(9.13) \quad v_\lambda = g_{\lambda\kappa} v^\kappa; \quad v^\kappa = g^{\kappa\lambda} v_\lambda.$$

We use the same kernel for corresponding vectors and consider  $v^\kappa$  and  $v_\lambda$  as two different sets of components of the *same* quantity, called *vector*, that can be represented geometrically either by an arrow or by an oriented set of two parallel hyperplanes.<sup>1)</sup> The distance between the two hyperplanes is the inverse of the length of  $v^\kappa$ . The orientations of  $v^\kappa$  and  $v_\lambda$  are equal if  $v^\kappa$  lies in the  $+$ -region and opposite if  $v^\kappa$  lies in the  $--$ -region. The process (9.13) is called *raising* and *lowering of indices*.<sup>2)</sup> It can also be applied to one or more indices of a quantity of valence  $> 1$ , e.g.

$$(9.14) \quad P_{\cdot\lambda}^{\theta\kappa} = g^{\theta\kappa} P_{\kappa\lambda}; \quad P_{\kappa\lambda\sigma} = P_{\kappa\lambda}^{\cdot\sigma} g_{\sigma\sigma}.$$

The relation

$$(9.15) \quad g_{\lambda\kappa} = g_{\lambda\eta} g_{\kappa\sigma} g^{\eta\sigma}$$

justifies the use of the same kernel in  $g_{\lambda\kappa}$  and  $g^{\kappa\lambda}$ .

Because  $i_j^\kappa$  and  $i_\lambda^j$  have the same orientation for every value of  $j$ , we have

$$(9.16) \quad i_1^\kappa = -i^\kappa; \dots; i_s^\kappa = -i^\kappa; i_{s+1}^\kappa = +i^\kappa; \dots; i_n^\kappa = +i^\kappa$$

and accordingly

$$(9.17) \quad v^h = \begin{cases} -v_h & h = 1, \dots, s \\ +v_h & h = s+1, \dots, n. \end{cases}$$

The length of  $v^\kappa$  in cartesian coordinates is

$$(9.18) \quad |V(-v^1 v^1 - \dots - v^s v^s + v^{s+1} v^{s+1} + \dots + v^n v^n)|$$

and for the equation of the nullcone in these coordinates we get

$$(9.19) \quad -x^1 x^1 - \dots - x^s x^s + x^{s+1} x^{s+1} + \dots + x^n x^n = 0. \text{ } ^3)$$

<sup>1)</sup> Identifications of this kind and their influence on the choice of the coefficients  $\alpha, \beta$  in I § 7 are dealt with elaborately in T. P. 1951, 1, Ch. III.

<sup>2)</sup> This process will never be applied to geometric objects which are not quantities. Cf. footnote 2 on p. 10.

<sup>3)</sup> If cartesian coordinates are used we often write  $x$  instead of  $\xi$  for the kernel of the radius vector in a centred  $R_n$ .

For the  $n$ -vectors belonging to a cartesian coordinate system  $(h)$  we use the kernels  $I$  and  $i$  (cf. 7.5)

$$(9.20) \quad \begin{cases} I^{x_1 \dots x_n} \stackrel{(h)}{=} n! i^{[x_1 \dots x_n]}; & I^{1 \dots n} = +1 \\ i_{\lambda_1 \dots \lambda_n} \stackrel{(h)}{=} n! i_{[\lambda_1 \dots \lambda_n]}; & i_{1 \dots n} = +1. \end{cases}$$

They change sign if we pass from a right hand system to a left hand system and are invariant if  $(h)$  and  $(h')$  have the same screwsense. Geometrically they are represented by a *unit* volume with the screwsense of  $(h)$ . The component  $i_{1 \dots n}$  may be considered as the component of a scalar  $\Delta$ -density  $\tilde{i}$  of weight  $+1$

$$(9.21) \quad i[\kappa] = i_{1 \dots n} \stackrel{(h)}{=} \frac{I}{I^{1 \dots n}}.$$

If  $\tilde{\omega}$  is a  $W$ -scalar with the component  $+1$  with respect to all rectilinear coordinate systems with the same screwsense as  $(h)$ , the product

$$(9.22) \quad i \stackrel{\text{def}}{=} \tilde{\omega} \tilde{i}$$

is a scalar density of weight  $+1$  with a component  $+1$  with respect to *all* cartesian coordinate systems. This latter density is intimately connected with  $g_{\lambda x}$ . In fact

$$(9.23) \quad g[\kappa] \stackrel{\text{def}}{=} |\text{Det}(g_{\lambda x})| = n! |g_{[1 \dots n][1 \dots n]}|$$

is a scalar density of weight  $+2$  and  $g[h] = +1$ , hence

$$(9.24) \quad i = |g|^{\frac{1}{2}}.$$

This density has no screwsense and this is in accordance with the fact that  $g_{\lambda x}$  fixes length and angles but not a screwsense.

After introduction of a fundamental tensor we may ask whether any vectors exist which satisfy the equation

$$(9.25) \quad T_{\lambda x} v^x = \sigma v_\lambda$$

where  $T_{\lambda x}$  is some given *real* symmetric tensor and  $\sigma$  a suitably chosen scalar. If we write

$$(9.26) \quad (T_{\lambda x} - \sigma g_{\lambda x}) v^x = 0$$

we see that the rank of  $T_{\lambda x} - \sigma g_{\lambda x}$  must be  $< n$ . Hence  $\sigma$  must be a solution of

$$(9.27) \quad \text{Det}(T_{\lambda x} - \sigma g_{\lambda x}) = 0.$$

This algebraic equation of degree  $n$  in  $\sigma$  is the so-called S-equation, well known from the theory of quadrics in ordinary  $R_3$ . If  $g_{\lambda \kappa}$  is *definite* it can be proved that the solutions of (9.27) are all real.<sup>1)</sup> If  $\sigma_1$  and  $\sigma_2$  are two *different* solutions and  $\begin{smallmatrix} v^{\lambda} \\ 1 \end{smallmatrix}$  and  $\begin{smallmatrix} v^{\lambda} \\ 2 \end{smallmatrix}$  vectors satisfying (9.26) for  $\sigma = \sigma_1$  and  $\sigma = \sigma_2$ , we have

$$(9.28) \quad \begin{smallmatrix} v^{\lambda} \\ 1 \end{smallmatrix} T_{\lambda \kappa} \begin{smallmatrix} v^{\lambda} \\ 2 \end{smallmatrix} = \sigma_1 \begin{smallmatrix} v^{\lambda} \\ 1 \end{smallmatrix} g_{\lambda \kappa} \begin{smallmatrix} v^{\lambda} \\ 2 \end{smallmatrix} = \sigma_2 \begin{smallmatrix} v^{\lambda} \\ 2 \end{smallmatrix} g_{\lambda \kappa} \begin{smallmatrix} v^{\lambda} \\ 2 \end{smallmatrix};$$

hence  $\begin{smallmatrix} v^{\lambda} \\ 1 \end{smallmatrix}$  and  $\begin{smallmatrix} v^{\lambda} \\ 2 \end{smallmatrix}$  are perpendicular to each other. If  $\sigma$  is a solution of (9.27) with multiplicity  $m$  it can be proved that the vectors satisfying (9.26) for  $\sigma = \sigma_1$  are the vectors that lie in a definite real  $m$ -direction called the *principal multidirection* belonging to  $\sigma$ , and that principal multidirections belonging to different solutions are mutually perpendicular. If the contravariant basis vectors of a cartesian coordinate system  $(h)$  are chosen in these multidirections, we have

$$(9.29) \quad T_{ih} * \begin{smallmatrix} i^{\lambda} \\ i \end{smallmatrix} \begin{smallmatrix} i^{\lambda} \\ h \end{smallmatrix} T_{\lambda \kappa} = 0 \quad \text{for } i \neq h$$

and this proves the

*Theorem of principal axes of a symmetric tensor in  $R_n$* <sup>2)</sup>:

If  $T_{\lambda \kappa}$  is a **real** symmetric tensor and if the fundamental tensor is **definite**, the realways exists a **real** cartesian coordinate system  $(h)$  such that

$$(9.30) \quad T_{ih} * 0; \quad h \neq i.$$

The axes of this coordinate system are called *principal axes* of the symmetric tensor. They are uniquely determined if and only if the solutions of the S-equation are all different.

There is a corresponding theorem for *real* bivectors.<sup>1)</sup>

*Theorem of principal blades of a bivector in  $R_n$* <sup>3)</sup>:

If  $F^{\lambda \kappa}$  is a **real** bivector of rank  $r$  and if the fundamental tensor is **definite** there always exists a **real** cartesian system  $(h)$  such that all components  $F^{hi}$  vanish except

$$(9.31) \quad F^{12} = -F^{21}; \quad F^{34} = -F^{43}; \dots; \quad F^{r-1, r} = -F^{r, r-1}.$$

The  $\frac{1}{2}r$  simple bivectors with the 2-directions  $12; 34; \dots; r-1, r$  are called the *principal blades* of the bivector. They are uniquely determined if and only if the components  $F^{12}, F^{34}, \dots, F^{r-1, r}$  all have different absolute values.

<sup>1)</sup> Cf. e.g. DICKSON 1926, 1.

<sup>2)</sup> See p. 34.

<sup>3)</sup> See p. 36.

The theorems of principal axes and of principal blades do not hold if the fundamental tensor is indefinite because in this case a real symmetric tensor or bivector may possibly have a special position with respect to the real nullcone. But if the symmetric tensor or bivector does not have any such a special position the theorems remain valid.

Orthogonal *coordinate* transformations are transformations from one cartesian coordinate system  $(h)$  to another  $(h')$ . Because

$$(9.32) \quad A_{h'i'}^{hi} g_{hi} = g_{h'i'}$$

we have

$$(9.33) \quad A_i^i g_{hi} = A_h^h g_{h'i'}$$

and, because  $A_{i'}^i = i^i$ ;  $A_h^{h'} = i^{h'}$  this leads to the reciprocal relation between the covariant (contravariant) components of the contravariant (covariant) basis vectors of both systems

$$(9.34) \quad i_h^i = i_{i'}^i; \quad i^h = i^{i'}.$$

Orthogonal *point* transformations in a centred  $R_n$  are those that leave the fundamental tensor invariant. Hence if  $T_{\cdot\lambda}^\alpha$  is such a transformation we have

$$(9.35) \quad T_{\cdot\sigma}^\lambda g^{\alpha\sigma} = T_{\cdot\sigma}^{-1} g^{\alpha\lambda}$$

hence

$$(9.36) \quad T^{\lambda\alpha} = T^{\alpha\lambda}; \quad T_{\lambda\alpha} = T_{\alpha\lambda}^{-1}$$

from which we see that  $\text{Det}(T_{\cdot\lambda}^\alpha) = \pm 1$ . Orthogonal point transformations are called *rotations (reflexotations)* if  $\text{Det}(T_{\cdot\lambda}^\alpha)$  is equal to  $+1$  ( $-1$ ).

An infinitesimal point transformation is a transformation of the form  $A_\lambda^\alpha + F_{\cdot\lambda}^\alpha dt$ . Its inverse is  $A_\lambda^\alpha - F_{\cdot\lambda}^\alpha dt$ . Hence an infinitesimal point transformation is orthogonal if and only if  $F^{\alpha\lambda}$  is a *bivector*. Of course such a transformation is always a rotation. If the fundamental tensor is definite,  $F^{\alpha\lambda}$  can be split up into  $\frac{1}{2}r$  mutually perpendicular blades,  $r$  being the rank of  $F^{\alpha\lambda}$ . From this it follows that in this case the transformation consists of  $\frac{1}{2}r$  rotations in the  $\frac{1}{2}r R_2$ 's of the blades, leaving these  $R_2$ 's and the  $R_{n-r}$ , perpendicular to them invariant. The  $R_{n-r}$  plays the part of an "axis of rotation". Hence, if  $n$  is odd, there is always such an "axis".

In the indefinite case with index  $s$  we consider two real flat manifolds  $R_s$  and  $'R_s$  through the origin  $O$  both of which lie wholly in the  $--$ -region and who intersect in an  $R_z$ . Let a hypersphere with radius  $I$

and centre in  $O$  in  $R_s$  be projected orthogonally on  $'R_s$ . Then we get a quadratic hypersurface in  $'R_s$ . It always has a set of  $s$  principal axes and these are the projections of  $s$  mutually perpendicular unitvectors in  $R_s$  exactly  $z$  of which lie in  $R_z$ . The  $s$  angles between these unitvectors and their projections are called the *principal angles* between  $R_s$  and  $'R_s$ . Exactly  $z$  of them are zero. We now choose  $(h)$  in such a way that  $i_1^x, \dots, i_s^x$  lie in the direction of the  $s$  unitvectors and  $i_1^x, \dots, i_z^x$  lie in  $R_z$ . Then if  $p_a^x, a = 1, \dots, s$  is the projection of  $i_a^x$  we have the equations

$$(9.37) \quad \frac{i_a^x}{a} = \frac{p_a^x}{a} + \frac{q_a^x}{a}; \quad a = 1, \dots, s$$

where  $q_1^x, \dots, q_z^x$  are zero and  $q_{z+1}^x, \dots, q_s^x$  are perpendicular to  $'R_s$  and consequently lie in the  $+-$ -region. Now all non-vanishing vectors  $\frac{p_a^x}{a}$  are mutually perpendicular and also perpendicular to all vectors  $\frac{q_a^x}{a}$ . Hence, according to (9.37)

$$(9.38) \quad 0 = \left( \frac{p_a^x}{a} + \frac{q_a^x}{a} \right) \left( \frac{p_b^x}{b} + \frac{q_b^x}{b} \right) = \frac{p_a^x}{a} \frac{q_b^x}{b}; \quad a, b = 1, \dots, s; a \neq b$$

and from this it follows that the  $s - z$  planes of projection of  $i_{z+1}^x, \dots, i_s^x$  are mutually perpendicular. All these planes have a minkowskian geometry and the length of  $\frac{p_a^x}{a}$  is  $|\cos \varphi| \geq 1$ , where  $\varphi$  is the angle between  $\frac{p_a^x}{a}$  and  $i_a^x$ . Hence the principal angles between  $R_s$  and  $'R_s$  are all imaginary or zero and  $R_s$  can be transformed into  $'R_s$  by a continuous rotation consisting of  $s - z$  real minkowskian rotations in the  $s - z$  planes of projection of  $i_{z+1}^x, \dots, i_s^x$ . Note that during this rotation  $R_s$  remains in the  $--$ -region and that  $i_a^x$  is transformed into  $\frac{p_a^x}{a} \cos^{-1} \varphi$ .

If originally an  $s$ -dimensional screwsense is given in  $R_s$  the process of orthogonal projection fixes a screwsense in  $'R_s$ . We call two screwsenses in  $R_s$  and  $'R_s$  *the same* if each is the orthogonal projection of the other and *opposite* if not. But we still have to prove that if  $R_s$ ,  $'R_s$  and  $"R_s$  all lie in the  $--$ -region and each is provided with a screwsense and if the screwsenses of  $R_s$  and  $'R_s$  and also those of  $'R_s$  and  $"R_s$  are the same, then the screwsenses of  $R_s$  and  $"R_s$  are also the same. The screwsenses in  $R_s$  and  $"R_s$  are both orthogonal projections of the screwsense in  $'R_s$ . But as we have seen already,  $R_s$  can be transformed into  $"R_s$  by a real rotation. Since this process is continuous the projection of the screwsense in  $'R_s$  on the moving  $R_s$  during its way to  $"R_s$  can never change except if there is a moment where there is no screwsense

at all, i.e. where  $R_s$  contains a direction perpendicular to ' $R_s$ '. But at that moment  $R_s$  would contain a direction in the  $+$ -region. This is impossible. Hence it is possible to fix a screwsense in all real  $R_s$ 's in the  $--$ -region simultaneously by fixing a screwsense in one of them. The same can of course be proved for the real  $R_{n-s}$ 's in the  $++$ -region.

Now let ' $\overset{\infty}{i}_1^x, \dots, \overset{\infty}{i}_s^x$ ' be unitvectors in the direction and with the sense of ' $\overset{\infty}{p}_1^x, \dots, \overset{\infty}{p}_s^x$ :

$$(9.39) \quad \overset{\infty}{i}_a^x |\cos \varphi| = \overset{\infty}{p}_a^x; \quad a = 1, \dots, s.$$

We know that

$$(9.40) \quad s! \overset{\infty}{i}_1^{[x_1} \dots \overset{\infty}{i}_s^{x_s]} \overset{\infty}{i}_{1[x_1} \dots \overset{\infty}{i}_{s]x_s]} = (-I)^s$$

and from this and (9.37) it follows that

$$(9.41) \quad \left\{ \begin{array}{l} s! \overset{\infty}{i}_1^{[x_1} \dots \overset{\infty}{i}_s^{x_s]} \overset{\infty}{i}_{1[x_1} \dots \overset{\infty}{i}_{s]x_s]} \\ = s! \overset{\infty}{p}_1^{[x_1} \dots \overset{\infty}{p}_s^{x_s]} \overset{\infty}{i}_{1[x_1} \dots \overset{\infty}{i}_{s]x_s]} = (-I)^s |\cos \varphi_{z+1} \dots \cos \varphi_s| \end{array} \right.$$

and in this formula instead of ' $\overset{\infty}{i}_a^x$ ' and ' $\overset{\infty}{i}_a^x$ ' we may take two arbitrarily chosen sets of  $s$  mutually perpendicular unitvectors in  $R_s$  and ' $R_s$ ', provided that the screwsenses fixed by their orders are the same in the sense defined above.

Now let a coordinate system  $(h)$  be chosen in such a way that ' $\overset{\infty}{i}_1^x, \dots, \overset{\infty}{i}_s^x$ ' lie in an arbitrary way in  $R_s$  and let there be another coordinate system  $(h')$  such that ' $\overset{\infty}{i}'_1^x, \dots, \overset{\infty}{i}'_{s'}^x$ ' lie in an arbitrary way in ' $R_s$ '. Then the transvection

$$(9.42) \quad (-I)^s s! \overset{\infty}{i}_1^{[x_1} \dots \overset{\infty}{i}_s^{x_s]} \overset{\infty}{i}_{1[x_1} \dots \overset{\infty}{i}_{s]x_s]}$$

is  $\geq 1$  ( $\leq -1$ ) if the two screwsenses fixed in  $R_s$  and ' $R_s$ ' by the order of these two sets of unitvectors are the same (opposite).

If  $P_{\lambda}^x$  represents the real point transformation belonging to the coordinate transformation  $(h) \rightarrow (h')$ , i.e. the transformation transforming ' $\overset{\infty}{i}_1^x, \dots, \overset{\infty}{i}_s^x$ ' into ' $\overset{\infty}{i}'_1^x, \dots, \overset{\infty}{i}'_{s'}^x$ ', we have

$$(9.43) \quad P_{\lambda}^x \overset{\infty}{i}_i^x = \overset{\infty}{i}'_{i'}^x; \quad i = 1, \dots, n; \quad i' = 1', \dots, n'$$

or

$$(9.44) \quad P_{\lambda}^h \overset{*}{i}_i^h = \overset{*}{i}'_{i'}^h = A_{i'}^h.$$

Accordingly (9.41) can now be written as

$$(9.45) \quad \left\{ \begin{array}{l} (-I)^s s! i_1^{[j_1} \dots i_s^{j_s]} P_{\cdot j_1}^{h_1} \dots P_{\cdot j_s}^{h_s} i_{h_1} \dots i_{h_s} \\ = s! i_1^{[j_1} \dots i_s^{j_s]} P_{\cdot j_1}^{h_1} \dots P_{\cdot j_s}^{h_s} i_{h_1} \dots i_{h_s} \\ * s! P_{\cdot [1}^{[1} \dots P_{\cdot s]}^{s]} = \text{Det}(P_{\cdot b}^a); \quad a, b = 1, \dots, s. \end{array} \right.$$

Hence the screwsense in  $R_s$  and its transform in  $'R_s$  are the same (opposite) if and only if  $\text{Det}(P_{\cdot b}^a) \geq +1 (\leq -1)$ . But we still have to prove that if, instead of  $R_s$ , we take another  $R_s$  in the --region not in any way connected with the coordinate system  $(h)$ ,  $\text{Det}(P_{\cdot b}^a) \geq 1$  is also the necessary and sufficient condition for the screwsense in this new  $R_s$  to be the same as its transform. Let all  $R_s$ 's in the --region be provided with a screwsense and let all these screwsenses be the same in the sense defined before. Now let  $P_{\cdot \lambda}^x$  be applied to all these  $R_s$ 's and let there be, among all transforms, one with the same and one with the opposite screwsense. Because every  $R_s$  in the --region can be transformed into every other  $R_s$ , with the same screwsense, in that region by a continuous rotation, there must be a transform where the screwsense changes and this transform could not have any screwsense at all which is impossible. Hence we may state the theorem<sup>1)</sup>:

*If  $P_{\cdot \lambda}^x$  represents a real orthogonal transformation in an  $R_n$  with index  $s$  and if a real coordinate system  $(h)$  is chosen in such a way that  $i_1^{x_1}, \dots, i_s^{x_s}$  lie in the --region and the other unitvectors in the +-region,  $P_{\cdot \lambda}^x$  transforms every real  $R_s$ , with a given screwsense in the --region into an  $R_s$  with the same (opposite) screwsense in that region, if the subdeterminant  $\text{Det}(P_{\cdot b}^a); a, b = 1, \dots, s$  is  $\geq +1 (\leq -1)$  and every real  $R_{n-s}$  with a given screwsense in the +-region, into an  $R_{n-s}$  with the same (opposite) screwsense in that region if the subdeterminant  $\text{Det}(P_{\cdot y}^x); x, y = s+1, \dots, n$  is  $\geq +1 (\leq -1)$ .*

If we call a real orthogonal transformation *--reflexional (+-reflexional)* if it changes the screwsense of a real  $R_s (R_{n-s})$  in the --region (+-region), we have the following possibilities for real orthogonal transformations in  $R_n$

- A) non +-reflexional; non --reflexional: *proper rotation*,  $\Delta = +1$
- B) non +-reflexional; --reflexional: *reflexotation*,  $\Delta = -1$
- C) +-reflexional; non --reflexional: *reflexotation*,  $\Delta = -1$
- D) +-reflexional; --reflexional: *improper rotation*,  $\Delta = +1$ .

The group of all real orthogonal transformations splits up into four sets of transformations. Obviously a transformation of one set cannot be transformed by a continuous proper rotation into a trans-

<sup>1)</sup> Cf. BRAUER and WEYL 1935, 1; GIVENS 1940, 1 gave an elementary proof.

formation of another set. From these sets only the set ( $A$ ) and the four sets together form a group.

### Exercises.

I 9,1. Prove that for a definite fundamental tensor

$$\text{I 9,1}\alpha) \quad I_{\kappa_1 \dots \kappa_p \mu_1 \dots \mu_{n-p}} I_{\lambda_1 \dots \lambda_p}^{\mu_1 \dots \mu_{n-p}} = p! (n-p)! g_{[\kappa_1 \lambda_1} \dots g_{\kappa_p] \lambda_p]} .^1)$$

I 9,2. An  $E_p$  and an  $E_q$ ,  $p \leq q$ , in  $E_n$  were defined in I § 1 to be  $t/p$ -parallel if the  $E_p$  contains an  $E_t$  and no  $E_{t+1}$  parallel to  $E_q$ .<sup>2)</sup>

An  $E_p$  and an  $E_q$ ,  $p \leq q$ , in  $R_n$  are said to be  $t/p$ -perpendicular, if the  $E_p$  contains an  $E_t$  and no  $E_{t+1}$  perpendicular to  $E_q$ .<sup>3)</sup>

If  $v^{\kappa_1 \dots \kappa_p}$  and  $w^{\kappa_1 \dots \kappa_q}$  are simple multivectors in  $E_p$  and  $E_q$  prove that  $E_p$  and  $E_q$  are  $t/p$ -parallel if and only if

$$\text{I 9,2}\alpha) \quad v^{\kappa_1 \dots \kappa_u [\lambda_1 \dots \lambda_{p-u} w^{\lambda_1 \dots \lambda_q}] \begin{cases} \neq 0 & \text{for } u = t \\ = 0 & \text{for } u = t - 1. \end{cases}}$$

Prove that  $E_p$  and  $E_q$  are  $t/p$ -perpendicular if and only if

$$\text{I 9,2}\beta) \quad v^{\kappa_1 \dots \kappa_u \cdot \lambda_1 \dots \lambda_{p-u} w^{\lambda_1 \dots \lambda_q}] \begin{cases} \neq 0 & \text{for } u = t \\ = 0 & \text{for } u = t - 1. \end{cases}$$

I 9,3. The real symmetric tensors  $P_{\lambda\kappa}$  and  $Q_{\lambda\kappa}$  in an ordinary  $R_n$  have the same principal multidirections if and only if

$$\text{I 9,3}\alpha) \quad P_{[\kappa}^{\lambda} Q_{\mu]\lambda} = 0.$$

I 9,4. <sup>4)</sup> If  $g_{\lambda\kappa}$  is definite and  $v^{\kappa_1 \dots \kappa_p}$  real, the equations  $v^{\kappa_1 \dots \kappa_p} v_{\kappa_1 \dots \kappa_p} = 0$  and  $v^{\kappa_1 \dots \kappa_p} = 0$  are equivalent.

### § 10. Hybrid quantities.

Every set of  $N$  independent functions of the components of a quantity with  $N$  components could be considered as a kind of components of this quantity. But generally such components are not used because their transformation formulae are too complicated. There is one exception. In dealing with a quantity with not only real components, for instance a vector  $v^\kappa$ , the complex conjugates of the  $v^\kappa$  may be considered as another set of components of the same vector. But we may also look upon them as components of another quantity, a quantity of the second kind in contradistinction to the ordinary quantities that will be called of the first kind. If we write  $\bar{v}^{\tilde{\kappa}}$  for these complex conjugates, they transform as follows

$$(10.1) \quad v^{\tilde{\kappa}'} = A_{\tilde{\kappa}}^{\tilde{\kappa}'} v^{\tilde{\kappa}}$$

<sup>1)</sup> LIPKA 1922, 1, p. 243.

<sup>2)</sup> SCHOUTEN 1902, 1, p. 34.

<sup>3)</sup> SCHOUTEN 1902, 1, p. 49.

<sup>4)</sup> NARLIKAR 1938, 1 for  $p = 2$ .

where the  $A_{\bar{x}}^{\bar{x}'}$  stand for the complex conjugates of the  $A_x^{x'}$ . In the same way to every tensor for instance  $P_{\cdot \cdot \mu}^{x \lambda}$  a tensor of the second kind with components  $P_{\cdot \cdot \bar{\mu}}^{\bar{x} \bar{\lambda}} = \overline{P_{\cdot \cdot \mu}^{x \lambda}}$  can be constructed. But it is not possible to construct in this way components  $P_{\cdot \cdot \mu}^{\bar{x} \lambda}$ , because  $P_{\cdot \cdot \mu}^{x \lambda}$  can be written in several ways as a sum of products of vectors and these several ways would lead to different values for  $P_{\cdot \cdot \mu}^{\bar{x} \lambda}$ . Notwithstanding this it has a definite meaning to consider for instance a quantity with components  $Q_{\cdot \bar{\lambda} \mu}^{\bar{x}}$  transforming as follows

$$(10.2) \quad Q_{\cdot \bar{\lambda} \mu'}^{\bar{x}'} = A_{\bar{x} \bar{\lambda} \mu'}^{\bar{x}' \bar{\lambda} \mu} Q_{\cdot \bar{\lambda} \mu}^{\bar{x}}.$$

Such quantities we call *hybrid*.<sup>1)</sup> The indices with a bar are called indices of the second kind in contradistinction to those of the first kind without a bar. Hybrid  $\Delta$ -densities can be defined in the same way as ordinary  $\Delta$ -densities. If they get the factor  $\Delta^{-w} \bar{\Delta}^{-w'}$ ,  $w$  is the weight and  $w'$  the antiweight. If  $w = w'$  we have an ordinary density of weight  $2w$ . Hence ordinary densities as defined in § 2 are really a kind of hybrid quantities. Ordinary tensors and their complex conjugates may be considered as special cases of hybrid quantities but as a rule we use this latter term only if there are really two different kinds of indices.

To a co- or contravariant hybrid quantity of valence 2 for instance  $T^{\bar{x} \lambda}$  there belongs the complex conjugate  $T^{x \bar{\lambda}}$ . Now if

$$(10.3) \quad T^{\bar{x} \lambda} = +T^{\lambda \bar{x}}$$

$T^{\bar{x} \lambda}$  (and its conjugate) is called a *hermitian symmetric tensor* or *hermitian tensor* for short. But if

$$(10.4) \quad T^{\bar{x} \lambda} = -T^{\lambda \bar{x}}$$

$T^{\bar{x} \lambda}$  (and its conjugate) is called a *hermitian alternating tensor* or a *hybrid bivector*. The difference between hermitian tensors and hybrid bivectors is not so great as in the ordinary case, because if  $T^{\bar{x} \lambda}$  is hermitian,  $i T^{\bar{x} \lambda}$  is a hybrid bivector. The property is obviously invariant for coordinate transformations and also for multiplication with a *real* scalar.

A mixed hybrid quantity of valence 2,  $V_{\cdot \bar{\lambda}}^{\bar{x}}$  has the complex conjugate  $V_{\cdot \bar{\lambda}}^x$ . If its determinant is not zero the inverses  $\bar{V}_{\cdot \bar{\lambda}}^{\bar{x}}$ ,  $\bar{V}_{\cdot \bar{\lambda}}^x$  exist. Now if

$$(10.5) \quad V_{\cdot \bar{\lambda}}^{\bar{x}} = \pm \bar{V}_{\cdot \bar{\lambda}}^x$$

<sup>1)</sup> SCHOUTEN 1929, 1; SCHOUTEN and VAN DANTZIG 1930, 1; E I 1935, 1, p. 8. In these publications we wrote  $\bar{v}^{\bar{x}}$  and  $\bar{A}_{\bar{x}}^{\bar{x}'}$  instead of  $v^{\bar{x}}$  and  $A_{\bar{x}}^{\bar{x}'}$ . This is useful if at any time we wish to drop the indices as for instance in matrix calculus. Because otherwise, where the indices had been dropped, we could no longer make a difference between a quantity and its complex conjugate. But as long as we have no intention of dropping indices and no ambiguity arises we can simplify all notations considerably by dropping the bar over the kernels. It can always be reintroduced.

$V_{\lambda}^{\bar{\lambda}}$  (and its conjugate) is called *positive* or *negative invertible*.<sup>1)</sup> This property is invariant for coordinate transformations and also for multiplication with a factor of the form  $e^{i\varphi}$  where  $\varphi$  is real.<sup>2)</sup>

For hermitian tensors the following theorem holds<sup>3)</sup> (cf. I § 8):

If  $r$  is the rank of  $T_{\lambda\bar{\lambda}} = T_{\bar{\lambda}\lambda}$  ( $U^{\bar{\lambda}\lambda} = U^{\lambda\bar{\lambda}}$ ), the coordinate system can always be chosen such that the matrix of  $T_{\lambda\bar{\lambda}}$  ( $U^{\bar{\lambda}\lambda}$ ) has the diagonal form with  $s$  numbers  $-1, r-s$  numbers  $+1$  and  $n-r$  numbers  $0$  in the main diagonal.

The number  $s$ , called the *index*, is an invariant of the hermitian tensor. The sequence  $\dots - + + \dots$  with  $s$   $-$ -signs and  $(r-s)$   $+$ -signs is called the *signature*. The signature is said to be even (odd) if  $s$  is even (odd). The hermitian tensor is called *positive (negative) definite* if  $s=0(n)$  and  $r=n$  and *indefinite* in all other cases. It is called *positive (negative) semi-definite* if  $n-r > 0$  and  $s=0(r)$  (cf. I § 8).

As a corollary we get that two hermitian tensors are equiform (cf. I § 2) if and only if they have the same rank and the same index.

In a centred  $E_n$  a hermitian tensor  $T_{\lambda\bar{\lambda}}$  is represented by a subspace satisfying the equations

$$(10.6) \quad T_{\lambda\bar{\lambda}} x^\lambda x^{\bar{\lambda}} = \pm 1.$$

The nullcone of  $T_{\lambda\bar{\lambda}}$  has the equation

$$(10.7) \quad T_{\lambda\bar{\lambda}} x^\lambda x^{\bar{\lambda}} = 0.$$

If  $T_{\lambda\bar{\lambda}}$  is definite its nullcone degenerates into the origin because in this case there are no other points satisfying (10.7).

In dealing with hybrid quantities it is often convenient to consider an auxiliary  $E_{2n}$  with coordinates  $x^\alpha, x^{\bar{\alpha}}$  that are now supposed to be linearly independent and each to take all complex values. The allowable coordinate transformations in this  $E_{2n}$  are the linear homogeneous transformations of the  $x^\alpha$  and  $x^{\bar{\alpha}}$  allowable in the original  $E_n$ , hence the  $E_{2n}$  has a restricted transformation group. It contains the original  $E_n$  as a subspace given by the  $n$  equations

$$(10.8) \quad x^{\bar{\alpha}} = \bar{x}^\alpha.$$

This subspace, called the *principal*  $E_n$ , is invariant for the allowable coordinate transformations and the same holds for the subspaces reduced where necessary (cf. T. P. 1951, 1, Ch. X, also for the shorter notations of matrix calculus arising by dropping all indices of quantities with valences 1 and 2).

<sup>1)</sup> Canonical forms for  $V_{\lambda}^{\bar{\lambda}}$  were given for the general case by HAANTJES 1935, 1 and for the invertible case by JACOBSTHAL 1934, 1. Cf. VEBLEN, v. NEUMANN and GIVENS 1936, 1, p. 4. 23; SCHOUTEN 1949, 3, p. 218.

<sup>2)</sup> Invertible quantities were introduced by SCHOUTEN and HAANTJES 1935, 2, p. 179 and by VEBLEN, v. NEUMANN and GIVENS 1936, 1, p. 4.23. For the projective case these latter authors called them anti-involutions. They are important in spinospace. Cf. SCHOUTEN 1949, 2; 3; 4; 1950, 1.

<sup>3)</sup> Cf. for the proof DICKSON 1926, 1, p. 70.

presented by each of the two sets of  $n$  equations

$$(10.9) \quad \text{a)} \quad x^{\bar{x}} = \underset{1}{c^{\bar{x}}} = \text{const.} \quad \text{b)} \quad x^x = \underset{2}{c^x} = \text{const.}$$

Each set of equations represents a set of  $\infty^n E_n$ 's called the *invariant*  $E_n$ 's of the *first* and of the *second set* respectively. The  $x^x$  can be used as coordinates in every  $E_n$  of the first set and the  $x^{\bar{x}}$  play the same role for the  $E_n$ 's of the second set. Every  $E_n$  of one set is mapped on each  $E_n$  of the same set by means of the  $E_n$ 's of the other set. According to KLEIN'S principle instead of  $x^x$  and  $x^{\bar{x}}$  we may now also introduce general rectilinear coordinate systems subject to the full group  $G_{h\sigma}$  in  $E_{2n}$ , provided that we adjoin the two sets of invariant  $E_n$ 's and the principal  $E_n$  (10.8).

Instead of an ordinary symmetric tensor  $g_{\lambda x}$  a hermitian tensor  $a_{\lambda\bar{x}}$  of rank  $n$  can be introduced as fundamental tensor. Lowering and raising of indices can then be effected by means of  $a_{\lambda\bar{x}}$  and its inverse denoted by  $a^{\bar{x}\lambda}$ . The determinant of  $a_{\lambda\bar{x}}$  is a scalar density of weight +2. An  $E_n$  with a fundamental tensor  $a_{\lambda\bar{x}}$  is called an  $\tilde{R}_n$ <sup>1)</sup> and it is said to be *ordinary* if  $a_{\lambda\bar{x}}$  is positive definite. In an  $\tilde{R}_n$ ,  $v^x$  and  $v^{\bar{x}}$  each get two kinds of components, viz.  $v^x, v_{\bar{x}}$  and  $v^{\bar{x}}, v_{\lambda}$ . So in  $\tilde{R}_n$  the difference between co- and contravariant vectors does not vanish, as it does in  $R_n$ . There remain in  $\tilde{R}_n$  two kinds of vectors, called *kets* and *bras* by DIRAC. The ket (bra) is a contravariant vector of the first (second) kind, equivalent to a covariant vector of the second (first) kind<sup>2)</sup>. Two vectors  $v^x$  and  $w^x$  are said to be (*unitary*) *perpendicular* to each other if  $v^\lambda w^{\bar{x}} a_{\lambda\bar{x}} = 0$ . This is equivalent to  $w^\lambda v^{\bar{x}} a_{\lambda\bar{x}} = 0$ . The *norm* of a vector  $v^x$  is the expression  $v^\lambda v^{\bar{x}} a_{\lambda\bar{x}}$  that is always real. The vectors with a positive (negative) norm fill the +(-)-region of the  $\tilde{R}_n$ . A vector is called *unitvector* if its norm equals  $\pm 1$ . A unitvector remains unitvector if it is multiplied by  $e^{i\varphi}$  with real  $\varphi$ .

According to the theorem stated above there exists at least one coordinate system ( $h$ ) with basis vectors  $\underset{i}{u^x}; \underset{j}{u_\lambda}$  such that

$$(10.10) \quad a_{\lambda\bar{x}} = -\underset{1}{u_\lambda} \underset{1}{u^{\bar{x}}} - \cdots - \underset{s}{u_\lambda} \underset{s}{u^{\bar{x}}} + \underset{s+1}{u_\lambda} \underset{s+1}{u^{\bar{x}}} + \cdots + \underset{n}{u_\lambda} \underset{n}{u^{\bar{x}}}.$$

The  $\underset{j}{u_\lambda}$  are mutually perpendicular unitvectors and they satisfy the equations

$$(10.11) \quad \begin{cases} u_i^x = -\underset{i}{u^x} = -\underset{i}{u_\lambda} a^{\bar{x}\lambda} & i = 1, \dots, s; \\ u_j^x = +\underset{j}{u^x} = +\underset{j}{u_\lambda} a^{\bar{x}\lambda} & j = s+1, \dots, n. \end{cases}$$

They form a *unitary cartesian* coordinate system. All these systems are transformed into each other by the transformations of the *unitary*

<sup>1)</sup> In E I 1935, 1, p. 60 we wrote  $U_n$  instead of  $\tilde{R}_n$  (cf. Ch. VIII).

<sup>2)</sup> DIRAC 1947, 1; cf. T. P. 1951, 1, p. 243.

*orthogonal* group. This is the group of homogeneous linear transformations characterized by the invariance of  $a_{\lambda\bar{\lambda}}$  (cf. 9.36):

$$(10.12) \quad \text{a)} \ U_{\cdot\lambda}^e U_{\cdot\bar{\lambda}}^{\bar{e}} a_{\lambda\bar{\lambda}} = a_{\lambda\bar{\lambda}} \quad \text{or} \quad \text{b)} \ U_{\bar{\lambda}\bar{\lambda}} = U_{\bar{\lambda}\bar{\lambda}}^{-1} \quad \text{or} \quad \text{c)} \ U_{\cdot\lambda}^* = U_{\cdot\lambda}^{*\cdot}$$

From this we see that  $\text{Det}(U_{\cdot\lambda}^*)$  has the form  $e^{i\varphi}$  with real  $\varphi$  and that if  $U_{\cdot\lambda}^*$  represents a unitary orthogonal transformation, the same holds for  $e^{i\psi} U_{\cdot\lambda}^*$  with real  $\psi$ .<sup>1)</sup>

The subspace represented by

$$(10.13) \quad a_{\lambda\bar{\lambda}} x^\lambda x^{\bar{\lambda}} = \pm 1 (0)$$

is called the *fundamental figure* (the *nullcone*) in the  $\tilde{R}_n$ . But (10.13) can also be interpreted as the equation of a hyperquadric in the auxiliary  $E_{2n}$  that makes this  $E_{2n}$  to an  $R_{2n}$ . In this  $R_{2n}$  the  $a_{\lambda\bar{\lambda}}$  are the components of a fundamental tensor whose components  $a_{\lambda\lambda}$  and  $a_{\bar{\lambda}\bar{\lambda}}$  are zero. Because of this the hyperquadric has a very peculiar position with respect to the two sets of invariant  $E_n$ 's. In fact, the section of the hyperquadric (nullcone) with an  $E_n$  of the first set is an  $E_{n-1}$  with the  $n+1$  linear equations

$$(10.14) \quad \text{a)} \ x^{\bar{\lambda}} = c_{\bar{\lambda}}^{\bar{\lambda}} \quad \text{b)} \ 2a_{\lambda\bar{\lambda}} x^\lambda c_{\bar{\lambda}}^{\bar{\lambda}} = 1 (0)$$

in which  $n$  independent parameters occur. Hence the hyperquadric is built up of two sets of  $\infty^n E_{n-1}$ 's and each  $E_{n-1}$  of one set has at most one point in common with an  $E_{n-1}$  of the other set.<sup>2)</sup>

If  $T_{\lambda\bar{\lambda}}$  is an arbitrary hermitian tensor, we may ask whether any vectors exist satisfying the relation

$$(10.15) \quad v^\lambda T_{\lambda\bar{\lambda}} = \sigma v_{\bar{\lambda}}.$$

$\sigma$  is a solution of the equation

$$(10.16) \quad \text{Det}(T_{\lambda\bar{\lambda}} - \sigma a_{\lambda\bar{\lambda}}) = 0.$$

If  $a_{\lambda\bar{\lambda}}$  is *definite*, it can be proved that the solutions are all *real* and that to every solution with multiplicity  $m$  there belongs an  $m$ -direction perpendicular to all multidirections belonging to other solutions.

<sup>1)</sup> Instead of  $a_{\lambda\bar{\lambda}}$  a positive or negative invertible hybrid quantity (cf. 10.5) could be introduced as fundamental quantity in  $E_n$ . This would lead to a geometry with a group leaving this quantity invariant. This geometry seems to have escaped attention and even its group seems not to have got a name.

<sup>2)</sup> It is well known that a quadric in  $E_{2n}$  contains  $\infty^{\binom{n+1}{2}} E_{n-1}$ 's. But a quadric in  $E_{2n}$  does not fix two preferred sets of  $E_n$ 's. This means that the  $E_{2n}$  is not an ordinary  $R_{2n}$  but an  $R_{2n}$  that contains besides a quadric also two sets of invariant  $E_n$ 's. If these  $E_n$ 's are fixed it contains moreover the principal  $E_n$  with the equation (10.8). This is the real reason why there is something new in the geometry of an  $\tilde{R}_n$  and why this geometry is not the ordinary geometry of an  $R_{2n}$  described in a queer way.

These multidirections are called the *principal multidirections* of the hermitian tensor.<sup>1)</sup> The unitvectors of  $(h)$  can be chosen in these multidirections, hence (cf. I § 9)

*Theorem of principle axes of a hermitian tensor in  $\tilde{R}_n$ :*

*If  $T_{\lambda\bar{\lambda}}$  is a hermitian tensor and if the fundamental tensor is definite there always exists a unitary cartesian system  $(h)$  such that*

$$(10.17) \quad T_{i\bar{h}} = 0; \quad h \neq i.$$

The axes of this coordinate system are called *principal axes* of the hermitian tensor. They are uniquely determined if and only if the solutions of the equation (10.16) are all different. The theorem of principal axes does not hold if the fundamental tensor is indefinite because in this case the tensor  $T_{\lambda\bar{\lambda}}$  may possibly have a special position with respect to the nullcone. If this is not the case the theorem remains valid.

### Exercises.

I 10,1. The infinitesimal transformation

$$I 10,1 \alpha) \quad 'x^\sigma = x^\sigma + P_{\sigma\lambda}^* x^\lambda dt; \quad t \text{ real}$$

belongs to the unitary orthogonal group if and only if  $P_{\bar{\lambda}\lambda}$  is hermitian alternating. Then  $iP_{\bar{\lambda}\lambda}$  is hermitian symmetric and every principal direction of this tensor is invariant for the transformation.

I 10,2<sup>2)</sup>). If  $T_{\lambda\bar{\lambda}}$  is hermitian symmetric (alternating) and  $S_\lambda^*$  arbitrary, prove that

$$I 10,2 \alpha) \quad S_\lambda^\sigma T_{\sigma\bar{\lambda}} S_{\bar{\lambda}}^{\bar{\sigma}}$$

is also hermitian symmetric (alternating).

If  $T_{\cdot\lambda}^{\bar{\sigma}}$  is positive (negative) invertible and  $S_\lambda^*$  arbitrary, prove that

$$I 10,2 \beta) \quad \bar{S}_{\lambda}^{\sigma} T_{\sigma\bar{\lambda}}^* S_{\bar{\lambda}}^{\bar{\sigma}}$$

is also positive (negative) invertible. If  $'x^\sigma = S_\lambda^* x^\lambda$  is a transformation of the unitary orthogonal group, in both formulae  $S_{\bar{\lambda}}^{\bar{\sigma}}$  can be replaced by  $\bar{S}_{\cdot\bar{\lambda}}^{\bar{\sigma}}$ .

I 10,3<sup>2)</sup>). In an ordinary  $\tilde{R}_2$  the most general unitary orthogonal transformation can be written in the form

$$I 10,3 \alpha) \quad \begin{cases} 'x = x e^{i\alpha} \cos \varphi + y e^{i\beta} \sin \varphi \\ 'y = -x e^{i\gamma} \sin \varphi + y e^{i(\gamma-\alpha+\beta)} \cos \varphi \end{cases}$$

with four real parameters  $\alpha, \beta, \gamma, \varphi$  with respect to a unitary cartesian coordinate system.

<sup>1)</sup> Cf. e.g. DICKSON 1926, 1.

<sup>2)</sup> Cf. E I 1935, 1, § 5.

## § 11. Abridged notations.

In tensor calculus there sometimes appears a great number of indices. Many authors therefore have tried to get rid of them by introducing abridged notations. Most of these attempts failed because an abridged notation always means a new kind of stenography that has to be learned by the reader, and because most readers are overbusy and seldom willing. It should be held in mind that as a rule abbreviations make the reading more difficult and the calculus less foolproof. Authors should therefore be very careful and they should introduce them only when necessary. We mention here some of the most interesting abbreviations.

*1. Collecting indices.* Some of the indices of a quantity can be replaced by one upper or lower index. For instance  $P_A^A$ , or  $P_A$ , may be written for  $P_{\cdot \cdot \mu \nu}^{\times \lambda}$ . Even transvections over collected indices can be made possible, for instance  $P_A^A, Q_{A_\ell}^{\cdot \cdot \sigma}$  for  $P_{\cdot \cdot \mu \nu}^{\times \lambda}, Q_{x \cdot \cdot \ell}^{\cdot \cdot \mu \sigma}$ . This abbreviation can be very useful if during an investigation or even during some part of it only the indices not collected are important. It is often used and there seems to be no objection provided its use is not exaggerated.

*2. Representative indices.* Let  $\lambda_p, \lambda_q$  where  $q < p$ , always stand for  $\lambda_p \dots \lambda_{q+1} \lambda_q \dots \lambda_1$ .<sup>1)</sup> Then we are able to write  $P_{[\lambda_p] \lambda_q}$  for  $P_{[\lambda_p \dots \lambda_{q+1}] \lambda_q \dots \lambda_1}$ . This notation is useful in investigations where repeated differentiations occur often, for instance  $\partial_{\lambda_p} w_{\lambda_q}$  stands for  $\partial_{\lambda_p} \dots \partial_{\lambda_{q+1}} w_{\lambda_q \dots \lambda_1}$ . But it needs to be handled carefully and this may be the reason that it has not been accepted. The same holds for the notation  $v_{\cdot \cdot \lambda_1 \cdot \cdot \lambda_q}^{\times_1 \dots \times_p}$  for  $v_{\cdot \cdot \lambda_1 \dots \lambda_q}^{\times_1 \dots \times_p}$ ,<sup>2)</sup> that has the same disadvantages and is less practical. A less developed version of this latter notation that has come out recently<sup>3)</sup> is to write  $P_{\cdot \cdot \lambda \dots \lambda}^{\times \dots \times}$  instead of  $P_{\cdot \cdot \lambda_1 \dots \lambda_q}^{\times_1 \dots \times_p}$ . If this version should find approval this would demonstrate the fact that only very uncomplicated abbreviations have any chance of being accepted.

*3. The notation of GIVENS.*<sup>4)</sup> GIVENS proposed to write  $(\lambda)$  instead of  $p$  alternated neighbouring indices  $[\lambda_1 \dots \lambda_p]$  wherever they might occur and  $|(\lambda)|$  for their number  $p$ . This notation makes it easy to write down transvections of multivectors. Though GIVENS' results were interesting his notation was not accepted.

*4. VITALI's method.* VITALI<sup>5)</sup> introduced a kind of indices that can take the values  $1, \dots, n$ ,  $11, 12, \dots, 1n, 21, 22, \dots, 2n, \dots, nn, 111, \dots, nnn, \dots$  and so on till all combinations (with repetition) from the numbers  $1, \dots, n$ . The notation is to be used in connexion

<sup>1)</sup> SCHOUTEN 1925, 1.

<sup>2)</sup> E I 1935, 1, p. 33.

<sup>3)</sup> CRAIG 1943, 1.

<sup>4)</sup> VEBLEN, V. NEUMANN and GIVENS 1936, 1; GIVENS 1937, 1.

<sup>5)</sup> VITALI 1923, 1.

with higher differentiations. Though strongly advocated by BORTOLOTTI<sup>1)</sup> it never came in general use. The method is outside of the realm of this book.

**5. CRAIG's extensors.** Another notation, also connected with higher differentiations is the method of extensors introduced by CRAIG.<sup>2)</sup> Though the double-indices used are in the beginning somewhat bewildering the notation is often used especially by Japanese authors and it seems to have a fair chance of being accepted for those investigations where higher differentiations play an important role. A. KAWAGUCHI<sup>3)</sup> and M. KAWAGUCHI<sup>4)</sup> gave generalizations. The method is outside of the realm of this book.

**6. Ideal vectors** (cf. III § 4). A tensor, for instance  $P_{\alpha; \lambda}^{\gamma}$ , is generally not a product of vectors. But we may write  $P_{\alpha; \lambda}^{\gamma} = u_{\alpha} v_{\lambda} w^{\gamma}$  and look upon  $u_{\alpha}$ ,  $v_{\lambda}$  and  $w^{\gamma}$  as symbolic factors. If this process is applied to a symmetric tensor the factors can be chosen equal, for instance  $w_{\lambda_1 \dots \lambda_q} = w_{(\lambda_1 \dots \lambda_q)} = w_{\lambda_1} \dots w_{\lambda_q}$  and the transvection  $w_{\lambda_1 \dots \lambda_q} x^{\lambda_1} \dots x^{\lambda_q}$  can then be written  $(w_{\lambda} x^{\lambda})^q$ . So we get the CLEBSCH-ARONHOLD symbolism well known from the theory of algebraic invariants. If  $w_{\lambda_1 \dots \lambda_q}$  is a  $q$ -vector we may write  $w_{\lambda_1 \dots \lambda_q} = w_{\lambda_1} \dots w_{\lambda_q}$  if we assume that  $w_{\lambda_1} w_{\lambda_2} = - w_{\lambda_2} w_{\lambda_1}$  for every choice of  $\lambda_1$  and  $\lambda_2$ . Then we arrive at WEITZENBÖCK's "Komplexsymbolik".<sup>5)</sup> Of course in both cases in all terms that contain a non-symbolical factor more than once, separate and distinguishable ideal vectors must be introduced for every factor in order to avoid ambiguity. This is all well known from algebra. Now both symbolisms are very useful in purely algebraic investigations as so many publications on algebraic invariants prove and they can of course be equally useful in tensor algebra. But as soon as differentiations are introduced, as in differential geometry, many difficulties arise with respect to the differentiation of ideal vectors. In some earlier publications<sup>6)</sup> an attempt was made to introduce ideal vectors in differential geometry. But the simplifications to be got in this way did not appear important enough to compensate for the introduction of symbolic differentiations of ideal vectors and so ideal vectors never got a proper place outside of the purely algebraic domain.

**7. Systems of "direct" calculus.** An invariant equation between tensors does not depend on the choice of the coordinates or the letters

<sup>1)</sup> BORTOLOTTI 1930, 1; 1937, 1; also for more literature.

<sup>2)</sup> CRAIG 1943, 1.

<sup>3)</sup> A. KAWAGUCHI 1939, 1.

<sup>4)</sup> M. KAWAGUCHI 1952, 1.

<sup>5)</sup> WEITZENBÖCK 1908, 1.

<sup>6)</sup> SCHOUTEN 1918, 1; 1921, 1; SCHOUTEN and STRUIK 1919, 1; 1921, 2; 3; 1922, 1; 2; STRUIK 1922, 1. Cf. III § 4.

used for the indices. It is a message from the author to the reader concerning certain properties of tensors independent of any coordinate system and this message is readable as soon as we know the *skeleton* of the formula that is the combination of the *kernel*, the *places* of the indices for every manifold concerned and the *places* where the transvections act. Hence it must be possible to invent notations that represent in some way this skeleton without the use of indices. Such a notation is often called a "direct" system of calculus. Well known examples are GRASSMANN's "Ausdehnungslehre", HAMILTON's quaternion calculus, ordinary "vectoranalysis", "tensoranalysis", matrix calculus etc. A direct system must have a notation for the kernels, mostly a letter of some special alphabet, in order to distinguish between tensors and scalars, a special method of marking the places of upper and lower indices that makes it possible to drop these indices and a number of special signs for the most frequently used multiplications and transvections. For instance in the most developed direct system<sup>1)</sup> the vectors  $v^\alpha$  and  $w_\lambda$  were written  $v'$  and  $w$  and the tensor  $P^\alpha_{\cdot \lambda}{}^{\mu\nu}$  was represented by  $P''''$ . The transvections  $v^\alpha w_\alpha$ ,  $P^\alpha_{\cdot \lambda}{}^{\mu\nu} w_\nu$ ,  $w_\alpha P^\alpha_{\cdot \lambda}{}^{\mu\nu}$  were written  $v'!w$ ,  $P''''!w$ ,  $w!P'''$  and other multiplication signs were given for a number of other transvections. This system worked rather efficiently and it was used not only for the translation of well known results (as many of these systems were) but especially for the establishment of new theorems. From the beginning there was a great difficulty, notwithstanding the great number of multiplication symbols the use of ideal vectors for the representation of all necessary transvections could not be avoided. For instance  $w_\mu P^\alpha_{\cdot \lambda}{}^{\mu\nu}$  could only be represented by  $(w!r)p'qs'$  after introduction of ideal vectors:  $P'''' = p'qr's'$ . Now in view of this difficulty one could get the idea to improve the direct calculus by introducing instead of the two signs ' and ' to mark the places of the indices, as many signs as there are places. Then for instance a tensor  $P^\alpha_{\cdot \lambda}{}^{\mu\nu}$  could be written  $P'_{\cdot \times}{}^{\circ *}$  and it would be easy to write all possible transvections, for instance  $w.P'_{\cdot \times}{}^{\circ *}$ ;  $w_o P'_{\cdot \times}{}^{\circ *}$ ;  $P'_{\cdot \times}{}^{\circ *}v^\alpha$  without the necessity of introducing ideal vectors. This would certainly be an improvement but then the next step would be the remark: "why not  $\alpha\lambda\mu\nu$  instead of  $\cdot \times \circ *$ ?" and this would lead us back to RICCI's calculus. The real cause of this remarkable situation is that tensor calculus has two different aspects. The *running* indices are on the one hand only marks labelling certain places and constituting together with the kernels the skeleton of the formula. From this point of view tensor calculus is a highly efficient "direct" calculus because it enables us

<sup>1)</sup> SCHOUTEN 1918, 1; STRUIK 1922, 1. In SCHOUTEN and STRUIK 1922, 1 (the first edition of E I 1935, 1) all formulae were given in duplo, once indexfree and once with indices.

to represent all kinds of multiplications and transvections without the help of any auxiliary device. But on the other hand as soon as a definite coordinate system is chosen, the running indices may be replaced by *fixed* ones, for instance  $\lambda$  by  $1, \dots, n$  and if this is done the equations can be read as equations between components. The “directness” of the calculus lies in the fact that the formulae *can* be read (and *should* be read) as relations between quantities and not between components.<sup>1)</sup> The process leading from the direct formulae to the formulae in components (in most kinds of “direct” calculus rather a difficult process!) happens automatically here because the same formulae *can* also be read (and *should* be read were desirable) as relations between components. Of course this is only exactly true if the kernel-index method is used, that is, if every object gets its own kernel and if every coordinate system gets its own kind of running indices, for instance  $x, x', h$  with their corresponding fixed indices,  $1, \dots, n; 1', \dots, n'; 1, \dots, n$ .

The greatest difficulties for all systems without indices lie in the construction of the necessary equivalent for the places of these indices and in the representation of all possible multiplications and transvections. These difficulties increase strongly if the valences of the quantities occurring become higher. Therefore indexfree systems only succeeded for valences 0, 1 and 2. In fact vector analysis in  $R_3$  with valences 0 and 1 only, and matrix calculus<sup>2)</sup> in  $R_n$  with valences 0, 1 and 2 only, are rather satisfactory. But the vector analysis that can be constructed for  $E_3$  or  $R_4$  is already much more complicated and systems of this kind for  $E_n$  and  $R_n$  are very amusing for the inventor himself and perhaps for some of his friends but they never will find general application.<sup>3)</sup>

*8. The method of the radiusvector in  $E_n$  and  $R_n$ .* For differential geometry in  $E_n$  and  $R_n$  often a middlecourse between tensor calculus and indexfree calculus is followed by writing  $r$  (instead of  $r^*$ ) for the radiusvector and conserving all other indices. In this way quantities

<sup>1)</sup> All students of tensor calculus should be encouraged from the beginning to try to read all invariant formulae as formulae between geometric objects and not as formulae between components only.

<sup>2)</sup> Matrix calculus in connexion with tensor calculus was dealt with elaborately in E I 1935, 1, p. 35ff. and in T. P. 1951, 1, p. 39ff. and Ch. X (DIRAC's matrix calculus).

<sup>3)</sup> The present author has some experience in this matter. He wrote his paper 1922, 3 on different kinds of connexions in differential geometry, in his own index-free system. But following the advice of F. KLEIN who warned him that the paper would never be read, he translated it in terms of Ricci calculus. The advice was a good one, the paper was much referred to and it had its influence on the development of modern differential geometry.

of valence  $\rho$  derived from  $r$  may get  $\rho - 1$ -indices. The method is certainly efficient and it is very often used, of course only in euclidean spaces.

*9. CARTAN's method.* Though from the point of view of RICCI calculus CARTAN's method could be considered as an abbreviation we prefer to consider it as a full sized method in its own right. It is not necessary to discuss it here because it will be dealt with elaborately at many places in this book.

*10. Abbreviations ad hoc.* Sometimes it is convenient to introduce for a short time during an investigation some abbreviation that is specially adapted to this part of the calculation but will be dropped later on. For instance in II § 8 we write  $\Omega$  with suppressed indices for any quantity and  $-o$  for any operation of the kind considered there. This is certainly justified because only in this way could such a general formula be written down. So abbreviations ad hoc may have their advantages. But if a book contains too many of them it becomes unreadable. Young authors especially, who can not yet foresee all the consequences, *should abstain as far as possible from introducing ad hoc abbreviations.* It is inexusable to spend valuable time on the writing of a paper that will not actually be read.

## II. Analytic preliminaries.

### § 1. The arithmetic $n$ -dimensional manifold $\mathfrak{A}_n$ .<sup>1)</sup>

A set of  $n$  real or complex values of  $n$  ordered variables  $\xi$  is called an *arithmetic point* and the totality of all these points an *arithmetic manifold*  $\mathfrak{A}_n$ . The  $\xi$  are called the *components* (Bestimmungszahlen) of the arithmetic point. “Point  $\xi$ ” means “point with components  $\xi$ ”.

The set of all arithmetic points satisfying the inequalities

$$(1.1) \quad |\xi - \xi_0| < \beta,$$

where  $\xi$  are  $n$  arbitrarily given numbers and  $\beta$  are  $n$  arbitrarily given positive numbers, is called a *polycylinder* (or *box*) of  $\mathfrak{A}_n$ .

If a set of arithmetic points is given, satisfying the conditions:

1. every point of the set belongs to at least one polycylinder consisting of points of the set;

1) For § 1 and § 2 of this Chapter cf. VEBLEN and WHITEHEAD 1932, 1; BEHNKE and THULLEN 1934, 1; BOCHNER and MARTIN 1948, 1.

2. for two points of the set there always exists a finite chain of polycylinders, all consisting of points of the set, such that the first point lies in the first and the second point in the last polycylinder and that consecutive polycylinders have at least one point in common, this set is called a *region* of  $\mathfrak{A}_n$ . Every polycylinder is a region and the whole  $\mathfrak{A}_n$  is a region. But not every region is a polycylinder. Every region is called a *neighbourhood* of each of its points. For “neighbourhood of  $\xi$ ” we write  $\mathfrak{N}(\xi)$ .

## § 2. The geometric $n$ -dimensional manifold $X_n$ .

Let the elements of a set  $M$  be in one to one correspondence with the points of a region  $\mathfrak{N}_0$  of  $\mathfrak{A}_n$ . We make no presuppositions concerning the “nature” of these elements. They may be, for instance, homogeneous linear forms in  $n$  variables or polynomials of degree  $n-1$  in one variable but they may be also just elements not connected in any way with other notions. We call the one to one correspondence between the elements of  $M$  and the points of  $\mathfrak{N}_0$  a *coordinate system defined over  $M$* .

If the point  $\xi^*$  corresponds to a certain element of  $M$ , we call the  $\xi^*$  also *coordinates* of this element *with respect to the coordinate system* ( $\mathfrak{x}$ ). If the  $\xi^*$  are considered in this way we write  $\xi^*$  instead of  $\xi$  in order to emphasize this fact.

We consider now a point transformation in  $\mathfrak{A}_n$

$$(2.1) \quad \xi^* = f(\xi)$$

satisfying the conditions that the  $f$  are analytic<sup>1)</sup> at  $\xi_0^*$  and that the functional determinant

$$(2.2) \quad \Delta \stackrel{\text{def}}{=} \text{Det} (\partial_{\lambda} \xi^*); \quad \partial_{\lambda} \stackrel{\text{def}}{=} \frac{\partial}{\partial \xi^*}$$

is  $\neq 0$  for  $\xi^* = \xi_0^*$ . Then it is a well established fact that there is an inverse transformation

$$(2.3) \quad \xi^* = f(\xi)$$

---

<sup>1)</sup> A function defined in an  $\mathfrak{N}(\xi_0^*)$  is said to be *analytic* at  $\xi_0^*$  if there exists an  $\mathfrak{N}(\xi)$  where it can be expanded into a power series in  $\xi - \xi_0^*$ , convergent in this latter  $\mathfrak{N}(\xi)$ . Instead we may only require that the functions  $f$  are of class  $u$  in some  $\mathfrak{N}(\xi)$ . A function is said to be of class  $u$  in some region if it is continuous and has continuous derivatives up to the order  $u$  at each point of that region. This case has been dealt with by VEBLEN and WHITEHEAD 1932, 1, p. 36.

with functions analytic at  $\xi \stackrel{\text{def}}{=} f(\tilde{\xi})$  and that there exists a neighbourhood  $\mathfrak{R}$  of  $\xi$  and a neighbourhood  $\mathfrak{R}'$  of  $\tilde{\xi}$  for the points of which (2.1) and (2.3) establish a one to one correspondence. Suppose now that  $\mathfrak{R}$  lies in  $\mathfrak{R}'$ . Then there is a subset  $R$  of  $M$  whose points are in one to one correspondence with the points of  $\mathfrak{R}$  and also with the points of  $\mathfrak{R}'$ . But this latter correspondence is another coordinate system  $(\xi')$  over  $R$ . For the  $\xi'$ , considered as coordinates of the elements of  $R$ , we write  $\xi''$ .

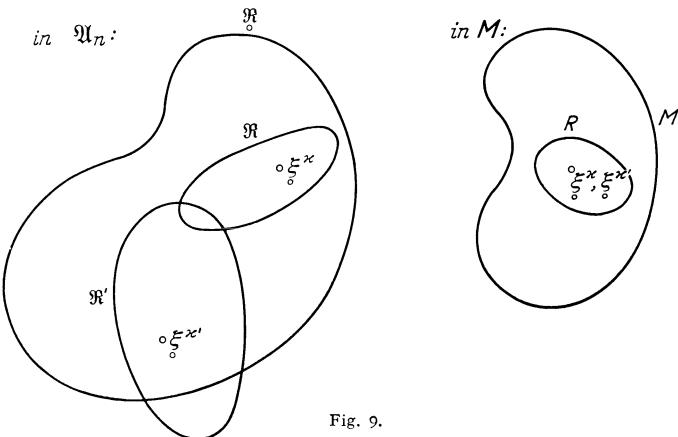


Fig. 9.

Fig. 9<sup>1)</sup> illustrates what is going on in  $\mathfrak{A}_n$  and in  $M$ . In  $\mathfrak{A}_n$  we have only one kind of variables and a transformation means a point transformation  $\xi \rightarrow \xi'; \tilde{\xi} \rightarrow \tilde{\xi}; \mathfrak{R} \rightarrow \mathfrak{R}'$ . In  $M$  we have two kinds of variables, the coordinates  $\xi''$  and  $\xi'''$  and the transformation

$$(2.4) \quad \xi'' = f(\xi'''); \quad \xi''' = f(\tilde{\xi}'); \quad \text{Det}(A_{\alpha}^{\beta}) \neq 0; \quad A_{\alpha}^{\beta} \stackrel{\text{def}}{=} \partial \xi''^{\alpha} / \partial \tilde{\xi}^{\beta}$$

corresponding to (2.1, 3) transforms the coordinates only and leaves the elements of  $M$  unchanged. As we see, our notation agrees with the principles of the kernel-index method (cf. I § 1). In  $\mathfrak{A}_n$  the transformation (2.1, 3) is considered as a point transformation and accordingly the kernel changes. But in  $M$  the same transformation is a coordinate transformation and accordingly in (2.4) the index changes and the kernel is invariant.

*Collecting results we see that a coordinate system over a set of elements  $R$  is a one to one correspondence between the elements of  $R$  and the points*

<sup>1)</sup> It is not necessary that  $\xi$  lies in  $\mathfrak{R}$ .

of a region of  $\mathfrak{A}_n$  and that transformation of coordinates in  $R$  means passing to another one to one correspondence between these elements and the points of another region of  $\mathfrak{A}_n$ .<sup>1)</sup>

Until now we had only the set  $R$  with a coordinate system over it and the possibility of introducing other coordinate systems by means of point transformations in  $\mathfrak{A}_n$ . Now everything depends on the choice of these point transformations. If we allow all analytic transformations (or transformations of class  $u$ ) with  $\mathfrak{R} = \mathfrak{R}' = \mathfrak{R}_0$  then these transformations form a group (cf. I § 1). But if we allow the set  $\mathfrak{P}_n$  of all transformations of points of  $\mathfrak{A}_n$  that are analytic (or of class  $u$ ) each in a certain region, but without the condition that these regions coincide, the set does not constitute a group. Because, if  $\mathfrak{R} \rightarrow \mathfrak{R}'$  and  $\mathfrak{R}' \rightarrow \mathfrak{R}''$  are two transformations of  $\mathfrak{P}_n$ , the second can only be performed after the first if  $\mathfrak{R}'$  and  $\mathfrak{R}''$  have some region in common. Such a set of transformations, satisfying the first condition for groups (cf. I § 1) only if the second transformation can really be effected after the first is called a *pseudo-group*.<sup>2)</sup>,<sup>3)</sup>

The set  $M$ , provided with the pseudo-group  $\mathfrak{P}_n$  and with all *allowable coordinate systems*, viz. all systems that can be derived from one originally specified system by transformations of  $\mathfrak{P}_n$  is called a *general geometric n-dimensional manifold* or  $X_n$ . The elements of  $M$  are called *geometric points* or *points of  $X_n$* . In the following "point" means always "geometric point" and "point  $\xi^x$ " means "point with coordinates  $\xi^x$  with respect to ( $x$ )".

Instead of  $\mathfrak{P}_n$  some subgroup or sub-pseudo-group of  $\mathfrak{P}_n$  can be taken. If for instance  $\mathfrak{R} = \mathfrak{A}_n$  and if  $\mathfrak{P}_n$  is replaced by the group  $G_a$  or  $G_{ho}$  we get instead of  $X_n$ , the  $E_n$  or the centred  $E_n$  (cf. I § 1). If we take  $\mathfrak{R} = \mathfrak{A}_n$  and the group  $G_o$ , of all orthogonal transformations or the group  $G_{ro}$  of all proper rotations we get the  $R_n$  or the  $R_n$  with a fixed screwsense (cf. I § 9). In all these cases the geometry is fixed wholly by the choice of the group or pseudo-group of allowable coordinate transformations. As was pointed out by F. KLEIN in 1872<sup>4)</sup> the geometry of a figure in some space is the theory of the invariants of this figure

<sup>1)</sup> VEBLEN and WHITEHEAD 1932, 1, p. 22. We consider here only ordinary coordinates. For supernumerary coordinates cf. R. S. 1947, 1; P. P. 1949, 1, II § 9.

<sup>2)</sup> VEBLEN and WHITEHEAD 1932, 1, p. 38.

<sup>3)</sup> For a discussion of the foundation of pseudo-groups cf. WAGNER 1950, 1; PENSOV 1950, 1. NIJENHUIS 1952, 1, succeeded in replacing the pseudo-group by a groupoid in the sense of BRANDT, by taking as the elements the transformations of the neighbourhoods of points instead of the transformations of regions.

<sup>4)</sup> KLEIN 1872, 1.

with respect to the underlying set of allowable coordinate transformations.<sup>1)</sup>

If we wish to make an  $E_n$  from a given  $X_n$ , we have to choose one coordinate system and consider all coordinate systems that can be derived from this chosen one by means of transformations of  $G_a$ . Then the  $X_n$  provided with this preferred set of coordinate systems is an  $E_n$  and the geometry of a figure in this  $E_n$  is, in  $X_n$ , the geometry of this figure together with the set of all preferred coordinate systems. This is an application of the principle of F. KLEIN<sup>2)</sup>:

*If in any space with a group  $G_1$  the subgroup  $G_2$  is introduced, consisting of all transformations of  $G_1$  which leave a figure  $F_1$  invariant, then the geometry of a figure  $F_2$  with respect to  $G_2$  is identical with the geometry of the set of figures  $F_1, F_2$  with respect to  $G_1$ .*

Starting from  $X_n$  we took for  $F_1$  the set of all preferred coordinate systems in order to get  $E_n$ . But this is not always necessary, we could also have chosen some figure from which these preferred coordinate systems might be derived and that is uniquely determined by these systems. For instance if we start from  $E_n$  in order to get  $R_n$ , we can use for  $F_1$  the set of all orthogonal coordinate systems, but we can as well use (cf. I §9) the quadratic hypersurface representing the fundamental tensor.

The allowable coordinate systems in  $E_n$  are the rectilinear systems used in Ch. I. This does not mean in any way that we are not allowed to introduce more general coordinate systems in  $E_n$ . But in doing this we consider the  $E_n$  as an  $X_n$  in which the allowable coordinate systems of  $E_n$  play the part of preferred coordinate systems. In the same way in an  $R_n$  besides the allowable orthogonal coordinate systems, general rectilinear or curvilinear systems can be used, but then we look upon the  $R_n$  as an  $E_n$  or  $X_n$  with preferred coordinate systems.

In an  $X_n$  we can not use the notion of polycylinder because contrary to  $\mathfrak{A}_n$  there is no preferred coordinate system in  $X_n$ . Instead we use cells, a *cell* being defined as a set of points given by the inequalities

$$(2.5) \quad |\xi^x| < 1$$

with respect to some allowable coordinate system. A pointset  $R$  in  $X_n$  is called a *region*, if there exists an allowable coordinate system, determining a one to one correspondence between the points of  $R$  and the arithmetic points of a region  $\mathfrak{R}$  in  $\mathfrak{A}_n$ .<sup>3)</sup> Every cell is a region, but not

<sup>1)</sup> Cf. for more general views on geometry SCHOUTEN and v. DANTZIG 1934, 1 and NIJENHUIS 1952, 1.

<sup>2)</sup> See footnote 4, p. 64.

<sup>3)</sup> A region in  $X_n$  can also be defined in the same way as a region in  $\mathfrak{A}_n$  by using cells instead of polycylinders.

every region need be a cell. Also the  $X_n$  is itself a region and every region in  $X_n$  is itself an  $X_n$ . Every  $X_n$  which is a region of another  $X_n$  is said to be *imbedded* in the latter. Every region in  $X_n$  is called a *neighbourhood* of every one of its points. For “neighbourhood of  $\xi^{\alpha}$ ” we write  $\mathfrak{N}_0(\xi^{\alpha})$ .

If  $\mathfrak{P}_n$  consists of analytical transformations only, a field  $\rho$  is said to be *analytic in the region R* if for some choice of  $(\alpha)$  we have  $\rho = f(\xi^{\alpha})$  and if  $f(\xi^{\alpha})$  is analytic at all arithmetic points of the region  $\mathfrak{R}$  of  $\mathfrak{A}_n$  corresponding to  $R$  by means of  $(\alpha)$ . Because in this case  $f(f(\xi^{\alpha}))$  is analytic at all points of the region  $\mathfrak{R}'$  of  $\mathfrak{A}_n$  corresponding to  $R$  by means of  $(\alpha')$ , the analytic nature of  $\rho$  is an invariant notion.<sup>1)</sup>

If  $\mathfrak{P}_n$  consists of transformations of class  $u$  an analytic function in a region of  $X_n$  can not be defined. But with the same reasoning as above we may call  $\rho$  a function of *class u* in some region  $R$  of  $X_n$ , if with respect to some allowable coordinate system  $(\alpha)$  we have  $\rho = f(\xi^{\alpha})$  and if  $f(\xi^{\alpha})$  is of class  $u$  in the region  $\mathfrak{R}$  corresponding to  $R$  by means of  $(\alpha)$ . In both cases the choice of the allowable coordinate system is not important.

### Exercises.

II 2,1. Prove that (cf. Exerc. I 1,1)

$$\text{II 2,1 } \alpha) \quad A_{\alpha}^{\lambda} \partial_{\mu} A_{\lambda}^{\alpha'} = \partial_{\mu} \log \text{Det}(A_{\lambda}^{\alpha'}).$$

II 2,2 (cf. Exerc. I 8,1). If  $P_{\alpha\lambda}$  is a tensor of rank  $n$ ,  $\bar{P}^{-1}_{\alpha\lambda}$  its inverse and  $\mathfrak{P}$  its determinant (scalar density of weight +2), prove that

$$\text{II 2,2 } \alpha) \quad \bar{P}^{-1}_{\alpha\lambda} \partial_{\mu} P_{\lambda\alpha} = \partial_{\mu} \log \mathfrak{P}.$$

II 2,3 (cf. V § 5). If  $h_{\lambda}^{\alpha}$  is a tensor, prove that

$$\text{II 2,3 } \alpha) \quad H_{\mu}^{\alpha\lambda} = 2 h_{[\mu}^{\alpha\rho} \partial_{|\rho|} h_{\lambda]}^{\alpha} - 2 h_{\rho}^{\alpha} \partial_{[\mu} h_{\lambda]}^{\rho}$$

is also a tensor.<sup>2)</sup>

II 2,4. If the eigenvalues of  $h_{\lambda}^{\alpha}$  of Exerc. II 2,3 are all different and if  $(h)$  is chosen in such a way that  $e_1^{\alpha}, \dots, e_n^{\alpha}$  are eigenvectors of  $h_{\lambda}^{\alpha}$ , prove that any  $\rho$  of them are  $X_{\rho}$ -forming (cf. II § 5) if and only if  $H_{\gamma i}^{\alpha} = 0; h, i, j \neq .$ <sup>2)</sup>

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<sup>1)</sup> In general in this book we only consider fields in a neighbourhood of a point and not “in the large” though these fields play a very important role in “global” differential geometry, cf. e.g. BOCHNER 1950, 1; 1951, 1.

<sup>2)</sup> NIJENHUIS 1951, 1; 2.

II 2,5. If  $P^{\kappa\lambda}$  and  $Q^{\kappa\lambda}$  are tensors, prove that

$$\text{II 2,5 a)} \quad P^{\mu[\kappa_1} \partial_{\mu} Q^{\kappa_2 \kappa_3]} + P^{[\kappa_1 | \mu]} \partial_{\mu} Q^{\kappa_2 \kappa_3]} - Q^{\mu[\kappa_1} \partial_{\mu} P^{\kappa_2 \kappa_3]} - Q^{[\kappa_1 | \mu]} \partial_{\mu} P^{\kappa_2 \kappa_3]}$$

is a tensor and that the same holds for () instead of [ ].<sup>1)</sup>

### § 3. Geometric objects and quantities in $X_n$ .

A geometric object<sup>2), 3)</sup> at a point  $\xi^{\kappa}$  of  $X_n$  is a correspondence between the allowable coordinate systems in regions containing  $\xi^{\kappa}$  and the ordered sets of  $N$  numbers, satisfying the conditions:

1. To every coordinate system ( $\kappa$ ) there belongs one and only one ordered set of  $N$  numbers.

2. If  $\Phi_A; A = 1, \dots, N$  corresponds to ( $\kappa$ ) and  $\Phi_{A'}; A' = 1', \dots, N'$  to ( $\kappa'$ ), the  $\Phi_{A'}$  are functions of the  $\Phi_A$  and functionals of the functions  $f^{\kappa'}$  in a neighbourhood of the point  $\xi^{\kappa'}$ . Instead we may also require that the  $\Phi_{A'}$  are functions of the  $\Phi_A$  and of the values of the  $\xi^{\kappa'}$  and of the derivatives  $A_{\kappa}' = \partial_{\kappa} \xi^{\kappa'}, \partial_{\mu} A_{\kappa}', \partial_{\nu} \partial_{\mu} A_{\kappa}', \dots$  up to a certain order  $u$  at the point  $\xi^{\kappa}$ . If the  $f^{\kappa'}$  are analytic and  $u = \infty$  both definitions are equivalent, but in all other cases the first is more general. But we prefer here the second since it is sufficiently general for our purposes and avoids the use of functionals.<sup>4)</sup> Then the transformation formula of a general geometric object takes the form

$$(3.1a) \quad \Phi_{A'}\{\xi\} = \delta_{A'}^A F_A(\Phi_A\{\xi\}, \xi^{\kappa}, \xi^{\kappa'}, A_{\kappa}'\{\xi\}, \partial_{\mu} A_{\kappa}'\{\xi\}, \dots)$$

where  $\{\xi\}$  means that the values of the whole foregoing expression must be taken at the point  $\xi^{\kappa}$ . Note that in expressions as  $\Phi_A\{\xi\}$  it would not be better to write  $\xi^{\kappa}$  instead of  $\xi$ , because  $\Phi_A\{\xi^{\kappa}\}$  would have the same meaning as  $\Phi_A\{\xi'\}$ . The form of the functions  $F_A$  does not depend on the values at  $\xi^{\kappa}$  mentioned between the parentheses. Hence the inversion of (3.1a)

$$(3.1b) \quad \Phi_A\{\xi\} = F_A(\Phi_{A'}\{\xi\}, \xi^{\kappa'}, \xi^{\kappa}, A_{\kappa}'\{\xi\}, \partial_{\mu} A_{\kappa}'\{\xi\}, \dots)$$

contains the same functions  $F_A$ .

<sup>1)</sup> SCHOUTEN 1940, 2 also for general valences of  $P$  and  $Q$ .

<sup>2)</sup> Cf. for more general definitions SCHOUTEN and HAANTJES 1937, 1, and P. P. 1949, 1.

<sup>3)</sup> Further literature: WUNDHEILER 1937, 1; GOLAB 1938, 1; 1946, 1; 1947, 1; 1948, 1; 1950, 1; 2; NAKAE 1944, 1; WAGNER 1945, 1; 1949, 1; 1950, 1; PENSOV 1946, 1; 1948, 1; GEORGHIU 1949, 1; LIBER 1951, 1; VASILIEV 1951, 1; EHRESMANN 1952, 1; HAANTJES and LAMAN 1953, 1; NIJENHUIS 1952, 1, gave a development from the point of view of the representation theory of a groupoid.

<sup>4)</sup> Of course the pseudo-group  $\mathfrak{P}_n$  used in the definition of the  $X_n$  must only contain transformations of class  $\geq u$ .

The  $\Phi_A$  are called the *components of the geometric object* with respect to  $(\kappa)$ .

If we wish to express that a scalar  $\phi$  is a function of the  $\xi^\kappa$  we must write  $\phi = f(\xi^\kappa)$  and not  $\phi = \phi(\xi^\kappa)$ . In fact, if a new coordinate system is introduced we can write  $\phi = f'(\xi'^\kappa)$  but we could not write  $\phi = \phi(\xi'^\kappa)$  because the same letter  $\phi$  can not be used as a function symbol for two different functions. The same holds for more general geometric objects as  $\Phi_A$  and this is the reason why we introduced the notation  $\Phi_A\{\xi\}$  for the values of  $\Phi_A$  at the point  $\xi^\kappa$ .

For all geometric objects we may make use of a notation first introduced by NIJENHUIS.<sup>1)</sup> If the components  $\Phi_A$  of a geometric object with respect to the coordinate system  $(\kappa)$  are expressed as functions of the coordinates  $\xi'^\kappa$  he uses  ${}_{(\kappa')}\Phi_A$  as function symbol. Hence

$$(3.2) \quad \Phi_A = {}_{(\kappa')}\Phi_A(\xi'^\kappa) = {}_{(\kappa)}\Phi_A(\xi^\kappa); \quad \Phi_{A'} = {}_{(\kappa')}\Phi_{A'}(\xi'^\kappa) = {}_{(\kappa)}\Phi_{A'}(\xi^\kappa).$$

It should be remarked that  ${}_{(\kappa')}\Phi_A$  is not a kernel in our sense because the change of a kernel always means a change of the object. If the object is not a scalar it is convenient to write for shortage  $\Phi_A$  instead of  ${}_{(\kappa)}\Phi_A$  and  $\Phi_{A'}$  instead of  ${}_{(\kappa')}\Phi_{A'}$ , though by doing this we use a symbol for components as a function symbol.

It may happen that two or more geometric objects have the same kernel letter. If, in some investigation, this is *not* the case it is often convenient to denote a geometric object in the text by its kernel letter only. Objects with the same number of components and the same manner of transformation are said to be *of the same kind*.

If, for every choice of  $(\kappa)$  and  $(\kappa')$ , the  $\Phi_{A'}$  are functions of the  $\Phi_A$  and the  $A_\kappa'$  only, linear homogeneous in the  $\Phi_A$  and homogeneous algebraic in the  $A_\kappa'$  the geometric object is called a *geometric quantity* or briefly a *quantity*.

Geometric objects may be defined with respect to two or more spaces<sup>2)</sup>, for instance an  $X_n$  with the coordinates  $\xi^\kappa$  and an  $X_m$  with the coordinates  $\eta^a$ ;  $a = 1, \dots, m$ . The components may be functions of  $\xi^\kappa$  and  $\eta^a$ . Objects of this kind are often called *connecting objects*. There may be relations between the  $X_m$  and the  $X_n$ , for instance the  $\xi^\kappa$  may be functions of the  $\eta^a$ . If a connecting object is a quantity it often happens that its components have two different kinds of indices each belonging to one of the underlying spaces. For instance  $B_b^\kappa = \partial \xi^\kappa / \partial \eta^b$ .

In dealing with objects with components with respect to more than one coordinate system, in most cases it is more convenient to introduce

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<sup>1)</sup> NIJENHUIS 1952, 1, p. 44 ff.

<sup>2)</sup> Such quantities were introduced incidentally by BOMPIANI 1921, 1 and more systematically by VAN DER WAERDEN 1927, 1. Cf. SCHOUTEN 1927, 2; DUSCHEK and MAYER 1930, 1, Ch. VII; TUCKER 1931, 1; RUSE 1931, 1; E I 1935, 1, p. 9.

special function symbols, for instance  $A_\lambda^{\kappa'} = F_\lambda^{\kappa'}(\xi') = G_\lambda^{\kappa'}(\xi')$ . But if symbols for components are used as function symbols in this more general case, it must always be made very clear to which coordinate system or systems the function symbols introduced in this way belong.

*Pseudo-quantities* are a special kind of connecting quantities. They are defined with respect to the  $X_n$  of the  $\xi^\kappa$  and the  $X_1$  of a variable  $\xi^0$  with the transformation group (cf. I § 2 under 5):

$$(3.3) \quad \begin{cases} \xi^{\kappa'} = f^{\kappa'}(\xi^\kappa) \\ \xi^{0'} = \sigma(\xi^\kappa) \xi^0. \end{cases}$$

Their components may be functions of the  $\xi^\kappa$  and of  $\xi^0$ . A *pseudo-scalar of class c* has one component, depending only on the  $\xi^\kappa$  and has the transformation

$$(3.4) \quad \overset{(0)}{\mathfrak{p}} = \sigma^c \overset{(0)}{\mathfrak{p}}.$$

Hence  $\xi^0$  itself is a pseudo-scalar of class 1. Other pseudo-quantities arise if a pseudo-scalar is multiplied by an ordinary quantity. All these pseudo-quantities can be multiplied and transvected. Addition is only possible if the pseudo-quantities are defined with respect to the same  $(n+1)$ st variable and if the valences, the weights and the class are the same.

If we have some quantity or some set of quantities each defined to within an entirely free scalar factor, we might consider them as pseudo-quantities, each defined with respect to its own  $(n+1)$ st auxiliary variable. But as there are no connections between these auxiliary variables there is in general no use in introducing them. In this case we use the sign  $\lfloor \rfloor$  (cf. I § 2). For instance  $\lfloor v^\kappa \rfloor$  is a vector defined to within a scalar factor. Note that it is no use asking whether  $\lfloor v^\kappa \rfloor$  is a vector or a vector density because the influence of a factor  $\Delta$  is nullified by the entirely free scalar factor.  $\lfloor v^\kappa \rfloor$  is neither a vector nor a vector density but a pseudo-vector though an auxiliary variable  $\xi^0$  is not mentioned.

The  $\xi^\kappa$  represent a geometric object which is no quantity. But according to (2.4) the  $d\xi^\kappa$  undergo a homogeneous linear transformation  $d\xi^{\kappa'} = A_\kappa^{\kappa'} d\xi^\kappa$ . Hence the  $d\xi^\kappa$  at the point  $\xi^\kappa$  represent a quantity at that point. But from the same equation we see that every allowable coordinate transformation in an  $\mathfrak{N}(\xi^\kappa)$  induces one and only one transformation of the group  $G_{h_0}$  at  $\xi^\kappa$  and that to every transformation of this group at least one allowable coordinate transformation in  $X_n$  can be found inducing this very transformation of  $G_{h_0}$  at  $\xi^\kappa$ . This means that to every point  $\xi^\kappa$  of  $X_n$  there belongs a centred  $E_n$ , the *tangent space* or *tangent E<sub>n</sub>* of the point  $\xi^\kappa$ .<sup>1)</sup>

1) In former publications we used the term *local E<sub>n</sub>*.

From the definition it follows that a quantity of  $X_n$  at  $\xi^*$  is the same as a quantity as defined in I § 2 in the tangent  $E_n$  of  $\xi^*$ . Hence all quantities of  $E_n$  considered in Ch. I may occur as quantities of  $X_n$  at every point of  $X_n$ . All equations of transformation like (I 2.1), (I 2.4), (I 2.13), (I 2.19) and (I 2.22, 23) are valid in  $X_n$ , but the  $A_{\lambda}^{*\nu}$ ,  $A_{\lambda}^{\nu}$  and  $\Delta$  are no longer constants.

If the components of a geometric object are defined in some region of  $X_n$  as functions of the  $\xi^*$ , the object is said to form an *object field*. A field is said to be *analytic* (of class  $u$ ) in a region if all its components are analytic (of class  $u$ ) in that region.

As a first example we take a scalar field  $\phi = f(\xi^*)$ . If  $\phi$  is analytic or at least of class 1 in the region considered we may form the derivatives  $\partial_{\lambda}\phi$ . Because of

$$(3.5) \quad \partial_{\lambda'}\phi = A_{\lambda'}^{\lambda} \partial_{\lambda}\phi$$

these derivatives are the components of a covariant vector, the *natural derivative* or *gradient* of  $\phi$ . If we use the abridged notation of I § 7, we write  $\text{Grad } \phi$  or  $D\phi$  for the gradient of  $\phi$  and also  $\bar{\phi}$  if the vector occurs in the abridged form (I 7.1) of an alternating product.

Now we take a vector field  $v^* = f^*(\xi^*)$  and form its derivatives  $\partial_{\lambda}v^*$ . Because of <sup>1)</sup>

$$(3.6) \quad \partial_{\lambda'}v^{*\nu} = A_{\lambda'}^{\lambda} \partial_{\lambda}A_{\nu}^{*\nu} v^* = A_{\lambda'}^{\lambda} A_{\nu}^{*\nu} \partial_{\lambda}v^* + A_{\lambda'}^{\lambda} v^* \partial_{\lambda}A_{\nu}^{*\nu}$$

these derivatives are neither the components of a tensor nor of any geometric object, since the  $\partial_{\lambda}v^*$  depend not only on  $\partial_{\lambda}v^*$ ,  $A_{\nu}^{*\nu}$  and  $\partial_{\lambda}A_{\nu}^{*\nu}$  but also on  $v^*$ . But the  $n+n^2$  expressions  $v^*$ ,  $\partial_{\lambda}v^*$  are the components of a geometric object whose transformation involves the  $\partial_{\lambda}A_{\nu}^{*\nu}$ . The same is true for the derivatives of all geometric quantities except scalars and  $W$ -scalars.

To every coordinate system ( $\nu$ ) in  $X_n$  there belong  $n$  scalar fields  $\xi^*$ . The transformation  $\xi^* \rightarrow \xi^{*\nu}$  is a coordinate transformation in  $X_n$ , accompanied by the point transformations  $\xi^* \rightarrow \xi^*$  in  $\mathfrak{U}_n$ . But this latter transformation can be considered also as an object transformation in  $X_n$ , viz. the transformation of the scalar fields  $\xi^*$  into the scalar fields  $\xi^{*\nu}$ . The gradients of the scalar fields  $\xi^*$  are the vectors of the *covariant basis*

<sup>1)</sup> In dealing with operators like  $\partial_{\mu}$  (and also  $\nabla_{\mu}$  in the next chapter, cf. footnote 3 on p. 132) we agree that the operator works by differentiation on all quantities to the right of it in the same term till the first closing bracket, whose corresponding opening bracket stands at the left of the operator. If these operators are applied several times in succession we usually prefer to write the kernel letter  $\partial$  (or  $\nabla$ ) only once, for instance  $\partial_{\mu_1} \dots \partial_{\mu_n}$  instead of  $\partial_{\mu_1} \dots \partial_{\mu_n}$ .

belonging to  $(\alpha)$ .<sup>1)</sup> Their components with respect to  $(\alpha)$  have the values 1 and 0 only

$$(3.7) \quad \tilde{e}_\lambda \stackrel{\text{def}}{=} \partial_\lambda \xi^* \stackrel{*}{=} \delta_\lambda^*$$

but the components with respect to  $(\alpha')$  may have quite other values

$$(3.8) \quad \tilde{e}_{\lambda'} = \partial_{\lambda'} \xi^* \stackrel{*}{=} A_{\lambda'}^*.$$

The vectors of the *contravariant basis* belonging to  $(\alpha)$  arise from differentiation of the coordinates  $\xi^*$  in  $X_n$  with respect to the  $\xi$

$$(3.9) \quad \tilde{e}_\lambda^* \stackrel{\text{def}}{=} \frac{\partial \xi^*}{\partial \lambda} \stackrel{*}{=} \delta_\lambda^*; \quad \tilde{e}_{\lambda'}^* = \frac{\partial \xi^*}{\partial \lambda'} \stackrel{*}{=} A_{\lambda'}^*.$$

The equations (3.7–9) and

$$(3.10) \quad A_\lambda^* = \partial_\lambda \xi^*; \quad A_{\lambda'}^* = \partial_{\lambda'} \xi^*$$

$$(3.11) \quad \delta_\lambda^* = \frac{\partial \xi^*}{\partial \lambda}$$

show clearly the difference between the tensor  $A_\lambda^*$ , the two bases each consisting of  $n$  vectors  $\tilde{e}_\lambda^*$  and  $\tilde{e}_\lambda$  and the set of  $n^2$  scalars  $\delta_\lambda^*$ . We also see the difference between the living indices, transforming only with coordinate transformations in  $X_n$ , and the dead indices, transforming only with the object transformations in  $X_n$  corresponding to point transformations in  $\mathfrak{A}_n$ .<sup>2)</sup>

At every point every quantity field has a rank with respect to each set of indices, as defined in I § 5. If such a rank is  $r$  at  $\xi^*$ , this means that a certain matrix formed from the components of the quantity has rank  $r$  at that point. Hence all subdeterminants with more than  $r$  rows vanish at  $\xi^*$  and there is at least one non vanishing subdeterminant with  $r$  rows. Now let the quantity be analytic in a region  $R$  containing  $\xi^*$  and let  $\xi^*$  be a point where  $r$  has a maximum value. Because all subdeterminants are analytic, there exists an  $\mathfrak{N}(\xi^*)$  such that the subdeterminants not vanishing at  $\xi^*$  can not vanish at any point of  $\mathfrak{N}(\xi^*)$ . Hence the rank has to be  $r$  at all points of  $\mathfrak{N}(\xi^*)$ . We call  $\mathfrak{N}(\xi^*)$  a region of *constant rank* with respect to the index or the indices concerned.

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<sup>1)</sup> The  $\tilde{e}_\lambda^*$  and  $\tilde{e}_\lambda$  were formerly called the contra- and covariant *measuring vectors* belonging to  $(\alpha)$ .

<sup>2)</sup> All these differences should not be neglected. Neglecting them leads to many difficulties especially after the introduction of special kinds of processes of differentiation.

Now let there be two subregions of constant rank in  $R$  in which the rank, always taken with respect to the same indices, takes the values  $r_1$  and  $r_2 \geq r_1$ . Then in the first region all subdeterminants with  $r_1 + 1$  rows vanish. But this implies that they vanish at all points of  $R$ . Hence  $r_1 = r_2$ . This proves that a quantity analytic in  $R$  has the same rank with respect to some given indices in all subregions of constant rank in  $R$  and that there is no point where it can have a higher rank. This does however not imply that there are no points in  $R$  where the rank is lower than in the subregions of constant rank. But these points can not form a region. For instance a vector field  $v^\alpha$  analytic in  $R$  and not vanishing at all points of  $R$  has rank zero at all points where  $v^\alpha = 0$  and rank 1 at all other points. Points where  $v^\alpha = 0$  may occur but they can not form a region because in that case  $v^\alpha$  would vanish all over  $R$ . If  $v^\alpha = 0$  at  $\xi^\alpha$ , in every  $\mathfrak{N}(\xi^\alpha)$  there is at least one point where  $v^\alpha \neq 0$ .

### Exercise.

II 3,1<sup>1)</sup>). Let  $\Phi_A$  be a tensor field with the collecting index  $A$ . If  $\Psi_Q$  is a tensor field with the collecting index  $Q$  whose components are functions of the  $\xi^\alpha$  and the  $\Phi_A$ , prove that the  $\partial \Psi_Q / \partial \Phi_A$ , where the  $\xi^\alpha$  are supposed to be left constant, are components of a tensor field. The same holds mutatis mutandis for tensor densities.

### § 4. The $X_m$ in $X_n$ .<sup>2)</sup>

Let  $N$  functions  $F^a(\xi^\alpha)$ ;  $a = 1, \dots, N$  be analytic in  $\mathfrak{N}(\xi^\alpha)$ . The matrix

$$(4.1) \quad \parallel \partial_\lambda F^a \parallel$$

is the *functional matrix* of the set of functions and its rank  $r$  at  $\xi^\alpha$  is the *rank* of the set at that point.  $r$  is also the maximum number of linearly independent differentials among the  $dF^a$ , hence  $r \leq n$ ;  $r \leq N$ . It is well known that the functions  $F^a$  are independent in  $\mathfrak{N}(\xi^\alpha)$  if and only if  $r = N$  at at least one point of  $\mathfrak{N}(\xi^\alpha)$ .

Let  $r = N \leq n$ . Then we can take  $m = n - N$  auxiliary functions  $F^{N+1}(\xi^\alpha), \dots, F^n(\xi^\alpha)$  analytic in  $\mathfrak{N}(\xi^\alpha)$ , such that the rank of the set  $F^h$ ;  $h = 1, \dots, n$  equals  $n$  at all points of  $\mathfrak{N}(\xi^\alpha)$ . Then the transformation

$$(4.2) \quad \xi^h = F^h(\xi^\alpha); \quad h = 1, \dots, n$$

is an allowable coordinate transformation in  $\mathfrak{N}(\xi^\alpha)$ . That proves the

<sup>1)</sup> Cf. HORAK 1929, 1.

<sup>2)</sup> Cf. for a more elaborate treatment P. P. 1949, 1, Ch. II; R. S. 1947, 1, § 2.

*Theorem of adaptation.*

If the functions  $F^a(\xi^x)$ ;  $a = 1, \dots, N$ , are analytic in  $\mathfrak{N}(\xi^x)$  and if  $r = N \leq n$ , there exists in  $\mathfrak{N}(\xi^x)$  an allowable coordinate system  $\xi^h$ ;  $h = 1, \dots, n$ , such that  $\xi^a = F^a(\xi^x)$ ;  $a = 1, \dots, N$ .

Let the system of equations

$$(4.3) \quad F^a(\xi^x) = 0$$

be consistent. We call  $r$  the *rank of the system*. Every point  $\xi^x$  of  $\mathfrak{N}(\xi^x)$  satisfying (4.3) is called a *null point* of (4.3) and the set of all null points is called the *null manifold* of (4.3). (4.3) is said to be a *null form* of this manifold. Two systems of equations having the same null points in  $\mathfrak{N}(\xi^x)$  are said to be *equivalent in that region*. Two equivalent systems need not have the same rank at all null points. E.g.  $\xi^1 = 0$ ;  $\xi^2 = 0$  and  $\xi^1 \xi^1 = 0$ ;  $\xi^2 = 0$  at the point  $\xi^x = 0$ .

Now we consider a system of this kind

$$(4.4) \quad C^x(\xi^x) = 0; \quad x = m + 1, \dots, n; \quad (C^x \text{ analytic})$$

with rank  $n - m$  at the null point  $\xi^x$ . Then there exists an  $\mathfrak{N}(\xi^x)$  such that the rank is  $n - m$  at all its points. In this case (4.4) is called *minimal regular of dimension m at  $\xi^x$* . The number  $m$  is called the *dimension* of the null manifold and we have  $0 \leq m \leq n$ . From the definition we see that every subsystem of a system which is minimal regular at  $\xi^x$  is itself minimal regular at  $\xi^x$  and we see also that there exists an  $\mathfrak{N}(\xi^x)$  such that the system is minimal regular and the null manifold  $m$ -dimensional at all null points in  $\mathfrak{N}(\xi^x)$ . The notions "minimal regular" and "dimension" are invariant for all allowable coordinate transformations. Of the following four examples of systems of equations only the last one is minimal regular at  $\xi^x = 0$ .

$$(4.5) \quad \begin{cases} \text{a)} & \xi^1 \xi^2 = 0; \quad \xi^1 \xi^3 = 0 \\ \text{b)} & \xi^1 \xi^1 = 0; \quad \xi^2 = 0 \\ \text{c)} & \xi^1 \xi^1 = 0; \quad \xi^1 = 0; \quad \xi^2 = 0 \\ \text{d)} & \xi^1 = 0; \quad \xi^2 = 0 \end{cases}$$

and only (4.5 b) and (4.5 c) are equivalent to a minimal regular system, viz. (4.5 d). The null manifold of (4.5 a) has no dimension at all at  $\xi^x = 0$ . If the  $\xi^x$  are rectilinear coordinates in  $E_n$ , (4.5 a) represents a figure consisting of an  $E_{n-1}$  and an  $E_{n-2}$  through the origin.

If (4.3) is minimal regular of dimension  $m$  at  $\mathfrak{N}(\xi^x)$ , according to the theorem of adaptation there exists a coordinate system  $(h)$ ;  $h = 1, \dots, n$  such that (4.3) is equivalent to the system

$$(4.6) \quad \xi^x = 0; \quad x = m + 1, \dots, n.$$

Accordingly the  $\xi^a$ ;  $a = 1, \dots, m$  can be used as coordinates in the null manifold. Now the pseudo-group  $\mathfrak{P}_n$  contains as a sub-pseudo-group the pseudo-group  $\mathfrak{P}_m$  of all analytic and invertible transformations of the  $\xi^a$  only, the  $\xi^x$  being left invariant. Hence the null manifold is an  $X_m$ . Such an  $X_m$  is said to be *imbedded in* an  $X_n$  or an  $X_m$  in  $X_n$ . An  $X_m$  in  $X_n$  is often called an *m-dimensional surface* and for  $m = 1, 2$  and  $n - 1$  it is also called *curve*, *surface* and *hypersurface* respectively.<sup>1)</sup>

The quantity

$$(4.7) \quad C_x^x \stackrel{\text{def}}{=} \partial_x C^x; \quad x = m + 1, \dots, n$$

with rank  $n - m$  is the *covariant connecting quantity* of the  $X_m$  in  $X_n$  (cf. I § 2 point 7 and II § 3).

If a function  $s(\xi^x)$  is analytic in  $\mathfrak{N}(\xi^x)$  and zero at all null points of (4.4) in  $\mathfrak{N}(\xi^x)$ , there always exists an  $\mathfrak{N}(\xi^x)$  such that  $s$  satisfies in this  $\mathfrak{N}(\xi^x)$  an equation of the form

$$(4.8) \quad s = \Phi_x(\xi^x) C^x; \quad x = m + 1, \dots, n$$

with analytic functions  $\Phi_x$ . This is the so-called *first base theorem*.<sup>2)</sup> A set of  $n - m$  functions  $C^{x'}$ ;  $x' = (m + 1)', \dots, n'$  which take the value zero at every null point of (4.4) and whose differentials  $dC^{x'}$  are linearly independent at  $\xi^x$ , constitutes a *base* of the  $X_m$  at  $\xi^x$  and consequently at all null points in some  $\mathfrak{N}(\xi^x)$ .<sup>3)</sup> Hence the  $C^x$  in (4.4) constitute a base. As a consequence of the first base theorem we have equations of the form

$$(4.9) \quad C^{x'} = C_x^{x'} C^x; \quad x = m + 1, \dots, n; \quad x' = (m + 1)', \dots, n'$$

valid in an  $\mathfrak{N}(\xi^x)$  with coefficients  $C_x^{x'}$  analytic in this region. Hence

$$(4.10) \quad dC^{x'} = C_x^{x'} dC^x; \quad x = m + 1, \dots, n; \quad x' = (m + 1)', \dots, n'$$

<sup>1)</sup> Note that in consequence of the definition an  $X_m$  in  $X_n$  never has singularities. An  $X_2$  or surface in our sense in ordinary space is never a surface in the ordinary sense with singular points and curves, but it can be a part of such a surface, free from singularities.

<sup>2)</sup> Cf. for a proof of this well known theorem for instance P. P. 1949, 1, p. 37; R. S. 1947, 1, § 3.

<sup>3)</sup> We define a base in a different way as KÄHLER, who also allows sets of more than  $n - m$  functions. Cf. 1934, 1, p. 13.

at all null points in this region. We see from this that  $C^x = 0$  is a null form of the  $X_m$  in some  $\mathfrak{N}(\xi^\alpha)$  which is minimal regular in this region.

This is the *second base theorem*.<sup>1)</sup> It is easy to see that  $\text{Det}(C_x^x) \neq 0$  in  $\mathfrak{N}(\xi^\alpha)$ .

Every transformation of the form (4.9) with  $\text{Det}(C_x^x) \neq 0$  is called a *base transformation*. If a coordinate transformation and a base transformation are effected simultaneously we have

$$(4.11) \quad C_{\lambda'}^x = A_{\lambda'}^\lambda C_x^\lambda C_\lambda^x$$

valid at the null points of the region considered.

There is another way of defining an  $X_m$  in  $X_n$ . Let the  $\eta^a$ ;  $a = 1, \dots, m$  be allowable coordinates of some  $X_m$  which has no point in common with  $X_n$  and let there be given a system of equations

$$(4.12) \quad \xi^\alpha = B^\alpha(\eta^a); \quad a = 1, \dots, m$$

where the  $B^\alpha$  are analytic in an  $\mathfrak{N}(\eta^a)$  and the matrix of

$$(4.13) \quad B_b^\alpha \stackrel{\text{def}}{=} \partial_b \xi^\alpha; \quad \partial_b \stackrel{\text{def}}{=} \frac{\partial}{\partial \eta^b}; \quad b = 1, \dots, m$$

has rank  $m$  in this region. Then (4.12) fixes a one to one correspondence between the points of  $X_m$  in an  $\mathfrak{N}(\eta^a)$  and certain points of  $X_n$  in an  $\mathfrak{N}(\xi^\alpha)$ ;  $\xi^\alpha \stackrel{\text{def}}{=} B^\alpha(\eta^a)$ . Hence every point of this  $\mathfrak{N}(\eta^a)$  can be identified with its corresponding point in  $\mathfrak{N}(\xi^\alpha)$ . This process is called the *imbedding of the  $X_m$  in  $X_n$* . (4.12) is called the *parametric form* (or *parametric representation*) of the  $X_m$  in  $X_n$ , *minimal regular of dimension m at  $\eta^a$* , and  $B_b^\alpha$  is the *contravariant connecting quantity of the  $X_m$  in  $X_n$* .

In order to prove that this second definition of an  $X_m$  in  $X_n$  is in agreement with the first one, we start from the null form (4.4) and use the theorem of adaptation. According to this theorem the coordinate system ( $\alpha$ ) can always be chosen in such a way that the null form takes the form  $\xi^\zeta = 0$ ;  $\zeta = m+1, \dots, n$ . Now, writing these equations in the form

$$(4.14) \quad \xi^\alpha = \xi^\alpha; \quad \xi^\zeta = 0; \quad \alpha = 1, \dots, m; \quad \zeta = m+1, \dots, n$$

and considering the  $\xi^\alpha$  as parameters we see that we have got a minimal regular parametric form. Now let us start from the parametric form (4.12). Since  $B_b^\alpha$  has rank  $m$ , the indices  $1, \dots, n$  can always be rearranged in such a way that  $B_b^\alpha$ ;  $\alpha = 1, \dots, m$ ;  $b = 1, \dots, m$  has rank  $m$ .

<sup>1)</sup> Cf. for a proof of this well known theorem for instance P. P. 1949, 1, p. 37; R. S. 1947, 1, § 3.

Then the transformation  $\xi^x \rightarrow \xi^h$ ,  $h = 1, \dots, n$ , defined by

$$(4.15) \quad \xi^x = B^x(\xi^\alpha) + \delta_x^\alpha \xi^x; \quad \alpha = 1, \dots, m; \quad x = m + 1, \dots, n$$

is an allowable coordinate transformation in the region concerned. If this transformation is applied to (4.12) we get the system

$$(4.16) \quad \left\{ \begin{array}{ll} \text{a)} & B^\alpha(\xi^\alpha) = B^\alpha(\eta^\alpha) \\ & \alpha = 1, \dots, m; \quad x = m + 1, \dots, n \\ \text{b)} & B^\zeta(\xi^\alpha) + \delta_x^\zeta \xi^x = B^\zeta(\eta^\alpha); \\ & \alpha = 1, \dots, m; \quad \zeta = m + 1, \dots, n. \end{array} \right.$$

From (4.16a) it follows that  $\xi^\alpha = \eta^\alpha$  and consequently that (4.16b) is equivalent to the system  $\xi^x = 0$ , constituting a minimal regular null form.<sup>1)</sup>

To every allowable coordinate system ( $\alpha$ ) there belong for every value of  $m < n$  just  $\binom{n}{m}$  sets of  $\infty^{n-m} X_m$ 's with equations of the form

$$(4.17) \quad \xi^\zeta = c^\zeta; \quad \zeta = m + 1, \dots, n$$

where the  $c^\zeta$  are  $n - m$  parameters. These  $X_m$ 's are called the *coordinate-* $X_m$ 's (-*curves*, -*surfaces*, -*hypersurfaces* for  $m = 1, 2, n - 1$ ) of ( $\alpha$ ). Hence the theorem of adaptation proves that every  $X_m$  in  $X_n$  is coordinate- $X_m$  of some suitable allowable coordinate system.  $\infty^{n-m} X_m$ 's with equations of the form (4.17) are said to constitute a *normal system* of  $X_m$ 's. The most important property of a normal system is that *every point of the region considered lies in one and only one  $X_m$  of the system*. From this we see once more that an  $X_m$  in our sense must always be considered in a certain region only. For instance the straight lines through the origin of a centred  $E_n$  form a normal system of  $X_1$ 's for all regions not containing the origin and for these regions only.

The coordinate- $X_m$ 's are said to constitute a *net*, the *net of the coordinate system*. A net contains for instance  $n$  normal systems of  $X_1$ 's whose directions at every point are linearly independent. Not every net can belong to a coordinate system (cf. II § 9).

The  $\infty^{n-m} X_m$ 's of a normal system themselves constitute an  $(n - m)$ -dimensional manifold. If the equations have the form (4.17) the  $\xi^\zeta$  can be used as coordinates in this manifold. Now the pseudo-group  $\mathfrak{P}_n$  in  $\mathfrak{N}(\xi^x)$  contains a sub-pseudo-group  $\mathfrak{P}_{n-m}$  of all analytic and invertible transformations of the  $\xi^\zeta$  only, the  $\xi^\alpha$ ;  $\alpha = 1, \dots, m$ , being left invariant. Hence the manifold is an  $X_{n-m}$ . It is called the *reduction (Zusammenlegung) of the  $X_n$  with respect to the normal system of*

<sup>1)</sup> In P. P. 1949, I, II § 4 it has been proved that it is also possible to derive a minimal regular parametric form from a minimal regular null form and vice versa without using coordinate transformations.

$X_m$ 's. This process of reduction is a generalization of the process defined in Ch. I § 1.

If an  $X_m$  in  $X_n$  is given by its null form and its parametric form, the connecting quantities  $B_b^x$  and  $C_\lambda^x$  are known at every point of  $X_m$ . In the tangent  $E_n$  of each of these points they fix an  $E_m$  called the *tangent  $E_m$  of  $X_m$* . It is a sub-space of the tangent  $E_n$  at that point. It can be identified with the tangent  $E_m$  of  $X_m$ , as defined in II § 3.  $B_b^x$  and  $C_\lambda^x$  can now be used in the way described in I § 4 to form

1° the section of any covariant tensor of  $X_n$ , called here the *section with  $X_m$* , e.g.  $'w_b = B_b^\lambda w_\lambda$  (cf. I 4.3, 14a);

2° the reduction of any contravariant tensor of  $X_n$ , called here the *reduction with respect to  $X_m$* , e.g.  $'v^x = C_\lambda^x v^\lambda$  (cf. I 4.7, 15a);

3° the  $X_n$ -components of a contravariant tensor of  $X_m$ , e.g.  $p^x = B_a^x p^a$  (cf. I 4.5, 16a);

4° the  $X_n$ -components of a covariant tensor in the  $E_{n-m}$  arising from the tangent  $E_n$  by reduction with respect to the tangent  $E_m$ , e.g.  $s_\lambda = C_\lambda^y s_y$  (cf. I 4.9, 17a).

An  $X_m$  in  $X_n$  is called *rigged* if all its tangent  $E_m$ 's are rigged (cf. I § 4), that is if, at each of its points, an  $E_{n-m}$  in the tangent  $E_n$  is given having no direction in common with the tangent  $E_m$ . In this case which occurs frequently in differential geometry the unity tensors  $B_b^a$  and  $C_\lambda^x$  of the  $X_m$  and of the local  $E_{n-m}$  also have components  $B_b^x$ ,  $B_\lambda^a$ ,  $B_\lambda^x$  and  $C_\lambda^x$ ,  $C_\lambda^y$ ,  $C_\lambda^z$ . The quantities can be used in the way described in I § 4 to form:

1° the *section with  $E_{n-m}$*  of any covariant tensor of  $X_n$ , e.g.  $'w_y = C_\lambda^y w_\lambda$  (cf. I 4.14b);

2° the *reduction with respect to  $E_{n-m}$*  of any contravariant tensor of  $X_n$ , e.g.  $'v^a = B_a^x v^x$  (cf. I 4.15b);

3° the  $X_n$ -components of a covariant tensor of  $X_m$ , e.g.  $q_\lambda = B_\lambda^b q_b$  (cf. I 4.16b);

4° the  $X_n$ -components of a contravariant tensor in the  $E_{n-m}$ , e.g.  $r^x = C_x^y r^y$  (cf. I 4.17b);

5° the  $X_m$ -part and the  $E_{n-m}$ -part of any tensor of  $X_n$ , e.g. (cf. I 4.21)

$$(4.18) \quad \begin{cases} 'v^x = B_\lambda^x v^\lambda; & 'w_\lambda = B_\lambda^x w_x \\ ''v^x = C_\lambda^x v^\lambda; & ''w_\lambda = C_\lambda^x w_x \\ v^x = B_\lambda^x v^\lambda + C_\lambda^x v^\lambda; & w_\lambda = B_\lambda^x w_x + C_\lambda^x w_x. \end{cases}$$

If a field, for instance a scalar field  $\rho$ , is defined in an  $\mathfrak{N}(\eta^a)$  over the  $X_m$  with the parametric equations  $\xi^x = B^x(\eta^a)$ , it is often useful to consider a field, analytic (or of a certain class  $u$ ) at all points of an  $\mathfrak{N}(\xi^x)$ ;

$\xi^a = B^a_b(\eta^a)$ , having at all points of  $\mathfrak{N}(\eta^a)$  the same values as the given field. This process is called a *prolongation* of the field over  $X_n$  and it is convenient to denote the prolonged field with the same kernel. Then  $\partial_\lambda \phi$  also has a sense and the equation  $\partial_b \phi = B^a_b \partial_\mu \phi$  holds. This enables us to use the operator  $\partial_\lambda$  freely. Of course all those results obtained in this way, that can be expressed in terms of quantities and operators of  $X_m$  only, are independent of the manner of prolongation.

### § 5. The $E_m$ -field or $X_n^m$ in $X_n$ . Systems of linear partial differential equations.

If we take the tangent  $E_m$ 's of the  $X_m$ 's belonging to a normal system we get an  $E_m$ -field. But this is a very special field because its  $E_m$ 's can be arranged in  $\infty^{n-m}$  sets of  $\infty^m E_m$ 's each set consisting of the tangent  $E_m$ 's of an  $X_m$ .

Instead of this special field we may now define a general  $E_m$ -field by giving, in each tangent  $E_n$  of a point of a region, an  $E_m$  through the contactpoint by means of its connecting quantities  $B^x_b$  and (or)  $C^x_\lambda$ . Such a general field is called an  $X_n^m$  in  $X_n$ .<sup>1)</sup> The  $E_m$  at a point is called tangent  $E_m$  because it lies in the tangent  $E_n$ . If a  $b$ -transformation and an  $x$ -transformation is effected in all  $E_n$ 's simultaneously we have

$$(5.1) \quad \begin{cases} \text{a)} & B^x_b = B^b_{b'} B^x_{b'}; \quad \text{b)} & C^x_\lambda = C^x_{\lambda'} C^{\lambda'}_\lambda \\ b = 1, \dots, m; & x = m + 1, \dots, n \\ b' = 1', \dots, m'; & x' = (m + 1)', \dots, n' \end{cases}$$

but the  $B^b_{b'}$  and the  $C^x_{\lambda'}$  are now functions of the  $\xi^a$ . The  $E_m$  at each point is spanned by the contravariant vectors  $B^x_1, \dots, B^x_m$  and also by the covariant vectors  $C^{\lambda+1}_\lambda, \dots, C^n_\lambda$ . A  $b$ -transformation can be interpreted as a linear homogeneous transformation of the  $m$  vectors  $B^x_b$ ;  $b = 1, \dots, m$  and an  $x$ -transformation as a transformation of the same kind of the  $n - m$  vectors  $C^x_\lambda$ ;  $x = m + 1, \dots, n$ . But the  $b$ -transformations here are not connected in any way with a coordinate transformation in some variables  $\eta^a$  (cf. II § 4).

By means of  $B^x_b$  and  $C^x_\lambda$  we are now able to define the *section with  $X_n^m$*  of any covariant tensor of  $X_n$ , the *reduction with respect to  $X_n^m$*  of

<sup>1)</sup> Cf. for the elder literature on the  $X_n^m$  and anholonomic coordinate systems VRANCEANU 1926, 1; 2; 1927, 1; 2; 1928, 1; 2; 3; 4; 1929, 1; 2; 1930, 1; 1931, 1; 1932, 1; 1933, 1; 1934, 1; 1936, 1; HLAVATY 1924, 1; 1928, 1; 1930, 1; 2; T. Y. THOMAS 1926, 1; HORAK 1927, 1; 2; SCHOUTEN 1928, 1; 1929, 3; SYNGE 1928, 1; SCHLESINGER 1929, 1; SCHOUTEN and v. KAMPEN 1930, 2; BORTOLOTTI 1931, 1; 2; 1932, 1; 1933, 1; 1937, 2; 3; 1940, 1; DIENES 1932, 1. Cf. for the newer literature Ch. V § 7.

any contravariant tensor of  $X_n$  and the  $X_n$ -components of any contravariant tensor of  $X_n^m$  and any covariant tensor in the local  $E_{n-m}$  arising from reduction of the tangent  $E_n$  with respect to the  $X_n^m$ . The formulae are the same as (I 4.14a, 15a, 16a, 17a). If, besides the  $X_n^m$  we give an  $X_n^{n-m}$  such that the tangent  $E_m$  and  $E_{n-m}$  have no direction in common, the  $X_n^m$  (and also the  $X_n^{n-m}$ ) are said to be *rigged*. In this case we have  $B_\lambda^a, B_b^a, B_b^x, B_\lambda^x$  and  $C_\lambda^x, C_y^x, C_y^z, C_\lambda^z$  at every point. By means of these we are now able to construct also the section with  $X_n^{n-m}$  of any covariant tensor of  $X_n$ , the reduction with respect to  $X_n^{n-m}$  of any contravariant tensor of  $X_n$ , the  $X_n$ -components of every tensor of  $X_n^m$  and of  $X_n^{n-m}$  and the  $X_n^m$ -part and  $X_n^{n-m}$ -part of every tensor of  $X_n$ .

Now the question arises whether a given  $E_m$ -field belongs to some normal system of  $X_m$ 's. But this is only a special case of a far more general question. An  $X_p$  is said to *envelop* an  $E_m$ -field if its tangent  $E_p$  contains, at every point, the tangent  $E_m$  of the field and it is said to be *enveloped by* an  $E_m$ -field if its tangent  $E_p$  is contained at every point in the tangent  $E_m$ . Hence there are two problems, the *outer problem* concerning all enveloping  $X_p$ 's and the *inner problem* concerning all enveloped  $X_p$ 's.<sup>1)</sup> We require especially all enveloped normal systems of  $X_p$ 's for the maximum value of  $p$  and also all enveloping normal systems of  $X_p$ 's for the minimum value of  $p$ . According to the definition an enveloping  $X_m$  is at the same time enveloped, hence the two problems are identical for  $p = m$ . But for all other values of  $p$  they are essentially different and require quite different mathematical methods.

The inner problem is much more difficult than the outer problem. Only the case  $m = n - 1$ , known as PFAFF's *problem*, can be dealt with by fairly elementary means as will be shown in II § 7.

The outer problem is closely connected with the theory of systems of linear partial differential equations of the first order. We give first a definition of totally integrable systems of general partial differential equations.

If a system of  $N$  partial differential equations of the first order with  $n$  independent variables  $\xi^x$  and  $M$  unknown variables  $\underline{z}_b$ ;  $b = 1, \dots, M$

$$(5.2) \quad F^i(\xi^x, \underline{z}_b, \partial_x \underline{z}_b) = 0; \quad i = 1, \dots, N; \quad b = 1, \dots, M$$

is given, every set of functions  $\underline{z}$  satisfying (5.2) is called a *solution* of (5.2) and every set of values  $\xi^x, \underline{\eta}_b^*, \underline{\eta}_\lambda^*$  satisfying the equations

$$(5.3) \quad F^i(\xi^x, \underline{\eta}_b^*, \underline{\eta}_\lambda^*) = 0$$

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<sup>1)</sup> Cf. P. P. 1949, 1 Ch. III, VIII, also for literature.

is called a *null point* of (5.2). Now the system (5.2) is said to be *totally integrable*<sup>1)</sup> in an  $\mathfrak{R}(\xi^*, \frac{\eta}{b}, \frac{\eta_\lambda}{b})$  if for every null point  $\xi^*, \frac{\eta}{b}, \frac{\eta_\lambda}{b}$  in this region there exists at least one solution of (5.2) satisfying the condition that for  $\xi^* = \xi^*$

$$(5.4) \quad z = \frac{\eta}{b}; \quad \partial_\lambda \frac{z}{b} = \frac{\eta_\lambda}{b}.$$

The necessary and sufficient conditions for a system to be totally integrable are called its first *conditions of integrability*.

Now let  $p$  be the smallest number for which a normal system of  $X_p$ 's exists enveloping the  $E_m$ -field  $B_b^x, C_x^x$  and let

$$(5.5) \quad F^x(\xi^*) = c^x; \quad \xi^* = \Phi^x(\eta^a, c^x); \quad \begin{array}{l} a = \bar{1}, \dots, \bar{p}; \\ x = \overline{p+1}, \dots, \bar{n} \end{array}$$

be the null form and the parametric form of the enveloping normal system. Then the  $F^x$  are  $n-p$  independent solutions (i.e. solutions with linearly independent gradients) of the homogeneous linear equations

$$(5.6) \quad B_b^\mu \partial_\mu f = 0; \quad b = 1, \dots, m$$

and every linear element  $d\xi^*$  satisfying the equations

$$(5.7) \quad C_\mu^x d\xi^\mu = 0; \quad x = m+1, \dots, n$$

also satisfies the equations

$$(5.8) \quad d\xi^\mu \partial_\mu F^x = 0; \quad x = \overline{p+1}, \dots, \bar{n}.^2)$$

The equations (5.6) and (5.7) are said to be *adjoint to each other*. The first gives the condition for the tangent  $E_p$ 's of an enveloping  $X_p$  and the second the condition for its linear elements. If  $F$  is a solution of (5.6),  $F$  is called an *integral function* of (5.7) and the equation  $F = \text{const.}$  an *integral* of (5.7).

If the operator  $B_b^\mu \partial_\mu$  is applied twice in succession we get, after alternation,

$$(5.9) \quad (B_{[\nu}^\mu \partial_{\mu]} B_b^\lambda) \partial_\lambda f = 0$$

because  $\partial_{[\nu} \partial_{\mu]} f = 0$ . Hence every solution of (5.6) must satisfy the  $\binom{m}{2}$  linear homogeneous equations (5.9). If, among these equations, there are exactly  $\mu_1$  linearly independent of (5.6) and of each other, we get

<sup>1)</sup> For the definition of total integrability for systems of higher order see for instance P. P. 1949, 1, p. 89.

<sup>2)</sup> Cf. for the conditions for one single enveloping  $X_p$ , P. P. 1949, 1, p. 85f.

a system of  $m_1 \stackrel{\text{def}}{=} m + \mu_1$  equations

$$(5.10) \quad B_{b_1}^\mu \partial_\mu f = 0; \quad b_1 = 1, \dots, m_1$$

called the *first derived system* of (5.6). This system has the same solutions as (5.6).  $\mu_1 = 0$  if and only if

$$(5.11) \quad C_{c;b}^{x;\mu} \stackrel{\text{def}}{=} -B_{[c}^\nu \partial_{\nu]} B_{b]}^\mu C_\mu^x = +B_c^\nu B_b^\mu \partial_{[\nu} C_{\mu]}^x = 0.$$

It is remarkable that  $C_{c;b}^{x;\mu}$  is a tensor though  $B_{[c}^\nu \partial_{\nu]} B_{b]}^\mu$  does not have this property. If  $\mu_1 = 0$  the system (5.6) is said to be *complete*. It can be proved that a complete system of  $m$  equations is totally integrable, that it has just  $n - m$  independent solutions and that its integrability conditions<sup>1)</sup> are identically satisfied. Hence if (5.11) is satisfied and only in that case there exists an enveloped and enveloping normal system of  $X_m$ 's. The  $E_m$ -field is then called  $X_m$ -*forming*.

The solution of the complete system is equivalent to the determination of this set of  $\infty^{n-m}$   $X_m$ 's. If the  $X_m$ 's are known, every function of the  $\xi^r$  that is constant on each of these  $X_m$ 's is a solution.

If the first derived system is complete there exists a normal system of  $X_{m+\mu_1}$ 's. If it is not complete a second derived system with  $m + \mu_1 + \mu_2$  equations can be found, and so on. At last we get a complete system having the same solutions as (5.6). If this last system consists of  $n$  equations, (5.6) has no solutions except the trivial one  $f = \text{const}$ . The numbers  $\mu_1, \mu_2, \dots$  are arithmetic invariants of the  $E_m$ -field.

By

$$(5.12) \quad v^{\kappa_1 \dots \kappa_m} \propto B_1^{[\kappa_1} \dots B_m^{\kappa_m]}$$

a contravariant pseudo- $m$ -vector is given with the  $m$ -direction of the tangent  $E_m$ . In the same way by

$$(5.13) \quad w_{\lambda_1 \dots \lambda_{n-m}} \propto C_{[\lambda_1}^{m+1} \dots C_{\lambda_{n-m}]}^n$$

a covariant pseudo- $(n - m)$ -vector is introduced with the same  $m$ -direction. The *n.a.s.* conditions (5.11) for the  $E_m$ -field to be  $X_m$ -forming can now be written as

$$(5.14) \quad (\partial_{\kappa_1} v^{\kappa_1 \dots \kappa_m}) w_{\kappa_2 \lambda_2 \dots \lambda_{n-m}} = 0 \text{ } ^2),$$

from which we may easily derive the following equivalent forms

$$(5.15) \quad (\partial_{\kappa_1} v^{\kappa_1 \kappa_2 \dots [\kappa_m]} v^{\nu_1 \dots \nu_m}) = 0 \text{ } ^3)$$

$$(5.16) \quad (\partial_{\kappa_1} w_{\kappa_2 \lambda_2 \dots \lambda_{n-m}}) v^{\kappa_1 \dots \kappa_m} = 0 \text{ } ^4), \text{ } ^2)$$

$$(5.16a) \quad (\partial_{[\kappa_1} w_{\kappa_{m+1} \dots \kappa_n]} w_{\kappa_2] \lambda_2 \dots \lambda_{n-m}} = 0.$$

<sup>1)</sup> Cf. for the exact formulation and the proof of the existence theorem for instance P. P. 1949, 1, p. 108.

<sup>2)</sup> SCHOUTEN 1918, 2; SCHOUTEN and STRUIK 1919, 1; STRUIK 1922, 1, p. 53ff.

<sup>3)</sup> v. WEBER 1900, 1, p. 73ff. has an equivalent form.

<sup>4)</sup> v. WEBER 1900, 1, p. 103 has an equivalent form.

To every simple contravariant pseudo-vector field  $v^{x_1 \dots x_m}$  that is  $X_m$ -forming there belongs a complete system with the equations

$$(5.17) \quad v^{x_1 \dots x_m} \partial_{x_i} f = 0 \quad \text{or} \quad w_{[\lambda_1 \dots \lambda_{n-m}]} \partial_{\lambda_j} f = 0,$$

expressing the fact that the  $(n-1)$ -direction of  $\partial_{\lambda_j} f$  contains the tangent  $m$ -direction, and the adjoint system with the equations

$$(5.18) \quad v^{[x_1 \dots x_m]} d\xi^x = 0 \quad \text{or} \quad w_{\lambda_1 \dots \lambda_{n-m}} d\xi^{\lambda_1} = 0$$

expressing the fact that the direction of  $d\xi^x$  lies in the tangent  $m$ -direction.

If the  $E_m$ -field is  $X_m$ -forming and if

$$(5.19) \quad F^x(\xi^x) = c^x; \quad x = m+1, \dots, n$$

are the equations of the normal system of  $X_m$ 's, the tangent  $E_m$  at every point lies in the tangent  $E_{n-1}$  of each of the  $n-m$   $X_{n-1}$ 's (5.19). Hence

$$(5.20) \quad w_{\lambda_1 \dots \lambda_{n-m}} \propto (\partial_{[\lambda_1} F^{m+1}) \dots \partial_{\lambda_{n-m}]} F^n$$

and this equation expresses the fact that  $w_{\lambda_1 \dots \lambda_{n-m}}$  is a *gradient product* (alternating product of gradients) to within a scalar factor. This proves that (5.16a) is the necessary and sufficient condition for  $w_{\lambda_1 \dots \lambda_{n-m}}$  to be a gradient product to within a scalar factor.

The theory of homogeneous linear equations of the first order can be generalized for non homogeneous linear equations of the form

$$(5.21) \quad B_b^\mu \partial_\mu f = \varphi_b(\xi^x); \quad b = 1, \dots, m,$$

where  $B_b^\mu$  has rank  $m$ . It can be proved<sup>1)</sup> that such a system is totally integrable and that it has just  $n-m+1$  independent solutions if and only if (5.6), called the *reduced system* of (5.21), is complete, that is, if there exist functions  $\sigma_{cb}^a(\xi^x)$  such that

$$(5.22) \quad B_{[c}^\mu \partial_{|\mu|} B_{b]}^\nu = \sigma_{cb}^a B_a^\nu$$

and if moreover the  $\varphi_b$  satisfy the equations

$$(5.23) \quad B_{[c}^\mu \partial_{|\mu|} \varphi_{b]} = \sigma_{cb}^a \varphi_a.$$

If the integrability conditions (5.22, 23) are satisfied, (5.21) is called *complete*. If  $f^1, \dots, f^{n-m}$  are  $n-m$  independent solutions of the reduced system, and if  $\psi(f^1, \dots, f^{n-m})$  is one solution of the complete system (5.21), the general solution is  $f + \psi(f^1, \dots, f^{n-m})$  where  $\psi$  denotes an arbitrary analytic function.<sup>2)</sup>

<sup>1)</sup> Cf. for instance P. P. 1949, 1, III § 14.

<sup>2)</sup> Cf. for the exact formulation and the proof of the existence theorem for instance P. P. 1949, 1, p. 111.

We apply this theory first to the gradient equation

$$(5.24) \quad \partial_\mu p = w_\mu(\xi^*).$$

If this system is written in the form

$$(5.25) \quad A_\lambda^\mu \partial_\mu p = w_\lambda$$

we see that the reduced system is complete and that the coefficients  $\sigma$  are all zero. Hence here the only integrability conditions are

$$(5.26) \quad \partial_{[\mu} w_{\lambda]} = 0.$$

In fact, if a solution  $\overset{\theta}{p}$  of (5.24) is known, the general solution is  $p = \overset{\theta}{p} + c$  and this means that the value of  $p$  at any point  $\xi^*$  can be given arbitrarily.

#### Exercise.

II 5.1. If a vector field  $v^*$  is given in  $\mathfrak{N}(\xi^*)$ , there always exists a coordinate system  $(x)$  in an  $\mathfrak{N}(\xi^*)$  such that  $v^* = e_1^*$ . Every covariant  $(n-1)$ -vector in  $X_n$  is product of a scalar and a gradient product.<sup>1)</sup>

## § 6. The invariant operators Rot and Div.

As we have seen in II § 3, the derivatives of the components of a tensor with valence  $> 0$  with respect to the  $\xi^*$  in an  $X_n$  are not components of a tensor. But if we form the *alternated* derivative of a *covariant*  $q$ -vector, the additional term in

$$(6.1) \quad \left\{ \begin{array}{l} \partial_{[\mu'} w_{\lambda'_1 \dots \lambda'_q]} = A_{\mu' \lambda'_1 \dots \lambda'_q}^{\mu \lambda_1 \dots \lambda_q} \partial_{[\mu} w_{\lambda_1 \dots \lambda_q]} + w_{\lambda_1 \dots \lambda_q} \partial_{[\mu'} A_{\lambda'_1 \dots \lambda'_q]}^{\lambda_1} \\ = A_{\mu' \lambda'_1 \dots \lambda'_q}^{\mu \lambda_1 \dots \lambda_q} \partial_{[\mu} w_{\lambda_1 \dots \lambda_q]} \end{array} \right.$$

vanishes because  $\partial_{[\mu'} A_{\lambda'_1]}^{\lambda_q]} = 0$ . We call this alternated derivative *multiplied by  $q+1$*  the *natural derivative* (cf. II § 3) and for  $q > 0$  also the *rotation* of  $w_{\lambda_1 \dots \lambda_q}$ . If we use the abridged notation of I § 7, we write  $\text{Rot } w$  and also  $(q+1) D w$ .  $\text{Rot } w$  vanishes identically if  $q=n$ . As a consequence of the definition the process of forming the natural derivative applied twice in succession, always leads to zero:

$$(6.2) \quad \left\{ \begin{array}{l} \text{Rot Grad } p = 2 D D p = 0 \\ \text{Rot Rot } w = (q+2)(q+1) D D w = 0. \end{array} \right.$$

All this is also true for  $W$ -scalars and covariant  $W$ - $q$ -vectors. If the corresponding  $(n-q)$ -vector  $\Delta$ -density  $\tilde{w}^{x_1 \dots x_p}$ ;  $p=n-q$  is formed from

<sup>1)</sup> GOURSAT 1922, 1, p. 117.

$w$  by means of  $\tilde{\mathfrak{G}}^{\kappa_1 \dots \kappa_p}$  (cf. I § 7) it follows that for  $p \geq 1$

$$(6.3) \quad \left\{ \begin{array}{l} \partial_\mu \tilde{w}^{\mu \kappa_2 \dots \kappa_p} = (-1)^{-\beta_q} \frac{1}{q!} \partial_\mu w_{\lambda_1 \dots \lambda_q} \tilde{\mathfrak{G}}^{\lambda_1 \dots \lambda_q \mu \kappa_2 \dots \kappa_p} \\ = (-1)^{q-\beta_q} \frac{1}{(q+1)!} (q+1) \partial_{[\mu} w_{\lambda_1 \dots \lambda_q]} \tilde{\mathfrak{G}}^{\mu \lambda_1 \dots \lambda_q \kappa_2 \dots \kappa_p}. \end{array} \right.$$

Hence,  $\partial_\mu \tilde{w}^{\mu \kappa_2 \dots \kappa_p}$  is a  $(p-1)$ -vector  $A$ -density of weight +1. It is called the *natural derivative* or the *divergence* of  $\tilde{w}^{\kappa_1 \dots \kappa_p}$  and it is written  $\text{Div } \tilde{w}$  in abridged notation.  $\text{Div } \tilde{w}$  is by definition zero if  $p=0$ . From (6.3) we see that if

$$(6.4) \quad W = \text{Rot } w; \quad \tilde{\mathfrak{W}} = \text{Div } \tilde{w},$$

$\tilde{\mathfrak{W}}$  corresponds to  $(-1)^{q-\beta_q+\beta_{q+1}} W$ . From the definition it follows that the rule concerning the result of two natural derivations in succession holds also in this case

$$(6.5) \quad \text{Div Div } \tilde{w} = 0.$$

All this is also true for contravariant  $p$ -vector densities of weight +1.

Let  $W_{\lambda_1 \dots \lambda_{q+1}}$  be a *simple*  $(q+1)$ -vector whose rotation vanishes in an  $\mathfrak{N}_\theta(\xi^\kappa)$ . Then according to (5.16 a) the  $E_{p-1}$ -field with the  $(p-1)$ -direction of  $W$  is  $X_{p-1}$ -forming and this means that there exists an  $\mathfrak{N}_\theta(\xi^\kappa)$  where  $W$  is a gradient product to within a scalar factor

$$(6.6) \quad W_{\lambda_1 \dots \lambda_{q+1}} = s (\partial_{[\lambda_1} s) \dots \partial_{\lambda_{q+1}] q+1} s \quad \text{or} \quad W = s [\bar{s}_1 \dots \bar{s}_{q+1}].$$

Taking the rotation we get

$$(6.7) \quad (\partial_{[\lambda} s) (\partial_{\lambda_1} s) \dots (\partial_{\lambda_{q+1}] q+1} s) = 0 \quad \text{or} \quad [\bar{s}_1 \bar{s}_2 \dots \bar{s}_{q+1}] = 0$$

and this equation expresses the fact that  $s$  can be written as a function of  $s_1, \dots, s_{q+1}$ . If we take  $s_1, \dots, s_{q+1}$  as the first  $q+1$  coordinates  $\xi^1, \dots, \xi^{q+1}$  of  $(\kappa)$ , we have in the region considered

$$(6.8) \quad W_{\lambda_1 \dots \lambda_{q+1}} = s \overset{I}{e}_{[\lambda_1} \dots \overset{q+1}{e}_{\lambda_{q+1}]}$$

where  $s$  is a function of  $\xi^1, \dots, \xi^{q+1}$  only. Now in the  $X_{q+1}$  of these variables we take a solution of the non homogeneous equation

$$(6.9) \quad \overset{I}{e}^\beta \partial_\beta z = s; \quad \beta = 1, \dots, q+1.$$

Then

$$(6.10) \quad W_{\lambda_1 \dots \lambda_{q+1}} = z \overset{2}{e}_{\lambda_1} \dots \overset{q+1}{e}_{\lambda_{q+1}}; \quad z_\lambda \stackrel{\text{def}}{=} \partial_\lambda z$$

and this proves that a simple covariant  $(q+1)$ -vector is a gradient product and consequently the rotation of some  $q$ -vector in an  $\mathfrak{N}_\theta(\xi^\kappa)$

if and only if its rotation vanishes in an  $\mathfrak{N}(\xi^*)$ . As a corollary it follows that every  $n$ -vector given in an  $\mathfrak{N}(\xi^*)$  is a gradient product in an  $\mathfrak{N}(\xi^*)$ .

If the  $(q+1)$ -vector is not simple it is still true that it is the rotation of a  $q$ -vector if and only if its rotation vanishes. This can be proved by induction. Let the theorem be proved for every value of  $q$ , ( $1 \leq q+1 \leq n-1$ ) in an  $X_{n-1}$ . Then it can be proved that it is also true in an  $X_n$ .<sup>1)</sup> Now if  $n \leq 3$  the  $(q+1)$ -vector is always simple and the theorem has been proved for simple  $(q+1)$ -vectors. Hence it is true for all values of  $n$ . The  $q$ -vector is determined to within an additive term whose rotation vanishes and it can be found by quadratures only.<sup>2)</sup> We give here the exact formulation of the theorem for  $(q+1)$ -vectors and for  $(p-1)$ -vector  $\Delta$ -densities:<sup>3)</sup>

*If  $W_{\lambda_1 \dots \lambda_{q+1}}$  is a covariant  $(q+1)$ -vector,  $1 \leq q+1 \leq n$ , analytic in an  $\mathfrak{N}(\xi^*)$ , it can be written in some  $\mathfrak{N}(\xi^*)$  as the gradient of a scalar for  $q=0$  and as the rotation of a  $q$ -vector for  $q > 0$ , if and only if there exists an  $\mathfrak{N}(\xi^*)$  where  $\text{Rot } W = 0$ . This condition is identically satisfied for  $q+1=n$ .*

*If  $\tilde{W}^{\lambda_1 \dots \lambda_{p-1}}$  is a contravariant  $(p-1)$ -vector  $\Delta$ -density of weight +1;  $0 \leq p-1 \leq n-1$ , analytic in an  $\mathfrak{N}(\xi^*)$ , it can be written in some  $\mathfrak{N}(\xi^*)$  as the divergence of a  $p$ -vector  $\Delta$ -density of weight +1, if and only if there exists an  $\mathfrak{N}(\xi^*)$  where  $\text{Div } \tilde{W} = 0$ . This condition is identically satisfied for  $p-1=0$ .*

There are of course theorems of exactly the same form for covariant  $W$ - $(q+1)$ -vectors and  $(p-1)$ -vector densities of weight +1.

Both theorems express conditions of integrability. For the equations

$$(6.11) \quad \partial_{[\mu} w_{\lambda_1 \dots \lambda_q]} = W_{\mu \lambda_1 \dots \lambda_q}$$

<sup>1)</sup> P. P. 1949, 1, II § 11. The whole theorem is proved there without using the theory of linear partial differential equations.

<sup>2)</sup> GOURSAT 1922, 1, p. 105 ff. Using the homotopy operator known from the theory of graded rings (H. CARTAN 1949, 1, p. 107) a very elegant expression can be found. If

$$w_{\lambda_1 \dots \lambda_q} = f_{\lambda_1 \dots \lambda_q}(\xi^*)$$

and

$$u_{\lambda_1 \dots \lambda_{q-1}} \stackrel{\text{def}}{=} \frac{1}{q} \xi^{\lambda_q} \int_{t=0}^{t=1} f_{\lambda_1 \dots \lambda_q}(\xi^* t) s t^{q-1} dt$$

it follows immediately that  $w = \text{Rot } u$ . If we write  $u = \text{Op } w$  it is proved easily that

$$\text{Rot Op } w - \text{Op Rot } w = w$$

for every field  $w$ . (From a personal communication of N. H. KUIPER.)

<sup>3)</sup> VOLTERRA 1889, 1 for  $R_n$ ; BROUWER 1906, 1, p. 22 for  $R_n$ .

the conditions are

$$(6.12) \quad \partial_{[\nu} W_{\mu\lambda_1\dots\lambda_q]} = 0$$

and for the equations

$$(6.13) \quad \partial_\mu \tilde{W}^{\mu\nu_2\dots\nu_p} = \tilde{W}^{\nu_2\dots\nu_p}$$

we have the conditions

$$(6.14) \quad \partial_\nu \tilde{W}^{\nu\nu_2\dots\nu_p} = 0.$$

We remark that these integrability conditions are independent of the manner of transformation of the objects concerned. For instance if  $W_{\mu\lambda_1\dots\lambda_q}$ , alternating in  $\mu\lambda_1\dots\lambda_q$ , is any geometric object whose components with respect to  $(\kappa)$  satisfy (6.12), in general no such relation holds for the components with respect to any other coordinate system  $(\kappa')$ . But nevertheless there exist functions  $w_{\lambda_1\dots\lambda_q}$  of the  $\xi^\kappa$  such that (6.11) holds.

Here the question arises whether there exist solutions of equations of the form

$$(6.15) \quad \partial_{[\mu} \pi_{\lambda_1\dots\lambda_q]A} = \Pi_{\mu\lambda_1\dots\lambda_q A}$$

where  $A$  is a collective index standing for any set of co- and contravariant indices and where  $\pi_{\lambda_1\dots\lambda_q A}$  and  $\Pi_{\mu\lambda_1\dots\lambda_q A}$  are alternating in the first  $q$  and  $q+1$  indices respectively. The equations are *not* invariant and  $\pi$  and  $\Pi$  need not be tensors. But they are closely connected with invariant equations to be dealt with in the next chapter. In order to show what is happening we take the simple example

$$(6.16) \quad \partial_{[\mu} \pi_{\lambda]}\cdot^\kappa_\varrho = \Pi_{\mu\lambda}\cdot^\kappa_\varrho.$$

We may consider (6.16) as a set of  $n^2$  equations by giving  $\kappa$  and  $\varrho$  all values from 1 to  $n$ . For instance if we take  $\kappa=1, \varrho=2$ , the integrability conditions of this special equation are

$$(6.17) \quad \partial_{[\nu} \Pi_{\mu\lambda]\cdot^1_2} = 0$$

and this proves that the integrability conditions of (6.16) are

$$(6.18) \quad \partial_{[\nu} \Pi_{\mu\lambda]\cdot^\kappa_\varrho} = 0.$$

It can be proved in the same way that the equation

$$(6.19) \quad \partial_\nu \pi^{\nu\nu_2\dots\nu_p A} = \Pi^{\nu_2\dots\nu_p A}$$

where  $A$  is a collective index and where  $\pi$  and  $\Pi$  are alternating in the first  $p$  and  $p-1$  indices respectively, has the integrability conditions

$$(6.20) \quad \partial_\nu \Pi^{\nu\nu_2\dots\nu_p A} = 0.$$

If  $q=0$  we can go a step further and consider the case

$$(6.21) \quad \partial_\mu \pi_B = \Pi_{\mu B}(\xi^\kappa, \pi_C)$$

where the right hand side depends also on the  $\pi_B$  (the  $\pi_B$  need not be components of a geometric object!).<sup>1)</sup>,<sup>2)</sup> (6.21) can be written in the form of a system of total differential equations

$$(6.22) \quad d\pi_B - \Pi_{\mu B}(\xi^*, \pi_C) d\xi^\mu = 0.$$

Now let  $\Omega(\xi^*, \pi_B)$  be an integral function (hence  $\Omega = \text{const.}$  an integral) of this system, then  $\Omega$  is a solution (II § 5) of the homogeneous linear system

$$(6.23) \quad \frac{\partial \Omega}{\partial \xi^\mu} + \frac{\partial \Omega}{\partial \pi_B} \Pi_{\mu B} = 0$$

with the independent variables  $\xi^*$  and  $\pi_B$ . This system is complete if and only if

$$(6.24) \quad (\bar{\partial}_{[\nu} \Pi_{\mu]B}) \frac{\partial}{\partial \pi_B} + \Pi_{[\nu|C]} \frac{\partial \Pi_{\mu]B}}{\partial \pi_C} \frac{\partial}{\partial \pi_B}$$

(where  $\bar{\partial}_\nu$  denotes differentiation with respect to  $\xi^\nu$ , the  $\pi_B$  being left constant) depends linearly on the operators  $\bar{\partial}_\mu + \Pi_{\mu B} \frac{\partial}{\partial \pi_B}$  in the left hand side of (6.23). But this is only possible if

$$(6.25) \quad \partial_{[\nu} \Pi_{\mu]B} = \bar{\partial}_{[\nu} \Pi_{\mu]B} + \Pi_{[\nu|C]} \frac{\partial \Pi_{\mu]B}}{\partial \pi_C} = 0.$$

Now let these equations, which do not contain any derivatives of  $\pi_B$ , be symbolized by

$$(6.26) \quad \overset{I}{F}(\pi_B, \xi^*) = 0.$$

If we apply  $\partial_\omega$  to (6.26) and eliminate  $\partial_\omega \pi_A$  by means of (6.21) we get a second set of the same kind, symbolized by

$$(6.27) \quad \overset{2}{F}(\pi_B, \xi^*) = 0$$

and this process can be continued ad infinitum. We need not consider the case when a relation between the  $\xi^*$  can be derived from all these equations, because in that case we could diminish the number of independent variables. So there are only two possibilities. Either after  $i+1$  steps the  $(i+1)$ th set contains an equation which is not consistent with the former ones, in which case (6.21) has no solutions. Or the  $(j+1)$ th set depends on the foregoing sets. For this latter case VEBLEN and J. M. THOMAS<sup>3)</sup> have proved that then all further sets depend on the first  $i$  ones. Collecting results we get the rule:

*In order to find the integrability conditions of equations of the form (6.21), where the  $\Pi_{\mu A}$  depend on the  $\xi^*$  and the  $\pi_B$ , we have to form the*

<sup>1)</sup> GUREWITCH considered 1933, 1 equations of the form  $\partial_{[\mu} w_{\lambda_1 \dots \lambda_q]} = p_{[\mu} w_{\lambda_1 \dots \lambda_q]}$ .

<sup>2)</sup> Cf. for literature SCHOUTEN 1925, 1, p. 442ff.

<sup>3)</sup> VEBLEN and J. M. THOMAS 1926, 1; cf. T. Y. THOMAS and LEVINE 1934, 1; T. Y. THOMAS 1934, 2, Ch. X.

integrability conditions (6.26), (6.27), and so on. If these conditions are inconsistent, (6.21) has no solutions. If the first  $i$  sets do not contain inconsistent equations and if the  $(i+1)$ th set depends on the foregoing sets, the equation (6.21) together with the  $i$  first sets form a totally integrable system.<sup>1)</sup> This means here that if  $\pi_B$  and  $\Pi_{\mu B}$  satisfy  $\Pi_{\mu B} = \Pi_{\mu B}(\pi_A, \xi^\alpha)$  and the first  $i$  conditions of integrability for  $\xi^\alpha = \xi^\alpha$ , there exists exactly one solution of (6.21) such that  $\pi_B = \pi_B$  and  $\partial_\mu \pi_B = \Pi_{\mu B}$  at  $\xi^\alpha$ .

If besides (6.21) a set of equations is given in  $\pi_B$  and  $\xi^\alpha$  which does not contain derivatives of  $\pi_B$ , this set must be adjoined to the first integrability conditions.<sup>2)</sup>

As we see the construction of the integrability conditions depends always on the possibility of eliminating at each step the derivatives of the  $\pi_B$ . Hence the same method can in general not be used for an equation of the form (6.15) if the right hand side depends on the  $\xi^\alpha$  and the  $\pi$ , because the derivatives of  $\pi$  arising at every step do not in general appear in the form  $\partial_{[\mu} \pi_{\lambda_1 \dots \lambda_q]} B$ . Only if they happen to appear in this form the elimination can be effected and in that case the integrability conditions can be constructed in the way described. But in the more general case the problem whether a set of equations has solutions can only be solved either by the theory of CARTAN or by the theory of RIQUIER.<sup>3)</sup>

It often happens that the  $\Pi_{\mu B}$  in (6.21) are linear homogeneous in the  $\pi_B$

$$(6.28) \quad \Pi_{\mu B} = \Pi_{\mu B}^A(\xi^\alpha) \pi_A.$$

Then for total integrability the equation

$$(6.29) \quad \{\partial_{[\mu} \Pi_{\lambda]}^A B + \Pi_{[\mu | C |}^A \Pi_{\lambda]}^C B\} \pi_A = 0$$

must be satisfied identically. Accordingly the conditions of integrability are in this case

$$(6.30) \quad \partial_{[\mu} \Pi_{\lambda]}^A B + \Pi_{[\mu | C |}^A \Pi_{\lambda]}^C B = 0.$$

If an  $X_m$  in  $X_n$  is given, the section of a field  $w_{\lambda_1 \dots \lambda_q}; q \leq m$  with  $X_m$  is

$$(6.31) \quad 'w_{b_1 \dots b_q} = B_{b_1 \dots b_q}^{\lambda_1 \dots \lambda_q} w_{\lambda_1 \dots \lambda_q}; \quad B_b^\lambda \stackrel{\text{def}}{=} \partial_b \xi^\lambda$$

and the section of its rotation is

$$(6.32) \quad 'W_{c b_1 \dots b_q} = (q+1) B_c^{\mu \lambda_1 \dots \lambda_q} \partial_{[\mu} w_{\lambda_1 \dots \lambda_q]} = (q+1) B_{[c b_1 \dots b_q]}^{\mu \lambda_1 \dots \lambda_q} \partial_\mu w_{\lambda_1 \dots \lambda_q}.$$

<sup>1)</sup> Cf. II § 5.

<sup>2)</sup> J. M. THOMAS 1927, 1.

<sup>3)</sup> Cf. also for literature P. P. 1949, 1, Ch. X.

We prove that *the rotation of the section equals the section of the rotation.*  
In fact

$$(6.33) \quad \left\{ \begin{array}{l} (q+I) \partial_{[c} w_{b_1 \dots b_q]} = (q+I) \partial_{[c} B_{b_1 \dots b_q]}^{\lambda_1 \dots \lambda_q} w_{\lambda_1 \dots \lambda_q} \\ \quad = (q+I) B_{[c}^{\mu \lambda_1 \dots \lambda_q} \partial_{[\mu} w_{\lambda_1 \dots \lambda_q]} \end{array} \right.$$

because  $\partial_{[c} B_{b]}^{\lambda} = 0$ .

### Exercises.

II 6.1. If  $w_{\lambda_1 \dots \lambda_{p-1} n} = 0$  and  $Dw = 0$  the field  $w$  does not depend on  $\xi^n$ .<sup>1)</sup>

II 6.2. <sup>2)</sup> If for every choice of the  $q$ -vector  $P_{\lambda_1 \dots \lambda_q}$  the rotation of  $P_{\lambda_1 \dots \lambda_q}$  vanishes always if the rotation of  $Q_{[\lambda}^{\lambda_q} P_{\lambda_1 \dots \lambda_q]}$  is zero, it follows that

$$\text{II 6.2 a)} \quad Q_{\lambda}^{;\nu} = \alpha A_{\lambda}^{\nu}; \quad \alpha = \text{constant}.$$

## § 7. PFAFF's problem.<sup>3)</sup>

A field  $w_{\lambda}$  in  $X_n$ , analytic in some  $\mathfrak{N}_{\theta}(\xi^{\nu})$ , fixes an  $E_{n-1}$ -field or  $X_n^{n-1}$ . For such a field the outer problem is trivial because it is immediately clear that the  $E_{n-1}$ -field is  $X_{n-1}$ -forming if and only if  $w_{\lambda}$  is a gradient to within a scalar factor. Only the inner problem is interesting, in fact we have here the most simple form of this problem.

We form the rotation  $W_{\mu \lambda} = 2 \partial_{[\mu} w_{\lambda]}$  and the multivectors

$$(7.1 \text{ a}) \quad \overset{1}{I}_{\lambda} = w_{\lambda}; \quad \overset{2}{I}_{\mu \lambda} = W_{\mu \lambda}; \quad \overset{3}{I}_{\nu \mu \lambda} = w_{[\nu} W_{\mu \lambda]}; \quad \overset{4}{I}_{\omega \nu \mu \lambda} = W_{[\omega \nu} W_{\mu \lambda]}; \quad \text{etc.}$$

or, using the abridged notation of I § 7

$$(7.1 \text{ b}) \quad \overset{1}{I} = w; \quad \overset{2}{I} = W; \quad \overset{3}{I} = [w W]; \quad \overset{4}{I} = [W W]; \quad \text{etc.}$$

Then if

$$(7.2) \quad \overset{K}{I} \neq 0; \quad \overset{K+1}{I} = 0,$$

and consequently  $\overset{s}{I} \neq 0$  for  $s \leq K$  and  $\overset{s}{I} = 0$  for  $s > K$ , we call  $K$  the *class* of  $w$ .  $K$  is the  $\lambda$ -rank of the set  $w_{\lambda}, W_{\mu \lambda}$ . The rank  $2\varrho$  of  $W$  is called the *rotation class* and the  $\lambda$ -rank  $k$  of  $w_{\lambda}, w_{[\nu} W_{\mu \lambda]}$  the *similarity class* of  $w$ . There are two cases:

In case I  $w$  does not belong to the domain of  $W$  (cf. I § 5). The  $\lambda$ -domain of the set  $w_{\lambda}, W_{\mu \lambda}$  contains both the domain of  $W$  and the vector  $w$  outside it, hence  $K = 2\varrho + 1$  and  $k = K$ .  $\overset{2\varrho}{I}$  and  $\overset{2\varrho+1}{I}$  are both

1) CARTAN 1922, 1, p. 72.

2) NAGABHUSHANAM 1949, 1.

3) We give here only a brief summary of results. Cf. for a detailed treatment for instance P. P. 1949, 1, Ch. IV, also for literature.

simple multivectors and gradient products because their rotations vanish (cf. II § 6). Hence  $\overset{2\varrho}{I}$  is  $X_{n-2\varrho}$ -forming and  $\overset{k}{I}$  is  $X_{n-k}$ -forming and the corresponding complete systems are

$$(7.3) \quad (S'_1) : [\overset{2\varrho}{I} \bar{f}] = 0$$

$$(7.4) \quad (S'_4) : [\overset{k}{I} \bar{f}] = 0; \quad (k = K)^1.$$

The  $X_{n-2\varrho}$ 's are called the *supports of rotation* of  $w$  and the  $X_{n-K}$ 's the *supports* and also the *characteristics* of  $w$ . Because  $\overset{2\varrho}{I}$  and  $\overset{k}{I}$  are both gradient products and  $\overset{2\varrho}{I}$  lies in the domain of  $\overset{k}{I}$ , there exists a scalar  $\phi$  such that

$$(7.5) \quad \overset{k}{I} = [w \overset{2\varrho}{I}] = [\bar{\phi} \overset{2\varrho}{I}].$$

In case II  $w$  belongs to the domain of  $W$ , hence  $K = 2\varrho$  and  $k = 2\varrho - 1$ . To prove this take the coordinate system at a definite point such that  $w = \overset{1}{e}$ ,  $W = [\overset{1}{e} \overset{2}{e}] + \dots + [\overset{k}{e} \overset{k+1}{e}]$ . The rotation of  $\overset{k}{I}$  does not vanish because  $\overset{2\varrho}{I} = 2D\overset{k}{I}$ . But now it follows from (5.16a) that  $\overset{k}{I}$  is  $X_{n-k}$ -forming.  $\overset{2\varrho}{I}$  is simple and a gradient product, hence  $X_{n-2\varrho}$ -forming, because its rotation vanishes. The corresponding complete systems are

$$(7.6) \quad (S'_4) : [\overset{k}{I} \bar{f}] = 0$$

$$(7.7) \quad (S'_1) : [\overset{2\varrho}{I} \bar{f}] = 0.$$

The  $X_{n-2\varrho}$ 's are called the *supports of rotation* and (because  $K = 2\varrho$ ) also the *supports* of  $w$  and the  $X_{n-k}$ 's are called the *characteristics* of  $w$ . Because  $\overset{k}{I}$  lies in the domain of  $\overset{2\varrho}{I}$  there must exist a vector  $u_\lambda$  such that  $\overset{2\varrho}{I} = [u \overset{k}{I}]$ . Moreover we can prove that it is always possible to make  $u_\lambda$  a gradient. Let

$$(7.8) \quad F(\xi^\alpha) = \text{const}; \quad \alpha = \bar{1}, \dots, \bar{k}$$

be the equations of the  $X_{n-k}$ 's formed by  $\overset{k}{I}$ . Then every tangent  $E_{n-k}$  of them has the  $(n-k)$ -direction of  $\overset{k}{I}$ . Hence there exists an equation of the form

$$(7.9) \quad \overset{k}{I} = \tau [\bar{F} \dots \bar{F}]$$

from which it follows that

$$(7.10) \quad \overset{2\varrho}{I} = 2[\bar{\tau} \bar{F} \dots \bar{F}] = 2[\log \tau \overset{k}{I}].$$

---

<sup>1)</sup> As in P. P. 1949, I, IV § 2 we denote the complete systems (7.3, 4) by  $S'_1$  and  $S'_4$  and their adjoint systems by  $S_1$  and  $S_4$ .

Note that in both cases  $2\varrho$  is the greatest even number  $\leq K$  and  $k$  the greatest odd number  $\leq K$ .

Transformations of the form

$$(7.11) \quad 'w_\lambda = \sigma w_\lambda; \quad \sigma \neq 0$$

are called *similarity transformations* of the field  $w$ . Such a transformation transforms  $W$  into

$$(7.12) \quad 'W = \sigma W + 2[\bar{\sigma} w]$$

and accordingly  $'I^{\frac{2\varrho}{2}}$  and  $'I^{\frac{k}{2}}$  into

$$(7.13) \quad 'I^{\frac{2\varrho}{2}} = \sigma^{\frac{2\varrho}{2}} I^{\frac{2\varrho}{2}} + 2\varrho \sigma^{2-1} [\bar{\sigma} I^{\frac{2\varrho-1}{2}}]$$

$$(7.14) \quad 'I^{\frac{k}{2}} = \sigma^{\frac{k+1}{2}} I^{\frac{k}{2}}.$$

Hence  $k$  and the characteristics are invariant for similarity transformations.

In case I  $'I^{\frac{K}{2}} \neq 0$  and  $'I^{\frac{K+2}{2}} = 0$ . For  $'I^{\frac{K+1}{2}}$  we get

$$(7.15) \quad 'I^{\frac{K+1}{2}} = (2\varrho + 2) \sigma^{\frac{K}{2}} [\bar{\sigma} I^{\frac{K}{2}}].$$

If  $\sigma$  is a solution of  $(S'_4)$  we get  $'I^{\frac{K+1}{2}} = 0$ , but if it is not a solution,  $'I^{\frac{K+1}{2}}$  does not vanish and we pass from case I to case II. Of course this latter possibility can never occur for  $K=n$ .

In case II  $'I^{\frac{k}{2}} \neq 0$  and  $'I^{\frac{k+2}{2}} = 0$ . From (7.13) we see that  $'I^{\frac{2\varrho}{2}}$  vanishes if and only if  $\sigma$  is a solution of the non homogeneous linear equation

$$(7.16) \quad 2\varrho [\overline{\log \sigma} I^{\frac{2\varrho-1}{2}}] = -'I^{\frac{2\varrho}{2}}.$$

This equation implies that  $\sigma$  is a solution of  $(S'_1)$  and not a solution of  $(S'_4)$ , but these conditions are only necessary and not sufficient. Comparing (7.16) with (7.10), it follows that  $\sigma = \tau^{-\frac{1}{\varrho}}$  is a solution of (7.16). Collecting results we see that by means of a similarity transformation it is always possible to pass from case II to case I with  $'K = K - 1$  and for  $K < n$  also from case I to case II with  $'K = K + 1$ .

A transformation of the form

$$(7.17) \quad 'w_\lambda = w_\lambda + \partial_\lambda u; \quad (u \text{ not constant})$$

is called a *gradient transformation* of the field  $w$ . Such a transformation leaves  $W$  invariant and accordingly transforms  $'I^{\frac{2\varrho}{2}}$  and  $'I^{\frac{k}{2}}$  into

$$(7.18) \quad 'I^{\frac{2\varrho}{2}} = I^{\frac{2\varrho}{2}}$$

$$(7.19) \quad 'I^{\frac{k}{2}} = I^{\frac{k}{2}} + [\bar{u} I^{\frac{k-1}{2}}].$$

Hence  $2\varrho$  and the supports of rotation are invariant for gradient transformations.

In case I  $'I^{\frac{2\varrho}{K}} \neq 0$  and  $'I^{\frac{2\varrho+2}{K}} = 0$ . Because  $K=k$  we get  $'I^K = 0$  if and only if  $u$  is a solution of the non homogeneous linear equation

$$(7.20) \quad [\bar{u} I^{\frac{2\varrho}{K}}] = -I^K.$$

This equation implies that  $u$  is a solution of  $(S'_4)$  and not a solution of  $(S'_1)$ , but these conditions are only necessary and not sufficient. In this case it follows  $'K = 2\varrho = K-1$  so that we get from case I to case II. Comparing (7.20) with (7.5) it follows that  $u = -\varrho$  is a solution of (7.20).

In case II  $'I^K \neq 0$  and  $'I^{K+2} = 0$ . For  $'I^{K+1}$  we get

$$(7.21) \quad 'I^{K+1} = [\bar{u} I^K].$$

If  $u$  is a solution of  $(S'_1)$  we get  $'I^{K+1} = 0$  but if  $u$  is not a solution,  $'I^{K+1}$  does not vanish and we pass from case II to case I. Of course this latter possibility can never occur for  $K=n$ . Collecting results we see that by means of a gradient transformation it is always possible to pass from case I to case II with  $'K = K-1$  and, for  $K < n$ , also to pass from case II to case I with  $'K = K+1$ .

Here is a table of results<sup>1)</sup>:

#### Similarity transformations

$$(7.22) \quad K = 2\varrho + 1 \begin{cases} 'K = K+1; \sigma \text{ is not a solution of } (S'_4) \text{ (never for } K=n) \\ 'K = K; \sigma \text{ is a solution of } (S'_4) \text{ (always for } K=n) \end{cases}$$

$$(7.23) \quad K = 2\varrho \quad \begin{cases} 'K = K; \sigma \text{ is not a solution of (7.16)} \\ 'K = K-1; \sigma \text{ is a solution of (7.16)} \end{cases}$$

#### Gradient transformations

$$(7.24) \quad K = 2\varrho \quad \begin{cases} 'K = K+1; u \text{ is not a solution of } (S'_1) \text{ (never for } K=n) \\ 'K = K; u \text{ is a solution of } (S'_1) \text{ (always for } K=n) \end{cases}$$

$$(7.25) \quad K = 2\varrho + 1 \begin{cases} 'K = K; u \text{ is not a solution of (7.20)} \\ 'K = K-1; u \text{ is a solution of (7.20).} \end{cases}$$

If a field  $w$  is given,  $K$  can be found by means of differentiations and algebraic operations only. Then by alternately applying a gradient transformation and a similarity transformation the class can be reduced

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<sup>1)</sup> FROBENIUS 1879, 1.

to zero after  $K$  steps. But this proves that  $w$  can always be written in the form

$$(7.26) \quad \begin{cases} w_\lambda = \varepsilon s_\lambda^\theta + z^1 s_\lambda^1 + \cdots + z^K s_\lambda^K; & \varepsilon = \begin{cases} 1 & \text{for } K \text{ odd} \\ 0 & \text{for } K \text{ even} \end{cases} \\ s_\lambda^a \stackrel{\text{def}}{=} \partial_\lambda s_\lambda^a; & a = 1, \dots, \varrho. \end{cases}$$

This form is called a *canonical form* for  $w$ . There can not be a canonical form with less than  $\varrho + \varepsilon$  terms, because in that case we see, by forming the rotation, that the class had to be smaller than  $K$ . The  $K$  variables occurring in (7.26) must be independent functions of the  $\xi^x$ . Because otherwise  $w$  could be expressed in terms of less than  $K$  variables and that would imply that  $I = 0$ . But these  $K$  variables are not uniquely determined. For instance for  $K$  odd

$$(7.27) \quad w_\lambda = (s_\lambda^\theta + \partial_\lambda(z^1 s_\lambda^1)) - s^1 z_\lambda^1 + z^2 s_\lambda^2 + \cdots + z^K s_\lambda^K$$

is another canonical form for  $w$ .

Taking the  $K$  variables in (7.26) as the first  $K$  coordinates of ( $x$ ) we get

$$(7.28) \quad \begin{cases} w_\lambda = e_\lambda^1 + \xi^2 e_\lambda^2 + \cdots + \xi^{K-1} e_\lambda^K & \text{for } K \text{ odd} \\ w_\lambda = \xi^1 e_\lambda^2 + \cdots + \xi^{K-1} e_\lambda^K & \text{for } K \text{ even.} \end{cases}$$

We now describe the process leading to a canonical form in more detail. If  $K$  is odd we have first to find a gradient transformation ' $w_\lambda = w_\lambda - \partial_\lambda \hat{s}$ ', diminishing  $K$  by 1. According to (7.25)  $-\hat{s}$  is an arbitrary solution of the non-homogeneous equation (7.20). If  $K$  is even we have first to find a similarity transformation ' $w_\lambda = z^{-1} w_\lambda$ ' diminishing  $K$  by 1. According to (7.23)  $z^{-1}$  is a solution of the non-homogeneous equation (7.16). Now it seems that (7.20) and (7.16) are each a set of  $\binom{n}{K}$  equations instead of one equation. But this is not true. Because this is a property independent of the choice of the coordinate system we may suppose that ( $x$ ) is chosen such that (7.28) holds. Then for  $K$  odd (7.20) takes the form

$$(7.29) \quad \partial_1 \hat{s}^\theta = I; \quad \partial_b \hat{s}^\theta = 0; \quad b = K + 1, \dots, n$$

with the general solution

$$(7.30) \quad \hat{s}^\theta = \xi^1 + \Phi(\xi^2, \dots, \xi^K)$$

with the arbitrary function  $\Phi$ . If  $K$  is even (7.16) takes the form

$$(7.31) \quad \begin{cases} 1 - z^{-1} (\xi^1 \partial_1 z^1 + \xi^3 \partial_3 z^1 + \cdots + \xi^{K-1} \partial_{K-1} z^1) = 0; \\ \partial_b z^1 = 0; \quad b = K + 1, \dots, n \end{cases}$$

whose general solution is any function of  $\xi^1, \dots, \xi^K$  which is homogeneous of degree one in  $\xi^1, \xi^2, \dots, \xi^{K-1}$ . Of course the equations (7.29) and (7.31) do not require any integrations. But we must remember that in order to get the special coordinate system we need a canonical form and that this form can not be formed without integrations.

If  $w$  is  $X_m$ -enveloping, there exists in the region considered an enveloped normal system of  $X_m$ 's with equations of the form

$$(7.32) \quad F^x(\xi^x) = c^x; \quad x = m + 1, \dots, n$$

and because  $w_\lambda$  is, at every point, tangent to the local  $X_m$ , it can be expressed linearly in the  $\partial_\lambda F^x$

$$(7.33) \quad w_\lambda = \alpha_x \partial_\lambda F^x; \quad x = m + 1, \dots, n.$$

From this equation it follows that  $K = 2\varrho + \varepsilon \leq 2(n - m)$  or  $m \leq n - \varrho - \varepsilon$ . But from (7.26) we see that

$$(7.34) \quad \varepsilon \frac{\partial}{\partial s} = \varepsilon \frac{\partial}{\partial c}; \quad \frac{\partial}{\partial s} = \frac{\partial}{\partial c}; \dots; \quad \frac{\partial}{\partial s} = \frac{\partial}{\partial c}$$

represents an enveloped normal system of  $X_{n-\varrho-\varepsilon}$ 's. Hence  $\nu \stackrel{\text{def}}{=} n - \varrho - \varepsilon$  is the maximum value of  $m$ . This proves

*Every field  $w_\lambda$  of class  $K = 2\varrho + \varepsilon$  is  $X_\nu$ -enveloping but not  $X_{\nu+1}$ -enveloping;  $\nu = n - \varrho - \varepsilon$ .*

If (7.28) holds we have

$$(7.35) \quad \begin{cases} \varepsilon = 1 \\ I = 2^\varrho \varphi! [\bar{\xi}^1 \dots \bar{\xi}^K] \\ I = 2^\varrho \varphi! [\bar{\xi}^2 \dots \bar{\xi}^K] \end{cases} \quad \begin{cases} \varepsilon = 0 \\ I = 2^\varrho \varphi! [\bar{\xi}^1 \dots \bar{\xi}^K] \\ I = 2^{\varrho-1} (\varphi-1)! \{ \xi^1 [\bar{\xi}^2 \dots \bar{\xi}^K] + \\ + \xi^2 [\bar{\xi}^1 \bar{\xi}^2 \bar{\xi}^4 \dots \bar{\xi}^K] + \dots \\ + \xi^{K-1} [\bar{\xi}^1 \dots \bar{\xi}^{K-2} \bar{\xi}^K] \} \end{cases}$$

and

$$(7.36) \quad \begin{cases} \text{supports=characteristics:} & \xi^1 = \text{const.}; \dots; \xi^K = \text{const.} \\ \text{supports of rotation:} & \xi^2 = \text{const.}; \dots; \xi^K = \text{const.} \\ & \text{for } \varepsilon = 1 \\ \text{supports=supports of rotation: } \xi^1 = \text{const.}; \dots; \xi^K = \text{const.} & \\ \text{characteristics:} & \left\{ \begin{array}{l} \xi^1: \xi^3: \dots: \xi^{K-1} = \text{const.}; \\ \xi^2 = \text{const.}; \xi^4 = \text{const.}; \dots; \xi^K = \text{const.} \end{array} \right. \\ & \text{for } \varepsilon = 0 \end{cases}$$

**Exercise.**

II 7,1<sup>1)</sup>. Prove that the class of

- a)  $x^1 x^3 dx^2 + x^1 x^2 dx^3 + (x^1 + x^3 x^5) dx^4 + x^3 x^4 dx^5$
- b)  $x^2 dx^1 + x^1 dx^2 - x^3 x^5 dx^4 - x^3 x^4 dx^5 + x^2 dx^6$

is 4 and 5 respectively.

**§ 8. The theorem of STOKES.**

In  $X_n$  we consider an  $X_{q+1}$  with the equations

$$(8.1) \quad \xi^{q+2} = 0; \dots; \quad \xi^n = 0$$

with respect to some coordinate system  $(\alpha)$  and in this  $X_{q+1}$  we consider a  $(q+1)$ -dimensional volume  $\tau_{q+1}$  and its boundary  $\tau_q$ , such that from the curves on  $X_{q+1}$

$$(8.2) \quad \xi^1 = \text{const.}; \dots; \quad \xi^{q-1} = \text{const.}; \quad \xi^{q+1} = \text{const.}; \dots; \quad \xi^{q+1} = \text{const.}$$

for every value of  $\alpha$  from 1 up to  $q+1$  each intersects  $\tau_q$  in at most two points. Let an *inner* orientation be fixed on  $\tau_{q+1}$  by  $e_1^\alpha, \dots, e_{q+1}^\alpha$  in this order. Then the volume element  $d\xi^{x_1 \dots x_{q+1}}$  with this orientation may be written in the form

$$(8.3) \quad d\xi^{x_1 \dots x_{q+1}} = (q+1)! \underset{1}{d}\xi^{[x_1} \dots \underset{q+1}{d}\xi^{x_{q+1}]}.$$

If the  $q+1$  vectors  $\underset{1}{d}\xi^\alpha, \dots, \underset{q+1}{d}\xi^\alpha$  are chosen in such a way that

$$(8.4) \quad \left\{ \begin{array}{l} \underset{1}{d}\xi^1 = d\xi^1; \quad \underset{1}{d}\xi^2 = 0; \dots; \quad \underset{1}{d}\xi^n = 0 \\ \underset{2}{d}\xi^1 = 0; \quad \underset{2}{d}\xi^2 = d\xi^2; \quad \underset{2}{d}\xi^3 = 0; \dots; \quad \underset{2}{d}\xi^n = 0 \\ \vdots \\ \underset{q+1}{d}\xi^1 = 0; \dots; \quad \underset{q+1}{d}\xi^q = 0; \quad \underset{q+1}{d}\xi^{q+1} = d\xi^{q+1}; \quad \underset{q+1}{d}\xi^{q+2} = 0; \dots; \quad \underset{q+1}{d}\xi^n = 0 \end{array} \right.$$

we have

$$(8.5) \quad d\xi^1 \dots \underset{q+1}{d}\xi^{q+1} = d\xi^1 \dots d\xi^{q+1}$$

and all components of  $d\xi^{x_1 \dots x_q}$  which have an index  $q+2, \dots, n$  vanish. Let one of the curves (8.2) for  $\alpha=1$  intersect the boundary  $\tau_q$  at the two points

$$(8.6) \quad \left\{ \begin{array}{l} P_1: \quad \xi^1 = \underset{1}{\xi^1} \\ P_2: \quad \xi^1 = \underset{2}{\xi^1} \end{array} \right\} \quad \xi^2 = \underset{0}{\xi^2}; \dots; \quad \xi^{q+1} = \underset{0}{\xi^{q+1}}; \quad \underset{2}{\xi^1} > \underset{1}{\xi^1}.$$

<sup>1)</sup> CARTAN 1899, 1, p. 259 and 265.

The points of  $\tau_{q+1}$  satisfying the equations

$$(8.7) \quad \begin{array}{l} \xi^1 \leq \xi^1 \leq \xi^1 \\ \hline 1 \quad 2 \end{array}; \quad \begin{array}{l} \xi^a \leq \xi^a \leq \xi^a + d\xi^a \\ \hline 0 \quad 0 \end{array}; \quad a = 2, \dots, q+1$$

constitute a  $(q+1)$ -dimensional tube in  $\tau_{q+1}$  cutting off a  $q$ -dimensional element from  $\tau_q$  at each of the points  $P_1$  and  $P_2$ .

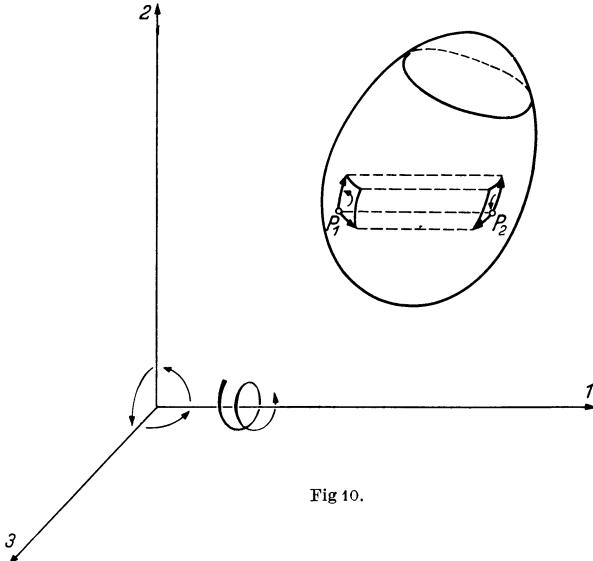


Fig 10.

Fig. 10 illustrates the case  $n=3, q=2$ . For convenience the coordinate- $X_1$ 's are drawn in this figure as straight lines but this has no real significance. If we fix now an inner orientation all over  $\tau_q$  by  $e_2^\kappa, \dots, e_{q+1}^\kappa$  in this order at  $P_2$ , then this orientation is opposite to the orientation of  $e_2^\kappa, \dots, e_{q+1}^\kappa$  at  $P_1$ . The orientations in  $\tau_{q+1}$  and  $\tau_q$  are now chosen in such a way that the direction from a point of  $\tau_{q+1}$  towards the boundary followed by the orientation of  $\tau_q$  gives the orientation of  $\tau_{q+1}$ . Accordingly we have for the  $q$ -dimensional element of  $\tau_q$  at  $P_2$

$$(8.8) \quad \overset{2}{d}f^{2 \dots q+1} \underset{*}{=} d\xi^2 \dots d\xi^{q+1}$$

and at  $P_1$

$$(8.9) \quad \overset{1}{d}f^{2 \dots q+1} \underset{*}{=} -d\xi^2 \dots d\xi^{q+1}.$$

After these preliminaries we now consider a covariant  $q$ -vector field  $v_{\lambda_1 \dots \lambda_q}$  satisfying the following conditions:

- a) the  $v_{\lambda_1 \dots \lambda_q}$  are continuous in  $\tau_{q+1}$  and on  $\tau_q$ ;
- b) the derivatives which occur in  $\partial_{[\mu} v_{\lambda_1 \dots \lambda_q]}$  exist at all points of  $\tau_{q+1}$ ;

c) these derivatives are continuous at all points of  $\tau_{q+1}$  except at the points of a finite number of  $X_q$ 's.

We consider the integral of  $\partial_1 v_{2 \dots q+1}$  over  $\tau_{q+1}$  (cf. 8.5)

$$(8.10) \quad \int_{\tau_{q+1}} \partial_1 v_{2 \dots q+1} d f^{1 \dots q+1} \stackrel{*}{=} \int_{\tau_{q+1}} \partial_1 v_{2 \dots q+1} d \xi^1 \dots d \xi^{q+1}.$$

As a consequence of the conditions for the field  $v_{\lambda_1 \dots \lambda_q}$ , the part of the integral over the tube is

$$(8.11) \quad \left\{ \begin{array}{l} (\overset{2}{v}_{2 \dots q+1} - \overset{1}{v}_{2 \dots q+1}) d \xi^2 \dots d \xi^{q+1} \\ \stackrel{*}{=} \overset{2}{v}_{2 \dots q+1} \overset{2}{d} f^{2 \dots q+1} + \overset{1}{v}_{2 \dots q+1} \overset{1}{d} f^{2 \dots q+1} \end{array} \right.$$

where  $\overset{2}{v}_{2 \dots q+1}$  and  $\overset{1}{v}_{2 \dots q+1}$  are the values of  $v_{2 \dots q+1}$  at  $P_2$  and at  $P_1$  respectively. Hence

$$(8.12) \quad \int_{\tau_{q+1}} \partial_1 v_{2 \dots q+1} d f^{1 \dots q+1} \stackrel{*}{=} \int_{\tau_q} v_{2 \dots q+1} d f^{2 \dots q+1}.$$

By using the other  $q$  sets of coordinate- $X_1$ 's, equations similar to (8.12) but with  $\partial_2, \dots, \partial_{q+1}$  instead of  $\partial_1$ , can be derived. By addition we get

$$(8.13) \quad \boxed{\int_{\tau_{q+1}} \partial_{[\mu} v_{\lambda_1 \dots \lambda_q]} d f^{\mu \lambda_1 \dots \lambda_q} = \int_{\tau_q} v_{\lambda_1 \dots \lambda_q} d f^{\lambda_1 \dots \lambda_q}} \text{ 1).}$$

In order to get rid of the condition of intersection imposed on the form of  $\tau_q$  we consider the case where  $\tau_{q+1}$  can be split up into a finite number of  $\tau_{q+1}$ 's such that the condition of intersection holds for the boundary of each of them. Then (8.13) holds for each of these  $\tau_{q+1}$ 's and by summation of the integrals all integrals over boundaries between two adjacent  $\tau_{q+1}$ 's cancel each other out, because the orientations in the common boundary are opposite. Hence (8.13) is proved now for this more general case.

(8.13) is a form of the generalized theorem of STOKES (in Germany also named after GAUSS and in France after OSTROGRADSKI). Many other forms can be deduced from it. First  $\tilde{v}^{\kappa_1 \dots \kappa_p}$ ,  $\tilde{f}_{\lambda_1 \dots \lambda_{p-1}}$  and  $\tilde{f}_{\lambda_1 \dots \lambda_p}$  ( $p = n - q$ ) can be introduced instead of  $v_{\lambda_1 \dots \lambda_q}$ ,  $f^{\mu \kappa_1 \dots \kappa_q}$  and  $f^{\lambda_1 \dots \lambda_q}$ . Then we get four other forms

$$(8.14) \quad p \int_{\tau_{q+1}} \partial_{[\mu} v_{\lambda_1 \dots \lambda_q} d \tilde{f}_{\kappa_1 \dots \kappa_{p-1}] \mu} = \int_{\tau_q} v_{[\lambda_1 \dots \lambda_q} d \tilde{f}_{\kappa_1 \dots \kappa_{p-1} \mu]}$$

$$(8.15) \quad p \int_{\tau_{q+1}} \partial_{\mu} \tilde{v}^{\lambda_1 \dots \lambda_{p-1} \mu} d \tilde{f}_{\lambda_1 \dots \lambda_{p-1}} = \int_{\tau_q} \tilde{v}^{\lambda_1 \dots \lambda_p} d \tilde{f}_{\lambda_1 \dots \lambda_p}$$

<sup>1)</sup> POINCARÉ 1887, 1; 1895, 1; BROUWER 1906, 1; 2; 1919, 1; WEITZENBÖCK 1923, 1, p. 398ff., also for literature; WEYSENHOFF 1937, 1; SCHOUTEN and v. DANTZIG 1940, 1; LEVI CIVITA 1940, 1.

$$(8.16) \quad \left\{ \begin{array}{l} \frac{n-q}{q+1} \int_{\tau_{q+1}} \partial_\mu \tilde{h}^{[\lambda_1 \dots \lambda_{p-1} | \mu]} d\tilde{f}^{\lambda_p \kappa_1 \dots \kappa_q} \\ = \int_{\tau_{q+1}} \partial_\mu \tilde{v}^{[\lambda_1 \dots \lambda_p} df^{|\mu| \kappa_1 \dots \kappa_q]} = \int_{\tau_q} \tilde{v}^{[\lambda_1 \dots \lambda_p} d\tilde{f}^{\kappa_1 \dots \kappa_q]} \end{array} \right.$$

which all have the same geometrical signification as (8.13). If the elements of  $\tau_q$  and  $\tau_{q+1}$  have *outer* orientation (this is the case that occurs most frequently in physics) and if these orientations are chosen in such a way that the orientation of  $\tau_{q+1}$  followed by the direction from a point of  $\tau_{q+1}$  towards the boundary gives the orientation of  $\tau_q$  we get the following five equations for a field  $\tilde{v}_{\lambda_1 \dots \lambda_q}$

$$(8.17) \quad \int_{\tau_{q+1}} \partial_{[\mu} \tilde{v}_{\lambda_1 \dots \lambda_q]} d\tilde{f}^{\mu \lambda_1 \dots \lambda_q} = \int_{\tau_q} \tilde{v}_{\lambda_1 \dots \lambda_q} d\tilde{f}^{\lambda_1 \dots \lambda_q}$$

$$(8.18) \quad p \int_{\tau_{q+1}} \partial_{[\mu} \tilde{v}_{\lambda_1 \dots \lambda_q} d\tilde{f}_{\kappa_1 \dots \kappa_{p-1}]} = \int_{\tau_q} \tilde{v}_{[\lambda_1 \dots \lambda_q} d\tilde{f}_{\kappa_1 \dots \kappa_{p-1} \mu]}$$

$$(8.19) \quad p \int_{\tau_{q+1}} \partial_\mu v^{\lambda_1 \dots \lambda_{p-1} \mu} d\tilde{f}_{\lambda_1 \dots \lambda_{p-1}} = \int_{\tau_q} v^{\lambda_1 \dots \lambda_p} d\tilde{f}_{\lambda_1 \dots \lambda_p}$$

$$(8.20) \quad \left\{ \begin{array}{l} \frac{n-q}{q+1} \int_{\tau_{q+1}} \partial_\mu v^{[\lambda_1 \dots \lambda_{p-1} | \mu]} d\tilde{f}^{\lambda_p \kappa_1 \dots \kappa_q} \\ = \int_{\tau_{q+1}} \partial_\mu v^{[\lambda_1 \dots \lambda_p} d\tilde{f}^{|\mu| \kappa_1 \dots \kappa_q]} = \int_{\tau_q} v^{[\lambda_1 \dots \lambda_p} d\tilde{f}^{\kappa_1 \dots \kappa_q]}. \text{ } ^1 \end{array} \right.$$

All integrals which have occurred up till now in this section have been scalars or  $n$ -vector  $\Delta$ -densities of weight  $\pm 1$ . In fact only these quantities can be added even if they belong to different points. But other forms of STOKES' theorem may arise if we restrict ourselves to less general transformations of coordinates. For instance, if only those coordinate transformations with  $\Delta > 0$  are allowed,  $W$ -scalars also can be added and thus besides (8.13) we have in that case the invariant formula

$$(8.21) \quad \int_{\tau_{q+1}} \partial_{[\mu} \tilde{v}_{\lambda_1 \dots \lambda_q]} d\tilde{f}^{\mu \lambda_1 \dots \lambda_q} = \int_{\tau_q} \tilde{v}_{\lambda_1 \dots \lambda_q} d\tilde{f}^{\lambda_1 \dots \lambda_q}.$$

If transformations of  $G_a$  only are allowed, that is, if we are in an  $E_n$ , also quantities that belong to *different* points can be added, provided that they have the same manner of transformation. Accordingly in  $E_n$  the following equations hold

$$(8.22) \quad \int_{\tau_{q+1}} d\tilde{f}^{\mu \lambda_1 \dots \lambda_q} \partial_\mu -\circ \Omega = \int_{\tau_q} d\tilde{f}^{\lambda_1 \dots \lambda_q} -\circ \Omega$$

$$(8.23) \quad p \int_{\tau_{q+1}} d\tilde{f}_{[\lambda_1 \dots \lambda_{p-1}} \partial_{\lambda_p]} -\circ \Omega = \int_{\tau_q} d\tilde{f}_{\lambda_1 \dots \lambda_p} -\circ \Omega$$

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<sup>1</sup> Cf. T. P. 1951, 1, p. 71 f. for the formulae occurring most frequently in physics.

$$(8.24) \quad \int_{\tau_{q+1}} d\tilde{f}^{\mu \lambda_1 \dots \lambda_q} \partial_\mu - \circ \Omega = \int_{\tau_q} d\tilde{f}^{\lambda_1 \dots \lambda_q} - \circ \Omega$$

$$(8.25) \quad p \int_{\tau_{q+1}} d\mathfrak{f}_{[\lambda_1 \dots \lambda_{p-1}] \partial_{\lambda_p}] - \circ \Omega = \int_{\tau_q} d\mathfrak{f}_{\lambda_1 \dots \lambda_p} - \circ \Omega$$

where  $\Omega$  is a symbol with suppressed indices standing for any quantity and where  $- \circ$  symbolizes an operation performed on the indices  $\lambda_1, \dots, \lambda_p$  or  $\lambda_1, \dots, \lambda_q$  and the indices of  $\Omega$ , which is composed of alternation, mixing and transvection<sup>1)</sup>.

For instance if

$$(8.26) \quad v_\lambda = \partial_\mu \mathfrak{P}_\lambda^\mu$$

we have in  $E_3$

$$(8.27) \quad \int_{\tau_3} v_\lambda d\mathfrak{f} = \int_{\tau_2} \mathfrak{P}_\lambda^\mu d\mathfrak{f}_\mu$$

but this equation has no invariant meaning in  $X_3$ .

The STOKES integrals may be looked upon from a quite different point of view, which is very important for many investigations. Take for instance (8.13). The  $v_{\lambda_1 \dots \lambda_q}$  may be considered as  $\binom{n}{q}$  functions of the  $\xi^\alpha$  that need not be components of a  $q$ -vector or of any geometric object. Then we are not able to give a corresponding formula for another coordinate system ( $\alpha'$ ) but the formula (8.13) remains true. This enables us to use STOKES' formulae in all cases where the manner of transformation of a set of variables is not known or not yet known. Of course in this case the integrals are not scalars because transformation of coordinates is not allowed.

### § 9. Anholonomic coordinates (cf. II § 5).

If at every point of  $X_n$  we introduce a set of  $n$  linearly independent contravariant vectors  $e_i^\alpha$  and the reciprocal set  $\dot{e}_\lambda^h$ ;  $h, i = 1, \dots, n$ , in such a way that these fields are analytic (or of class  $u$ ) in the region considered, these vectors are contra- and covariant basis vectors of a coordinate system  $\xi^h$  if and only if the rotations  $\partial_{[\mu} e_{\lambda]}^h$  vanish at all points. In fact  $\partial_{[\mu} \dot{e}_{\lambda]}^h = 0$  are the integrability conditions of the system  $\partial_\lambda \xi^h = \dot{e}_\lambda^h$ . If these conditions are not satisfied, the system of reference formed by the fields  $e_i^\alpha, \dot{e}_\lambda^h$  is called an *anholonomic* coordinate system ( $h$ )<sup>2)</sup> in contradistinction to the coordinate systems used before, that we call

<sup>1)</sup> E I 1935, 1, p. 130.

<sup>2)</sup> Cf. for literature the footnote 1 on p. 78 and Ch. V § 7. Though we often prefer latin running indices for anholonomic coordinate systems, this is not obligatory; also ( $\alpha$ ) can be taken anholonomic.

*holonomic*. The  $\xi^i$  determine a *net* of  $X_1$ 's (cf. II § 4), called the *net of the anholonomic coordinate system*. But in general an anholonomic coordinate system has no coordinate- $X_m$ 's,  $1 < m < n$ . The components of any quantity with respect to  $(h)$  can be derived from the components with respect to  $(\alpha)$  by using the intermediate components  $A_\lambda^h$ ,  $A_i^x$  of the unity tensor

$$(9.1) \quad A_\lambda^h \stackrel{\text{def}}{=} e_\lambda^h e_\lambda^i \stackrel{*}{=} e_\lambda^h; \quad A_i^x \stackrel{\text{def}}{=} e_{i_h}^h e_i^x \stackrel{*}{=} e_i^x$$

just in the same way as if  $(h)$  were holonomic. But the components of a more general geometric object (as defined in II § 3) with respect to  $(h)$  can not be derived by using  $A_\lambda^h$ ,  $A_i^x$  and  $\partial_j \stackrel{\text{def}}{=} A_j^\mu \partial_\mu$  just as if  $(h)$  were holonomic. In some way the components with respect to  $(h)$  must first be fixed by the introduction of a condition.<sup>1)</sup>

From now on we sometimes use the sign  $\stackrel{h}{=}$  in order to emphasize the fact that an equation is only valid with respect to holonomic coordinate systems. We write  $(d\xi)^h$  instead of  $d\xi^h$  for the components of the linear element  $d\xi^x$  with respect to an anholonomic coordinate system  $(h)$  because variables  $\xi^h$  do not exist.

The expression

$$(9.2) \quad Q_{ji}^h \stackrel{\text{def}}{=} A_{[j}^\mu A_{i]}^\lambda \partial_{[\mu} A_\lambda^h \stackrel{*}{=} e_{j_h}^\mu e_i^\lambda \partial_{[\mu} e_{\lambda]}^h$$

is called the *object of anholonomy*. It is a geometric object all of whose components vanish with respect to all holonomic systems and only with respect to such systems. For two holonomic or anholonomic systems  $(h)$  and  $(h')$  it follows from this definition that

$$(9.3) \quad Q_{j'i'}^{h'} = A_{j'}^{i'} A_{i'}^{h'} Q_{ji}^h + A_{[j'}^{i'} A_{i']}^{h'} \partial_{i'} A_i^{h'}$$

(9.2) is a special case of this formula.

If any expression with respect to holonomic coordinates is transformed with respect to an anholonomic system, correction terms appear, all containing the object of anholonomy. For instance<sup>2)</sup>

$$(9.4) \quad \partial_{[j} \partial_{i]} p = A_{[j}^\mu \partial_{|\mu|} A_{i]}^\lambda \partial_\lambda p = (A_{[j}^\mu \partial_{|\mu|} A_{i]}^\lambda) A_\lambda^h \partial_h p = -Q_{ji}^h \partial_h p.$$

If an  $X_m$  in  $X_n$  is given by its parametric equations

$$(9.5) \quad \xi^x = f^x(\eta^\alpha); \quad \alpha, \beta, \gamma, \delta = \dot{1}, \dots, \dot{m}$$

we may introduce besides  $(\alpha)$  and  $(\alpha)$  the anholonomic coordinate systems  $(h)$ ;  $h, i, j, k = 1, \dots, n$  and  $(a)$ ;  $a, b, c, d = \dot{1}, \dots, \dot{m}$  in  $X_n$  and  $X_m$  respectively. If  $Q_{cb}^a$  is the object of anholonomy of the  $X_m$  and if the  $X_m$  is rigged, we have

<sup>1)</sup> For an example cf. III § 9. Here the condition for the geometric object  $\Gamma$  is that the equations (III 9.1) hold for an anholonomic coordinate system.

<sup>2)</sup> Cf. Exerc. II 9.1.

$$(9.6) \quad B_c^{jih} \Omega_{ji}^h = B_c^{\mu\lambda} \partial_{[\mu} A_{\lambda]}^h = B_h^a \partial_{[c} B_{b]}^h - B_{[c}^{\mu} B_{|\lambda}^a \partial_{\mu]} B_{b]}^{\lambda} = B_h^a \partial_{[c} B_{b]}^h + ' \Omega_{cb}^a$$

where

$$(9.6a) \quad B_c^\mu \stackrel{\text{def}}{=} B_c^\beta B_\beta^\mu; \quad B_c^j \stackrel{\text{def}}{=} B_c^\beta B_\beta^\mu A_\mu^j; \quad B_\lambda^a \stackrel{\text{def}}{=} B_\alpha^a B_\lambda^\alpha; \quad B_i^a \stackrel{\text{def}}{=} B_\alpha^a B_\lambda^\alpha A_i^\lambda.$$

If in this case we choose at all points of  $X_m$  the first  $m$  vectors  $e_i^x$  coinciding with the vectors  $e_b^x$  and the last  $n-m$  vectors  $e_i^x$  in the  $E_{n-m}$  of the rigging, we have, taking  $a, b, c, d=1, \dots, m$  and  $x, y, z, u=m+1, \dots, n$

$$(9.7) \quad \begin{cases} A_\lambda^h * e_\lambda^h; & A_i^x * e_i^x; & B_\beta^a * e_\beta^a; & B_b^x * e_b^x; & B_c^j * e_c^\mu A_\mu^j = e_c^j * \delta_c^j; \\ B_c^\mu * e_c^\beta B_\beta^\mu = e^\mu; & B_\lambda^a * e_\alpha^a B_\lambda^\alpha = e_\lambda^a; & B_i^a * e_\lambda^a A_i^\lambda = e_i^a * \delta_i^a, \end{cases}$$

hence

$$(9.8) \quad ' \Omega_{cb}^a * \Omega_{cb}^a.$$

Anholonomic coordinate systems are frequently used when dealing with an  $X_n^m$  in  $X_n^{n-1}$ ). If the  $X_n^m$  is rigged it is often convenient to take at each point the  $e_b^x$ ;  $a, b, c, d=1, \dots, m$  in the tangent  $E_m$  and the  $e_y^x$ ;  $x, y, z, u=m+1, \dots, n$  in the tangent  $E_{n-m}$ . For this choice we prove that the  $X_n^m$  is tangent to  $\infty^{n-m} X_m$ 's if and only if  $\Omega_{cb}^x * 0$ . In fact, the fields  $e_b^x$  are  $X_m$ -forming if and only if the  $[c^\mu \partial_{[\mu} e_b^x]_b$  depend linearly on the  $e_b^x$ , that is if

$$(9.9) \quad 0 = e^\mu (\partial_{[\mu} e_\lambda^\lambda]_b) \dot{e}_\lambda = - [e^\mu e^\lambda \partial_\mu \dot{e}_\lambda] * - \Omega_{cb}^x.$$

All other applications of anholonomic coordinates will be postponed until after the introduction of a linear connexion in Ch. III.

Besides the  $\Omega_{ji}^h$ , components of another kind are often used, these are the intermediate components (( $\varkappa$ ) holonomic)

$$(9.10) \quad \Omega_{\mu\lambda}^h \stackrel{\text{def}}{=} A_{\mu\lambda}^{ij} \Omega_{ji}^h = \partial_{[\mu} A_{\lambda]}^h * \partial_{[\mu} \dot{e}_{\lambda]}^h.$$

The  $n$  objects  $\Omega_{\mu\lambda}^{(h)}$  resulting from strangling the index  $h$  are bivectors.

### Exercises.

II 9.1. Prove the formulae for ( $\varkappa$ ) holonomic

$$\text{II 9.1 } \alpha) \quad \partial_{[\mu} w_{\lambda]} = A_{\mu\lambda}^{ij} (\partial_{[i} w_{j]} + w_h \Omega_{ji}^h);$$

$$\text{II 9.1 } \beta) \quad \partial_\mu v^\mu = \Delta (\partial_j v^j - 2 \Omega_{ji}^j v^i); \quad \Delta = \text{Det}(A_{ij}^h)$$

where  $w_\lambda$  is a covariant vector and  $v^\mu$  a vector  $\Delta$ -density of weight +1.

II 9.2<sup>2)</sup>. Prove the identity

$$\text{II 9.2 } \alpha) \quad \partial_{[k} \Omega_{ji]}^h - 2 \Omega_{[ki]}^l \Omega_{ij]}^h = 0.$$

<sup>1)</sup> Cf. for applications to anholonomic dynamical systems and references to literature T. P. 1951, 1, Ch. VIII.

<sup>2)</sup> Cf. (II 12.10 b).

### § 10. The LIE derivative.<sup>1)</sup>,<sup>2)</sup>

Let the points of a region  $R$  be subject to a point transformation

$$(10.1) \quad \eta^{\alpha} = f^{\alpha}(\xi^{\lambda}); \quad \text{Det}(\partial_{\lambda} f^{\alpha}) \neq 0.$$

The functions  $f^{\alpha}$  are supposed to be analytic in the region considered and to establish a one to one correspondence between the points of  $R$  and those of some other region  $R'$ . Another coordinate system  $(\xi')$  is now introduced such that the transform in  $R'$  has the same coordinates with respect to  $(\xi')$  as the original point in  $R$  had with respect to  $(\alpha)$ . Hence if

$$(10.2) \quad \xi^{\alpha} = \varphi^{\alpha}(\eta^{\lambda})$$

is the inverse of (10.1) and if  $\eta^{\alpha}$  and  $\eta'^{\alpha}$  are the coordinates with respect to  $(\alpha)$  and to  $(\xi')$  of the transform of  $\xi^{\alpha}$ , we have

$$(10.3) \quad \eta'^{\alpha} = \delta_{\alpha}^{\alpha'} \xi^{\alpha} = \delta_{\alpha}^{\alpha'} \varphi^{\alpha}(\eta^{\lambda}).$$

Accordingly the transformation of  $\xi^{\alpha}$  into  $\xi'^{\alpha}$  and vice versa is given by the equations

$$(10.4) \quad \xi'^{\alpha} = \delta_{\alpha}^{\alpha'} \varphi^{\alpha}(\xi^{\lambda}); \quad \xi^{\alpha} = f^{\alpha}(\xi'^{\lambda}).$$

This process is called the *dragging along*<sup>3)</sup> of the coordinate system  $(\alpha)$  by the point transformation (10.1) and  $(\xi')$  is called the *coordinate system dragged along*. The dragging along of a coordinate system  $(\alpha)$  can be interpreted geometrically in the following way. Let the region considered be filled up with gelatine and let a coordinate net of  $(\alpha)$  with sufficiently small meshes be made visible by red curves in the gelatine. Then, if the material points of the gelatine suffer the point transformation (10.1), the red curves in their new position form a coordinate net of the system dragged along  $(\xi')$ .

Now let the components of a geometric object  $\Phi_A$  as defined in II § 3 be analytic in  $R$ . Then we form a field  $\overset{m}{\Phi}_A$  in  $R'$  whose components  $\overset{m}{\Phi}_{A'}$  with respect to  $(\xi')$  at each point of  $R'$  are equal to the  $\Phi_A$  at the corresponding point of  $R$ . This process is called the *dragging along of a field* by the point transformation (10.1) and the field  $\overset{m}{\Phi}_A$  is called the

<sup>1)</sup> This section was written in collaboration with A. NIJENHUIS. For literature cf. SLEBODZINSKI 1931, 1; SCHOUTEN and v. KAMPEN 1933, 1; E I 1935, 1, p. 142; E II 1938, 2, p. 161; P. P. 1949, 1, Ch. II; T. P. 1951, 1, Ch. IV; NIJENHUIS 1952, 1.

<sup>2)</sup> The LIE derivative is defined here for an  $X_n$ . It can be defined for more general spaces for instance path-space, that is an  $X_n$  in which a system of curves is given such that through any two points there passes one and only one curve of the system. See YANO 1945, 1; KOSAMBI 1952, 1.

<sup>3)</sup> "Mitschleppen" in SCHOUTEN and v. KAMPEN 1933, 1 and E I 1935, 1.

*field dragged along*<sup>1)</sup>. The dragging along of a geometric object can be interpreted geometrically. Every geometric object as defined in II § 3 can be represented by some figure or figures in  $X_n$ , for instance a contravariant vector in a point by an arrow on an infinitesimal scale. Let the points of these figures be marked blue in the gelatine filling up the region. Then, if the gelatine is transformed, the transformed figures represent the object dragged along.

From the definition of dragging along it follows that (cf. II § 3)

$$(10.5) \quad \overset{\text{m}}{\Phi}_{A'}\{\eta\} = \delta_{A'}^A \Phi_A\{\xi\},$$

hence, making use of (3.1 b)

$$(10.6) \quad \begin{cases} \overset{\text{m}}{\Phi}_A\{\eta\} = F_A(\overset{\text{m}}{\Phi}_M\{\eta\}, \eta^*, \eta^*, A_\lambda^*\{\eta\}, \partial_\mu A_\lambda^*\{\eta\}, \dots) \\ = F_A(\Phi_M\{\xi\}, \xi^*, f^*\{\xi\}, \partial_\lambda f^*\{\xi\}, \partial_\mu \partial_\lambda f^*\{\xi\}, \dots). \end{cases}$$

If the transformation is infinitesimal:  $\xi^* \rightarrow \xi^* + v^* dt$ , we use the term *dragging along over  $v^* dt$* . The process of dragging along of a contravariant vector field  $u^* ds$  over  $v^* dt$  is illustrated in Fig. 11. The vector  $\overset{\text{m}}{u}^* ds$  in  $\xi^* + v^* dt$  results from the displacement of beginning-point and endpoint of the vector  $u^* ds$  in  $\xi^*$  over  $v^* dt$ . In general  $\overset{\text{m}}{u}^* ds$  and  $u^* ds$  in  $\xi^* + v^* dt$  do not coincide and so there arises a pentagon.

We now consider a vector field  $v^* = \psi^*(\xi^*)$  and the system of  $n$  ordinary differential equations

$$(10.7) \quad \frac{d\eta^*}{dt} = \psi^*(\eta^*)$$

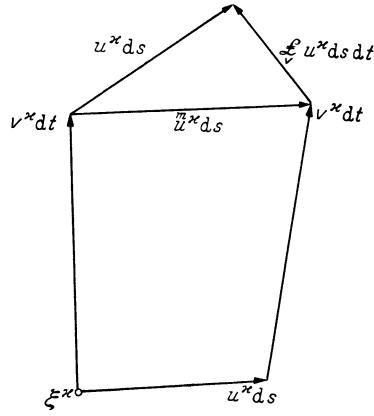


Fig. 11.

with the independent variable  $t$  and the unknown variables  $\eta^*$ , with functions  $\psi^*$  analytic in the region considered and with the initial conditions

$$(10.8) \quad \eta^* = \xi^* \quad \text{for } t = 0.$$

The solution is

$$(10.9) \quad \begin{cases} \eta^* = \xi^* + t v^* + \frac{t^2}{2!} v^\mu \partial_\mu v^* + \frac{t^3}{3!} (v^\mu \partial_\mu)^2 v^* + \dots \\ = \left(1 + t v^\mu \partial_\mu + \frac{t^2}{2!} (v^\mu \partial_\mu)^2 + \dots\right) \xi^* \end{cases}$$

<sup>1)</sup> "Mitschleppen" in SCHOUTEN and v. KAMPEN 1933, 1 and E I 1935, 1.

or symbolically

$$(10.10) \quad \eta^\alpha = e^{tX} \xi^\alpha; \quad X \stackrel{\text{def}}{=} v^\mu \partial_\mu.$$

This solution represents a point transformation, conveniently indicated by  $\xi^\alpha \xrightarrow{tv} \eta^\alpha$  or  $\eta^\alpha = \overset{tv}{T} \xi^\alpha$ . Its geometrical significance follows from (10.7). According to this equation  $t$  is a parameter fixed by the field  $v^\alpha$  on every streamline of this field if the point  $t=0$  on each streamline is chosen arbitrarily. Now  $\xrightarrow{tv}$  indicates the point transformation that on every streamline transforms a point with parameter  $t$  into the point with parameter  $t+\tau$ . We call this transformation the *displacement over  $\tau v^\alpha$* . From this it follows immediately that

$$(10.11) \quad \overset{t_1 v}{T} \overset{t_2 v}{T} = \overset{t_3 v}{T}; \quad t_3 \stackrel{\text{def}}{=} t_1 + t_2$$

and that accordingly the transformations  $\overset{tv}{T}$  form a one-parameter abelian group (cf. Ch. IV).

Neglecting second and higher powers of  $t$  and writing  $dt$  instead of  $t$  we get the infinitesimal transformation

$$(10.12) \quad \eta^\alpha \stackrel{*}{=} \xi^\alpha + v^\alpha dt \quad \text{or} \quad \eta^\alpha = \overset{v dt}{T} \xi^\alpha; \quad \xi^\alpha \stackrel{*}{=} \eta^\alpha - v^\alpha dt \quad \text{or} \quad \xi^\alpha = \overset{-v dt}{T} \eta^\alpha$$

which is said to *generate* the group of finite transformations (10.10). According to (10.1, 2, 4) the corresponding transformations of coordinates are

$$(10.13) \quad \begin{cases} \xi^{\alpha'} \stackrel{*}{=} \delta_{\alpha'}^\alpha (\xi^\alpha - v^\alpha dt) & \text{or} \quad \xi^{\alpha'} = \overset{-v dt}{T} \xi^\alpha \\ \xi^\alpha \stackrel{*}{=} \delta_\alpha^\alpha \xi^{\alpha'} + v^\alpha dt & \text{or} \quad \xi^\alpha = \overset{v dt}{T} \xi^{\alpha'} \end{cases}$$

and  $(\alpha')$  is the coordinate system *dragged along over  $v^\alpha dt$* . By differentiation of (10.13) we get

$$(10.14) \quad \begin{cases} \text{a)} \quad A_\lambda^{\alpha'} \stackrel{*}{=} \delta_{\alpha'}^\alpha (A_\lambda^\alpha - \partial_\lambda v^\alpha dt) \\ \text{b)} \quad \partial_\mu A_\lambda^{\alpha'} \stackrel{*}{=} -\delta_{\alpha'}^\alpha \partial_\mu \partial_\lambda v^\alpha dt \end{cases}$$

$$(10.15) \quad \begin{cases} \text{a)} \quad A_\lambda^{\alpha'} \stackrel{*}{=} \delta_{\alpha'}^\alpha (A_\lambda^\alpha + \partial_\lambda v^\alpha dt) \\ \text{b)} \quad \partial_{\mu'} A_\lambda^{\alpha'} \stackrel{*}{=} \delta_{\mu'}^\mu \delta_{\alpha'}^\alpha \partial_\mu \partial_\lambda v^\alpha dt. \end{cases}$$

The expression

$$(10.16) \quad \underset{v}{\mathfrak{L}} \Phi \stackrel{\text{def}}{=} \lim_{t \rightarrow 0} \frac{\Phi - \overset{t}{\Phi}}{t}$$

will be called the LIE derivative of the field  $\Phi$  with respect to the field  $v^\alpha$  and  $\underset{v}{\mathfrak{L}} \Phi dt$  the LIE differential of  $\Phi$  with respect to the infinitesimal displacement  $v^\alpha dt$ <sup>1)</sup>. This can be expressed in another way:

<sup>1)</sup> SCHOUTEN and VAN KAMPEN 1933, 1; YANO 1946, 5; 1949, 1; YANO and TAKANO 1946, 6; SCHOUTEN 1949, 5 and for more general geometric objects LAP-

If a field  $\Phi$  is dragged along over  $v^\kappa dt$  the variation of the field at  $\xi^\kappa$  is the negative LIE differential of  $\Phi$  at that point.

If  $\Phi_A$  is a geometric object, also  $(\Phi_A, \partial_\mu \Phi_A)$  are components of a geometric object. From the definition of dragging along it follows that

$\widehat{\Phi_A, \partial_\mu \Phi_A} = \overset{m}{\underset{v}{\mathfrak{L}}} \Phi_A, \partial_\mu \overset{m}{\underset{v}{\mathfrak{L}}} \Phi_A$ . But this implies that  $\overset{m}{\underset{v}{\mathfrak{L}}} (\Phi_A, \partial_\mu \Phi_A) = (\overset{m}{\underset{v}{\mathfrak{L}}} \Phi_A, \partial_\mu \overset{m}{\underset{v}{\mathfrak{L}}} \Phi_A)$ . Now on the other hand this same LIE derivative can also be written  $(\overset{m}{\underset{v}{\mathfrak{L}}} \Phi_A, \overset{m}{\underset{v}{\mathfrak{L}}} \partial_\mu \Phi_A)$ , and this proves that in the expression of the LIE derivative of the object consisting of a geometric object and its first derivative,  $\overset{m}{\underset{v}{\mathfrak{L}}} \partial_\mu \Phi_A$  may be replaced by  $\partial_\mu \overset{m}{\underset{v}{\mathfrak{L}}} \Phi_A$ :

$$(10.17) \quad \overset{m}{\underset{v}{\mathfrak{L}}} \partial_\mu \Phi_A = \partial_\mu \overset{m}{\underset{v}{\mathfrak{L}}} \Phi_A^{-1}.$$

Of course the expression  $\overset{m}{\underset{v}{\mathfrak{L}}} \partial_\mu \Phi_A$  can not be considered as the LIE derivative of  $\partial_\mu \Phi_A$  but only as a set of components of the LIE derivative of  $(\Phi_A, \partial_\mu \Phi_A)$ , because  $\partial_\mu \Phi_A$  is not a geometric object. In the same way  $\overset{m}{\underset{v}{\mathfrak{L}}} u^I$  is not the LIE derivative of the component  $u^I$  but only the 1-component of the LIE derivative  $\overset{m}{\underset{v}{\mathfrak{L}}} u^\kappa$ . Obviously (10.17) is also true if (x) is anholonomic <sup>2)</sup>.

According to the definition we have for a contravariant vector field  $u^\kappa$  and the field  $\overset{m}{\underset{v}{\mathfrak{L}}} u^\kappa$  dragged along over  $v^\kappa dt$

$$(10.18a) \quad \overset{m}{\underset{v}{\mathfrak{L}}} u^\kappa \{ \xi + v dt \} \stackrel{*}{=} \delta_\kappa^\lambda u^\lambda \{ \xi \}$$

or

$$(10.18b) \quad \overset{m}{\underset{v}{\mathfrak{L}}} u^\kappa \{ \xi \} \stackrel{*}{=} \delta_\kappa^\lambda u^\lambda \{ \xi - v dt \} = \delta_\kappa^\lambda (u^\lambda \{ \xi \} - v^\mu \partial_\mu u^\lambda \{ \xi \} dt),$$

hence, according to (10.15)

$$(10.19) \quad \left\{ \begin{array}{l} \overset{m}{\underset{v}{\mathfrak{L}}} u^\kappa = (A_\lambda^\kappa + \partial_\lambda v^\kappa dt) (u^\lambda - v^\mu \partial_\mu u^\lambda dt) \\ \qquad \qquad \qquad = u^\kappa - v^\mu \partial_\mu u^\kappa dt + u^\mu \partial_\mu v^\kappa dt \end{array} \right.$$

and, according to (10.16)

$$(10.20a) \quad \overset{m}{\underset{v}{\mathfrak{L}}} u^\kappa = v^\mu \partial_\mu u^\kappa - u^\mu \partial_\mu v^\kappa.$$

Returning now to Fig. 9 we see that  $\overset{m}{\underset{v}{\mathfrak{L}}} u^\kappa ds dt$  is the fifth side of the pentagon and that  $\overset{m}{\underset{v}{\mathfrak{L}}} u^\kappa$  vanishes if and only if the fields  $u^\kappa$  and  $v^\kappa$  are  $X_2$ -forming (cf. II § 5).

TEW 1938, 1; DE DONDER and GÉHÉNIAU 1942, 1; RACHEVSKI 1949, 1; TASHIRO 1950, 1; WAGNER 1950, 1; NIJENHUIS 1952, 1, Ch. II. The name LIE derivative occurred first in V. DANTZIG 1932, 1; 2.

<sup>1)</sup> NIJENHUIS 1952, 1, p. 95.

<sup>2)</sup> Cf. for LIE derivatives in anholonomic coordinates NIJENHUIS 1952, 1, II § 6 and Exerc. II 10, 6; 7.

From the definition of dragging along it follows that for any product or transvection of geometric objects we have  $\overset{m}{\Phi} \Psi = \overset{m}{\Phi} \overset{m}{\Psi}$  and from this we see that the rule of LEIBNIZ

$$(10.21) \quad \overset{v}{\mathcal{L}} \Phi \Psi = \Phi \overset{v}{\mathcal{L}} \Psi + (\overset{v}{\mathcal{L}} \Phi) \Psi$$

holds for  $\overset{v}{\mathcal{L}}$  applied to products and transvections provided that  $\Phi, \Psi$  and  $\Phi \Psi$  are geometric objects. It can be proved in a similar way that

1) the LIE derivative of a sum of geometric objects equals the sum of the LIE derivatives of the summands, provided that the sum is a geometric object;

2) the LIE derivative of a contraction equals the contraction of the LIE derivative.

If  $\Phi$  is a quantity,  $\overset{m}{\Phi}$  is a quantity of the same kind and therefore  $\overset{v}{\mathcal{L}} \Phi$  is also a quantity of the same kind. The same is not true for every geometric object.

According to the definition we have for a scalar

$$(10.22) \quad \overset{v}{\mathcal{L}} s = v^\mu \partial_\mu s$$

and from this and the rule of LEIBNIZ for transvections it can be proved that for a covariant vector

$$(10.20b) \quad \overset{v}{\mathcal{L}} w_\lambda = v^\mu \partial_\mu w_\lambda + w_\mu \partial_\lambda v^\mu$$

because  $v^\lambda w_\lambda$  is a scalar.

Because every tensor can be written as a sum of products of vectors we get for the LIE derivative of a tensor, for instance  $P_{\cdot\lambda}^x$ :

$$(10.23) \quad \overset{v}{\mathcal{L}} P_{\cdot\lambda}^x = v^\mu \partial_\mu P_{\cdot\lambda}^x - P_{\cdot\lambda}^\mu \partial_\mu v^x + P_{\cdot\mu}^x \partial_\lambda v^\mu.$$

For a tensor with higher valences the LIE derivative is constructed in the same way. Each upper (lower) index gives rise to a term with a negative (positive) sign. As a special case we mention the contravariant and covariant  $p$ -vector field

$$(10.24) \quad \begin{cases} a) & \overset{v}{\mathcal{L}} u^{x_1 \dots x_p} = v^\mu \partial_\mu u^{x_1 \dots x_p} - p u^{\mu [x_2 \dots x_p} \partial_\mu v^{x_1]} \\ b) & \overset{v}{\mathcal{L}} w_{\lambda_1 \dots \lambda_p} = v^\mu \partial_\mu w_{\lambda_1 \dots \lambda_p} + p w_{\mu [\lambda_2 \dots \lambda_p} \partial_{\lambda_1] v^\mu}. \end{cases}$$

From (10.24b) we get for  $p=n$

$$(10.25) \quad \begin{cases} \overset{v}{\mathcal{L}} w_{\lambda_1 \dots \lambda_n} = v^\mu \partial_\mu w_{\lambda_1 \dots \lambda_n} + n w_{\mu [\lambda_2 \dots \lambda_n} \partial_{\lambda_1] v^\mu} \\ \quad = v^\mu \partial_\mu w_{\lambda_1 \dots \lambda_n} + w_{\lambda_1 \dots \lambda_n} \partial_\mu v^\mu = \partial_\mu v^\mu w_{\lambda_1 \dots \lambda_n} \end{cases}$$

and because  $w_{\lambda_1 \dots \lambda_n}$  may be considered as the component of a scalar  $\Delta$ -density of weight +1, it follows that for such a quantity  $\tilde{q}$

$$(10.26) \quad \overset{v}{\mathcal{L}} \tilde{q} = v^\mu \partial_\mu \tilde{q} + \tilde{q} \partial_\mu v^\mu = \partial_\mu \tilde{q} v^\mu = \text{Div}(\tilde{q} v).$$

Since every scalar density  $q$  of weight +1 is a product of the  $W$ -scalar  $\tilde{\omega}$  (cf. I § 2) with a scalar  $A$ -density of weight +1 we have

$$(10.27) \quad \frac{L}{v} q = v^\mu \partial_\mu q + q \partial_\mu v^\mu = \partial_\mu q v^\mu = \text{Div}(q v).$$

From this we get for a scalar density  $s$  of weight  $w$

$$(10.28) \quad \frac{L}{v} s = v^\mu \partial_\mu s + w s \partial_\mu v^\mu$$

and for a tensor density of weight  $w$  for instance  $\mathfrak{P}_{\cdot\lambda}^x$ ,

$$(10.29) \quad \frac{L}{v} \mathfrak{P}_{\cdot\lambda}^x = v^\mu \partial_\mu \mathfrak{P}_{\cdot\lambda}^x - \mathfrak{P}_{\cdot\lambda}^x \partial_\mu v^\mu + \mathfrak{P}_{\cdot\mu}^x \partial_\lambda v^\mu + w \mathfrak{P}_{\cdot\lambda}^x \partial_\mu v^\mu.$$

Formulae of the same form hold for scalar  $A$ -densities and tensor  $A$ -densities.

It can be proved for more general geometric objects as defined in II § 3 that the LIE derivative of a geometric object together with the object itself form a geometric object and that the LIE derivative itself is always a geometric object if and only if the transformation of the given geometric object is *linear* (not necessary homogeneous) in the components with coefficients depending on  $\xi^x, \xi^{x'}, A_x^{x'}, \partial_\mu A_x^{x'}, \dots$ .<sup>1)</sup>

A field is said to be *absolutely invariant with respect to the field  $v^*$*  if its LIE derivative with respect to  $v^*$  vanishes. The necessary and sufficient condition is that the field does not change if it is dragged along by the point transformations of the one-parameter groups generated by  $v^* dt$ . Obviously sums, products and transvections of absolutely invariant fields are absolutely invariant.

A field is said to be *absolutely invariant with respect to the streamlines of  $v^*$*  if it is absolutely invariant with respect to  $\sigma v^*$  for every choice of the scalar  $\sigma$ .

The additional condition for this stronger invariance for an absolutely invariant covariant  $p$ -vector field  $w_{\lambda_1 \dots \lambda_p}$  is

$$(10.30) \quad v^\mu w_{\mu \lambda_1 \dots \lambda_p} = 0.$$

This follows immediately from (10.24b).<sup>2)</sup> The LIE derivative of every geometric object  $\Phi_A$  as defined in II § 3 can be found from the definition (10.16) and the form  $\overset{m}{\Phi}_A$  in (10.6). But the computing may be rather cumbersome if the transformation of the object is complicated. As an example we deal here with a rather simple geometric object without using (10.6). Let the transformation of the object be<sup>3)</sup>

$$(10.31) \quad \Pi_{\mu' \lambda'}^{x'} = A_{\mu' \lambda'}^{\mu \lambda} \Pi_{\mu \lambda}^x - A_{\mu' \lambda'}^{\mu \lambda} \partial_\mu A_{\lambda}^{x'}.$$

<sup>1)</sup> NIJENHUIS 1952, 1, Ch. II. Cf. SCHOUTEN and VAN KAMPEN 1933, 1; E I 1935, 1; TASHIRO 1950, 1.

<sup>2)</sup> Cf. for more particulars concerning absolutely and relatively invariant fields P. P. 1949, 1, p. 43ff. and T. P. 1951, 1, p. 78.

<sup>3)</sup> Objects of this kind occur frequently in differential geometry, cf. Ch. III.

For  $\overset{m}{\Pi}_{\mu\lambda}^{\nu}$  we get

$$(10.32) \quad \overset{m}{\Pi}_{\mu\nu}^{\lambda}\{\xi\} = \delta_{\nu}^{\lambda} \overset{m}{\Pi}_{\mu\lambda}^{\nu}\{\xi - v dt\} = \delta_{\nu}^{\lambda} \overset{m}{\Pi}_{\mu\lambda}^{\nu}\{\xi\} - v^{\sigma} \partial_{\sigma} \overset{m}{\Pi}_{\mu\lambda}^{\nu}\{\xi\} dt$$

hence, according to (10.14, 15)

$$(10.33) \quad \left\{ \begin{aligned} \overset{m}{\Pi}_{\mu\lambda}^{\nu} &= (A_{\mu}^{\tau} - \partial_{\mu} v^{\tau} dt) (A_{\lambda}^{\sigma} - \partial_{\lambda} v^{\sigma} dt) (A_{\rho}^{\nu} + \partial_{\rho} v^{\nu} dt) \times \\ &\quad \times (\Pi_{\tau\sigma}^{\rho} - v^{\omega} \partial_{\omega} \Pi_{\tau\sigma}^{\rho} dt) - (A_{\mu}^{\tau} - \partial_{\mu} v^{\tau} dt) (A_{\lambda}^{\sigma} - \partial_{\lambda} v^{\sigma} dt) \partial_{\tau} \partial_{\sigma} v^{\nu} dt. \end{aligned} \right.$$

$$(10.34) \quad \underset{v}{\mathfrak{L}} \overset{m}{\Pi}_{\mu\lambda}^{\nu} = v^{\sigma} \partial_{\sigma} \overset{m}{\Pi}_{\mu\lambda}^{\nu} - \Pi_{\mu\lambda}^{\rho} \partial_{\rho} v^{\nu} + \Pi_{\mu\sigma}^{\nu} \partial_{\lambda} v^{\sigma} + \Pi_{\rho\lambda}^{\nu} \partial_{\mu} v^{\rho} + \partial_{\mu} \partial_{\lambda} v^{\nu}.$$

We see from (10.31) that the difference of two objects of this kind is always a tensor, because the second term to the right in (10.31) does not contain  $\overset{m}{\Pi}_{\mu\lambda}^{\nu}$ . Now  $\overset{m}{\Pi}_{\mu\lambda}^{\nu}$  is an object of the same kind as  $\overset{m}{\Pi}_{\mu\lambda}^{\nu}$  and this implies that  $\underset{v}{\mathfrak{L}} \overset{m}{\Pi}_{\mu\lambda}^{\nu}$  is a tensor.

If a field  $v^{\nu}$  is given, the LIE operator  $\underset{v}{\mathfrak{L}}$  can be used to find the field  $\overset{m}{\Phi}_A$  that results from a field of a quantity  $\Phi_A$  if this field is dragged along by a finite point transformation  $\overset{t\nu}{T}$  belonging to the one-parameter group generated by the infinitesimal point transformation  $\xi^{\nu} \rightarrow \xi^{\nu} + v^{\nu} dt$ . Let  $(\nu)$  be chosen in such a way that  $v^{\nu} = e_I^{\nu}$  (cf. Exerc. II 5,1). Then  $\underset{v}{\mathfrak{L}} e_I^{\nu} = 0$  for every value of  $\lambda$  and this has as a consequence that the transformation  $\overset{t\nu}{T}$  takes the form

$$(10.35) \quad ' \xi^{\nu} \stackrel{*}{=} \xi^{\nu} + t e_I^{\nu}$$

and that

$$(10.36) \quad \overset{m}{\Phi}_A \{\xi\} \stackrel{*}{=} \Phi_A \{\xi - t e_I\}.$$

Now for this special choice of  $(\nu)$  we have

$$(10.37) \quad \underset{v}{\mathfrak{L}} \Phi_A \stackrel{*}{=} \partial_1 \Phi_A,$$

hence

$$(10.38) \quad \overset{m}{\Phi}_A = e^{-t \underset{v}{\mathfrak{L}}} \Phi_A = \Phi_A - t \underset{v}{\mathfrak{L}} \Phi_A + \frac{t^2}{2!} \underset{v}{\mathfrak{L}}^2 \Phi_A - \dots {}^1).$$

Because this equation has the invariant form it is independent of the choice of  $(\nu)$ . Every term represents the values at  $\xi^{\nu}$  of the components of a quantity of the same kind as  $\Phi_A$ .

It can be proved that (10.38) also holds for all *geometric objects* whose transformation is *linear*<sup>2)</sup>.

<sup>1)</sup> This series occurs in KOSAMBI 1946, 1 but no proof is given. SHABBAR's justification of (10.38) in 1942, 1 really shows only the correctness of the first two terms. In 1951, 1 KOSAMBI gave for the first time a complete proof. The proof given here is from NIJENHUIS 1952, 1.

<sup>2)</sup> NIJENHUIS 1952, 1. GOLAB 1950, 1 called these objects "purely differential". The  $\overset{m}{\Pi}_{\mu\lambda}^{\nu}$  in (10.31) are components of such an object.

In Ch. IV we need the following formula<sup>1)</sup>

$$(10.39) \quad \boxed{e^{-s\frac{\xi}{u}} e^{-t\frac{\xi}{v}} e^{+s\frac{\xi}{w}} \Phi = e^{-\frac{\xi}{w}} \bar{\Phi}; \quad w^\kappa \stackrel{\text{def}}{=} e^{-s\frac{\xi}{u}} v^\kappa}$$

valid for all geometric objects whose transformation is linear. First we prove for objects of this kind

$$(10.40) \quad * \psi \stackrel{\text{def}}{=} e^{-s\frac{\xi}{u}} \underline{\xi}_v e^{+s\frac{\xi}{w}} \psi = \underline{\xi}_w \psi.$$

For the derivatives of  $* \psi$  we get (cf. Exerc. II 10,9)

$$(10.41) \quad \left\{ \begin{array}{l} \frac{d * \psi}{ds} = -e^{-s\frac{\xi}{u}} (\underline{\xi}_u \underline{\xi}_v - \underline{\xi}_v \underline{\xi}_u) e^{+s\frac{\xi}{w}} \psi = -e^{-s\frac{\xi}{u}} \underline{\xi}_w e^{+s\frac{\xi}{w}} \psi \\ \frac{d^2 * \psi}{ds^2} = e^{-s\frac{\xi}{u}} \underline{\xi}_w \underline{\xi}_v e^{+s\frac{\xi}{w}} \psi; \quad w^\kappa \stackrel{\text{def}}{=} \underline{\xi}_u v^\kappa; \quad w^\kappa \stackrel{\text{def}}{=} \underline{\xi}_w w^\kappa = \underline{\xi}_v v^\kappa. \end{array} \right.$$

Hence we have

$$(10.42) \quad * \psi = \underline{\xi}_v \psi - s \underline{\xi}_w \psi + \frac{s^2}{2!} \underline{\xi}_v \psi - \dots$$

and because of

$$(10.43) \quad \underline{\xi}_{u+v} \psi = \underline{\xi}_u \psi + \underline{\xi}_v \psi$$

and the definition of  $w^\kappa$ , it follows that  $* \psi = \underline{\xi}_w \psi$ . Now writing  $\chi$  for the left hand side of (10.39) we have according to (10.40)

$$(10.44) \quad \frac{d \chi}{dt} = -e^{-s\frac{\xi}{u}} \underline{\xi}_v e^{-t\frac{\xi}{v}} e^{+s\frac{\xi}{w}} \Phi = -e^{-s\frac{\xi}{u}} \underline{\xi}_v e^{+s\frac{\xi}{w}} \chi = -\underline{\xi}_w \chi$$

and if this differential equation is solved by means of a series, we get (10.39).

### Exercises.

II 10,1. Prove that  $\underline{\xi}_v v^\kappa = 0$  and that the LIE derivative of  $A_\lambda^\kappa$ ,  $\tilde{G}^{\kappa_1 \dots \kappa_n}$  and  $\tilde{e}_{\lambda_1 \dots \lambda_n}$  vanish for every choice of  $v^\kappa$ .

II 10,2. Prove that the additional conditions for absolute invariance of a  $\phi$ -vector density (or  $\Delta$ -density)  $w^{\kappa_1 \dots \kappa_p}$  of weight +1 is

$$\text{II 10,2 } \alpha) \quad w^{[\kappa_1 \dots \kappa_p} v^{\kappa]} = 0.$$

II 10,3<sup>2)</sup>). Prove the identities for the geometrical object  $\xi^\kappa$  and for the scalar fields  $\tilde{\xi}$

$$\text{II 10,3 } \alpha) \quad \underline{\xi}_v \xi^\kappa = 0$$

$$\text{II 10,3 } \beta) \quad \underline{\xi}_v \tilde{\xi} = v^\mu \partial_\mu \tilde{\xi} = v^\mu \tilde{e}_\mu = \tilde{v}.$$

<sup>1)</sup> NIJENHUIS 1952, 1, II § 8.

<sup>2)</sup> NIJENHUIS 1952, 1, II § 4.

II 10,4<sup>1)</sup>. Prove that the operators  $\mathfrak{L}_v$  and  $D$  are commutative, for instance

$$\text{II 10,4 } \alpha) \quad \mathfrak{L}_v \partial_{[\mu} w_{\lambda_1 \dots \lambda_q]} = \partial_{[\mu} \mathfrak{L}_v w_{\lambda_1 \dots \lambda_q]}$$

$$\text{II 10,4 } \beta) \quad \mathfrak{L}_v \partial_\mu w^{\mu \kappa_2 \dots \kappa_p} = \partial_\mu \mathfrak{L}_v w^{\mu \kappa_2 \dots \kappa_p}.$$

II 10,5<sup>2)</sup>. Prove that

$$\text{II 10,5 } \alpha) \quad \mathfrak{L}_v \mathbf{w} = \mathbf{v} \cdot \text{Rot } \mathbf{w} + \text{Rot } \mathbf{v} \cdot \mathbf{w}$$

$$\text{II 10,5 } \beta) \quad \mathfrak{L}_v^k \mathbf{w} = (\mathbf{v} \cdot \text{Rot})^k \mathbf{w} + (\text{Rot } \mathbf{v})^k \mathbf{w}$$

$$\text{II 10,5 } \gamma) \quad \mathfrak{L}_v \mathbf{w} = \text{Div } \mathbf{v} \cdot \mathbf{w} - \mathbf{v} \cdot \text{Div } \mathbf{w}$$

$$\text{II 10,5 } \delta) \quad \mathfrak{L}_v^k \mathbf{w} = (\text{Div } \mathbf{v})^k \mathbf{w} + (-\mathbf{v} \cdot \text{Div})^k \mathbf{w},$$

where  $\mathbf{w}$  is a covariant  $q$ -vector and  $\mathbf{w}$  a contravariant  $p$ -vector density of weight  $+1$ . Rot has to be changed into Grad if working on a scalar and  $\mathbf{v}$  is defined by:

$$\text{II 10,5 } \epsilon) \quad \mathbf{v} \cdot \mathbf{w} : \quad v^\mu w_{\mu \lambda_2 \dots \lambda_q}$$

$$\text{II 10,5 } \zeta) \quad \mathbf{v} \cdot \mathbf{w} : \quad (p+1) v^{[\kappa} w^{\kappa_1 \dots \kappa_p]}.$$

II 10,6. Prove that<sup>3)</sup>

$$\text{II 10,6 } \alpha) \quad \mathfrak{L}_v u^h = v^j \partial_j u^h - (\partial_j v^h - 2v^i \Omega_{ji}^h) u^j$$

$$\text{II 10,6 } \beta) \quad \mathfrak{L}_v w_i = v^j \partial_j w_i + (\partial_i v^h - 2v^j \Omega_{ij}^h) w_h$$

$$\text{II 10,6 } \gamma) \quad \mathfrak{L}_v \Omega_{ji}^h = 0.$$

$$\text{II 10,6 } \delta) \quad \mathfrak{L}_v \mathfrak{p}^{(h)} = v^j \partial_j \mathfrak{p}^{(h)} (\partial_j v^j + 2v^j \Omega_{ji}^i).$$

II 10,7. Prove that<sup>3)</sup>

$$\text{II 10,7 } \alpha) \quad \mathfrak{L}_{\frac{j}{i}} e_i^h = -2 \Omega_{ji}^h$$

$$\text{II 10,7 } \beta) \quad \mathfrak{L}_{\frac{j}{i}} e_i^x = -2 \Omega_{ji}^h A_x^h.$$

where  $\mathfrak{L}_j$  denotes the LIE derivative with respect to  $e_j^x$ .

II 10,8. Prove that for any three vector fields  $t^x, u^x, v^x$ <sup>4)</sup>

$$\text{II 10,8 } \alpha) \quad \mathfrak{L}_t \mathfrak{L}_u v^x + \mathfrak{L}_u \mathfrak{L}_t v^x + \mathfrak{L}_v \mathfrak{L}_t u^x = 0.$$

II 10,9. Prove that for any quantity  $\Phi$  (indices suppressed) and any vector fields  $u^x, v^x$ <sup>5)</sup>

<sup>1)</sup> P. P. 1949, 1, p. 76. This is only a special case of the commutativity of  $\mathfrak{L}_v$  and  $\partial_\mu$  [cf. (10.17)].

<sup>2)</sup> E. J. Post, personal communication.

<sup>3)</sup> NIJENHUIS 1952, 1, II § 6.

<sup>4)</sup> DAVIES 1938, 1.

<sup>5)</sup> NIJENHUIS 1952, 1, II § 8; Taschiro 1952, 1.

$$\text{II 10,9 } \alpha) \quad (\underline{\xi} \underline{\xi} - \underline{\xi} \underline{\xi}) \Phi = \underline{\xi} \Phi; \quad w^\kappa \stackrel{\text{def}}{=} \underline{\xi} v^\kappa = - \underline{\xi} u^\kappa.$$

II 10,10. Prove that  $\underline{\xi} v^\kappa = 0$  if and only if <sup>1)</sup>

$$\text{II 10,10 } \alpha) \quad \overset{\overset{\overset{su}{\overset{tv}{T}}}{\overset{tu}{T}}} \xi^\kappa = \overset{\overset{tv}{\overset{su}{T}}}{\overset{tu}{T}} \xi^\kappa.$$

II 10,11<sup>2)</sup>. In an  $X_n$  a covariant  $q$ -vector field  $w_{\lambda_1 \dots \lambda_q}$  and a set of  $\infty^1 X_q$ 's depending on a parameter  $t$  are given. Let a set of curves intersect each  $X_q$  in one point, such that  $t$  is also a parameter on each of these curves. If then  $\tau_q$  is a  $q$ -dimensional part of one  $X_q$  there is a one to one correspondence between the points of this  $\tau_q$  and  $\infty^1 \tau_q$ 's on the  $\infty^1 X_q$ 's and

$$\text{II 10,11 } \alpha) \quad \Phi(t) \stackrel{\text{def}}{=} \int_{\tau_q} w_{\lambda_1 \dots \lambda_q} d^{\lambda_1 \dots \lambda_q}$$

is a function of  $t$ . Prove that

$$\text{II 10,11 } \beta) \quad \frac{d\Phi}{dt} = \int_{\tau_q} \underline{\xi} w_{\lambda_1 \dots \lambda_q} d^{\lambda_1 \dots \lambda_q}; \quad v^\kappa \stackrel{\text{def}}{=} \frac{d\xi^\kappa}{dt}.$$

II 10,12. Prove that  $\underline{\xi} \partial_j u^h = \partial_j \underline{\xi} u^h$  for (h) anholonomic.

## § 11. The LAGRANGE derivative.

Let  $\Phi_A$ ;  $A=1, \dots, N$  be a set of functions of the  $\xi^\kappa$ . They may be scalars or components of geometric objects or just quite arbitrary functions for which no manner of transformation is defined. Let the index  $A$  be suppressed and the derivatives of the  $\Phi_A$  with respect to the  $\xi^\kappa$  be denoted by  $\Phi_\mu, \Phi_{\nu\mu}$ , etc.

If  $\mathfrak{L}$  is a function of the  $\Phi, \Phi_\mu, \Phi_{\nu\mu}$ , etc. up to a certain definite order, then, in a certain region of  $X_n$ ,  $\mathfrak{L}$  is a function of the  $\xi^\kappa$ . We consider the volume integral (cf. II 8.5)

$$(11.1) \quad \int_{\tau_n} \mathfrak{L} d\xi^1 \dots d\xi^n$$

over an arbitrary region  $\tau_n$  where the  $\Phi$  are analytic.

Let the field  $\Phi$  be subjected to a variation  $\overset{\circ}{d}\Phi$  such that the variations of  $\Phi$  and of all its derivatives occurring in  $\mathfrak{L}$  vanish at the boundary  $\tau_{n-1}$  of  $\tau_n$ .

Then we have for the variation of  $\mathfrak{L}$

$$(11.2) \quad \overset{\circ}{d}\mathfrak{L} = \frac{\partial \mathfrak{L}}{\partial \Phi} \overset{\circ}{d}\Phi + \frac{\partial \mathfrak{L}}{\partial \Phi_\mu} \overset{\circ}{d}\Phi_\mu + \dots$$

<sup>1)</sup> A. NIJENHUIS, personal communication.

<sup>2)</sup> E. J. POST, personal communication.

and consequently

$$(11.3) \quad \overset{v}{d} \int_{\tau_n} \mathfrak{L} d\xi^1 \dots d\xi^n = \int_{\tau_n} \left( \frac{\partial \mathfrak{L}}{\partial \Phi} \overset{v}{d}\Phi + \frac{\partial \mathfrak{L}}{\partial \Phi_\mu} \overset{v}{d}\Phi_\mu + \dots \right) d\xi^1 \dots d\xi^n.$$

Let now  $\tau_{n-1}$  have a form such that the theorem of STOKES is valid for  $\tau_n$  and  $\tau_{n-1}$ . Then we have, integrating by parts and using this theorem in the form (8.20) for  $p=1^1)$

$$(11.4) \quad \left\{ \begin{array}{l} \overset{v}{d} \int_{\tau_n} \mathfrak{L} d\xi^1 \dots d\xi^n = \int_{\tau_n} \left( \frac{\partial \mathfrak{L}}{\partial \Phi} \overset{v}{d}\Phi + \frac{\partial \mathfrak{L}}{\partial \Phi_\mu} \partial_\mu \overset{v}{d}\Phi + \dots \right) d\xi^1 \dots d\xi^n \\ = \int_{\tau_n} \left( \frac{\partial \mathfrak{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathfrak{L}}{\partial \Phi_\mu} + \partial_\nu \partial_\mu \frac{\partial \mathfrak{L}}{\partial \Phi_{\nu\mu}} - \dots \right) \overset{v}{d}\Phi d\xi^1 \dots d\xi^n \end{array} \right.$$

hence

$$(11.5) \quad \overset{v}{d} \int_{\tau_n} \mathfrak{L} d\xi^1 \dots d\xi^n = \int_{\tau_n} [\mathfrak{L}] \overset{v}{d}\Phi d\xi^1 \dots d\xi^n$$

where

$$(11.6) \quad [\mathfrak{L}] \stackrel{\text{def}}{=} \frac{\partial \mathfrak{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathfrak{L}}{\partial \Phi_\mu} + \partial_\nu \partial_\mu \frac{\partial \mathfrak{L}}{\partial \Phi_{\nu\mu}} - \dots$$

$[\mathfrak{L}]$  is called the LAGRANGE derivative of  $\mathfrak{L}$ .  $[\mathfrak{L}]$  has of course a suppressed upper index  $A$  just as  $\Phi$  has a suppressed lower index  $A$ . If  $[\mathfrak{L}] = 0$  the variation (11.3) vanishes for every choice of  $\tau_n$ , provided that the variation of  $\Phi$  satisfies the boundary conditions.

The equation  $[\mathfrak{L}] = 0$  is called the LAGRANGE equation. In classical dynamics LAGRANGE equations with  $n=1$  occur. The  $\xi^a$  reduce then to one variable  $t$  and the  $\Phi_A$  are the coordinates of the dynamical system.<sup>2)</sup>

If the  $\Phi_A$  are components of a geometric object with a given manner of transformation, and if  $\mathfrak{L}$  is computed from the  $\Phi_A$  and their derivatives in the same way for all coordinate systems it may happen that  $\mathfrak{L}$  is a scalar  $A$ -density of weight +1. In that case the integral (11.3) is a scalar and the process of forming the LAGRANGE derivative is invariant for all coordinate transformations. If moreover  $\Phi$  is a tensor with the valences  $p, q$ ,  $[\mathfrak{L}]$  is a tensor  $A$ -density of weight +1 with the valences  $q, p$ . If only coordinate transformations with  $A > 0$  are considered we may replace  $A$ -density by density in the above statements.

If  $\Phi$  is a tensor and  $\mathfrak{L}$  a scalar  $A$ -density (or density for  $A > 0$ ) of weight +1, a very important relation exists between  $\Phi$ ,  $[\mathfrak{L}]$  and their first derivatives. In order to obtain this relation we suppose

<sup>1)</sup> Cf. the remark at the end of II § 8.

<sup>2)</sup> Cf. T. P. 1951, 1, p. 198.

that the variation  $\overset{v}{d}\Phi$  is due to the dragging along of the field  $\Phi$  over  $v^\kappa dt$ , hence

$$(11.7) \quad \overset{v}{d}\Phi = - \underset{v}{\mathfrak{L}}\Phi dt$$

and that  $v^\kappa, \partial_\mu v^\kappa, \partial_\mu \partial_\lambda v^\kappa, \dots$  are zero on  $\tau_{n-1}$ . Let  $\bar{\Phi}$  be the tensor  $P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\lambda_1 \dots \lambda_q}$ . Then we have

$$(11.8) \quad \left\{ \begin{array}{l} P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} = P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} - v^\mu \partial_\mu P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} dt + \\ \quad + P_{\cdot \cdot \cdot}^{\mu \kappa_2 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} \partial_\mu v^{\kappa_1} dt + \dots \\ \quad - P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\mu \lambda_2 \dots \lambda_q} \partial_{\lambda_1} v^\mu dt - \dots \end{array} \right.$$

and thus

$$(11.9) \quad \partial_\nu P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} = \partial_\nu P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} + *$$

where  $*$  denotes terms each of which contains  $v^\kappa$  or a first or second derivative of  $v^\kappa$  as a factor. Hence  $\overset{v}{d}P$  and  $\overset{v}{d}\partial_\mu P$  vanish on  $\tau_{n-1}$  and the same can be proved for the variations of all higher derivatives of  $P$ .

By substituting (11.7) in (11.4) we get using STOKES' theorem

$$(11.10) \quad \left\{ \begin{array}{l} \overset{v}{\int}_{\tau_n} \mathcal{L} d\xi^1 \dots d\xi^n = - \int [\mathcal{L}]_{\kappa_1 \dots \kappa_p}^{\cdot \cdot \cdot \lambda_1 \dots \lambda_q} \underset{v}{\mathfrak{L}} P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} dt d\xi^1 \dots d\xi^n \\ = - \int [\mathcal{L}]_{\kappa_1 \dots \kappa_p}^{\cdot \cdot \cdot \lambda_1 \dots \lambda_q} \{ v^\mu \partial_\mu P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} - P_{\cdot \cdot \cdot}^{\mu \kappa_2 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} \partial_\mu v^{\kappa_1} - \dots \\ \quad + P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\mu \lambda_2 \dots \lambda_q} \partial_{\lambda_1} v^\mu + \dots \} dt d\xi^1 \dots d\xi^n \\ = - \int [\mathcal{L}]_{\kappa_1 \dots \kappa_p}^{\cdot \cdot \cdot \lambda_1 \dots \lambda_q} \{ \partial_\mu P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} + A_{\mu}^{\kappa_1} \partial_\varrho P_{\cdot \cdot \cdot}^{\varrho \kappa_2 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} + \dots \\ \quad - \partial_{\lambda_1} P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\mu \lambda_2 \dots \lambda_q} - \dots \} + \\ \quad + P_{\cdot \cdot \cdot}^{\varrho \kappa_2 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} \partial_\varrho [\mathcal{L}]_{\mu \kappa_2 \dots \kappa_p}^{\cdot \cdot \cdot \lambda_1 \dots \lambda_q} + \dots \\ \quad - P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\mu \lambda_2 \dots \lambda_q} \partial_{\lambda_1} [\mathcal{L}]_{\kappa_1 \dots \kappa_p}^{\cdot \cdot \cdot \lambda_1 \dots \lambda_q} - \dots \} v^\mu dt d\xi^1 \dots d\xi^n \end{array} \right.$$

but this variation must be zero because the field dragged along has exactly the same components with respect to  $(\kappa')$  as the original field with respect to  $(\kappa)$ . Hence

$$(11.11) \quad \left\{ \begin{array}{l} [\mathcal{L}]_{\kappa_1 \dots \kappa_p}^{\cdot \cdot \cdot \lambda_1 \dots \lambda_q} \{ \partial_\mu P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} + A_{\mu}^{\kappa_1} \partial_\varrho P_{\cdot \cdot \cdot}^{\varrho \kappa_2 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} + \dots \\ \quad - \partial_{\lambda_1} P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\mu \lambda_2 \dots \lambda_q} - \dots \} + P_{\cdot \cdot \cdot}^{\varrho \kappa_2 \dots \kappa_p}_{\lambda_1 \dots \lambda_q} \partial_\varrho [\mathcal{L}]_{\mu \kappa_2 \dots \kappa_p}^{\cdot \cdot \cdot \lambda_1 \dots \lambda_q} + \dots \\ \quad - P_{\cdot \cdot \cdot}^{\kappa_1 \dots \kappa_p}_{\mu \lambda_2 \dots \lambda_q} \partial_{\lambda_1} [\mathcal{L}]_{\kappa_1 \dots \kappa_p}^{\cdot \cdot \cdot \lambda_1 \dots \lambda_q} - \dots = 0. \end{array} \right.$$

For  $p=0, q=2$  the identity takes the simple form

$$(11.12) \quad [\mathcal{L}]^{1\kappa} (\partial_\mu P_{\lambda\kappa} - \partial_\lambda P_{\mu\kappa} - \partial_\kappa P_{\lambda\mu}) - P_{\mu\kappa} \partial_\lambda [\mathcal{L}]^{\lambda\kappa} - P_{\lambda\mu} \partial_\kappa [\mathcal{L}]^{\lambda\kappa} = 0$$

which is of special importance in the theory of relativity. (11.11) is not the only relation between a quantity, its LAGRANGE derivative and the derivatives of these two, as has already been pointed out by BUCHDAHL<sup>1)</sup> and SCHRÖDINGER.<sup>2)</sup> BUCHDAHL derived identities of this kind for a density as a function of the fundamental tensor and its derivatives up to a certain order. SCHRÖDINGER considered a function of the parameters of a linear connexion and their first derivatives.

We give here only an example, the generalization being then obvious.<sup>3)</sup> Let  $P_{\cdot\lambda}^{\kappa}$  be a tensor field and  $\mathfrak{L}$  a density of weight +1 depending on the  $P_{\cdot\lambda}^{\kappa}$  and their first and second derivatives

$$(11.13) \quad \overset{(x)}{\mathfrak{L}} = F(P_{\cdot\lambda}^{\kappa}, \partial_{\mu} P_{\cdot\lambda}^{\kappa}, \partial_{\nu} \partial_{\mu} P_{\cdot\lambda}^{\kappa}).$$

Then we have

$$(11.14) \quad F(P_{\cdot\lambda'}^{\kappa'}, \partial_{\mu'} P_{\cdot\lambda'}^{\kappa'}, \partial_{\nu'} \partial_{\mu'} P_{\cdot\lambda'}^{\kappa'}) = \Lambda^{-1} F(P_{\cdot\lambda}^{\kappa}, \partial_{\mu} P_{\cdot\lambda}^{\kappa}, \partial_{\nu} \partial_{\mu} P_{\cdot\lambda}^{\kappa})$$

and in this equation the expressions

$$(11.15) \quad \begin{cases} a) & P_{\cdot\lambda'}^{\kappa'} = A_{\kappa' \lambda'}^{\lambda} P_{\cdot\lambda}^{\kappa} \\ b) & \partial_{\mu'} P_{\cdot\lambda'}^{\kappa'} = (\partial_{\mu'} A_{\kappa' \lambda'}^{\lambda}) P_{\cdot\lambda}^{\kappa} + A_{\kappa' \lambda' \mu'}^{\lambda \mu} \partial_{\mu} P_{\cdot\lambda}^{\kappa} \\ c) & \partial_{\nu'} \partial_{\mu'} P_{\cdot\lambda'}^{\kappa'} = (\partial_{\nu'} \partial_{\mu'} A_{\kappa' \lambda'}^{\lambda}) P_{\cdot\lambda}^{\kappa} + (\partial_{\mu'} A_{\kappa' \lambda'}^{\lambda}) A_{\nu'}^{\nu} \partial_{\nu} P_{\cdot\lambda}^{\kappa} + \\ & + (\partial_{\nu'} A_{\kappa' \lambda' \mu'}^{\lambda \mu}) \partial_{\mu} P_{\cdot\lambda}^{\kappa} + A_{\kappa' \lambda' \mu' \nu'}^{\lambda \mu \nu} \partial_{\nu} \partial_{\mu} P_{\cdot\lambda}^{\kappa} \end{cases}$$

can be substituted. After this substitution (11.14) is an identity in the variables  $P_{\cdot\lambda}^{\kappa}$ ,  $\partial_{\mu} P_{\cdot\lambda}^{\kappa}$ ,  $\partial_{\nu} \partial_{\mu} P_{\cdot\lambda}^{\kappa}$ ,  $A_{\lambda}^{\kappa'}$ ,  $\partial_{\mu} A_{\lambda}^{\kappa'}$ ,  $\partial_{\nu} \partial_{\mu} A_{\lambda}^{\kappa'}$  and these may be considered as *independent*. If this identity is differentiated with respect to  $\partial_{\nu} \partial_{\mu} P_{\cdot\lambda}^{\kappa}$  we get

$$(11.16) \quad A_{\kappa' \lambda' \mu' \nu'}^{\lambda \mu \nu} \frac{\partial \overset{(x)}{\mathfrak{L}}}{\partial (\partial_{\nu'} \partial_{\mu'} P_{\cdot\lambda'}^{\kappa'})} = \Lambda^{-1} \frac{\partial \overset{(x)}{\mathfrak{L}}}{\partial (\partial_{\nu} \partial_{\mu} P_{\cdot\lambda}^{\kappa})}$$

and this proves that

$$(11.17) \quad \mathfrak{L}_{\cdot\cdot\cdot\cdot\kappa}^{\nu\mu\lambda} \stackrel{\text{def}}{=} \frac{\partial \overset{(x)}{\mathfrak{L}}}{\partial (\partial_{\nu} \partial_{\mu} P_{\cdot\lambda}^{\kappa})}$$

is a *tensor density of weight +1*. Similarly by differentiating the identity with respect to  $\partial_{\mu} P_{\cdot\lambda}^{\kappa}$  and with respect to  $P_{\cdot\lambda}^{\kappa}$  it can be proved once more that  $[\mathfrak{L}]_{\kappa}^{\lambda}$  is a tensor density of weight +1. But new identities arise if we *first* differentiate with respect to  $A_{\lambda}^{\kappa'}$ ,  $\partial_{\mu} A_{\lambda}^{\kappa'}$  and  $\partial_{\nu} \partial_{\mu} A_{\lambda}^{\kappa'}$  and *afterwards* substitute  $A_{\lambda}^{\kappa'} = \delta_{\lambda}^{\kappa'}$ ,  $\partial_{\mu} A_{\lambda}^{\kappa'} = 0$ ,  $\partial_{\nu} \partial_{\mu} A_{\lambda}^{\kappa'} = 0$ . This treatment, which is in reality equivalent to effecting an infinitesimal coordinate transformation, is rather complicated and it is much simpler to replace

<sup>1)</sup> BUCHDAHL 1948, 1; 1951, 1; 2; 3.

<sup>2)</sup> SCHRÖDINGER 1948, 2; 1950, 1; 1951, 1.

<sup>3)</sup> NIJENHUIS 1952, 1.

the coordinate transformation by a point transformation. Then we can use the LIE derivative. The LIE derivative of  $\mathfrak{L}$  can be computed in two ways (cf. 10.27 and 10.17)

$$(11.18) \quad \left\{ \begin{array}{l} \text{a)} \frac{\mathfrak{L}}{v} = \partial_\mu \mathfrak{L} v^\mu \\ \text{b)} \frac{\mathfrak{L}}{v} = \frac{\partial \mathfrak{L}}{\partial P_{\cdot\lambda}^x} \frac{\mathfrak{L}}{v} P_{\cdot\lambda}^x + \frac{\partial \mathfrak{L}}{\partial (\partial_\mu P_{\cdot\lambda}^x)} \partial_\mu \frac{\mathfrak{L}}{v} P_{\cdot\lambda}^x + \frac{\partial \mathfrak{L}}{\partial (\partial_\nu \partial_\mu P_{\cdot\lambda}^x)} \partial_\nu \partial_\mu \frac{\mathfrak{L}}{v} P_{\cdot\lambda}^x. \end{array} \right.$$

Equating these expressions we get

$$(11.19) \quad \left\{ \begin{array}{l} \left( \frac{\partial \mathfrak{L}}{\partial P_{\cdot\lambda}^x} - \partial_\mu \frac{\partial \mathfrak{L}}{\partial (\partial_\mu P_{\cdot\lambda}^x)} + \partial_\nu \partial_\mu \frac{\partial \mathfrak{L}}{\partial (\partial_\nu \partial_\mu P_{\cdot\lambda}^x)} \right) \frac{\mathfrak{L}}{v} P_{\cdot\lambda}^x + \\ + \partial_\mu \left\{ \frac{\partial \mathfrak{L}}{\partial (\partial_\mu P_{\cdot\lambda}^x)} \frac{\mathfrak{L}}{v} P_{\cdot\lambda}^x - 2 \left( \partial_\nu \frac{\partial \mathfrak{L}}{\partial (\partial_\nu \partial_\mu P_{\cdot\lambda}^x)} \right) \frac{\mathfrak{L}}{v} P_{\cdot\lambda}^x - \mathfrak{L} v^\mu \right\} + \\ + \partial_\nu \partial_\mu \frac{\partial \mathfrak{L}}{\partial (\partial_\nu \partial_\mu P_{\cdot\lambda}^x)} \frac{\mathfrak{L}}{v} P_{\cdot\lambda}^x = 0 \end{array} \right.$$

or [cf. (11.17)]

$$(11.20) \quad \left\{ \begin{array}{l} [\mathfrak{L}]_{\cdot\lambda}^{\cdot\lambda} \frac{\mathfrak{L}}{v} P_{\cdot\lambda}^x + \partial_\mu \left\{ \left( \frac{\partial \mathfrak{L}}{\partial (\partial_\mu P_{\cdot\lambda}^x)} - 2 \partial_\nu \mathfrak{L}_{\cdot\cdot\lambda}^{\mu\lambda} \right) \frac{\mathfrak{L}}{v} P_{\cdot\lambda}^x - \mathfrak{L} v^\mu \right\} + \\ + \partial_\nu \partial_\mu \mathfrak{L}_{\cdot\cdot\lambda}^{\mu\lambda} \frac{\mathfrak{L}}{v} P_{\cdot\lambda}^x = 0 \end{array} \right.$$

and after some calculation

$$(11.21) \quad \left\{ \begin{array}{l} v^\mu \left( [\mathfrak{L}]_{\cdot\lambda}^{\cdot\lambda} \partial_\mu P_{\cdot\lambda}^x - \partial_\lambda [\mathfrak{L}]_{\cdot\lambda}^{\cdot\lambda} P_{\cdot\mu}^x + \partial_\lambda P_{\cdot\lambda}^x [\mathfrak{L}]_{\mu}^{\cdot\lambda} \right) + \\ + \partial_\lambda ([\mathfrak{L}]_{\cdot\lambda}^{\cdot\lambda} P_{\cdot\mu}^x v^\mu - P_{\cdot\lambda}^x [\mathfrak{L}]_{\mu}^{\cdot\lambda} v^\mu) + \\ + \partial_\mu \left[ -\mathfrak{L} v^\mu + v^\nu \left\{ \left( \frac{\partial \mathfrak{L}}{\partial (\partial_\mu P_{\cdot\lambda}^x)} - 2 \partial_\nu \mathfrak{L}_{\cdot\cdot\lambda}^{\mu\lambda} \right) \partial_\nu P_{\cdot\lambda}^x - \right. \right. \\ \left. \left. - \partial_\lambda P_{\cdot\nu}^x \left( \frac{\partial \mathfrak{L}}{\partial (\partial_\mu P_{\cdot\lambda}^x)} - 2 \partial_\nu \mathfrak{L}_{\cdot\cdot\lambda}^{\mu\lambda} \right) \right. \right. \\ \left. \left. + \partial_\lambda P_{\cdot\lambda}^x \left( \frac{\partial \mathfrak{L}}{\partial (\partial_\mu P_{\cdot\nu}^x)} - 2 \partial_\nu \mathfrak{L}_{\cdot\cdot\lambda}^{\mu\nu} \right) \right\} + \right. \\ \left. + \partial_\lambda \left\{ \left( \frac{\partial \mathfrak{L}}{\partial (\partial_\mu P_{\cdot\lambda}^x)} - 2 \partial_\nu \mathfrak{L}_{\cdot\cdot\lambda}^{\mu\lambda} \right) P_{\cdot\nu}^x v^\nu - \right. \right. \\ \left. \left. - P_{\cdot\lambda}^x \left( \frac{\partial \mathfrak{L}}{\partial (\partial_\mu P_{\cdot\nu}^x)} - 2 \partial_\nu \mathfrak{L}_{\cdot\cdot\lambda}^{\mu\nu} \right) v^\nu \right\} \right] + \\ + \partial_\nu \partial_\mu [v^\rho (\mathfrak{L}_{\cdot\cdot\lambda}^{\mu\lambda} \partial_\rho P_{\cdot\lambda}^x - \partial_\lambda \mathfrak{L}_{\cdot\cdot\lambda}^{\mu\lambda} P_{\cdot\rho}^x + \partial_\lambda P_{\cdot\lambda}^x \mathfrak{L}_{\cdot\cdot\rho}^{\mu\lambda}) + \\ + \partial_\lambda (\mathfrak{L}_{\cdot\cdot\lambda}^{\mu\lambda} P_{\cdot\rho}^x v^\rho - P_{\cdot\lambda}^x \mathfrak{L}_{\cdot\cdot\rho}^{\mu\lambda} v^\rho)] = 0. \end{array} \right.$$

This is an identity of the form

$$(11.22) \quad \mathfrak{p}_x v^\lambda + \partial_\lambda q_{\cdot\lambda}^x v^\lambda + \partial_\mu \partial_\lambda r_{\cdot\cdot\lambda}^{\mu\lambda} v^\lambda + \partial_\nu \partial_\mu \partial_\lambda s_{\cdot\cdot\lambda}^{\nu\mu\lambda} v^\lambda = 0$$

valid for all values of the field  $v^\kappa$  and its derivatives. Now such an identity has as a consequence that

$$(11.23) \quad \left\{ \begin{array}{l} (\mathfrak{p}_\kappa + \partial_\lambda q_{\cdot\kappa}^\lambda + \partial_\mu \partial_\lambda r_{\cdot\cdot\kappa}^{(\mu\lambda)} + \partial_\nu \partial_\mu \partial_\lambda \tilde{s}_{\cdot\cdot\kappa}^{(\nu\mu\lambda)}) v^\kappa + \\ \quad + (q_{\cdot\kappa}^\lambda + 2 \partial_\mu r_{\cdot\cdot\kappa}^{(\mu\lambda)} + 3 \partial_\nu \partial_\mu \tilde{s}_{\cdot\cdot\kappa}^{(\nu\mu\lambda)}) \partial_\lambda v^\kappa + \\ \quad + (r_{\cdot\cdot\kappa}^{(\mu\lambda)} + 3 \partial_\nu \tilde{s}_{\cdot\cdot\kappa}^{(\nu\mu\lambda)}) \partial_\mu \partial_\lambda v^\kappa + \tilde{s}_{\cdot\cdot\kappa}^{(\nu\mu\lambda)} \partial_\nu \partial_\mu \partial_\lambda v^\kappa = 0, \end{array} \right.$$

hence

$$(11.24) \quad \tilde{s}_{\cdot\cdot\kappa}^{(\nu\mu\lambda)} = 0; \quad r_{\cdot\cdot\kappa}^{(\mu\lambda)} = 0; \quad q_{\cdot\kappa}^\lambda = 0; \quad \mathfrak{p}_\kappa = 0.$$

Applying this to (11.22) we get four identities

$$(11.25) \quad \left\{ \begin{array}{l} \text{a) } [\mathfrak{L}]_{\kappa}^{\cdot\lambda} \partial_\mu P_{\cdot\lambda}^\kappa - \partial_\lambda [\mathfrak{L}]_{\kappa}^{\cdot\lambda} P_{\cdot\mu}^\kappa + \partial_\lambda P_{\cdot\kappa}^\lambda [\mathfrak{L}]_{\mu}^{\cdot\kappa} = 0 \\ \text{b) } - \mathfrak{L} A_{\nu}^{\lambda} + [\mathfrak{L}]_{\kappa}^{\cdot\lambda} P_{\cdot\nu}^\kappa - P_{\cdot\kappa}^\lambda [\mathfrak{L}]_{\nu}^{\cdot\kappa} + \left( \frac{\partial \mathfrak{L}}{\partial (\partial_\lambda P_{\cdot\mu}^\kappa)} - 2 \partial_\nu \mathfrak{L}_{\cdot\cdot\kappa}^{\nu\mu} \right) \partial_\nu P_{\cdot\mu}^\kappa - \\ \quad - \partial_\mu P_{\cdot\nu}^\kappa \left( \frac{\partial \mathfrak{L}}{\partial (\partial_\lambda P_{\cdot\mu}^\kappa)} - 2 \partial_\nu \mathfrak{L}_{\cdot\cdot\kappa}^{\nu\mu} \right) + \partial_\mu P_{\cdot\kappa}^\mu \left( \frac{\partial \mathfrak{L}}{\partial (\partial_\lambda P_{\cdot\nu}^\kappa)} - 2 \partial_\nu \mathfrak{L}_{\cdot\cdot\kappa}^{\nu\lambda} \right) = 0 \\ \text{c) } \left( \frac{\partial \mathfrak{L}}{\partial (\partial_\mu P_{\cdot\lambda}^\kappa)} - 2 \partial_\nu \mathfrak{L}_{\cdot\cdot\kappa}^{\nu\lambda} \right) P_{\cdot\sigma}^\kappa - A_{\nu}^{(\mu} P_{\cdot\kappa}^{\lambda)} \left( \frac{\partial \mathfrak{L}}{\partial (\partial_\nu P_{\cdot\sigma}^\kappa)} - 2 \partial_\rho \mathfrak{L}_{\cdot\cdot\sigma}^{\rho\kappa} \right) + \\ \quad + \mathfrak{L}_{\cdot\cdot\kappa}^{(\lambda\mu)\nu} \partial_\sigma P_{\cdot\nu}^\kappa - \partial_\nu \mathfrak{L}_{\cdot\cdot\kappa}^{(\lambda\mu)\nu} P_{\cdot\sigma}^\kappa + \partial_\nu P_{\cdot\kappa}^\nu \mathfrak{L}_{\cdot\cdot\sigma}^{(\lambda\mu)\kappa} = 0 \\ \text{d) } \mathfrak{L}_{\cdot\cdot\kappa}^{(\nu\mu\lambda)} P_{\cdot\sigma}^\sigma - \mathfrak{L}_{\cdot\cdot\kappa}^{(\nu\mu|\rho|} P_{\cdot\rho}^{\lambda)} = 0. \end{array} \right.$$

Of these (11.25 a) is a special case of (11.11). (11.25 d) is an algebraic identity and the others are differential identities.

Similarly, if a density  $\mathfrak{L}$  of weight +1 is formed from a tensor  $P$  (with arbitrary valences) and its derivatives up to the order  $p$ , an equation of the form (11.23) arises with  $p+2$  terms and thus we get  $p+2$  identities whose total number of equations is

$$(11.26) \quad n + n^2 + n \binom{n+1}{2} + \dots + n \binom{n+p}{p+1} = n \binom{n+p+1}{p+1}.$$

The first  $n$  equations can be found in the same way in which (11.11) is found and the last  $n \binom{n+p}{p+1}$  are algebraic identities linear homogeneous in  $P$  and in the derivatives of  $\mathfrak{L}$  with respect to  $\partial_{\mu_1} \dots \partial_{\mu_n} P$ .

If instead of  $P_{\cdot\lambda}^\kappa$  a tensor  $g_{\lambda\kappa}$  of rank  $n$  is taken, the second and third term of (11.25 b) are

$$(11.27) \quad - [\mathfrak{L}]^{\kappa\nu} g_{\nu\lambda} - [\mathfrak{L}]^{\nu\kappa} g_{\lambda\nu} = - 2 [\mathfrak{L}]^{\kappa\nu} g_{\nu\lambda}.$$

This is also true when higher derivatives of  $g_{\lambda\kappa}$  occur. Hence by transvection with the inverse of  $g_{\lambda\kappa}$  the LAGRANGE derivative can be expressed in terms of  $\mathfrak{L}$ ,  $g_{\lambda\kappa}$  and their derivatives.<sup>1)</sup>

<sup>1)</sup> BUCHDAHL 1951, 3.

If  $P$  is a geometric object but not a quantity [for instance the  $\Pi_{\mu\lambda}^{\kappa}$  in (10.31)] the LIE derivative of  $\Pi$  contains not only first but also *second* derivatives of  $v^\kappa$ . If for instance  $\Pi$  and its first derivatives occur in  $\mathfrak{L}$ , one finds  $n \binom{n+3}{3}$  equations of which the last  $n \binom{n+2}{3}$  are algebraic.<sup>1)</sup>

### Exercise.

II 11,1. Prove that the left hand side of (11.11) is a covariant vector  $\Delta$ -density of weight +1 if  $[\mathfrak{L}]$  is replaced by an arbitrary tensor  $\Delta$ -density  $\mathfrak{Q}$  of valences  $q$  and  $p$  and weight +1. For  $q=1$ ,  $p=0$  and  $q=0$ ,  $p=1$  this gives the differential concomitants

$$\text{II 11,1 } \alpha) \quad 2\mathfrak{Q}^\lambda \partial_{[\mu} P_{\lambda]} - P_\mu \partial_\lambda \mathfrak{Q}^\lambda$$

$$\text{II 11,1 } \beta) \quad \mathfrak{Q}_\lambda \partial_\mu P^\lambda + \partial_\lambda (\mathfrak{Q}_\mu P^\lambda).$$

In an  $R_3$  with signature + + + these are the concomitants written in ordinary vector notation:

$$\text{II 11,1 } \gamma) \quad \mathbf{Q} \times \operatorname{rot} \mathbf{P} - \mathbf{P} \operatorname{div} \mathbf{Q}$$

$$\text{II 11,1 } \delta) \quad \operatorname{grad}(\mathbf{P} \cdot \mathbf{Q}) - \mathbf{P} \times \operatorname{rot} \mathbf{Q} + \mathbf{Q} \operatorname{div} \mathbf{P}.$$

## § 12. CARTAN'S symbolical method.

CARTAN'S symbolical method is intimately connected with anholonomic coordinate systems. It is in fact an abridged notation specially suited to dealing with these systems. If the  $A_\lambda^h$  of an anholonomic system ( $h$ ) are fixed, these components may be considered as  $n$  sets of components of the  $n$  basis vectors  $\overset{1}{e}_\lambda, \dots, \overset{n}{e}_\lambda$  of ( $h$ ) or (if we like) of the corresponding pfaffians  $\overset{h}{e}_\lambda d\xi^\lambda$ . Then any geometry introduced into  $X_n$  can be expressed in terms of these vectors or these pfaffians. This is the point of view of CARTAN. In his thesis of 1894<sup>2)</sup> he developed a very ingenious method to deal with pfaffians and their generalizations and this method was used in connection with some kind of tensor calculus in nearly all his papers on differential geometry. The introduction of a tensor calculus will always be necessary if non alternating quantities are considered, because the symbolical method is made for alternating quantities only. In 1934 KÄHLER<sup>3)</sup> gave a systematic account of the possibilities of the method as far as alternating quantities only were concerned and under his influence some notations were slightly changed.

<sup>1)</sup> SCHRÖDINGER 1951, 1.

<sup>2)</sup> CARTAN 1894, 1; 1901, 1.

<sup>3)</sup> KÄHLER 1934, 1.

In this section we try to give an exposition of the abridged notation in such a way that it becomes fully adapted to the notations of tensor calculus.<sup>1)</sup> Using the kernel-index method this can be attained very easily by *using the same kernels in both sets of formulae*. Then a simple rule for translation can be established.

If  $d\xi^{\kappa_1}, \dots, d\xi^{\kappa_p}$  are linearly independent linear elements and  $w_{\lambda_1 \dots \lambda_p}$  a covariant  $p$ -vector, all at the same point, the expression

$$(12.1) \quad [d\xi^{\kappa_1} \dots d\xi^{\kappa_p}] \stackrel{\text{def}}{=} \alpha_p d\xi^{\kappa_1} \dots d\xi^{\kappa_p},$$

where  $\alpha_p$  is a constant that will be fixed later on, is a contravariant  $p$ -vector and

$$(12.2) \quad \begin{cases} w \stackrel{\text{def}}{=} w_{\lambda_1 \dots \lambda_p} [d\xi^{\lambda_1} \dots d\xi^{\lambda_p}] = \alpha_p w_{\lambda_1 \dots \lambda_p} d\xi^{\lambda_1} \dots d\xi^{\lambda_p} \\ = \alpha_p w_{\lambda'_1 \dots \lambda'_p} d\xi^{\lambda'_1} \dots d\xi^{\lambda'_p} = \alpha_p w_{i_1 \dots i_p} (d\xi)^{i_1} \dots (d\xi)^{i_p} \end{cases}$$

is a generalization of a pfaffian, called an *alternating differential form*. For  $p=1$  we take  $\alpha_1=1$  and

$$(12.3) \quad w = w_\lambda d\xi^\lambda = w_{\lambda'} d\xi^{\lambda'} = w_i (d\xi)^i.$$

The product of two forms  $\overset{1}{w}$  and  $\overset{2}{w}$  of valence  $p$  and  $q$  may be defined as follows

$$(12.4) \quad [\overset{1}{w} \overset{2}{w}] \stackrel{\text{def}}{=} \alpha_{p+q} \overset{1}{w}_{[\lambda_1 \dots \lambda_p} \overset{2}{w}_{\lambda_1 \dots \lambda_q]} d\xi^{\lambda_1} \dots d\xi^{\lambda_p} d\xi^{\lambda_{p+1}} \dots d\xi^{\lambda_{p+q}}; \quad p+q \leq n.$$

Then we have (cf. I 7.1)

$$(12.5) \quad [\overset{1}{w} \overset{2}{w}] = (-1)^{pq} [\overset{2}{w} \overset{1}{w}].$$

In all these expressions it is quite unimportant whether we consider  $w$  as an alternating form or as an abridged notation of a multivector. The results are the same and the interpretation is only a question of personal predilection.<sup>2)</sup>

<sup>1)</sup> This is very important. All authors who are using, where convenient, CARTAN's abridged calculus, always use also notations of tensor calculus and they seem to have no difficulties with the latter. But as a matter of fact most authors using tensor calculus have the greatest difficulties with CARTAN's abridged calculus, even in translating results. This is a great pity. Both methods have their advantages (CARTAN himself performed miracles with his method!) and both should be used by every author, each on the place where it suits best. But this ideal can only be attained if there is a convenient bridge between them. We try to build this bridge here.

<sup>2)</sup> Just the same thing occurs in linear algebra. We may look upon  $w$  as an abridged notation for a covariant vector  $w_\lambda$  or as a linear form  $w_\lambda x^\lambda$ . The old theory of invariants preferred the latter point of view but nowadays many people think more easily in terms of vectors.

If  $s$  is a scalar, an invariant differentiation of the  $p$ -vector  $s [d\xi^{\kappa_1} \dots d\xi^{\kappa_p}]$  can be defined as follows

$$(12.6) \quad \left\{ \begin{array}{l} [d\{s [d\xi^{\kappa_1} \dots d\xi^{\kappa_p}]\}] \stackrel{\text{def}}{=} \alpha_{p+1} (\partial_\lambda s) d\xi^{\lambda}_I d\xi^{\kappa_1} \dots d\xi^{\kappa_p} \\ = (\partial_\lambda s) [d\xi^{\lambda} d\xi^{\kappa_1} \dots d\xi^{\kappa_p}]. \end{array} \right.$$

From this definition and (12.2) it follows that

$$(12.7) \quad [dw] = (\partial_\lambda w_{\lambda_1 \dots \lambda_p}) [d\xi^\lambda d\xi^{\lambda_1} \dots d\xi^{\lambda_p}] = \alpha_{p+1} \partial_{[\lambda} w_{\lambda_1 \dots \lambda_p]} d\xi^\lambda d\xi^{\lambda_1} \dots d\xi^{\lambda_p}]$$

The differentiation of a product we easily find then to be

$$(12.8) \quad [d\overset{1}{w}\overset{2}{w}] = [[d\overset{1}{w}\overset{2}{w}]] + (-1)^p [\overset{1}{w}[d\overset{2}{w}]]; \quad p = \text{valence of } \overset{1}{w}.$$

CARTAN always took  $\alpha_p = p!$  and he used the product notation  $[\overset{1}{w}\overset{2}{w}]$ . But instead of  $[dw]$ , at first he wrote  $w'$  and more recently under the influence of KÄHLER  $d w$ . Now this latter notation may lead to ambiguity because  $[dw]$  is certainly not an ordinary differential. KÄHLER wrote  $d(\xi^{\kappa_1}, \dots, \xi^{\kappa_p})$  for what we have written  $\alpha_p d\xi^{\kappa_1} \dots d\xi^{\kappa_p}$  and  $[d\xi^{\kappa_1} \dots d\xi^{\kappa_p}]$ , and he did not fix the values of  $\alpha_p$ . In fact, if only symbolic expressions are used, and if we never wish to translate into notations of tensor calculus, the constants  $\alpha_p$  never occur. For  $[\overset{1}{w}\overset{2}{w}]$  KÄHLER wrote  $\overset{1}{w}\overset{2}{w}$  without brackets. But in dealing with less simple problems the notation with brackets should be preferred. As to the choice of the  $\alpha_p$  we remark that, according to (12.4),  $[\overset{1}{w}\overset{2}{w}]$  does not correspond to the ordinary alternating product of the multivectors corresponding to  $\overset{1}{w}$  and  $\overset{2}{w}$  but to  $\alpha_{p+q} \alpha_p^{-1} \alpha_q^{-1}$  times this product. So, if we wish to establish a reasonable adaptation we have to take  $\alpha_p = 1$  for all values of  $p$ . Doing this and *always using the same kernels in both notations* we get the following simple scheme of translation. Let  $\Pi_{\lambda_1 \dots \lambda_p A}$  be components of a geometric object, alternating in the first  $p$  indices and carrying a collective index  $A$  standing for any number of co- and contravariant indices, each belonging to any holonomic or anholonomic coordinate system in  $X_n$  or any other space. Let the objects  $\Pi_{\lambda_1 \dots \lambda_p (A)}$  with  $p$  living indices  $\lambda_1 \dots \lambda_p$ , resulting from strangling all indices symbolized by  $A$ , be covariant  $p$ -vectors. As an example we take the intermediate components  $A_\lambda^h$  and  $\Omega_{\mu\lambda}^h$  (cf. II § 9). In fact  $A_\lambda^{(h)} = e_\lambda^h$  are  $n$  vectors and  $\Omega_{\mu\lambda}^{(h)}$  are  $n$  bivectors. Now  $\Pi_A$  denotes the  $p$ -vectors, resulting in this way:

$$(12.9) \quad \boxed{\Pi_A : \left\{ \begin{array}{l} \text{and also } \Pi_{\lambda_1 \dots \lambda_p (A)} \\ \Pi_{\lambda_1 \dots \lambda_p (A)} d\xi^{\lambda_1} \dots d\xi^{\lambda_p} \end{array} \right.}.$$

The product  $[\overset{1}{\prod}_A \overset{2}{\prod}_B]$  stands for

$$(12.10) \quad [\overset{1}{\prod}_A \overset{2}{\prod}_B] : \begin{cases} \overset{1}{\prod}_{[\lambda_1 \dots \lambda_p | (A)]} \overset{2}{\prod}_{\kappa_1 \dots \kappa_q} (B) \\ \text{and also for} \\ \overset{1}{\prod}_{[\lambda_1 \dots \lambda_p | (A)]} \overset{2}{\prod}_{\kappa_1 \dots \kappa_q} (B) d\xi^{\lambda_1} \dots d\xi^{\lambda_p} d\xi^{\kappa_1} \dots d\xi^{\kappa_q} \end{cases}$$

and the expression  $[d\overset{1}{\prod}_A]$  symbolizes

$$(12.11) \quad [d\overset{1}{\prod}_A] : \begin{cases} \text{and also } \partial_{[\mu} \overset{1}{\prod}_{\lambda_1 \dots \lambda_p] (A)} \\ \partial_{[\mu} \overset{1}{\prod}_{\lambda_1 \dots \lambda_p] (A)} d\xi^\mu d\xi^{\lambda_1} \dots d\xi^{\lambda_p} \end{cases} .$$

With this scheme every abridged formula can immediately be translated into a formula of tensor calculus. There is one point that is at first bewildering for one accustomed to tensor calculus only. For instance  $w$  does not stand for  $w_\lambda$  only, but also for  $w_\kappa$  or  $w_i$  and even for  $w_\lambda d\xi^\lambda$ ,  $w_\lambda d\xi^\kappa$  and  $w_i (d\xi)^i$ , and  $[dw]$  denotes not only  $\partial_{[\mu} w_{\lambda]} d\xi^\mu$  but also  $\partial_{[\mu} w_{\lambda]} + \partial_{[\mu} w_{i]} + w_h \Omega_{ji}^h$  (cf. Exerc. II 9,1) and even the alternating differential forms obtained by transvecting these expressions by  $d\xi^\mu d\xi^\lambda$ ,  $d\xi^\mu d\xi^\kappa$  and  $(d\xi)^i (d\xi)^j$  respectively. This means that in *passing to the abridged notation the coordinate system belonging to the alternating part of the object is set free and that at the same time the indices not belonging to this part are all strangled*.

From now on we mark all formulae in abridged notation where they appear together with the non abridged formula by the sign **C** (a sign taken from music) and give both formulae the same number. We give here two examples, concerning the object of anholonomy [cf. (9.10) and Exerc. II 9,2]:

$$(12.12) \quad \partial_{[\mu} A_{\lambda]}^h = A_{\mu\lambda}^i \Omega_{ji}^h = \Omega_{\mu\lambda}^h; \quad ((\mathcal{N}) \text{ holonomic})$$

$$(12.12) \quad \mathbf{C} \quad [dA^h] = \Omega^h = \Omega_{ji}^h [A^j A^i]$$

$$(12.13) \quad 0 = A_{\nu\mu\lambda}^{kji} \{ \partial_{[k} \Omega_{ji]}^h - 2 \Omega_{[kj}^l \Omega_{i]l}^h \}$$

$$(12.13) \quad \mathbf{C} \quad 0 = [(d\Omega_{ji}^h) A^j A^i] - 2 \Omega_{kj}^l \Omega_{il}^h [A^k A^j A^i].$$

So far we have seen that certain (not all) formulae of tensor calculus can be written in the abridged form and that by doing this a lot of indices can be saved. On the other hand the abridged calculus requires

more attention during the calculation and so it is merely a question of personal inclination whether one or the other calculus will be preferred. But in order to give abridged calculus fair play we draw the attention to some cases where it has certainly advantages.

Let us take a vector field  $w_\lambda$  and its rotation  $W_{\mu\lambda}$ :

$$(12.14) \quad 2\partial_{[\mu} w_{\lambda]} = W_{\mu\lambda}$$

$$(12.14) \quad \text{C} \quad 2[dw] = W$$

and let us suppose that we wish to consider  $w_\lambda$  not only as a function of the  $\xi^\alpha$  but also of certain parameters  $\eta^\alpha; \alpha = \dot{1}, \dots, \dot{r}$ . Then we introduce in the  $X_{n+r}$ , of  $\xi^\alpha, \eta^\alpha$  a field  $w_B; B = 1, \dots, n, \dot{1}, \dots, \dot{r}$  with the  $n+r$  components  $w_\lambda, w_\beta = 0$ . If this  $X_{n+r}$  is reduced with respect to the normal system of  $X_r$ 's  $\xi^\alpha = \text{const.}$  the field  $w_B$  is reduced to the original field  $w_\lambda$  in the  $X_n$ , depending on the parameters  $\eta^\alpha$ . In tensor-calculus we have now

$$(12.15) \quad \begin{cases} W_{\mu\lambda} = 2\partial_{[\mu} w_{\lambda]}, & \mu, \lambda = 1, \dots, n \\ W_{\gamma\lambda} \stackrel{\text{def}}{=} \partial_{\gamma} w_{\lambda}, & \beta, \gamma = \dot{1}, \dots, \dot{r} \\ W_{\gamma\beta} \stackrel{\text{def}}{=} 2\partial_{[\gamma} w_{\beta]} = 0 \end{cases}$$

or

$$(12.16) \quad W_{CB} = 2\partial_{[C} w_{B]}; \quad B, C = 1, \dots, n, \dot{1}, \dots, \dot{r}$$

but in the abridged calculus this equation is symbolized by the same equation (12.14 C). Hence *the abridged calculus gives the possibility of introducing at any moment auxiliary variables without changing the form of the formulae.*

Some other examples of the use of the abridged calculus will be found in Ch. III.

### III. Linear connexions.

#### § 1. Parallel displacement in an $E_n$ .<sup>1)</sup>

Let  $v^\alpha$  be a vector field in an  $X_n$ . As we have already seen in II § 3,  $dv^\alpha$  is not a vector and  $\partial_\mu v^\alpha$  not a tensor. But if the  $X_n$  is an  $E_n$  and if  $(h)$  is a rectilinear coordinate system,  $dv^h$  and  $\partial_h v^h$  are a vector and a tensor because if  $(h')$  is another rectilinear system, the  $A_h^{h'}$  are constants.

<sup>1)</sup> Tensor calculus was founded, developed and applied by Ricci in a great number of papers. We mention here 1884, 1 and 1886, 1; 2 (all preliminary); 1887, 1; 2; 1888, 1; 2; 1889, 1; 2; 1892, 1; 1893, 1; 2; 1894, 1; 2; 1895, 1; 2; 1897,

$dv^h$  is the difference between the vector  $v^h + dv^h$  at  $\xi^h + d\xi^h$  and a vector with the components  $v^h$  at the same point. This latter vector can be derived from the vector  $v^h$  at  $\xi^h$  by the well known process of parallel displacement.

Now let  $(\alpha)$  be a general coordinate system in  $E_n$ . Then the  $A_\alpha^h$  are not necessarily constants and we have

$$(1.1) \quad dv^h = d(v^\alpha A_\alpha^h) = A_\alpha^h dv^\alpha + v^\lambda \partial_\mu A_\alpha^h d\xi^\mu = A_\alpha^h (dv^\alpha + \Gamma_{\mu\lambda}^\alpha v^\lambda d\xi^\mu)$$

where

$$(1.2) \quad \Gamma_{\mu\lambda}^\alpha \stackrel{\text{def}}{=} A_\lambda^h \partial_\mu A_\alpha^h = - A_\lambda^h \partial_\mu A_\alpha^h.$$

Hence the components of the vector  $dv^h$  with respect to  $(\alpha)$  are

$$(1.3) \quad \delta v^\alpha \stackrel{\text{def}}{=} A_\alpha^h dv^h = dv^\alpha + \Gamma_{\mu\lambda}^\alpha v^\lambda d\xi^\mu.$$

We call  $\delta v^\alpha$  the *covariant differential* of  $v^\alpha$  and the  $\Gamma_{\mu\lambda}^\alpha$  the *parameters* of the parallel displacement in  $E_n$  with respect to  $(\alpha)$ . If we write  $\delta v^\alpha = V_\mu v^\alpha d\xi^\mu$ , we get

$$(1.4) \quad V_\mu v^\alpha \stackrel{\text{def}}{=} \partial_\mu v^\alpha + \Gamma_{\mu\lambda}^\alpha v^\lambda.$$

$V_\mu v^\alpha$  is called the *covariant derivative* of  $v^\alpha$ . If  $(\alpha)$  is rectilinear, the  $\Gamma_{\mu\lambda}^\alpha$  are all zero and we have  $\delta v^\alpha \equiv dv^\alpha$  and  $V_\mu v^\alpha \equiv \partial_\mu v^\alpha$ .

Instead of  $v^h$  we can take a covariant vector field  $w_i$ . Then for the components of  $w_i$  and  $\partial_j w_i$  with respect to  $(\alpha)$  we get

$$(1.5) \quad \delta w_\lambda \stackrel{\text{def}}{=} dw_\lambda - \Gamma_{\mu\lambda}^\alpha w_\alpha d\xi^\mu,$$

$$(1.6) \quad V_\mu w_\lambda \stackrel{\text{def}}{=} \partial_\mu w_\lambda - \Gamma_{\mu\lambda}^\alpha w_\alpha.$$

But these formulae can also be derived by using the fact that the transvection  $v^\varrho w_\varrho$  is invariant if both vectors undergo the same parallel displacement. Hence we have  $d(v^\varrho w_\varrho) = \delta(v^\varrho w_\varrho)$  and the equation

$$(1.7) \quad w_\varrho dv^\varrho + v^\varrho dw_\varrho = (dv^\varrho + \Gamma_{\mu\lambda}^\varrho v^\lambda d\xi^\mu) w_\varrho + v^\varrho \delta w_\varrho$$

can only be true for every choice of the field  $v^\alpha$  if (1.5) holds.

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1; 2; 3; 1898, 1; 2; 1902, 1; 2; 1903, 1; 1904, 1; 1905, 1; 1910, 1; 2; 1912, 1; 2; 1918, 1; 2. It found its first systematic representation in RICCI and LEVI CIVITA 1901, 1. His “derivazione covariante” got a geometric interpretation in the “pseudo-parallel” or “geodesic” displacement (now called parallel displacement) introduced by LEVI CIVITA 1917, 1 and independently by SCHOUTEN 1918, 1. Cf. also BROUWER 1906, 2, p. 121 who gave a first example of a displacement of a vector in an  $S_n$  (cf. III § 5).

## § 2. Parallel displacements in $X_n$ <sup>1), 2).</sup>

We have seen in III § 1 that the construction of the covariant differential in  $E_n$  only depends on the existence of parallel displacements in  $E_n$ . We do not need all displacements but only those from one point to a “neighbouring” point. Hence if in an  $X_n$  a definition of a displacement of quantities were given for every point of the region considered and for every linear element in such a point, it must be possible to construct some kind of a covariant differential.

Let  $\Phi$  (indices suppressed) be a field of *quantities* defined in some region of  $X_n$ . Then at  $\xi^x$  and  $\xi^x + d\xi^x$  we have the field values  $\Phi$  and  $\Phi + d\Phi$ . Let a parallel displacement for these quantities be defined for every choice of  $\xi^x$  and  $d\xi^x$  in the region considered and let  $\Phi + \overset{*}{d}\Phi$  be the field value at  $\xi^x + d\xi^x$  after a parallel displacement from  $\xi^x$  to  $\xi^x + d\xi^x$ . Then  $d\Phi - \overset{*}{d}\Phi$  is a covariant differential  $\delta\Phi$  of  $\Phi$  belonging to this displacement. Of course we make the condition that if the components of  $\Phi + \overset{*}{d}\Phi$  are given for one coordinate system ( $x$ ) its com-

1) Parallel displacements (formerly often called pseudo-parallel displacements) are considered here only with respect to the tangent  $E_n$ 's of an  $X_n$ . KÖNIG generalized 1920, 1 this idea for a set of  $E_N$ 's each of which is associated with a point of  $X_n$ . Cf. KÖNIG 1932, 1; SIBATA 1934, 1; HLAVATY 1934, 1; 1935, 1; KÖNIG and PESCHL 1934, 1; KÖNIG and WEISE 1935, 1. As we proved 1924, 2; 3 the conformal and projective connexions introduced by CARTAN 1923, 2 and 1924, 1 can be looked upon as special cases of KÖNIG connexions. There is much literature on these general connexions. We mention here only a few papers where also other literature can be found: SCHOUTEN 1924, 1, p. 62; 1926, 1; SCHLESINGER 1928, 1; MORINAGA 1934, 2; 3; TUCKER 1935, 1; YANO and MUTO 1935, 1; 1936, 1; WAGNER 1943, 1; YANO 1945, 2; 3.

2) There is a large literature on connexions and parallelism. We mention here only HESSENBERG 1916, 1 (preliminary); LEVI CIVITA 1917, 1; SCHOUTEN 1918, 1; 1922, 3; 1923, 1; 1926, 1 (also non linear) 1928, 1; FRIESEKE 1925, 1 (non linear); EISENHART 1927, 1; STRUIK 1928, 1 (literature); SCHLESINGER 1928, 1; 1929, 1; WUNDHEILER 1929, 1 (non linear); DUSCHEK 1930, 2 (axiomatic); BORTOLOTTI 1930, 3 (literature); 1930, 2 and 1931, 3 (degenerate fundamental tensor); 1931, 4 (also non linear and parallel directions) 1935, 1; 1936, 1 (non linear); 1937, 3; 1939, 1 (also for literature) 1942, 1; CONFORTO 1931, 1; 2 (in function space); A. KAWAGUCHI 1932, 1 (also non linear) 1933, 1; 1933, 2 (also non linear); 1934, 1; 1935, 1 (in function space and literature); 1952, 1 (non linear, general theory and literature); VANDERSLICE 1934, 1; HOKARI 1934, 1 (for bivectors); BORTOLOTTI and HLAVATY 1936, 2 (also literature); URABI 1940, 1; MIKAMI 1941, 1; WAGNER 1943, 1 and 1945, 2 (general geom. obj.); NORDEN 1945, 1 and 1950, 1 (space with two connexions); DALLA VOLTA 1946, 1 and 2 (asymmetric); BOMPIANI 1945, 1; 1946, 1 (for tensors); 1952, 1 (conn. non posizionali); ROZENFEL'D 1947, 1 and 1948, 1 (generalizations); JOHNSON 1948, 1; LIBER 1950, 1 (classification of connexions in  $L_2$ ); EHRESMANN 1950, 1 (in fibre space); 1952, 2 (higher order); KNEBELMANN 1951, 1 (parallelism relative to a continuous curve); WINOGRADSKI 1951, 1 (axiomatic); TAKASU 1953, 1.

ponents with respect to  $(\xi')$  can be derived by the rules valid for the transformation of  $\Phi + d\Phi$ . That means that we introduce as a first condition

C I) *The covariant differential of a quantity  $\Phi$  transforms in the same way as  $\Phi$ .*

Now there are many possibilities for the definition of a displacement. Here we consider only those displacements that satisfy (C I) and the following four conditions (all quantities considered are supposed to be given as fields in a certain region):

C II) *The covariant differential of a sum of quantities is the sum of the covariant differentials of the terms.*

C III) *For the covariant differential of a product or a transvection of quantities the rule of LEIBNIZ holds, for instance (indices suppressed)*

$$\delta(\Phi\Psi) = (\delta\Phi)\Psi + \Phi\delta\Psi.$$

C IV) *The covariant differential of a quantity is linear homogeneous in the  $d\xi^\mu$ :*

$$\delta\Psi = V_\mu\Psi d\xi^\mu.$$

C V) *The covariant differential of a scalar or W-scalar is identical with the ordinary differential.*

For a contravariant vector  $v^\mu$  we have  $\delta v^\mu = dv^\mu - \overset{*}{d}v^\mu$ , hence according to (C IV)  $\overset{*}{d}v^\mu$  is linear homogeneous in  $d\xi^\mu$

$$(2.1) \quad \overset{*}{d}v^\mu = \varphi_\mu^\lambda(v^\lambda) d\xi^\mu.$$

But then it follows from (C II) that  $\varphi_\mu^\lambda(v^\lambda)$  is linear homogeneous in  $v^\lambda$ . Hence  $\delta v^\mu$  and  $V_\mu v^\mu$  have the form

$$(2.2) \quad \delta v^\mu = dv^\mu + \Gamma_{\mu\lambda}^\nu v^\lambda d\xi^\mu,$$

$$(2.3) \quad V_\mu v^\mu = \partial_\mu v^\mu + \Gamma_{\mu\lambda}^\nu v^\lambda.$$

According to (C I) we get for the transformation  $(\xi) \rightarrow (\xi')$

$$(2.4) \quad A_{\mu'}^{\nu'}(dv^\mu + \Gamma_{\mu\lambda}^\nu v^\lambda d\xi^\mu) = dv'^\nu + \Gamma_{\mu'\lambda'}^{\nu'} v'^\lambda d\xi'^\mu$$

from which we easily deduce the transformation formulae

$$(2.5) \quad I_{\mu'\lambda'}^{\nu'} = A_{\mu'\lambda'}^{\mu\lambda} I_{\mu\lambda}^\nu + A_{\lambda'}^{\nu'} \partial_{\mu'} A_\lambda^\mu = A_{\mu'\lambda'}^{\mu\lambda} I_{\mu\lambda}^\nu - A_{\mu'\lambda'}^{\mu\lambda} \partial_\mu A_\lambda^\nu$$

showing that the  $I_{\mu\lambda}^\nu$  do not constitute a quantity but a more general geometric object of the kind already considered in II § 10.

The  $n^3$  parameters  $I_{\mu\lambda}^\nu$  can be given arbitrarily for any system  $(\xi)$ . Then by (2.5) the parameters with respect to every allowable coordinate

system are known. They are said to fix a *linear displacement* or a *linear connexion* in  $X_n$ .<sup>1)</sup> An  $X_n$  provided with a linear connexion is called an  $L_n$ . If a vector in an  $L_n$  is displaced in such a way that  $\delta v^\kappa = 0$  the displacement is said to be *parallel*.

Using (C V) it can be proved in the same way as in III § 1 that for a covariant field  $w_\lambda$

$$(2.6) \quad \delta w_\lambda = dw_\lambda - \Gamma_{\mu\lambda}^\kappa w_\kappa d\xi^\mu,$$

$$(2.7) \quad \nabla_\mu w_\lambda = \partial_\mu w_\lambda - \Gamma_{\mu\lambda}^\kappa w_\kappa.$$

Every tensor can be written as a sum of general products of vectors (cf. I § 3). Hence as a consequence of (C III) we have for instance for  $P_{\cdot\cdot\mu}^{\kappa\lambda}$

$$(2.8) \quad \delta P_{\cdot\cdot\mu}^{\kappa\lambda} = dP_{\cdot\cdot\mu}^{\kappa\lambda} + \Gamma_{v\mu}^\kappa P_{\cdot\cdot\mu}^{v\lambda} d\xi^v + \Gamma_{v\mu}^\lambda P_{\cdot\cdot\mu}^{v\kappa} d\xi^v - \Gamma_{v\mu}^q P_{\cdot\cdot\mu}^{v\lambda} d\xi^v,$$

$$(2.9) \quad \nabla_v P_{\cdot\cdot\mu}^{\kappa\lambda} = \partial_v P_{\cdot\cdot\mu}^{\kappa\lambda} + \Gamma_{v\mu}^\kappa P_{\cdot\cdot\mu}^{v\lambda} + \Gamma_{v\mu}^\lambda P_{\cdot\cdot\mu}^{v\kappa} - \Gamma_{v\mu}^q P_{\cdot\cdot\mu}^{v\lambda}.$$

For every upper (lower) index there is a term with a  $+-(--)$ -sign. Every  $W$ -tensor is the product of a tensor with a  $W$ -scalar. Hence, according to (C V) the formulae for the covariant differentiation of tensors hold also for  $W$ -tensors.

The covariant differential is the difference between the field value at  $\xi^\kappa + d\xi^\kappa$  and the field value of the field that has been displaced parallel from  $\xi^\kappa$  to  $\xi^\kappa + d\xi^\kappa$ . This can be said in another way; *the covariant differential is the ordinary differential from the point of view of a local coordinate system that is displaced parallel* from  $\xi^\kappa$  to  $\xi^\kappa + d\xi^\kappa$ .

A quantity is called *allround covariant stationary at a point* of an  $L_n$  if its covariant differential vanishes at that point for every direction. It is called *covariant stationary at a point in an m-direction* if the covariant differential vanishes at that point for all directions lying in this  $m$ -direction. It is said to be *covariant constant over  $L_n$*  or *covariant constant* for short if it is allround covariant stationary at all points of  $L_n$ .<sup>2)</sup>

If a quantity is defined over an  $X_m$  in  $L_n$  it is called *covariant constant over  $X_m$*  if at all points of  $X_m$  it is covariant stationary in the tangent  $m$ -direction of  $X_m$ . If the quantity is also defined in some

<sup>1)</sup> GALVANI has shown that connexions often can be realized by taking for points certain subspaces of a flat space, 1942, 1; 2; 1943, 1; 2; 1945, 1; 1946, 1. Cf. also for other choices of the elements of space ROZENFEL'D 1947, 1; 1948, 1; YANO and HIRAMATU (sets of hypersurfaces) 1951, 1.

<sup>2)</sup> The existence of covariant constant fields *in the large* is quite another problem that does not belong to the realm of this book. We mention here only a few papers where also other literature can be found: BORTOLOTTI 1928, 1; 1929, 1; T. Y. THOMAS 1936, 1; MAYER and T. Y. THOMAS, 1937, 1.

neighbourhood of  $X_m$  it may happen that it is allround covariant stationary in  $L_n$  at all points of  $X_m$ . In this case it is called *allround covariant stationary* in  $L_n$  over  $X_m$ . Of course it is then also covariant constant over  $X_m$ . The following scheme gives the relations between these five cases

$$\begin{array}{ccc} \curvearrowleft & \text{allround cov. stat.} & \curvearrowright \\ & \text{at point of } X_m & \\ \text{cov. constant} & \cap & \cap \\ \text{over } L_n & & \\ \curvearrowleft & \text{allround cov. stat.} & \curvearrowright \\ & \text{in } L_n \text{ over } X_m & \\ & & \text{cov. constant over } X_m^{\text{(1)}} \end{array}$$

A field that is at every point and for every direction proportional to its covariant differential is called *recurrent*. A recurrent vector field is also called *parallel*.

If we wish to write out a covariant derivative in terms of its intermediate components (cf. I § 2) for instance between  $(\varkappa)$  and  $(\varkappa')$  it is convenient to introduce the notations

$$(2.10) \quad \Gamma_{\mu'\lambda}^{\varkappa} \stackrel{\text{def}}{=} A_{\mu'}^{\mu} \Gamma_{\mu\lambda}^{\varkappa}; \quad \Gamma_{\mu\lambda'}^{\varkappa'} \stackrel{\text{def}}{=} A_{\mu}^{\mu'} \Gamma_{\mu'\lambda'}^{\varkappa'}.$$

Then for  $P_{\cdot\cdot\mu}^{\varkappa\lambda}$  for instance we have the formulae

$$(2.11) \quad V_{\nu} P_{\cdot\cdot\mu}^{\varkappa\lambda} = \partial_{\nu} P_{\cdot\cdot\mu}^{\varkappa\lambda} + \Gamma_{\nu\lambda}^{\varkappa} P_{\cdot\cdot\mu}^{\lambda} + \Gamma_{\nu\mu}^{\lambda} P_{\cdot\cdot\mu}^{\varkappa} - \Gamma_{\nu\mu}^{\rho} P_{\cdot\cdot\mu}^{\varkappa\rho},$$

$$(2.12) \quad V_{\nu'} P_{\cdot\cdot\mu}^{\varkappa\lambda} = \partial_{\nu'} P_{\cdot\cdot\mu}^{\varkappa\lambda} + \Gamma_{\nu'\lambda}^{\varkappa} P_{\cdot\cdot\mu}^{\lambda} + \Gamma_{\nu'\mu}^{\lambda} P_{\cdot\cdot\mu}^{\varkappa} - \Gamma_{\nu'\mu}^{\rho} P_{\cdot\cdot\mu}^{\varkappa\rho},$$

equivalent to (2.9). They each contain only intermediate components of the same kind.<sup>2)</sup> The  $\Gamma_{\mu'\lambda}^{\varkappa}$  and  $\Gamma_{\mu\lambda'}^{\varkappa'}$  might be considered as a kind of intermediate components of the geometric object  $\Gamma_{\mu\lambda}^{\varkappa}$ . But in using them we have always to remember that they can only be formed by a transvection over the index  $\mu$  and that  $\Gamma_{\mu'\lambda}^{\varkappa} \neq \Gamma_{\mu'\lambda'}^{\varkappa'} A_{\lambda'}^{\lambda}$ . That is the reason why, in general, we use intermediate components of quantities only.

From (2.5) it follows that

$$(2.13) \quad S_{\mu\lambda}^{\cdot\cdot\varkappa} \stackrel{\text{def}}{=} \Gamma_{[\mu\lambda]}^{\varkappa}$$

is a tensor. It is called the *tensor of asymmetry* or *torsion*<sup>3)</sup> of the  $L_n$ . The connexion is said to be *symmetric* and the  $L_n$  is called an  $A_n$  if  $S_{\mu\lambda}^{\cdot\cdot\varkappa} = 0$  at all points of the region considered. It is said to be *semi-symmetric* if  $S_{\mu\lambda}^{\cdot\cdot\varkappa}$  has the form

$$(2.14) \quad S_{\mu\lambda}^{\cdot\cdot\varkappa} = S_{[\mu} A_{\lambda]}^{\varkappa} \text{ } ^4).$$

<sup>1)</sup> Cf. E I 1935, 1, p. 77f. The term "allround" introduced here is new.

<sup>2)</sup> Formulae of this kind were introduced first by R. LAGRANGE 1926, 1, p. 10.

<sup>3)</sup> Mc CONNELL used 1928, 2; 3 the word torsion in  $V_n$  in another sense connected with a geodesic.

<sup>4)</sup> Cf. FRIEDMANN and SCHOUTEN 1924, 1.

From (2.14) we easily deduce

$$(2.15) \quad S_\mu = \frac{2}{n-1} S_{\mu\lambda}^{\cdot\lambda}$$

and this equation will also be taken as the definition of the concomitant  $S_\mu$  if (2.14) does not hold.  $S_{\mu\lambda}^{\cdot\lambda}$  is intimately connected with the operation Rot (cf. II § 6). The covariant derivative of a  $q$ -vector  $w_{\lambda_1\dots\lambda_q}$

$$(2.16) \quad \left\{ \begin{array}{l} \nabla_\mu w_{\lambda_1\dots\lambda_q} = \partial_\mu w_{\lambda_1\dots\lambda_q} - \Gamma_{\mu\lambda}^\kappa w_{\kappa\lambda_2\dots\lambda_q} - \dots - \Gamma_{\mu\lambda_q}^\kappa w_{\lambda_1\dots\lambda_{q-1}\kappa} \\ = \partial_\mu w_{\lambda_1\dots\lambda_q} - q \Gamma_{\mu[\lambda_q}^\kappa w_{\lambda_1\dots\lambda_{q-1}]\kappa}, \end{array} \right.$$

hence

$$(2.17) \quad \nabla_{[\mu} w_{\lambda_1\dots\lambda_q]} = \partial_{[\mu} w_{\lambda_1\dots\lambda_q]} - q S_{[\mu\lambda_q}^{\cdot\lambda}\ w_{\lambda_1\dots\lambda_{q-1}]\kappa}$$

and for the semi-symmetric case

$$(2.18) \quad \nabla_{[\mu} w_{\lambda_1\dots\lambda_q]} = \partial_{[\mu} w_{\lambda_1\dots\lambda_q]} - q S_{[\mu\lambda_q}^{\cdot\lambda}\ w_{\lambda_1\dots\lambda_{q-1}]\kappa}$$

from which it follows that only in an  $A_n$  the symbols  $\partial$  and  $\nabla$  can be interchanged in the expression  $\nabla_{[\mu} w_{\lambda_1\dots\lambda_q]}$ .

If two linear elements  $\underset{1}{d\xi^\kappa}$  and  $\underset{2}{d\xi^\kappa}$  at  $\xi^\kappa$  are displaced parallel along each other, at  $\xi^\kappa + \underset{1}{d\xi^\kappa}$  we get the vector  $\underset{2}{d\xi^\kappa} - \Gamma_{\mu\lambda}^\kappa \underset{2}{d\xi^\lambda} \underset{1}{d\xi^\mu}$  and at  $\xi^\kappa + \underset{2}{d\xi^\kappa}$  the vector  $\underset{1}{d\xi^\kappa} - \Gamma_{\mu\lambda}^\kappa \underset{1}{d\xi^\lambda} \underset{2}{d\xi^\mu}$ , neglecting all terms of third and higher orders. The figure obtained is in general not a parallelogram but a pentagon with the closing vector

$$(2.19) \quad 2S_{\mu\lambda}^{\cdot\lambda} \underset{1}{d\xi^\mu} \underset{2}{d\xi^\lambda}.$$

Hence in an  $L_n$  with an asymmetric connection, in general no infinitesimal parallelograms exist. In the semi-symmetrical case the closing vector is

$$(2.20) \quad 2S_\mu A_\lambda^{\kappa} \underset{1}{d\xi^{[\mu}} \underset{2}{d\xi^{\lambda]}} = 2S_\mu \underset{1}{d\xi^{[\mu}} \underset{2}{d\xi^{\lambda]}}$$

and accordingly infinitesimal parallelograms only exist in the 2-directions contained in the  $(n-1)$ -direction of  $S_\lambda$ . In the symmetric case there exist infinitesimal parallelograms in every 2-direction.<sup>1)</sup>

There is another geometric interpretation of  $S_{\mu\lambda}^{\cdot\lambda}$  due to CARTAN. Thus far we have only displaced the *vectors* and not the *points* of the tangent  $E_n$ . This comes to the same thing as if we had assumed that

<sup>1)</sup> Cf. CARTAN 1922, 2; FRIEDMANN and SCHOUTEN 1924, 1; J. M. THOMAS 1926, 1. Cf. for other geometric interpretations of  $S_{\mu\lambda}^{\cdot\lambda}$ , PINL 1951, 1; BOMPIANI 1951, 1.

the contact-point always remained a contact-point. But another displacement of points can be fixed by the assumption that the contact-point of the tangent  $E_n$  of  $\xi^x$  has after its arrival in the tangent  $E_n$  of  $\xi^x + d\xi^x$  the radiusvector  $-d\xi^x$ . This we call a CARTAN *displacement*. If the  $X_n$  is an  $E_n$  all tangent  $E_n$ 's coincide and the CARTAN displacement is the displacement zero. The covariant differential  $\delta v^x$  of a vector  $v^x$  is the differential with respect to a parallel displaced local coordinate system. But the covariant differential of a point is the differential with respect to a local coordinate system that is undergoing a CARTAN displacement. Hence, if  $v^x$  is the radiusvector of a field of points, each of which lies in a tangent  $E_n$ , its covariant differential is  $d\xi^x + \delta v^x$ . Vice versa, if a vector is displaced parallel over  $d\xi^x$ , its ordinary differential is  $-d\xi^\mu \Gamma_{\mu\lambda}^x v^\lambda$ , but if the point with radiusvector  $v^x$  undergoes a CARTAN displacement, its ordinary differential is  $-d\xi^x - d\xi^\mu \Gamma_{\mu\lambda}^x v^\lambda$ .<sup>1)</sup> Now let us suppose that there is a closed curve in  $X_n$  through  $\xi^x$  and that the local  $E_n$  at  $\xi^x$  undergoes a CARTAN displacement along this curve. The points of the curve may be considered as a point field having in each local coordinate system along the curve the radius-vector zero. During the motion the moving  $E_n$  coincides at every moment with one of the tangent  $E_n$ 's of the points of the curve. Hence at every moment the place of this point can be marked in the moving  $E_n$ . Then in this  $E_n$  we get a curve and it can be proved that if the original curve is contracted into a point and if terms of third and higher orders in the differentials are neglected, the image in the moving  $E_n$  is always closed in the symmetric case and is closed in the semi-symmetric case if and only if the 2-direction of the enclosed surface element is lying in the  $(n-1)$ -direction of  $S_1$ . In an  $E_n$  the curve and its image coincide. A rigorous proof of theorems of this kind always requires a rather detailed discussion. Therefore we only give an illustration here, that, without being a rigorous proof, may have some heuristic value. Let the curve be the quadrilateral of the points  $\xi^x, \xi^x + \underset{1}{d\xi^x}, \xi^x + \underset{1}{d\xi^x} + \underset{2}{d\xi^x}$  and  $\xi^x + \underset{2}{d\xi^x}$ . Starting from the first contact-point  $\xi^x$  the radiusvector of this point with respect to the second contact-point  $\xi^x + \underset{1}{d\xi^x}$  is after the arrival of the  $E_n$  at this latter point equal to  $-d\xi^x$ . Going from  $\xi^x + \underset{1}{d\xi^x}$  to the third contact-point  $\xi^x + \underset{1}{d\xi^x} + \underset{2}{d\xi^x}$  the point  $\xi^x + \underset{1}{d\xi^x}$

<sup>1)</sup> From the large amount of literature on the CARTAN displacement in various spaces we mention here only a few titles: DIENES 1934, 1; WAGNER 1942, 1; TACHIBANA 1949, 1; SASAKI and YANO 1949, 1; ÔTSUKI 1950, 1; KANITANI 1950, 1. Of course the CARTAN displacement can also be defined for the more general linear connexions mentioned in footnote 1, 2 on p. 123 and especially for conformal and projective connexions (cf. CARTAN 1937, 1, p. 163 ff., 278 ff.).

gets the radiusvector  $-d_2 \xi^x$  with respect to this third contact-point but at the same time the vector  $-d_1 \xi^x$  is displaced parallel over  $d_2 \xi^x$ . Hence  $\delta d_1 \xi^x = d_1 d_2 \xi^x + d_2 \xi^\mu \Gamma_{\mu\lambda}^x d_1 \xi^\lambda = 0$  and this has as a consequence that the point that had a radiusvector zero with respect to the first contact-point and after the first CARTAN displacement a radiusvector  $-d_1 \xi^x$  with respect to the second contact-point gets after the second displacement a radiusvector  $-d_1 \xi^x - d_2 \xi^x + d_2 \xi^\mu \Gamma_{\mu\lambda}^x d_1 \xi^\lambda$  with respect to the third contact-point. Going via  $\xi^x + d_2 \xi^x$  we get in the same way a radiusvector  $-d_2 \xi^x - d_1 \xi^x + d_1 \xi^\mu \Gamma_{\mu\lambda}^x d_2 \xi^\lambda$ . Hence, if we go from  $\xi^x$  round the quadrilateral back to  $\xi^x$ , the original contact-point has after its return in  $\xi^x$  the radiusvector  $-2 d_1 \xi^\mu d_2 \xi^\lambda S_{\mu\lambda}^{xx}$ .

Applying (2.16) for  $q=n$  we get

$$(2.21) \quad \begin{cases} V_\mu w_{1\dots n} = \partial_\mu w_{1\dots n} - n \Gamma_{\mu[n}^x w_{1\dots n-1]x} \\ \quad = \partial_\mu w_{1\dots n} - \Gamma_{\mu x}^x w_{1\dots n}. \end{cases}$$

Hence, if  $w_{\lambda_1\dots\lambda_n}$  is displaced parallel we have .

$$(2.22) \quad d w_{1\dots n} = \overset{*}{d} w_{1\dots n} = \Gamma_\mu w_{1\dots n} d\xi^\mu; \quad \Gamma_\mu \stackrel{\text{def}}{=} \Gamma_{\mu x}^x.$$

$\Gamma_\mu$  is a geometric object with the transformation [cf. Exerc. II 2.1 and (2.5)]

$$(2.23) \quad \Gamma_{\mu'} = A_{\mu'}^\mu \Gamma_\mu - A_{\mu'}^\lambda \partial_{\mu'} A_\lambda^{xx} = A_{\mu'}^\mu \Gamma_\mu - \partial_{\mu'} \log \Delta.$$

Now  $w_{1\dots n}$  may be considered as the component of a scalar  $\Delta$ -density of weight +1. Accordingly we have for such a density  $\tilde{p}$

$$(2.24) \quad \delta \tilde{p} = d \tilde{p} - \Gamma_\mu \tilde{p} d \xi^\mu,$$

$$(2.25) \quad V_\mu \tilde{p} = \partial_\mu \tilde{p} - \Gamma_\mu \tilde{p}$$

and for a  $\Delta$ -density  $\tilde{q}$  of weight  $w$

$$(2.26) \quad \delta \tilde{q} = d \tilde{q} - w \Gamma_\mu \tilde{q} d \xi^\mu,$$

$$(2.27) \quad V_\mu \tilde{q} = \partial_\mu \tilde{q} - w \Gamma_\mu \tilde{q}.$$

If a  $\Delta$ -density is multiplied by a  $W$ -scalar we get an ordinary density. Hence, according to (C V) the formulae for the covariant differentiation of ordinary scalar densities and scalar  $\Delta$ -densities are the same.<sup>1)</sup>

1) Cf. for covariant differentiation of densities VEBLEN and T. Y. THOMAS 1924, 1; HLAVATY 1927, 2; SCHOUTEN and HLAVATY 1929, 2.

Because every tensor density can be written as the product of a tensor and a scalar density, in the formulae for the covariant differential and the covariant derivative we get an extra term with  $\Gamma_\mu$ , for instance [cf. (2.8, 9)]

$$(2.28) \quad \left\{ \begin{array}{l} \delta \mathfrak{P}_{.. \mu}^{\lambda} = d \mathfrak{P}_{.. \mu}^{\lambda} + \Gamma_{\nu \rho}^{\lambda} \mathfrak{P}_{.. \mu}^{\rho} d \xi^\nu + \Gamma_{\nu \rho}^{\lambda} \mathfrak{P}_{.. \mu}^{\rho} d \xi^\nu - \\ \qquad \qquad \qquad - \Gamma_{\nu \rho}^{\rho} \mathfrak{P}_{.. \nu}^{\lambda} d \xi^\nu - w \mathfrak{P}_{.. \mu}^{\lambda} \Gamma_\nu d \xi^\nu, \end{array} \right.$$

$$(2.29) \quad \nabla_\nu \mathfrak{P}_{.. \mu}^{\lambda} = \partial_\nu \mathfrak{P}_{.. \mu}^{\lambda} + \Gamma_{\nu \rho}^{\lambda} \mathfrak{P}_{.. \mu}^{\rho} + \Gamma_{\nu \rho}^{\lambda} \mathfrak{P}_{.. \mu}^{\rho} - \Gamma_{\nu \mu}^{\rho} \mathfrak{P}_{.. \rho}^{\lambda} - w \mathfrak{P}_{.. \mu}^{\lambda} \Gamma_\nu.$$

For a  $p$ -vector density of weight +1 we have

$$(2.30) \quad \left\{ \begin{array}{l} \nabla_\mu \mathfrak{w}^{\mu \kappa_2 \dots \kappa_p} = \partial_\mu \mathfrak{w}^{\mu \kappa_2 \dots \kappa_p} + \Gamma_{\mu \lambda}^\mu \mathfrak{w}^{\lambda \kappa_2 \dots \kappa_p} + \Gamma_{\mu \lambda}^{\kappa_2} \mathfrak{w}^{\mu \lambda \kappa_3 \dots \kappa_p} + \dots \\ \qquad \qquad \qquad \dots + \Gamma_{\mu \lambda}^{\kappa_p} \mathfrak{w}^{\mu \kappa_2 \dots \kappa_{p-1} \lambda} - \Gamma_{\mu \lambda}^\lambda \mathfrak{w}^{\mu \kappa_2 \dots \kappa_p} \\ \qquad \qquad \qquad = \partial_\mu \mathfrak{w}^{\mu \kappa_2 \dots \kappa_p} + 2 S_{\nu \mu}^{.. \nu} \mathfrak{w}^{\mu \kappa_2 \dots \kappa_p} + \\ \qquad \qquad \qquad + (\phi - 1) S_{\mu \lambda}^{.. \lambda} \mathfrak{w}^{[\mu \lambda] \kappa_3 \dots \kappa_p} \end{array} \right.$$

from which we see that in the expression  $\nabla_\mu \mathfrak{w}^{\mu \kappa_2 \dots \kappa_p}$  in an  $A_n$  the symbols  $\partial$  and  $\nabla$  can be interchanged but that in a general  $L_n$  a correction term containing  $S_\lambda$  occurs and for  $\phi > 1$  also one containing  $S_{\mu \lambda}^{.. \lambda}$ .

In order to establish the covariant differentiation of pseudo-quantities as defined in II § 3 with respect to an auxiliary variable  $\xi^\theta$ , we consider a pseudo-scalar  $\mathfrak{p}$  of class 1. Its ordinary differential is not a pseudo-scalar because

$$(2.31) \quad d^{(\theta)} \mathfrak{p} = d\sigma \mathfrak{p} = \sigma d^{(\theta)} \mathfrak{p} + \mathfrak{p}^{(\theta)} \partial_\mu \sigma d\xi^\mu. \text{ } ^1)$$

From the conditions (CI–IV) we see that the covariant differential must have the form

$$(2.32) \quad \delta \mathfrak{p} = d \mathfrak{p} - A_\mu \mathfrak{p} d\xi^\mu$$

with parameters  $A_\mu$  subject to the transformation

$$(2.33) \quad A_{\mu'} = A_\mu^{\mu'} A_\mu + \partial_{\mu'} \log \sigma.$$

This implies that in order to get the covariant differential of a pseudo-quantity  $\Phi$  (indices suppressed) of class  $c$  we have first to write down all terms that would occur if  $\Phi$  were a quantity and then to add the term  $-c \Phi A_\mu d\xi^\mu$ . For instance the covariant derivative of a pseudo-tensor density  $\mathfrak{p}_{.. \lambda}^{\lambda}$  of weight  $w$  and class  $c$  is

$$(2.34) \quad \nabla_\mu \mathfrak{p}_{.. \lambda}^{\lambda} = \partial_\mu \mathfrak{p}_{.. \lambda}^{\lambda} + \Gamma_{\mu \rho}^{\lambda} \mathfrak{p}_{.. \rho}^{\rho} - \Gamma_{\mu \lambda}^{\sigma} \mathfrak{p}_{.. \sigma}^{\lambda} - w \mathfrak{p}_{.. \lambda}^{\lambda} \Gamma_\mu - c \mathfrak{p}_{.. \lambda}^{\lambda} A_\mu.$$

If a vector field  $v^\kappa$  is given we may require the field ' $\Phi_A$ ' that arises from a given field  $\Phi_A$  if this field is displaced parallel along the streamlines of  $v^\kappa$  over  $v^\kappa t$  (cf. II § 10). In order to find ' $\Phi_A$ ' we take the co-

<sup>1)</sup> If  $\sigma$  also depends on  $\xi^\theta$  there is another term  $\mathfrak{p}^{(\theta)} \partial_0 \sigma d\xi^\theta$ .

ordinate system  $(x)$  such that  $v^\kappa = e_i^\kappa$  (cf. Exerc. II 5,1) and introduce an anholonomic system  $(h)$  such that the  $e_i^\kappa$ ;  $h, i = 1, \dots, n$  are covariant constant along the streamlines of  $v^\kappa$ . Let the components of  $\Phi_A$  with respect to  $(h)$  be symbolized by  $\Phi_L$ . Then we have

$$(2.35) \quad v^l \nabla_l \Phi_L \stackrel{*}{=} v^l \partial_l \Phi_L = v^\mu \partial_\mu \Phi_L \stackrel{*}{=} \partial_1 \Phi_L.$$

Since the  $e_i^\kappa$  are covariant constant along the streamlines,  $'\Phi_L(\xi^\kappa)$  equals  $\Phi_L(\xi^\kappa - t e_1^\kappa)$ . Hence

$$(2.36) \quad ' \Phi_L(\xi^\kappa) = \Phi_L(\xi^\kappa - t e_1^\kappa) = e^{-t \partial_1} \Phi_L(\xi^\kappa) = \exp(-t v^l \nabla_l) \Phi_L(\xi^\kappa)$$

and, because this result has the invariant form

$$(2.37) \quad ' \Phi_A(\xi^\kappa) = e^{-t v^\mu \nabla_\mu} \Phi_A(\xi^\kappa) \quad .^1)$$

### Exercises.

III 2,1. Prove that  $A_\lambda^\kappa$  is covariant constant.

III 2,2. Prove (2.5) starting from  $\nabla_\mu A_\lambda^\kappa = 0$ .

III 2,3. The vector  $S_{\mu\lambda}{}^\kappa d\xi^\mu d\xi^\lambda$  lies in the 2-direction of  $d\xi^\kappa$  and  $d\xi^\lambda$  for every choice of these latter vectors if and only if the connexion is semi-symmetric (cf. Exerc. I 3,6).

III 2,4. Prove that for every given point there exists always a coordinate system  $(x)$  such that  $E_{\lambda_1 \dots \lambda_n}^{x_1 \dots x_n}$  is covariant stationary at this point.

III 2,5. Prove that  $\tilde{E}_{\lambda_1 \dots \lambda_n}^{x_1 \dots x_n}$  and  $\tilde{e}_{\lambda_1 \dots \lambda_n}$  are covariant constant (cf. Exerc. II 10,1).

III 2,6. In the identity (II 11.11) for  $[\mathfrak{L}]$  in  $A_n$  it is allowed to replace  $\partial_\mu$  by  $\nabla_\mu$ . Prove this for the special case (II 11.12).

### § 3. A linear connexion expressed in terms of $S_{\mu\lambda}{}^\kappa$ , an auxiliary symmetric tensor field $g_{\lambda\kappa}$ of rank $n$ and $\nabla_\mu g_{\lambda\kappa}$ .

Let  $g_{\lambda\kappa}$  be a symmetric tensor,  $g_{\lambda\kappa} = g_{\kappa\lambda}$  and let its inverse be written  $g^{\kappa\lambda}$ . At first we do not consider  $g_{\lambda\kappa}$  as a fundamental tensor in tangent space, it is just an auxiliary symmetric tensor and nothing more. But we will use it to raise and lower indices in the way described in I § 9. If we write

$$(3.1) \quad \nabla_\mu g^{\kappa\lambda} \stackrel{\text{def}}{=} Q_\mu^{\kappa\lambda}$$

<sup>1)</sup> This formula was first found by MORINAGA 1934, 2, the proof given here is by NIJENHUIS (personal communication). Cf. also the footnote 1 on p. 340 on product integrals VII § 1 and DIENES 1934, 1; JOHNSON 1948, 1; NIJENHUIS 1952, 1, p. 174.  $\Phi_A$  and  $'\Phi_A$  are here used as function symbols, cf. II § 3.

it follows from  $g^{\kappa\sigma} g_{\sigma\lambda} = A_{\lambda}^{\kappa}$  that

$$(3.2) \quad \nabla_{\mu} g_{\lambda\kappa} = -Q_{\mu\lambda\kappa}.$$

By writing out these equations

$$(3.3) \quad \partial_{\mu} g_{\lambda\kappa} = \Gamma_{\mu\lambda}^{\kappa} g_{\kappa\kappa} + \Gamma_{\mu\kappa}^{\kappa} g_{\kappa\lambda} - Q_{\mu\lambda\kappa}$$

permuting the indices  $\mu\lambda\kappa$  and taking the sum of the permutations  $\mu\lambda\kappa, \kappa\mu\lambda$  diminished by the permutation  $\lambda\kappa\mu$  we get by addition

$$(3.4) \quad \left\{ \begin{array}{l} 2\Gamma_{(\kappa\mu)}^{\kappa} g_{\sigma\lambda} = \partial_{\mu} g_{\lambda\kappa} + \partial_{\kappa} g_{\mu\lambda} - \partial_{\lambda} g_{\mu\kappa} + Q_{\mu\kappa\lambda} + Q_{\kappa\mu\lambda} - Q_{\lambda\kappa\mu} - \\ \quad - 2S_{\mu\lambda}^{;\kappa} g_{\kappa\kappa} - 2S_{\kappa\lambda}^{;\mu} g_{\mu\mu} \end{array} \right.$$

or

$$(3.5) \quad \boxed{\Gamma_{\mu\lambda}^{\kappa} = \{_{\mu\lambda}^{\kappa}\} + S_{\mu\lambda}^{;\kappa} - S_{\lambda\mu}^{;\kappa} + S_{\kappa\mu}^{;\lambda} + \frac{1}{2}(Q_{\mu\lambda}^{;\kappa} + Q_{\lambda\mu}^{;\kappa} - Q_{\kappa\mu}^{;\lambda})}$$

where

$$(3.6) \quad \{_{\mu\lambda}^{\kappa}\} \stackrel{\text{def}}{=} \frac{1}{2}g^{\kappa\sigma}(\partial_{\mu} g_{\lambda\sigma} + \partial_{\lambda} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\lambda})$$

is the so called CHRISTOFFEL symbol belonging to the tensor  $g_{\lambda\kappa}$ .<sup>1)</sup> In fact (3.5) satisfies the equation (3.3).

The  $\frac{1}{2}g^{\kappa\sigma}\partial_{\mu}g_{\lambda\sigma}$ ,  $S_{\mu\lambda}^{;\kappa}$  and  $\frac{1}{2}Q_{\mu\lambda}^{;\kappa}$  occur in (3.5) in an analogous way. This can be made clearer by using the abbreviation (the place of the indices is important!)

$$(3.7) \quad \psi_{\{\mu\lambda\kappa\}} \stackrel{\text{def}}{=} \psi_{\mu\lambda\kappa} - \psi_{\lambda\kappa\mu} + \psi_{\kappa\mu\lambda}$$

for any expression  $\psi_{\mu\lambda\kappa}$ . Then, if we write

$$(3.8) \quad \chi_{\mu\lambda\kappa} \stackrel{\text{def}}{=} \frac{1}{2}\partial_{\mu}g_{\lambda\kappa}$$

(3.5) takes the form

$$(3.9) \quad \boxed{\Gamma_{\mu\lambda}^{\kappa} = g^{\kappa\sigma}(\chi_{\{\mu\sigma\lambda\}} - S_{\{\mu\sigma\lambda\}} + \frac{1}{2}Q_{\{\mu\sigma\lambda\}})}.$$

If  $Q_{\mu\lambda}^{;\kappa} = 0$  the connexion is called metric<sup>2)</sup> with respect to  $g_{\lambda\kappa}$ . In this case the covariant differential of  $g_{\lambda\kappa}$  vanishes and this implies that the processes of covariant differentiation and of raising and lowering of indices are commutative, for instance

$$(3.10) \quad \nabla_{\mu} g_{\lambda\kappa} v^{\kappa} = g_{\lambda\kappa} \nabla_{\mu} v^{\kappa} = \nabla_{\mu} v_{\lambda}.^3)$$

<sup>1)</sup> CHRISTOFFEL 1869, 1.

<sup>2)</sup> The equation (3.9) can also be used to establish relations between two different non-metric connexions. Cf. SEN 1944, 1; 1945, 1; 1946, 1; 1948, 1; 1949, 1; 2; 1950, 1; 2; NORDEN 1945, 1.

<sup>3)</sup> Most authors accept the convention that the range of differentiation of  $\nabla$  stretches as far as the first closing bracket whose corresponding opening bracket stands at the left hand side of  $\nabla$  (cf. footnote 1 on p. 70).

If the connexion is *metric* and if  $g_{\lambda\nu}$  is *real* and introduced as a fundamental tensor (cf. I § 9) in each of the local  $E_n$ 's, the  $X_n$  is called a  $U_n$  and  $g_{\lambda\nu}$  is said to be the *fundamental tensor* of the  $U_n$ . A  $U_n$  with a *symmetric* connexion is called a  $V_n$ .<sup>1)</sup> This is the case of *riemannian geometry*. A  $U_n$  or  $V_n$  is called *ordinary* if  $g_{\lambda\nu}$  is positive definite. In a  $V_n$  (3.5) takes the form

$$(3.11) \quad \Gamma_{\mu}^{\nu} = \{_{\mu\nu}^{\nu}\} \quad ^2)$$

and from this it follows that

$$(3.12) \quad \Gamma_{\mu} = \frac{1}{2} g^{\nu\lambda} \partial_{\mu} g_{\nu\lambda} = \frac{1}{2} \partial_{\mu} \log g; \quad g \stackrel{\text{def}}{=} |\text{Det}(g_{\lambda\nu})|.$$

If

$$(3.13) \quad Q_{\mu}^{\nu\lambda} = Q_{\mu} g^{\lambda\nu}$$

the connexion is called *semi-metric*. From (3.2) it follows that in this case

$$(3.14) \quad \nabla_{\mu} g_{\lambda\nu} = - Q_{\mu} g_{\lambda\nu}.$$

If the connexion is *semi-metric* and *symmetric*, and if  $g_{\lambda\nu}$  is *real*, the  $X_n$  is called a  $W_n$ . This is the case of WEYL's geometry<sup>3)</sup>. Let a vector  $v^{\nu}$  in a  $W_n$  be displaced parallel. Then  $\delta v^{\nu} = 0$  and

$$(3.15) \quad \delta g_{\lambda\nu} v^{\lambda} v^{\nu} = - g_{\lambda\nu} v^{\lambda} v^{\nu} Q_{\mu} d\xi^{\mu}.$$

Hence in a  $W_n$  the length of a vector is not invariant for parallel displacement but the ratio of the length of two vectors at the same point is invariant. If  $g_{\lambda\nu}$  in a  $W_n$  undergoes a conformal transformation

$$(3.16) \quad 'g_{\lambda\nu} = \sigma(\xi^{\nu}) g_{\lambda\nu}$$

we have

$$(3.17) \quad \nabla_{\mu} 'g_{\lambda\nu} = (-\sigma Q_{\mu} + \partial_{\mu} \sigma) g_{\lambda\nu} = (-Q_{\mu} + \partial_{\mu} \log \sigma) 'g_{\lambda\nu}.$$

Hence if we take ' $g_{\lambda\nu}$ ' as fundamental tensor instead of  $g_{\lambda\nu}$ , and if at the same time  $Q_{\lambda}$  is transformed into ' $Q_{\lambda} = Q_{\lambda} - \partial_{\lambda} \log \sigma$ ' we get the

<sup>1)</sup> An  $A_n$  may be considered as a  $V_n$  if and only if there exists a covariant constant symmetric tensor  $g_{\lambda\nu}$  of rank  $n$ . The conditions were investigated by T. Y. THOMAS and LEVINE 1934, 1; T. Y. THOMAS 1935, 1; 1936, 2; LEVINE 1948, 1. Several authors investigated the geometries with a fundamental tensor of rank  $< n$ . We mention here only BORTOLOTTI 1930, 2; MOISIL 1940, 1; VRANCEANU 1942, 1; NORDEN 1945, 2.

<sup>2)</sup> Note that  $\{_{\mu\nu}^{\nu}\}$  does not change if  $g_{\lambda\nu}$  gets a *constant* factor. Introducing such a factor in fact means a change of gauge in the whole space without changing the geometry.

<sup>3)</sup> WEYL 1918, 1, p. 400; R. K. 1924, 1, Ch. 6; HLAVATY 1929, 2. The semi-metric semi-symmetric case appeared in the classification R. K. 1924, 1, p. 75 and was applied to mechanics by LICHNEROWICZ 1941, 1.

same connexion. That implies that in a  $W_n$  the fundamental tensor is only fixed to within an arbitrary scalar factor. But a  $W_n$  is not the same as a conformal space. In the latter only  $|g_{\lambda\mu}|$  is given (cf. I § 2) but no linear connexion can be fixed by  $|g_{\lambda\mu}|$  alone. Therefore we need the vector field  $Q_\lambda$ , given for one of the possible fields  $g_{\lambda\mu}$ . Of course we must exclude the case when  $Q_\lambda$  is a gradientvector because in that case it would be possible to fix ' $g_{\lambda\mu}$ ' by choosing  $\sigma$  in such a way that ' $Q_\lambda$ ' vanishes, and that would mean that we did not really have a  $W_n$  but a  $V_n$  with an undetermined gauge in which  $g_{\lambda\mu}$  could have been given in a more practical way (determined to within a *constant* scalar factor). In one dimensional space every vector is a gradientvector. Hence every  $W_1$  is a  $V_1$  with an undetermined gauge. In a  $W_n$  (3.5) takes the form

$$(3.18) \quad \Gamma_{\mu\lambda}^\kappa = \{_{\mu\lambda}^\kappa\} + \frac{1}{2} (Q_\mu A_\lambda^\kappa + Q_\lambda A_\mu^\kappa - Q^\kappa g_{\mu\lambda})$$

and for  $\Gamma_\mu$  we have the formula

$$(3.19) \quad \Gamma_\mu = \frac{1}{2} \partial_\mu \log g + \frac{n}{2} Q_\mu.$$

In a  $W_n$  the process of covariant differentiation does not commute with the process of raising and lowering of indices. But, since the fundamental tensor is fixed only to within an arbitrary scalar factor, this latter process is not invariant and can be effected only if a preference is given to one of the possible fundamental tensor fields. If this can be done in an invariant way we go back to the  $V_n$  and in the other case it is better not to use the non invariant process at all.

The geometry in  $W_n$  is based on two different transformation groups, the group of the coordinate transformations and the group of the conformal transformations of  $g_{\lambda\mu}$  (the "Umeichung" of WEYL). But in the above these two groups are not dealt with in the same way; the coordinate transformations are fundamental because they form the foundation for the definition of all objects and the conformal transformations come in after the introduction of a fundamental tensor to make this tensor variable. Though quite correct mathematically, this method seems incongruous from a methodical and aesthetic point of view.<sup>1)</sup> It is just the same as if, in dealing with some geometry, we first worked out everything with respect to one definite coordinate system and afterwards introduced coordinate transformations and raised the question of invariance.<sup>2)</sup> Now it is possible to handle both groups in the same

<sup>1)</sup> My attention was drawn for the first time to this point by G. LYRA in his letter of 27. 6. 1949. Cf. LYRA 1951, 1. The same point of view occurs in HLAVATY 1949, 1; 1952, 1.

<sup>2)</sup> In many 19th century publications this was actually done!

way by introducing at the beginning a fundamental pseudo-tensor  $g_{\lambda x}$  of class +1, defined with respect to an auxiliary variable  $\xi^{\theta}$  with the transformation  $\xi^{\theta} = \sigma \xi^{\theta}$  (cf. II § 3).<sup>1)</sup> Then, using the formula [cf. (2.34)] for the covariant differentiation of pseudo-quantities we get

$$(3.20) \quad V_{\mu} g_{\lambda x} = \partial_{\mu} g_{\lambda x} - \Gamma_{\mu \lambda}^{\theta} g_{\theta x} - \Gamma_{\mu x}^{\theta} g_{\theta \lambda} - g_{\lambda x} A_{\mu}.$$

Equating this expression to zero, taking  $S_{\mu}^{\cdot} \lambda^x = 0$  and using  $g_{\lambda x}$  and its inverse  $g^{x \lambda}$  for raising and lowering of indices we get

$$(3.21) \quad \Gamma_{\mu \lambda}^x = \{_{\mu \lambda}^x\}^* - \frac{1}{2} (A_{\mu} A_{\lambda}^x + A_{\lambda} A_{\mu}^x - A^x g_{\mu \lambda})$$

$$(3.22) \quad I_{\mu} = \frac{1}{2} \partial_{\mu} \log \mathfrak{G} - \frac{n}{2} A_{\mu}; \quad \mathfrak{G} \stackrel{\text{def}}{=} |\text{Det}(g_{\lambda x})|$$

where

$$(3.23) \quad \{_{\mu \lambda}^x\}^* \stackrel{\text{def}}{=} \frac{1}{2} g^{x \sigma} (\partial_{\mu} g_{\lambda \sigma} + \partial_{\lambda} g_{\mu \sigma} - \partial_{\sigma} g_{\mu \lambda}).$$

$\mathfrak{G}$  is a pseudo-scalar density of weight +2 and class  $n$ . From (3.18, 19) and (3.21–23) we see that  $A_{\lambda}$  plays now the same role as  $-Q_{\lambda}$  before. The only difference is that we have now two definitely fixed geometric objects  $g_{\lambda x}$  and  $A_{\lambda}$  instead of the variable quantities  $g_{\lambda x}$  and  $Q_{\lambda}$ .  $g_{\lambda x}$  being fixed, it can be used for raising or lowering of indices and according to  $V_{\mu} g_{\lambda x} = 0$  this process is commutative with the process of covariant differentiation. Note however that by raising (lowering) of an index the class increases with  $-1$  (+1).<sup>2)</sup>

It is very remarkable that in fixing a linear displacement the tensor  $g_{\lambda x}$  plays such a fundamental part. Why not any quantity with another valence and another condition of symmetry? This has been made clear by WEYL. First let an arbitrary linear displacement be given by its parameters  $I_{\mu \lambda}^x$ . At a point  $\overset{\theta}{\xi}^x$  we choose an arbitrary symmetric tensor  $\overset{\theta}{g}_{\lambda x}$  of rank  $n$ . Now we try to form a field  $g_{\lambda x}$  in  $\mathfrak{N}(\overset{\theta}{\xi}^x)$  such that  $g_{\lambda x} = \overset{\theta}{g}_{\lambda x}$  at  $\overset{\theta}{\xi}^x$  and that the given displacement from  $\overset{\theta}{\xi}^x$  to any point  $\overset{\theta}{\xi}^x + d\xi^x$  is the result of

1° the displacement from  $\overset{\theta}{\xi}^x$  to  $\overset{\theta}{\xi}^x + d\xi^x$  belonging to the Riemannian geometry with  $g_{\lambda x}$  as fundamental tensor;

2° a rotation of all vectors at  $\overset{\theta}{\xi}^x + d\xi^x$  with respect to  $\overset{\theta}{g}_{\lambda x} + dg_{\lambda x}$  as fundamental tensor.

<sup>1)</sup> NEWMAN 1927, 1; SCHOUTEN and HLAVATY 1929, 2, p. 426; E I 1935, 1, p. 87.

<sup>2)</sup> Note that the geometry of a  $W_n$  is not the conformal geometry. In a conformal geometry we have only the pseudo-tensor  $g_{\lambda x}$  and no  $A_{\lambda}$ . That implies that there is no fixed linear connexion.

Such a displacement WEYL calls a “congruent transplantation” (kongruente Verpflanzung). If  $v^\kappa$  is a vector at  $\xi^x$ , it changes first into

$$(3.24) \quad v^\kappa - \{_{\mu\lambda}^x\} v^\lambda d\xi^\mu$$

where  $\{_{\mu\lambda}^x\}$  is the CHRISTOFFEL symbol belonging to  $g_{\lambda\kappa}$  and afterwards into

$$(3.25) \quad v^\kappa - \{_{\mu\lambda}^x\} v^\lambda d\xi^\mu - F_{\mu}^{\cdot\kappa\nu} g_{\nu\lambda} v^\lambda d\xi^\mu$$

where  $F_{\mu}^{\cdot\kappa\nu} d\xi^\mu$  is the bivector of the rotation (cf. I § 9). Hence at  $\xi^x$

$$(3.26) \quad F_{\mu\lambda}^x = \{_{\mu\lambda}^x\} - F_{\mu\lambda\nu} g^{\nu\kappa}.$$

Now  $F_{\mu\lambda\nu}$  is alternating in  $\lambda\nu$  and from this it follows that

$$(3.27) \quad g_{\kappa\nu} F_{\mu\lambda}^{\nu} + g_{\lambda\nu} F_{\mu\kappa}^{\nu} = \partial_\mu g_{\lambda\kappa}$$

or

$$(3.28) \quad \nabla_\mu g_{\lambda\kappa} = 0.$$

Now let a field  $g_{\lambda\kappa}$  of rank  $n$  be given in  $\mathfrak{N}(\xi^x)$ . Then all linear connexions representing congruent transplantations with respect to  $g_{\lambda\kappa}$  satisfy equations of the form (3.26) where  $F_{\mu\lambda\nu}$  is an arbitrary tensor, alternating in  $\lambda\nu$ . Among these connexions there exists one and only one symmetric connexion because  $F_{\mu\lambda\nu}$  vanishes if  $F_{[\mu\lambda]\nu} = 0$ . Collecting results we have two theorems:

I. *If a linear connexion and a symmetric tensor  $g_{\lambda\kappa}$  of rank  $n$  at a point  $\xi^x$  be given, it is always possible to find a field  $g_{\lambda\kappa}$  in an  $\mathfrak{N}(\xi^x)$  such that  $g_{\lambda\kappa} = \overset{\theta}{g}_{\lambda\kappa}$  at  $\xi^x$  and that the linear displacement from  $\xi^x$  to every point  $\xi^x + d\xi^x$  is a congruent transplantation with respect to  $g_{\lambda\kappa}$ .*

II. *If a symmetric tensor field  $g_{\lambda\kappa}$  of rank  $n$  be given there exists one and only one symmetric connexion representing a congruent transplantation with respect to  $g_{\lambda\kappa}$ .*

The first theorem was proved by WEYL<sup>1)</sup>. The second theorem deals with a special case of the general formula (3.5).

Now one could try to generalize the notion of congruent transplantation by using instead of the group of rotations some other subgroup of the group  $G_h$  which left the volume invariant. Then, if the following conditions are introduced:

1. If the subgroup and a linear connexion be given, it is always possible to define the “generalized rotation” in an  $\mathfrak{N}(\xi^x)$  in such a

<sup>1)</sup> WEYL 1921, 1, p. 131; cf. WEYL 1922, 2, p. 117.

way that the connexion represents a congruent transplantation from  $\xi^{\kappa}$  to  $\underset{\theta}{\xi}{}^{\kappa} + d\xi^{\kappa}$  for every  $d\xi^{\kappa}$ ;

2. if the generalized rotations are given at each point of  $\mathfrak{N}(\xi^{\kappa})$ , there exists one and only one symmetric connexion representing a congruent transplantation with respect to this definition of rotation,

it has been proved<sup>1)</sup> that there exists one and only one subgroup of  $G_{ho}$  that leaves the volume invariant and also satisfies these conditions, and that this subgroup is the group that leaves invariant a symmetric tensor of valence 2 and rank  $n$ . This theorem makes clear why in the problem of fixing a symmetric connexion, a symmetric tensor of valence 2 and rank  $n$  plays such an important part.

### Exercises.

III 3,1<sup>2)</sup>. The symmetric part ' $\Gamma_{\mu\lambda}^{\kappa} \stackrel{\text{def}}{=} \Gamma_{(\mu\lambda)}^{\kappa}$ ' depends not only on  $Q_{\mu\lambda}^{\kappa}$  but also on  $S_{\mu\lambda}^{\kappa}$ . The ' $\Gamma_{\mu\lambda}^{\kappa}$ ' are parameters of another connexion for which

$$\text{III 3,1 a)} \quad \nabla_{\mu} g_{\lambda\kappa} = 2S_{\mu(\lambda\kappa)} - Q_{\mu\lambda\kappa}.$$

III 3,2<sup>2)</sup>. If the connexion  $\Gamma_{\mu\lambda}^{\kappa}$  is metric, the symmetric connexion ' $\Gamma_{\mu\lambda}^{\kappa} = \Gamma_{(\mu\lambda)}^{\kappa}$ ' is also metric, if and only if  $S_{\mu\lambda\kappa}$  is a trivector.

III 3,3<sup>2)</sup>. If  $g_{\lambda\kappa}$  and  $\nabla_{\mu} g_{\lambda\kappa}$  are known,  $\Gamma_{\mu}$  is fixed without it being necessary to give  $S_{\mu\lambda}^{\kappa}$ .

III 3,4. In a metric connexion length and angles are invariant for all parallel displacements.

III 3,5. In a  $W_n$  angles are invariant for parallel displacement.

III 3,6. Quantities in  $L_n$  form a ring (in the sense of abstract algebra) with respect to addition, multiplication and the operators  $\nabla_{\mu}$  and  $\delta$ . Prove that pseudo-quantities in  $W_n$  defined with respect to  $\xi^{\theta}$  with a class equal to one half of the difference between covariant and contravariant valence form a ring with respect to addition, multiplication and the operators  $(\xi^{\theta})^{-\frac{1}{2}} \nabla_{\mu}$  and  $(\xi^{\theta})^{-\frac{1}{2}} \delta$ .

III 3,7<sup>3)</sup>. A semi-symmetric linear connexion is for  $n > 2$  fixed by the covariant derivative  $Q_{\mu}^{\cdot\kappa\lambda}$  of some symmetric tensor  $g^{\kappa\lambda}$  of rank  $n$  and the derivative  $I^{\kappa} = \nabla_{\mu} f^{\mu\kappa}$  of some bivector  $f^{\mu\kappa}$  of rank  $n$ . This derivative can be replaced by the derivative  $F_{\mu\lambda\kappa} = 6\nabla_{[\mu} f_{\lambda\kappa]}$  of some bivector  $f_{\lambda\kappa}$  of rank  $n$ .

<sup>1)</sup> WEYL 1922, 2; CARTAN gave another proof, 1923, 1; cf. also KOSAMBI 1952 1, p. 6.

<sup>2)</sup> E I 1935, 1, p. 84.

<sup>3)</sup> E I 1935, 1, p. 88.

### § 4. Curvature.

We consider a field  $v^\kappa$  or  $w_\lambda$  in an  $\mathfrak{N}(\xi^\kappa)$  and an infinitesimal simple contravariant bivector  $df^{\mu\lambda}$  at  $\xi^\kappa$  represented by a part of an  $E_2$  (with inner orientation) through  $\xi^\kappa$  with a definitely fixed boundary curve passing through  $\xi^\kappa$ . Now let the vector  $v^\kappa$  or  $w_\lambda$  at  $\xi^\kappa$  be displaced parallel along this curve till it returns to  $\xi^\kappa$ . Then it can be proved that the difference between the final value and the initial value can be written in the form

$$(4.1) \quad \text{a)} - \frac{1}{2} R_{\nu\mu}^{\cdot\cdot\lambda} v^\lambda df^{\mu\nu}; \quad \text{b)} + \frac{1}{2} R_{\nu\mu}^{\cdot\cdot\lambda} w_\lambda df^{\mu\nu}$$

where  $R_{\nu\mu}^{\cdot\cdot\lambda}$  is a tensor depending on the  $\Gamma_{\mu\lambda}^\kappa$  and their first derivatives

$$(4.2) \quad R_{\nu\mu}^{\cdot\cdot\lambda} \stackrel{\text{def}}{=} 2\partial_{[\nu} \Gamma_{\mu]\lambda}^\kappa + 2\Gamma_{[\nu|\rho|}^\kappa \Gamma_{\mu]\lambda}^\rho.$$

The rigorous proof of this theorem will be given after (4.9b).<sup>1)</sup> Here we give a short illustration that is not meant to be a proof. For the curve we take the quadrilateral of the points  $\xi^\kappa, \xi^\kappa + d\xi^\kappa, \xi^\kappa + d\xi^\kappa + d\xi^\kappa, \xi^\kappa + d\xi^\kappa + d\xi^\kappa + d\xi^\kappa$  and  $\xi^\kappa + d\xi^\kappa$ . Then, displacing  $v^\kappa$  from  $\xi^\kappa$  to  $\xi^\kappa + d\xi^\kappa + d\xi^\kappa$  via the point  $\xi^\kappa + d\xi^\kappa$  we get at  $\xi^\kappa + d\xi^\kappa$  the field value

$$(4.3) \quad v^\kappa - \Gamma_{\mu\lambda}^\kappa v^\lambda d\xi^\mu$$

neglecting terms of second and higher orders. At  $\xi^\kappa + d\xi^\kappa + d\xi^\kappa + d\xi^\kappa$  we get

$$(4.4) \quad v^\kappa - \Gamma_{\mu\lambda}^\kappa v^\lambda d\xi^\mu - \Gamma_{\mu\lambda}^\kappa d\xi^\mu (v^\lambda - \Gamma_{\nu\lambda}^\kappa v^\nu d\xi^\nu) - (\partial_\nu \Gamma_{\mu\lambda}^\kappa) v^\lambda d\xi^\nu d\xi^\mu.$$

Going via  $\xi^\kappa + d\xi^\kappa$  we get in the same way

$$(4.5) \quad v^\kappa - \Gamma_{\nu\lambda}^\kappa v^\lambda d\xi^\mu - \Gamma_{\nu\lambda}^\kappa d\xi^\mu (v^\lambda - \Gamma_{\mu\lambda}^\kappa v^\mu d\xi^\mu) - (\partial_\mu \Gamma_{\nu\lambda}^\kappa) v^\lambda d\xi^\mu d\xi^\mu$$

and the difference of these two values is

$$(4.6) \quad -2(\partial_{[\nu} \Gamma_{\mu]\lambda}^\kappa + \Gamma_{[\nu|\rho|}^\kappa \Gamma_{\mu]\lambda}^\rho) v^\lambda d\xi^\nu d\xi^\mu = -\frac{1}{2} R_{\nu\mu}^{\cdot\cdot\lambda} v^\lambda df^{\mu\nu}$$

because  $df^{\mu\nu} = 2d\xi^{[\nu} d\xi^{\mu]}$ . If (4.1) is proved to be true for every field  $v^\kappa$  (or  $w_\lambda$ ) and every bivector  $f^{\mu\nu}, R_{\nu\mu}^{\cdot\cdot\lambda}$  must be a tensor because every

<sup>1)</sup> Cf. TIETZE 1923, 1; SYNGE 1924, 1; T. Y. THOMAS 1925, 1; SCHLESINGER 1928, 1; 1931, 1; 1932, 1; McCONNELL 1928, 1; AGOSTINELLI 1933, 1; MORINAGA 1934, 1; GUGINO 1935, 1; JOHNSON 1948, 1; GRAEBER 1950, 1. If instead of vectors other quantities are displaced, other quantities arise instead of  $R_{\nu\mu}^{\cdot\cdot\lambda}$ , cf. for instance HOKARI 1934, 1; HOMBU 1936, 1; TAKENO 1942, 1; BOMPIANI 1946, 1; PETROV 1948, 1; BOCHNER 1951, 2.

expression of this kind with four indices alternating in  $\nu\mu$ , whose transvection with every bivector  $f^\nu{}^\mu$  and every vector  $v^\lambda$  (or  $w_\lambda$ ) is a vector, is necessarily a tensor.

$R_{\nu\mu}^{\cdot\cdot\lambda}{}^\kappa$  is called the RIEMANN-CHRISTOFFEL *tensor* or *curvature tensor* of the connexion  $\Gamma_{\mu\lambda}^\kappa$ . If it is zero at all points, the connexion is said to be *integrable*.

Connecting these results with those obtained in III § 2 we see that if a point with radiusvector  $v^\lambda$  at  $\xi^\lambda$  undergoes a CARTAN displacement along the boundary curve of the infinitesimal bivector  $df^\nu{}^\mu$  in the sense of this bivector, the point will after its return have the radiusvector

$$(4.7) \quad v^\lambda - \frac{1}{2} R_{\nu\mu}^{\cdot\cdot\lambda}{}^\kappa v^\lambda df^\nu{}^\mu = S_{\nu\mu}^{\cdot\cdot\lambda}{}^\kappa df^\nu{}^\mu.$$

If the process of covariant differentiation is applied twice in succession we get, after alternation over the indices of differentiation, an expression containing  $S_{\mu\lambda}^{\cdot\cdot\kappa}$  and  $R_{\nu\mu}^{\cdot\cdot\lambda}{}^\kappa$ . For instance

$$(4.8) \quad \left\{ \begin{aligned} V_{[\nu} V_{\mu]} w_\lambda &= V_{[\nu} (\partial_{\mu]} w_\lambda - \Gamma_{\mu]}^\rho{}_\lambda w_\rho) \\ &= \partial_{[\nu} \partial_{\mu]} w_\lambda - (\partial_{[\nu} \Gamma_{\mu]}^\rho{}_\lambda) w_\rho - \Gamma_{[\mu}^\rho{}_{|\lambda]} \partial_{\nu]} w_\rho - \Gamma_{[\nu}^\rho{}_{|\mu]} \partial_\rho w_\lambda - \\ &\quad - \Gamma_{[\nu}^\rho{}_{|\lambda]} \partial_{\mu]} w_\rho + \Gamma_{[\nu}^\rho{}_{|\mu]} \Gamma_{\rho\lambda}^\sigma w_\sigma + \Gamma_{[\nu}^\rho{}_{|\lambda]} \Gamma_{\mu]}^\sigma w_\sigma \end{aligned} \right.$$

or

$$(4.9a) \quad \boxed{V_{[\nu} V_{\mu]} w_\lambda = -\frac{1}{2} R_{\nu\mu}^{\cdot\cdot\lambda}{}^\kappa w_\kappa - S_{\nu\mu}^{\cdot\cdot\lambda}{}^\kappa V_\kappa w_\lambda}$$

and in the same way

$$(4.9b) \quad \boxed{V_{[\nu} V_{\mu]} v^\lambda = \frac{1}{2} R_{\nu\mu}^{\cdot\cdot\lambda}{}^\kappa v^\lambda - S_{\nu\mu}^{\cdot\cdot\lambda}{}^\kappa V_\kappa v^\lambda}.$$

Using (4.9b) we can now give a rigorous proof of (4.1a). Let  $t^\nu$  and  $u^\nu$  be two vector fields with the parameters  $t$  and  $u$  and let these fields satisfy the equation  $\oint_t u^\nu = 0$ .

Then for any field  $v^\lambda$  it follows from (4.9b) that

$$(4.10) \quad (t^\nu V_\nu u^\mu V_\mu - u^\nu V_\nu t^\mu V_\mu) v^\lambda = t^\nu u^\mu R_{\nu\mu}^{\cdot\cdot\lambda}{}^\kappa v^\lambda + (\oint_t u^\rho) V_\rho v^\lambda = t^\nu u^\mu R_{\nu\mu}^{\cdot\cdot\lambda}{}^\kappa v^\lambda.$$

Now instead of displacing a vector at  $\xi^\lambda$  parallel along an infinitesimal quadrilateral as was done in the illustration given above we give the whole vector field  $v^\lambda$  a finite parallel displacement over  $t^\nu t$  (cf. III § 2). We follow this with another displacement over  $u^\nu u$ , then over  $-t^\nu t$  and finally over  $-u^\nu u$ . Because  $\oint_t u^\nu = 0$ <sup>1)</sup>, after these four displacements every point is back at its original place, and according to (2.37), the new field

<sup>1)</sup> Cf. Exerc. II 10,10.

value  $'v^\kappa$  is

$$(4.11) \quad \left\{ \begin{array}{l} 'v^\kappa = \exp(u u^\mu V_\mu) \exp(t t^\mu V_\mu) \exp(-u u^\mu V_\mu) \exp(-t t^\mu V_\mu) v^\kappa \\ \quad = v^\kappa - t u (t^\nu V_\nu u^\mu V_\mu - u^\nu V_\nu t^\mu V_\mu) v^\kappa + \dots \\ \quad = v^\kappa - t u t^\nu u^\mu R_{\nu\mu}^{\lambda} v^\lambda + \dots \end{array} \right.$$

which proves (4.1a). <sup>1)</sup>

The operator  $V_{[\nu} V_{\mu]}$  has the very important property that it satisfies the rule of LEIBNIZ. In fact, if  $\Phi$  and  $\Psi$  are two quantities with suppressed indices, then we have (cf. footnote 3 on p. 132).

$$(4.12) \quad \left\{ \begin{array}{l} V_{[\nu} V_{\mu]} \Phi \Psi = V_{[\nu} (V_{\mu]} \Phi) \Psi + V_{[\nu} \Phi V_{\mu]} \Psi \\ \quad = (V_{[\nu} V_{\mu]} \Phi) \Psi + (V_{[\mu} \Phi) V_{\nu]} \Psi + (V_{[\nu} \Phi) V_{\mu]} \Psi + \Phi V_{[\nu} V_{\mu]} \Psi \\ \quad = (V_{[\nu} V_{\mu]} \Phi) \Psi + \Phi V_{[\nu} V_{\mu]} \Psi. \end{array} \right.$$

Now every tensor can be written as a sum of products of vectors and from this we get immediately a formula for the operator  $V_{[\nu} V_{\mu]}$  applied to a tensor. In this formula there appears a term with  $R_{\nu\mu}^{\lambda} v^\lambda$  with the  $+ - (-)$ -sign for each upper (lower) index. For instance, for the tensor  $P_{\cdot\cdot\omega}^{\alpha\lambda}$

$$(4.13) \quad \left\{ \begin{array}{l} V_{[\nu} V_{\mu]} P_{\cdot\cdot\omega}^{\alpha\lambda} = \frac{1}{2} R_{\nu\mu}^{\alpha\lambda} P_{\cdot\cdot\omega}^{\rho} + \frac{1}{2} R_{\nu\mu}^{\alpha\lambda} P_{\cdot\cdot\omega}^{\sigma} - \\ \quad - \frac{1}{2} R_{\nu\mu}^{\alpha\sigma} P_{\cdot\cdot\sigma}^{\lambda} - S_{\nu\mu}^{\alpha\rho} V_\rho P_{\cdot\cdot\omega}^{\lambda}. \end{array} \right.$$

If  $V_{[\nu} V_{\mu]}$  is applied to a scalar density or  $\Delta$ -density  $\wp$  of weight +1 we may deduce in the same way as (4.9a)

$$(4.14) \quad V_{[\nu} V_{\mu]} \wp = -\wp \partial_{[\nu} \Gamma_{\mu]} - S_{\nu\mu}^{\lambda} V_\lambda \wp.$$

But, although  $\Gamma_\mu$  is not a vector, the expression  $\partial_{[\nu} \Gamma_{\mu]}$  is always a bivector because

$$(4.15) \quad V_{\nu\mu} \stackrel{\text{def}}{=} R_{\nu\mu}^{\lambda} = 2\partial_{[\nu} \Gamma_{\mu]} + 2\Gamma_{[\nu|}^\lambda \Gamma_{\mu]}^\sigma = 2\partial_{[\nu} \Gamma_{\mu]}^{\lambda \sigma}.$$

Hence, we get for a scalar density or  $\Delta$ -density  $\wp$  of weight  $w$

$$(4.16) \quad V_{[\nu} V_{\mu]} \wp = -\frac{1}{2} w \wp V_{\nu\mu} - S_{\nu\mu}^{\lambda} V_\lambda \wp$$

and for a tensor density or  $\Delta$ -density of weight  $w$ , for instance  $\mathfrak{P}_{\cdot\lambda}^{\alpha}$ , according to the rule of LEIBNIZ

$$(4.17) \quad V_{[\nu} V_{\mu]} \mathfrak{P}_{\cdot\lambda}^{\alpha} = \frac{1}{2} R_{\nu\mu}^{\alpha\lambda} \mathfrak{P}_{\cdot\lambda}^{\rho} - \frac{1}{2} R_{\nu\mu}^{\alpha\lambda} \mathfrak{P}_{\cdot\sigma}^{\sigma} - S_{\nu\mu}^{\alpha\rho} V_\rho \mathfrak{P}_{\cdot\lambda}^{\lambda} - \frac{1}{2} w V_{\nu\mu} \mathfrak{P}_{\cdot\lambda}^{\alpha}.$$

In dealing with pseudo-quantities for which a covariant differentiation (that is a set of parameters  $A_\lambda$  besides the  $\Gamma_\mu^\lambda$ ) has been defined

<sup>1)</sup> Apart from a slight change this is the proof given by MORINAGA 1934, 1.

<sup>2)</sup> Cf. III § 6 for the geometric meaning of the bivector  $V_{\nu\mu}$ .

(cf. III § 2) we get in the same way an additional term containing the bivector

$$(4.18) \quad l_{\nu\mu} \stackrel{\text{def}}{=} 2\partial_{[\nu} A_{\mu]}$$

instead of  $V_{\nu\mu}$  and  $c$  (class) instead of  $w$  (weight). For instance for a pseudo-vector density  $\mathfrak{v}^\kappa$

$$(4.19) \quad V_{[\nu} V_{\mu]} \mathfrak{v}^\kappa = \frac{1}{2} R_{\nu\mu}{}^{\lambda\kappa} \mathfrak{v}^\lambda - S_{\nu\mu}{}^{\lambda} V_\lambda \mathfrak{v}^\kappa - \frac{1}{2} (w V_{\nu\mu} + c l_{\nu\mu}) \mathfrak{v}^\kappa.$$

If a symmetric tensorfield  $g_{\lambda\kappa}$  of rank  $n$ , its covariant derivative  $-Q_{\mu\lambda\kappa}$  and  $S_{\mu\lambda}{}^{\kappa}$  are given, the curvature tensor can be expressed in terms of these quantities, their first ordinary derivatives and the second ordinary derivatives of  $g_{\lambda\kappa}$ . If we write [cf. (3.9)]

$$(4.20) \quad \Gamma_{\mu\lambda}^\kappa = \{_{\mu\lambda}^\kappa\} + T_{\mu\lambda}{}^{\kappa}; \quad T_{\mu\lambda}{}^{\kappa} \stackrel{\text{def}}{=} g^{\kappa\varrho} (-S_{\{\mu\varrho\lambda\}} + \frac{1}{2} Q_{\{\mu\varrho\lambda\}})$$

and

$$(4.21) \quad K_{\nu\mu}{}^{\lambda\kappa} \stackrel{\text{def}}{=} 2\partial_{[\nu} \{_{\mu}^{\kappa}\}_{\lambda]} + 2\{_{[\nu}^{\kappa}\}_{[\mu]} \{_{\lambda]}^{\varrho}\}.$$

$K_{\nu\mu}{}^{\lambda\kappa}$  is the curvature tensor of the riemannian connexion belonging to  $g_{\lambda\kappa}$ , and  $T_{\mu\lambda}{}^{\kappa}$  is a tensor representing the difference between the connexion  $\Gamma_{\mu\lambda}^\kappa$  and this riemannian connexion. If we indicate the covariant differentiation of the connexion  $\{_{\mu\lambda}^\kappa\}$  by  $\overset{*}{V}$  we get from (4.2) and (4.20)

$$(4.22a) \quad R_{\nu\mu}{}^{\lambda\kappa} = K_{\nu\mu}{}^{\lambda\kappa} + 2\overset{*}{V}_{[\nu} T_{\mu]}{}^{\lambda\kappa} + 2T_{[\nu|\varrho|}{}^{\lambda\kappa} T_{\mu]}{}^{\varrho}$$

and also

$$(4.22b) \quad R_{\nu\mu}{}^{\lambda\kappa} = K_{\nu\mu}{}^{\lambda\kappa} + 2V_{[\nu} T_{\mu]}{}^{\lambda\kappa} - 2T_{[\nu|\varrho|}{}^{\lambda\kappa} T_{\mu]}{}^{\varrho} + 2S_{\nu\mu}{}^{\varrho} T_{\varrho}{}^{\lambda\kappa}.$$

From (4.22a) we may derive the relations

$$(4.23) \quad \begin{cases} R_{\mu\lambda} = K_{\mu\lambda} + \overset{*}{V}_\mu T_{\lambda}{}^{\kappa} - \overset{*}{V}_\lambda T_{\mu}{}^{\kappa} + T_{\kappa\varrho}{}^{\lambda} T_{\mu\lambda}{}^{\varrho} - T_{\mu\varrho}{}^{\lambda} T_{\kappa\lambda}{}^{\varrho}; \\ R_{\mu\lambda} \stackrel{\text{def}}{=} R_{\kappa\mu\lambda}{}^{\kappa}; \quad K_{\mu\lambda} = K_{\lambda\mu} \stackrel{\text{def}}{=} K_{\kappa\mu\lambda}{}^{\kappa} \end{cases}$$

$$(4.24) \quad V_{\nu\mu} = 2\overset{*}{V}_{[\nu} T_{\mu]}{}^{\kappa} = 2\partial_{[\nu} T_{\mu]}{}^{\kappa}; \quad (\text{cf. 4.15}).$$

$R_{\mu\lambda}$  ( $K_{\mu\lambda}$ ) is called the RICCI tensor of the  $L_n$  ( $V_n$ ).

(4.22) is a special case of a more general formula. If  $A_{\mu\lambda}{}^{\kappa}$  is an arbitrary tensor,  $'\Gamma_{\mu\lambda}^\kappa = \Gamma_{\mu\lambda}^\kappa + A_{\mu\lambda}{}^{\kappa}$  represents another connexion and the relation between the curvature tensors of these two connexions is easily proved to be

$$(4.25) \quad 'R_{\nu\mu}{}^{\lambda\kappa} = R_{\nu\mu}{}^{\lambda\kappa} + 2V_{[\nu} A_{\mu]}{}^{\lambda\kappa} + 2S_{\nu\mu}{}^{\varrho} A_{\varrho}{}^{\lambda\kappa} - 2A_{[\nu|\lambda|}{}^{\varrho} A_{\mu]}{}^{\lambda\kappa}.$$

In a WEYL connexion we have according to (3.18)

$$(4.26) \quad T_{\mu\lambda}{}^{\kappa} = \frac{1}{2} (Q_\mu A_\lambda^\kappa + Q_\lambda A_\mu^\kappa - Q^\kappa g_{\lambda\mu})$$

and from this it follows after some calculation

$$(4.27) \quad \left\{ \begin{array}{l} R_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda} = K_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda} + V_{[\nu} Q_{\mu]} A_{\lambda]}^{\cdot\cdot\cdot} - g_{[\nu[\sigma} (2V_{\mu]} Q_{\lambda]} + Q_{\mu]} Q_{\lambda]} - \\ - \frac{1}{2} Q_{[\sigma|} Q^{\sigma} g_{\mu]\lambda]} \end{array} \right. g^{\sigma\lambda}.$$

In a metric and semi-symmetric connexion we have according to (2.14) and (3.5)

$$(4.28) \quad R_{\mu\lambda}^{\cdot\cdot\cdot\lambda} = - S_{\lambda} A_{\mu}^{\cdot\cdot\cdot} + S^{\cdot\cdot\cdot} g_{\lambda\mu}$$

and this leads without difficulties to

$$(4.29) \quad R_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda} = K_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda} + g^{\sigma\sigma} g_{[\nu[\sigma} (4V_{\mu]} S_{\lambda]} - 4S_{\mu]} S_{\lambda]} + 2S_{[\sigma|} S^{\sigma} g_{\mu]\lambda]}).$$

An  $L_n$  is said to possess *teleparallelism*<sup>1)</sup>, <sup>2)</sup> or *absolute parallelism* (Fernparallelismus) in  $\mathfrak{N}_{\theta}(\xi^{\mu})$  if for any two points  $P$  and  $Q$  of this region the parallel displacement of any quantity from  $P$  to  $Q$  along a curve of  $\mathfrak{N}_{\theta}(\xi^{\mu})$  connecting  $P$  and  $Q$  gives a result at  $Q$  which is independent of the choice of the curve. According to (4.1) the curvature tensor vanishes in this case. Naturally an  $E_n$  possesses teleparallelism. In order to prove that every  $A_n$  with teleparallelism is an  $E_n$ , we take  $n$  linearly independent covariant vectors  $\overset{h}{e}_{\lambda}$ ;  $h=1, \dots, n$  at  $\xi^{\mu}$  and by parallel displacement we construct  $n$  covariant constant fields  $\overset{h}{e}_{\lambda}$  in an  $\mathfrak{N}_{\theta}(\xi^{\mu})$ . Because  $\partial_{[\mu} \overset{h}{e}_{\lambda]} = V_{[\mu} \overset{h}{e}_{\lambda]} = 0$ , the  $\overset{h}{e}_{\lambda}$  are gradientfields. If  $\overset{h}{e}_{\lambda} \neq \partial_{\lambda} \xi^h$ , the  $\xi^h$  are independent because their gradients are linearly independent. If we take them as new coordinates we have

$$(4.30) \quad 0 = V_j \overset{h}{e}_i = \partial_j \overset{h}{e}_i - \Gamma_{ji}^h \overset{h}{e}_k = - \Gamma_{ji}^h$$

and this proves that  $(h)$  is a rectilinear coordinate system in an  $E_n$ . The same does not hold for an  $L_n$ . In fact an  $L_n$  can be endowed with teleparallelism without being an  $E_n$ . <sup>3)</sup>

Here is another proof showing the relations of  $R_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda}$  to integrability conditions. We look for a *holonomic* coordinate system  $(h)$  in  $L_n$  such that the  $\Gamma_{ji}^h$  vanish. The differential equations are [cf. (2.5)]

$$(4.31) \quad \left\{ \begin{array}{l} \text{a)} \quad \partial_{\mu} \xi^h = A_{\mu}^h \\ \text{b)} \quad \partial_{\mu} A_{\lambda}^h = A_{\lambda}^h \Gamma_{\mu\lambda}^{\cdot\cdot\cdot} \end{array} \right.$$

<sup>1)</sup> Cf. WEITZENBÖCK 1923, 1; VITALI 1924, 1; GRISS 1925, 1; CARTAN and SCHOUTEN 1926, 3; 4; CARTAN 1927, 2; BORTOLOTTI 1927, 1; 2; 1928, 1; 1929, 1; EINSTEIN 1928, 1; WEITZENBÖCK 1928, 1; SCHOUTEN 1929, 4; E I 1935, 1, p. 88; HAIMOVICI 1943, 1.

<sup>2)</sup> For literature on absolute parallelism in the large, cf. footnote 2 on p. 125.

<sup>3)</sup> It is proved in Chapter IV that every non-trivial group space is such an  $L_n$ .

with the integrability conditions

$$(4.32) \quad \begin{cases} \text{a)} & 0 = \partial_{[\nu} A_{\mu]}^h = A_{\nu}^h S_{\mu}^{;\lambda} \\ \text{b)} & 0 = \partial_{[\nu} \partial_{\mu} A_{\lambda]}^h = (\partial_{[\nu} A_{\mu]}^h) \Gamma_{\mu]}^{\lambda} + A_{\nu}^h \partial_{[\nu} \Gamma_{\mu]}^{\lambda} = \frac{1}{2} A_{\nu}^h R_{\mu}^{;\lambda} \end{cases}$$

These conditions are satisfied identically if and only if  $S_{\mu}^{;\lambda} = 0$  and  $R_{\mu}^{;\lambda} = 0$ .<sup>1)</sup>

If the fundamental tensor in a  $V_n$  is written as a product of two equal ideal factors (cf. I § 11),  $g_{\lambda\alpha} = a_{\lambda} a_{\alpha} = b_{\lambda} b_{\alpha} = \dots$ , we have

$$(4.33) \quad 0 = V_{\mu} g_{\lambda\alpha} = V_{\mu} a_{\lambda} a_{\alpha} = 2(V_{\mu} a_{(\lambda}) a_{\alpha)}.$$

The expression  $(V_{\mu} a_{\lambda}) a_{\alpha}$  has no meaning by itself, only the sum of  $(V_{\mu} a_{\lambda}) a_{\alpha}$  and  $(V_{\mu} a_{\alpha}) a_{\lambda}$  is known to be zero. This enables us to simplify the calculations by taking  $(V_{\mu} a_{\lambda}) a_{\alpha} = 0$ . Then we get for instance

$$(4.34) \quad V_{\mu} v^{\alpha} = V_{\mu} v^{\lambda} a_{\lambda} a^{\alpha} = a^{\alpha} \partial_{\mu} v^{\lambda} a_{\lambda} = \partial_{\mu} v^{\alpha} + v^{\lambda} a^{\alpha} \partial_{\mu} a_{\lambda},$$

hence

$$(4.35) \quad \Gamma_{\mu\lambda}^{\alpha} = a^{\alpha} \partial_{\mu} a_{\lambda} = - a_{\lambda} \partial_{\mu} a^{\alpha}$$

and

$$(4.36) \quad 2V_{[\nu} V_{\mu]} v^{\alpha} = 2V_{[\nu} V_{\mu]} v^{\lambda} a_{\lambda} a^{\alpha} = 2v^{\lambda} a_{\lambda} V_{[\nu} V_{\mu]} a^{\alpha},$$

hence

$$(4.37) \quad K_{\nu\mu\lambda}^{;\alpha} = 2a_{\lambda} V_{[\nu} V_{\mu]} a^{\alpha}.$$

Using this symbolism many calculations can be shortened considerably (cf. I § 11 under 6, footnote 6, p. 58).

### Exercises.

III 4,1<sup>2)</sup>. A man is moving on the surface of the earth always facing one definite point, say Jerusalem or Mekka or the North pole. Prove that this displacement is semi-symmetric and metric and compute  $S_{\lambda}$ .<sup>3)</sup> During the mathematical congress in Moscow in 1934 one evening some of us invented the "Moscow displacement". The streets of Moscow are approximately straight lines through the Kremlin and concentric circles around it. Now let a person walk in the street always facing the Kremlin. Prove that also this displacement is semi-symmetric and metric and compute  $S_{\lambda}$ .

III 4,2. Prove that for a symmetric connexion the  $\lambda$ -rank of  $R_{\nu\mu\lambda}^{;\alpha}$  can not be greater than the  $\nu$ -rank. For a metric and symmetric connexion the rank of  $K_{\nu\mu\lambda}^{;\alpha}$  is the same for all indices.

<sup>1)</sup> RIEMANN 1861, 1, p. 402, LIPSCHITZ 1869, 1, p. 94; 1874, 1, p. 109; RICCI 1884, 1, p. 142, all for  $V_n$ .

<sup>2)</sup> Cf. E I 1935, 1, p. 127, Exerc. 11,9.

<sup>3)</sup> HESSENBERG 1924, 1 gave other examples of teleparallelism on a sphere.

III 4,3. Prove that (cf. 4.20)

$$\text{III 4,3 } \alpha) \quad \frac{1}{2} R_{\nu\mu}^{\cdot\cdot\lambda} - V_{[\nu} T_{\mu]}^{\cdot\lambda} - S_{\nu\mu}^{\cdot\sigma} T_{\sigma}^{\cdot\lambda} = \frac{1}{2} K_{\nu\mu}^{\cdot\cdot\lambda} + \overset{*}{V}_{[\nu} T_{\mu]}^{\cdot\lambda}.$$

III 4,4. If the vector  $v^\lambda$  is displaced parallel along the boundary of the infinitesimal simple bivector  $d\Gamma^\mu$  the difference between the final value and the initial value lies in the plane of  $d\Gamma^\mu$  if and only if

$$\text{III 4,4 } \alpha) \quad (n-1) R_{\nu\mu}^{\cdot\cdot\lambda} = 2 A_{[\nu}^{\lambda} R_{\mu]}^{\cdot\cdot 1}.$$

III 4,5. Prove that in an  $L_n$  teleparallelism for scalar densities exists if and only if  $V_{\nu\mu} = 0$  (cf. III § 6). The connexion is called in that case *volume preserving*<sup>2)</sup>.

## § 5. The identities for the curvature tensor.<sup>3)</sup>

From the definition (4.2) we get the *first identity*

$$(5.1) \quad \boxed{R_{(\nu\mu)\lambda}^{\cdot\cdot\lambda} = 0}$$

and by alternation of (4.2) over  $\nu\mu\lambda$  we get the *second identity*

$$(5.2) \quad \boxed{R_{[\nu\mu]\lambda}^{\cdot\cdot\lambda} = 2 V_{[\nu} S_{\mu]\lambda}^{\cdot\cdot\lambda} - 4 S_{[\nu\mu}^{\cdot\cdot\sigma} S_{\lambda]\sigma}^{\cdot\cdot\lambda}}$$

taking for *semi-symmetric* connexions the form

$$(5.3) \quad R_{[\nu\mu]\lambda}^{\cdot\cdot\lambda} = 2 A_{[\nu}^{\lambda} V_{\mu]} S_{\lambda]}^{\cdot\cdot\lambda} \quad 4)$$

and for the *symmetric* case the form

$$(5.4) \quad \boxed{R_{[\nu\mu]\lambda}^{\cdot\cdot\lambda} = 0}.$$

If we write (cf. 4.15, 23)

$$(5.5) \quad R_{\mu\lambda} \stackrel{\text{def}}{=} R_{\nu\mu}^{\cdot\cdot\nu} \lambda; \quad V_{\nu\mu} \stackrel{\text{def}}{=} R_{\nu\mu}^{\cdot\cdot\lambda} \lambda$$

we get from (5.2) by contraction

$$(5.6) \quad \boxed{2 R_{[\mu\lambda]} = - V_{\mu\lambda} + 2 V_\sigma S_{\mu\lambda}^{\cdot\cdot\sigma} + 2(n-1) V_{[\mu} S_{\lambda]} + 2(n-1) S_{\mu\lambda}^{\cdot\cdot\sigma} S_\sigma}$$

from which for the *semi-symmetric* case

$$(5.7) \quad \boxed{2 R_{[\mu\lambda]} = - V_{\mu\lambda} + 2(n-2) V_{[\mu} S_{\lambda]}}$$

<sup>1)</sup> COSSU 1949, 1.

<sup>2)</sup> VEBLEN 1923, 2; SCHOUTEN 1923, 1; EISENHART 1923, 3. Cf. GOLAB 1931, 1.

<sup>3)</sup> Cf. for complete sets of identities for different cases T. Y. THOMAS 1934, 2; LEVINE 1936, 1; SUN 1937, 1; BLUM 1947, 1.

<sup>4)</sup> The case where (5.3) holds for a not semi-symmetric connexion was investigated by v. d. KULK 1939, 1.

and for the *symmetric* case

$$(5.8) \quad \boxed{2R_{[\mu\lambda]} = -V_{\mu\lambda}}.$$

By differentiation and alternation we get from (3.2) and (4.13)

$$(5.9) \quad V_{[\nu} V_{\mu]} g_{\lambda\kappa} = -R_{\nu\mu(\lambda\kappa)} - S_{\nu\mu}^{\cdot\cdot\varrho} V_{\varrho} g_{\lambda\kappa}$$

from which follows the *third identity*

$$(5.10) \quad \boxed{R_{\nu\mu(\lambda\kappa)} = V_{[\nu} Q_{\mu]\lambda\kappa} + S_{\nu\mu}^{\cdot\cdot\varrho} Q_{\varrho\lambda\kappa}} \text{ } ^1).$$

This identity takes the following forms for special cases

*semi-symmetric connexion*

$$(5.11) \quad R_{\nu\mu(\lambda\kappa)} = V_{[\nu} Q_{\mu]\lambda\kappa} + S_{[\nu} Q_{\mu]\lambda\kappa}$$

*semi-metric connexion*

$$(5.12) \quad R_{\nu\mu(\lambda\kappa)} = (V_{[\nu} Q_{\mu]} + S_{\nu\mu}^{\cdot\cdot\varrho} Q_{\varrho}) g_{\lambda\kappa}$$

*metric connexion*

$$(5.13) \quad \boxed{R_{\nu\mu(\lambda\kappa)} = 0}.$$

Every quantity  $Q_{\nu\mu\lambda\kappa}$ , alternating in  $\nu\mu$  satisfies the identity

$$(5.14) \quad \left\{ \begin{array}{l} Q_{\lambda\kappa\nu\mu} = Q_{\nu\mu\lambda\kappa} - \frac{3}{2}(Q_{[\nu\lambda\kappa]\mu} + Q_{[\kappa\mu\nu]\lambda} + Q_{[\mu\lambda\nu]\kappa} + Q_{[\lambda\kappa\mu]\nu}) + \\ + Q_{\nu\kappa(\mu\lambda)} + Q_{\lambda\nu(\mu\kappa)} + Q_{\kappa\mu(\lambda\nu)} + Q_{\mu\lambda(\kappa\nu)} + Q_{\mu\nu(\kappa\lambda)} + Q_{\lambda\kappa(\mu\nu)}. \end{array} \right.$$

If (5.2) and (5.10) are substituted in (5.14) for  $Q_{\nu\mu\lambda\kappa} = R_{\nu\mu\lambda\kappa}$  we get  $R_{\nu\mu\lambda\kappa} - R_{\lambda\kappa\nu\mu}$  expressed in terms of  $S_{\mu\lambda}^{\cdot\cdot\kappa}$ ,  $V_{[\nu} S_{\mu]\lambda}^{\cdot\cdot\kappa}$ ,  $Q_{\mu\lambda\kappa}$  and  $V_{[\nu} Q_{\mu]\lambda\kappa}$ . This rather complicated formula is the *fourth identity*. For the *metric* and *symmetric* case it takes a very simple form. For this special case (riemannian connexion) we collect the four identities

$$(5.15) \quad \left\{ \begin{array}{ll} \text{I)} & K_{(\nu\mu)\lambda\kappa} = 0 \\ \text{II)} & K_{[\nu\mu\lambda]\kappa} = 0 \\ \text{III)} & K_{\nu\mu(\lambda\kappa)} = 0 \\ \text{IV)} & K_{\nu\mu\lambda\kappa} = K_{\lambda\kappa\nu\mu}. \end{array} \right.$$

From (II, III, IV) it follows that

$$(5.16) \quad \boxed{K_{[\nu\mu\lambda\kappa]} = 0}.$$

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\* 1) CASTOLDI 1947, 1 finds only one term,  $\overset{*}{V}_{[\nu} Q_{\mu]\lambda\kappa}$  at the right hand side, where  $\overset{*}{V}$  is the operator of the riemannian connexion belonging to  $g_{\lambda\kappa}$ . This seems to be a mistake.

It can be proved that (I, II, III, IV) is equivalent to (I, II', III, IV). The identities (I, II', III, IV) can be used to evaluate the number of independent components of  $K_{\nu\mu\lambda\kappa}$ . According to (I, III, IV)  $K_{\nu\mu\lambda\kappa}$  is a symmetric bivector tensor (I § 8 under B) i.e. a symmetric tensor in the  $\binom{n}{2}$ -dimensional space of all bivectors.<sup>1)</sup> Now a symmetric bivector tensor has  $\frac{1}{2} \binom{n}{2} \left\{ \binom{n}{2} + 1 \right\} = \frac{1}{8} (n^2 - n)(n^2 - n + 2)$  components. But (II') stands for  $\binom{n}{4}$  independent relations. Hence the number of independent components is  $\frac{1}{12} n^2(n^2 - 1)$ , that is 1 for  $n = 2$ ; 6 for  $n = 3$  and 20 for  $n = 4$  (the case of relativity).

From (5.15) it follows that  $V_{\nu\mu}$  is always zero in a  $V_n$  and that  $K_{\mu\lambda}$  is always symmetric there.  $K_{\mu\lambda}$  is the Ricci tensor of the  $V_n$ . The principal directions of  $K_{\mu\lambda}$  are called the *principal directions of the  $V_n$* .<sup>2)</sup>

The equations (4.2)

$$(5.17) \quad \partial_{[\nu} \Gamma_{\mu]\lambda}^\kappa = - \Gamma_{[\nu|\varrho|}^\kappa \Gamma_{\mu]\lambda}^\varrho + \frac{1}{2} R_{\nu\mu|\lambda}^\kappa$$

have the form (II 6.15) with a right hand member that depends on the  $\Gamma_{\mu\lambda}^\kappa$ . As we have seen in II § 6 the construction of the integrability conditions depends on the possibility of being able to eliminate the derivatives of the  $\Gamma_{\mu\lambda}^\kappa$  at each step. Now for (5.17) this possibility exists and we get for the first set of integrability conditions

$$(5.18) \quad \left\{ \begin{array}{l} 0 = \Gamma_{[\omega|\sigma|}^\kappa \Gamma_{\nu|\varrho|}^\sigma \Gamma_{\mu]\lambda}^\varrho - \frac{1}{2} R_{[\nu|\varrho|}^\kappa \Gamma_{\mu]\lambda}^\varrho + \Gamma_{[\nu|\varrho|}^\kappa \Gamma_{\omega|\sigma|}^\varrho \Gamma_{\mu]\lambda}^\sigma - \\ \qquad - \frac{1}{2} \Gamma_{[\nu|\varrho|}^\kappa R_{\omega\mu|\lambda}^{\dot{\nu}\dot{\mu}\dot{\lambda}} + \frac{1}{2} \partial_{[\omega} R_{\nu|\mu]\lambda}^{\dot{\nu}\dot{\mu}\dot{\lambda}} \\ \qquad = \frac{1}{2} V_{[\omega} R_{\nu|\mu]\lambda}^{\dot{\nu}\dot{\mu}\dot{\lambda}} + S_{[\omega|\nu|}^\sigma R_{|\mu]\sigma|\lambda}^{\dot{\nu}\dot{\mu}\dot{\lambda}}. \end{array} \right.$$

If  $\partial_\varrho$  is applied to the right hand side and if we alternate over  $\varrho\omega\nu\mu$  the expression is identically zero.<sup>3)</sup> Hence the only integrability conditions are

$$(5.19) \quad \boxed{V_{[\omega} R_{\nu|\mu]\lambda}^{\dot{\nu}\dot{\mu}\dot{\lambda}} = 2 S_{[\omega|\nu|}^\sigma R_{|\mu]\sigma|\lambda}^{\dot{\nu}\dot{\mu}\dot{\lambda}}}$$

known as BIANCHI's *identity*.<sup>4)</sup> The identity can be proved in many other ways, for instance by computing  $V_{[\omega} V_\nu V_{\mu]} w_\lambda$  from  $V_{[\omega} V_\nu] V_\mu w_\lambda$

<sup>1)</sup> The geometric properties of this quantity were for  $n=4$  investigated by CHURCHILL 1932, 1; RUSE 1936, 1; 1944, 1; 2; 1946, 1; 2; 1948, 1; LANZOS 1938, 1; cf. WRONA 1948, 1 for general values of  $n$ .

<sup>2)</sup> RICCI 1903, 1; 1904, 1. Cf. EISENHART 1922, 1. Other principal directions were defined by SYNGE 1922, 1.

<sup>3)</sup> This result will be obtained later in a much shorter way.

<sup>4)</sup> An identity of this kind was first considered by VOSS 1880, 1. Then RICCI found it independently and sent it to PADOVA, who published it in 1889, 1. In 1902, 1 it was proved independently by BIANCHI. Therefore italien mathematiciens

and from  $\nabla_\omega \nabla_{[\nu} \nabla_{\mu]} w_\lambda$ . For a *semi-symmetric* connexion it takes the form

$$(5.20) \quad \boxed{\nabla_{[\omega} R_{\nu\mu]\lambda}^{\cdot\cdot\cdot\cdot} = 2S_{[\omega} R_{\nu\mu]\lambda}^{\cdot\cdot\cdot\cdot}}$$

and for a *symmetric* connexion we have

$$(5.21) \quad \boxed{\nabla_{[\omega} R_{\nu\mu]\lambda}^{\cdot\cdot\cdot\cdot} = 0} \quad [1), 2).$$

By contraction of (5.19) over  $\omega\lambda$  we get for the general case:

$$(5.22) \quad \text{a)} \quad \boxed{\nabla_{[\omega} V_{\nu\mu]} = 2S_{[\omega}^{\cdot\cdot\cdot\cdot} V_{\mu]\rho}^{\cdot\cdot\cdot\cdot}} ; \quad (\text{equivalent to b)} \quad \partial_{[\omega} V_{\nu\mu]} = 0,$$

for the *semi-symmetric* case:

$$(5.23) \quad \boxed{\nabla_{[\omega} V_{\nu\mu]} = 2S_{[\omega}^{\cdot\cdot\cdot\cdot} V_{\mu]\nu}^{\cdot\cdot\cdot\cdot}}$$

and for the *symmetric* case:

$$(5.24) \quad \boxed{\nabla_{[\omega} V_{\nu\mu]} = 0} .$$

By contraction of (5.19) over  $\omega\alpha$  we get for the general case:

$$(5.25) \quad \boxed{\nabla_\alpha R_{\nu\mu}\lambda^{\cdot\cdot\cdot\cdot} - 2\nabla_{[\nu} R_{\mu]\lambda}^{\cdot\cdot\cdot\cdot} = 4S_\alpha^{\cdot\cdot\cdot\cdot} R_{\mu]\rho}\lambda^{\cdot\cdot\cdot\cdot} + 2S_\nu^{\cdot\cdot\cdot\cdot} R_{\rho\lambda}^{\cdot\cdot\cdot\cdot}} ;$$

for the *semi-symmetric* case:

$$(5.26) \quad \boxed{\nabla_\alpha R_{\nu\mu}\lambda^{\cdot\cdot\cdot\cdot} - 2\nabla_{[\nu} R_{\mu]\lambda}^{\cdot\cdot\cdot\cdot} = 4S_\alpha^{\cdot\cdot\cdot\cdot} R_{\mu]\lambda}^{\cdot\cdot\cdot\cdot} + 2S_{[\nu} R_{\mu]\lambda}^{\cdot\cdot\cdot\cdot}}$$

and for the *symmetric* case:

$$(5.27) \quad \boxed{\nabla_\alpha R_{\nu\mu}\lambda^{\cdot\cdot\cdot\cdot} - 2\nabla_{[\nu} R_{\mu]\lambda}^{\cdot\cdot\cdot\cdot} = 0} .$$

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often speak of the identity of RICCI-BIANCHI. All these publications consider the metric symmetric case only. BACH gave 1921, 1 the proof for a WEYL connexion and the author at the Jena congress 1921 did the same for a general symmetric connexion, cf. 1923, 2. In 1922, 1 VEBLEN dealt with the symmetric case and in 1923, 1 WEITZENBÖCK published the formula (5.19). Cf. SCHOUTEN-STUIK 1923, 3. As a more recent publication we mention LEVINE 1936, 1. WAGNER gave 1945, 2 a generalization for general geometric objects.

1) For an  $A_n$  the second identity and BIANCHI's identity can be proved in a very short way by using normal coordinates (cf. III § 7, Exerc. III 7,4). Another method is used in III § 9.

2) BLUM proved 1946, 1 that BIANCHI's identity in a  $V_n$  consists of  $\frac{1}{24} n^2(n^2 - 1) \times (n - 2)$  independent equations. Cf. BLUM 1947, 1 for complete sets of identities

For a riemannian connexion we get from (5.27) by transvection with  $g^{\mu\lambda}$

$$(5.28) \quad V_\nu K_\mu^\nu - V_\nu K + V_\mu K_\nu^\mu = 0; \quad K \stackrel{\text{def}}{=} K_{\mu\lambda} g^{\mu\lambda}$$

or

$$(5.29) \quad \boxed{V_\mu G_{\nu\lambda}^\mu = 0; \quad G_{\nu\lambda} \stackrel{\text{def}}{=} K_{\nu\lambda} - \frac{1}{2} K g_{\nu\lambda}} \quad ^1).$$

The scalar  $\kappa \stackrel{\text{def}}{=} \frac{1}{n(n-1)} K$  is called the *scalar curvature*<sup>2)</sup> of the  $V_n$ . For  $n=2$  we have  $G_{\nu\lambda} = 0$ , and  $\kappa$  is equal to the *gaussian curvature*. For  $n=4$ ,  $G_{\nu\lambda}$  is the tensor of momentum and energy in general relativity and (5.29) is the well known equation of conservation of momentum and energy.

An  $L_n$  in which  $V_\nu R_{\mu\lambda} = 0$  is called a *Ricci space*. If  $R_{\mu\lambda} = 0$  it is said to be a *special Ricci space*. A  $V_n$ ,  $n > 1$  in which  $K_{\mu\lambda}$  differs from  $g_{\mu\lambda}$  only by a scalar factor is called an *Einstein space*.<sup>3)</sup> If  $K_{\mu\lambda} = 0$  it is said to be a *special Einstein space*. According to the definition of  $K$  and  $\kappa$  we have in this case

$$(5.30) \quad K_{\mu\lambda} = \frac{1}{n} K g_{\mu\lambda} = (n-1) \kappa g_{\mu\lambda}.$$

For  $n > 2$  it follows from (5.29) that the scalar curvature of an *Einstein* space is constant. But a  $V_n$  with a constant scalar curvature need not be an *Einstein* space.<sup>4)</sup> This is a theorem of HERGLOTZ.<sup>5)</sup> An *Einstein* space is for  $n > 2$  a special case of a *Ricci* space.<sup>6)</sup>

A  $V_n$ ,  $n > 2$  in which  $K_{\nu\mu\lambda\zeta}$  differs from  $g_{[\nu[\lambda} g_{\mu]\zeta]}$  only by a scalar factor is called an  $S_n$  or a *space of constant curvature*. In this case we have

$$(5.31) \quad K_{\nu\mu\lambda\zeta} = \frac{-2}{n(n-1)} K g_{[\nu[\lambda} g_{\mu]\zeta]} = -2\kappa g_{[\nu[\lambda} g_{\mu]\zeta]}.$$

For  $n=2$  an  $S_2$  is defined as a  $V_2$  in which  $\kappa$  is constant.

<sup>1)</sup> BIANCHI 1902, 1; RICCI 1903, 1; for  $n=3$ .

<sup>2)</sup> WEYL 1921, 2 used the term "scalar curvature" in another sense. He called an  $S_n$  a "space of scalar curvature". Cf. footnote 4 and VI p. 289, footnote 1.

<sup>3)</sup> Cf. E I 1935, 1, p. 125; FIALKOW 1938, 1; 1939, 1; 1942, 1; T. Y. THOMAS 1938, 1 (invariants); HAANTJES and WRONA 1939, 1 (and  $C_n$ ); YANO 1943, 1 (conf. and concirc); WRONA 1947, 1 ( $C_n$ ); KUIPER 1950, 1; 2 (conf. and proj.); 1951, 1 (conf.); TACHIBANA 1951, 1 (parallel fields).

<sup>4)</sup> The opposite assertion in LOVELL 1934, 1 is erroneous and perhaps caused by a wrong interpretation of WEYL's term "scalar curvature". Cf. footnote 2 and footnote 1 Ch. VI, p. 289.

<sup>5)</sup> HERGLOTZ 1916, 1, p. 203. Cf. for generalizations WRONA 1941, 1; BOMPIANI 1950, 1; VARGA 1943, 1.

<sup>6)</sup> There are many other examples of spaces whose curvature tensor satisfies special identities. We mention here only a paper of v. D. KULK 1939, 1 where the case  $(n+1) R_{\nu(\mu\dot{\lambda})}^{\cdot\cdot\cdot} = (V_{\nu(\mu} - R_{\nu(\mu}) A_{\lambda)}^{\cdot\cdot\cdot}$  and especially  $R_{\nu(\mu\dot{\lambda})}^{\cdot\cdot\cdot} = 0$  or  $R_{\nu\mu\dot{\lambda}}^{\cdot\cdot\cdot} = R_{[\nu\mu\dot{\lambda}}^{\cdot\cdot\cdot}$  was investigated.

An  $S_n$  is a special case of an EINSTEIN space.

The second identity and the identity of BIANCHI can be put into another form by introducing another invariant differential operator.

Let  $P_{\lambda_1 \dots \lambda_p A}$  be a tensor alternating in the first  $p$  indices and let  $A$  be a collecting index standing for any number of co- and contravariant indices. Then we write

$$(5.32) \quad \overset{p}{V}_{[\mu} P_{\lambda_1 \dots \lambda_p] A} \stackrel{\text{def}}{=} \overset{p}{V}_{[\mu} P_{\lambda_1 \dots \lambda_p] A} + p S_{[\mu} \overset{p}{\lambda}_1 \dots \overset{p}{\lambda}_p] A.$$

As we see the significance of  $\overset{p}{V}_{[\mu \dots]}$  is that the alternating part  $S_{\mu \lambda}^{\nu \zeta}$  of  $\Gamma_{\mu \lambda}^{\nu}$  does not apply to the *alternated* indices  $\lambda_1 \dots \lambda_p$ . In an  $A_n$ , the only difference between  $\overset{p}{V}_{\mu}$  and  $\overset{0}{V}_{\mu}$  is that  $\overset{p}{V}_{\mu}$  can be used only if we alternate over the first  $p$  indices. In  $\overset{0}{V}_{\mu}$  there is no alternation, hence  $\overset{0}{V}_{\mu}$  is identical with  $\overset{p}{V}_{\mu}$ . For this new operator the rule of LEIBNIZ holds in the form

$$(5.33) \quad \left\{ \begin{array}{l} \overset{p+q}{V}_{[\mu} P_{\lambda_1 \dots \lambda_p | A]} Q_{\kappa_1 \dots \kappa_q] B} \\ = (\overset{p}{V}_{[\mu} P_{\lambda_1 \dots \lambda_p | A]}) Q_{\kappa_1 \dots \kappa_q] B} + P_{[\lambda_1 \dots \lambda_p | A]} \overset{q}{V}_{\mu} Q_{\kappa_1 \dots \kappa_q] B} \end{array} \right.$$

and for the operator  $\overset{p+1}{V}_{[\nu} \overset{p}{V}_{\mu \dots]}$  the rule of LEIBNIZ takes the form

$$(5.34) \quad \left\{ \begin{array}{l} \overset{p+1}{V}_{[\nu} \overset{p}{V}_{\mu} P_{\lambda_1 \dots \lambda_s | A]} Q_{\lambda_{s+1} \dots \lambda_p] B} \\ = (\overset{s+1}{V}_{[\nu} \overset{s}{V}_{\mu} P_{\lambda_1 \dots \lambda_s | A]}) Q_{\lambda_{s+1} \dots \lambda_p] B} + P_{[\lambda_1 \dots \lambda_s | A]} \overset{p-s+1}{V}_{\nu} \overset{p-s}{V}_{\mu} Q_{\lambda_{s+1} \dots \lambda_p] B}. \end{array} \right.$$

These identities and the identities

$$(5.35) \quad \boxed{\begin{aligned} \overset{1}{V}_{[\nu} \overset{0}{V}_{\mu]} v^{\lambda} &= \frac{1}{2} R_{\nu \mu}^{\lambda \kappa} v^{\lambda} \\ \overset{1}{V}_{[\nu} \overset{0}{V}_{\mu]} w_{\lambda} &= -\frac{1}{2} R_{\nu \mu}^{\lambda \kappa} w_{\lambda} \end{aligned}}$$

that are valid in  $L_n$  (not only in  $A_n$ ) and simplify the equations (4.9), are easily verified and from them and (5.32) we get the identity

$$(5.36) \quad \left\{ \begin{array}{l} \overset{p+1}{V}_{[\nu} \overset{p}{V}_{\mu} P_{\lambda_1 \dots \lambda_p] \lambda}^{\lambda \dots \tau \dots} \\ = \frac{1}{2} R_{[\nu \mu}^{\lambda \rho} P_{\lambda_1 \dots \lambda_p] \rho}^{\lambda \dots \tau \dots} + \dots - \frac{1}{2} R_{[\nu \mu}^{\lambda \sigma} P_{\lambda_1 \dots \lambda_p] \sigma}^{\lambda \dots \tau \dots} - \dots \end{array} \right.$$

This is a simplification and generalization of (4.13).<sup>1)</sup>

<sup>1)</sup> The new operator was introduced by CARTAN. The general idea was in the first edition 1928, 1, p. 211 and the operator itself, symbolized by D, appeared in the second edition 1946, 1, p. 209.

In order to find the geometrical meaning of the new operator we consider an  $X_{p+1}$  through  $\xi^\alpha$  and an *infinitesimal* (properly chosen) closed  $X_p$  in this  $X_{p+1}$  surrounding  $\xi^\alpha$ . Taking for simplicity the field  $P_{\lambda_1 \dots \lambda_p}^* \dot{\lambda}_{p+1}^\alpha$ , the field value at a point  $\xi^\alpha$  of  $X_p$  is

$$(5.37) \quad P_{\lambda_1 \dots \lambda_p}^* \dot{\lambda}_{p+1}^\alpha + (\xi^\mu - \xi^\alpha) \partial_\mu P_{\lambda_1 \dots \lambda_p}^* \dot{\lambda}_{p+1}^\alpha.$$

If  $d f^{\lambda_1 \dots \lambda_p}$  is the infinitesimal volume element of the  $X_p$  with a screw-sense that is the same for all points, then

$$(5.38) \quad d f^{\lambda_1 \dots \lambda_p} \{ P_{\lambda_1 \dots \lambda_p}^* \dot{\lambda}_{p+1}^\alpha + (\xi^\mu - \xi^\alpha) \partial_\mu P_{\lambda_1 \dots \lambda_p}^* \dot{\lambda}_{p+1}^\alpha \}$$

is a well defined tensor at  $\xi^\alpha$ . But the formation of the integral over  $X_p$  of this expression has no geometrical sense because addition of tensors at different points is not an invariant process. Now if we displace each of them parallel to  $\xi^\alpha$  we get

$$(5.39) \quad \left\{ \begin{aligned} & d f^{\lambda_1 \dots \lambda_p} \{ P_{\lambda_1 \dots \lambda_p}^* \dot{\lambda}_{p+1}^\alpha + (\xi^\mu - \xi^\alpha) \partial_\mu P_{\lambda_1 \dots \lambda_p}^* \dot{\lambda}_{p+1}^\alpha + \\ & + (\xi^\mu - \xi^\alpha) \Gamma_{\mu \rho}^\alpha P_{\lambda_1 \dots \lambda_p}^* \dot{\lambda}_{p+1}^\rho - (\xi^\mu - \xi^\alpha) \Gamma_{\mu \tau}^\sigma P_{\lambda_1 \dots \lambda_p}^* \dot{\lambda}_{p+1}^\tau \} \end{aligned} \right.$$

and since these tensors can now be added we can form the integral over  $X_p$ . Then, using the theorem of STOKES we can form the integral over the part  $\tau_{p+1}$  of  $X_{p+1}$  bordered by  $X_p$ . Neglecting terms containing  $\xi^\alpha - \xi^\alpha$  in the result, we get

$$(5.40) \quad \left\{ \begin{aligned} & \int_{\tau_{p+1}} \{ \partial_{[\nu} P_{\lambda_1 \dots \lambda_p]}^* \dot{\lambda}_{p+1}^\nu + \Gamma_{[\nu | \rho]}^\alpha P_{\lambda_1 \dots \lambda_p}^* \dot{\lambda}_{p+1}^\rho - \Gamma_{[\nu | \tau]}^\sigma P_{\lambda_1 \dots \lambda_p}^* \dot{\lambda}_{p+1}^\tau \} d f^{\lambda_1 \dots \lambda_p} \\ & = \int_{\tau_{p+1}}^p \bar{V}_{[\nu} P_{\lambda_1 \dots \lambda_p]}^* \dot{\lambda}_{p+1}^\nu d f^{\lambda_1 \dots \lambda_p} \end{aligned} \right.$$

if  $d f^{\lambda_1 \dots \lambda_p}$  is the infinitesimal volume element of  $\tau_{p+1}$ . This result also holds *mutatis mutandus* if there are more contravariant and covariant indices  $\alpha$  and  $\tau$ . The integral is zero for every choice of the  $(p+1)$ -direction of  $X_{p+1}$  at  $\xi^\alpha$  if and only if  $\bar{V}_{[\nu} P_{\lambda_1 \dots \lambda_p]}^* \dot{\lambda}_{p+1}^\nu = 0$  at  $\xi^\alpha$ . In that case we say that the field  $P_{\lambda_1 \dots \lambda_p}^* \dot{\lambda}_{p+1}^\alpha$  is *in equilibrium* at  $\xi^\alpha$ .

If we compare (5.32) with (5.2) and (5.19) we see that these equations can now be written in the simple form

$$(5.41) \quad R_{[\nu \mu] \lambda}^* \dot{\lambda}_{p+1}^\alpha = 2 \bar{V}_{[\nu} S_{\mu] \lambda}^* \dot{\lambda}_{p+1}^\alpha \quad (\text{second identity})$$

$$(5.42) \quad \bar{V}_{[\omega} R_{\nu \mu] \lambda}^* \dot{\lambda}_{p+1}^\alpha = 0 \quad (\text{identity of BIANCHI}).$$

Hence BIANCHI's identity expresses the fact that the curvature tensor is in equilibrium at each point of  $L_n$ .

In order to obtain the geometrical meaning of (5.41, 42) we have to make use of CARTAN displacements. We choose an  $X_3$  through  $\xi^*$  and in this  $X_3$  we choose a closed  $X_2$  surrounding  $\xi^*$ . Let  $v^*$  be a vector at  $\xi^*$  and let  $\Gamma_{\mu\lambda}^*$ ,  $R_{\nu\mu\lambda}^*$  and  $S_{\mu\lambda}^*$  be field values at  $\xi^*$ . Let  $df^{\mu}$  be the bivector of an infinitesimal surface-element of  $X_2$  whose bordering  $X_1$  goes through a point  $\xi^*$  of  $X_2$ . The point with radiusvector  $v^*$  at  $\xi^*$  now undergoes a CARTAN displacement from  $\xi^*$  to  $\xi^*$ , along the border of  $df^{\mu}$  in the sense of  $df^{\mu}$  back to  $\xi^*$  and finally from  $\xi^*$  to  $\xi^*$ . From  $\xi^*$  to  $\xi^*$  the radiusvector changes into

$$(5.43) \quad v^* - (\xi^* - \xi^*) - (\xi^* - \xi^*) \Gamma_{\omega\lambda}^* v^\lambda.$$

At  $\xi^*$  we have the field values  $S_{\mu\lambda}^* + (\xi^* - \xi^*) \partial_\omega S_{\mu\lambda}^*$  and  $R_{\nu\mu\lambda}^* + (\xi^* - \xi^*) \partial_\omega R_{\nu\mu\lambda}^*$ . Hence the change of the moving point due to the displacement around  $df^{\mu}$  is given by the vector (cf. 4.7)

$$(5.44) \quad \left\{ \begin{aligned} & -S_{\nu\mu}^* - (\xi^* - \xi^*) \partial_\omega S_{\nu\mu}^* - \frac{1}{2} R_{\nu\mu\lambda}^* v^\lambda + \frac{1}{2} R_{\nu\mu\omega}^* (\xi^* - \xi^*) + \\ & + \frac{1}{2} R_{\nu\mu\lambda}^* (\xi^* - \xi^*) \Gamma_{\omega\lambda}^* v^\omega - \frac{1}{2} (\xi^* - \xi^*) \partial_\omega R_{\nu\mu\lambda}^* v^\lambda \end{aligned} \right\} df^{\mu}.$$

For the integration of this vector over  $X_2$  we must first displace it parallel to  $\xi^*$  in order to make addition possible. Then at  $\xi^*$  we get the vector

$$(5.45) \quad \left\{ \begin{aligned} & -S_{\nu\mu}^* - \frac{1}{2} R_{\nu\mu\lambda}^* v^\lambda + (\xi^* - \xi^*) (-S_{\nu\mu}^* \Gamma_{\omega\lambda}^* - \frac{1}{2} R_{\nu\mu\lambda}^* v^\lambda \Gamma_{\omega\lambda}^* - \\ & - \partial_\omega S_{\nu\mu}^* + \frac{1}{2} R_{\nu\mu\lambda}^* + \frac{1}{2} R_{\nu\mu\sigma}^* \Gamma_{\omega\lambda}^* v^\lambda - \frac{1}{2} v^\lambda \partial_\omega R_{\nu\mu\lambda}^*) \end{aligned} \right\} df^{\mu}.$$

According to the theorem of STOKES the integral of this vector over  $X_2$  is equal to

$$(5.46) \quad \left\{ \begin{aligned} & \int_{\tau_3} \left\{ -S_{[\nu\mu}^* \Gamma_{\omega]\lambda}^* - \frac{1}{2} R_{[\nu\mu|\lambda]}^* v^\lambda \Gamma_{\omega]\lambda}^* - \partial_{[\omega} S_{\nu\mu]}^* + \right. \\ & \quad \left. + \frac{1}{2} R_{[\omega\nu\lambda]}^* + \frac{1}{2} R_{[\nu\mu|\sigma]}^* \Gamma_{\omega]\sigma}^* v^\sigma - \frac{1}{2} v^\lambda \partial_{[\omega} R_{\nu\mu]\lambda}^* \right\} df^{\omega\mu} \\ & = \int_{\tau_3} \left\{ -\bar{V}_{[\omega} S_{\nu\mu]}^* + \frac{1}{2} R_{[\omega\nu\mu]}^* - \frac{1}{2} v^\lambda \bar{V}_{[\omega} R_{\nu\mu]\lambda}^* \right\}_{\xi^*=\xi^*} df^{\omega\mu}, \end{aligned} \right.$$

and this integral is zero in consequence of the second identity (5.41) and BIANCHI's identity (5.42). Hence these two identities together express the following geometric fact:

If an infinitesimal closed  $X_2$  in an  $X_3$  through  $\xi^*$  in  $X_n$  surrounds  $\xi^*$  and if a point with radiusvector  $v^*$  in the local  $E_n$  of  $\xi^*$  undergoes a CARTAN displacement from  $\xi^*$  to a point  $\xi^*$  of  $X_2$ , after that along the border of a surfaceelement  $df^\mu$  of  $X_2$  in the sense of  $df^\mu$ , and finally back from  $\xi^*$  to  $\xi^*$ , the difference between the end value and the beginning value at  $\xi^*$  be called the CARTAN-deviation of the point  $v^*$  with respect to the surface-element  $df^\mu$ . The integral of the CARTAN-deviation over  $X_2$  is zero for every (proper) choice of  $X_3$  and  $X_2$ , to within infinitesimal quantities of an order higher than 1 in  $\xi^* - \xi^*$ .

There is an identity connecting the curvature tensor with the LIE derivative of  $\Gamma_{\mu\lambda}^*$

$$(5.47) \quad \boxed{\frac{\mathfrak{L}}{v} \Gamma_{\mu\lambda}^* = V_\mu v_{\lambda}^{*\mu} + v^\sigma R_{\sigma\mu\lambda}^{*\mu} ; \quad v_{\lambda}^{*\mu} \stackrel{\text{def}}{=} V_\lambda v^\mu - 2S_{\lambda\varrho}^{*\mu} v^\varrho}$$

that can be derived immediately from (II 10.34). The quantity  $v_{\lambda}^{*\mu}$  is a covariant derivative of  $v^\mu$  formed after changing the order of the lower indices of  $\Gamma_{\mu\lambda}^*$ . It frequently occurs in problems of deformation. In the general formula for the operation of  $\frac{\mathfrak{L}}{v}$  on a quantity in  $A_n$ , for instance (II 10.29), it is allowed to replace  $\partial_\mu$  by  $V_\mu$  in all terms at the same time. This is clear because in an  $A_n$  the coordinates can always be chosen in such a way that  $\Gamma_{\mu\lambda}^* = 0$  at some given point (cf. III § 7). But in an  $L_n$  the same rule holds provided that  $V_\mu v^\mu$  is changed into  $v_{\mu}^{*\mu}$ . For instance

$$(5.48) \quad \begin{cases} \text{a) } \frac{\mathfrak{L}}{v} u^\mu = v^\mu \partial_\mu u^\mu - u^\mu \partial_\mu v^\mu = v^\mu V_\mu u^\mu - v^\mu \Gamma_{\mu\lambda}^* u^\lambda - u^\mu \partial_\mu v^\mu = v^\mu V_\mu u^\mu - u^\mu v_{\mu}^{*\mu} \\ \text{b) } \frac{\mathfrak{L}}{v} w_\lambda = v^\mu \partial_\mu w_\lambda + w_\mu \partial_\lambda v^\mu = v^\mu V_\mu w_\lambda + w_\mu v_{\lambda}^{*\mu}. \end{cases}$$

Now, making use of the commutability of  $\frac{\mathfrak{L}}{v}$  and  $\partial_\mu$  (cf. II § 10) we get for  $\frac{\mathfrak{L}}{v} V_\mu u^\mu$

$$(5.49) \quad \frac{\mathfrak{L}}{v} V_\mu u^\mu = \frac{\mathfrak{L}}{v} \partial_\mu u^\mu + \frac{\mathfrak{L}}{v} \Gamma_{\mu\lambda}^* u^\lambda = \partial_\mu \frac{\mathfrak{L}}{v} u^\mu + \Gamma_{\mu\lambda}^* \frac{\mathfrak{L}}{v} u^\lambda + u^\lambda \frac{\mathfrak{L}}{v} \Gamma_{\mu\lambda}^*,$$

hence

$$(5.50) \quad \boxed{\frac{\mathfrak{L}}{v} V_\mu u^\mu - V_\mu \frac{\mathfrak{L}}{v} u^\mu = u^\lambda \frac{\mathfrak{L}}{v} \Gamma_{\mu\lambda}^*}$$

is the commutation formula for  $\frac{\mathfrak{L}}{v}$  and  $V_\mu$ . If this formula is written out we get

$$(5.51) \quad \left\{ \begin{array}{l} v^\varrho V_\varrho V_\mu u^\mu + (V_\varrho u^\mu) v_{\mu}^{*\varrho} - (V_\mu u^\varrho) v_{\varrho}^{*\mu} \\ = (V_\mu v^\varrho) V_\varrho u^\mu + v^\varrho V_\mu V_\varrho u^\mu - (V_\mu u^\varrho) v_{\varrho}^{*\mu} - u^\varrho V_\mu v_{\varrho}^{*\mu} + u^\lambda \frac{\mathfrak{L}}{v} \Gamma_{\mu\lambda}^* \end{array} \right.$$

from which (5.47) follows, according to (4.9.b).

### Exercises.

III 5,1. Prove (5.2) by writing out  $\nabla_{[\nu} \nabla_\mu \nabla_{\lambda]} \phi$  in two ways.

III 5,2. If a vector  $v^\kappa$  in  $V_2$  is displaced parallel along a closed curve, the angle between the beginning position and the end position is equal to the surface integral of  $\kappa$  over the area bounded by the curve.

III 5,3. In a  $V_n$  the expression  $-\frac{1}{4} K_{\nu\mu\lambda\kappa} f^{\mu\nu} f^{\lambda\kappa}$ , where  $f^{\mu\nu}$  is a simple bivector with the area 1, is called the *riemannian curvature* of the  $V_n$  with respect to the 2-direction of  $f^{\mu\nu}$  at the point considered. For  $n=2$  it is equal to  $\kappa$ . The principal directions of the symmetric tensor  $K_{\mu\lambda}$  are the principal directions of the  $V_n$ . If they are determined uniquely and if  $(h)$  is an orthogonal system whose basis vectors are the unitvectors  $i_1^\kappa, \dots, i_n^\kappa$  in these directions, prove that  $K_{11}$  is the sum of the riemannian curvatures with respect to  $n-1$  arbitrary mutually perpendicular 2-directions through the direction of  $i_1^\kappa$  and that  $-G_{11}$  is the sum of  $\binom{n-1}{2}$  arbitrary mutually and to the direction of  $i_1^\kappa$  perpendicular 2-directions.<sup>1)</sup>

III 5,4. (Theorem of SCHUR.) A  $V_n$  is an  $S_n$  if and only if at every point the riemannian curvature with respect to a 2-direction is independent of the choice of this 2-direction.<sup>2)</sup>

III 5,5. A  $V_n$  is an  $S_n$  if and only if the components  $K_{kjih}$  with three different indices with respect to every local cartesian coordinate system at every point vanish.<sup>3)</sup>

III 5,6. Prove the identity for  $V_n$ <sup>4)</sup>

$$\text{III 5,6 } \alpha) \quad \nabla_{[\varrho\sigma]} K_{\nu\mu\lambda\kappa} + \nabla_{[\nu\mu]} K_{\lambda\kappa\varrho\sigma} + \nabla_{[\lambda\kappa]} K_{\varrho\sigma\nu\mu} = 0 \quad \nabla_{[\varrho\sigma]} \stackrel{\text{def}}{=} \nabla_{[\varrho} \nabla_{\sigma]}.$$

III 5,7. Prove (5.41) by applying  $\overset{2}{V}$  to the identity

$$\text{III 5,7 } \alpha) \quad \overset{1}{\nabla}_{[\mu} A_{\lambda]}^\kappa = S_{\mu\lambda}^\kappa.$$

Check (III 5,7  $\alpha$ ). Prove (5.42) by computing  $\overset{2}{\nabla}_{[\omega} \overset{1}{\nabla}_{\nu} \nabla_{\mu]} w_\lambda$  in two ways.

### § 6. Integrability conditions in $L_n$ .

In II § 6 we have developed integrability conditions for sets of partial differential equations in  $X_n$ . In an  $L_n$  differential equations

<sup>1)</sup> HERGLOTZ 1916, 1. Cf. E I 1935, 1, p. 117.

<sup>2)</sup> SCHUR 1886, 1; cf. E I 1935, 1, p. 126; WRONA 1941, 1 and 1947, 1, also for generalizations: EISENHART 1949, 1, p. 83. Cf. footnote 5 on p. 148; BOMPIANI 1950, 1.

<sup>3)</sup> R. K. 1924, 1, p. 171; E I 1935, 1, p. 126.

<sup>4)</sup> Cf. WALKER 1950, 1 and (VII 4.45).

occur containing the covariant instead of the ordinary differentiation. We consider equations of the first order formed by covariant differentiation and alternation. A very general form is

$$(6.1) \quad \overset{\rho}{V}_{[\nu} P_{\mu_1 \dots \mu_p]}^{\kappa} \dot{\lambda}^{..} = Q_{\nu \mu_1 \dots \mu_p}^{\kappa} \dot{\lambda}^{..}$$

where  $Q$  depends on the  $\xi^\kappa$  only and not on the components of  $P$ . If this equation is written out, on the left hand side we get  $\partial_{[\nu} P_{\mu_1 \dots \mu_p]}^{\kappa} \dot{\lambda}^{..}$  and on the right hand side we get terms which are linear in the components of  $P$  and contain a factor  $\Gamma_{\mu_1}^\kappa$ . But these are equations of the form (II 6.15); the only difference being that the right hand side is not independent of the components of  $P$ . As we have seen in Chapter II the integrability conditions can only be found easily if the derivatives of  $P$  arising at every step can be eliminated by means of (6.1). Now this is just the case here, hence, proceeding in this way we must get at the first step equations containing first derivatives of  $Q$  and no derivatives of  $P$ . The equations are obtained by applying  $\partial_\omega$  to (6.1) and alternating over  $\omega \nu \mu_1 \dots \mu_p$ . But instead of  $\partial_\omega$  we can just as well use  $\overset{\rho+1}{V}_\omega$  because the difference vanishes by virtue of (6.1). But if we use this operator we have nothing to compute because the result is already known from the preceding section (5.36):

$$(6.2) \quad \overset{\rho+1}{V}_{[\omega} Q_{\nu \mu_1 \dots \mu_p]}^{\kappa} \dot{\lambda}^{..} = \frac{1}{2} R_{[\omega \nu | \rho}^{\kappa} P_{\mu_1 \dots \mu_p]}^{\sigma} \dot{\lambda}^\rho + \dots - \frac{1}{2} R_{[\omega \nu | \lambda]}^{\kappa} P_{\mu_1 \dots \mu_p]}^{\sigma} \dot{\lambda}^\lambda - \dots$$

For  $\rho = 0$  we get

$$(6.3) \quad \overset{\theta}{V}_{[\nu} P_{\mu_1 \dots \mu_p]}^{\kappa} \dot{\lambda}^{..} = V_\nu P_{\mu_1 \dots \mu_p]}^{\kappa} \dot{\lambda}^{..} = Q_{\nu \mu_1 \dots \mu_p}^{\kappa} \dot{\lambda}^{..}$$

with the first integrability conditions

$$(6.4) \quad \overset{1}{V}_{[\omega} Q_{\nu \mu_1 \dots \mu_p]}^{\kappa} \dot{\lambda}^{..} = \frac{1}{2} R_{\omega \nu \rho}^{\kappa} P_{\mu_1 \dots \mu_p]}^{\sigma} \dot{\lambda}^\rho + \dots - \frac{1}{2} R_{\omega \nu \lambda}^{\kappa} P_{\mu_1 \dots \mu_p]}^{\sigma} \dot{\lambda}^\lambda - \dots$$

and if  $P$  is a scalar

$$(6.5) \quad V_\lambda P = \partial_\lambda P = Q_\lambda,$$

(6.4) takes the form

$$(6.6) \quad \overset{1}{V}_{[\mu} Q_{\lambda]} = \partial_{[\mu} Q_{\lambda]} = 0.$$

Applying the general rule to the equation

$$(6.7) \quad V_\nu v^{\kappa_1 \dots \kappa_n} = 0; \quad v^{\kappa_1 \dots \kappa_n} = v^{[\kappa_1 \dots \kappa_n]}$$

we get the first integrability conditions

$$(6.8) \quad R_{\nu \mu \rho}^{\kappa_1 \dots \kappa_n} v^{\rho \kappa_2 \dots \kappa_n} = \frac{1}{n} R_{\nu \mu \rho}^{\kappa_1 \dots \kappa_n} v^{\kappa_1 \dots \kappa_n} = \frac{1}{n} V_{\nu \mu} v^{\kappa_1 \dots \kappa_n} = 0.$$

Hence in an  $L_n$  a covariant constant  $n$ -vector field (or a scalar density-field) is possible if and only if  $V_{\nu\mu} = 0$ . This can also be proved by using (5.22b). It is in accordance with the property of volume preserving of an  $L_n$  with  $V_{\nu\mu} = 0$ , mentioned already in Exerc. III 4,5.<sup>1)</sup>

### § 7. Geodesics and normal coordinates<sup>2)</sup>.

A curve  $\xi^\kappa = f^\kappa(t)$  in  $L_n$  is called a *geodesic* if the tangent vector  $d\xi^\kappa/dt$  at a point remains tangent if it is displaced parallel along the curve.

The n. a. s. conditions are

$$(7.1) \quad \frac{\delta}{dt} \frac{d\xi^\kappa}{dt} = \alpha(t) \frac{d\xi^\kappa}{dt}$$

or

$$(7.2) \quad \frac{d^2\xi^\kappa}{dt^2} + \Gamma_{\mu\lambda}^\kappa \frac{d\xi^\mu}{dt} \frac{d\xi^\lambda}{dt} = \alpha(t) \frac{d\xi^\kappa}{dt}$$

from which we see that the geodesics only depend on  $\Gamma_{(\mu\lambda)}^\kappa$ . If a new parameter  $z$  is introduced

$$(7.3) \quad z = z(t); \quad t = t(z)$$

the equation (7.2) takes the form

$$(7.4) \quad \frac{d^2\xi^\kappa}{dz^2} + \Gamma_{\mu\lambda}^\kappa \frac{d\xi^\mu}{dz} \frac{d\xi^\lambda}{dz} = - \frac{d\xi^\kappa}{dz} \left( \frac{d^2z}{dt^2} - \alpha \frac{dz}{dt} \right) \left( \frac{dz}{dt} \right)^{-2}$$

and the right hand side of these equations reduces to zero if  $z$  is a solution of the ordinary differential equation of second order

$$(7.5) \quad \frac{d^2z}{dt^2} - \alpha(t) \frac{dz}{dt} = 0.$$

If this is the case,  $z$  is called *an affine parameter* on the geodesic. The general solution of (7.5) has the form

$$(7.6) \quad z = C_1 \int e^{\int \alpha dt} dt + C_2; \quad C_1 \neq 0; \quad C_1, C_2 \text{ constants}$$

1) Cf. for a generalization URABE 1940, 1.

2) General references: RIEMANN 1854, 1 (for  $V_\mu$ ); VEBLEN 1922, 1; 1927, 1; BIRKHOFF 1923, 1; VEBLEN and T. Y. THOMAS 1923, 1; 1924, 1; VEBLEN and J. M. THOMAS 1925, 1; T. Y. THOMAS 1925, 2; 1926, 2 and 1927, 3 (projective); 1928, 1; 1936, 3; J. M. THOMAS 1926, 2 (projective); HLAVATY 1926, 1; 2; 3; 1927, 1; 1930, 3; T. Y. THOMAS and MICHAL 1927, 2; EISENHART 1927, 1, § 41 and 1930, 1 (projective); DUBOURDIEU 1927, 1; 1928, 1; NALLI 1928, 1; BORTOLOTTI 1929, 2; 1930, 3 (asymmetric); 1940, 2; MICHAL 1930, 1; 2; 1931, 1; 2; MATTIOLI 1930, 3; LEVY 1930, 1; WHITEHEAD and WILLIAMS 1930, 1; RUSE 1930, 1; 1931, 2; 3; DOUGLAS 1931, 1; WHITEHEAD 1931, 1; EI 1935, 1, p.100ff.; SASAKI 1941, 1 (conformal); BOMPIANI 1945, 1; BLUM 1947, 1; NARLIKAR and PRASAD 1948, 1 (relativity); VARGA 1950, 1; PINL 1951, 1 (asymmetric); HARTMANN 1951, 1.

and accordingly the affine parameter is fixed to within the place of the nullpoint and a constant factor. This implies that two segments on the same geodesic have an invariant ratio of "length" that can be measured by means of one parameter chosen arbitrarily from all possible affine parameters.

In an ordinary  $V_n$  the length  $s$  on a *real* geodesic is always an affine parameter because  $d\xi^\mu/ds$  is a unitvector and the covariant differential of a unitvector is always perpendicular to the vector. Hence, if we write  $i^\mu$  for  $d\xi^\mu/ds$ , the equation of a geodesic in  $V_n$  takes the form

$$(7.7) \quad i^\mu V_\mu i^\lambda = \frac{d^2 \xi^\lambda}{ds^2} + \Gamma_{\mu\lambda}^\mu \frac{d\xi^\mu}{ds} \frac{d\xi^\lambda}{ds} = 0^1).$$

We may ask whether it is possible to transform a connexion  $\Gamma_{\mu\lambda}^\nu$  in such a way that all geodesics remain geodesics. If

$$(7.8) \quad \Gamma_{\mu\lambda}^\nu \rightarrow \Gamma_{\mu\lambda}^\nu + P_{\mu\lambda}^{\nu}$$

$P_{\mu\lambda}^{\nu}$  is necessarily a tensor (cf. II § 10) and we may take  $P_{[\mu\lambda]}^{\nu} = 0$  because the alternating part does not affect the geodesics. According to (7.2) we must have

$$(7.9) \quad P_{\mu\lambda}^{\nu} \frac{d\xi^\mu}{dt} \frac{d\xi^\lambda}{dt} = \beta(t) \frac{d\xi^\nu}{dt}$$

for every choice of  $d\xi^\nu/dt$  with a function  $\beta(t)$  that depends on this choice. But this is only possible (cf. Exerc. I 3,4) if  $P_{\mu\lambda}^{\nu}$  has the form

$$(7.10) \quad P_{\mu\lambda}^{\nu} = p_\mu A_\lambda^\nu + p_\lambda A_\mu^\nu$$

If (7.10) is satisfied the new affine parameter ' $z$ ' has the form

$$(7.11) \quad 'z = C_1 \int e^{2 \int p_\mu d\xi^\mu} dz + C_2; \quad C_1 \neq 0; \quad C_1, C_2 \text{ constants}$$

and from this it follows that  $z$  is an affine parameter for the transformed connexion if and only if  $p_\lambda d\xi^\lambda = 0$  at all points of the geodesic considered, i.e. if at all points  $p_\lambda$  is tangent to the geodesic.  $z$  remains affine parameter for all geodesics if and only if  $p_\lambda = 0$ , hence *a symmetric connexion is wholly determined by its geodesics and the affine parameters on them*.

As we have already seen in III § 4 a coordinate system in an  $A_n$  with the  $\Gamma_{\mu\lambda}^\nu$  vanishing at *all* points exists if and only if the  $A_n$  is an  $E_n$ . But it is easy to find a coordinate system in  $A_n$  for which the  $\Gamma_{\mu\lambda}^\nu$  vanish at *one* arbitrarily given point  $\xi^\mu$ . Such a coordinate system is called

<sup>1)</sup> The expression  $V_\mu i^\nu$  has a sense only if the field  $i^\nu$  is prolonged (cf. II § 4) but  $i^\mu V_\mu i^\nu$  always has a sense.

<sup>2)</sup> Cf. EISENHART 1926, 1, p. 132; T. Y. THOMAS 1934, 2, p. 9.

geodesic at  $\xi^*$ . VEBLEN has given the following general method for the construction of a coordinate system of this kind, that has moreover many other useful properties. At  $\xi^*$  we take an arbitrary vector  $t^* \neq 0$  depending on  $n - 1$  parameters, in such a way that to every direction at  $\xi^*$  there belongs one of the vectors  $t^*$  with that direction. Then we consider all geodesics through  $\xi^*$  and on each of these geodesics we fix an affine parameter  $z$  such that at  $\xi^*$

$$(7.12) \quad \frac{d\xi^*}{dz} = t^*; \quad z = 0.$$

Now the equation of a geodesic enables us to compute for each geodesic the value of  $d^2\xi^*/dz^2$  at  $\xi^*$

$$(7.13) \quad \frac{d^2\xi^*}{dz^2}(\xi^*) = -\Gamma_{\mu\lambda}^*(\xi^*) t^\mu_0 t^\lambda_0$$

and by repeated differentiation of the equation of a geodesic we can find all derivatives of the  $\xi^*$  with respect to  $z$  at  $\xi^*$

$$(7.14) \quad \frac{d^{p+1}\xi^*}{dz^{p+1}}(\xi^*) = -\Gamma_{\mu_p \dots \mu_1 \lambda}^*(\xi^*) t^{\mu_p}_0 \dots t^{\mu_1}_0 t^\lambda_0$$

where<sup>1)</sup>

$$(7.15) \quad \Gamma_{\mu_p \dots \mu_1 \lambda}^* \stackrel{\text{def}}{=} \partial_{(\mu_p} \Gamma_{\mu_{p-1} \dots \mu_1 \lambda)}^* - p \Gamma_{(\mu_p \mu_{p-1}}^* \Gamma_{\mu_{p-2} \dots \mu_1 \lambda) \tau}^*.$$

Then  $\xi^*$  can be expanded in a series

$$(7.16) \quad \left\{ \begin{array}{l} \xi^* = \xi^*_0 + \left( \frac{d\xi^*}{dz} \right)_{z=0} z + \frac{1}{2!} \left( \frac{d^2\xi^*}{dz^2} \right)_{z=0} z^2 + \dots \\ = \xi^*_0 + z t^*_0 - \frac{1}{2!} \Gamma_{\mu\lambda}^*(\xi^*) z^2 t^\mu_0 t^\lambda_0 - \frac{1}{3!} \Gamma_{\mu_2 \mu_1 \lambda}^*(\xi^*) z^3 t^{\mu_2}_0 t^{\mu_1}_0 t^\lambda_0 - \dots \end{array} \right.$$

If this series is convergent the expressions  $z t^*$  can be used in an  $\mathfrak{N}(\xi^*)$  as new coordinates. Taking  $\xi^* = 0$  for simplicity and writing  $\xi^h$ ;  $h = 1, \dots, n$  for these new coordinates

$$(7.17) \quad \xi^h \stackrel{\text{def}}{=} \delta_{\mu}^h t^{\mu} z$$

we get

$$(7.18) \quad \xi^* = \delta_h^* \xi^h - \frac{1}{2!} \Gamma_{\mu\lambda}^*(0) \delta_j^{\mu} \xi^j \xi^i - \frac{1}{3!} \Gamma_{\mu_2 \mu_1 \lambda}^*(0) \delta_{j_2}^{\mu_2} \delta_{j_1}^{\mu_1} \xi^{j_2} \xi^{j_1} \xi^i - \dots$$

<sup>1)</sup> BORTOLOTTI and HLAVATY remarked 1936, 2 that the construction of  $\Gamma_{\mu_p \dots \mu_1}^*$  is analogous to the construction of a covariant derivative for the covariant indices only.

and the inverse

$$(7.19) \quad \xi^h = \delta_{\alpha}^h \left( \xi^{\alpha} + \frac{1}{2!} A_{\mu\lambda}^{\alpha} (0) \xi^{\mu} \xi^{\lambda} + \frac{1}{3!} A_{\mu_2 \mu_1 \lambda}^{\alpha} (0) \xi^{\mu_2} \xi^{\mu_1} \xi^{\lambda} + \dots \right)$$

where

$$(7.20) \quad \begin{cases} A_{\mu\lambda}^{\alpha} \stackrel{\text{def}}{=} \Gamma_{\mu\lambda}^{\alpha} \\ A_{\mu_2 \mu_1 \lambda}^{\alpha} \stackrel{\text{def}}{=} \Gamma_{\mu_2 \mu_1 \lambda}^{\alpha} + 3 \Gamma_{(\mu_2 \mu_1}^{\sigma} \Gamma_{\lambda)\sigma}^{\alpha} \\ A_{\mu_3 \mu_2 \mu_1 \lambda}^{\alpha} \stackrel{\text{def}}{=} \Gamma_{\mu_3 \mu_2 \mu_1 \lambda}^{\alpha} + 4 \Gamma_{(\mu_3 \mu_2 \mu_1}^{\sigma} \Gamma_{\lambda)\sigma}^{\alpha} + 6 \Gamma_{(\mu_3 \mu_2}^{\sigma} A_{\mu_1 \lambda)\sigma}^{\alpha} - 3 \Gamma_{\sigma\varrho}^{\alpha} \Gamma_{(\mu_3 \mu_2}^{\sigma} \Gamma_{\mu_1 \lambda)}^{\varrho} \end{cases}$$

By differentiation of (7.18) and (7.19) we get at  $\xi^{\alpha}_0$

$$(7.21) \quad \begin{cases} \text{a)} \quad A_i^{\alpha} \left( \xi_0 \right) = \delta_i^{\alpha}; & A_{\lambda}^h \left( \xi_0 \right) = \delta_{\lambda}^h \\ \text{b)} \quad \partial_j A_i^{\alpha} \left( \xi_0 \right) = - \delta_j^{\mu} \Gamma_{\mu\lambda}^{\alpha} \left( \xi_0 \right); & \partial_{\mu} A_{\lambda}^h \left( \xi_0 \right) = \delta_{\alpha}^h \Gamma_{\mu\lambda}^{\alpha} \left( \xi_0 \right) \\ \text{c)} \quad \partial_k \partial_j A_i^{\alpha} \left( \xi_0 \right) = - \delta_k^{\nu} \delta_j^{\mu} \Gamma_{\nu\mu\lambda}^{\alpha} \left( \xi_0 \right); & \partial_{\nu} \partial_{\mu} A_{\lambda}^h \left( \xi_0 \right) = \delta_{\alpha}^h A_{\nu\mu\lambda}^{\alpha} \left( \xi_0 \right) \end{cases}$$

and accordingly at  $\xi^{\alpha}_0$

$$(7.22) \quad \Gamma_{ji}^h \left( \xi_0 \right) = \delta_j^{\mu} \delta_{i\alpha}^h \Gamma_{\mu\lambda}^{\alpha} \left( \xi_0 \right) - A_{j\alpha}^{\mu} \partial_{\mu} A_{\lambda}^h \left( \xi_0 \right) = 0.$$

Now the choice of the system  $(\alpha)$  is entirely free. Hence, if we take  $(\alpha)$  equal to  $(h)$  it follows from (7.18) that at  $\xi^{\alpha}_0$

$$(7.23) \quad \Gamma_{j_p \dots j_1 i}^h \left( \xi_0 \right) = 0$$

and in consequence of (7.15) at  $\xi^{\alpha}_0$

$$(7.24) \quad \partial_{(j_p+1} \Gamma_{j_p \dots j_1 i}^h \left( \xi_0 \right) = 0;$$

$$(7.25) \quad \partial_{(j_p} \partial_{j_{p-1}} \dots \partial_{j_1} \Gamma_{j i}^h \left( \xi_0 \right) = 0.$$

The coordinates  $(h)$  defined in this way are called by VEBLEN the *normal coordinates in  $A_n$  with respect to  $\xi^{\alpha}$  and to the coordinate system  $(\alpha)$* .

They are a generalization of the normal coordinates introduced in a  $V_n$  by RIEMANN<sup>1)</sup>. If we use normal coordinates, not only the  $\Gamma_{ji}^h$  vanish at  $\xi^{\alpha}_0$  but so do the symmetrized derivatives of  $\Gamma_{ji}^h$  of every order also.

If the coordinates  $\xi^{\alpha}$  are transformed into  $\xi^{\alpha'}$  we get the new normal coordinates  $\xi^{h'}$  belonging to the same point  $\xi^{\alpha}_0$

$$(7.26) \quad \xi^{h'} \stackrel{\text{def}}{=} \delta_{\alpha}^{h'} t^{\alpha} z = \delta_{\alpha}^{h'} A_{\alpha}^{\alpha'} \left( \xi_0 \right) t^{\alpha} z = A_h^{h'} \xi^h$$

<sup>1)</sup> RIEMANN 1854, 1, p. 279.

where the  $A_h^k$  are constants. Hence, if the  $\xi^k$  are transformed, the normal coordinates with respect to  $\xi^k$  and  $(\alpha)$  undergo a linear homogeneous transformation with *constant* coefficients and these coefficients are the values of  $A_h^k$  at  $\xi^k$ .

By means of the normal coordinates  $\xi^h$  the points of an  $\mathfrak{N}(\xi^k)$  are mapped on the points of the tangent  $E_n$  of  $\xi^k$  in such a way that the point of  $\mathfrak{N}(\xi^k)$  that has normal coordinates  $\xi^h$  is mapped on the point of  $E_n$  with rectilinear coordinates  $\xi^h$  with respect to the local coordinate system of  $E_n$  belonging to the coordinates  $\xi^k$ .

From (7.21a) we see that the basis vectors of  $(\alpha)$  and of  $(h)$  at  $\xi^k$  coincide. But that implies that the components of a *quantity* (not of every geometric object, consider for instance  $\Gamma_{\mu\lambda}^\alpha$ ) at  $\xi^k$  with respect to  $(\alpha)$  and to  $(h)$  are numerically equal. This proves the following proposition:

*If a set of numbers be given at  $\xi^k$  with respect to  $(h)$  and if we know that they transform like the components of a quantity if  $(h)$  is transformed into  $(h')$ , these numbers are the components of a quantity at  $\xi^k$  with respect to  $(\alpha)$  and  $(\alpha')$  and numerically equal to the components of this quantity with respect to  $(h)$  and  $(h')$  respectively.*

This proposition has been used by VEBLEN and T. Y. THOMAS<sup>1)</sup> for a very elegant construction of a set of quantities called the normal tensors. The (not symmetrized) derivatives of order  $p$  of the  $\Gamma_{ij}^h$  with respect to the  $\xi^k$  at  $\xi^k$

$$(7.27) \quad N_{j_p \dots j_1; i}^{h \dots h} (\xi) \stackrel{\text{def}}{=} \partial_{j_p} \dots \partial_{j_1} \Gamma_{i i}^h (\xi)$$

transform obviously like the components of a tensor with valences  $1, p+2$  if  $(h)$  is transformed into  $(h')$ . Hence the  $N_{j_p \dots j_1; i}$  are components of a tensor. These tensors are called the *normal tensors* belonging to the connexion  $\Gamma_{\mu\lambda}^\alpha$  and the point  $\xi^k$ . They occur in the series

$$(7.28) \quad \Gamma_{ij}^h (\xi) = \xi^k N_{k_j; i}^{h \dots h} (\xi) + \frac{1}{2!} \xi^{k_2} \xi^{k_1} N_{k_2 k_1; i}^{h \dots h} + \dots; \quad (\xi^k = 0)^2.$$

They are all symmetric in the *first*  $p$  and in the *last* 2 lower indices and vanish identically by mixing over *all* lower indices according to (7.25). The normal tensors belonging to the connexion  $\Gamma_{\mu\lambda}^\alpha$  can be constructed at every point and thus form tensor fields. But if we do this in the way described, at every point we get the components of these fields with respect to a normal coordinate system belonging to *this*

<sup>1)</sup> VEBLEN and T. Y. THOMAS 1923, 1.

<sup>2)</sup> The convergence of this series was discussed by T. Y. THOMAS 1934, 2, p. 126.

point and we should like to have the components with respect to a general coordinate system ( $\xi$ ). These can of course be computed by repeated differentiation of [cf. (2.5)]

$$(7.29) \quad \Gamma_{ji}^h = A_{j\mu i\nu}^{\mu\lambda h} \Gamma_{\mu\lambda}^\nu - A_{j\mu i}^{\mu\lambda} \partial_\mu A_{\lambda}^h$$

using (7.21). In this way we find for the first normal tensor at  $(\xi)$

$$(7.30) \quad N_{kjj;i}^{::h} = \delta_{k\mu i}^{\nu\mu\lambda h} (\partial_\nu \Gamma_{\mu\lambda}^\nu - \Gamma_{\nu\mu\lambda}^\nu - 2 \Gamma_{\nu(\mu}^o \Gamma_{\lambda)\rho}^\nu)$$

and accordingly for the general components

$$(7.31) \quad N_{\nu\mu\lambda}^{::\nu} = \partial_\nu \Gamma_{\mu\lambda}^\nu - \Gamma_{\nu\mu\lambda}^\nu - 2 \Gamma_{\nu(\mu}^o \Gamma_{\lambda)\rho}^\nu$$

valid at all points of  $\mathfrak{M}(\xi)$ . The same could be done for all normal tensors but as we shall see later there is a much shorter way using the curvature tensor and its covariant derivatives.

If we start from the  $\Gamma_{j_p \dots j_i i}^h$  (zero at  $\xi$ ) other sets of generalized normal tensors can be formed since the  $\partial_{k_q} \dots \partial_{k_1} \Gamma_{j_p \dots j_i i}^h$  transform for  $(h) \rightarrow (h')$  like the components of a tensor. All these tensor fields are differential concomitants of the connexion  $\Gamma_{\mu\lambda}^\nu$  and there exists a great number of relations between them and their covariant derivatives. Much information can be found in the publications of VEBLEN and T. Y. THOMAS<sup>1)</sup>.

The process applied here to the  $\Gamma_{ji}^h$  etc. can also be applied to any quantity. In fact if we take the  $(h)$ -components  $\Phi_L$  ( $L$  is a collecting index), the derivatives  $\partial_{j_1} \Phi_L, \partial_{j_2} \partial_{j_1} \Phi_L$  at  $\xi$  transform for  $(h) \rightarrow (h')$  like the components of a quantity and this gives rise to the construction of a series of quantities, called by VEBLEN the first, second, etc. *extension* of  $\Phi_L$ . We denote them by the operators  $\overset{\epsilon}{V}_\mu, \overset{\epsilon}{V}_{\mu_2 \mu_1},$  etc. The first extension is of course identical with the first covariant derivative but this is not true for the higher extensions. For instance for a vector field  $v^\nu$  we have at  $\xi$ :

$$(7.32) \quad \left\{ \begin{array}{l} \text{a)} \quad \overset{\epsilon}{V}_\mu v^\nu(\xi) = A_{\mu h}^{j_\nu} \overset{\epsilon}{V}_j v^h(\xi) = A_{\mu h}^{j_\nu} \partial_j v^h(\xi) = A_{\mu h}^{j_\nu} \overset{\epsilon}{V}_j v^h(\xi) = \overset{\epsilon}{V}_\mu v^\nu(\xi), \\ \text{b)} \quad \overset{\epsilon}{V}_{\mu_2 \mu_1} v^\nu(\xi) = A_{\mu_2 \mu_1 h}^{j_\nu} \overset{\epsilon}{V}_{j_1} v^h(\xi) = A_{\mu_2 \mu_1 h}^{j_\nu} (\partial_{j_2} \partial_{j_1} v^h + v^i \partial_{j_2} \Gamma_{j_1 i}^h)(\xi) \\ \qquad \qquad \qquad = A_{\mu_2 \mu_1 h}^{j_\nu} (\overset{\epsilon}{V}_{j_2 j_1} v^h + v^i N_{j_2 j_1 i}^h)(\xi) \\ \qquad \qquad \qquad = \overset{\epsilon}{V}_{\mu_2 \mu_1} v^\nu(\xi) + v^\lambda N_{\mu_2 \mu_1 \lambda}^h(\xi) \end{array} \right.$$

<sup>1)</sup> VEBLEN and T. Y. THOMAS 1924, 1; T. Y. THOMAS 1934, 2. When dealing with all these sets of quantities the way is open for "orgies of formalism" but it must be said that these authors have restricted themselves in an exemplary way to those relations that are really interesting.

and because these results have the invariant form and  $\xi^\alpha$  is arbitrary

$$(7.33) \quad \begin{cases} \text{a)} & \nabla_\mu v^\alpha = \overset{\epsilon}{\nabla}_\mu v^\alpha \\ \text{b)} & \nabla_{\mu_2 \mu_1} v^\alpha = \overset{\epsilon}{\nabla}_{\mu_2 \mu_1} v^\alpha + v^\lambda N_{\mu_2 \mu_1 \lambda}^\alpha \end{cases}$$

valid at all points of  $\mathfrak{N}(\xi)$ . Conversely all extensions can be expressed in terms of normal tensors and the covariant derivatives of the same and lower orders.

The most interesting relations are those between the simplest normal tensors defined by (7.27) and the curvature tensor  $R_{\nu \mu \lambda}^\alpha$  and its derivatives. With respect to (h) we have at  $\xi^\alpha$  according to (7.31)

$$(7.34) \quad R_{k \cdot i}^h(\xi) = 2 \partial_{[k} \Gamma_{j]i}^h(\xi) = 2 N_{[k \cdot j]i}^h(\xi),$$

hence everywhere

$$(7.35) \quad \boxed{R_{\nu \mu \lambda}^\alpha = 2 N_{[\nu \mu] \lambda}^\alpha}.$$

But, because of  $N_{\nu [\mu \lambda]}^\alpha = 0$  and  $N_{(\nu \mu \lambda)}^\alpha = 0$ ,  $N_{\nu \mu \lambda}^\alpha$  can be solved from (7.35):

$$(7.36) \quad \boxed{N_{\nu \mu \lambda}^\alpha = \frac{2}{3} R_{\nu (\mu \lambda)}^\alpha}.$$

If the higher covariant derivatives of  $R_{\nu \mu \lambda}^\alpha$  are written out we get the general formula

$$(7.37) \quad \nabla_{\nu_p \dots \nu_1} R_{\nu \mu \lambda}^\alpha = 2 \partial_{\nu_p} \dots \partial_{\nu_1} \partial_{[\nu} \Gamma_{\mu] \lambda}^\alpha + *$$

where \* stands for terms containing only the  $\Gamma_{\mu \lambda}^\alpha$  and their derivatives up to the order  $p$ . Using the coordinate system (h) we get at  $\xi^\alpha$

$$(7.38) \quad \nabla_{k_p \dots k_1} R_{k \cdot i}^h(\xi) = 2 N_{k_p \dots k_1 [k \cdot j] i}^h(\xi) + *$$

where \* symbolizes terms containing only normal tensors with a valence lower than  $p+4$ . Note that for  $p=0$  and  $p=1$  these additional terms vanish because  $\Gamma_{ii}^h(\xi)=0$ . Going back to general coordinates we get from (7.38)

$$(7.39) \quad \boxed{\nabla_{\nu_p \dots \nu_1} R_{\nu \mu \lambda}^\alpha = 2 N_{\nu_p \dots \nu_1 [\nu \mu] \lambda}^\alpha + *}$$

where \* symbolizes the same quantities as in (7.38), valid at all points of  $\mathfrak{N}(\xi)$ . Using the symmetry of the normal tensor with valence  $p+4$  in the first  $p+1$  and in the last 2 lower indices and the fact that mixing over all  $p+3$  lower indices leads to zero, from (7.38) we get an equation of the form

$$(7.40) \quad \boxed{N_{\nu_p \dots \nu_1 \nu \mu \lambda}^\alpha = L(\nabla_{\nu_p \dots \nu_1} R_{\nu \mu \lambda}^\alpha) + *}$$

where  $L()$  denotes a polynomial linear homogeneous in the isomers of  $\nabla_{v_p \dots v_1} R_{v \mu \lambda}^*$  with constant scalar coefficients and where  $*$  stands for an expression in terms of  $R_{v \mu \lambda}^*$  and its covariant derivatives up to the order  $p - 1$ .

In fact the equation

$$(7.41) \quad N_{(v_p \dots v_1) v \mu \lambda}^* = 0$$

contains three kinds of terms: 1. those with  $\mu \lambda$  as the last two lower indices; 2. those with one  $v$ -index among the last two lower indices and 3. those with two  $v$ -indices as the last two lower indices. Because of the symmetry of  $N$  in the first  $p+1$  and in the last 2 lower indices the place of the indices in each of these two sets of indices is not important. Now according to (7.39) a term of the second kind can be written as a sum of  $R$ , a term of the first kind and terms containing only  $N$ 's with lower valences. In the same way a term of the third kind can be written as a sum of an isomer of  $R$ , a term of the second kind and terms containing only  $N$ 's with lower valences. If the term of the second kind arising in this way is expressed in the way described above we have finally in (7.41) only a sum of isomers of  $R$  with constant coefficients, terms of the first kind, and terms containing only  $N$ 's with lower valences. If these  $N$ 's are dealt with in the same way, we get (7.40). It is possible to give some more information concerning the form of the equations (7.39, 40). Let  $\overset{v}{R}$  and  $\overset{v}{N}$  symbolize the  $(v-4)$ th covariant derivative of  $R_{v \mu \lambda}^*$  and the normal tensor of valence  $v$  respectively. Then it is clear that (7.39) expresses  $\overset{4}{R}$  in terms of  $\overset{4}{N}$ , and  $\overset{5}{R}$  in terms of  $\overset{5}{N}$ , but that in the expression for  $\overset{6}{R}$  we have not only  $\overset{6}{N}$  but also a transvection of two factors  $\overset{4}{N}$ . The general scheme is as follows

$$(7.42) \quad \left\{ \begin{array}{l} \overset{4}{R}: \overset{4}{N} \\ \overset{5}{R}: \overset{5}{N} \\ \overset{6}{R}: \overset{6}{N}, \overset{4}{N}^4 \\ \overset{7}{R}: \overset{7}{N}, \overset{5}{N}^4 \\ \overset{8}{R}: \overset{8}{N}, \overset{6}{N}^4, \overset{5}{N}^5, \overset{4}{N}^4, \overset{4}{N}^4 \\ \overset{9}{R}: \overset{9}{N}, \overset{7}{N}^4, \overset{6}{N}^5, \overset{5}{N}^4, \overset{4}{N}^4 \\ \text{etc.} \end{array} \right.$$

There is a similar scheme belonging to (7.40) and derived from (7.42) by interchanging the letters  $R$  and  $N$ .

From these schemes we see 1° that a term with  $\overset{v_1}{N} \dots \overset{v_u}{N} (\overset{v_1}{R} \dots \overset{v_u}{R})$  occurs in  $\overset{v}{R}(\overset{v}{N})$  if and only if the following conditions hold

$$(7.43) \quad v = v_1 + \dots + v_u - 2(u-1);$$

2° that there are exactly  $u-1$  transvections in this term, every  $N(R)$  being transvected with every other  $N(R)$  over at most one index and 3° that every  $N(R)$  in a product is transvected at least once.

The components of the normal tensors at  $\overset{\xi^x}{\emptyset}$  with respect to  $(h)$  are the derivatives of the  $\overset{h}{\Gamma}_{ij}^k$  with respect to the  $\overset{\xi^h}{\emptyset}$ . Hence (7.40) gives very valuable information about the  $\overset{h}{\Gamma}_{ji}^k$  as functions of the  $\overset{\xi^h}{\emptyset}$  in  $\mathfrak{N}(\overset{\xi^x}{\emptyset})$ .

We consider the case when all covariant derivatives of  $R_{\nu\mu\lambda}^{;;;\alpha}$  of odd orders are zero at  $\overset{\xi^x}{\emptyset}$ . Then according to (7.40, 43) all  $N$ 's with an odd valence vanish at  $\overset{\xi^x}{\emptyset}$  and it follows from (7.28) that all terms with an even number of factors  $\overset{\xi^h}{\emptyset}$  vanish at that point and that consequently

$$(7.44) \quad \overset{h}{\Gamma}_{ji}^k(\overset{\xi^h}{\emptyset}) = -\overset{h}{\Gamma}_{ji}^k(-\overset{\xi^h}{\emptyset}).$$

But this means that the  $A_n$  is invariant for the transformation  $\overset{\xi^h}{\emptyset} \rightarrow -\overset{\xi^h}{\emptyset}$ , that is a reflection at the point  $\overset{\xi^x}{\emptyset}$ . In fact a vector field with a field value  $v^h$  at  $\overset{\xi^h}{\emptyset}$  has a field value  $-v^h$  at  $-\overset{\xi^h}{\emptyset}$  after the reflection. Now a linear element  $d\xi^h$  at  $\overset{\xi^h}{\emptyset}$  has for reflection the element  $-d\xi^h$  at  $-\overset{\xi^h}{\emptyset}$ , hence (7.44) has as a consequence that the reflection of  $\delta v^h$  at  $\overset{\xi^h}{\emptyset}$  is a vector with the same components but with opposite sign at  $-\overset{\xi^h}{\emptyset}$ . An  $A_n$  with this property, is called *symmetric with respect to  $\overset{\xi^x}{\emptyset}$* . If an  $A_n$  has this property all  $N$ 's with an odd valence must vanish in (7.28) and then it follows from (7.39, 43) that all covariant derivatives of  $R_{\nu\mu\lambda}^{;;;\alpha}$  of odd orders vanish at  $\overset{\xi^x}{\emptyset}$ . Hence *the vanishing at  $\overset{\xi^x}{\emptyset}$  of all derivatives of  $R_{\nu\mu\lambda}^{;;;\alpha}$  of odd orders is necessary and sufficient for the symmetry of the  $A_n$  with respect to  $\overset{\xi^x}{\emptyset}$* .

An  $A_n$  is called *symmetric* if it is symmetric with respect to all points of an  $\mathfrak{N}(\overset{\xi^x}{\emptyset})$ . It follows from the above that this occurs if and only if the first and consequently all covariant derivatives of  $R_{\nu\mu\lambda}^{;;;\alpha}$  vanish in  $\mathfrak{N}(\overset{\xi^x}{\emptyset})$ :

*An  $A_n$  is symmetric if and only if  $R_{\nu\mu\lambda}^{;;;\alpha}$  is covariant constant.<sup>1)</sup>*

<sup>1)</sup> (Cf. also VII § 5.) CARTAN 1926, 1; 2; 1927, 1, p. 87; 1932, 1; LEVY 1926, 1; 2; WHITEHEAD 1932, 1; WALKER 1944, 1; 2; RASCHEVSKI 1950, 1; ROZENFEL'D and ABRAMOV 1950, 1; SCHIROKOV 1950, 1; BOREL and LICHNEROWICZ 1952, 1 (also for  $\tilde{U}_n$ ).

Any quantity whose components can be expressed in terms of the  $\Gamma_{\mu\lambda}^{\kappa}$  and their derivatives up to a certain order by means of equations that are invariant for coordinate transformations will be called a *differential concomitant* of the connexion  $\Gamma_{\mu\lambda}^{\kappa}$ . For instance  $R_{\nu\mu\lambda}^{\kappa}$ ,  $R_{\mu\lambda}$ ,  $V_{\mu\lambda}$ ,  $V_{\nu}R_{\mu\lambda}$ . In general we only consider concomitants whose components can be expressed *algebraically* in terms of the  $\Gamma_{\mu\lambda}^{\kappa}$  and their derivatives. Special cases are scalars and scalar densities (or  $\Delta$ -densities), often called *absolute differential invariants* and *relative differential invariants* of the connexion respectively. If we use first a normal coordinate system ( $h$ ) with respect to a point  $\xi^{\kappa}$ , we get at this point equations containing only the derivatives of the  $\Gamma_{ji}^h$  with respect to the  $\xi^h$  and these derivatives are the components of the normal tensors. Since the expressions are invariant for coordinate transformations we can go back now to a general coordinate system ( $\kappa$ ), and by using (7.40) we may change these expressions into expressions in  $R_{\nu\mu\lambda}^{\kappa}$  and its covariant derivatives. Hence we have proved the *first reduction theorem* for differential concomitants (cf. I § 3):

*All differential concomitants of a symmetric connexion are ordinary concomitants of  $R_{\nu\mu\lambda}^{\kappa}$  and its covariant derivatives.*<sup>1)</sup>

Any quantity whose components can be expressed in terms of the  $\Gamma_{\mu\lambda}^{\kappa}$ , their derivatives up to a certain order, the components of a set of quantities  $\Phi_1, \dots, \Phi_M$  (indices suppressed) and their (ordinary) derivatives up to a certain order by means of equations that are invariant for coordinate transformations, are called *differential concomitants* of the set  $\Phi_1, \dots, \Phi_M$  and the connexion  $\Gamma_{\mu\lambda}^{\kappa}$ . As before in general we only consider concomitants whose components can be expressed *algebraically*. *Absolute* and *relative invariants* are defined as before. If we use a normal coordinate system ( $h$ ) for the point  $\xi^{\kappa}$  first, we get at this point equations containing only the derivatives of the  $\Gamma_{ji}^h$  with respect to the  $\xi^h$ , and the ( $h$ )-components of  $\Phi_1, \dots, \Phi_M$  and their derivatives with respect to the  $\xi^h$ . Now these latter derivatives are the components of the extensions of  $\Phi_1, \dots, \Phi_M$  and as we have seen, these can be expressed in terms of the covariant derivatives of  $\Phi_1, \dots, \Phi_M$  and the normal tensors. Since the expressions are invariant for coordinate transformations we can now go back to a general coordinate system ( $\kappa$ ) and by means of (7.40) we may change all normal tensors into expressions in  $R_{\nu\mu\lambda}^{\kappa}$  and its covariant derivatives. This proves the *second reduction theorem* for differential concomitants:

<sup>1)</sup> CHRISTOFFEL 1869, 1.

All differential concomitants of a set of quantities  $\Phi_1, \dots, \Phi_M$  (indices suppressed) and the symmetric connexion  $\Gamma_{\mu\lambda}^{\kappa}$  are ordinary concomitants of  $\Phi_1, \dots, \Phi_M, R_{\nu\mu\lambda}^{\kappa}$  and their covariant derivatives.<sup>1)</sup>

If the connexion  $\Gamma_{\mu\lambda}^{\kappa}$  is not symmetric we have only to consider  $S_{\mu\lambda}^{\kappa}$  as one of the set of given quantities. Then all differential concomitants of the connexion  $\Gamma_{\mu\lambda}^{\kappa}$  and the set  $\Phi_1, \dots, \Phi_M$  are ordinary concomitants of  $\Phi_1, \dots, \Phi_M, S_{\mu\lambda}^{\kappa}$  and  $R_{\nu\mu\lambda}^{\kappa}$  (the curvature tensor for the connexion  $\Gamma_{\mu\lambda}^{\kappa}$ ) and their covariant derivatives with respect to the connexion  $\Gamma_{\mu\lambda}^{\kappa}$ . But they are of course also ordinary concomitants of  $\Phi_1, \dots, \Phi_M, S_{\mu\lambda}^{\kappa}$  and  $R_{\nu\mu\lambda}^{\kappa}$  (the curvature tensor for the connexion  $\Gamma_{\mu\lambda}^{\kappa}$ ) and their covariant derivatives with respect to the connexion  $\Gamma_{\mu\lambda}^{\kappa}$ .<sup>1)</sup>

### Exercises.

III 7,1<sup>2)</sup>. If an  $X_m$  in  $A_n$  is formed by  $\infty^{m-1}$  geodesics through  $\xi_{\theta}^{\kappa}$  tangent to an  $E_m$  in the tangent space of that point, its equations in the normal coordinates belonging to  $\xi_{\theta}^{\kappa}$  are linear. (Note that an  $X_m$  in our sense can not be singular at  $\xi_{\theta}^{\kappa}$ , cf. Ch. II, footnote 1 on p. 74.)

III 7,2)<sup>2)</sup>. If the  $\xi^h$  are normal coordinates with respect to  $\xi_{\theta}^{\kappa}$ , prove that

$$\text{III 7,2 } \alpha) \quad \Gamma_{ji}^h \xi^j \xi^i = 0$$

is true at all points of  $\mathfrak{N}(\xi_{\theta}^{\kappa})$  (use 7.25).

III 7,3<sup>3)</sup>. If the  $\xi^h$  are normal coordinates in a  $V_n$  with respect to  $\xi_{\theta}^{\kappa}$ , prove that

$$\text{III 7,3 } \alpha) \quad \left\{ \begin{aligned} ds^2 &= (g_{ih} \{ \xi_{\theta}^i \} + N_{kjih} \{ \xi_{\theta}^i \} \xi^k \xi^j + \dots) d\xi^i d\xi^h \\ &= (g_{ih} \{ \xi_{\theta}^i \} + \frac{1}{3} K_{kijh} \{ \xi_{\theta}^i \} \xi^k \xi^j + \dots) d\xi^i d\xi^h. \end{aligned} \right.$$

III 7,4. Prove the second identity and the identity of BIANCHI using normal coordinates.

<sup>1)</sup> RICCI and LEVI CIVITA 1901, 1; WEITZENBÖCK 1923, 1; T. Y. THOMAS 1926, 2. Cf. T. Y. THOMAS and MICHAL 1927, 2; T. Y. THOMAS 1929, 1; 1934, 2, Ch. V. Cf. for the projective and the conformal case T. Y. THOMAS 1927, 3; LEVINE 1942, 1 and for a more general case NOETHER 1918, 1.

<sup>2)</sup> E I 1935, 1, p. 102.

<sup>3)</sup> E I 1935, 1, p. 104, 138.

III 7.5. Prove that

$$\text{III 7.5 } \alpha) \quad N_{\omega \nu \mu \lambda}^{\cdot \cdot \cdot \cdot} = \frac{5}{6} V_{(\omega} R_{\nu)} (\dot{\mu} \dot{\lambda})^{\cdot \cdot} - \frac{1}{6} V_{(\mu} R_{\lambda)} (\dot{\omega} \dot{\nu})^{\cdot \cdot}$$

using normal coordinates.

III 7.6<sup>1)</sup>). Prove that in normal coordinates with respect to  $\xi^x$

$$(V_k V_j P_{\cdot i})_0 = (\partial_k \partial_j P_{\cdot i}^h + N_{k \cdot m}^{\cdot h} P_{\cdot i}^m - N_{k j i}^{\cdot m} P_{\cdot m}^h)_0.$$

### § 8. FERMI coordinates<sup>2)</sup>.

If a curve is given in  $A_n$ , there exist coordinate systems ( $h$ ) such that the  $\Gamma_{ji}^h$  vanish at *every* point of the curve. FERMI proved this for a  $V_n$ <sup>3)</sup> and EISENHART gave the generalization for  $A_n$ .<sup>4)</sup> Let

$$(8.1) \quad \xi^x = f^x(t); \quad \xi^x = f^x(0)$$

be the equations of a curve and let

$$(8.2) \quad \xi^x = \varphi^x(\alpha_1, \dots, \alpha_{n-2})$$

be a set of  $\infty^{n-2}$  vectors at  $\xi^x$  in an  $E_{n-1}$  that does not contain the tangent of the curve, such that there is one vector in every direction of  $E_{n-1}$ . If now the vectors  $t^x$  are displaced parallel along the curve we get a set of  $\infty^{n-2}$  vector fields

$$(8.3) \quad t^x = \psi^x(\alpha_1, \dots, \alpha_{n-2}, t); \quad t^x = \psi^x(\alpha_1, \dots, \alpha_{n-2}, 0)$$

whose covariant differentials vanish for displacements along the curve:

$$(8.4) \quad d\xi^\mu V_\mu t^x = 0.$$

We only consider points of the curve where the  $t^x$  span an  $E_{n-1}$  that does not contain the tangent direction. We now take all geodesics that are tangent to one of the vectors  $t^x$  at an arbitrary point  $\xi^x$  of the curve in  $\mathfrak{N}(\xi^x)$ , and on each of these geodesics we choose an affine parameter  $z$  such that  $d\xi^x/dz = t^x$  at  $\xi^x$ . Then according to (7.16)

$$(8.5) \quad \xi^x = \xi^x + z t^x - \frac{1}{2!} \Gamma_{\mu \lambda}^x \{ \xi \} z^2 t^\mu t^\lambda - \frac{1}{3!} \Gamma_{\mu \nu \mu \lambda}^x \{ \xi \} z^3 t^\mu t^\nu t^\lambda - \dots$$

expressing  $\xi^x$  in terms of  $t^x$  and  $zt^x$ . Hence, taking the general point  $f^x(t)$  of the curve for  $\xi^x$ , we get the expansion in  $\mathfrak{N}(f^x)$

$$(8.6) \quad \xi^x = f^x(t) + z t^x - \frac{1}{2!} \Gamma_{\mu \lambda}^x \{ f^x(t) \} z^2 t^\mu t^\lambda - \dots$$

<sup>1)</sup> E I 1935, 1, p. 105 also for other examples.

<sup>2)</sup> FERMI 1922, 1; EISENHART 1927, 1, p. 64; BORTOLOTTI 1929, 2; 1930, 4; 1934, 1; DIENES 1933, 1; 2; E I 1935, 1, p. 106.

<sup>3)</sup> FERMI 1922, 1.

<sup>4)</sup> EISENHART 1927, 1, p. 64.

expressing  $\xi^x$  in terms of the  $n+1$  variables  $t, zt^x$  and valid in a neighbourhood of the curve. If now a set of  $n$  linearly independent vectors  $\overset{\theta}{e}_i^x; i=1, \dots, n$  is introduced at  $\xi^x$ , such that  $\overset{\theta}{e}_1^x, \dots, \overset{\theta}{e}_n^x$  span the  $E_{n-1}$  of the  $\overset{\theta}{t}^x$ , we have at  $\xi^x$

$$(8.7) \quad \overset{\theta}{t}^x \underset{x}{\equiv} t^x \overset{\theta}{e}_x^x; \quad x = 2, \dots, n$$

where the  $t^x$  are functions of the parameters  $\alpha_1, \dots, \alpha_{n-2}$ . If the vectors  $\overset{\theta}{e}_i^x$  are displaced parallel along the curve we get at each point of the curve a set  $e_i^x$  such that

$$(8.8) \quad t^x \underset{x}{\equiv} t^x e_i^x; \quad x = 2, \dots, n$$

where the  $t^x$  are *constant* along the curve. The  $e_i^x$  are known as functions of  $t$ . Introducing the values (8.8) in (8.6) we get

$$(8.9) \quad \begin{cases} \xi^x \underset{x}{\equiv} f^x(t) + e^x_z z t^x - \frac{1}{2!} e^{\mu}_y e^{\lambda}_z \Gamma_{\mu\lambda}^x(t) z^2 t^x t^y - \\ \quad - \frac{1}{3!} e^{\mu_2}_z e^{\mu_1}_y e^{\lambda}_x \Gamma_{\mu_2\mu_1\lambda}^x(t) z^3 t^x t^y t^z - \dots; \quad x, y, z = 2, \dots, n. \end{cases}$$

Now we introduce the new coordinate system  $(h)$  of  $A_n$  by means of the equations

$$(8.10) \quad \begin{cases} \xi^1 \underset{x}{\equiv} \omega^1(t); & \omega^1(0) \underset{x}{\equiv} 0; \quad t = \theta(\xi^1) \\ \xi^x \underset{x}{\equiv} \omega^x(t) + z t^x; & \omega^x(0) \underset{x}{\equiv} 0; \quad x = 2, \dots, n \end{cases}$$

and try to determine the as yet unknown functions  $\omega^h(t); h=1, \dots, n$  in such a way that the contravariant basis vectors of  $(h)$  at all points of the curve are identical with the  $e_i^x$  already defined. The equations of the curve are

$$(8.11) \quad \xi^x - \omega^x(\theta(\xi^1)) \underset{x}{\equiv} 0$$

in these new coordinates and the  $e_i^x$  and  $\Gamma_{\mu\lambda}^x$  along the curve may now be considered as functions of  $\xi^1$ . It follows from (8.9) and (8.10) that

$$(8.12) \quad \begin{cases} \xi^x \underset{x}{\equiv} f^x(\theta(\xi^1)) + e^x \{\xi^1\} (\xi^x - \omega^x(\theta(\xi^1))) - \\ \quad - \frac{1}{2!} e^{\mu} \{\xi^1\} e^{\lambda} \{\xi^1\} \Gamma_{\mu\lambda}^x(\xi^1) (\xi^x - \omega^x(\theta(\xi^1))) (\xi^y - \omega^y(\theta(\xi^1))) - \dots \end{cases}$$

and, by differentiation with respect to  $\xi^h$  for all points of the curve, that

$$(8.13) \quad \partial_1 \xi^x \underset{x}{\equiv} \frac{df^x}{dt} \left( \frac{d\omega^1}{dt} \right)^{-1} - e^x \frac{d\omega^x}{dt} \left( \frac{d\omega^1}{dt} \right)^{-1}; \quad \partial_y \xi^x \underset{y}{\equiv} e^x; \quad x, y = 2, \dots, n.$$

Hence the  $e^x$  are, in fact, basis vectors of  $(h)$ , and in order to make  $e^x$  also such a basis vector,  $\omega^1(t)$  and  $\omega^x(t)$  must be chosen such that they satisfy the equations

$$(8.14) \quad \frac{df^x}{dt} - e^x \frac{d\omega^x}{dt} = e^x \frac{d\omega^1}{dt}.$$

Now according to our assumptions  $d f^x/dt$  and  $e^x$  are linearly independent vectors and they are known as functions of  $t$  along the curve. That means that there exists an equation of the form

$$(8.15) \quad \frac{df^x}{dt} = \alpha_1 e^x + \alpha_x e^x; \quad \alpha_1 \neq 0$$

valid at all points of the curve and that  $\alpha_1, \alpha_x$  are known as functions of  $t$ . But then it follows from (8.14, 15) that

$$(8.16) \quad \frac{d\omega^1}{dt} = \alpha_1; \quad \frac{d\omega^x}{dt} = \alpha_x,$$

from which the functions  $\omega^1(t)$  and  $\omega^x(t)$  can be found by means of quadratures. If the curve happens to be a geodesic and if for  $t$  we choose an affine parameter,  $e^x$  can be chosen such that  $\alpha_1 = 1, \alpha_x = 0$ . Then we get as a solution  $\xi^1 = t, \xi^x = zt^x$ .

In order to prove that the  $\Gamma_{ji}^h$  vanish at all points of the curve we remark that (8.12) holds for every choice of  $(x)$ , hence it holds if we take  $(h)$  for  $(x)$ . From (8.14) we then get  $f^h = \omega^h$  at all points of the curve, which is in accordance with (8.10). (8.12) changes into

$$(8.17) \quad \left\{ \begin{array}{l} \xi^h = \omega^h(\theta(\xi^1)) + e^h (\xi^x - \omega^x(\theta(\xi^1))) - \\ - \frac{1}{2!} e^j e^i \Gamma_{ji}^h(\xi^1) (\xi^x - \omega^x(\theta(\xi^1))) (\xi^y - \omega^y(\theta(\xi^1))) - \dots \end{array} \right.$$

and from this equation we get by differentiation

$$(8.18) \quad \left\{ \begin{array}{l} \delta_y^h = \partial_y \xi^h = e^h - \Gamma_{yx}^h (\xi^x - \omega^x(\theta(\xi^1))) - \dots \\ x, y = 2, \dots, n; h = 1, \dots, n; \end{array} \right.$$

hence  $\Gamma_{yx}^h = 0$  at all points of the curve. Moreover we know that the fields  $e^x$  are covariant constant along the curve. Hence we have on the curve [cf. (8.15)]

$$(8.19) \quad \theta = \frac{df^j}{dt} V_j e^h = \alpha_1 \Gamma_{1i}^h + \alpha_x \Gamma_{yi}^h$$

or

$$(8.20) \quad \alpha_1 \Gamma_{11}^h + \alpha_x \Gamma_{1x}^h = 0; \quad \alpha_1 \Gamma_{1x}^h = 0; \quad x, y = 2, \dots, n;$$

and because of  $\alpha \neq 0$  this also proves that  $\Gamma_{y1}^h = \Gamma_{1y}^h \equiv 0$  and  $\Gamma_{11}^h \equiv 0$  at all points of the curve. The coordinates  $(h)$  defined in this way are called *FERMI coordinates* for the curve considered. By repeated differentiation it can be proved that [cf. (7.15)]

$$(8.21) \quad \begin{cases} \Gamma_{z_p \dots z_1 y}^h \equiv 0; & y, z_1, \dots, z_{p+1} = 2, \dots, n; \quad h = 1, \dots, n \\ \partial_{(z_{p+1})} \Gamma_{z_p \dots z_1 y}^h \equiv 0 \end{cases}$$

at all points of the curve.

If we know of an  $X_m$  that there exist  $n$  linearly independent vector fields  $e_i^*$ , *allround covariant stationary* in  $A_n$ <sup>1)</sup> over  $X_m$ , it can easily be proved that there exists a coordinate system  $(h)$  such that the  $\Gamma_{ji}^h$  vanish at all points of  $X_m$ .<sup>2)</sup> In the case of an  $X_1$  we have proved more, the fields  $e_i^*$  being only *covariant constant* over  $X_1$ . It is not yet known whether a system  $(h)$  with  $\Gamma_{ji}^h$  vanishing over  $X_m$  can be constructed if only covariant constant fields  $e_i^*$  are supposed to exist over  $X_m$ .

## § 9. Linear connexions expressed in anholonomic coordinates (cf. II § 9).

If we write for the covariant derivative of a vector with respect to an anholonomic coordinate system  $(h)$

$$(9.1) \quad \begin{cases} a) \quad V_j v^h = \partial_j v^h + \Gamma_{ji}^h v^i; & \partial_j \stackrel{\text{def}}{=} A_j^\mu \partial_\mu \\ b) \quad V_j w_i = \partial_j w_i - \Gamma_{ji}^h w_h \end{cases}$$

and if  $(\alpha)$  is a holonomic system, we get for the  $\Gamma_{ji}^h$  the transformation [cf. (2.5)]

$$(9.2) \quad \Gamma_{ji}^h = A_j^\mu \Gamma_{\mu\lambda}^\alpha A_\lambda^h,$$

hence (cf. II § 9)

$$(9.3) \quad \Gamma_{[ji]}^h = S_{ji}^h - \Omega_{ji}^h,$$

which implies that  $\Gamma_{[ji]}^h$  need not be zero if the connexion is symmetric. From (9.2) follows

$$(9.4) \quad \Gamma_j = A_j^\mu \Gamma_\mu - A_{ji}^{\mu\lambda} \partial_\mu A_\lambda^i.$$

The equations (9.2) and (9.4) also hold if  $(h)$  and  $(\alpha)$  are both anholonomic.

<sup>1)</sup> Cf. III § 2.

<sup>2)</sup> E I 1935, 1, p. 106. Read on that page  $A_n$  instead of  $L_n$ .

From (9.3) we get the general formulae for Rot and Div with  $\nabla$  in anholonomic components (cf. 2.17, 2.30)

$$(9.5) \quad \nabla_{[j} w_{i_1 \dots i_q]} = \partial_{[j} w_{i_1 \dots i_q]} - q S_{[j i_q}^{\dots h} w_{i_1 \dots i_{q-1}]h} + q Q_{[j i_q}^h w_{i_1 \dots i_{q-1}]h};$$

$$(9.6) \quad \left\{ \begin{array}{l} \nabla_j w^{j h_2 \dots h_p} = \partial_j w^{j h_2 \dots h_p} - 2 S_{j i}^{i h} w^{j h_2 \dots h_p} + (\rho - 1) S_{j i}^{i [h_2} w^{j i] h_3 \dots h_p} + \\ \quad + 2 Q_{j i}^i w^{j h_2 \dots h_p} - (\rho - 1) Q_{j i}^{[h_2} w^{j i] h_3 \dots h_p]; \\ w^{h_1 \dots h_p} = \rho\text{-vector density of weight } +1. \end{array} \right.$$

From these equations another very simple proof of the equations in Exerc. II 9.1 can be derived.

If we try to express a linear connexion in terms of  $S_{j i}^{i h}$ ,  $g_{i h}$  and  $\nabla_j g_{i h}$  the only difference from III § 3 is that (9.3) occurs instead of (2.13). Hence, if we write

$$(9.7) \quad Q_{j i h} \stackrel{\text{def}}{=} Q_{j i}^k g_{k h}^{-1})$$

we get

$$(9.8) \quad \boxed{I_{j i}^h = g^{h k} (\chi_{\{j k i\}} - S_{\{j k i\}} + Q_{\{j k i\}} + \frac{1}{2} Q_{\{j k i\}}); \quad \chi_{j k i} \stackrel{\text{def}}{=} \frac{1}{2} \partial_j g_{k i}}$$

instead of (3.9).

This equation takes a very simple form in an ordinary  $V_n$  if we introduce an anholonomic system consisting of mutually perpendicular unit-vectors  $i_i^x, i_\lambda^h$  at every point. Then the components of  $g_{i h}$  are all 1 or 0 and  $S_{j i}^{i h}$  and  $Q_{j i}^{i h}$  vanish, hence<sup>2)</sup>

$$(9.9) \quad I_{j i}^h \stackrel{*}{=} + g^{h k} Q_{\{j k i\}} = - 2 g^{h k} Q_{k(j i)} - Q_{j i}^h$$

and

$$(9.10) \quad I_j \stackrel{*}{=} - Q_{j i}^i + g_{i h} Q_{j l}^h g^{i l} = 0.$$

In this case the  $I_{j i}^h$  have a simple geometric meaning. If we write

$$(9.11) \quad \Gamma_{j i h} \stackrel{\text{def}}{=} I_{j i}^k g_{k h}$$

it follows from  $\nabla_j g_{i h} = 0$  that

$$(9.12) \quad I_{j(i h)} \stackrel{*}{=} 0.$$

Now the covariant differential of  $i_i^h$  is for  $g_{i h}$  positive definite

$$(9.13) \quad \delta i_i^h \stackrel{*}{=} \Gamma_{j k}^h i_k^h (d\xi)^j \stackrel{*}{=} \Gamma_{j i h} (d\xi)^j \stackrel{*}{=} \Gamma_{j k h} i_k^h (d\xi)^j$$

<sup>1)</sup> Note that the  $Q_{j i h}$  are not components of a geometric object.

<sup>2)</sup> VRANCEANU 1932, 1.

and this means that for every fixed value of  $i$  the vector  $\dot{e}_i^h$  undergoes an infinitesimal rotation with respect to a parallel moving coordinate system, with  $\Gamma_{jkh}(d\xi)^j$  as “bivector” of rotation (cf. I § 9). As such the  $\Gamma_{jih}$  were discovered by RICCI<sup>1)</sup> long before anholonomic coordinates came into use. He wrote

$$(9.14) \quad \gamma_{ihj} \stackrel{\text{def}}{=} \dot{\Gamma}_{jih} = \Gamma_{jih}^k g_{kh}$$

and called the  $\gamma_{ihj}$  the *coefficients of rotation* of the orthogonal net consisting of the  $n$  congruences of streamlines of the  $n$  fields  $\dot{e}_i^x$ . The  $\gamma_{ihj}$  alternate in the *first two indices*.<sup>2)</sup>

In an  $L_n$  facts are quite analogous. Having introduced the anholonomic system  $(h)$  the covariant differential of  $e_i^h$  is

$$(9.15) \quad \delta e_i^h \stackrel{*}{=} \Gamma_{jih}^h (d\xi)^j \stackrel{*}{=} \Gamma_{jik}^h e_i^k (d\xi)^j.$$

Hence, for every fixed value of  $i$  the vector  $e_i^h$  undergoes a homogeneous affine transformation with respect to a parallel moving coordinate system, with  $\Gamma_{jih}^h(d\xi)^j$  as “tensor” of the transformation.<sup>3)</sup>

The rotation coefficients  $\Gamma_{jih}$  contain a lot of information about the orthogonal net of  $(h)$ . A congruence  $i^x$  is geodesic (cf. III § 7, V § 5) if and only if  $i^\mu V_\mu i^h = 0$ . Hence the n.a.s. conditions for the congruence  $i^x$  to be geodesic are

$$(9.16) \quad \Gamma_{nnh} \stackrel{*}{=} -\Gamma_{nhn} \stackrel{*}{=} 0.$$

A congruence is  $V_{n-1}$ -normal if and only if (cf. II 5.16, V § 5)  $i_{[\nu} V_\mu i_{\lambda]} = 0$ , hence the n.a.s. conditions for the congruence  $i^x$  to be  $V_{n-1}$ -normal are

$$(9.17) \quad \Gamma_{[cb]n} \stackrel{*}{=} 0; \quad b, c = 1, \dots, n-1.$$

Taking (9.16) and (9.17) together we get for a geodesic and  $V_{n-1}$ -normal congruence  $i^x$  the n.a.s. conditions

$$(9.18) \quad \Gamma_{[jih]n} \stackrel{*}{=} 0.$$

All congruences  $i^x$  are geodesic and  $V_{n-1}$ -normal if and only if  $\Gamma_{[jih]n} \stackrel{*}{=} 0$ .

But since the  $\Gamma_{jih}$  are alternating in  $ih$  this is only possible if  $\Gamma_{jih} \stackrel{*}{=} 0$ ,

<sup>1)</sup> RICCI 1895, 1, p. 303. Cf. SCHOUTEN-STRUJK 1919, 1; LEVY 1925, 1; HLA-VATY 1928, 1; VRANCEANU 1929, 3; MATTIOLI 1930, 1; 2; DIENES 1933, 1; 2; E I 1935, 1, p. 85; E II 1938, 2, p. 34. The coefficients of rotation are intimately connected with the theory of the “trièdre mobile” of DARBOUX, cf. Ricci and LEVI CIVITA 1901, 1, p. 157.

<sup>2)</sup> Of course, if we have introduced the anholonomic system  $(h)$  and thus the  $\dot{e}_i^h$  and  $\Gamma_{jih}$ , it is incorrect from a methodical point of view to introduce *besides* these the rotation  $\gamma_{ihj}$  also, as many authors still do.

<sup>3)</sup> Cf. SLEBODZINSKI 1930, 1. In this case also it would be incorrect to introduce some kind of “generalized coefficients of rotation” *besides* the  $\Gamma_{jih}^h$ .

that is if the  $I_{ji}^h$  vanish at all points. But this implies according to (9.3) that  $(h)$  is holonomic and that consequently  $K_{kji}^{;;h}$  vanishes. Hence *an orthogonal net of geodesic and  $V_{n-1}$ -normal congruences is only possible in an  $R_n$ .*

In order to find the expression for the anholonomic components  $R_{kji}^{;;h}$  of the curvature tensor we start from (9.1b) and use (9.3) and (II 9.4). Then we find

$$(9.19) \quad V_{[k} V_{j]} w_i = - \{ \partial_{[k} I_{j]}^h + I_{[k|l]}^h I_{j]}^l + \Omega_{kj}^l I_{l|i}^h \} w_h - S_{kji}^{;;h} V_h w_i.$$

Comparing this result with (4.9a) we get

$$(9.20) \quad R_{kji}^{;;h} = 2 \partial_{[k} I_{j]}^h + 2 I_{[k|l]}^h I_{j]}^l + 2 \Omega_{kj}^l I_{l|i}^h$$

instead of (4.2) and, instead of (4.15) we get

$$(9.21) \quad V_{kj} = 2 \partial_{[k} I_{j]} + 2 \Omega_{kj}^l I_l.$$

The equations (9.20, 21) could also be derived from (4.2), (4.15) and (9.2).

## § 10. CARTAN'S SYMBOLICAL METHOD USED FOR CONNEXIONS.

Using the symbolic notation explained in II § 12 we write<sup>1)</sup>

$$(10.1) \quad \left\{ \begin{array}{ll} A^h \text{ for } A_\lambda^h & \text{and also for } A_\lambda^h d\xi^\lambda \\ \Omega^h \text{ for } \Omega_{\mu\lambda}^h = \Omega_{ji}^h A_\mu^j A_\lambda^i & \text{and also for } \Omega_{\mu\lambda}^h d\xi^\mu d\xi^\lambda \\ I_i^h \text{ for } I_{\mu i}^h = I_{ji}^h A_\mu^j & \text{and also for } I_{\mu i}^h d\xi^\mu \\ S^h \text{ for } S_{\mu\lambda}^{;;h} & \text{and also for } S_{\mu\lambda}^{;;h} d\xi^\mu d\xi^\lambda \\ R_i^{;;h} \text{ for } R_{\mu\lambda i}^{;;h} & \text{and also for } R_{\mu\lambda i}^{;;h} d\xi^\mu d\xi^\lambda. \end{array} \right.$$

In II § 12 we had already for  $\omega$  holonomic (cf. II 12.9):

$$(10.2) \quad \left\{ \begin{array}{ll} \text{a)} & \partial_{[\mu} A_{\lambda]}^h = \Omega_{ji}^h A_{\mu\lambda}^{ji} = \Omega_{\mu\lambda}^h \\ \text{b)} & \mathbf{C} [dA^h] = \Omega^h. \end{array} \right.$$

<sup>1)</sup> CARTAN writes  $A_{ji}^h$  and  $A_{ikj}^h$  or  $R_{ikj}^h$  for  $S_{ji}^{;;h}$  and  $R_{kji}^{;;h}$  but  $\omega^h$ ,  $\omega_i^h$ ,  $\Omega^h$  and  $\Omega_i^h$  for what we have written  $A^h$ ,  $I_i^h$ ,  $S^h$  and  $R_i^{;;h}$ . This gives rise to a duplication of formulae that could have been avoided. The same holds for all publications where the abridged calculus is used, except those dealing with alternating forms only. T. Y. THOMAS gave 1927, 1 a severe criticism and used such expressions as "superabundance of geometrical terminology" and "a lack of content". This is certainly too severe. The first objection can be countered by the change of notation we have proposed here, that makes duplication of formulae unnecessary and the second one is only true where the abridged calculus is used at places where it is not appropriate to do so.

Now we know from (9.3) that

$$(10.3) \quad \begin{cases} \text{a)} & Q_{\mu\lambda}^h = -A_{\mu\lambda}^{ij}(\Gamma_{[j}^h - S_{j|i}^h) = -\Gamma_{[\mu|k]}^h A_{\lambda]}^k + S_{\mu\lambda}^{i|h}; \\ \text{b)} & \mathcal{Q}^h = [A^i \Gamma_i^h] + S^h. \end{cases}$$

Hence, from (10.2) and (10.3)

$$(10.4) \quad \begin{cases} \text{a)} & \partial_{[\mu} A_{\lambda]}^h = A_{[\mu}^i \Gamma_{\lambda]}^h + S_{\mu\lambda}^{i|h} \\ \text{b)} & [d A^h] = [A^i \Gamma_i^h] + S^h. \end{cases}$$

Moreover we know from (9.20) that

$$(10.5) \quad \begin{cases} \text{a)} & \partial_{[v} \Gamma_{\mu]}^h i + \Gamma_{[v|l]}^h \Gamma_{\mu]}^l i = \frac{1}{2} R_{v\mu|i}^{i|h} \\ \text{b)} & [d \Gamma_i^h] + [\Gamma_l^h \Gamma_i^l] = \frac{1}{2} R_i^{i|h}. \end{cases}$$

CARTAN calls (10.4b) and (10.5b) the *structural equations* (équations de structure) of the  $L_n$ . According to the definition, repeated application of  $\partial_{[v\dots]}$  or  $[d\dots]$  leads to zero, hence from (10.4), giving the calculation in full for the symbolical method

$$(10.6) \quad \begin{cases} \text{a)} & 0 = -\frac{1}{2} A_{[\omega}^i R_{v\mu]}^{i|h} + \partial_{[\omega} S_{v\mu]}^{i|h} + S_{[\omega|v}^i \Gamma_{\mu]}^h \\ \text{b)} & \mathcal{C} \quad 0 = [[d A^i] \Gamma_i^h] - [A^i [d \Gamma_i^h]] + [d S^h] \\ & = [A^i \Gamma_l^i \Gamma_i^h] + [S^i \Gamma_i^h] + [A^i \Gamma_l^h \Gamma_i^l] - \frac{1}{2} [A^i R_i^{i|h}] + [d S^h] \\ & = -\frac{1}{2} [A^i R_i^{i|h}] + [d S^h] + [S^i \Gamma_i^h], \end{cases}$$

or

$$(10.7) \quad \begin{cases} \text{a)} & \boxed{A_{[\omega}^i R_{v\mu]}^{i|h} - 2 \overset{2}{V}_{[\omega} S_{v\mu]}^{i|h} = 0} \\ \text{b)} & \mathcal{C} \quad \boxed{[A^i R_i^{i|h}] - 2 \overset{2}{V} S^h = 0^1} \end{cases} \quad (\text{second identity, cf. 5.2, 41 and Exerc. III 5,7}).$$

In the same way, we get from (10.5)

$$(10.8) \quad \begin{cases} \text{a)} & 0 = \frac{1}{2} \partial_{[\omega} R_{v\mu]}^{i|h} - \frac{1}{2} \Gamma_{[\omega|i]}^l R_{v\mu]}^{i|h} + \frac{1}{2} \Gamma_{[\omega|l]}^h R_{v\mu]}^{i|h} \\ \text{b)} & \mathcal{C} \quad 0 = \frac{1}{2} [d R_i^{i|h}] - [[d \Gamma_l^h] \Gamma_i^l] + [\Gamma_l^h [d \Gamma_i^l]] \\ & = \frac{1}{2} [d R_i^{i|h}] + [\Gamma_k^h \Gamma_l^k \Gamma_i^l] - \frac{1}{2} [\Gamma_l^l R_i^{i|h}] - \\ & \quad - [\Gamma_l^h \Gamma_k^l \Gamma_i^k] + \frac{1}{2} [\Gamma_l^h R_i^{i,l}] \end{cases}$$

or

$$(10.9) \quad \begin{cases} \text{a)} & \boxed{\overset{2}{V}_{[\omega} R_{v\mu]}^{i|h} = 0} \\ \text{b)} & \mathcal{C} \quad \boxed{\overset{2}{V} R_i^{i|h} = 0^1} \end{cases} \quad (\text{identity of BIANCHI, cf. 5.19, 42 and Exerc. III 5,7}).$$

<sup>1)</sup> CARTAN calls (10.7b) and (10.9b) "théorème de conservation de la courbure et de la torsion".

In these formulae we have used the operator  $\overset{\rho}{V}[\dots]$  corresponding to the operator  $\overset{\rho}{V}_{[\omega\dots]}$  defined in III § 5. The working of the first operator on any quantity  $P_{\lambda_1\dots\lambda_p}^{h_1\dots h_q} v_{i_1\dots i_r}$  or its corresponding alternating differential form (cf. II § 12) is defined for  $\rho > 0$  by the equation (cf. 5.32)

$$(10.10) \quad \left\{ \begin{aligned} \overset{\rho}{V} P_{\cdot}^{h_1\dots h_q} v_{i_1\dots i_r} &= [d P_{\cdot}^{h_1\dots h_q} v_{i_1\dots i_r}] + [\Gamma_l^{h_1} P_{\cdot}^{l h_2\dots h_q} v_{i_1\dots i_r}] + \dots \\ &\quad - [\Gamma_{i_1}^l P_{\cdot}^{h_1\dots h_q} v_{i_2\dots i_r}] - \dots \end{aligned} \right.$$

In order to make comparison between tensor calculus and CARTAN's notation possible in a more complicated case we now translate the proof of the geometrical meaning of the two identities (10.7, 9) given in III § 5. Writing  $d\xi^x$  for  $\xi^x - \xi^x$ ,  $d\xi^{[p} d\xi^{\mu]}$  for  $d f^{p\mu}$ ,  $d\xi^{[\omega} d\xi^{\nu} d\xi^{\mu]}$  for  $d f^{\omega\nu\mu}$  and passing to general anholonomic coordinates we get for the change of the radiusvector  $v^h$  from  $\xi^x$  to  $\xi^x$

$$(10.11) \quad \mathcal{C}_v v^h = A^h - \Gamma_i^h v^i \quad (5.43)$$

and for the change after moving around  $d f^{p\mu}$

$$(10.12) \quad \mathcal{C}_S S^h + d S^h + \frac{1}{2} R_i^h v^i - \frac{1}{2} R_i^h A^i - \frac{1}{2} R_i^h \Gamma_j^i v^j + \frac{1}{2} v^i d R_i^h \quad (5.44)$$

After parallel displacement we get

$$(10.13) \quad \left\{ \begin{aligned} \mathcal{C}_S S^h + \frac{1}{2} R_i^h v^i + S^i \Gamma_i^h + \frac{1}{2} R_i^j v^i \Gamma_j^h + d S^h - \\ (5.45) \quad &- \frac{1}{2} R_i^h A^i - \frac{1}{2} R_i^h \Gamma_j^i v^j + \frac{1}{2} v^i d R_i^h, \end{aligned} \right.$$

and after applying the theorem of STOKES

$$(10.14) \quad \left\{ \begin{aligned} \mathcal{C}_S \int_{\gamma_s} [S^i \Gamma_i^h] + \frac{1}{2} v^i [R_i^j \Gamma_j^h] + [d S^h] - \frac{1}{2} [A^i R_i^h] - \\ (5.46) \quad &- \frac{1}{2} [R_i^h \Gamma_j^i] v^i + \frac{1}{2} v^i [d R_i^j] \\ &= \int_{\gamma_s} \{[\overset{\rho}{V} S^h] - \frac{1}{2} [A^i R_i^h] + \frac{1}{2} v^i [\overset{\rho}{V} R_i^h]\}_{\xi^x = \xi^x}. \end{aligned} \right.$$

The abridged calculus has often been used by CARTAN to solve problems concerning point transformations transforming fields into each other. We consider first the question whether  $n$  given linearly independent vector fields  $\overset{\rho}{w}_\lambda$  in an  $X_n$  with coordinates  $\xi^x$  can be transformed by a transformation  $\xi^x \rightarrow \xi^x$  into  $n$  given linearly independent

vector fields  $'\vec{w}_\lambda'$  in another  $X_n$  with coordinates  $\xi^\kappa'$

$$(10.15) \quad \begin{cases} \text{a)} & \vec{w}_\lambda d\xi^\lambda = 'w_{\lambda'}^h d\xi^{\lambda'} {}^1) \\ \text{b)} & \mathcal{C} \quad \vec{w} = 'w. \end{cases}$$

By differentiation we get

$$(10.16 \text{ a}) \quad \begin{cases} \alpha) & \partial_{[\mu} \vec{w}_{\lambda]}^h = u_{j;i}^h \vec{w}_{[\mu}^j \vec{w}_{\lambda]}^i; \\ \beta) & \partial_{[\mu'} 'w_{\lambda']}^h = 'u_{j;i}^h 'w_{[\mu'}^j 'w_{\lambda']}^i; \end{cases}$$

$$(10.16 \text{ b}) \quad \mathcal{C} \quad \begin{cases} \alpha) & [d\vec{w}] = u_{j;i}^h [\vec{w}^j \vec{w}^i]; \\ \beta) & [d 'w] = 'u_{j;i}^h ['w^j 'w^i]; \end{cases}$$

where  $u_{j;i}^h$  and  $'u_{j;i}^h$  are known functions of  $\xi^\kappa$  and  $\xi^{\kappa'}$  respectively. From (10.15) and (10.16) there follows as a first condition the set of non-differential equations between  $\xi^\kappa$  and  $\xi^{\kappa'}$

$$(10.17) \quad u_{j;i}^h(\xi^\kappa) = 'u_{j;i}^h(\xi^{\kappa'}).$$

If these equations are inconsistent, no transformation  $\xi^\kappa \rightarrow \xi^{\kappa'}$  transforming the fields  $\vec{w}$  into the fields  $'\vec{w}$  is possible. If they are consistent we get by differentiation (defining the operator  $\partial_j$  in both  $X_n$ 's by  $\partial_\mu = \vec{w}_\mu^j \partial_j$ ;  $\partial_{\mu'} = 'w_{\mu'}^j \partial_j$ ):

$$(10.18 \text{ a}) \quad \begin{cases} \alpha) & \partial_\mu u_{j;i}^h = \vec{w}_\mu^k \partial_k u_{j;i}^h; \\ \beta) & \partial_{\mu'} 'u_{j;i}^h = 'w_{\mu'}^k \partial_k 'u_{j;i}^h; \end{cases}$$

$$(10.18 \text{ b}) \quad \mathcal{C} \quad \begin{cases} \alpha) & d u_{j;i}^h = \vec{w}^k \partial_k u_{j;i}^h; \\ \beta) & d 'u_{j;i}^h = 'w^k \partial_k 'u_{j;i}^h; \end{cases}$$

and from this we get as a second set of non-differential equations between  $\xi^\kappa$  and  $\xi^{\kappa'}$

$$(10.19) \quad \partial_k u_{j;i}^h = \partial_k 'u_{j;i}^h.$$

In the same way we can derive

$$(10.20) \quad \partial_l \partial_k u_{j;i}^h = \partial_l \partial_k 'u_{j;i}^h$$

and so on till we get inconsistent equations or a set depending on the former sets. In this latter case all following sets depend on these former sets and we have got a finite number of equations from which the  $\xi^{\kappa'}$  can be solved as functions of the  $\xi^\kappa$  and a finite number of parameters. These functions give the transformation  $\xi^\kappa \rightarrow \xi^{\kappa'}$  required.

<sup>1)</sup> CARTAN 1946, 1, p. 315 ff.

The result just obtained can be applied to the question of whether there exists a transformation  $\xi^\alpha \rightarrow \xi^{\alpha'}$  mapping an  $L_n$  with coordinates  $\xi^\alpha$  and a connexion  $\Gamma_{\mu\lambda}^\alpha$  on another  $L_n$ , called hereafter ' $L_n$ ' with coordinates  $\xi^{\alpha'}$  and a connexion ' $\Gamma_{\mu'\lambda'}^{\alpha'}$ '.<sup>1)</sup> In  $L_n$  we take an arbitrary anholonomic coordinate system ( $h$ ) and in ' $L_n$ ' an arbitrary anholonomic coordinate system ( $h'$ ). Then at each point of  $L_n$  the  $A_\lambda^h$  depend on  $r$  ( $= n^2$ ) parameters  $\eta^\alpha$ ,  $\alpha = \dot{1}, \dots, \dot{r}$  and at each point of ' $L_n$ ' the  $A_{\lambda'}^h$  depend on  $r$  parameters  $\eta^{\alpha'}$ ,  $\alpha' = \dot{1}', \dots, \dot{r}'$ .<sup>2)</sup> Then we have to investigate whether it is possible to choose the systems ( $h$ ) in  $L_n$  and ' $L_n$ ' such that  $\Gamma_{ji}^h = \Gamma_{j'i}^{h'}$  at corresponding points. We remark that, though  $\Gamma_{\mu\lambda}^\alpha$ ,  $S_{\mu\lambda}^{\cdot\alpha}$  and  $R_{\nu\mu\lambda}^{\cdot\alpha}$  are independent of the  $\eta^\alpha$ , the components  $\Gamma_{ji}^h$ ,  $S_{ji}^{\cdot h}$  and  $R_{kji}^{\cdot h}$  also depend on these parameters, and that the same holds mutatis mutandis in ' $L_n$ '. In  $L_n$  the equations (10.4) and (10.5) are valid but  $A_\lambda^h$ ,  $\Gamma_{ji}^h$ ,  $S_{\mu\lambda}^{\cdot h}$  and  $R_{\nu\mu\lambda}^{\cdot h}$  now depend not only on the  $\xi^\alpha$  but also on the  $\eta^\alpha$ . That means that we are now working in the  $X_{n+}$ , of the variables  $\xi^\alpha$ ,  $\eta^\alpha$  from which the  $L_n$  can be formed by reduction (cf. II § 4). If a field  $v^\alpha$  or  $w_\lambda$  in  $L_n$  depends on the parameters  $\eta^\alpha$ , it may be considered as the reduction of a field  $v^\alpha$ ,  $v^\alpha = \text{undetermined}$  or as the section of a field  $w_\lambda$ ,  $w_\beta = 0$  in the  $X_{n+}$ . Hence the covariant multivectors  $A_\lambda^h$ ,  $S_{\mu\lambda}^{\cdot h}$  and  $R_{\nu\mu\lambda}^{\cdot h}$  may be considered as the sections of the multivectors  $A_B^h$ ,  $S_{CB}^{\cdot h}$  and  $R_{D\dot{C}\dot{B}}^{\cdot h}$ ;  $A, B, C, D = 1, \dots, n, \dot{1}, \dots, \dot{r}$  satisfying the conditions

$$(10.21) \quad \begin{cases} A_\beta^h = 0; & S_{\dot{\gamma}\lambda}^{\cdot h} = -S_{\lambda\dot{\gamma}}^{\cdot h} = 0; & S_{\gamma\beta}^{\cdot h} = 0; & R_{\delta\gamma i}^{\cdot h} = 0; \\ & R_{\delta\mu i}^{\cdot h} = -R_{\mu\delta i}^{\cdot h} = 0; & \alpha, \beta, \gamma, \delta = \dot{1}, \dots, \dot{r}. \end{cases}$$

In the  $X_{n+}$ , of  $\xi^\alpha$ ,  $\eta^\alpha$  the allowable coordinate transformations are  $\xi^{\alpha'} = f^{\alpha'}(\xi^\alpha)$ ;  $\eta^{\alpha'} = \varphi^{\alpha'}(\eta^\alpha)$ , hence

$$(10.22) \quad A_\beta^{\alpha'} = 0; \quad A_{\beta'}^{\alpha} = 0; \quad A_\lambda^{\alpha'} = 0; \quad A_{\lambda'}^{\alpha} = 0.$$

Accordingly the components of some connexion  $\Gamma_{CB}^A$  in  $X_{n+}$ , transform as follows

$$(10.23) \quad \begin{cases} \Gamma_{\mu'\lambda'}^{\alpha'} = A_{\mu'\lambda'}^{\mu\lambda} \Gamma_{\mu\lambda}^\alpha + A_{\lambda'}^{\alpha'} \partial_{\mu'} A_\lambda^\alpha; \\ \Gamma_{\mu'\lambda'}^{\alpha'} = A_{\mu'\lambda'}^{\mu\lambda} \Gamma_{\mu\lambda}^\alpha; \\ \Gamma_{\gamma'\lambda'}^{\alpha'} = A_{\gamma'\lambda'}^{\gamma\lambda} \Gamma_{\gamma\lambda}^\alpha; \quad \Gamma_{\mu'\beta'}^{\alpha'} = A_{\mu'\beta'}^{\mu\beta} \Gamma_{\mu\beta}^\alpha, \\ \Gamma_{\gamma'\beta'}^{\alpha'} = A_{\gamma'\beta'}^{\gamma\beta} \Gamma_{\gamma\beta}^\alpha; \\ \Gamma_{\mu'\beta'}^{\alpha'} = A_{\mu'\beta'}^{\mu\beta} \Gamma_{\mu\beta}^\alpha; \quad \Gamma_{\gamma'\lambda'}^{\alpha'} = A_{\gamma'\lambda'}^{\gamma\lambda} \Gamma_{\gamma\lambda}^\alpha \\ \Gamma_{\gamma'\beta'}^{\alpha'} = A_{\gamma'\beta'}^{\gamma\beta} \Gamma_{\gamma\beta}^\alpha + A_\beta^{\alpha'} \partial_{\gamma'} A_\gamma^\beta. \end{cases}$$

<sup>1)</sup> CARTAN has dealt with this problem for the case of two  $V_n$ 's.

<sup>2)</sup> In the case of two  $V_n$ 's considered by CARTAN the local coordinate systems can be taken orthogonal and then  $r = \binom{n}{2}$ .

We see from (10.23) that the  $\Gamma_{\mu\lambda}^{\alpha}$  transform just like the parameters of a connexion in  $L_n$  and that of the other seven sets of indices six undergo a linear homogeneous transformation. Hence in  $X_{n+r}$ , a connexion can be partially fixed by taking the  $\Gamma_{\mu\lambda}^{\alpha}$  equal to the  $\Gamma_{\mu\lambda}^{\alpha}$  of the connexion in  $L_n$  and

$$(10.24) \quad \Gamma_{\mu\lambda}^{\alpha} = 0; \quad \Gamma_{\gamma\lambda}^{\alpha} = 0; \quad \Gamma_{\mu\beta}^{\alpha} = 0; \quad \Gamma_{\gamma\beta}^{\alpha} = 0; \quad \Gamma_{\mu\beta}^{\alpha} = 0; \quad \Gamma_{\gamma\lambda}^{\alpha} = 0$$

and leaving the  $\Gamma_{\gamma\beta}^{\alpha}$  undetermined. This agrees with the fact that there is no connexion given in the space of the parameters  $\eta^{\alpha}$ .

Passing now to the anholonomic system  $(h, \alpha)$  in  $X_{n+r}$  (the  $\eta^{\alpha}$  remain unchanged) we get

$$(10.25) \quad \begin{cases} \text{a)} & \Gamma_{ji}^h = A_j^{\mu\lambda} \Gamma_{\mu\lambda}^{\alpha} - A_j^{\mu\lambda} \partial_{\mu} A_{\lambda}^h; \\ \text{b)} & \Gamma_{\gamma i}^h = -A_i^{\lambda} \partial_{\gamma} A_{\lambda}^h; \\ \text{c)} & \Gamma_{j\beta}^h = 0; \quad \text{d)} \quad \Gamma_{\gamma\beta}^h = 0; \quad \text{e)} \quad \Gamma_{ji}^{\alpha} = 0; \quad \text{f)} \quad \Gamma_{\gamma i}^{\alpha} = 0; \quad \text{g)} \quad \Gamma_{j\beta}^{\alpha} = 0 \end{cases}$$

and this means that the  $\Gamma_{ji}^h$  transform in the ordinary way, that the only new components are the  $\Gamma_{\gamma i}^h$ , and that the  $\Gamma_{\gamma\beta}^h$  remain undetermined.

Using (10.25) we get besides (10.4)

$$(10.26) \quad \begin{cases} \text{a)} & \partial_{[\gamma} A_{\lambda]}^h = A_{[\gamma}^i \Gamma_{\lambda]}^h + S_{\gamma\lambda}^{i\cdot h}; \\ \text{b)} & 0 = \partial_{[\gamma} A_{\beta]}^h = A_{[\gamma}^i \Gamma_{\beta]}^h + S_{\gamma\beta}^{i\cdot h}; \end{cases}$$

or, taking (10.4) and (10.26) together

$$(10.27) \quad \begin{cases} \text{a)} & \partial_{[C} A_{B]}^h = A_{[C}^i \Gamma_{B]}^h + S_{CB}^{i\cdot h}; \quad B, C = 1, \dots, n, \dot{1}, \dots, \dot{r}; \\ \text{b)} & \mathbf{C} [dA^h] = [A^i \Gamma_i^h] + S^h, \end{cases}$$

in accordance with CARTAN's general principle that an abridged equation remains in some way valid if the number of independent variables is extended by the introduction of parameters. According to this principle we should now have also next to (10.5)

$$(10.28) \quad \begin{cases} \text{a)} & \partial_{[D} \Gamma_{C]i}^h + \Gamma_{[D|I|}^h \Gamma_{C]i}^l = \frac{1}{2} R_{D\dot{C}\dot{i}}^{i\cdot h}; \\ \text{b)} & \mathbf{C} [d\Gamma_i^h] + [\Gamma_i^h \Gamma_i^l] = \frac{1}{2} R_i^{i\cdot h}. \end{cases}$$

In fact the  $\Gamma_{\mu\lambda}^{\alpha}$  are independent of the parameters, hence according to (10.25) after some calculation

$$(10.29) \quad 0 = \partial_{[\delta} \Gamma_{\mu]\lambda}^{\alpha} = 2A_{\lambda}^i A_{\mu}^{\alpha} (\partial_{[\delta} \Gamma_{\mu]i}^h + \Gamma_{[\delta|I|}^h \Gamma_{\mu]i}^l)$$

and this proves the validity of (10.28a) for  $D = \delta$ ,  $C = \mu$ ,  $R_{\delta\mu i}^{i\cdot h} = 0$ . From (10.25b) we get

$$(10.30) \quad \partial_{[\delta} \Gamma_{\gamma]i}^h = -(\partial_{[\delta} A_{|i|}^{\lambda}) \partial_{\gamma]} A_{\lambda}^h = -\Gamma_{[\delta|I|}^h \Gamma_{\gamma]i}^l.$$

Hence (10.28a) also holds for  $D = \delta$ ,  $C = \gamma$ ,  $R_{\delta\gamma}{}^h = 0$ . Since the equations (10.4b) and (10.5b) are now valid in the more general sense of (10.27b) and (10.28b) it stands to reason that the same must hold for the second identity (10.7b) and BIANCHI's identity (10.9b), but a direct proof could be given.

In  $X_{n+r}$ , we now have the  $n+r$  vectors (or Pfaffians)  $A^h, \Gamma_i^h$  and their rotations given by (10.27, 28) and in the ' $X_{n+r}$ ' we have in the same way vectors (or Pfaffians) ' $A^h, \Gamma_i^h$ ' whose rotations satisfy equations of the same form. The problem is how to find a transformation  $\xi^\alpha, \eta^\alpha \rightarrow \xi'^\alpha, \eta'^\alpha$  such that  $A^h = 'A^h, \Gamma_i^h = '\Gamma_i^h$ . If we write (10.27) and (10.28) in the form

$$(10.31) \quad \begin{cases} \text{a)} & \partial_{[C} A_{B]}^h = A_{[C}^i \Gamma_{B]}^h + S_{ji}{}^h A_{[C}^j A_{B]}^i \\ \text{b)} & \mathbf{C} [dA^h] = [A^i \Gamma_i^h] + S_{ji}{}^h [A^j A^i]; \end{cases}$$

$$(10.32) \quad \begin{cases} \text{a)} & \partial_{[D} \Gamma_{C]}^h = -\Gamma_{[D}^h \Gamma_{C]}^l + \frac{1}{2} R_{kji}{}^h A_{[D}^k A_{C]}^j \\ \text{b)} & \mathbf{C} [d\Gamma_i^h] = -[\Gamma_l^h \Gamma_i^l] + \frac{1}{2} R_{kji}{}^h [A^k A^j] \end{cases}$$

we have got equations of the form (10.16α) and equations of the same form, corresponding to (10.16β) can be derived for ' $X_{n+r}$ '. Comparison with (10.16) shows that many of the coefficients corresponding to the  $u_{ji}{}^h$  and the ' $u_{ji}{}^h$ ' vanish or are simple constants and that the remaining ones are the anholonomic components of  $S, R, 'S$  and ' $R$ '.

Now the holonomic components of any tensor of  $X_n ('X_n)$  are independent of the  $\eta^\alpha (\eta'^\alpha)$ . Hence, for instance for a tensor  $P_\lambda{}^\alpha$  we have

$$(10.33) \quad \begin{cases} \text{a)} & dP_i{}^h = d\xi^\mu \partial_\mu P_i{}^h + d\eta^\nu (\partial_\nu A_i{}^h) A_\nu{}^h P_\lambda{}^\alpha + d\eta^\nu A_i{}^h (\partial_\nu A_\nu{}^\alpha) P_\lambda{}^\alpha \\ & = d\xi^\mu A_\mu{}^j \nabla_j P_i{}^h + d\xi^\mu \Gamma_{\mu i}^l P_i{}^h - d\xi^\mu \Gamma_{\mu l}^h P_i{}^l + \\ & \quad + d\eta^\nu \Gamma_{\nu i}^l P_i{}^h - d\eta^\nu \Gamma_{\nu l}^h P_i{}^l; \\ \text{b)} & \mathbf{C} dP_i{}^h = A^j \nabla_j P_i{}^h + \Gamma_i^l P_i{}^h - \Gamma_l^h P_i{}^{h-1}. \end{cases}$$

Applying this formula to  $S_{ji}{}^h, R_{kji}{}^h$  and their covariant derivatives in  $L_n$  we get

$$(10.34) \quad \mathbf{C} dS_{ji}{}^h = A^k V_k S_{ji}{}^h + \Gamma_i^l S_{li}{}^h + \Gamma_l^h S_{ji}{}^h - \Gamma_l^h S_{ji}{}^{h-1};$$

$$(10.35) \quad \mathbf{C} dV_k S_{ji}{}^h = A^{k_1} V_{k_1} V_k S_{ji}{}^h + \Gamma_k^l V_l S_{ji}{}^h + \Gamma_j^l V_k S_{ji}{}^h + \Gamma_i^l V_k S_{ji}{}^h - \Gamma_l^h V_k S_{ji}{}^{h-1}$$

$$(10.36) \quad \mathbf{C} dR_{kji}{}^h = A^{k_1} V_{k_1} R_{kji}{}^h + \Gamma_k^l R_{lj}{}^h + \Gamma_j^l R_{ki}{}^h + \Gamma_i^l R_{kj}{}^h - \Gamma_l^h R_{kji}{}^{h-1}$$

---

<sup>1)</sup> E. CARTAN 1946, 1, p. 320 introduced this formula for  $K_{kji}{}^h, V_l K_{kji}{}^h$  in  $V_n$  as "evident". This optimism we don't share but it has to be granted that the general formula (10.33) after having been proved rigorously gives excellent means to use the abridged calculus for the calculations.

Hence, dealing with the  $n+r = n+n^2$  vector fields  $A_B^h, T_{C_i}^h$  in  $X_{n+r}$ , in the same way as with the  $n$  vector fields  $w_\lambda$  in  $X_n$  and using (10.34–36) we get the n.a.s. conditions for the mapping of the  $L_n$  and ' $L_n$ ' on each other in the form

$$(10.37) \quad \begin{cases} \text{a)} \ S_{ji}^{::h} = {}'S_{ji}^{::h}; \\ \text{b)} \ \nabla_k S_{ji}^{::h} = {}'\nabla_k {}'S_{ji}^{::h} \\ \vdots \end{cases}$$

$$(10.38) \quad \begin{cases} \text{a)} \ R_{kji}^{::h} = {}'R_{kji}^{::h}; \\ \text{b)} \ \nabla_{k_1} R_{kji}^{::h} = {}'\nabla_{k_1} {}'R_{kji}^{::h} \\ \vdots \end{cases}$$

in which all quantities on the left hand side are expressed as functions of  $\xi^\alpha, \eta^\alpha$  and all on the right hand side as functions of  $\xi'^\alpha, \eta'^\alpha$ . Either after a finite number of steps we get inconsistent equations or after a certain numbers of steps all following steps lead to equations dependent on the equations already derived. Only in the latter case there may exist transformations  $\xi^\alpha, \eta^\alpha \rightarrow \xi'^\alpha, \eta'^\alpha$  giving the desired mapping and they can be found by solving  $\xi^\alpha, \eta^\alpha$ , as functions of  $\xi'^\alpha, \eta'^\alpha$  and some parameters, from the equations (10.37, 38).<sup>1)</sup>

## § 11. Linear connexions depending on a non-symmetric fundamental tensor<sup>2)</sup>.

In III §3 we have seen that a fundamental tensor together with  $S_{\mu\lambda}^{\cdot\cdot\cdot\cdot}$  determines uniquely a linear connexion. In order to obtain a unified field theory EINSTEIN started 1945<sup>3)</sup> from a non-symmetric fundamental tensor.<sup>4)</sup>

<sup>1)</sup> This example is very instructive, because it reveals at the same time the strength and the weakness of the abridged calculus. The strength lies in the possibility of reading one and the same formula in different aspects, the weakness in the necessity to go back often to non abridged calculus in order to derive the formulae indispensable for further progress or for the formulation of the results.

<sup>2)</sup> This section deals with a fundamental tensor whose symmetric part is not zero. There is another line of thought starting from a fundamental bivector. We mention here only LEE 1943, 1; 1945, 1; 1947, 1; CHERN and WANG 1947, 1; VRANCEANU 1949, 1; GUGGENHEIMER 1951, 1 (relations with the  $\tilde{V}_{2m}$ ).

<sup>3)</sup> EINSTEIN 1945, 1.

<sup>4)</sup> Cf. EINSTEIN and STRAUS 1946, 1; KOSAMBI 1947, 1; 1949, 1; SCHRÖDINGER 1947, 1; 1948, 1; 2; 1950, 1; 1951, 2; EINSTEIN 1948, 1; 1950, 1; JORDAN 1948, 1; 2; 1950, 1; STRAUS 1949, 1; TONNELAT 1949, 1; 1950, 1; 2; 3; 4; 1951, 1; 2; 3; 4; UDESCHINI 1950, 1; 1951, 1; 2; 3; INGRAHAM 1950, 1; 1951, 1; 2; EISENHART 1951, 1; 1952, 1; HLAVATY 1952, 2; 3; 4; 6; 1953, 1; 2; 3; 4; 5; 1954, 1; LICHNEROWICZ 1953, 1. The idea of the deduction given here goes back to SCHRÖDINGER but its very simple form is due to NIJENHUIS (personal communication).

Let this tensor be denoted by  $s_{\lambda\kappa}$ , the isomer of its inverse  $\bar{s}^{1\kappa\lambda}$  (cf. I § 8) by  $S^{\kappa\lambda}$ , hence  $s_{\lambda\mu} S^{\kappa\mu} = A_{\lambda}^{\kappa}$ , and let  $\mathfrak{s}$  stand for  $\text{Det}(s_{\lambda\kappa})$ . Then we introduce the symmetric and alternating part of  $s_{\lambda\kappa}$

$$(11.1) \quad h_{\lambda\kappa} \stackrel{\text{def}}{=} s_{(\lambda\kappa)}; \quad k_{\lambda\kappa} \stackrel{\text{def}}{=} s_{[\lambda\kappa]}$$

and use  $h_{\lambda\kappa}$ , supposed to be of rank  $n$ , and its inverse  $h^{\kappa\lambda}$  for raising and lowering of indices. Let  $\mathfrak{h}$  stand for  $\text{Det}(h_{\lambda\kappa})$ . Of course generally  $s^{\kappa\lambda} \neq S^{\kappa\lambda}$  and  $\mathfrak{h} \neq \mathfrak{s}$ .

Let there also be a connexion  $\Gamma_{\mu\lambda}^{\kappa}$  and let  $S_{\mu\lambda}^{\cdot\kappa}$  not necessarily be zero. Then we establish relations between  $s_{\lambda\kappa}$  and  $\Gamma_{\mu\lambda}^{\kappa}$  by means of a variational equation. *We require that the variation*

$$(11.2) \quad d \int_{\tau_n}^v R_{\mu\lambda} \mathfrak{S}^{\mu\lambda} d\xi^1 \dots d\xi^n; \quad \mathfrak{s}^{\mu\lambda} \stackrel{\text{def}}{=} \mathfrak{S}^{\frac{1}{2}} S^{\mu\lambda}; \quad R_{\mu\lambda} = R_{\nu\mu}^{\cdot\cdot\lambda} v$$

should vanish if  $s_{\lambda\kappa}$  and  $\Gamma_{\mu\lambda}^{\kappa}$  are varied independently. Hence

$$(11.3) \quad \begin{cases} \text{a)} & \int_{\tau_n}^v \mathfrak{S}^{\mu\lambda} dR_{\mu\lambda} d\xi^1 \dots d\xi^n = 0; \\ \text{b)} & \int_{\tau_n}^v R_{\mu\lambda} d\mathfrak{S}^{\mu\lambda} d\xi^1 \dots d\xi^n = 0. \end{cases}$$

Now we have

$$(11.4) \quad \begin{aligned} dR_{\mu\lambda} &= \partial_{\nu} d\Gamma_{\mu\lambda}^{\nu} - \partial_{\mu} d\Gamma_{\nu\lambda}^{\nu} + 2(d\Gamma_{[\nu|\rho]}^{\nu}) \Gamma_{\mu]\lambda}^{\rho} + 2\Gamma_{[\nu|\rho]}^{\nu} d\Gamma_{\mu]\lambda}^{\rho} \\ &= V_{\nu} d\Gamma_{\mu\lambda}^{\nu} - V_{\mu} d\Gamma_{\nu\lambda}^{\nu} + 2S_{\nu\mu}^{\cdot\cdot\rho} d\Gamma_{\rho\lambda}^{\nu}, \end{aligned}$$

because  $d\Gamma_{\mu\lambda}^{\nu}$  is a tensor. Hence

$$(11.5) \quad \mathfrak{S}^{\mu\lambda} dR_{\mu\lambda} = \partial_{\nu} (\mathfrak{S}^{\mu\lambda} d\Gamma_{\mu\lambda}^{\nu} - \mathfrak{S}^{\nu\mu} d\Gamma_{\lambda\mu}^{\lambda}) - (\mathfrak{G}_{\cdot\cdot\kappa}^{\mu\lambda} - A_{\kappa}^{\mu} \mathfrak{G}_{\cdot\cdot\sigma}^{\sigma\lambda}) d\Gamma_{\mu\lambda}^{\kappa};$$

where

$$(11.6) \quad \begin{cases} \mathfrak{G}_{\cdot\cdot\kappa}^{\mu\lambda} \stackrel{\text{def}}{=} (\partial_{\kappa} \mathfrak{S}^{\mu\lambda} + \Gamma_{\rho\kappa}^{\mu} \mathfrak{S}^{\rho\lambda} + \Gamma_{\kappa\rho}^{\lambda} \mathfrak{S}^{\mu\rho} - \Gamma_{\rho\kappa}^{\rho} \mathfrak{S}^{\mu\lambda}) + \\ \quad + \frac{2}{n-1} S_{\rho\kappa}^{\cdot\cdot\mu} \mathfrak{S}^{\rho\lambda} A_{\kappa}^{\mu}. \end{cases}$$

The terms in brackets in the right hand side of (11.6) do not represent exactly the covariant derivative  $V_{\kappa} \mathfrak{S}^{\mu\lambda}$  but they differ from this derivative only because of the fact that the term belonging to the first index of  $\mathfrak{S}^{\mu\lambda}$  and the term owing its origin to the density-character of  $\mathfrak{S}^{\mu\lambda}$  have  $\Gamma_{\lambda\mu}^{\kappa}$  used instead of  $\Gamma_{\mu\lambda}^{\kappa}$ . Slightly generalizing a notation of EINSTEIN we therefore write

$$(11.7) \quad \begin{cases} V_{\kappa} \bar{\mathfrak{S}}^{\mu\lambda} \stackrel{\text{def}}{=} \partial_{\kappa} \mathfrak{S}^{\mu\lambda} + \Gamma_{\rho\kappa}^{\mu} \mathfrak{S}^{\rho\lambda} + \Gamma_{\kappa\rho}^{\lambda} \mathfrak{S}^{\mu\rho} - \Gamma_{\rho\kappa}^{\rho} \mathfrak{S}^{\mu\lambda} \\ \quad = V_{\kappa} \mathfrak{S}^{\mu\lambda} + 2S_{\rho\kappa}^{\cdot\cdot\mu} \mathfrak{S}^{\rho\lambda} - 2S_{\rho\kappa}^{\cdot\cdot\lambda} \mathfrak{S}^{\mu\rho}. \end{cases}$$

Making the usual condition that the variations of the  $\Gamma_{\mu\lambda}^{\kappa}$  vanish at the boundary, we get from (11.5)

$$(11.8) \quad \mathfrak{G}_{..z}^{\mu\lambda} - A_z^{\mu} \mathfrak{G}_{..o}^{\sigma\lambda} = 0$$

but this implies that  $(1-n) \mathfrak{G}_{..o}^{\sigma\lambda} = 0$  and that (11.8) is consequently equivalent to

$$(11.9) \quad \mathfrak{G}_{..z}^{\mu\lambda} = \nabla_z \bar{\mathfrak{S}}_{..+}^{\mu\lambda} + \frac{2}{n-1} S_{\cdot o}^{\cdot\lambda} \mathfrak{S}^{o\lambda} A_z^{\mu} = 0.$$

If the  $\mathfrak{S}^{*\lambda}$  are given there are  $n^3$  algebraic equations for the  $\Gamma_{\mu\lambda}^{\kappa}$ . But they cannot be independent because if  $\overset{\theta}{I}_{\mu\lambda}^{\kappa}$  is a solution  $\overset{\theta}{I}_{\mu\lambda}^{\kappa} + t_{\mu} A_{\lambda}^{\kappa}$  is also a solution for every choice of  $t_{\mu}$ . This is seen very easily if we put (11.9) into the form

$$(11.10) \quad \partial_z \mathfrak{S}^{*\lambda} = P_{z\cdot o}^{\mu\lambda\tau\sigma} I_{\tau\sigma}^o$$

where

$$(11.11) \quad P_{z\cdot o}^{\mu\lambda\tau\sigma} \stackrel{\text{def}}{=} \mathfrak{S}^{*\lambda} \left( -A_{z\tau\omega}^{\sigma\mu\tau} + \frac{2}{n-1} A_{z\cdot o}^{\mu[\sigma\tau]} \right) + \mathfrak{S}^{*\omega} \left( -A_{o\tau\omega}^{\lambda\tau\sigma} + A_{o\cdot\omega}^{\tau\sigma\lambda} \right).$$

In fact  $P_{z\cdot o}^{\mu\lambda\tau\sigma} = 0$  and this means that the  $\tau_{\theta}^{\sigma}$ -rank of  $P$  is  $\leq n^3 - n$ . But then the  $\mu^{\lambda}$ -rank is also  $\leq n^3 - n$  and this is confirmed by the identity  $P_{\mu\cdot o}^{\mu\lambda\tau\sigma} = 0$ . This implies that (11.10) can only have solutions  $I_{\tau\sigma}^o$  if

$$(11.12) \quad \boxed{\partial_{\mu} \mathfrak{S}^{[\mu\lambda]} = 0}.$$

Note that this is an *invariant* equation because  $\mathfrak{S}^{*\lambda}$  is a bivector density of weight +1.

If a solution  $\overset{\theta}{I}_{\mu\lambda}^{\kappa}$  is known the vector  $t_{\mu}$  mentioned above can be chosen in one and only one way so that for  $*I_{\mu\lambda}^{\kappa} \stackrel{\text{def}}{=} \overset{\theta}{I}_{\mu\lambda}^{\kappa} + t_{\mu} A_{\lambda}^{\kappa}$  the equation (cf. 2.15)

$$(11.13) \quad \boxed{*S_{\mu} = \frac{2}{n-1} *S_{\mu\lambda}^{\cdot\lambda} = 0}$$

holds. If this identity is introduced as a *condition*, the equations (11.10) have one and only one solution  $*I_{\mu\lambda}^{\kappa}$  if and only if the  $\tau_{\theta}^{\sigma}$ -rank of  $P$  is exactly  $n^3 - n$ . It is an algebraic problem to find the n.a.s. conditions for this special value of the  $\tau_{\theta}^{\sigma}$ -rank. But there is a special case where these conditions are not needed. Because of (11.13) we get for  $*V_{\mu}$  instead of (11.9) the simpler equation

$$(11.14) \quad *V_{\mu} \bar{\mathfrak{S}}_{..+}^{\kappa\lambda} \stackrel{\text{def}}{=} *V_{\mu} \mathfrak{S}^{*\lambda} - 2 *S_{\mu\cdot o}^{\cdot\kappa} \mathfrak{S}^{o\lambda} = 0,$$

from which it is easily deduced that

$$(11.15) \quad *V_\mu \mathfrak{s} = 0$$

and consequently

$$(11.16) \quad \boxed{\text{a)} \quad *V_\mu S_{\lambda\mu}^{\alpha\lambda} = 0; \quad \text{b)} \quad *V_\mu s_{\lambda\mu}^{\alpha\lambda} = 0}.$$

From (11.16b) it follows that

$$(11.17) \quad \left\{ \begin{array}{ll} \text{a)} & \partial_\mu s_{\lambda\mu} - *I_{\lambda\mu}^\rho s_{\rho\lambda} - *I_{\mu\lambda}^\rho s_{\lambda\rho} = 0; \\ \text{b)} & \partial_\lambda s_{\alpha\mu} - *I_{\alpha\lambda}^\rho s_{\rho\mu} - *I_{\lambda\mu}^\rho s_{\alpha\rho} = 0; \\ \text{c)} & -\partial_\alpha s_{\mu\lambda} + *I_{\mu\lambda}^\rho s_{\rho\alpha} + *I_{\alpha\lambda}^\rho s_{\mu\rho} = 0; \end{array} \right.$$

For  $s_{[\lambda\alpha]} = 0$  this gives immediately

$$(11.18) \quad \text{a)} \quad *I_{\mu\lambda}^\alpha = \frac{1}{2} S^{\alpha\sigma} (\partial_\mu s_{\sigma\lambda} + \partial_\lambda s_{\mu\sigma} - \partial_\sigma s_{\mu\lambda}) \quad \text{b)} \quad *S_{\mu\lambda}^{\alpha\lambda} = 0.$$

But this proves that the  $\tau_\rho^\alpha$ -rank of  $P$  in (11.11) is exactly  $n^3 - n$  if  $s_{\lambda\mu}^{\alpha\lambda}$  is symmetric. As this rank can never be  $> n^3 - n$  it must be exactly  $n^3 - n$  for small values of  $s_{[\lambda\alpha]}$ .<sup>1)</sup> Hence we see that the equations (11.13, 16) determine the  $*I_{\mu\lambda}^\alpha$  uniquely if the fundamental tensor  $s_{\lambda\mu}^{\alpha\lambda}$  differs only little from a symmetric tensor.

If  $*I_{\mu\lambda}^\alpha$  is known every other solution of (11.9) satisfies the relations

$$(11.19) \quad I_{\mu\lambda}^\alpha = *I_{\mu\lambda}^\alpha + S_\mu A_\lambda^\alpha$$

because  $*S_\mu = 0$ .

The equation (11.3 b) leads to

$$(11.20) \quad \left\{ \begin{array}{l} R_{\mu\lambda} d^\nu \mathfrak{S}^{\mu\lambda} = \mathfrak{s}^{\frac{1}{2}} R_{\mu\lambda} d^\nu S^{\mu\lambda} + \frac{1}{2} R_{\mu\lambda} S^{\mu\lambda} \mathfrak{s}^{-\frac{1}{2}} \frac{\partial \mathfrak{s}}{\partial s_{\rho\sigma}} d^\nu s_{\rho\sigma} \\ = \mathfrak{s}^{\frac{1}{2}} R_{\mu\lambda} d^\nu S^{\mu\lambda} - \frac{1}{2} R_{\rho\sigma} S^{\rho\sigma} \mathfrak{s}^{\frac{1}{2}} s_{\mu\lambda} d^\nu S^{\mu\lambda} \\ = \mathfrak{s}^{\frac{1}{2}} (R_{\mu\lambda} - \frac{1}{2} R s_{\mu\lambda}) d^\nu S^{\mu\lambda}; \quad R = R_{\rho\sigma} S^{\rho\sigma}; \end{array} \right.$$

hence this part of the variation leads to

$$(11.21) \quad G_{\mu\lambda} \stackrel{\text{def}}{=} R_{\mu\lambda} - \frac{1}{2} R s_{\mu\lambda} = 0.$$

Here the general tensor  $G_{\mu\lambda}$  plays the role of the symmetric tensor  $G_{\mu\lambda}$  defined in (5.29) and it is identical with this latter quantity if  $s_{[\lambda\alpha]} = 0$  and  $S_\lambda = 0$ . For  $n \neq 2$  (11.21) is equivalent to

$$(11.22) \quad R_{\mu\lambda} = 0$$

<sup>1)</sup> This remark was made by Mr. SCHRÖDINGER in a letter to Mr. NIJENHUIS of 12th November 1951.

and because  $R_{\mu\lambda} = *R_{\mu\lambda} - 2\partial_{[\mu} S_{\lambda]}$ , as follows from (11.19), also with

$$(11.23) \quad [*R_{\mu\lambda} - 2\partial_{[\mu} S_{\lambda]}] = 0.$$

Now we have for the  $4 + 16 + 60 = 80$  unknowns  $S_\lambda$ ,  $s_{\lambda\mu}$  and  $*I_{\mu\lambda}^\nu$  the  $4 + 60 + 16 = 80$  equations (11.12), (11.16b) and (11.23). This number is by four too much because the equations must be invariant for coordinate transformations and a coordinate transformation contains four arbitrary functions. Hence there must be four relations between the left hand sides of these equations and their derivatives, and these relations must hold independent of the vanishing or not vanishing of these left hand sides. They can be obtained by taking in (11.5) and (11.20) only those variations of  $I_{\mu\lambda}^\nu$  and  $S^{\nu\lambda}$  that arise from dragging along these fields over some arbitrary  $v^\nu dt$ . Because by dragging along a solution we get another solution, provided that  $v^\nu$  and its derivatives vanish at the boundary. According to II § 10, III § 5 we get

$$(11.24) \quad \frac{d}{dt} I_{\mu\lambda}^\nu = -\frac{v}{v} I_{\mu\lambda}^\nu = -\nabla_\mu \nabla_\lambda v^\nu + 2\nabla_\mu S_{\lambda\mu}^\nu v^\rho - v^\rho R_{\rho\mu\lambda}^\nu$$

$$(11.25) \quad \frac{d}{dt} S^{\nu\lambda} = -\frac{v}{v} S^{\nu\lambda} = -v^\mu \partial_\mu S^{\nu\lambda} + S^{\mu\lambda} \partial_\mu v^\nu + S^{\nu\mu} \partial_\mu v^\lambda.$$

From these variations the four equations can be derived in the ordinary way. All divergences have to be reduced to zero by means of STOKES' theorem and therefore it is profitable to introduce  $*\nabla$  instead of  $\nabla$  because  $*\nabla_\mu S^{\mu\nu} = 0$  and  $*\nabla_\mu p^\mu = \partial_\mu p^\mu$  for any vector density  $p^\nu$  of weight +1.

If the  $*I_{\mu\lambda}^\nu$  are solved from (11.16b) as functions of the  $s_{\lambda\mu}$  and their derivatives these solutions can be substituted in (11.23). Then we get the following  $4 + 10 + 4 = 18$  equations

$$(11.12) \quad \partial_\mu \mathfrak{S}^{[\mu\lambda]} = 0$$

$$(11.26) \quad *R_{(\mu\lambda)} = 0$$

$$(11.27) \quad \partial_{[\nu} *R_{\mu\lambda]} = 0$$

for the 16 unknowns  $s_{\lambda\mu}$ . There must be six relations. Two of them are trivial

$$(11.28) \quad \partial_\lambda \partial_\mu \mathfrak{S}^{[\mu\lambda]} = 0$$

$$(11.29) \quad \partial_{[\omega} \partial_{\nu} *R_{\mu\lambda]} = 0$$

and the other four can be obtained in the way described above. In the symmetric case only the 10 equations (11.26) remain for the 10 unknowns  $s_{\lambda\mu}$ . In this symmetric case  $R_{\mu\lambda} \frac{d}{dt} \mathfrak{S}^{\mu\lambda}$  vanishes because the

$I_{\mu\lambda}^*$  are already considered as functions of the  $s_{\mu\lambda}$ . From (11.20, 25) we get now for  $S_\lambda = 0$ , neglecting terms that vanish after integration

$$(11.30) \quad \left\{ \begin{array}{l} R_{\mu\lambda} \overset{\circ}{d} S^{\mu\lambda} = \frac{1}{2} G_{\mu\lambda} (S^{\rho\lambda} \nabla_\rho v^\mu + S^{\mu\rho} \nabla_\rho v^\lambda) dt \\ = -2 (\nabla_\rho \frac{1}{2} G_{\mu\lambda} S^{\rho\lambda}) v^\mu dt = -2 (\nabla_\rho G_{\mu\lambda}) v^\mu dt \end{array} \right.$$

from which

$$(11.31) \quad \nabla_\mu G^{\mu\nu} = 0.$$

The condition (11.13) that ensures at least in the simplest cases the uniqueness of the connexion  $*I_{\mu\lambda}^*$ , means geometrically that this connexion behaves as a symmetric one as far as scalar densities are concerned. But it has yet another meaning as was pointed out by EINSTEIN. If  $s_{\lambda\mu}$  is changed into  $s_{\nu\lambda}$ , that is, if the alternating part of the fundamental tensor gets another sign while the symmetric part is unchanged,  $*I_{\mu\lambda}^*$  will change into  $*I_{\lambda\mu}^*$  as is seen quite easily from (11.17a). Now let this change induce a change of  $*R_{\mu\lambda}$  into  $*R'_{\mu\lambda}$ . Then we find

$$(11.32) \quad \left\{ \begin{array}{l} a) *R_{(\mu\lambda)} - *R'_{(\mu\lambda)} = -2 \partial_{(\mu} *S_{|\nu|\lambda)} + 2 *S_{\nu|\lambda} *I_{(\mu\lambda)}^\rho; \\ b) *R_{[\mu\lambda]} + *R'_{[\mu\lambda]} = -2 \partial_{[\mu} *S_{|\nu|\lambda]} + 2 *S_{\nu|\lambda} *S_{\mu\lambda}^\rho; \end{array} \right.$$

hence,  $*S_\lambda = 0$  is sufficient (but not necessary) for the tensor  $*R_{\mu\lambda}$  to change into  $*R_{\lambda\mu}$  if  $s_{\lambda\mu}$  is changed into  $s_{\nu\lambda}$ . In the original form of the theory given by EINSTEIN in 1945,  $s_{(\lambda\mu)}$  was taken real and  $s_{[\lambda\mu]}$  imaginary. Then  $s_{\lambda\mu}$  could be considered in a way as "hermitian" and the condition that  $*R_{\lambda\mu}$  also should be "hermitian" could be fulfilled by taking  $*S_\lambda = 0$ . This condition however presents itself in a quite natural way if we wish that  $*R_{\lambda\mu}$  and  $s_{\lambda\mu}$  should differ only by a scalar factor just as  $K_{\lambda\mu}$  and  $g_{\lambda\mu}$  do in the ordinary theory of relativity.

The connexion derived in this way can be used for  $n=4$  to build a unified field theory and the authors mentioned above have got some remarkable results in this line. But it should be remarked that in the end there is nothing but an  $X_4$  with only two fields  $s_{(\lambda\mu)}$  and  $s_{[\lambda\mu]}$  and that according to the second reduction theorem (III § 7) the differential concomitants of these fields are ordinary concomitants of  $s_{(\lambda\mu)}$ ,  $s_{[\lambda\mu]}$ , the curvature tensor of the *symmetric connexion belonging to*  $s_{(\lambda\mu)}$  and the covariant derivatives of these quantities with respect to *this* connexion. So, from a mathematical point of view the new connexion derived from (11.10) can never produce anything that could not be expressed in terms of  $s_{(\lambda\mu)}$ ,  $s_{[\lambda\mu]}$  and the ordinary symmetric connexion belonging to  $s_{(\lambda\mu)}$ . It may be useful to introduce this new connexion in order to get some heuristic principles that may suggest for instance a choice of the Hamiltonian or a desirable form of the field equations. But nothing really new can ever arise from following this course. This

remark is not valid if the introduction of a non-symmetric fundamental tensor is combined with a change of dimension. In projective relativity in five coordinates a non-symmetric fundamental tensor does not seem to have been used till yet but in conformal relativity in six coordinates INGRAHAM<sup>1)</sup> has recently tried to generalize SCHRÖDINGER's ideas.

By transvecting (11.17a) with  $S^{\lambda\mu}$  we get

$$(11.33) \quad \partial_\mu \log s = S^{\lambda\mu} \partial_\mu s_{\lambda\mu} = {}^*I_{\mu\mu}^\lambda + {}^*I_{\mu\lambda}^\lambda,$$

hence

$$(11.34) \quad \partial_{[\nu} {}^*I_{\mu]\lambda}^\lambda = \partial_{[\nu} {}^*S_{\mu]}^\lambda \dot{\lambda}^\lambda$$

and according to (4.15)

$$(11.35) \quad {}^*V_{\nu\mu} = {}^*R_{\nu\mu}^{\lambda\lambda} = 2\partial_{[\nu} {}^*I_{\mu]}^\lambda = 2\partial_{[\nu} {}^*S_{\mu]}^\lambda \dot{\lambda}^\lambda.$$

This proves that the condition (11.13) implies that  ${}^*V_{\nu\mu}$  vanishes, that is, that for the connexion  ${}^*I_{\mu\lambda}^\lambda$  a covariant constant scalar density field is possible (cf. III § 6).

From (5.6) we get according to the condition (11.13)

$$(11.36) \quad {}^*R_{[\mu\lambda]} = {}^*V_\sigma {}^*S_{\mu\lambda}^{\sigma\sigma}$$

and it is easy to show that this is in accordance with (11.32b).

## IV. LIE groups and linear connexions.

### § 1. Finite continuous groups<sup>2)</sup>.

An  $r$ -parameter finite continuous group in the sense of LIE is a set of elements between which a "multiplication" is defined and that satisfies the following conditions:

- a) The elements are in one to one correspondence with the points of an  $\mathfrak{N}(\eta^\alpha)$  in an  $X$ , with the coordinates  $\eta^\alpha; \alpha = 1, \dots, r$ .
- b) If the element  $T_\eta$  belongs to  $\eta^\alpha$  and  $T_\zeta$  to  $\zeta^\alpha$ , the product  $T_\theta = T_\eta T_\zeta$  belongs to the set and the  $\theta^\alpha$  are analytic functions<sup>3)</sup> of the  $\eta^\alpha$  and  $\zeta^\alpha$ .
- c) There is an element  $I$  called "unity" corresponding to  $\eta^\alpha$  such that  $I T_\eta = T_\eta I = T_\eta$  for every choice of  $T_\eta$ .

<sup>1)</sup> L. c. p. 179 footnote 4.

<sup>2)</sup> General references also for literature: LIE and ENGEL 1888, 1; BIANCHI 1918, 1; EISENHART 1933, 1; VRANCEANU 1947, 1. According to the general character of this book we only consider the "group germ", that means we only are interested in local properties. There is a vast literature on properties in the large. We mention here only also for literature CARTAN 1927, 2; 1930, 2; 1936, 1; MAYER and T. Y. THOMAS 1935, 1; CHEVALLEY 1946, 1.

<sup>3)</sup> See footnote 1 on page 186.

d) To every  $T_\eta$  there belongs an element  $T_\eta^{-1}$  such that  $T_\eta T_\eta^{-1} = I$ . Its coordinates are analytic functions<sup>1)</sup> of the  $\eta^\alpha$ .

e)  $(T_\eta T_\zeta) T_\theta = T_\eta (T_\zeta T_\theta)$ .

The elements are often transformations in  $n$  variables but this is not necessary. From the definition we see that only elements of the group are considered in a neighbourhood of unity. This neighbourhood is called the *group germ* (Gruppenkeim).

We identify each element with its corresponding point in  $X$ . This  $X$ , is called *group space*.

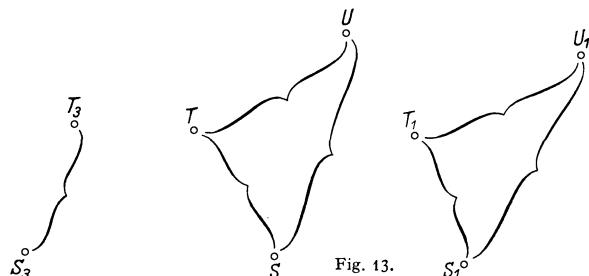
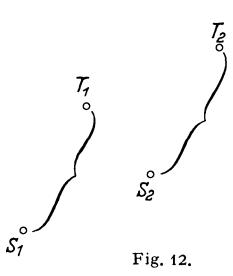
To every pair of elements  $S, T$  there belong two elements  $TS^{-1}$  and  $S^{-1}T$ . Two pairs  $S, T$  and  $S_1, T_1$  are called

- (+)-equipollent if  $TS^{-1} = T_1 S_1^{-1}$
- (-)-equipollent if  $S^{-1}T = S_1^{-1}T_1$ .

From this we get the following deductions:

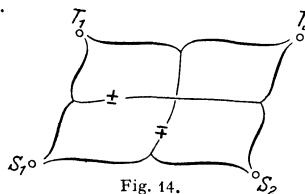
1. If  $S_1, T_1; S_2, T_2$  are  $(\pm)$ -equipollent, three of these elements determine the fourth uniquely.

2. If  $S_1, T_1; S_2, T_2$  and  $S_2, T_2; S_3, T_3$  are both  $(\pm)$ -equipollent, the same holds for  $S_1, T_1; S_3, T_3$  (Fig. 12).



3. If  $S, T; S_1, T_1$  and  $T, U; T_1, U_1$  are  $(\pm)$ -equipollent, the same holds for  $S, U; S_1, U_1$  (Fig. 13).

4. If  $S_1, T_1; S_2, T_2$  are  $(\pm)$ -equipollent, then  $S_1, S_2; T_1, T_2$  are  $(\mp)$ -equipollent (Fig. 14).



<sup>1)</sup> Though in the following all functions will be supposed to be analytic, many theorems remain valid if only it is assumed that all functions are of class  $u$ .

5. An  $\mathfrak{N}(T_1)$  can always be mapped continuously on an  $\mathfrak{N}(T_2)$  such that  $T_2$  is the image of  $T_1$  and all pointpairs of  $\mathfrak{N}(T_1)$  are mapped  $(\pm)$ -equipollent on the pointpairs of  $\mathfrak{N}(T_2)$ .

6. If  $\mathfrak{N}(T_1)$  is mapped in this way on  $\mathfrak{N}(T_2)$  and  $\mathfrak{N}(T_2)$  in the same way on  $\mathfrak{N}(T_3)$  the result is the same as if  $\mathfrak{N}(T_1)$  were mapped directly on  $\mathfrak{N}(T_3)$ .

If a pointpair  $T_\eta, T_{\eta+d\eta}$  is given, there exists for every choice of  $T_\eta$  in an  $\mathfrak{N}(\eta^\alpha)$  a  $(\pm)$ -equipollent pointpair  $T_\eta, T_{\eta+d\eta}$ . Hence, using for instance the  $(+)$ -equipollence we get for every choice of an infinitesimal vector  $v^\alpha dt$  at  $\eta^\alpha$  a vector field  $v^\alpha dt$  in  $\mathfrak{N}(\eta^\alpha)$ . That means that the  $(+)$ -equipollence fixes a parallel displacement (cf. III § 2) for contravariant vectors in  $\mathfrak{N}(\eta^\alpha)$ . According to the conditions 1, 2, 3, 5 and 6 the parameters of this displacement establish a connexion that is linear in the sense defined in III § 2. Let  $e^\alpha_b, \dot{e}_\beta^a; a, b = 1, \dots, r^1$  be fields of basis vectors covariant constant for this connexion or shortly  $(+)$ -constant. Let the covariant differentiation symbol of the connexion be  $\overset{+}{V}$  and the parameters  $\overset{+}{\Gamma}_{\gamma\beta}^\alpha$ , then we have  $\overset{+}{V}_\gamma^a e^\alpha_b = 0$ , hence

$$(1.1) \quad \overset{+}{\Gamma}_{\gamma\beta}^\alpha = - \dot{e}_\beta^a \partial_\gamma e^\alpha_b = + e^\alpha_b \partial_\gamma \dot{e}_\beta^a.$$

Let this connexion be called the  $(+)$ -connexion. The  $(-)$ -equipollence leads in the same way to the  $(-)$ -connexion with the symbol  $\overset{-}{V}$  and the parameters  $\overset{-}{\Gamma}_{\gamma\beta}^\alpha$

$$(1.2) \quad \overset{-}{\Gamma}_{\gamma\beta}^\alpha = - \overset{B}{e}_\beta^a \partial_\gamma e^\alpha_B = + \overset{B}{e}^\alpha_B \partial_\gamma \overset{B}{e}_\beta^a$$

where  $e^\alpha_B, \dot{e}_\beta^A; A, B = 1, \dots, r$  are fields of basis vectors covariant constant for this connexion or shortly  $(-)$ -constant.<sup>2)</sup>

Vectors at different points that can be transformed into each other by  $(\pm)$ -equipollence are said to be  $(\pm)$ -parallel.

As both connexions are integrable (cf. III § 4 and deduction 2) their curvature tensors vanish

$$(1.3) \quad \overset{\pm}{R}_{\delta\gamma\beta}^\alpha \stackrel{\text{def}}{=} 2\partial_{[\delta}\overset{\pm}{\Gamma}_{\gamma]\beta}^\alpha + 2\overset{\pm}{\Gamma}_{[\delta|e]}^\alpha \overset{\pm}{\Gamma}_{\gamma]\beta}^e = 0$$

<sup>1)</sup> Or  $i, \dots, r$  if the indices  $1, 2, \dots$  are already used for another purpose, cf. for instance VII § 4 and 5.

<sup>2)</sup> These connexions were introduced by CARTAN and SCHOUTEN 1926, 3; 4, cf. CARTAN 1927, 1; SCHOUTEN 1927, 1; 1929, 4.

but they are in general not symmetric (cf. III, p. 142, footnote 3)

$$(1.4) \quad S_{\gamma\beta}^{\pm; \alpha} \stackrel{\text{def}}{=} \bar{\Gamma}_{[\gamma\beta]}^{\pm\alpha} \neq 0.$$

If a curve contains a pointpair  $R, S$  and the first point of a (+)-equipollent pointpair  $T, U$ , it may happen that it also contains  $U$ . If this is the case for every choice of  $R, S$  and  $T$  the curve is said to be *geodesic*. Now  $R, T$  and  $S, U$  are  $(-)$ -equipollent pointpairs, hence, if we use in the definition  $(-)$ -equipollence instead of  $(+)$ -equipollence, we get the same geodesics. If a geodesic contains  $T$  and  $VT$  it also contains  $V^2T, V^3T$ , etc. Taking  $V$  infinitesimal different from  $I$  we see that a geodesic as defined here is identical with a geodesic of the  $(+)$ -connexion and that for every choice of  $V$  the points  $T, VT; VT, V^2T$  etc. have the same distance if measured by means of an affine parameter on the geodesic (cf. III § 7). The same reasoning holds for the geodesics of the  $(-)$ -connexion. Moreover, if  $VT = TW$  it follows that  $V^2T = TW^2$  etc. Hence the  $(+)$ - and  $(-)$ -geodesics are not only identical but the affine parameter on them is also the same for both connexions. Using this parameter, the equation of the geodesic must have the form

$$(1.5) \quad \frac{d^2 \eta^\alpha}{dt^2} + \bar{\Gamma}_{\gamma\beta}^{\pm\alpha} \frac{d\eta^\gamma}{dt} \frac{d\eta^\beta}{dt} = 0$$

and from this it follows that

$$(1.6) \quad \bar{\Gamma}_{\gamma\beta}^{\alpha} \stackrel{\text{def}}{=} \bar{\Gamma}_{(\gamma\beta)}^{\alpha} = \bar{\Gamma}_{(\gamma\beta)}^{\alpha}$$

are the parameters of a symmetric connexion that can be obtained by symmetrizing the  $(+)$ -connexion and the  $(-)$ -connexion. This  $(0)$ -connexion is in general not integrable (cf. III § 4) and its curvature tensor

$$(1.7) \quad R_{\delta\gamma\beta}^{\alpha} \stackrel{\text{def}}{=} 2\partial_{[\delta}\bar{\Gamma}_{\gamma]\beta}^{\alpha} + 2\bar{\Gamma}_{[\delta|e|}^{\alpha}\bar{\Gamma}_{\gamma]\beta}^e$$

plays an important role.

Besides the operators  $\bar{V}, \bar{V}$  and  $\nabla$  we also need the LIE derivative (cf. II § 10) with respect to the fields  $e^\alpha$  and  $e^\alpha$ , for instance

$$(1.8) \quad \mathfrak{L}_b P_{.\beta}^{\alpha} \stackrel{\text{def}}{=} e^\gamma \partial_\gamma P_{.\beta}^{\alpha} - P_{.\beta}^\gamma \partial_\gamma e^\alpha + P_{.\gamma}^\alpha \partial_\beta e^\gamma.$$

The LIE derivative  $\mathfrak{L}_v P_{.\beta}^{\alpha}$  with respect to the  $(+)$ -constant field  $v^\alpha = v^\alpha A_a^\alpha$  with constant  $v^\alpha$  is then given by

$$(1.9) \quad \mathfrak{L}_v P_{.\beta}^{\alpha} = \overset{a}{v} \mathfrak{L}_a P_{.\beta}^{\alpha}.$$

If in deduction 4 for  $S_1, T_1; S_2, T_2$  we take elements that differ from  $J$  only in infinitesimals as illustrated in Fig. 15, the quadrangle is built up by means of the vector fields  $\frac{e^\alpha}{b} dt$  and  $\frac{e^\alpha}{B} dt'$ . This means that the

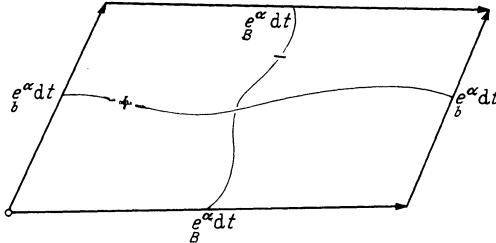


Fig. 15.

field  $\frac{e^\alpha}{b}$  is absolutely invariant (cf. II § 10) with respect to the field  $\frac{e^\alpha}{B}$  and vice versa for every choice of  $B$  and  $b$ :

$$(1.10) \quad \text{a)} \quad \frac{\mathcal{L}}{b} \frac{e^\alpha}{B} = \frac{e^\gamma}{b} \partial_\gamma \frac{e^\alpha}{B} - \frac{e^\gamma}{B} \partial_\gamma \frac{e^\alpha}{b} = 0; \quad \text{b)} \quad \frac{\mathcal{L}}{B} \frac{e^\alpha}{b} = 0.$$

From this it follows that

$$(1.11) \quad 0 = \frac{e^\gamma}{b} \bar{\nabla}_\gamma \frac{e^\alpha}{B} - \frac{e^\gamma}{B} \bar{\nabla}_\gamma \frac{e^\alpha}{b} = \frac{\mathcal{L}}{b} \frac{e^\alpha}{B} + (\bar{\Gamma}_{\gamma\beta}^\alpha - \bar{\Gamma}_{\beta\gamma}^\alpha) \frac{e^\gamma}{b} \frac{e^\beta}{B};$$

hence

$$(1.12) \quad \bar{\Gamma}_{\gamma\beta}^\alpha = \bar{\Gamma}_{\beta\gamma}^\alpha.$$

The symmetric part of (1.12) leads back to (1.6) and the alternating part is

$$(1.13) \quad \overset{+}{S}_{\gamma\beta}^\alpha = - \overset{-}{S}_{\gamma\beta}^\alpha$$

hence, for every quantity  $\Phi$  (indices suppressed)

$$(1.14) \quad 2 \bar{\nabla}_\gamma \Phi = \overset{+}{\bar{\nabla}_\gamma} \Phi + \overset{-}{\bar{\nabla}_\gamma} \Phi.$$

It follows from (1.1, 2, 10, 12) that for any fields  $v^\alpha, w_\beta$

$$(1.15) \quad \text{a)} \quad \begin{cases} \frac{\mathcal{L}}{b} v^\alpha = \frac{e^\gamma}{b} \bar{\nabla}_\gamma v^\alpha; \\ \frac{\mathcal{L}}{b} w_\beta = \frac{e^\gamma}{b} \bar{\nabla}_\gamma w_\beta; \end{cases} \quad \text{b)} \quad \begin{cases} \frac{\mathcal{L}}{B} v^\alpha = \frac{e^\gamma}{B} \overset{+}{\bar{\nabla}_\gamma} v^\alpha; \\ \frac{\mathcal{L}}{B} w_\beta = \frac{e^\gamma}{B} \overset{+}{\bar{\nabla}_\gamma} w_\beta; \end{cases}$$

and this implies that for every quantity  $\Phi$  (indices suppressed)

$$(1.16) \quad \text{a)} \quad \frac{\mathcal{L}}{b} \Phi = \frac{e^\gamma}{b} \bar{\nabla}_\gamma \Phi; \quad \text{b)} \quad \frac{\mathcal{L}}{B} \Phi = \frac{e^\gamma}{B} \overset{+}{\bar{\nabla}_\gamma} \Phi$$

expressing the fact that dragging along over a  $(\pm)$ -constant infinitesimal field is the same as a  $(\mp)$ -displacement over the same field. Note that this is just what we see in Fig. 15 for an infinitesimal contravariant vector.

Because the fields  $\overset{\alpha}{e}_\beta$  are (+)-constant, we have

$$(1.17) \quad 0 = \overset{+}{V}_\gamma \overset{\alpha}{e}_\beta = \partial_\gamma \overset{\alpha}{e}_\beta - \overset{+}{I}_{\gamma\beta}^\alpha \overset{\alpha}{e}_\alpha;$$

hence

$$(1.18) \quad \boxed{\partial_{[\gamma} \overset{\alpha}{e}_{\beta]} = -\frac{1}{2} c_{\gamma\beta}^{\alpha} \overset{\alpha}{e}_\alpha}; \quad c_{\gamma\beta}^{\alpha} \stackrel{\text{def}}{=} -2 \overset{+}{S}_{\gamma\beta}^{\alpha};$$

or, introducing  $n^3$  scalar fields  $c_{cb}^a \stackrel{*}{=} c_c^a b^a$ :

$$(1.19) \quad \boxed{\partial_{[\gamma} \overset{\alpha}{e}_{\beta]} = -\frac{1}{2} c_{cb}^a \overset{\epsilon}{e}_\gamma^c \overset{b}{e}_\beta^b}.$$

Now since the field  $\overset{\alpha}{e}_\beta$  is invariant for dragging along over  $\overset{\alpha}{e}_B dt$ , the rotation  $\partial_{[\gamma} \overset{\alpha}{e}_{\beta]}$  must have the same property. Hence, according to (1.16b)<sup>1)</sup>

$$(1.20) \quad 0 = \underset{B}{\mathcal{L}} c_{cb}^a \overset{\epsilon}{e}_\gamma^c \overset{b}{e}_\beta^b = (\underset{B}{\mathcal{L}} c_{cb}^a) \overset{\epsilon}{e}_\gamma^c \overset{b}{e}_\beta^b = (\underset{B}{e^\delta} \partial_\delta c_{cb}^a) \overset{\epsilon}{e}_\gamma^c \overset{b}{e}_\beta^b$$

for every value of  $b, c$  and  $B$ . This is only possible if

$$(1.21) \quad \boxed{c_{cb}^a = \text{const}}.$$

Since the fields  $\overset{\alpha}{e}_b, \overset{\alpha}{e}_\beta$  are (+)-constant it follows from (1.21) that

$$(1.22) \quad \overset{+}{V}_\delta \overset{+}{S}_{\gamma\beta}^{\alpha} = -\frac{1}{2} \overset{+}{V}_\delta c_{\gamma\beta}^{\alpha} = -\frac{1}{2} \overset{+}{V}_\delta c_{cb}^a \overset{\epsilon}{e}_\gamma^c \overset{b}{e}_\beta^b e^\alpha = 0.$$

In the same way we get

$$(1.23) \quad \boxed{\partial_{[\gamma} \overset{A}{e}_{\beta]} = +\frac{1}{2} c_{\gamma\beta}^{\alpha} \overset{\alpha}{e}_\alpha^A}; \quad c_{\gamma\beta}^{\alpha} \stackrel{\text{def}}{=} +2 \overset{-}{S}_{\gamma\beta}^{\alpha} = -2 \overset{+}{S}_{\gamma\beta}^{\alpha}.$$

$$(1.24) \quad \boxed{c_{CB}^A = \text{const.}}; \quad c_{CB}^A \stackrel{*}{=} c_C^A B^B c_{CB}^a.$$

For convenience we always choose the basis vectors  $\overset{\alpha}{e}_b$  and  $\overset{\alpha}{e}_B$  in such a way that they are identical at  $\underset{0}{\eta}{}^\alpha: \overset{\alpha}{e}_b = \overset{\alpha}{e}_B$ . Then at all points of  $X$ , we have

$$(1.25) \quad c_{CB}^A \stackrel{*}{=} \delta_a^A \overset{a}{c}_C^b \overset{b}{c}_B^a.$$

The  $\overset{+}{I}_{cb}^a$  and  $\overset{-}{I}_{CB}^A$  are all zero because the fields  $\overset{\alpha}{e}_b$  and  $\overset{\alpha}{e}_B$  are (+)- and (-)-constant respectively. Hence it follows from (1.21) and also from (1.24) that  $c_{\gamma\beta}^{\alpha}$  is (+)- and (-)-constant and (0)-constant

$$(1.26) \quad \text{a)} \quad \overset{\pm}{V}_\delta c_{\gamma\beta}^{\alpha} = -2 \overset{\pm}{V}_\delta \overset{+}{S}_{\gamma\beta}^{\alpha} = 2 \overset{\pm}{V}_\delta \overset{-}{S}_{\gamma\beta}^{\alpha} = 0; \quad \text{b)} \quad \overset{\pm}{V}_\delta c_{\gamma\beta}^{\alpha} = 0.$$

<sup>1)</sup> The difference between the  $n^3$  scalars  $c_{cb}^a$  and the tensor  $c_{cb}^{a\alpha}$  is important here.

From this and (1.3) and the second identity for  $\overset{+}{R}_{\delta\gamma\beta}^{\alpha}$  (cf. III 5.2) we get

$$(1.27) \quad 0 = \overset{+}{R}_{[\delta\gamma\beta]}^{\alpha} = -\overset{+}{V}_{[\delta} c_{\gamma\beta]}^{\alpha} - c_{[\delta} \overset{+}{c}_{\gamma}^{\epsilon} c_{\beta]}^{\alpha} = -c_{[\delta} \overset{+}{c}_{\gamma}^{\epsilon} c_{\beta]}^{\alpha};$$

hence

$$(1.28) \quad \text{a)} \quad c_{[\delta} \overset{+}{c}_{\gamma}^{\epsilon} c_{\beta]}^{\alpha} = 0; \quad \text{b)} \quad c_{[d} \overset{+}{c}^{\epsilon} c_{b]}^{\alpha} = 0; \quad \text{c)} \quad c_{[D} \overset{+}{c}^E c_{B]}^{\alpha} = 0.$$

By (1.26) and (1.28) we are now able to compute the curvature tensor  $R_{\delta\gamma\beta}^{\alpha}$  of the symmetric connexion  $\Gamma_{\gamma\beta}^{\alpha}$  (cf. III, 4.25)

$$(1.29) \quad \left\{ \begin{array}{l} R_{\delta\gamma\beta}^{\alpha} = 2\partial_{[\delta}\Gamma_{\gamma]\beta}^{\alpha} + 2\Gamma_{[\delta|\epsilon|}^{\alpha}\Gamma_{\gamma]\beta}^{\epsilon} = \overset{+}{R}_{\delta\gamma\beta}^{\alpha} - 2\partial_{[\delta}\overset{+}{S}_{\gamma]\beta}^{\alpha} - \\ \quad - 2\overset{+}{S}_{[\delta|\epsilon|}^{\alpha}\overset{+}{\Gamma}_{\gamma]\beta}^{\epsilon} - 2\overset{+}{\Gamma}_{[\delta|\epsilon|}^{\alpha}\overset{+}{S}_{\gamma]\beta}^{\epsilon} - 2\overset{+}{S}_{[\delta|\epsilon|}^{\alpha}\overset{+}{S}_{\gamma]\beta}^{\epsilon} \\ \quad = -2\overset{+}{V}_{[\delta} S_{\gamma]\beta}^{\alpha} - 2\overset{+}{S}_{\delta}^{\alpha} \overset{+}{S}_{\gamma\beta}^{\epsilon} - 2\overset{+}{S}_{[\delta|\epsilon|}^{\alpha}\overset{+}{S}_{\gamma]\beta}^{\epsilon} \\ \quad = -\frac{1}{4} c_{\delta}^{\alpha} c_{\gamma}^{\epsilon} c_{\beta}^{\epsilon}. \end{array} \right.$$

Hence  $R_{\delta\gamma\beta}^{\alpha}$  is an algebraical concomitant of  $c_{\gamma\beta}^{\alpha}$  and according to (1.26) this implies

$$(1.30) \quad \overset{+}{V}_{\epsilon} R_{\delta\gamma\beta}^{\alpha} = V_{\epsilon} R_{\delta\gamma\beta}^{\alpha} = 0$$

from which we see that every group space is a symmetric A, with respect to its (0)-connexion (cf. III § 7). Moreover we see that all differential concomitants of group space are ordinary concomitants of  $c_{\gamma\beta}^{\alpha}$  (cf. III § 7). This has as a consequence that all these differential concomitants have constant components with respect to the anholonomic systems (a) and (A).

### Exercise.

IV 1.1. Prove (1.3) using (1.1, 2).

## § 2. The parameter-groups and the adjoint group of a finite continuous group.

The equation

$$(2.1) \quad 'T = U T$$

represents a transformation of the elements of a group by which a one to one correspondence between the elements and their transforms is established. If for  $U$  all the elements are taken we get a group of element transformations, called the *first parameter-group* of the given group, and the transformations of this new group are in one to one correspondence to the elements of the original group. The following

propositions can easily be proved for the transformations of the first parameter-group:

a 1) If  $S, T$  is transformed into  $'S, 'T$ , the pairs  $S, T; 'S, 'T$  are  $(-)$ -equipollent and the pairs  $S, 'S; T, 'T$  are  $(+)$ -equipollent.

a 2) If a transformation of elements transforms every pair of elements into a  $(-)$ -equipollent pair, the transformation belongs to the first parameter-group.

b) If  $S_1, T_1; S_2, T_2$  are  $(\pm)$ -equipollent, their transforms are also  $(\pm)$ -equipollent.

In the same way, starting from the transformations  $'T = TU$  the second parameter-group can be defined. For this group the same propositions hold as did for the first group if  $(\pm)$  is changed into  $(\mp)$  (see Fig. 16).

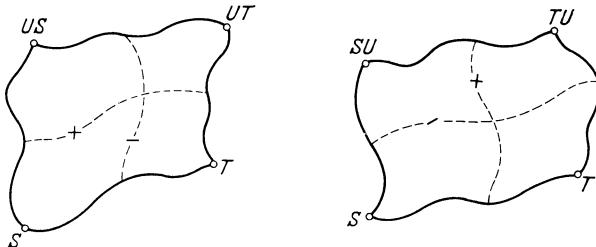


Fig. 16.

Given a pair of elements  $\eta^\alpha, \zeta^\alpha$  there is exactly *one* transformation of the first and also exactly *one* transformation of the second parameter-group transforming  $\eta^\alpha$  into  $\zeta^\alpha$ , namely  $T_\zeta T_\eta^{-1}$  and  $T_\eta^{-1} T_\zeta$  respectively. Now let there be a transformation of the first parameter-group carrying the element  $\eta^\alpha$  into  $\eta^\alpha + d\eta^\alpha$  and the element  $\eta^\alpha$  into  $\eta^\alpha + d\eta^\alpha$ . Then the point pairs  $\eta_1^\alpha, \eta_1^\alpha + d\eta_1^\alpha$  and  $\eta_2^\alpha, \eta_2^\alpha + d\eta_2^\alpha$  are  $(+)$ -equipollent. Hence, if  $d\eta^\alpha = e^\alpha A_a^\alpha \{ \eta^\alpha \} dt$  it follows that  $d\eta^\alpha = e^\alpha A_a^\alpha \{ \eta^\alpha \} dt$  and the constants  $e^\alpha dt$  are the components of  $d\eta^\alpha$  at  $\eta_1^\alpha$  and  $d\eta^\alpha$  at  $\eta_2^\alpha$  with respect to the anholonomic system  $(a)$ . This proves that the transformations

$$(2.2) \quad \eta^\alpha \rightarrow \eta^\alpha + e^\alpha_b dt$$

are  $r$  linearly independent infinitesimal transformations of the first parameter-group. The same reasoning holds for the second parameter-group and the system  $(A)$ . Hence

$$(2.3) \quad \eta^\alpha \rightarrow \eta^\alpha + e^\alpha_B dt$$

are  $r$  linearly independent infinitesimal transformations of the second parameter-group. Taking  $S = \eta^\alpha$ ,  $T = \eta^\alpha + e_\beta^\alpha dt$ ,  $US = \eta^\alpha + e_b^\alpha dt$  in Fig. 16 we get back the situation of Fig. 15 and again it is proved that the fields  $e_\beta^\alpha$  ( $e_b^\alpha$ ) remain invariant for the infinitesimal transformations (2.2) ((2.3)) of the first (second) parameter-group.

*Apart from any considerations about groups* we now ask whether it is possible to have in an  $X$ , an anholonomic system (a) satisfying the condition

$$(2.4) \quad 2\partial_{[\gamma}^a e_{\beta]}^b = -c_{cb}^a \dot{e}_\gamma^b; \quad c_{(c}^a b) = 0$$

with  $\frac{1}{2}r^2(r-1)$  given (not necessarily constant) coefficients  $c_{cb}^a$ , and the condition that the fields  $\dot{e}_\beta^a$  are invariant for  $r$  linearly independent infinitesimal transformations. (2.4) expresses the fact that (cf. II 9.2)

$$(2.5) \quad \Omega_{cb}^a \equiv -\frac{1}{2}c_{cb}^a.$$

But from this equation and Exerc. II 9.2 we see that the first integrability conditions of (2.4) can be written in the form

$$(2.6) \quad \partial_{[d} c_{cb]e}^a + c_{[d}^e c_{b]e}^a = 0; \quad \partial_d \stackrel{\text{def}}{=} A_d^\delta \partial_\delta.$$

The invariance of the  $\dot{e}_\beta^a$  for an infinitesimal transformation  $\eta^\alpha \rightarrow \eta^\alpha + v^\alpha dt$  is expressed by [cf. Exerc. II 10.6 and (2.5)]

$$(2.7) \quad 0 = \mathfrak{L}_{v^\alpha} \dot{e}_\beta^a \equiv v^c \partial_c \dot{e}_\beta^a - (\partial_c v^\alpha - v^d c_{dc}^a) \dot{e}_\beta^c = -(\partial_c v^\alpha - v^d c_{dc}^a) \dot{e}_\beta^c;$$

and the first integrability condition of this equation is (cf. II 9.4)

$$(2.8) \quad \Omega_{ec}^b \partial_b v^a \equiv -\partial_{[e} v^d c_{|d|c]}^a;$$

or

$$(2.9) \quad \frac{1}{2} c_{ec}^b c_{b|d}^a v^d \equiv c_{[e|f]}^d c_{|d|c]}^a v^f - v^d \partial_{[e} c_{|d|c]}^a.$$

Because this must hold for  $r$  linearly independent choices of  $v^\alpha$  we get

$$(2.10) \quad 2\partial_{[e} c_{c]d}^a = 3c_{[e|d}^b c_{|c]}^a.$$

Now (2.6) and (2.10) together are equivalent to

$$(2.11) \quad \text{a)} \quad c_{cb}^a = \text{const.}; \quad \text{b)} \quad c_{[d}^e c_{b]e}^a = 0.$$

If the conditions are satisfied, to every set of values  $\dot{e}_\beta^a$ ,  $v^\alpha$  chosen at a given point  $\eta^\alpha$  of  $X$ , there exist fields  $\dot{e}_\beta^a$ ,  $v^\alpha$  in an  $\mathfrak{N}(\eta^\alpha)$  satisfying (2.4) and (2.7). If  $r$  linearly independent vectors  $v^\alpha$  are chosen as the  $r$  contravariant basis vectors  $e_\beta^\alpha$  of an anholonomic coordinate system ( $A$ )

we have

$$(2.12) \quad 0 = \frac{\xi}{B} \dot{e}_\beta = e^\gamma \partial_\gamma \dot{e}_\beta + \dot{e}_\gamma \partial_\beta \frac{e^\gamma}{B};$$

and accordingly

$$(2.13) \quad (\partial_\beta \frac{e^\alpha}{B}) \dot{e}_\alpha = - \frac{e^\gamma}{B} (\partial_\gamma \dot{e}_\beta) e_\alpha^A \dot{e}_\alpha^A;$$

hence

$$(2.14) \quad 2 \partial_{[\gamma} \dot{e}_{\beta]}^A = - 'c_{CB}^A \dot{e}_\gamma^C \dot{e}_\beta^B;$$

where

$$(2.15) \quad 'c_{CB}^A \stackrel{\text{def}}{=} - c_{cb}^a \dot{e}_\gamma^c \dot{e}_\beta^b e_\gamma^\nu e_\beta^\delta \dot{e}_\delta^A.$$

As the  $\dot{e}_\beta^A$  are invariant for the transformations  $e^\alpha dt$ , the fields  $\dot{e}_\beta^A$  are also invariant for the transformations  $e^\alpha dt$ . But we have already proved that  $r$  linearly independent fields  $\dot{e}_\beta^A$  satisfying equations of the form (2.14) and invariant for  $r$  linearly independent infinitesimal transformations can only exist if the  $'c_{CB}^A$  are constants satisfying the equations  $'c_{[DC}^E 'c_{B]E}^A = 0$ .

Now if at some arbitrary point  $\eta^\alpha$  we choose the  $\dot{e}_\beta^A$  such that they coincide with the  $\dot{e}_\beta^A$ , we have at that point according to (2.15)

$$(2.16) \quad 'c_{CB}^A = - \delta_{CB}^a \dot{e}_a^A$$

and because the  $c_{cb}^a$  and  $'c_{CB}^A$  are constants, this holds at all points.

To the fields  $\dot{e}_\beta^A$ ,  $\dot{e}_\beta^A$  belongs a connexion, say  $\bar{\Gamma}_{\gamma\beta}^\alpha$ , for which these fields are covariant constant and in the same way a connexion  $\bar{\Gamma}_{\gamma\beta}^\alpha$  can be constructed with respect to the fields  $\dot{e}_\beta^A$ ,  $\dot{e}_\beta^A$ . Then it is easily proved from (2.4) and (2.14) that

$$(2.17) \quad \dot{S}_{cb}^A \stackrel{*}{=} - \frac{1}{2} c_{cb}^a; \quad \dot{S}_{CB}^A \stackrel{*}{=} - \frac{1}{2} 'c_{CB}^A$$

and from the invariance of the fields  $\dot{e}_\beta^A$  and  $\dot{e}_\beta^A$  with respect to each other that

$$(2.18) \quad \dot{S}_{\gamma\beta}^\alpha \stackrel{*}{=} - \bar{\Gamma}_{\gamma\beta}^\alpha.$$

This means that the geometry in  $X_r$  is just the same as in a group space of an  $r$ -parameter group. But the  $X_r$  is not yet a group space; the  $r$ -parameter group itself is still missing.

Now we have in fact two groups of point transformations in  $X_r$ , viz. the invertible transformations leaving invariant the fields  $\dot{e}_\beta^A$  and those that leave invariant the fields  $\dot{e}_\beta^A$ . Taking the first group we see that the transformation  $\eta^\alpha \rightarrow \eta^\alpha + e^b A_b^\alpha dt$  represents for all values of the constants  $e^b$  one of its infinitesimal transformations (cf. II § 10).

We also see that the set of finite transformations

$$(2.19) \quad ' \eta^\alpha = e^{t e^b \partial_b} \eta^\alpha; \quad \partial_b \stackrel{\text{def}}{=} A_b^\beta \partial_\beta;$$

depending on  $r$  parameters  $t e^b$  and consisting of the transformations of all one-parameter subgroups each generated by the infinitesimal transformation for some values of  $e^b$  (cf. II § 10), is contained in the set. But we do not yet know whether the set contains all the transformations of the group.

If we take the  $\eta^\alpha$  in (2.19) as fixed, say  $\eta^\alpha = \eta_1^\alpha$ , the point ' $\eta^\alpha$ ' describes a curve with the differential equation

$$(2.20) \quad \frac{d \eta^\alpha}{dt} = e^b A_b^\alpha$$

and this is a geodesic in  $X_r$  for the connexion  $\overset{+}{I}_{\gamma\beta}^\alpha$ ,  $\overset{-}{I}_{\gamma\beta}^\alpha$  and

$I_{\gamma\beta}^\alpha = \frac{1}{2} (\overset{+}{I}_{\gamma\beta}^\alpha + \overset{-}{I}_{\gamma\beta}^\alpha)$  because

$$(2.21) \quad \delta \frac{d \eta^\alpha}{dt} = \overset{+}{\delta} \frac{d \eta^\alpha}{dt} - \overset{-}{S}_{;\beta}^{\alpha} d \eta^\beta \frac{d \eta^\alpha}{dt} = \overset{+}{\delta} \frac{d \eta^\alpha}{dt} = 0.$$

The  $t e^b$  are normal coordinates in  $X_r$  with respect to the point  $\eta_1^\alpha$  and the connexion  $I_{\gamma\beta}^\alpha$  (cf. III § 7). Hence there is always one and only one choice of  $t e^b$  for which ' $\eta^\alpha = \eta_2^\alpha$ ' if  $\eta^\alpha$  is some arbitrarily given point in  $\mathfrak{N}(\eta_1^\alpha)$ . Moreover it follows that there is one and only one transformation in the set that transforms  $\eta_1^\alpha$  into an arbitrary point  $\eta_2^\alpha$  of  $\mathfrak{N}(\eta_1^\alpha)$ . This transformation belongs to the group but we have not yet proved that the group does not contain another transformation carrying  $\eta_1^\alpha$  into  $\eta_2^\alpha$ . Now every transformation of the group leaves the fields  $e_B^\alpha$  invariant. Hence it leaves invariant the set of all geodesics and also the natural parameters on them. Hence if  $\eta_1^\alpha$  is transformed into  $\eta_2^\alpha$  the set of all geodesics through  $\eta_1^\alpha$  is transformed into the set of all geodesics through  $\eta_2^\alpha$  and this transformation is fixed completely by the condition that the direction of a geodesic at  $\eta_1^\alpha$  is (+)-parallel to the direction at  $\eta_2^\alpha$  of its transform. Moreover the normal coordinates of a point with respect to  $\eta_1^\alpha$  must be the same as those of its transform with respect to  $\eta_2^\alpha$  and this condition not only fixes the transformation of the geodesics but also the transformation of all points in  $\mathfrak{N}(\eta_1^\alpha)$ .

This proves that every transformation in the group is also a transformation of the set and that accordingly the group is an  $r$ -parameter one with the infinitesimal transformations  $\eta^\alpha \rightarrow \eta^\alpha + e^b A_b^\alpha dt$ .

If now we associate with every point  $\eta^\alpha$  of  $X$ , the transformation of the group carrying the already chosen point  $\eta^\alpha$  into  $\eta^\alpha$ ,  $X$ , is the group space of this group. The group is now a group of point transformations in its own group space and accordingly it coincides with its first parameter-group.

*After this digression on sets of vector fields* in an  $X$ , we return to group space and the first and second parameter-group and ask whether the transformation  $\eta^\alpha \rightarrow \eta^\alpha + \lambda_b^b e^\alpha dt$  is an infinitesimal transformation of the first parameter-group. Necessary and sufficient is the invariance of the fields  $e^\alpha$ :

$$(2.22) \quad \lambda_b^b e^\beta \partial_\beta e^\alpha - e^\beta (\partial_\beta \lambda_b^b) e^\alpha - \lambda_b^b e^\beta \partial_\beta e^\alpha = 0$$

and this is only possible if the  $\lambda_b^b$  are constants.

In group theory it is usual to denote an infinitesimal transformation in some  $X_n$ ,  $\xi^\alpha \rightarrow \xi^\alpha + X^\mu dt = \xi^\alpha + X^\mu \partial_\mu \xi^\alpha dt$  by a symbol, for instance  $X$ , standing for the operator  $X^\mu \partial_\mu$ . Then we have for the transformation of  $\xi^\alpha$  and of any function  $f(\xi^\alpha)$

$$(2.23) \quad d\xi^\alpha = X \xi^\alpha dt; \quad df = X f dt.$$

If  $X$  and  $Y$  are two different symbols of this kind, we write  $(XY)$  for the operator  $XY - YX$ . This notation was introduced by LIE.

Using  $\partial_b \stackrel{\text{def}}{=} A_b^\alpha \partial_\alpha$  in this way as a symbol for the infinitesimal transformation  $\eta^\alpha \rightarrow \eta^\alpha + e^\alpha dt$  of the first parameter-group we get from (1.19)

$$(2.24) \quad c_{cb}^a = -2 e^\gamma e^\beta \partial_{[\gamma} \dot{e}_{\beta]}^a = 2 e^\gamma \dot{e}_{[\beta}^a \partial_{\gamma]} e^\beta \stackrel{*}{=} 2 A_\beta^a A_{[\gamma}^\gamma \partial_{\gamma]} A_{b]}^\beta$$

hence

$$(2.25) \quad (\partial_c \partial_b) = c_{cb}^{;a} \partial_a .$$

In the same way it is proved for the second parameter-group

$$(2.26) \quad (\partial_C \partial_B) = -c_{CB}^{;A} \partial_A ; \quad \partial_B \stackrel{\text{def}}{=} A_B^\alpha \partial_\alpha .$$

The equations (2.25, 26) are called the LIE *structural formulae* of the first and second parameter-group and the  $c_{cb}^{;a}$  its *structural constants*. On the other hand (1.19, 23) are called the CARTAN *structural formulae*.

of these same groups. These latter equations were established first by MAURER.<sup>1)</sup> LIE started from infinitesimal transformations and this is the reason why he looked at the matter from the contravariant side. But CARTAN started from systems of Pfaffians and his point of view was therefore entirely covariant. The rather fine dualism between the two points of view was pointed out by CARTAN.

Comparing the equations (1.19) and (2.25) it is obvious that (1.19) contains an invariant operation, viz. Rot, but that the operator  $\partial_c$  is only invariant if it is applied to scalars. Now we have according to (1.16)

$$(2.27) \quad \underline{c}_b e^\alpha = e^v \bar{V}_v \underline{c}_b e^\alpha = - 2 e^v \overset{+}{S}_{v\beta} \underline{e}_b^\alpha e^\beta = c_{cb}^a e_a^\alpha$$

and therefore, considering Exerc. II 10,9

$$(2.28) \quad (\underline{c}_b \underline{c}) \Phi = c_{cb}^a \underline{c}_a \Phi; \quad (\underline{c}_b \underline{c}) \stackrel{\text{def}}{=} \underline{c}_c \underline{c}_b - \underline{c}_b \underline{c}_c$$

for every quantity  $\Phi$  (indices suppressed). Hence

$$(2.29) \quad (\underline{c}_b \underline{c}) = c_{cb}^a \underline{c}_a \quad ?$$

and in the same way

$$(2.30) \quad (\underline{c}_C \underline{c}_B) = - c_{CB}^A \underline{c}_A.$$

(2.29) and (2.30) are equivalent to (2.25, 26) if the operators are applied to a scalar. But in contradistinction to (2.25, 26), in (2.29, 30) only those operators occur that can be applied to *all* quantities.

The finite transformations of the first and second parameter-group can be written in the form (cf. II § 10)

$$(2.31) \quad ' \eta^\alpha = e^{t \epsilon^b \partial_b} \eta^\alpha; \quad ' \eta^\alpha = e^{t \epsilon^B \partial_B} \eta^\alpha$$

and also in the more general form valid for any quantity  $\Phi$  (indices suppressed) dragged along (cf. II § 10)

$$(2.32) \quad \overset{m}{\Phi} = e^{-t \overset{b}{\epsilon} \overset{b}{\underline{e}}} \Phi; \quad \overset{m}{\Phi} = e^{-t \overset{B}{\epsilon} \overset{B}{\underline{e}}} \Phi; \quad \overset{b}{\epsilon} \stackrel{\text{def}}{=} \underline{e}_c e^c; \quad \overset{B}{\epsilon} \stackrel{\text{def}}{=} \underline{e}_c e^c.$$

We gather together the following results:

An  $r$ -parameter group being given, there exist in group space an holonomic coordinate systems ( $a$ ) and ( $A$ ), satisfying the equations

<sup>1)</sup> MAURER 1888, 1, p. 117; cf. for the solution of these equations WHITEHEAD 1932, 2 also for literature.

<sup>2)</sup> The identities (2.29, 30) are proved here only for quantities. But they are valid for all geometric objects having second LIE derivatives (cf. II § 10).

(1.19), (1.23), (1.21), (1.24), (2.25), (2.26). There are two groups of point transformations in  $X_r$ , the two parameter-groups, whose transformations are in one to one correspondence with the transformations of the given group. The first leaves invariant the fields of basis vectors of ( $A$ ) and the second those of ( $a$ ). Conversely if  $\frac{1}{2} r^2(r - 1)$  constants  $c_{cb}^a = c_{[cb]}^a$  be given that satisfy (1.28b), it is always possible to construct in an arbitrary  $X_r$  an anholonomic system ( $a$ ) satisfying (1.19) and (2.25) and another anholonomic system ( $A$ ) satisfying (1.23) and (2.26). The  $\partial_b$  are the symbols of infinitesimal transformations of a group of point transformations in  $X_r$  that leave invariant the fields of the basis vectors of ( $A$ ) and the same holds for the  $\partial_B$  with respect to the basis vectors of ( $a$ ). The transformations of these groups can be brought into one to one correspondence to the points of the  $X_r$ , thus making it a group space.

Besides the two parameter-groups there is a third transformation group connected with every group. The equation

$$(2.33) \quad {}'T = UTU^{-1}$$

represents a transformation of the elements  $T$  of this group. If for  $U$  all the elements are taken we get the *adjoint group* of the given group. Because of

$$(2.34) \quad V U T U^{-1} V^{-1} = (V U) T (V U)^{-1}$$

this group and the given group are homomorphic but the homomorphism need not be an isomorphism (that is one to one). In fact, two elements  $U_1$  and  $U_2$  may exist such that

$$(2.35) \quad U_1 T U_1^{-1} = U_2 T U_2^{-1}$$

for every  $T$ . This is the case if and only if there exists besides  $I$  another element that is commutative with respect to every element of the group. All elements of this kind form an invariant subgroup called the *centre* of the group. Let  $C$  be an element of the centre, then the transformation  $UTU^{-1}$  of the adjoint group corresponds not only to  $U$  but also to  $CU$ .

An element  $T$  and its transform  $UTU^{-1}$  are called *homologous*. A subgroup and its transform are also said to be *homologous*. If a subgroup is identical with its transform for every choice of  $U$  it is called an *invariant* or *normal subgroup*.

From the definition (2.33) it follows that a transformation of the adjoint group arises if a transformation of the first parameter-group carrying  $I$  into  $U$  is followed by a transformation of the second group

carrying  $I$  into  $U^{-1}$ . If  $s^\alpha$  is a vector at  $\eta^\alpha$  and  $\overset{+}{s}{}^\alpha$  and  $\overset{-}{s}{}^\alpha$  are the fields we can get from  $s^\alpha$  by  $(\pm)$ -displacement, the infinitesimal transformation of the adjoint group corresponding to the point  $\overset{+}{\eta}{}^\alpha + s^\alpha du$  arises if the  $(-)$ -parallel displacement over  $\overset{+}{s}{}^\alpha du$  is followed by the  $(+)$ -parallel displacement over  $-\overset{-}{s}{}^\alpha du$ . A vector  $e^a$  at  $\eta^\alpha$  is transformed by these transformations first into the vector  $e^a - \bar{\Gamma}_{cb}^a e^b s^c du$  and afterwards into the vector  $e^a - c_{cb}^{;a} e^b s^c du$ . Hence the infinitesimal transformation corresponding to the point  $\overset{+}{\eta}{}^\alpha + e^\alpha dt$  in group space, with the symbol  $e^a A_a^\beta \partial_\beta$ , is transformed by an infinitesimal transformation of the adjoint group in the following way

$$(2.36) \quad de^a = - c_{cb}^{;a} e^b s^c du = s^b E_b^a du; \quad E_b^a \stackrel{\text{def}}{=} e^c c_{cb}^{;a}.$$

This is an infinitesimal transformation of the  $e^a$  with the symbol

$$(2.37) \quad s^b E_b; \quad E_b \stackrel{\text{def}}{=} E_b^a \frac{\partial}{\partial e^a} = e^c c_{cb}^{;a} \frac{\partial}{\partial e^a}.$$

Because

$$(2.38) \quad (E_c E_b) = 2e^f c_{[c}^{;d} c_{|d|b]}^e \partial_e = c_{cb}^{;a} E_a;$$

the  $E_b$  and  $\partial_b$  satisfy equations with the same structural constants. But the  $E_b$  are independent if and only if the  $c$ -rank of  $c_{cb}^{;a}$  is equal to  $r$ .

Writing  $S_b^a \stackrel{\text{def}}{=} s^c c_{cb}^{;a}$  we get for a finite transformation of the one-parameter group generated by  $s^b E_b$  for some constant values of the  $s^b$

$$(2.39) \quad 'e^a = (e^{-u S})_b^a e^b;$$

where  $S$  symbolizes the matrix of  $S_b^a$ . The equation (2.39) represents, for variable values of the  $s^\alpha$ , a linear homogeneous group of transformations of the  $e^a$ , called the *linear adjoint group*. It is the adjoint group but only if considered as a group of transformations working on the infinitesimal transformations of the given group.

If we introduce the  $t e^a$  as normal coordinates  $\eta^a$  of  $X$ , with respect to the point  $\overset{+}{\eta}{}^\alpha$

$$(2.40) \quad \eta^a \stackrel{\text{def}}{=} \delta_a^\alpha t e^\alpha; \quad \alpha = \bar{1}, \dots, \bar{r}$$

the three coordinate systems  $(a)$ ,  $(A)$  and  $(\alpha)$  have the same basis vectors at  $\eta^\alpha$  and we see that the transformation of these new coordinates is also linear homogeneous. That proves the statement:

*The transformations of the adjoint group are linear homogeneous if they are expressed in the normal coordinates  $\eta^a$  with respect to  $\overset{+}{\eta}{}^\alpha$ .*

We write these transformations

$$(2.41) \quad \begin{cases} \text{a)} & d\eta^a = -S_b^a \eta^b du; \quad S_b^a \stackrel{\text{def}}{=} S_b^a \{\eta^\alpha\}^1 \\ \text{b)} & {}' \eta^a = (e^{-u} \overset{S}{\partial})_b^a \eta^b; \quad S = \text{matrix of } S_b^a. \end{cases}$$

(2.41 b) is the transformation  $U \dots U^{-1}$  if  $U$  corresponds to the point  $\eta^a = \delta_a^a us^a$  of group space. This transformation arises if the (+)-equipollence that transforms  $U$  into  $I$  is followed by the (-)-equipollence that transforms  $I$  into  $U$ . The first transforms the vectors  $e_b^\alpha$  at  $\delta_a^a us^a$  into the  $e_b^\alpha$  at  $\eta^\alpha$  coinciding with the  $e_B^\alpha$  at that point, and the second transforms these vectors into the  $e_B^\alpha$  at  $\delta_a^a us^a$ . Hence

$$(2.42) \quad \text{a)} \quad e_B^\alpha = \delta_B^b (e^{-u} \overset{S}{\partial})_\beta^a e_b^\beta \quad \text{or} \quad \text{b)} \quad A_B^\alpha = (e^{-u} \overset{S}{\partial})_B^a;$$

or in words:

*The vectors  $e_b^\alpha$  at a point of group space that corresponds to the transformation  $U$  of the group, transform into the vectors  $e_B^\alpha$  at that point by the same linear homogeneous transformation that represents the transformation  $UTU^{-1}$  of the adjoint group if applied to the normal coordinates  $\eta^a$ .*

### Exercises.

IV 2,1. If operators  $\mathfrak{L}_b$  are defined by  $\mathfrak{L}_b \stackrel{\text{def}}{=} \overset{a}{\partial}_b \mathfrak{L}_a$ , prove that

$$\text{IV 2,1 } \alpha) \quad (\mathfrak{L}_c \mathfrak{L}_b) = 0.$$

This is the reason why we do not use the operators  $\mathfrak{L}_b$ . The same difficulty does not arise with the operators  $\partial_b$  because  $\partial_b \overset{a}{\partial}_c = 0$  whereas  $\mathfrak{L}_b \overset{a}{\partial}_c \neq 0$ .

IV 2,2. Prove that

$$\text{IV 2,2 } \alpha) \quad \text{a)} \quad \overset{+}{I}_{cb}^a * 0; \quad \text{b)} \quad Q_{cb}^a * \overset{+}{S}_{cb}^a.$$

IV 2,3. Prove that the transformation  $\eta^\alpha \rightarrow \eta^\alpha + v^\alpha dt$  represents an infinitesimal transformation of the first parameter-group if and only if  $v^\alpha = \text{const.}$

IV 2,4. Prove that

$$\text{IV 2,4 } \alpha) \quad \overset{c}{\mathfrak{L}} \overset{B}{\mathfrak{L}} = \overset{B}{\mathfrak{L}} \overset{c}{\mathfrak{L}}$$

and

$$\text{IV 2,4 } \beta) \quad \overset{+}{V}_\gamma \overset{-}{V}_\beta = \overset{-}{V}_\beta \overset{+}{V}_\gamma.$$

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<sup>1)</sup> Note that the components  $S_b^a$  are constants but that the  $S_b^a$  equal the  $S_b^a$  at  $\eta^\alpha$  only.

IV 2,5. Prove that a quantity is (+)-constant if and only if its components with respect to (a) are constants.

IV 2,6<sup>1)</sup>. Prove that the (0)-connexion in group space is volume preserving.

### § 3. Finite continuous transformation groups.

So far we have not supposed that the group originally given was in any way a group of transformations.<sup>2)</sup>

Let us now take an r-parameter group of transformations in n variables  $\xi^\alpha$ ;  $\alpha = 1, \dots, n$

$$(3.1) \quad \xi^\alpha = f^\alpha(\xi^r, \eta^\alpha); \quad \alpha = i, \dots, r.$$

First we have to make sure that the  $\xi^\alpha$  can be solved from (3.1). The necessary and sufficient condition is that the matrix of the  $\partial_\lambda f^\alpha$  has rank n. Secondly it is necessary that the  $\eta^\alpha$  should be *essential*, that is, that there should be no equations of the form

$$(3.2) \quad f^\alpha(\xi^r, \eta^\alpha) = F^\alpha(\xi^r, \zeta^a(\eta^\alpha)); \quad a = i, \dots, r'; \quad r' < r.$$

From (3.2) it follows that

$$(3.3) \quad \partial_\beta f^\alpha = (\partial_\beta \zeta^a) \partial_a F^\alpha; \quad \partial_a = \frac{\partial}{\partial \zeta^a}$$

and because the rank of  $\partial_\beta \zeta^a$  is  $\leq r'$  this implies that there exists at least one set of functions  $U^\beta(\eta^\alpha)$  such that

$$(3.4) \quad U^\beta(\eta^\alpha) \partial_\beta f^\alpha(\xi^r, \eta^\alpha) = 0.$$

Conversely, if an equation of the form (3.4) holds, the  $f^\alpha(\xi^r, \eta^\alpha)$  must be solutions of the equation

$$(3.5) \quad U^\beta(\eta^\alpha) \partial_\beta \varphi(\eta^\alpha) = 0$$

containing parameters  $\xi^\alpha$ . But because an equation of this form has at most  $r - 1$  independent solutions (cf. II § 5) the  $\eta^\alpha$  cannot be essential. Hence the  $\eta^\alpha$  in (3.1) are essential if and only if no equation of the form (3.4) exists.

From now on we suppose that the  $\eta^\alpha$  are essential and that  $\eta^\alpha$  corresponds to the identical transformation. Then there is a one to one correspondence between the transformations of the group in a neighbourhood of the identical transformation and the points of the  $X_r$  of the  $\eta^\alpha$  in an  $\mathfrak{N}(\eta^\alpha)$ .

<sup>1)</sup> CARTAN 1927, 1, p. 67.

<sup>2)</sup> HAIMOVICI has proved 1945, 1; 1946, 1; 2; 1947, 1; 1948, 1; 2 that a connexion can also be established in the manifold of a set of transformations. Certain comitants of this connexion vanish if and only if the set is a group.

We only consider transformations of this group germ. For every definite choice  $\eta^{\alpha} = \underset{I}{\eta}^{\alpha}$  the equation (3.1)

$$(3.6) \quad ' \xi^{\alpha} = f^{\alpha}(\xi^{\alpha}, \underset{I}{\eta}^{\alpha})$$

represents a definite transformation  $T_{\underset{I}{\eta}}$  operating on the  $\xi^{\alpha}$ . Other values  $\eta^{\alpha} = \underset{2}{\eta}^{\alpha}$  give another transformation  $T_{\underset{2}{\eta}}$

$$(3.7) \quad '' \xi^{\alpha} = f^{\alpha}(\xi^{\alpha}, \underset{2}{\eta}^{\alpha})$$

of the  $\xi^{\alpha}$ . If  $\underset{I}{\eta}^{\alpha}$  is changed into  $\underset{I}{\eta}^{\alpha} + d\underset{I}{\eta}^{\alpha}$  the change of ' $\xi^{\alpha}$ ' is

$$(3.8) \quad d' \xi^{\alpha} = (\partial_{\beta} f^{\alpha})_{\eta^{\alpha} = \underset{I}{\eta}^{\alpha}} d\underset{I}{\eta}^{\beta}$$

and if in (3.8)  $\xi^{\alpha}$  is eliminated from the right hand side by means of (3.6), the resulting equation represents the transformation of ' $\xi^{\alpha}$ '  $= T_{\underset{I}{\eta}} \xi^{\alpha}$  into  $T_{\underset{I}{\eta} + d\underset{I}{\eta}} \xi^{\alpha} = T_{\underset{I}{\eta} + d\underset{I}{\eta}} T_{\underset{I}{\eta}}^{-1} ' \xi^{\alpha}$ , that is the infinitesimal transformation  $T_{\underset{I}{\eta} + d\underset{I}{\eta}} T_{\underset{I}{\eta}}^{-1}$  operating on the ' $\xi^{\alpha}$ '. This transformation corresponds to the infinitesimal vector  $d\underset{I}{\eta}^{\alpha}$  at the point  $\underset{I}{\eta}^{\alpha}$  in group space. In the same way

$$(3.9) \quad d'' \xi^{\alpha} = (\partial_{\beta} f^{\alpha})_{\eta^{\alpha} = \underset{2}{\eta}^{\alpha}} d\underset{2}{\eta}^{\beta}$$

after elimination of the  $\xi^{\alpha}$  by means of (3.7) represents the transformation  $T_{\underset{2}{\eta} + d\underset{2}{\eta}} T_{\underset{2}{\eta}}^{-1}$  operating on the variables '' $\xi^{\alpha}$  and it corresponds to the infinitesimal vector  $d\underset{2}{\eta}^{\alpha}$  at the point  $\underset{2}{\eta}^{\alpha}$  in group space. If  $d\underset{2}{\eta}^{\alpha}$  is now chosen in such a way that

$$(3.10) \quad T_{\underset{I}{\eta} + d\underset{I}{\eta}} T_{\underset{I}{\eta}}^{-1} = T_{\underset{2}{\eta} + d\underset{2}{\eta}} T_{\underset{2}{\eta}}^{-1}$$

then  $d\underset{I}{\eta}^{\alpha}$  at  $\underset{I}{\eta}^{\alpha}$  is (+)-equipollent to  $d\underset{2}{\eta}^{\alpha}$  at  $\underset{2}{\eta}^{\alpha}$ . Hence, introducing the coordinate system (a) we have  $(d\underset{I}{\eta})^a = (d\underset{2}{\eta})^a$ , and the two equations

$$(3.11) \quad d' \xi^{\alpha} = (\partial_{\beta} ' \xi^{\alpha})_{\eta^{\alpha} = \underset{I}{\eta}^{\alpha}} A_b^{\beta}(\underset{I}{\eta}^{\alpha}) (d\underset{I}{\eta})^b$$

$$(3.12) \quad d'' \xi^{\alpha} = (\partial_{\beta} '' \xi^{\alpha})_{\eta^{\alpha} = \underset{2}{\eta}^{\alpha}} A_b^{\beta}(\underset{2}{\eta}^{\alpha}) (d\underset{2}{\eta})^b$$

after elimination of the  $\xi^{\alpha}$  represent one and the same transformation, viz. the transformation corresponding to the point  $\underset{0}{\eta}^{\alpha} + A_b^{\alpha}(\underset{0}{\eta}^{\alpha}) (d\underset{I}{\eta})^b$  in group space, operating on the ' $\xi^{\alpha}$ ' in (3.11) and on the '' $\xi^{\alpha}$  in (3.12). But *this is only possible if the expressions*

$$(3.13) \quad \Xi_b^{\alpha} \stackrel{\text{def}}{=} (\partial_{\beta} ' \xi^{\alpha}) A_b^{\beta}(\underset{I}{\eta}^{\alpha})$$

<sup>1)</sup> Cf. for the use of  $A_b^{\beta}$  as a function symbol, II § 3.

after elimination of the  $\xi^\alpha$  by means of (3.6), depend only on the ' $\xi^\alpha$ ' and are independent of the  $\eta^\alpha$ .

The infinitesimal transformation corresponding to the point  $\eta^\alpha + e^\alpha A_a^\alpha dt$  of group space is given by

$$(3.14) \quad d\xi^\alpha = \Xi_a^\alpha(\xi^\nu) e^\alpha dt; \quad e^\alpha = \text{const.}$$

and to every choice of  $e^\alpha dt$  there belongs a definite infinitesimal transformation with  $d\xi^\alpha \neq 0$ . In fact no relation of the form

$$(3.15) \quad c^b \Xi_b^\alpha = 0$$

can exist with coefficients  $c^b$  independent of the  $\xi^\alpha$  because in that case there would be a relation

$$(3.16) \quad c^b A_b^\beta \partial_\beta f^\alpha = 0$$

and as we have proved already this would imply that the parameters were not essential [cf. (3.4)].

According to CAMPBELL<sup>1)</sup> we call a set of infinitesimal transformations *connected* if they satisfy a homogeneous linear equation, and *dependent* if in addition the coefficients in this relation are constants. It frequently happens that a set is independent but connected.

Collecting results we have the first part of the first fundamental theorem of LIE<sup>2)</sup>:

#### I.1. If the invertible transformations

$$(3.17) \quad ' \xi^\alpha = f^\alpha(\xi^\nu, \eta^\alpha); \quad \alpha = 1, \dots, r$$

with  $r$  essential parameters  $\eta^\alpha$  form a group, there exist  $r^2$  functions  $A_b^\alpha(\eta^\beta)$ ;  $\text{Det}(A_b^\alpha) \neq 0$ ;  $a, b = 1, \dots, r$  and  $nr$  functions  $\Xi_b^\alpha(\xi^\lambda)$  for which no equations exist of the form  $c^b \Xi_b^\alpha = 0$  with coefficients  $c^b$  independent of the  $\xi^\alpha$ , such that

$$(3.18) \quad \boxed{\partial_\beta ' \xi^\alpha = A_\beta^\alpha(\eta^\alpha) \Xi_b^\alpha(' \xi^\nu)}.$$

Starting from the inverse of (3.17)

$$(3.19) \quad \xi^\alpha = F^\alpha(' \xi^\nu, \eta^\alpha)$$

it can be proved in the same way that

$$(3.20) \quad \boxed{\partial_\beta \xi^\alpha = A_\beta^\alpha(\eta^\alpha) ' \Xi_B^\alpha(\xi^\lambda)}; \quad ' \Xi_B^\alpha \stackrel{\text{def}}{=} (\partial_\beta \xi^\alpha) A_\beta^\alpha(\eta^\alpha).$$

<sup>1)</sup> CAMPBELL 1903, 1.

<sup>2)</sup> In this and the following fundamental theorems all functions are supposed to be analytic in the region considered.

The finite transformations of the group are found as follows. The equation of a geodesic through  $\eta^\alpha$  is

$$(3.21) \quad \frac{d\eta^\alpha}{dt} = A_a^\alpha e^a; \quad \eta^\alpha = \eta_0^\alpha \text{ for } t = 0.$$

Hence at every point of this curve we have according to (3.18)

$$(3.22) \quad \frac{d'\xi^\alpha}{dt} = \Xi_b^\alpha('\xi^\lambda) e^b$$

and the solution of this equation with the condition  $'\xi^\alpha = \xi^\alpha$  for  $t = 0$  is (cf. II § 10)

$$(3.23) \quad ' \xi^\alpha = e^{tX} \xi^\alpha; \quad X \stackrel{\text{def}}{=} e^b X_b; \quad X_b \stackrel{\text{def}}{=} \Xi_b^\mu(\xi^\alpha) \partial_\mu.$$

From this, for any function  $f(\xi^\alpha)$  we get

$$(3.24) \quad f(' \xi^\alpha) = e^{tX} f(\xi^\alpha)$$

representing the transformation of the value at the point  $\xi^\alpha$  into the value at  $'\xi^\alpha$ . But if we have a scalar field  $p = f(\xi^\alpha)$  and if this field is dragged along by the point transformation (3.23) the new value at  $\xi^\alpha$  is the original value at  $e^{-tX} \xi^\alpha$  (cf. II § 10), hence this new value is

$$(3.25) \quad \overset{m}{p} = e^{-tX} p$$

and also

$$(3.26) \quad \overset{m}{p} = e^{-t\xi} p$$

where  $\xi$  is the LIE operator belonging to the field  $e^b \Xi_b^\alpha$ .

Unlike (3.25) the formula (3.26) can be generalized for any quantity  $\Phi$  (indices suppressed) as defined in II § 3 (cf. II 10.38)

$$(3.27) \quad \overset{m}{\Phi} = e^{-t\xi} \Phi.$$

If we introduce in  $X_r$  the normal coordinates  $\eta^a$ ;  $a = \bar{I}, \dots, \bar{r}$ , with respect to  $\eta^\alpha$  and to the coordinate system  $(a)$ , then the transformation (3.23) corresponds to the point in group space with the normal coordinates

$$(3.28) \quad \eta^a \stackrel{*}{=} \delta_a^a t e^a.$$

The system of equations (3.18) has the solutions

$$(3.29) \quad ' \xi^\alpha = f^\alpha(\xi^\nu, \eta^\alpha); \quad f^\alpha(\xi^\nu, \eta^\alpha) = \xi^\alpha$$

depending on the  $n$  parameters  $\xi^\alpha$ . Hence the system is totally integrable (cf. II § 5) and its integrability conditions must be satisfied identically.

These conditions are [cf. (2.4)]

$$(3.30) \quad \left\{ \begin{array}{l} 0 = A_{[\gamma}^c A_{\beta]}^b \Xi_c^\mu (' \xi^\lambda ) \frac{\partial \Xi_b^\nu (' \xi^\theta )}{d' \xi^\mu} + \Xi_b^\nu (' \xi^\lambda ) \partial_{[\gamma} A_{\beta]}^b \\ = A_{[\gamma}^c A_{\beta]}^b \left( \Xi_c^\mu (' \xi^\lambda ) \frac{\partial \Xi_b^\nu (' \xi^\theta )}{d' \xi^\mu} - \frac{1}{2} c_{cb}^{ca} \Xi_a^\nu (' \xi^\lambda ) \right) \end{array} \right.$$

or, because the choice of the variables ' $\xi^\lambda$ ' is free and consequently ' $\xi^\nu$ ' can be replaced by ' $\xi^\alpha$ '

$$(3.31) \quad (X_c X_b) = c_{cb}^{ca} X_a .$$

This identity only contains operators operating on scalar functions. But it can be generalized by generalizing the operator  $\mathfrak{L}_b$  defined in (1.8). Let from now on in this section  $\mathfrak{L}_b$  be the LIE operator in the  $(n+r)$ -dimensional space of the  $\xi^\alpha$  and  $\eta^\alpha$  belonging to the field  $e_b^\alpha \Xi_a^\alpha$ ,  $e^\alpha$  for every value of  $b$ .<sup>1)</sup> Then

$$(3.32) \quad (\mathfrak{L}_c \mathfrak{L}_b) = c_{cb}^a \mathfrak{L}_a$$

is true and equivalent to the combination of (2.25) and (3.31) so long as scalar functions only are operated upon. According to (2.28) it is also true for quantities in  $X_n$ . In order to prove that it also holds for quantities in  $X_n$  it can first be proved for co- and contravariant vectors and after that the rule of LEIBNIZ can be applied.<sup>2)</sup>

Of course besides (3.31, 32) we also have

$$(3.33) \quad (X_C X_B) = - c_{CB}^A X_A$$

$$(3.34) \quad (\mathfrak{L}_C \mathfrak{L}_B) = - c_{CB}^A \mathfrak{L}_A .$$

(3.31, 33) and also (3.32, 34) are called the LIE *structural formulae* of the group and the  $c_{cb}^{ca}$  its *structural constants*. So a group of transformations and both its parameter-groups have the same structural formulae and the same structural constants. This is of course trivial because these groups are isomorphic.

<sup>1)</sup> In the classical proof of (3.31) (cf. LIE-ENGEL 1888, 1, p. 158 ff.) the equations (3.18) are first brought back to the linear homogeneous system  $\mathfrak{L}_b \varphi(\xi^\alpha, \eta^\alpha) = 0$ , so the operator  $\mathfrak{L}_b$  really occurs there in its most general form, except that it is connected with scalar functions only.

<sup>2)</sup> It is also valid for all geometric objects having second LIE derivatives (cf. II § 10).

Collecting results we may now state the first part of the second fundamental theorem:

**II.1.** *If  $X_b$  are the symbols of  $r$  independent infinitesimal transformations of the group mentioned in I.1*

$$(3.35) \quad X_b = \Xi_b^\mu(\xi^\alpha) \partial_\mu$$

*equations of the form (3.31) with constant coefficients  $c_{cb}^a$ ,  $c_{(cb)}^a = 0$  hold and the first part of the third fundamental theorem:*

**III.1.** *The constants  $c_{cb}^a$  satisfy the equations (1.28b).*

The second part of the first fundamental theorem is the converse of the first part:

**I.2.** *If a set of transformations of the  $\xi^\alpha$  is given*

$$(3.36) \quad ' \xi^\alpha = f^\alpha(\xi^\nu, \eta^\alpha); \quad \alpha = 1, \dots, r;$$

*with  $\text{Det}(\partial_\lambda ' \xi^\alpha) \neq 0$  in an  $\mathfrak{N}(\xi^\alpha, \eta^\alpha)$ , depending on  $r$  essential parameters  $\eta^\alpha$  and containing the identical transformation ( $\eta^\alpha = \eta^\alpha$ ) and if there exist  $r^2$  functions  $A_b^\beta(\eta^\alpha)$ ;  $\text{Det}(A_b^\beta) \neq 0$  and  $nr$  functions  $\Xi_b^\alpha(\xi^\lambda)$  not satisfying an equation  $c^b \Xi_b^\alpha = 0$  with coefficients  $c^b$  independent of  $\xi^\alpha$ , such that*

$$(3.37) \quad \partial_\beta ' \xi^\alpha = \Xi_b^\alpha(' \xi^\lambda) A_\beta^\lambda(\eta^\alpha),$$

*these transformations form an  $r$ -parameter group germ.*

Proof: though the  $X_r$  is not yet proved to be a group space, the  $A_\beta^\alpha$  and their inverses  $A_b^\alpha$  fix an anholonomic coordinate system ( $a$ ) and a connexion  $\overset{+}{I}_{\gamma\beta}^\alpha$  satisfying the condition that the  $e_a^\alpha$ ,  $\overset{a}{e}_\beta$  are covariant constant. The geodesics of this connexion satisfy the equations

$$(3.38) \quad \frac{d\eta^\alpha}{dt} = e^b A_b^\alpha$$

with constants  $e^a$ .

Now let  $\overset{i}{\eta}^\alpha$  be a point of  $X_r$  in  $\mathfrak{N}(\eta^\alpha)$  corresponding to the arbitrary transformation  $T_i$  of the set. Every point  $'\eta^\alpha$  of  $\mathfrak{N}(\eta^\alpha)$  on a geodesic through  $\overset{i}{\eta}^\alpha$  is given by

$$(3.39) \quad ' \eta^\alpha = (e^{t e^b \partial_b} \eta^\alpha)_{,\alpha=1,\alpha}; \quad \partial_b = A_b^\alpha \partial_\alpha$$

for suitable values of  $t$  and  $e^a$ . If  $' \xi^\alpha = f^\alpha(\xi^\nu, '\eta^\alpha)$ , then, changing  $'\eta^\alpha$  into the point of the same geodesic  $'\eta^\alpha + d\eta^\alpha = '\eta^\alpha + e^a A_a^\alpha dt$  we get

$$(3.40) \quad d' \xi^\alpha = e^b \Xi_b^\alpha(' \xi^\lambda) dt$$

and accordingly the transformations of the set belonging to an arbitrary point  $\eta^\alpha$  of this geodesic is given by

$$(3.41) \quad {}' \xi^\alpha = (e^{t e^b X_b} \xi^\alpha)_{\xi^\alpha = \xi^\alpha}; \quad {}_I \xi^\alpha = f^\alpha(\xi^\nu, {}_I \eta^\alpha); \quad X_b \stackrel{\text{def}}{=} \Xi_b^\mu \partial_\mu$$

for some suitable value of  $t$ .

For fixed values of the  $e^b$  the transformations  $\xi^\alpha \rightarrow {}' \xi^\alpha$  form a one-parameter group with the parameter  $t$ . Every point  $\eta^\alpha$  of  $\mathfrak{N}(\eta^\alpha)$  lies on one of the geodesics through  $\eta^\alpha$ . Hence every transformation of the set can be obtained by the transformation  $T_\eta$  followed by a transformation of one of these one-parameter groups and conversely every transformation constructed in this way belongs to the set. As  $\eta^\alpha$  is entirely arbitrary we may take  $\eta^\alpha = \eta^\alpha$  and this proves that every transformation of the set belongs to one of these groups. But that implies that two transformations of the set applied one after the other give rise to a new transformation of the set. As the set also contains the identical transformation and the inverses, it is proved that the set is a group and the  $X$ , its group space.

The second part of the second fundamental theorem is the converse of the first part:

**II.2.** *If  $X_b = \Xi_b^\mu(\xi^\alpha) \partial_\mu$  are the symbols of  $r$  independent infinitesimal transformations of the  $\xi^\alpha$  satisfying the equations (3.31) with constant coefficients  $c_{cb}^\alpha, c_{(cb)}^\alpha = 0$ ; these transformations are the infinitesimal transformations of an  $r$ -parameter group germ.*

In order to prove this we remark first that the identities (cf. Exerc. IV 3,3)

$$(3.42) \quad (X_{[d} (X_c X_{b]}) = 0$$

can easily be verified and that they imply that the  $c_{cb}^\alpha$  satisfy the identities (2.11b). Now we have proved in IV §2 that according to these identities and the constancy of the  $c_{cb}^\alpha$  there exist in the  $X$ , of the auxiliary variables  $\eta^\alpha, r$  linearly independent vector fields  $\tilde{e}_\beta$  satisfying the equations (2.4). But then it follows from (3.31, 42) that the integrability conditions of (3.37) are satisfied identically. Hence these equations are totally integrable and they have solutions of the form

$$(3.43) \quad {}' \xi^\alpha = f^\alpha(\xi^\nu, \eta^\alpha); \quad \xi^\alpha = f^\alpha(\xi^\nu, {}_0 \eta^\alpha); \quad \text{Det}(\partial_\lambda {}' \xi^\alpha) \neq 0 \text{ for } \eta^\alpha = {}_0 \eta^\alpha$$

valid in an  $\mathfrak{N}(\eta^\alpha)$  and depending on  $n$  parameters  $\xi^\alpha$ . That finishes the proof because according to the fundamental theorem I.2, (3.43) represents an  $r$ -parameter group and the  $X_b$  are infinitesimal transformations of

this group. As we see an  $r$ -parameter group germ is wholly determined by  $r$  linearly independent infinitesimal transformations satisfying (3.31) with constants  $c_{cb}^a$ . It is said to be *generated* by them.

The second part of the third fundamental theorem is the converse of the first part:

III.2. *If  $\frac{1}{2}r^2(r-1)$  constants  $c_{cb}^a; c_{(cb)}^a = 0$  satisfy equations of the form (2.11b) there always exists an  $r$ -parameter group germ with  $r$  linearly independent infinitesimal transformations  $X_b$  satisfying (3.31).*

In order to prove this we construct in the  $X$ , of the auxiliary variables  $\eta^\alpha$  the fields  $\tilde{e}_\beta^a$  satisfying (2.4). In IV §2 we have already proved that it is possible to construct in  $X$ , an  $r$ -parameter group of point transformations with the infinitesimal transformations  $\eta^\alpha \rightarrow \eta^\alpha + e^b A_\beta^\alpha dt$  whose symbols  $X_b$  satisfy (3.31). But as the fundamental theorem II.2 is already known, it is only necessary to remark that these  $X_b$  satisfy (3.31) because then according to II.2 they generate an  $r$ -parameter group germ.<sup>1)</sup>,<sup>2)</sup>

The results obtained so far show that the properties of an  $r$ -parameter finite continuous group germ of an abstract group correspond to and are in fact identical with the algebraic properties of a tensor  $c_{cb}^a$ , which is alternating in  $cb$  and satisfies (1.28b) and also identical with the geometric properties of a very special  $A$ , which is symmetric with respect to all its points (cf. III § 7).

As soon as the elements are supposed to be transformations in  $n$  variables we also get properties in the space of these variables. But it makes a difference whether the elements are interpreted as point transformations or as coordinate transformations as the following example will show.

Let us assume first that the elements are point transformations in the  $X_n$  of the  $\xi^\alpha$ . Let  $T$  and  $U$  be two of these transformations

$$(3.44) \quad \begin{cases} T: \xi^\alpha \rightarrow ' \xi^\alpha \\ U: \xi^\alpha \rightarrow \bar{\xi}^\alpha; \quad ' \xi^\alpha \rightarrow ' \bar{\xi}^\alpha. \end{cases}$$

Then we have

$$(3.45) \quad ' \bar{\xi}^\alpha = U ' \xi^\alpha = U T \xi^\alpha = U T U^{-1} \bar{\xi}^\alpha.$$

<sup>1)</sup> In the proofs of the three fundamental theorems we made use of the existence theorems of partial differential equations. For our purposes this is the natural way because sets of these equations are just what we are interested in. FREUDENTHAL proved in 1933, 1 that it is possible to use only the existence theorem for ordinary differential equations.

<sup>2)</sup> E. CARTAN has shown 1930, 1; 1937, 2, p. 189 (with LERAY) that the original proofs of the fundamental theorems given by LIE fail if one does not confine oneself to the group germ. Of course the same holds for the proofs given here.

Hence, if  $T$  transforms  $\xi^\alpha$  into  $'\xi^\alpha$  we get the transformation of  $U\xi^\alpha$  into  $U'\xi^\alpha$  by applying the transformation  $U \dots U^{-1}$  of the adjoint group to  $T$ . Now let us assume that  $T$  is a point transformation and  $U$  a coordinate transformation

$$(3.46) \quad \begin{cases} T : \xi^\alpha \rightarrow '\xi^\alpha \\ U : \xi^\alpha \rightarrow \xi^{\alpha'}; \quad '\xi^\alpha \rightarrow '\xi^{\alpha'}. \end{cases}$$

Then we have

$$(3.47) \quad '\xi^{\alpha'} = U'\xi^\alpha = UT\xi^\alpha = UTU^{-1}\xi^{\alpha'}$$

and this means that  $T$  describes a point transformation with respect to  $(\alpha)$  and  $UTU^{-1}$  describes the same transformation with respect to  $(\alpha')$ .

In group space the transformations of the adjoint group are of course always point transformations as we have already seen in IV § 2.

### Exercises.

IV 3,1. Prove that

$$\text{IV 3,1 } \alpha) \quad 'E_B^\alpha(\xi^\nu) = -\delta_B^b E_b^\alpha(\xi^b).$$

IV 3,2. Prove that the functions  $E_b^\alpha(\xi^b)$  and  $A_\beta^b(\eta^\alpha)$  can be obtained by algebraic operations and differentiations from the equations (3.17). (Use the transformations  $'\xi^\alpha = f^\alpha(\xi^\nu, \eta^\alpha)$ ;  $"\xi^\alpha = f^\alpha(''\xi^\nu, \xi^\alpha) = f^\alpha(\xi^\nu, \theta^\alpha)$ .)

IV 3,3. Prove that

$$\text{IV 3,3 } \alpha) \quad \left( \frac{\partial}{\partial \xi^b} \left( \frac{\partial \xi^a}{\partial \eta^\alpha} \right) \right) = 0.$$

## § 4. The geometry of group space.

In group space we now have the general coordinates  $\eta^\alpha$ , the integrable but not symmetric (+)- and (-)-connexion and the anholonomic coordinate systems  $(a)$  and  $(A)$  belonging to them, the symmetric ( $\theta$ )-connexion and the normal coordinates  $\eta^a$  with respect to  $\eta^\alpha$ , the first and second parameter-group with the transformations  $T \rightarrow UT$  and  $T \rightarrow TU$  consisting of all (-)-equipollences and (+)-equipollences, and the adjoint group with the transformations  $T \rightarrow UTU^{-1}$  linear homogeneous if expressed in the  $\eta^a$ . In this section a number of important relations will be established.

In III § 7 the relations between general coordinates and normal coordinates were derived. If this time we take  $\alpha, \beta, \gamma = 1, \dots, r$  instead of  $\alpha, \lambda, \mu = 1, \dots, n$  and  $a, b, c = \bar{1}, \dots, \bar{r}$  instead of  $h, i, j = 1, \dots, n$  we

get from (III 7.18, 19)

$$(4.1) \quad \eta^\alpha = \delta_a^\alpha \eta^a - \frac{1}{2!} \Gamma_{\gamma\beta}^\alpha \{ \eta \} \delta_c^\gamma b^\beta \eta^c \eta^b - \frac{1}{3!} \Gamma_{\gamma_2 \gamma_1 \beta}^\alpha \{ \eta \} \delta_{c_2}^{c_1} c_1^\beta \eta^{c_2} \eta^{c_1} \eta^b - \dots$$

$$(4.2) \quad \eta^a = \delta_a^\alpha \eta^\alpha + \frac{1}{2!} \delta_a^\alpha A_{\gamma\beta}^\alpha \{ \eta \} \eta^\gamma \eta^\beta + \frac{1}{3!} \delta_a^\alpha A_{\gamma_2 \gamma_1 \beta}^\alpha \{ \eta \} \eta^{\gamma_2} \eta^{\gamma_1} \eta^\beta + \dots$$

where

$$(4.3) \quad \Gamma_{\gamma_p \dots \gamma_1 \beta}^\alpha \stackrel{\text{def}}{=} \partial_{(\gamma_p} \Gamma_{\gamma_{p-1} \dots \gamma_1 \beta)}^\alpha - p \Gamma_{(\gamma_p \gamma_{p-1} \dots \gamma_1 \beta) \delta}^\delta \Gamma_{\gamma_p \dots \gamma_1 \beta}^\alpha$$

$$(4.4) \quad \begin{cases} \alpha) & A_{\gamma\beta}^\alpha \stackrel{\text{def}}{=} \Gamma_{\gamma\beta}^\alpha \\ \beta) & A_{\gamma_2 \gamma_1 \beta}^\alpha \stackrel{\text{def}}{=} \Gamma_{\gamma_2 \gamma_1 \beta}^\alpha + 3 \Gamma_{(\gamma_2 \gamma_1 \beta)}^\delta \Gamma_{\gamma_2 \gamma_1}^\alpha \\ \gamma) & A_{\gamma_3 \gamma_2 \gamma_1 \beta}^\alpha \stackrel{\text{def}}{=} \Gamma_{\gamma_3 \gamma_2 \gamma_1 \beta}^\alpha + 4 \Gamma_{(\gamma_3 \gamma_2 \gamma_1 \beta)}^\delta \Gamma_{\gamma_3 \gamma_2 \gamma_1}^\alpha + 6 \Gamma_{(\gamma_3 \gamma_2 \gamma_1 \beta) \delta}^\alpha A_{\gamma_1 \beta}^\alpha - 3 \Gamma_{\delta \epsilon}^\alpha \Gamma_{(\gamma_3 \gamma_2 \gamma_1 \beta) \epsilon}^\delta \end{cases}$$

Instead of (III 7.23, 24, 25) we have here

$$(4.5) \quad \Gamma_{c_p \dots c_1 b}^\alpha \{ \eta \} = 0$$

$$(4.6) \quad \partial_{(c_{p+1}} \Gamma_{c_p \dots c_1 b)}^\alpha \{ \eta \} = 0$$

$$(4.7) \quad \partial_{(c_p} \dots \partial_{c_1} \Gamma_{c b)}^\alpha \{ \eta \} = 0.$$

Now  $A_b^\alpha$  and  $A_\beta^\alpha$  could be derived from (4.1, 2). But in group space we already have  $A_b^\alpha$  and  $A_\beta^\alpha$  and since it is very easy to find expressions for  $A_b^\alpha$  and  $A_\beta^\alpha$  the indirect way is better. We consider the geodesics through  $\eta_\theta^\alpha$ . At every point of such a geodesic we have on one hand

$$(4.8) \quad (d\eta)^a = e^a d t = \delta_a^\alpha d\eta^\alpha$$

and on the other hand

$$(4.9) \quad (d\eta)^a = A_a^\alpha d\eta^\alpha,$$

and (a) and (a) have the same basis vectors at  $\eta_\theta^\alpha$ . Hence, because  $d\eta^a \propto \eta^a$  we have

$$(4.10) \quad \eta^a A_a^\alpha = \eta^\alpha \delta_a^\alpha$$

though  $A_a^\alpha = \delta_a^\alpha$  at  $\eta_\theta^\alpha$  only. By differentiation of (4.10) we get

$$(4.11) \quad \eta^b \partial_c A_b^\alpha + A_c^\alpha = \delta_c^\alpha.$$

The identities (4.10) are true for every choice of the coordinate system ( $\alpha$ ), hence

$$(4.12) \quad \partial_{[c} A_{b]}^\alpha = -\frac{1}{2} c_{[c}^\epsilon A_{b]}^{c\alpha}$$

and if this is substituted in (4.11) we get

$$(4.13) \quad -\eta^b c_{cb}^a A_{cb}^{cb} + \eta^b \partial_b A_c^a + A_c^a = \delta_c^a.$$

For differentiation along the geodesic we have

$$(4.14) \quad \frac{d}{dt} = \frac{d\eta^b}{dt} \partial_b = \frac{d\eta^b}{dt} \partial_b = \delta_b^b e^b \partial_b = \frac{1}{t} \eta^b \partial_b;$$

hence, according to (2.36, 40) and (4.13, 14)

$$(4.15) \quad \frac{d}{dt} A_c^a + \frac{1}{t} (A_c^a - \delta_c^a) = c_{cb}^a A_c^c e^b = -E_c^a A_c^c; \quad E_b^a \stackrel{\text{def}}{=} e^c c_{cb}^a.$$

Because  $A_c^a \left\{ \eta \right\} = \delta_c^a$  this equation can be compared with the ordinary differential equation

$$(4.16) \quad \frac{d x}{dt} + \frac{1}{t} (x - \theta) = \alpha x; \quad x \left\{ \theta \right\} = \theta;$$

whose well known solution can be written as a series

$$(4.17) \quad x = \theta \frac{e^{\alpha t} - 1}{\alpha t} = \theta \left( 1 + \frac{1}{2!} \alpha t + \frac{1}{3!} \alpha^2 t^2 + \dots \right)$$

and from this we derive the solution of (4.15) in the form

$$(4.18) \quad A_b^a = \delta_b^b \left( \frac{e^{-Et} - A}{-Et} \right)_b^a = \delta_b^b \left( A - \frac{1}{2!} Et + \frac{1}{3!} E^2 t^2 - \dots \right)_b^a.$$

The inversion of (4.17)

$$(4.19) \quad x = \theta \frac{\alpha t}{e^{\alpha t} - 1} = x (1 + \lambda_1 \alpha t + \lambda_2 \alpha^2 t^2 + \dots)$$

with

$$(4.20) \quad \lambda_q = (-I)^q \frac{B_q}{q!},$$

where  $B_q$  are the BERNOULLI numbers<sup>1)</sup>  $B_0 = 1$ ;  $B_1 = \frac{1}{2}$ ;  $B_2 = \frac{1}{4}$ ;  $B_3 = 0$ ;  $B_4 = -\frac{1}{30}$ ;  $B_5 = 0$ ;  $B_6 = \frac{1}{42}$ ;  $B_7 = 0$ ;  $B_8 = -\frac{1}{30}$ , gives the inversion of (4.18)

$$(4.21) \quad A_b^a = \delta_a^a \left( \frac{-Et}{e^{-Et} - A} \right)_b^a = \delta_a^a (A - \lambda_1 Et + \lambda_2 E^2 t^2 - \dots)_b^a.$$

In the same way we find

$$(4.22) \quad A_b^A = \delta_b^A \left( \frac{e^{Et} - A}{Et} \right)_b^A = \delta_b^A \left( A + \frac{1}{2!} Et + \frac{1}{3!} E^2 t^2 + \dots \right)_b^A;$$

$$(4.23) \quad A_B^a = \delta_a^B \left( \frac{E t}{e^{Et} - A} \right)_B^a = \delta_a^B (A + \lambda_1 Et + \lambda_2 E^2 t^2 + \dots)_B^a.$$

<sup>1)</sup> Cf. for instance Pascal Repertorium I, p. 437.

From (4.18, 23) and (4.21, 22) we find by transvection once more that

$$(4.24) \quad A_B^a = \delta_B^b \left( \frac{e^{-E} t - A}{-E t} \cdot \frac{E t}{e^{E t} - A} \right)_b^a = \delta_B^b (e^{-E t})_b^a$$

$$(4.25) \quad A_b^A = \delta_a^A (e^{E t})_b^a$$

in agreement with (2.42b).

The matrix notation can be used to compute the  $\Gamma_{cb}^a$ . From (III 7.28) we get in group space

$$(4.26) \quad \Gamma_{cb}^a = \eta \frac{\partial}{\partial} N_{\dot{c}\dot{b}}{}^a + \frac{1}{2} \eta^{\dot{b}_2} \eta^{\dot{b}_1} N_{\dot{b}_2 \dot{b}_1 \dot{c}\dot{b}}{}^a + \dots$$

According to (III 7.36) and (1.29) the first term of the right hand side takes the form

$$(4.27) \quad \frac{2}{3} \eta^{\dot{b}} R_{\dot{b}(\dot{c}\dot{b})}{}^a = \frac{1}{6} \eta^{\dot{b}} c_{\dot{b}(\dot{c}}{}^{\dot{c}} c_{\dot{b})}{}^a.$$

The normal tensor of valence 5 is zero and the normal tensor of valence 6 can be written as the transvection of two tensors  $R_{\dot{b}\dot{c}\dot{b}}{}^a$  (cf. III § 7) from which four factors  $c_{\dot{c}\dot{b}}{}^a$  result. The result is a series with an even number of factors  $c_{\dot{c}\dot{b}}{}^a$  in each term. But this result can be obtained much more easily in the following way.

Because the  $\dot{e}_{\beta}^a$  are (+)-constant we have

$$(4.28) \quad \text{a)} \quad \Gamma_{cb}^a = \overset{+}{\Gamma}_{(c b)}^a = A_a^a \partial_{(c} A_{b)}^a; \quad \text{b)} \quad c_{\dot{c}\dot{b}}{}^a = -2 \overset{+}{S}_{\dot{c}\dot{b}}{}^a = -2 A_a^a \partial_{[c} A_{b]}^a.$$

Hence, writing  $P$  for the matrix  $e^c c_{\dot{c}\dot{b}}{}^a t = \eta^c \delta_c^{\dot{c}} c_{\dot{c}\dot{b}}{}^a$  and  $C_c$  for the matrix  $c_{\dot{c}\dot{b}}{}^a$ , we have

$$(4.29) \quad \partial_c P = \delta_c^{\dot{c}} C_{\dot{c}}$$

and (4.18) can be written

$$(4.30) \quad A_b^a = \delta_b^b \left( A - \frac{1}{2!} P + \frac{1}{3!} P^2 - \frac{1}{4!} P^3 + \dots \right)_b^a.$$

From this we get, taking into account the non-commutativity of  $C$  and  $P$

$$(4.31) \quad \begin{cases} \partial_c A_b^a = \delta_{c b}^a \left( -\frac{1}{2!} C_c + \frac{1}{3!} C_c P + \frac{1}{3!} P C_c - \right. \\ \left. -\frac{1}{4!} C_c P^2 - \frac{1}{4!} P C_c P - \frac{1}{4!} P^2 C_c + \dots \right)$$

and after some calculation (note that the factor  $A_a^a$  in (4.28a) comes in matrix notation at the right hand side)

$$(4.32) \quad \begin{cases} (\partial_c A_b^a) A_a^a = \delta_{b a}^a \left( -\frac{1}{2} C_c - \frac{1}{12} C_c P + \frac{1}{6} P C_c - \frac{1}{24} C_c P^2 + \right. \\ \left. + \frac{1}{24} P C_c P - \frac{1}{24} P^2 C_c + \dots \right)_b^a \\ = \delta_{b a}^a \left( -\frac{1}{2} c_{\dot{c}\dot{b}}{}^a - \frac{1}{12} c_{\dot{c}\dot{b}}{}^d \eta^{\dot{c}} \delta_{\dot{c}}^{\dot{c}} c_{\dot{c}\dot{b}}{}^a + \frac{1}{6} \eta^{\dot{c}} \delta_{\dot{c}}^{\dot{c}} c_{\dot{c}\dot{b}}{}^d c_{\dot{c}\dot{d}}{}^a - \right. \\ \left. - \frac{1}{24} c_{\dot{c}\dot{b}}{}^d \eta^{\dot{c}_1} \delta_{\dot{c}_1}^{\dot{c}_1} c_{\dot{c}_1\dot{b}}{}^d \eta^{\dot{c}_2} \delta_{\dot{c}_2}^{\dot{c}_2} c_{\dot{c}_2\dot{d}_2}{}^a + \right. \\ \left. + \frac{1}{24} \eta^{\dot{c}_1} \delta_{\dot{c}_1}^{\dot{c}_1} c_{\dot{c}_1\dot{b}}{}^d c_{\dot{c}\dot{d}_2}{}^d \eta^{\dot{c}_2} \delta_{\dot{c}_2}^{\dot{c}_2} c_{\dot{c}_2\dot{d}_2}{}^a - \right. \\ \left. - \frac{1}{24} \eta^{\dot{c}_1} \delta_{\dot{c}_1}^{\dot{c}_1} c_{\dot{c}_1\dot{b}}{}^d \eta^{\dot{c}_2} \delta_{\dot{c}_2}^{\dot{c}_2} c_{\dot{c}_2\dot{d}_1}{}^d c_{\dot{c}\dot{d}_2}{}^a + \dots \right). \end{cases}$$

Hence, making use of the fact that (a) and (a) have the same basis vectors at  $\eta^a$  we get

$$(4.33) \quad \left\{ \begin{aligned} \overset{+}{\Gamma}_{cb}^a &= -\frac{1}{2} c_{c:b}^{::a} - \frac{1}{12} \eta^e \overset{+}{c}_{c:b}^{::b} c_{e:b}^{::a} + \frac{1}{6} \eta^e \overset{+}{c}_{e:b}^{::b} c_{c:b}^{::a} - \\ &- \frac{1}{24} \eta^{c_1} \eta^{c_2} \overset{+}{c}_{c:b}^{::b_1} c_{c_1:b_r}^{::b_2} c_{e_2:b_2}^{::a} + \frac{1}{24} \eta^{c_1} \eta^{c_2} \overset{+}{c}_{e_1:b}^{::b_1} c_{c:b_1}^{::b_2} c_{e_2:b_2}^{::a} - \\ &- \frac{1}{24} \eta^{c_1} \eta^{c_2} \overset{+}{c}_{e_1:b}^{::b_1} c_{c_2:b_1}^{::b_2} c_{c:b_2}^{::a} + \dots \end{aligned} \right.$$

and by mixing and alternating over cb

$$(4.34) \quad \Gamma_{cb}^a = \overset{+}{\Gamma}_{(cb)}^a = \frac{1}{6} \eta^e \overset{+}{c}_{e(b}^{::b} \overset{+}{c}_{c)b}^{::a} + \text{terms of an odd degree } \geq 3 \text{ in } \eta^a$$

in accordance with (4.27) and

$$(4.35) \quad \left\{ \begin{aligned} c_{c:b}^{::a} &= -2 \overset{+}{\Gamma}_{[cb]}^a = c_{c:b}^{::a} + \frac{1}{12} \eta^{c_1} \eta^{c_2} \overset{+}{c}_{c:b}^{::b_1} c_{e_1:b_1}^{::b_2} c_{e_2:b_2}^{::a} - \\ &- \frac{1}{12} \eta^{c_1} \eta^{c_2} \overset{+}{c}_{e_1:[c}^{::b_1} c_{|e_2|b]}^{::b_2} c_{b_1:b_2}^{::a} \\ &+ \text{terms of an even degree } \geq 4 \text{ in } \eta^a. \end{aligned} \right.$$

By a reflexion at the point  $\eta^a = 0$  a field with the values  $v^a$  at  $\eta^a$  is changed into a field with the value  $u^a = -v^a$  at  $-\eta^a$ . If  $v^a$  are the (a)-components of the first field at  $\eta^a$ , it follows from (4.18, 22) that the  $u^a$  at  $-\eta^a$  are not equal to  $-v^a$  but that  $u^A = -\delta_a^A v^a$ . The reflection of a (+)-parallel displacement of  $v^a$  over  $d\eta^a$  at  $\eta^a$  is a (-)-parallel displacement of  $u^a$  over  $-d\eta^a$  at  $-\eta^a$  because (+)-equipollency transforms into (-)-equipollency by a reflexion at the origin. Hence the reflexion of a (0)-parallel displacement of  $v^a$  over  $d\eta^a$  at  $\eta^a$  is a (0)-parallel displacement of  $u^a$  over  $-d\eta^a$  at  $-\eta^a$ . Accordingly, taking these (0)-displacements we get

$$(4.36) \quad dv^a = -d\eta^c \Gamma_{cb}^a \{\eta\} v^b = -du^a = +d\eta^c \bar{\Gamma}_{cb}^a \{-\eta\} v^b$$

or

$$(4.37) \quad \Gamma_{cb}^a \{\eta\} = -\bar{\Gamma}_{cb}^a \{-\eta\};$$

and this proves that only terms of an *odd* degree in  $\eta^a$  occur in the series of  $\Gamma_{cb}^a$  in accordance with (4.34).

Taking a (+)-displacement at  $\eta^a$  and its reflexion at  $-\eta^a$  we get

$$dv^a = -d\eta^c \overset{+}{\Gamma}_{cb}^a \{\eta\} v^b = -du^a = +d\eta^c \bar{\Gamma}_{cb}^a \{-\eta\} v^b$$

from which

$$c_{c:b}^{::a} \{\eta\} = -2 \overset{+}{\Gamma}_{[cb]}^a \{\eta\} = 2 \bar{\Gamma}_{[cb]}^a \{-\eta\} = c_{c:b}^{::a} \{-\eta\};$$

and this proves that only terms of an *even* degree in  $\eta^a$  occur in the series of  $c_{c:b}^{::a}$  in accordance with (4.35).

### § 5. Invariants of a transformation group in the $X_n$ of the $\xi^\alpha$ .

In the  $X_n$  of the  $\xi^\alpha$  we may have invariant functions of the  $\xi^\alpha$ , invariant subspaces or sets of subspaces, invariant fields of quantities or geometric objects, and invariant systems of partial differential equations.

A *subspace* is called *invariant* if its points are interchanged by the transformations of the group; a set of subspaces if the subspaces are interchanged. A *field* is called *invariant* if it does not change when dragged along by the transformations of the group. A *system of differential equations* is called *invariant* if by dragging along all solutions pass into solutions.

We shall deal here only with the most important cases of invariance.

1. *Invariant functions*<sup>1)</sup>. A function  $\varphi(\xi^\alpha)$  is invariant if and only if

$$(5.1) \quad X_b \varphi = \Xi_b^\mu \partial_\mu \varphi = 0; \quad b = 1, \dots, r. \text{ } ^2)$$

The infinitesimal transformations  $X_b$  are independent but they may be connected. The number of linearly independent equations among (5.1) is equal to the  $b$ -rank of  $\Xi_b^\alpha$ . If this  $b$ -rank is  $q \leq r$ , (5.1) is equivalent to a system of  $q$  linearly independent equations and because of

$$(5.2) \quad 2\Xi_{[c}^\nu \partial_{|\nu|} \Xi_{b]}^\mu \partial_\mu = (X_c X_b) = c;_b^a X_a = c;_b^a \Xi_a^\mu \partial_\mu;$$

this system is complete (cf. II § 5) and has  $n - q$  independent solutions  $\overset{1}{\varphi}, \dots, \overset{n-q}{\varphi}$  of which every other solution is a function. The equations  $\overset{1}{\varphi} = \text{const.}, \dots, \overset{n-q}{\varphi} = \text{const.}$  represent a set of  $\infty^{n-q}$   $X_q$ 's and each  $X_q$  of this set is invariant for transformations of the group.

If the finite equations of the group

$$(5.3) \quad {}' \xi^\alpha = f^\alpha(\xi^\nu, \eta^\alpha)$$

are given we can see that there must be invariant  $X_q$ 's by giving the  $\xi^\alpha$  fixed values  $\overset{0}{\xi}^\alpha$  and letting the  $\overset{0}{\eta}^\alpha$  vary. Then an  $X_q$  is described because  $\partial_\beta f^\alpha$  has the rank  $q$ . (5.3) represents a system of  $n$  equations in  $2n + r$  variables, which is regular of dimension  $n + r$  and whose rank with respect to the variables  $\eta^\alpha$  is  $q$ . Hence according to the well-known theorem of elimination<sup>3)</sup> exactly  $2n + r - n - r - q = n - q$  equations not containing the  $\eta^\alpha$  can be derived from (5.3) and from these

<sup>1)</sup> LIE-ENGEL 1888, 1, p. 211 ff.

<sup>2)</sup> The invariance for finite transformations follows from the series (2.31) and also from the series (2.32) because (5.1) can also be written in the form  $\overset{b}{\xi} \varphi = 0$ .

<sup>3)</sup> P. P. 1949, 1, p. 42.

equations  $n-q$  of the ' $\xi^x$ ' can be solved

$$(5.4) \quad ' \xi^a = \psi^a (' \xi^x, \xi^x); \quad a = 1, \dots, n-q; \quad x = n-q+1, \dots, n.$$

Now we know that there exist  $n-q$  invariant functions  $\varphi^a$ ;  $a = 1, \dots, n-q$ ; and this proves that the set of  $n-q$  equations

$$(5.5) \quad \varphi^a (\xi^x) = \varphi^a (' \xi^x)$$

is equivalent to the set (5.4). Hence, if we give the ' $\xi^x$ ' in (5.4) constant values  $c^1, \dots, c^n$  the equations

$$(5.6) \quad \psi(c^x, \xi^x) = \text{const.}$$

represent the  $\infty^{n-q}$  invariant  $X_q$ 's and the left hand side gives a set of  $n-q$  independent invariant functions for each choice of the  $q$  constants  $c^x$ .

2. *Invariant subspaces*<sup>1)</sup>. A set of independent functions  $\varphi^a (\xi^x)$  is called *relative invariant*<sup>2)</sup> if the equations

$$(5.7) \quad X_b \varphi^a = \Xi_b^\mu \partial_\mu \varphi^a = 0; \quad a = 1, \dots, n-q;$$

are a consequence of  $\varphi^a = 0$ . This means that the  $X_q$  represented by  $\varphi^a = 0$  is invariant. For every choice of the constants  $e^b$  the  $X_q$  is built up by streamlines of the field  $e^b \Xi_b^x$ . Therefore the dimension  $q$  is minimal if and only if  $q$  is the rank of  $\Xi_b^x$ . Such an  $X_q$  through a point  $\xi^x_0$  is called the *smallest invariant subspace* through  $\xi^x_0$ . Hence, in order to find all invariant subspaces of a group we have only to construct all points where  $\Xi_b^x$  has a rank  $q$  for all values  $q \leq r$ .<sup>3)</sup>

The group interchanges the points of  $X_q$  and we may require the group of these transformations. Let

$$(5.8) \quad \xi^x = \Phi^x(y^h); \quad h = 1, \dots, q;$$

be the parametric equations of the  $X_q$  and let  $Y_b \stackrel{\text{def}}{=} Y_b^j \partial_j$  be the infinitesimal point transformations in  $X_q$  corresponding to  $X_b$ . Then, if  $f(\xi^x) = f(\Phi^x(y^h))$  is given at all points of  $X_q$  we have for the subgroup generated by  $e^b X_b$

$$(5.9) \quad \frac{df}{dt} = e^b \Xi_b^\mu \partial_\mu f$$

<sup>1)</sup> LIE-ENGEL 1888, 1, p. 222 ff.

<sup>2)</sup> EISENHART 1933, 1, p. 63.

<sup>3)</sup> LIE-ENGEL 1888, 1, p. 228, 237.

and for the subgroup generated by  $e^b Y_b$

$$(5.10) \quad \frac{df}{dt} = e^b Y_b^j B_j^\mu \partial_\mu f; \quad B_j^\kappa \stackrel{\text{def}}{=} \partial_j \xi^\kappa;$$

hence

$$(5.11) \quad E_b^\kappa = B_b^\kappa Y_b^h.$$

In general such an equation can not exist because the  $\kappa$ -domain of  $E_b^\kappa$  does not coincide with the  $\kappa$ -domain of  $B_b^\kappa$ . But here the  $X_q$  is invariant and this means that its tangent  $E_q$  is the support of both domains. This implies that the  $Y_b^h$  can be solved from (5.11). The easiest way to find this solution is to choose the coordinate system such that  $\xi^{q+1} = 0, \dots, \xi^n = 0$  are the equations of the  $X_q$  and to take  $\xi^1, \dots, \xi^q$  (suitably chosen) as parameters on the  $X_q$ . Then  $h = 1, \dots, q$  and

$$(5.12) \quad B_b^\kappa = \delta_b^\kappa; \quad Y_b^h = E_b^h; \quad E_b^{q+1} = 0; \dots; \quad E_b^n = 0.$$

Hence we get the  $Y_b^h \partial_h$  by dropping in  $E_b^\mu \partial_\mu$  all differentiations with respect to  $\xi^{q+1}, \dots, \xi^n$ . From this it follows immediately that the group of point transformations in  $X_q$  found in this way, and called the group *induced* in  $X_q$ , has  $r$  infinitesimal transformations  $Y_b$  satisfying the equations

$$(5.13) \quad (Y_c Y_b) = c_{cb}^a Y_a.$$

These  $r$  transformations *need not be independent*. Take for instance the case where the  $X_b$  are independent but connected. Then there exists at least one equation of the form  $\psi^b(\xi^\kappa) X_b = 0$ . It may happen that the  $\psi^b(\xi^\kappa)$  are constant over  $X_q$  and in that case the  $Y_b$  are not independent. Let the induced group be  $r'$ -parametrical, or in other words, let there exist exactly  $r - r'$  relations

$$(5.14) \quad \underset{p}{c^b} Y_b = 0; \quad p = 1, \dots, r - r'; \quad \underset{p}{c^b} = \text{const.}$$

Then there must also exist exactly  $r - r'$  relations

$$(5.15) \quad \underset{p}{c^b} X_b = 0$$

valid at all points of  $X_q$ . But that implies that there exist  $r - r'$  independent infinitesimal transformations of the original group that leave every point of  $X_q$  invariant. Of course these transformations form an  $(r - r')$ -parameter subgroup:

*The group induced in an invariant  $X_q$  is an  $r'$ -parameter group if and only if there exists an  $(r - r')$ -parameter subgroup of the original group, leaving each point of  $X_q$  individually invariant.*

A group is called *transitive* if the rank of  $E_b^\kappa$  is  $n$  at all points of an  $\mathfrak{N}(\xi^\kappa)$ . If the group is transitive, to every point pair  $P$  and  $Q$  in a

sufficiently small  $\mathfrak{N}(\xi^x)$  there is at least one transformation of the group transforming  $P$  into  $Q$ . Of course for a transitive group  $r \geq n$ . The group is called *simply transitive* if  $r = n$ ; then there exists exactly one transformation transforming  $P$  into  $Q$ .

*3. Invariant contravariant vector fields.* A contravariant vector field  $Z^x$  can be looked upon as an infinitesimal transformation  $\xi^x \rightarrow \xi^x + Z^x dt$  with the symbol  $Z \stackrel{\text{def}}{=} Z^\mu \partial_\mu$ . Hence invariance of infinitesimal transformations and invariance of contravariant vector fields is essentially the same problem. Besides the complete invariance defined in the beginning of this section, there are less restricted forms of invariance. We distinguish:

- a) invariance to within an arbitrary factor;
- b) invariance to within a constant factor;
- c) complete invariance.

a) If a field  $Z^x$  is subjected to the infinitesimal transformation  $X_b$  of a group, the increase of the field is equal to the negative LIE differential with respect to the field  $\overset{e^a}{\underset{b}{Z}} \underset{a}{Z}^x$ :

$$(5.16) \quad dZ^x = -dt \underset{b}{\mathfrak{L}} Z^x = -e^a (\underset{a}{Z}^\mu \partial_\mu Z^x dt - Z^\mu \partial_\mu \underset{a}{Z}^x dt)$$

or in another form, writing  $Z$  for the operator  $Z^\mu \partial_\mu$

$$(5.17) \quad dZ = e^a (Z X_a) dt.$$

Instead of one vector field we now take a set of  $p$  independent vector fields  $\underset{y}{Z}^x$ ;  $y = 1, \dots, p$  and consider the case where the  $\underset{b}{\mathfrak{L}} \underset{y}{Z}^x$  depend linearly on the  $\underset{y}{Z}^x$ . This is a generalization of invariance to within an arbitrary factor:

$$(5.18) \quad \underset{b}{\mathfrak{L}} \underset{y}{Z}^x = \alpha_{b,y}^x (\xi^x) \underset{x}{Z}^x$$

or in another form, introducing the LIE operators  $\underset{y}{\mathfrak{L}}$  belonging to the fields  $\underset{y}{Z}^x$

$$(5.19) \quad (\underset{b}{\mathfrak{L}} \underset{y}{\mathfrak{L}}) s = \alpha_{b,y}^x (\xi^x) \underset{x}{\mathfrak{L}} s^{-1}$$

for every scalar  $s$ . The equations (5.18, 19) express the fact that the  $E_p$ -field spanned by the  $p$  vectors  $\underset{y}{Z}^x$  is invariant. This  $E_p$ -field is  $X_p$ -forming if and only if the equations  $\underset{y}{Z}^\mu \partial_\mu f = 0$  form a complete system (cf. II § 5):

$$(5.20) \quad (\underset{y}{Z} \underset{z}{Z}) = \beta_{yz}^x (\xi^x) \underset{x}{Z}$$

<sup>1)</sup> Note that (5.19) in general does not hold if  $s$  is replaced by an arbitrary non-scalar because the  $\alpha_{b,y}^x$  are not constant.

or in another form

$$(5.21) \quad \left( \begin{smallmatrix} \xi & \xi \\ y & z \end{smallmatrix} \right) s = \beta_{yz}^x (\xi^x) \xi_x s$$

and the equations of the  $X_p$ 's are obtained by equating the solutions of  $Z^\mu \partial_\mu f = 0$  to constants. If  $f$  is such a solution, then we get from (5.19)

$$(5.22) \quad \left( \begin{smallmatrix} \xi & \xi \\ y & b \end{smallmatrix} \right) f = \left( \begin{smallmatrix} \xi & \xi \\ b & y \end{smallmatrix} \right) f - \alpha_{by}^x (\xi^x) \xi_x f$$

and this proves that  $\xi_b f$  is also a solution. Every solution of the complete system is a function of  $n-p$  independent solutions, hence (cf. II 10.38) the group transforms the set of all solutions into itself and also the set of all  $\infty^{n-p}$   $X_p$ 's (but not every  $X_p$  individually!).

If a group has an invariant complete system (that is an invariant set of  $\infty^{n-p}$   $X_p$ 's forming a normal system) it is called *imprimitive* and if not it is said to be *primitive*. If a group has an invariant  $X_p$ ,  $p < n$ , it can not be transitive, but an imprimitive group can be transitive.

b)<sup>1)</sup> If in (5.18, 19) all coefficients are constants we have a generalization of invariance to within a constant factor:

$$(5.23) \quad \text{a)} \quad \left( \begin{smallmatrix} \xi & Z^\alpha \\ b & y \end{smallmatrix} \right) = k_{by}^x Z^\alpha_x \quad \text{or} \quad \text{b)} \quad \left( \begin{smallmatrix} \xi & \xi \\ b & y \end{smallmatrix} \right) = k_{by}^x \xi_x; \quad k_{by}^x = \text{const.}$$

For  $p=1$  not only the streamlines of  $Z^\alpha$  are invariant but the field  $Z^\alpha$  gets a constant factor. For general values of  $p$  the infinitesimal transformation  $s_y^y Z$  with constant coefficients  $s_y^y$  generates a one-parameter group. These transformations together form a linear set (Schar) of infinitesimal transformations. If the  $Z_y^y$  are independent (though possibly connected) it can be proved<sup>2)</sup> that this linear set always generates  $\infty^p$  finite transformations. The statement is evidently true if the  $Z_y^y$  belong to one group (though not generating this group). In fact in this case each one-parameter group is represented by a geodesic through  $\eta_0^\alpha$  and these geodesics form an  $X_p$  in  $\mathfrak{N}(\eta^\alpha)$ . If the  $Z_y^y$  are arbitrary we have

only to prove that in the transformations  $'\xi^\alpha = e^{t s_y^y Z} \xi^\alpha$  the  $t s_y^y$  are essential parameters (cf. IV § 3). Now if these parameters were not essential there would exist equations of the form

$$(5.24) \quad 0 = \dot{\varphi}(t s_y^y) \frac{\partial}{\partial t s_y^y} \dot{s}_y^y e^{t s_y^y Z} \xi^\alpha = \dot{\varphi}_x^y e^{t s_y^y Z} \xi^\alpha = \dot{\varphi}_y^y Z' \xi^\alpha$$

and this is only possible if  $\dot{\varphi} = 0$  because the  $Z_y^y$  are supposed to be independent.

<sup>1)</sup> LIE-ENGEL 1888, 1, p. 246 ff.

<sup>2)</sup> LIE-ENGEL 1888, 1, p. 62.

If the infinitesimal transformations  $\overset{y}{\overset{x}{s}} Z$  are subjected to the infinitesimal transformation  $e^b X_b$  of the group, we have

$$(5.25) \quad \frac{d}{dt} \overset{y}{\overset{x}{s}} Z = - \overset{y}{\overset{x}{s}} e^b \overset{b}{\overset{x}{L}} Z = - \overset{y}{\overset{x}{s}} e^b k_{b,y}^x Z.$$

This equation expresses that the infinitesimal transformation characterized by the constants  $\overset{y}{\overset{x}{s}}$  is transformed into the transformation characterized by the constants  $\overset{y}{\overset{x}{s}} - \overset{b}{\overset{x}{e}} k_{b,y}^x dt$ . Hence, according to (II 10.38) we have for the transformation of the  $\overset{y}{\overset{x}{s}}$  under a finite transformation of the group

$$(5.26) \quad \overset{y}{\overset{x}{s}}(t) = (e^{-tK})_x^y \overset{y}{\overset{x}{s}}(0)$$

where  $K$  stands for the matrix of  $\overset{b}{\overset{x}{k}}_{b,y}^x$ . From this we see that the linear set of infinitesimal transformations  $\overset{y}{\overset{x}{s}} Z$  is invariant for all finite transformations of the group germ. A finite transformation of the one-parameter group generated by  $\overset{y}{\overset{x}{s}}(0) Z$  has the form

$$(5.27) \quad ' \xi^x = e^{u \overset{y}{\overset{x}{s}}(0) Z} \xi^x$$

with some parameter  $u$ . Now if a finite transformation of the group is applied, the transformation (5.27) is transformed into

$$(5.28) \quad ' \xi^x = e^{u \overset{y}{\overset{x}{s}}(t) Z} \xi^x$$

and this is a finite transformation of the group generated by  $\overset{y}{\overset{x}{s}}(t) Z$ . Hence the one-parameter groups generated by  $\overset{y}{\overset{x}{s}} Z$  for all different values of  $\overset{y}{\overset{x}{s}}$  are interchanged by the transformations of the group. This proves that (5.23) is the n.a.s. condition for the invariance of the linear set  $\overset{y}{\overset{x}{s}} Z$  for all transformations of the group and also for the invariance of the set of all one-parameter subgroups generated by  $\overset{y}{\overset{x}{s}} Z$  for different values of the  $\overset{y}{\overset{x}{s}}$ .

If the  $Z$  belong to the given group and if they generate a group, (5.23) is the n.a.s. condition for this group to be an invariant subgroup. We get as a corollary that a linear set generates a group if and only if it is invariant for all its own infinitesimal transformations.

c) In the case of complete invariance there is no need to consider  $p > 1$  because each vector field is dealt with separately. The equations are

$$(5.29) \quad \text{a)} \underset{b}{\overset{b}{L}} Z = 0 \quad \text{or} \quad \text{b)} (\underset{b}{\overset{b}{L}} \underset{Z}{\overset{Z}{L}}) = 0$$

and this proves that the one-parameter group generated by  $Z$  is an invariant subgroup of the group generated by the  $X_b$  and  $Z$ .

### § 6. Invariants of a group in group space.<sup>1)</sup>

At the end of IV § 1 we have seen that the differential concomitants of group space are the ordinary concomitants of  $c_{\gamma\beta}^{\alpha}$  and that accordingly they have constant components with respect to the anholonomic coordinate systems ( $a$ ) and ( $A$ ). In IV § 2 we have seen that the infinitesimal transformation of the linear adjoint group corresponding to the transformation  $s^b X_b$  ( $s^b = \text{const.}$ ) of the given group is

$$(6.1) \quad s^b E_b = s^b E_b^a \frac{\partial}{\partial e^a} = s^b e^c c_{cb}^{ca} \frac{\partial}{\partial e^a}; \quad E_b^a \stackrel{\text{def}}{=} e^c c_{cb}^{ca}.$$

This is a linear transformation in the variables  $e^a$  and these variables may be considered as affine coordinates in the tangent  $E_r$  of  $\eta^a$ .

We have mentioned already (cf. 2.42) the transformation of the  $e_b^\alpha$  into the  $e_B^\alpha$  in a tangent  $E_r$  of a point of group space as an example of a finite transformation of the linear adjoint group. Now the invariance of a quantity for a linear homogeneous vector transformation is the same as the invariance of its components for the corresponding coordinate transformation. Moreover we know from (2.42) that the components  $c_{cb}^{ca}$  are invariant for every coordinate transformation belonging to the linear adjoint group. Hence we know already that the quantity  $c_{cb}^{ca}$  is invariant for vector transformations belonging to this group. This can be checked in the following way. The transformation of a contra- and a covariant vector by (6.1) is

$$(6.2) \quad \begin{cases} \text{a)} & v^a \rightarrow (A_b^a - S_b^a dt) v^b \\ \text{b)} & w_b \rightarrow (A_b^a + S_b^a dt) w_a \end{cases} \quad S_b^a \stackrel{\text{def}}{=} s^c c_{cb}^{ca};$$

hence (cf. 1.28)

$$(6.3) \quad \begin{cases} c_{cb}^{ca} \rightarrow c_{cb}^{ca} + (S_c^d c_{db}^{ca} + S_b^d c_{cd}^{ca} - S_d^a c_{cb}^{ad}) dt \\ \quad = c_{cb}^{ca} + s^e (c_{ec}^d c_{db}^{ca} + c_{eb}^d c_{cd}^{ca} - c_{ed}^a c_{cb}^{ad}) dt = c_{cb}^{ca} \end{cases}$$

or in other words:

*The concomitants of  $c_{cb}^{ca}$  are invariant for the transformations of the linear adjoint group.*

There is a set of concomitants, that is of great importance for the theory:

$$(6.4) \quad \begin{cases} \text{a)} & g_b \stackrel{\text{def}}{=} c_{ab}^{ca} \\ \text{b)} & g_{ba} \stackrel{\text{def}}{=} c_{cb}^{cd} c_{da}^{ac} \\ \text{c)} & g_{cba} \stackrel{\text{def}}{=} c_{ac}^{ce} c_{eb}^{af} c_{fa}^{bc} \\ & \vdots \end{cases}$$

<sup>1)</sup> LIE-ENGEL 1888, 1, p. 270 ff.

They are all invariant for *cyclic* permutations of the free indices. This implies that  $g_{ba}$  is a symmetric tensor and that  $g_{cba}$  is the sum of a symmetric tensor and a trivector.

If a concomitant of  $c_{cb}^a$  is transvected with one or more factors  $e^a$  and if this expression is equated to zero we get an equation that is invariant for the linear adjoint group. This was proved by CARTAN. We prove this here for the transvection  $P_{..cd}^{ab} e^c e^d$  where  $P_{..cd}^{ab}$  is a concomitant. According to (6.2) we have

$$(6.5) \quad \frac{d}{dt} (P_{..cd}^{ab} e^c e^d) = - S_e^a P_{..cd}^{eb} e^c e^d - S_e^b P_{..cd}^{ae} e^c e^d.$$

Hence, because  $\frac{d}{dt} P_{..cd}^{ab} = 0$  we have

$$(6.6) \quad 0 = - P_{..cd}^{ab} S_e^c e^e e^d - P_{..cd}^{ab} S_e^d e^c e^e$$

and this shows that if  $e^a$  is a solution of the equation  $P_{..cd}^{ab} e^c e^d = 0$ , then  $e^a - S_b^a e^b dt$  is also a solution. This proves that the vectors  $e^a$  satisfying this equation are *interchanged* by the transformations of the adjoint group (6.2) and that consequently an equation of this kind represents a subspace in the tangent  $E_r$  of  $\eta^\alpha$  that consists of straight lines through  $\eta^\alpha$  and is invariant for transformations of this group.

To every point  $e^a$  in the tangent  $E_r$  of  $\eta^\alpha$  there corresponds the point of group space with the normal coordinates  $\eta^a = \delta_a^a e^a$ . Because the  $e^a$  and the  $\eta^a$  transform in the same way, to an  $X_p$  in  $E_r$  consisting of straight lines through  $\eta^\alpha$  there corresponds an  $X_p$  in group space consisting of geodesics through  $\eta^\alpha$ . Hence equations of the form considered represent in group space subspaces that consist of geodesics through  $\eta^\alpha$  and that are invariant for the adjoint group.

Now consider an  $E_p$  through the origin in  $E_r$ . To this  $E_p$  corresponds an  $X_p$  in group space consisting of  $\infty^{p-1}$  geodesics through  $\eta^\alpha$ , each of them representing a one-parameter subgroup. In normal coordinates this  $X_p$  is represented by  $r-p$  homogeneous linear equations. Let the normal coordinates be chosen such that these equations are

$$(6.7) \quad \eta^x = 0; \quad \xi, \eta, \zeta = \bar{p+1}, \dots, \bar{r}$$

so that the  $\eta^m$ ;  $m, n, p = \bar{1}, \dots, \bar{p}$  can be used as coordinates in the  $X_p$ . An  $X_p$  is said to be *geodesic* in  $X_r$  if and only if every contravariant vector in  $X_p$  remains in  $X_p$  if it is displaced parallel in a direction lying in  $X_p$ . Let  $v^a$  be the vector and  $d\eta^a$  the displacement,  $a = \bar{1}, \dots, \bar{r}$ . Then we have  $v^x = 0$ ;  $d\eta^x = 0$ ;  $\xi = \bar{p+1}, \dots, \bar{r}$  and

$$(6.8) \quad dv^a = - \Gamma_{cb}^a v^b d\eta^c = - \Gamma_{pn}^a v^n d\eta^p$$

and  $dv^x$  must be zero. Hence

$$(6.9) \quad \Gamma_{\mathfrak{p}n}^x = 0; \quad \mathfrak{p}, \mathfrak{n} = \bar{I}, \dots, \bar{p}; \quad \mathfrak{x} = \overline{\bar{p} + \bar{I}}, \dots, \bar{r};$$

at every point of  $X_p$  and this condition is n.a.s. Now making use of the expansion (4.34) we see that

$$(6.10) \quad \dot{c}_{m(\mathfrak{p})}^{\mathfrak{b}} \dot{c}_{n(\mathfrak{b})}^{\mathfrak{x}} = 0$$

is a *necessary* condition. But (6.10) expresses the fact that  $\dot{c}_{m(\mathfrak{p})}^{\mathfrak{b}} \dot{c}_{n(\mathfrak{b})}^{\mathfrak{x}}$  is alternating in  $\mathfrak{p}, \mathfrak{n}$  and because the same expression is also alternating in  $\mathfrak{m}, \mathfrak{p}$ , (6.10) expresses the fact that it is alternating in  $\mathfrak{m}, \mathfrak{p}, \mathfrak{n}$ . According to (1.28) this is only possible if it is zero. Hence, (6.10) is equivalent to

$$(6.11) \quad \dot{c}_{m(\mathfrak{p})}^{\mathfrak{b}} \dot{c}_{n(\mathfrak{b})}^{\mathfrak{x}} = 0.$$

Now we will prove that (6.11) is not only necessary but also sufficient. Every term of the expression (4.34) [cf. (4.32)] contains one factor  $C_c$  and an odd number of factors  $P$ , for instance  $P^2 C_c P^3$ . If this term is written out we get for its contribution to  $\Gamma_{\mathfrak{p}n}^x$ :

$$(6.12) \quad \left\{ \begin{array}{l} \eta^{q_1} \dot{c}_{q_1(\mathfrak{n})}^{\mathfrak{a}_1} \eta^{q_2} \dot{c}_{q_2(\mathfrak{a}_1)}^{\mathfrak{a}_2} \dot{c}_{\mathfrak{a}_2(\mathfrak{p})}^{\mathfrak{a}_3} \eta^{q_3} \dot{c}_{q_3(\mathfrak{a}_3)}^{\mathfrak{a}_4} \eta^{q_4} \dot{c}_{q_4(\mathfrak{a}_4)}^{\mathfrak{a}_5} \eta^{q_5} \dot{c}_{q_5(\mathfrak{a}_5)}^{\mathfrak{x}}; \\ \mathfrak{a}_1, \dots, \mathfrak{a}_5 = \bar{I}, \dots, \bar{r}; \\ \mathfrak{n}, \mathfrak{p}, q_1, \dots, q_5 = \bar{I}, \dots, \bar{p}; \\ \mathfrak{x} = \overline{\bar{p} + \bar{I}}, \dots, \bar{r}. \end{array} \right.$$

But in this expression  $\mathfrak{p}_2$  can be written instead of  $\mathfrak{a}_2$  according to (6.11). Then by the same argument  $\mathfrak{p}_4$  can be written instead of  $\mathfrak{a}_4$  and if this is done we see that the whole expression is zero because of the index  $\mathfrak{x}$  in the last factor. The reasoning is independent of the number of factors  $P$  (if odd) and of the place of the factor  $C_c$  in the product. Hence (6.10) has as a consequence that not only the first term in the expansion of  $\Gamma_{\mathfrak{p}n}^x$  vanishes but also all other terms.

The  $E_p$  at  $\eta^\alpha$  corresponding to the  $X_p$  (6.7) is spanned by the vectors  $e^\alpha$  at  $\eta^\alpha$ . Taking the basis vectors of the system  $(a)$  at  $\eta^\alpha$  such that they coincide with the basis vectors of  $(\mathfrak{a})$ , the one-parameter subgroups (geodesics) constituting the  $X_p$  are generated by the linear set of infinitesimal transformations  $e^m \partial_m$ ;  $m = 1, \dots, p$ . Writing out the expression  $(\partial_n (\partial_m \partial_p))$ ;  $m, n, p = 1, \dots, p$ ; we get

$$(6.13) \quad (\partial_n (\partial_m \partial_p)) = c_{m(\mathfrak{p})}^{\mathfrak{b}} c_{n(\mathfrak{b})}^{\mathfrak{a}} \partial_a = \delta_{m(\mathfrak{p}) n(\mathfrak{a})}^{\mathfrak{m}(\mathfrak{p}) n(\mathfrak{a})} \dot{c}_{m(\mathfrak{p})}^{\mathfrak{b}} \dot{c}_{n(\mathfrak{b})}^{\mathfrak{a}} \partial_a.$$

Hence (6.11) is the n.a.s. condition that  $(\partial_n (\partial_m \partial_p))$  depends only on  $\partial_1, \dots, \partial_{p-1}$ <sup>1)</sup>

<sup>1)</sup> CARTAN 1927, 1, p. 72ff.

It may happen that  $(\partial_n \partial_m)$  already depends on  $\partial_1, \dots, \partial_p$  only. The n.a.s. condition is that  $c_{nm}^{xx} = 0$ ;  $m, n = 1, \dots, p$ ;  $x = p+1, \dots, r$  or in another form that  $c_{nm}^{xx} = 0$ . That means that the  $X_p$  represents a subgroup with the structural constants  $c_{nm}^{xp}$ ;  $m, n, p = 1, \dots, p$ . Of course the  $X_p$  remains geodesic, but in this special case it has another remarkable geometric property. If the vector  $v^a, v^x = 0$  is displaced (+)- or (-)-parallel along the  $X_p$  we have

$$(6.14) \quad dv^a = -I_{cb}^a v^b d\eta^c \pm \frac{1}{2} c_{cb}^{aa} v^b d\eta^c = -I_{pn}^a v^n d\eta^p \pm \frac{1}{2} c_{pn}^{aa} v^n d\eta^p;$$

hence, according to (6.9)

$$(6.15) \quad dv^x = \pm \frac{1}{2} c_{pn}^{xx} v^n d\eta^p.$$

Now we use the expansion (4.35) [cf. (4.32)] and remark that all terms contain a factor  $C_c$  and an even number of factors  $P$ , for instance  $PC_cP$ . If this term is written out, its contribution to  $c_{pn}^{xx}$  is

$$(6.16) \quad \left\{ \begin{array}{ll} \eta^{q_1} \underset{\theta}{c}_{q_1} [\underset{\theta}{n}^{a_1} \underset{\theta}{c}_{p_1}^{a_1}] \underset{\alpha_1}{\eta}^{q_2} \underset{\theta}{c}_{q_2} \underset{\alpha_2}{\dot{c}}^x; & a_1, a_2 = \bar{I}, \dots, \bar{r}; \\ n, p, q_1, q_2 = \bar{I}, \dots, \bar{p}; & x = \overline{p + I}, \dots, \bar{r}. \end{array} \right.$$

In this expression  $a_1$  can be replaced by  $p_1$  according to  $c_{nm}^{xx} = 0$  and by the same argument  $a_2$  can be replaced by  $p_2$  and  $a_3$  by  $p_3$ . But then the whole expression is zero because of the index  $x$  in the last factor. The reasoning is independent of the number of factors  $P$  (if even) and of the place of the factor  $C_c$ . Hence *the  $X_p$  represents a subgroup if and only if it is not only geodesic, that is (0)-parallel in itself, but also (+)-parallel and (-)-parallel in itself*.

If the vectors  $e^\alpha$  spanning the  $E_p$  at  $\eta^\alpha$  are subjected to the transformations  $s^b E_b$  of the linear adjoint group corresponding to the transformation  $s^b X_b$  of the group, we get

$$(6.17) \quad \underset{m}{d} e^a = -S_b^a \underset{m}{e^b} dt = -s^c c_{cb}^{aa} \underset{m}{e^b} dt.$$

Hence  $c_{nm}^{xx} = 0$  or  $c_{nm}^{xx} = 0$  is the n.a.s. condition that the  $E_p$  is invariant for those transformations of the linear adjoint group that correspond to the transformation of the subgroup represented by  $E_p$ .

It may happen that not only  $(\partial_n \partial_m)$  but also  $(\partial_b \partial_m)$ ;  $b = 1, \dots, r$ , depends on  $\partial_1, \dots, \partial_p$  only. That means that the  $E_p$  is invariant for all transformations of the linear adjoint group. But we have already seen in IV § 5 that this means that  $\partial_1, \dots, \partial_p$  generate an invariant subgroup. In this case the  $X_p$  in  $X_r$  has yet another remarkable property. In  $X$ , we form a (+)-constant field  $v^a, v^x = 0$ . This is always possible

because the  $X_p$  is (+)-parallel in itself. Then  $\overset{+}{\delta}v^a = 0$  and

$$(6.18) \quad \bar{\delta}v^a = \overset{+}{\delta}v^a - (\overset{+}{T}_{cb}^a - \bar{T}_{cb}^a)v^b d\eta^c = c_{cb}^{;a} v^b d\eta^c,$$

hence

$$(6.19) \quad \begin{cases} \bar{\delta}v^m = c_{cn}^{;m} v^n d\eta^c \\ \bar{\delta}v^x = c_{cn}^{;x} v^n d\eta^c. \end{cases}$$

Now in the same way as above it can be proved that  $c_{cn}^{;x} = 0$  implies that also  $c_{cn}^{;x} = 0$ . Hence  $\bar{\delta}v^x = 0$  and this proves that an  $X_p$  through  $\eta^x$  represents an invariant subgroup if and only if it is geodesic and (+)-parallel and (−)-parallel in itself and if moreover every  $X_p$  that is (+)-parallel to it is also (−)-parallel to it.

This property is clear from the point of view of group theory. A (+)-parallel displacement drags along a (−)-constant field (cf. IV § 1) and this is a transformation of the second parameter-group. Hence the  $X_p$ 's (+)- or (−)-parallel to the one through  $\eta^x$  represent the left hand and right hand co-sets (Nebenklassen) of the invariant subgroup.

Every set of equations linear homogeneous in the  $e^a$ , formed by means of concomitants of  $c_{cb}^{;a}$  represents a flat manifold in  $E$ , that is invariant for all transformations of the linear adjoint group. Hence this manifold and the corresponding manifold in  $X$ , represent an invariant subgroup. The following invariant subgroups are most important (cf. 6.4):

a) *the group of  $g_b$*

$$(6.20) \quad g_b e^b = 0.$$

If  $g_b \neq 0$  this group is an  $(r - 1)$ -parameter group.

b) *The centre*

$$(6.21) \quad e^c c_{cb}^{;a} = 0.$$

This is the group of those transformations of the group that commute with all transformations of the group (cf. IV § 2). The centre is always abelian and it vanishes if the  $c$ -rank of  $c_{cb}^{;a}$  is equal to  $r$ , that is if the linear adjoint group is an  $r$ -parameter group (cf. IV § 2).

c) *The first derived group*, represented by the support of the  $a$ -domain of  $c_{cb}^{;a}$ . If  $q$  is the  $a$ -rank of  $c_{cb}^{;a}$  the equation is

$$(6.22) \quad c_{c_1 b_1}^{;} e^{[a_1} \dots c_{c_q b_q}^{;} e^{a_q]} = 0.$$

Its infinitesimal transformations are those transformations of the group that can be written as LIE brackets. This group is contained in the

group of  $g_b$ . In fact

$$(6.23) \quad c_c^{\cdot b}{}^a g_a = c_c^{\cdot b}{}^a c_{d a}^{\cdot d} = -c_{b d}^{\cdot a} c_{c a}^{\cdot d} - c_{d c}^{\cdot a} c_{b a}^{\cdot d} = 0.$$

It may happen that a group is identical with its first derived group. This is the case if and only if the  $a$ -rank of  $c_{cb}^{\cdot d}$  is equal to  $r$ . *The first derived group of a subgroup is a subgroup of the first derived group of the original group*, as follows from the definition. Let  $X_1, \dots, X_m$  be independent infinitesimal transformations of an  $m$ -parameter invariant subgroup. Then  $(X_b X_p)$ ;  $p = 1, \dots, m$  belongs to this subgroup and  $(X_p X_q)$ ;  $p, q = 1, \dots, m$  to its first derived group. Now in the identity

$$(6.24) \quad ((X_p X_q) X_b) + ((X_q X_b) X_p) + ((X_b X_p) X_q) = 0$$

the last two terms belong to the first derived group of the subgroup and this proves that *the first derived group of an invariant subgroup is an invariant subgroup of the group*.

The first derived group of the first derived group is called the *second derived group*, etc. All derived groups are *invariant subgroups* of the group under consideration.

d) *The group of  $g_{ba}$*

$$(6.25) \quad g_{ba} e^a = 0$$

is an  $(r-q)$ -parameter group if  $q$  is the rank of  $g_{ba}$  and vanishes if  $g_{ba}$  has rank  $r$ .

e) The group of  $c_{cb}^{\cdot d} g_{da}$  viz.  $c_{cb}^{\cdot d} g_{da} e^a = 0$  obviously contains the group of  $g_{ba}$ . Notice that  $c_{cb}^{\cdot d} g_{da}$  is always a trivector. In fact

$$(6.26) \quad \left\{ \begin{array}{l} c_{cb}^{\cdot d} g_{da} = c_{cb}^{\cdot d} c_{ed}^{\cdot f} c_{fa}^{\cdot e} = -c_{be}^{\cdot d} c_{cd}^{\cdot f} c_{fa}^{\cdot e} - c_{ec}^{\cdot d} c_{bd}^{\cdot f} c_{fa}^{\cdot e} \\ \quad = -2g_{[bc]a} = -2g_{[ab]c} \end{array} \right.$$

because  $g_{abc}$  is the sum of a trivector and a symmetric tensor.

## § 7. Properties of integrable groups.

If we form the successive derived groups of a given group it may happen that the process stops because the derived group of the  $i$ -th derived group is identical with this latter group. If this is not the case the last derived group is zero and the last but one is abelian. The group is then said to be *integrable*. Every abelian group is integrable and the same holds for every 2-parameter group.

We prove the well known theorem:

*An  $r$ -parameter group  $\mathfrak{G}$ , is integrable if and only if it is possible to find a sequence of groups  $\mathfrak{G}_1, \dots, \mathfrak{G}_{r-1}, \mathfrak{G}_r$ , such that  $\mathfrak{G}_p$  is a  $p$ -parameter group and an invariant subgroup of  $\mathfrak{G}_{p+1}$ ;  $p = 1, \dots, r-1$ .*

*Proof.* Let  $\mathfrak{G}_r$  be integrable and let its  $q$ -th derived group  $\mathfrak{G}^{(q)}$  be an  $s$ -parameter group and its  $(q-1)$ -th derived group  $\mathfrak{G}^{(q-1)}$  be a  $t$ -parameter group,  $t > s$ . Let  $X_1, \dots, X_s$  be independent infinitesimal transformations of  $\mathfrak{G}^{(q)}$  and  $X_1, \dots, X_t$  independent infinitesimal transformations of  $\mathfrak{G}^{(q-1)}$ . Then  $\mathfrak{G}^{(q)}$  is the derived group of  $\mathfrak{G}^{(q-1)}$  and accordingly  $c_{cb}^{ab} = 0$  for  $c, b = 1, \dots, t$ ;  $a = s+1, \dots, t$ . Hence  $X_1, \dots, X_{s+1}$  generate an  $(s+1)$ -parameter invariant subgroup of  $\mathfrak{G}^{(q-1)}$  and this subgroup contains  $\mathfrak{G}^{(q)}$  as an *invariant subgroup*. In the same way  $X_1, \dots, X_{s+2}$  generate an  $(s+2)$ -parameter invariant subgroup of  $\mathfrak{G}^{(q-1)}$  and this subgroup contains the  $(s+1)$ -parameter subgroup as an *invariant subgroup*. Proceeding in this way for all values of  $q$  we get at last a sequence of groups  $\mathfrak{G}_1, \dots, \mathfrak{G}_r$  such that  $\mathfrak{G}_p$  is always an invariant subgroup of  $\mathfrak{G}_{p+t}$  provided that no derived groups occur between  $\mathfrak{G}_p$  and  $\mathfrak{G}_{p+t}$ .

Conversely let there be a sequence  $\mathfrak{G}_1, \dots, \mathfrak{G}_r$  such that  $\mathfrak{G}_p$  is always an invariant subgroup of  $\mathfrak{G}_{p+1}$ . If  $X_1, \dots, X_p$  are independent infinitesimal transformations of  $\mathfrak{G}_p$  and  $X_1, \dots, X_p, X_{p+1}$  such transformations of  $\mathfrak{G}_{p+1}$  we have

$$(7.1) \quad c_{cb}^{ab}{}^{p+1} = 0; \quad c_{p+1;b}^{ab}{}^{p+1} = 0; \quad c, b = 1, \dots, p$$

because  $\mathfrak{G}_p$  is an invariant subgroup of  $\mathfrak{G}_{p+1}$ . But it follows from these equations that the derived group of  $\mathfrak{G}_{p+1}$  is contained in  $\mathfrak{G}_p$ . Hence the first derived group of  $\mathfrak{G}_r$  is a subgroup of  $\mathfrak{G}_{r-1}$ , the second is a subgroup of  $\mathfrak{G}_{r-2}$ , and so on. This proves that it can never happen that one of the derived groups is its own derived group and this implies that the group is integrable.

If the sequence  $\mathfrak{G}_1, \dots, \mathfrak{G}_r$  is known the coordinate system can be chosen such that  $X_1$  belongs to  $\mathfrak{G}_1$ ,  $X_1$  and  $X_2$  to  $\mathfrak{G}_2$  and so on. Then we have the following simple equations for the integrable group

$$(7.2) \quad \begin{cases} \text{a)} & (X_1 X_2) = c_{12}^{-1} X_1 \\ \text{b)} & (X_1 X_3) = c_{13}^{-1} X_1 + c_{13}^{-2} X_2; \quad (X_2 X_3) = c_{23}^{-1} X_2 + c_{23}^{-2} X_3 \end{cases}$$

or in a general formula

$$(7.3) \quad \begin{cases} (X_p X_{p+t}) = c_{pp+t}^{-q} X_q; \quad t > 0; \\ p = 1, \dots, r-1 \\ q = 1, \dots, p+t-1 < r. \end{cases}$$

Notice that the centre, the group of  $g_{ba}$  and the group of  $c_{cb}^{ab} g_{da}$  are always integrable.<sup>1)</sup> The proof of these propositions is rather long and makes use of the theory of elementary divisors applied to the matrix  $e^c c_{cb}^{ab}$ . So we can not show this proof here though we need the integrability of the group of  $g_{ba}$  in the next section.

<sup>1)</sup> Cf. EISENHART 1933, 1, p. 173.

### § 8. Simple and semi-simple groups.

A group is called *simple* if it does not contain any invariant subgroup. It is called *semi-simple* if it does not contain any integrable invariant subgroup. Hence a simple group is also semi-simple. As the group of  $g_{ba}$  is an integrable invariant subgroup (cf. IV § 7) it follows that for a semi-simple group  $g_{ba}$  has rank  $r$ . *As a corollary we get that the group space of a semi-simple group has a riemannian geometry.* But this condition is not only necessary but also sufficient for the group to be semi-simple. We prove that the rank of  $g_{ba}$  is always  $< r$  if the group contains an integrable invariant subgroup. If such a subgroup exists, either it is abelian itself or one of its derived groups is abelian. This latter abelian group is an invariant subgroup of the group in question. Let  $X_1, \dots, X_r$  be chosen in such a way that  $X_p; p = 1, \dots, m$  generate this abelian group. Then we have

$$(8.1) \quad c_{pq}^{::a} = 0; \quad c_{pb}^{::x} = 0; \quad p, q = 1, \dots, m; \quad x = m + 1, \dots, r$$

and consequently

$$(8.2) \quad g_{pb} = c_{cp}^{::a} c_{ab}^{::c} = c_{cp}^{::q} c_{qb}^{::c} = c_{sp}^{::q} c_{qb}^{::s} = 0; \quad p, q, s = 1, \dots, m$$

and this proves that the rank of  $g_{ba}$  is  $< r$ .

## V. Imbedding and Curvature.

### § 1. The $X_1$ in $V_n$ .

In a  $V_n$  the length  $ds$  of  $d\xi^x$ , called the *linear element* of the  $V_n$ , is given by

$$(1.1) \quad ds^2 = g_{\lambda x} d\xi^\lambda d\xi^x.$$

Hence a curve in  $V_n$  need not be a  $V_1$  because its linear element may have a length zero at some point or at all points. But a real curve in an ordinary  $V_n$  is always a  $V_1$ .<sup>1)</sup>

<sup>1)</sup> In this chapter a  $V_n$  is always supposed to be *ordinary* and only *real* subspaces are considered. This means that multi-directions with a singular position with respect to the nullcone do not occur. Cf. for the more general case EISENHART 1926, 1; E I 1935, 1; E II 1938, 2. Also GUICHARD 1928, 1; LENSE 1936, 1. There is a vast literature on isotropic subspaces, that are subspaces with isotropic tangent spaces. We mention here only LENSE 1926, 1; 2; 3; 1928, 1; 1932, 1; 1936, 1; 2; 1939, 1; 2; 1940, 1; 2; SYNGE and Mc CONNELL 1928, 2; SCHRENZEL 1929, 1; PINL 1932, 1; 2; 1935, 1; 1936, 1; 1937, 1; 2; 1940, 1; 2; 3; 1947, 1; 1948, 1; 1949, 1; ANDREOLI 1935, 1; BOMPIANI 1940, 1; 1941, 1; MOISIL 1940, 1; STRUBECKER

If  $\xi^*$  is a definitely chosen point of the curve and  $\xi^*$  an arbitrary point of it, the length of the curve from  $\xi_0^*$  to  $\xi^*$  is  $s = \int ds$  and the parametric equations of the curve can be written in the form

$$(1.2) \quad \xi^* = f^*(s).$$

The vector

$$(1.3) \quad j^* = \tilde{j}^* \stackrel{\text{def}}{=} \frac{d\xi^*}{ds}$$

is a unitvector because  $ds$  is the length of  $d\xi^*$ . The sense of  $j^*$  depends on the (arbitrary) choice of  $\xi^*$ . The unitvector  $j^*$  is only given at the points of the curve but by the process of prolongation (cf. II § 4) we may make it possible for the operator  $\nabla_\mu$  to be applied to fields of the  $V_1$ . Because  $j^*$  is a unitvector, the vector  $j^\mu \nabla_\mu j^*$  is either non zero and perpendicular to  $j^*$  or zero. If it is non zero and if  $\tilde{j}^*$  is a unitvector with the same direction *and* the same sense, we have

$$(1.4) \quad j^\mu \nabla_\mu \tilde{j}^* = k \tilde{j}^*; \quad k > 0.$$

If it is zero the same equation is valid with  $k=0$ .  $k$  is the *first curvature* and  $k j^*$  is the *curvature vector*<sup>1)</sup> of the curve. The direction of  $\tilde{j}^*$  is the *first normal* of the curve and  $j^*$  and  $\tilde{j}^*$  span its *osculating R<sub>2</sub>*. The sense of  $\tilde{j}^*$  is independent of the choice of  $\xi^*$ . At a point where  $k=0$  the tangent is said to be *stationary* (cf. III § 2) and  $\tilde{j}^*$  and the first normal can not be found by an equation of the form (1.4). If  $k \neq 0$  in the neighbourhood of the point, a normal may be fixed by a limiting process from one side or from the other side of the curve. If  $k=0$  at all points, (1.4) passes into

$$(1.5) \quad j^\mu \nabla_\mu \frac{d\xi^*}{ds} = \frac{d^2\xi^*}{ds^2} + \Gamma_{\mu\lambda}^\kappa \frac{d\xi^\mu}{ds} \frac{d\xi^\lambda}{ds} = 0$$

and this is the equation of a geodesic with the affine parameter  $s$  (cf. III § 7). If  $k$  is not zero at all points, the vector  $j^\mu \nabla_\mu j^*$  is perpendicular to  $j^*$ , hence it has the direction of  $\tilde{j}^*$  or there exists exactly one direc-

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1941, 1; 1942, 1; 2; 1944, 1; THALER 1941, 1; CHERN 1942, 1; MOGHE 1943, 1; WONG 1945, 2; PETKANTSCHIN 1948, 1; WALKER 1950, 2; 3; 4; PATTERSON and WALKER 1952, 3. Cf. for spaces with a fundamental tensor of rank  $< n$  Ch. III p. 133, footnote 1.

<sup>1)</sup> RICCI 1895, 1. p. 298 used the expression "curvatura geodetica".

tion perpendicular to  $\overset{\mu}{j}_1^*$  and  $\overset{\mu}{j}_2^*$  such that it lies in the  $R_2$  spanned by  $\overset{\mu}{j}_1^*$  and this direction. Because

$$(1.6) \quad 0 = j^\mu V_\mu (\overset{\mu}{j}_1^* \overset{\mu}{j}_2^*) = k_1 + \overset{\mu}{j}_1^* j^\mu V_\mu \overset{\mu}{j}_2^*$$

there exists a unitvector  $\overset{\mu}{j}_3^*$  perpendicular to  $\overset{\mu}{j}_1^*$  and  $\overset{\mu}{j}_2^*$  such that

$$(1.7) \quad j^\mu V_\mu \overset{\mu}{j}_3^* = -k_1 \overset{\mu}{j}_1^* + k_2 \overset{\mu}{j}_2^*; \quad k_2 \geq 0.$$

$k_2$  is the *second curvature*, the direction of  $\overset{\mu}{j}_3^*$  is the *second normal* and  $\overset{\mu}{j}_1^*$ ,  $\overset{\mu}{j}_2^*$  and  $\overset{\mu}{j}_3^*$  span the *osculating*  $R_3$ . The sense of  $\overset{\mu}{j}_3^*$  depends on the choice of  $\xi^*$  on the curve. If  $k_2 = 0$  at a point, the osculating  $R_2$  is said to be *stationary* (cf. III § 2) at that point. In this case  $\overset{\mu}{j}_3^*$  and the second normal can not be found by an equation of the form (1.7). If  $k_2 = 0$  at only one point, an osculating plane may be fixed at this point by a limiting process from one side or the other. But if  $k_2 = 0$  at all points the process stops.<sup>1)</sup> If  $k_2$  is not zero at all points we can go on. Let us suppose that the process stops after the construction of the  $(m-1)$ -th normal  $\overset{\mu}{j}_m^*$ . Then we have the equations

$$(1.8) \quad j^\mu V_\mu \overset{\mu}{j}_s^* = -k_{s-1} \overset{\mu}{j}_{s-1}^* + k_s \overset{\mu}{j}_{s+1}^*; \quad s = 1, \dots, m; \quad k_0 = 0; \quad k_m = 0;$$

in which the curvatures  $k_1, \dots, k_{m-1}$  are all  $> 0$  at ordinary points.<sup>2)</sup> It follows immediately from them that

$$(1.9) \quad j^\mu V_\mu \overset{\mu}{j}_1^{[s_1} \dots \overset{\mu}{j}_{m]}^{s_m]} = 0$$

and this means that the osculating  $R_m$  is displaced parallel along the curve. For a curve in an  $R_n$  it means that the curve lies in its osculating  $R_m$ . According to (1.9) it is possible to construct along the curve  $n-m$  fields of unitvectors  $\overset{\mu}{j}_{m+1}^*, \dots, \overset{\mu}{j}_n^*$  so that these vectors are perpendicular at each point to the osculating  $R_m$  and to each other, and that  $\delta \overset{\mu}{j}_{m+1}^* = 0, \dots, \delta \overset{\mu}{j}_n^* = 0$ . Now taking  $k_{m+1} = 0, \dots, k_n = 0$ , we get the general equations, valid in all cases

$$(1.10) \quad \left\{ \begin{array}{l} \frac{\delta}{ds} \overset{\mu}{j}_s^* = j^\mu V_\mu \overset{\mu}{j}_s^* = -k_{s-1} \overset{\mu}{j}_{s-1}^* + k_s \overset{\mu}{j}_{s+1}^*; \quad s = 1, \dots, n; \\ k_0 = k_m = k_{m+1} = \dots = k_n = 0. \end{array} \right.$$

<sup>1)</sup> CARTAN 1927, 3 has investigated this case.

<sup>2)</sup> We always take the signs of all curvatures positive. But it may be done in another way. Cf. v. d. WOODE 1921, 1; BLASCHKE 1921, 1, p. 9; SYNGE 1924, 2; EISENHART 1926, 1, p. 61.

These are known as the equations of FRENET.<sup>1)</sup> In these formulae the curvatures are all  $\geq 0$  and the senses of  $j_2^*, j_4^*, \dots$  are all fixed. But the senses of  $j_1^*, j_3^*, \dots$  are only fixed if  $\xi^*$  is chosen and the senses of  $j_3^*, j_5^*, \dots$  change if the sense of  $j_1^*$  changes. From this it follows that the screw-sense of  $j_1^*, \dots, j_n^*$  is fixed by the curve and the sense of  $j_1^*$ , and when  $\frac{1}{2}n(n+1)$  is even, by the curve alone (cf. I § 1).  $k_1, \dots, k_{n-1}$  can be computed as functions of  $s$  from the FRENET formulae if the curve is given in the form (1.2). Conversely *the curve is fixed if  $k_1, \dots, k_{n-1}$  are given as functions of  $s$  and if one point of the curve and the vectors  $j_1^*, \dots, j_n^*$  at that point are given.* Accordingly the equations expressing  $k_1, \dots, k_{n-1}$  as functions of  $s$  are called the *natural equations* of the curve.<sup>2)</sup> The proof can be given as follows.<sup>3)</sup> The equations of FRENET can be considered as a system of simultaneous ordinary differential equations of the first order with the unknowns  $\xi^*, j_1^*, \dots, j_n^*$  and the independent variable  $s$ , in which  $k_1, \dots, k_n$  are known functions of  $s$  and the  $\Gamma_{\mu\lambda}^{\nu}$  known functions of the  $\xi^*$ . If these functions are continuous and have continuous derivatives, these equations have one and only one solution such that for  $s=s_0$  the  $\xi^*$  take some prescribed values  $\xi_0^*$  and the  $j^*$  take some prescribed values satisfying the conditions  $\sum_s j_s^* j_t^* = \delta_{st}$ . Then these conditions are satisfied at all points of the curve because  $\frac{d}{ds} j_s^* j_t^* = 0$  is a consequence of the FRENET formulae. As a corollary we find that for  $V_n=R_n$  a curve is determined by its natural equations to within translations, rotations and reflexions.

If in an  $R_n$  the fundamental tensor  $g_{\lambda\mu}$  is transformed by multiplication with a *constant* real factor  $1/c^2$ ;  $c > 0$ , the new unit tangent vector at the curve is  $'j_1^* = c j_1^*$ . From (1.10) it follows that  $'j_s^* = c j_s^*$  and  $'k_s^* = c k_s$ . Hence the proportions of the  $n-1$  curvatures are invariant for this transformation of the fundamental tensor. As a corollary we

<sup>1)</sup> JORDAN 1874, 1; KÜHNE 1903, 1; BLASCHKE 1920, 1; JUVET 1921, 1; 1925, 1; STRUIK 1922, 1, p. 76; SYNGE 1924, 2; DUSCHEK and MAYER 1930, 1, II, p. 62; HLAVATY 1934, 2, p. 12; DAVIES 1938, 1. GOLAB gave 1945, 1 another form, generalizing a formula of LANCRET. If instead of  $j^*$  an arbitrary vector field is taken along the curve we get a generalization of the FRENET formulae considered first by MEYER 1911, 1. Cf. STRUIK 1922, 1, p. 77; BIANCHI 1922, 1; EISENHART 1926, 1, p. 105; Mc CONNELL 1928, 4; 5; BORTOLOTTI 1930, 4; KRAUS 1934, 1; WONG 1940, 3.

<sup>2)</sup> There are many publications on curves in  $R_n$  satisfying relations between the  $n-1$  curvatures. Cf. for instance HAYDEN 1930, 1; SYPTÁK 1932, 1; 1934, 1; E II 1938, 2, p. 21; WONG 1940, 3; 1941, 1; 2.

<sup>3)</sup> HLAVATY 1934, 2, p. 16; E II 1938, 2, p. 16.

get that a curve in  $R_n$  is fixed to within translations, rotations, reflexions and similarity transformations if the proportions of its curvatures are given as functions of  $s$ .

If a hypersurface

$$(1.11) \quad F(\xi^*) = 0$$

is given, the points of intersection with the curve (1.2) are determined by the solutions of

$$(1.12) \quad \theta = \Phi(s) \stackrel{\text{def}}{=} F(f^*(s)).$$

If  $s = s_0$  is a solution of (1.12) and if moreover

$$(1.13) \quad \frac{d\Phi}{ds}\{s_0\} = 0; \dots; \frac{d^u\Phi}{ds^u}\{s_0\} = 0; \frac{d^{u+1}\Phi}{ds^{u+1}}\{s_0\} \neq 0$$

there is said to be a *contact of order u* at the point of intersection  $\xi^* = f^*(s_0)$ .

If (1.12) is satisfied identically in  $s$  the curve lies on the hypersurface. If  $u=1$ , the direction of the curve at  $\xi^*$  lies in the  $(n-1)$ -direction of the hypersurface at that point. But if  $u=2$  the osculating  $R_2$  does not necessarily lie in this  $(n-1)$ -direction and a similar remark can be made for all higher values of  $u$ .

## § 2. The $X_1$ in $W_n$ and $L_n$ .<sup>1)</sup>

Curves in a space with a general linear connexion can only be dealt with in a satisfactory manner if it is possible to define in some invariant way a parameter and a tangent vector. If no connexion is given in a space it is not possible to define an invariant parameter and this is the reason why for instance an  $X_1$  in  $X_n$  has no osculating 2-direction.

For an  $X_1$  in  $W_n$  [cf. III § 3]<sup>2)</sup> the factor in  $g_{\lambda\kappa}$  can be fixed at a point  $\xi^*$  of the  $X_1$ . Then this fixed  $g_{\lambda\kappa}$  can be displaced parallel along the curve and at each point a tangent vector  $j^*$  can be constructed having unit length with respect to this tensor field. The field  $j^*$  is then fixed to within a *constant* scalar factor by the connexion in  $W_n$  and the curve. Taking  $j^* = d\xi^*/ds$  and  $s=0$  for  $\xi^* = \xi^*_0$  we get a parameter  $s$  on the curve which is also fixed to within a *constant* scalar factor. This parameter  $s$  can now be used to establish FRENET formulae in exactly the same way as was described in the preceding section. From these formulae we see then that all vectors  $j^*_s$  and all curvatures  $k_s$  are fixed to within the same constant scalar factor and that consequently the proportions

<sup>1)</sup> Cf. HLAVATY 1935, 2, I.

<sup>2)</sup> HLAVATY 1928, 3.

of the  $\frac{k}{s}$  and also the vectors  $\frac{k}{1} \dot{j}_1$  and  $\frac{k^{-1}}{1} \dot{j}_2^*$  are invariants of the curve and the connexion.<sup>1)</sup>

The case of an  $X_1$  in  $L_n$  is much more complicated.<sup>2)</sup> Let such a curve be given by its parametric equation

$$(2.1) \quad \xi^* = f^*(t).$$

Then we may form the vectors

$$(2.2) \quad \begin{aligned} v_1^* &\stackrel{\text{def}}{=} \frac{d\xi^*}{dt}; & v_2^* &\stackrel{\text{def}}{=} \frac{\delta}{dt} \frac{d\xi^*}{dt}; & \text{etc.} \end{aligned}$$

Let  $v_{p+1}^*$  be the first vector that depends linearly on the preceding vectors. Then it follows from (2.2) that

$$(2.3) \quad \frac{\delta}{dt} v_1^{*\alpha_1} \dots v_p^{*\alpha_p} = \alpha v_1^{*\alpha_1} \dots v_p^{*\alpha_p}$$

where  $\alpha$  is some determinable function of  $t$ . This equation expresses the fact that the  $p$ -direction of  $v_1^{*\alpha_1} \dots v_p^{*\alpha_p}$  is covariant constant along the curve. This  $p$ -direction is called the *osculating p-direction* of the curve. If the  $L_n$  happens to be an  $E_n$  it follows that the curve lies in an  $E_p$ .

Now let  $s$  be another parameter on the curve

$$(2.4) \quad t = t(s); \quad t' \stackrel{\text{def}}{=} \frac{dt}{ds}; \quad t'' \stackrel{\text{def}}{=} \frac{d^2t}{ds^2}.$$

Then we have

$$(2.5) \quad \left\{ \begin{array}{l} u_1^* \stackrel{\text{def}}{=} \frac{d\xi^*}{ds} = t' v_1^* \\ u_2^* \stackrel{\text{def}}{=} \frac{\delta}{ds} \frac{d\xi^*}{ds} = (t')^2 v_2^* + t'' v_1^* \\ \vdots \\ u_p^* \stackrel{\text{def}}{=} \left(\frac{\delta}{ds}\right)^{p-1} \frac{d\xi^*}{ds} = (t')^p v_p^* + \dots \end{array} \right.$$

<sup>1)</sup> The centre of the osculating circle has the radiusvector  $\frac{k^{-1}}{1} \dot{j}_2^*$  in the tangent space.

<sup>2)</sup> Imbedding in  $L_n$  is a generalization of the imbedding in  $E_n$  dealt with in the so called "affine geometry" (Affingeometrie). There is a vast literature on this subject. We mention here only some papers on the imbedding in  $A_n$  and  $L_n$  (cf. Ch. VI § 3 for projective imbedding): WEYL 1922, 3; R. K. 1924, 1, Ch. IV (also for liter.); HLAVATY 1926, 4; 1934, 2 (also for literature); JÄRNEFELT 1928, 1 ( $L_n$ ); MAYER 1928, 1 ( $L_n$ ); 1935, 2 ( $L_n$ ); BORTOLOTTI 1929, 3; 1931, 2; DIENES 1932, 1 ( $L_n$ ); 1934, 1; GOLAB 1934, 1 ( $L_n$ ); 2 (curves in  $X_m$  in  $L_n$ ); WEISE 1935, 1; E II 1938, 2, p. 21 ff.; T. Y. THOMAS 1938, 3; MAEDA 1939, 1; WAGNER 1946, 1 (variation  $X_m$  in  $X_n$ ); COSSU 1947, 1; COTTON 1948, 1; CASTOLDI 1948, 1; RŽHNINA 1950, 1; KLINGENBERG 1951, 1; 2; 1952, 1.

hence

$$(2.6) \quad u_1^{[\kappa_1} \dots u_p^{\kappa_p]} = (t')^{\binom{p+1}{2}} v_1^{[\kappa_1} \dots v_p^{\kappa_p]}$$

and

$$(2.7) \quad \frac{\delta}{ds} u_1^{[\kappa_1} \dots u_p^{\kappa_p]} = \left( \binom{p+1}{2} (t')^{\binom{p+1}{2}-1} t'' + (t')^{\binom{p+1}{2}+1} \alpha \right) v_1^{[\kappa_1} \dots v_p^{\kappa_p]}.$$

This proves that  $u_1^{[\kappa_1} \dots u_p^{\kappa_p]}$  is covariant constant along the curve if and only if

$$(2.8) \quad \binom{p+1}{2} t'' + \alpha (t')^2 = 0$$

or

$$(2.9) \quad \binom{p+1}{2} \frac{d^2 s}{dt^2} - \alpha \frac{ds}{dt} = 0$$

from which  $s$  can be solved in the form

$$(2.10) \quad s = C_1 \varphi(t) + C_2; \quad C_1, C_2 = \text{constants.}$$

containing two arbitrary constants. A parameter satisfying (2.10) is called an *affine parameter* of the curve. It is determined to within arbitrary linear transformations with constant coefficients.<sup>1)</sup> If  $p=1$ , the curve is a geodesic and the affine parameter is the same as that defined in III § 7. If  $p=n$ , the two  $n$ -vectors in (2.6) can be replaced by their corresponding  $\Delta$ -densities  $\tilde{u}$  and  $\tilde{v}$ :

$$(2.11) \quad \tilde{u} = (t')^{\binom{p+1}{2}} \tilde{v}$$

hence

$$(2.12) \quad s - \frac{s}{\theta} = \int_{t_0}^t \left( \frac{\tilde{v}}{\tilde{u}} \right)^{\frac{2}{n(n+1)}} dt$$

where  $\tilde{v}$  is the  $\Delta$ -density belonging to  $t$  and  $\tilde{u}$  an arbitrary  $\Delta$ -density of weight  $-1$ , that is covariant constant along the curve. In general  $\tilde{u}$  is determined to within a *constant* scalar factor. But in an  $L_n$  with a volume preserving connexion,  $V_{\nu\mu} = 0$ , scalar density fields exist which are covariant constant over the whole space (cf. III § 6). Hence if in this case (f. i. in an  $E_n$ ) such a field is given, this field can be taken for  $\tilde{u}$  and then  $s$  is determined to within an additive constant  $s_0$ . The difference of the values of  $s$  for two different points is the “affine length” (Affinlänge) known from affine geometry (in  $E_n$ ).<sup>2)</sup> If the coordinate system

<sup>1)</sup> HLAVATY 1934, 2.

<sup>2)</sup> Cf. footnote 2 on p. 232.

is chosen in such a way that  $E^{x_1 \dots x_n}$  corresponds to the covariant constant field  $\tilde{u}$ , the affine length between the points  $s$  and  $\tilde{s}$  takes the form

$$(2.13) \quad s - \tilde{s} = \int_{t_0}^t (v_1^{[1} \dots v_n^{n]})^{\frac{2}{n(n+1)}} dt$$

and it is invariant for transformations of coordinates with  $A = +1$ .<sup>1)</sup>

We return now to the general  $L_n$ . Once the affine parameter is known, other invariants may be deduced. Let  $\xi^x$  be a point of the curve with  $s=0$ . Then  $s$  is determined to within a constant scalar factor. For simplicity we take  $p=n$ . Then

$$(2.14) \quad \frac{\delta}{ds} u_1^{[x_1} \dots u_n^{x_n]} = 0$$

and from this equation and (2.5) it follows that

$$(2.15) \quad \frac{\delta}{ds} u_n^{x_n} + \kappa_1 u_{n-1}^{x_n} + \dots + \kappa_{n-1} u_1^{x_n} = 0$$

where  $\kappa_1, \dots, \kappa_{n-1}$  are functions of  $s$ . If  $s$  acquires a constant scalar factor  $c$ ,  $u_j^{x_j}$  gets a factor  $c^{-j}$  and  $\kappa_j$  a factor  $c^{-(j+1)}$ . In an  $E_n$  the vectors  $u_j^{x_j}$  and the  $\kappa_j$  are invariants for the special-affine group ( $A = +1$ , cf. I § 2). The  $\kappa_j$ 's could be called special-affine curvatures of the curve. In fact it could be proved that a curve in  $E_n$  is determined to within a special-affine transformation if  $\kappa_1, \dots, \kappa_{n-1}$  are given as functions of  $s$ . But there is one serious difficulty. To each point of the curve there belongs the invariant set of  $n$  vectors  $u_1^{x_1}, \dots, u_n^{x_n}$ . Now if these vectors are expressed in terms of the derivatives of  $\xi^x$  with respect to an *arbitrary* parameter  $t$ , it follows that these vectors may contain derivatives up to the orders  $n, n+1, \dots, 2n-1$  and that in the  $\kappa_1, \dots, \kappa_{n-1}$  derivatives up to the orders  $n+2, \dots, 2n$  may occur. But it is possible to find sets of  $n$  vectors and a set of "curvatures" belonging to them, that depend only on derivatives of lower order as was proved by WINTERNITZ for  $n=3$ .<sup>2)</sup>

Of course for a curve in  $L_n$  or  $E_n$  there are no equations similar to the equations of FRENET in a  $V_n$ , because perpendicularity does not exist in an  $E_n$ .

<sup>1)</sup> Cf. for  $E_n$  BERWALD 1923, 1, p. 95; BLASCHKE 1923, 1, p. 73.

<sup>2)</sup> Cf. BLASCHKE 1923, 1, p. 76ff.; HLAVATY 1929, 1; 1934, 2, p. 21ff.; E II 1938, 2, p. 26.

**Exercise.**

V 2,1. Let the tangent vector  $t^*$  and a normal vector  $n_\lambda$  of a real curve in an ordinary  $W_n$  satisfy the conditions

$$\text{V 2,1 } \alpha) \quad t^\mu n_\mu = 1; \quad n_\lambda t^\mu \nabla_\mu t^\lambda = 0.$$

Prove that the vector  $u_\lambda = t^\mu \nabla_\mu n_\lambda$  is perpendicular to  $t^*$  and invariant if  $t^*$  and  $n_\lambda$  are multiplied with a scalar as long as (V 2,1  $\alpha$ ) remains valid.  $u_\lambda$  is the curvature vector of the curve in  $W_n$ .

**§ 3. The  $X_{n-1}$  in  $A_n$ .<sup>1)</sup>**

Let an  $X_{n-1}$  in  $A_n$  be given by its nullform

$$(3.1) \quad C^n(\xi^*) = 0; \quad C_\lambda^n \stackrel{\text{def}}{=} \partial_\lambda C^n;$$

or by its parametric form

$$(3.2) \quad \xi^* = B^*(\eta^a); \quad B_b^* \stackrel{\text{def}}{=} \partial_b \xi^*; \quad a, b = 1, \dots, n - 1$$

(cf. II 4.4, 12) and let  $\xi^*_0$  be a point of  $X_{n-1}$ . Then at each point of the  $X_{n-1}$  in an  $\mathfrak{N}(\xi^*_0)$  we have the connecting quantities  $B_b^*$  and  $C_\lambda^n$  (cf. II § 4).  $C_\lambda^n$  is a covariant vector tangent to the  $X_{n-1}$  and determined to within a scalar factor, because here the upper index takes only one value  $n$ .

Let the  $X_{n-1}$  be rigged (cf. II § 5). To this end we introduce at each point of  $X_{n-1}$  an  $E_1$  (in the tangent  $E_n$ ) not lying in the tangent  $E_{n-1}$ . Then in the way described in I § 4 and II § 5 we have at each point of the  $X_{n-1}$  in  $\mathfrak{N}(\xi^*_0)$  the connecting quantities  $C_n^*$ ,  $B_\lambda^a$ ;  $a = 1, \dots, n - 1$ .  $C_n^*$  is a contravariant vector in the direction of the rigging, determined to within a scalar factor. As follows from I § 4 and II § 5 we have in this special case, writing  $n^*$  for  $C_n^*$  and  $t_\lambda$  for  $C_\lambda^n$ ,

$$(3.3) \quad \text{a)} \quad B_b^\lambda t_\lambda = 0 \quad \text{b)} \quad B_\lambda^a n^* = 0;$$

$$(3.4) \quad \text{a)} \quad B_b^\mu B_\mu^a = B_b^a \quad \text{b)} \quad \boxed{n^\mu t_\mu = 1};$$

$$(3.5) \quad \text{a)} \quad B_b^* B_\lambda^b = B_\lambda^* \quad \text{b)} \quad n^* t_\lambda = C_\lambda^*;$$

$$(3.6) \quad A_\lambda^* = B_\lambda^* + n^* t_\lambda.$$

We call  $n^*$  the *pseudo-normal vector* and  $t_\lambda$  the *tangent vector* of  $X_{n-1}$  and (3.4b) the *first normalizing condition*. If  $t_\lambda$  acquires a factor  $\sigma$ ,  $n^*$  gets a factor  $\sigma^{-1}$  but the quantities  $B_b^*$ ,  $B_\lambda^a$ ,  $B_\lambda^*$ ,  $B_b^a$  and  $C_\lambda^*$  do not change.

<sup>1)</sup> General references on  $X_{n-1}$  in  $A_n$ : SCHOUTEN 1923, 1; R. K. 1924, 1, p. 133 ff.; NOŻYNSKA 1950, 1 and many papers dealing also with  $X_m$  in  $A_n$  and  $L_n$ .

After the rigging every tensor with valence  $\geq 1$  of  $X_{n-1}$  has components with greek indices and also components with latin indices and every tensor with valence  $\geq 1$  of  $A_n$  at a point of  $X_{n-1}$  has an  $X_{n-1}$ -part and a normal part (cf. I § 4 and II § 5). If the valence is 1, the vector is the sum of its two parts:

$$(3.7) \quad \begin{cases} \text{a)} & v^\kappa = B_\lambda^\kappa v^\lambda + C_\lambda^\kappa v^\lambda = 'v^\kappa + ''v^\kappa \\ \text{b)} & w_\lambda = B_\lambda^\kappa w_\kappa + C_\lambda^\kappa w_\kappa = 'w_\lambda + ''w_\lambda. \end{cases}$$

If  $p$  is a scalar field of  $X_{n-1}$ , the expression  $\nabla_\lambda p = \partial_\lambda p$  has no meaning if the field is not prolonged (cf. II § 4). But  $B_b^\lambda \nabla_\lambda p = B_b^\lambda \partial_\lambda p = \partial_b p$  has a meaning independent of any prolongation. It is the gradient of  $p$  as a field of  $X_{n-1}$  and  $B_\mu^b \partial_b p$  are its  $A_n$ -components.

If a vector field  $v^a$  of  $X_{n-1}$  is prolonged we may define a covariant differentiation in  $X_{n-1}$  by means of the formula

$$(3.8) \quad 'V_c v^a \stackrel{\text{def}}{=} B_{c\kappa}^{\mu a} V_\mu v^\kappa.$$

This means that  $'V_c v^a$  is the  $X_{n-1}$ -part of  $V_\mu v^\kappa$ . But  $'V_c v^a$  is independent of the manner of prolongation as will be seen if we write out the right hand side of (3.8):

$$(3.9) \quad 'V_c v^a = B_c^\mu B_\kappa^\alpha (\partial_\mu v^\kappa + \Gamma_{\mu\lambda}^\kappa v^\lambda) = \partial_c v^a + B_{cb\kappa}^{\mu\lambda a} \Gamma_{\mu\lambda}^\kappa v^b - B_b^\kappa (\partial_c B_\kappa^\alpha) v^b.$$

Moreover this equation proves that the connexion in  $A_n$  induces in the rigged  $X_{n-1}$  a connexion with the parameters

$$(3.10) \quad \begin{cases} 'T_{cb}^a = B_{cb\kappa}^{\mu\lambda a} \Gamma_{\mu\lambda}^\kappa - B_b^\kappa \partial_c B_\kappa^\alpha \\ \quad = B_{cb\kappa}^{\mu\lambda a} \Gamma_{\mu\lambda}^\kappa + B_\kappa^\alpha \partial_c B_\kappa^\alpha \\ \quad = B_{cb\kappa}^{\mu\lambda a} \Gamma_{\mu\lambda}^\kappa + B_\kappa^\alpha \partial_c \partial_b \xi^\kappa \end{cases}$$

which are independent of the normalization of  $t_\lambda$ . Of course for this connexion we have  $'V_c p = \partial_c p$ . From the third line of (3.10) we see that  $'T_{[cb]}^a = 0$ , hence:

*A rigged  $X_{n-1}$  in  $A_n$  is an  $A_{n-1}$  whose connexion is uniquely determined by the connexion of the  $A_n$  and the rigging.*

Besides  $'T_{cb}^a$  other geometric objects are also induced in the  $A_{n-1}$ . The two tensors of  $A_{n-1}$

$$(3.11) \quad \boxed{h_{cb} \stackrel{\text{def}}{=} -B_{cb}^{\mu\lambda} V_\mu t_\lambda} = -B_{cb}^{\mu\lambda} \partial_\mu \partial_\lambda C^n - B_{cb}^{\mu\lambda} \Gamma_{\mu\lambda}^\kappa t_\kappa;$$

$$(3.12) \quad \boxed{l_c^a \stackrel{\text{def}}{=} -B_{c\kappa}^{\mu a} V_\mu n^\kappa}$$

get a factor  $\sigma$  and  $\sigma^{-1}$  respectively if  $t_\lambda$  acquires a factor  $\sigma$ . It follows from (3.11) that  $h_{cb}$  is a *symmetric tensor*. The directions in the null-

cone of  $h_{cb}$  are called the *principal tangents* of the rigged or non rigged  $X_{n-1}$ , and curves of  $X_{n-1}$  whose tangents are everywhere principal are called *asymptotic lines*. The directions of the eigenvectors of  $l_c^a$  are called the *directions of the principal curvature* of the rigged  $X_{n-1}$ , and curves of  $X_{n-1}$  whose tangent lies everywhere in a principal direction are called *lines of curvature*.

In order to derive ' $\nabla_d \nabla_c v^a$ ' we have first to take the  $A_n$ -components  $B_{\mu\sigma}^{\alpha\kappa} V_\nu v^\sigma$  of ' $\nabla_c v^a$ ' and to apply a process analogous to (3.8) to this quantity of valence 2. Then we get

$$(3.13) \quad 'V_d 'V_c v^a = B_{dc\kappa}^{v\mu a} h_{v\mu} n^\kappa V_\rho v^\kappa + B_{dc\kappa}^{v\mu a} l_v^{\cdot\kappa} h_{\mu\sigma} v^\sigma + B_{dc\kappa}^{v\mu a} V_\nu V_\mu v^\kappa$$

hence

$$(3.14) \quad 'R_{dc\kappa}^{\cdot\cdot b} v^b = B_{dc\kappa}^{v\mu\lambda a} R_{v\mu\lambda}^{\cdot\cdot\kappa} v^b + 2l_{[d}^a h_{c]b} v^b$$

or

$$(3.15) \quad \boxed{'R_{dc\kappa}^{\cdot\cdot b} = B_{dc\kappa}^{v\mu\lambda a} R_{v\mu\lambda}^{\cdot\cdot\kappa} + 2l_{[d}^a h_{c]b}}.$$

This is the *generalized equation of GAUSS* for a rigged  $A_{n-1}$  in  $A_n$ .

From (3.11) and (3.12) we get by differentiation and alternation

$$(3.16) \quad \left\{ \begin{array}{l} 2'V_{[d} h_{c]b} = -2B_{[dc]b}^{v\mu\lambda} V_\nu B_{\mu\lambda}^{\rho\sigma} V_\rho t_\sigma \\ = 2B_{[dc]b}^{v\mu\lambda} (V_\nu t_\lambda n^\sigma) V_\mu t_\sigma - 2B_{[dc]b}^{v\mu\lambda} V_\nu V_\mu t_\lambda \\ = -2B_{[d}^{\mu} h_{c]b} n^\sigma V_\mu t_\sigma + B_{dc\kappa}^{v\mu\lambda} R_{v\mu\lambda}^{\cdot\cdot\kappa} t_\kappa; \end{array} \right.$$

$$(3.17) \quad \left\{ \begin{array}{l} 2'V_{[d} l_{c]}^a = -2B_{[dc]\kappa}^{v\mu a} V_\nu B_{\mu\sigma}^{\rho\kappa} V_\rho n^\sigma \\ = 2B_{[dc]\kappa}^{v\mu a} (V_\nu t_\sigma n^\kappa) V_\mu n^\sigma - 2B_{[dc]\kappa}^{v\mu a} V_\nu V_\mu n^\kappa \\ = -2B_{[d}^{\mu} l_{c]}^a t_\sigma V_\mu n^\sigma - B_{dc\kappa}^{v\mu a} R_{v\mu\lambda}^{\cdot\cdot\kappa} n^\lambda. \end{array} \right.$$

Now in (3.15) all terms are invariant if the normalization of  $t_\lambda$  is changed. But in (3.16, 17) only the last terms of each of the right hand sides acquire a factor  $\sigma$  and  $\sigma^{-1}$  respectively and the other terms change in a more complicated way because  $\sigma$  need not be a constant. This is very inconvenient and we therefore introduce as a *second condition of normalization* the equations

$$(3.18) \quad \boxed{\text{a)} \quad B_b^\mu (V_\mu t_\lambda) n^\lambda = 0; \quad \text{b)} \quad B_b^\mu (V_\mu n^\kappa) t_\kappa = 0}$$

which are consequences of each other.<sup>1)</sup> If (3.18) is satisfied,  $t_\lambda$  and  $n^\kappa$  are fixed to within a *constant* scalar factor and in (3.16, 17) the first

<sup>1)</sup> BERWALD 1922, 1, p. 164 for  $A_n = E_n$ ; SCHOUTEN 1923, 1, p. 169.

term of each right hand side vanishes:

$$(3.19) \quad 2'V_{[d} h_{c]b} = B_{dc}^{\nu\mu\lambda} R_{\nu\mu\lambda}^{\cdot\cdot\cdot} t_{\nu}$$

$$(3.20) \quad 2'V_{[d} l_{c]a} = - B_{dc}^{\nu\mu a} R_{\nu\mu\lambda}^{\cdot\cdot\cdot} n^{\lambda} .$$

These are the *generalized equations of CODAZZI* for a rigged  $A_{n-1}$  in  $A_n$ . Moreover we have now from (3.11, 12, 18).

$$(3.21) \quad 2l_{[d}^{\cdot a} h_{c]a} = 2B_{[d|\nu|c]a}^{\nu\mu\lambda} (\nabla_{\nu} n^{\lambda}) V_{\mu} t_{\lambda} = 2B_{[d|c]}^{\nu\mu} (\nabla_{\nu} n^{\lambda}) V_{\mu} t_{\lambda} = B_{dc}^{\nu\mu} R_{\nu\mu\lambda}^{\cdot\cdot\cdot} t_{\lambda} n^{\lambda}$$

and by substituting this value in (3.15) and making use of (3.6)

$$(3.22) \quad 'R_{ac}^{\cdot\cdot\cdot a} = B_{dc}^{\nu\mu} R_{\nu\mu\lambda}^{\cdot\cdot\cdot} t_{\lambda}^{\lambda} .$$

Hence, if both normalizing conditions are satisfied, and if the connexion in  $A_n$  is volume preserving, the induced connexion in  $A_{n-1}$  has the same property.<sup>1)</sup>

We now have to investigate first whether it is possible to satisfy (3.18) if the direction of  $n^{\lambda}$  is given. Let  $t_{\lambda}$  and  $n^{\lambda}$  be normalized in any way so as to satisfy (3.4b). Then, if  $B_b^{\mu} (\nabla_{\mu} t_{\lambda}) n^{\lambda} \neq 0$  we try to choose  $\sigma$  such that

$$(3.23) \quad B_b^{\mu} (\nabla_{\mu} \sigma t_{\lambda}) \sigma^{-1} n^{\lambda} = 0 .$$

This gives the following differential equation for  $\sigma$

$$(3.24) \quad 'V_b \log \sigma = B_b^{\mu} (\nabla_{\mu} t_{\lambda}) n^{\lambda}$$

and after some calculation we find for the integrability conditions of this equation

$$(3.25) \quad -\frac{1}{2} B_{dc}^{\nu\mu} R_{\nu\mu\lambda}^{\cdot\cdot\cdot} t_{\lambda} n^{\lambda} + B_{dc}^{\nu\mu} (\nabla_{[\nu} t_{|\lambda|}) V_{\mu]} n^{\lambda} = 0 . \text{<sup>2)</sup>}$$

This is equivalent to (3.22) because of (3.6) and (3.15). Hence:

If the direction of rigging be given,  $t_{\lambda}$  and  $n^{\lambda}$  can be normalized according to (3.4b) and (3.18) if and only if (3.22) holds

and as a corollary

If the connexion in  $A_n$  is volume preserving and the direction of  $n^{\lambda}$  given,  $t_{\lambda}$  and  $n^{\lambda}$  can be normalized according to (3.4b) and (3.18) if and only if the connexion in  $A_{n-1}$  is volume preserving as well.

We ask now whether it is always possible to find for a given normalization of  $t_{\lambda}$  a vector  $n^{\lambda}$  such that (3.4b) and (3.18) both hold. It

<sup>1)</sup> SCHOUTEN 1923, 1, p. 172. On p. 187 this is generalized for  $A_m$  in  $A_n$ .

<sup>2)</sup> Cf. EISENHART 1927, 1, p. 153 (55.9).

follows from (3.18) that the direction of  $n^*$  must lie in the support of the  $\lambda$ -domain of  $B_c^\mu V_\mu t_\lambda$ , hence the n.a.s. condition is that this support is not wholly contained in the tangent  $E_{n-1}$  or in other words, that  $t_\lambda$  does not lie in the  $\lambda$ -domain of  $B_c^\mu V_\mu t_\lambda$ . Now if ' $r$ ' is the rank of  $h_{cb}$ , the rank  $r$  of  $B_c^\mu V_\mu t_\lambda$  is ' $r$ ' or ' $r+1$ ' and it is ' $r+1$ ' if and only if  $t_\lambda$  lies in the  $\lambda$ -domain of  $B_b^\mu V_\mu t_\lambda$ . Hence for ' $r=n-1$ ' the vector  $t_\lambda$  can never lie in this domain, because  $r$  can never be greater than  $n-1$ . Collecting results we get:

*If the normalization of  $t_\lambda$  be given, there exists a vector  $n^*$  that satisfies (3.4b) and (3.18) if and only if  $t_\lambda$  does not lie in the  $\lambda$ -domain of  $B_c^\mu V_\mu t_\lambda$ . If the rank of  $h_{cb}$  is equal to  $n-1$ ,  $t_\lambda$  never lies in this domain and  $n^*$  is uniquely determined and has the direction of the support of this domain.<sup>1)</sup>*

If a vector  $n^*$  satisfying (3.4b) and (3.18) has been found for a given normalization of  $t_\lambda$  and if this normalization is changed,  $t_\lambda \rightarrow \sigma t_\lambda$ , the new value of the normal vector may be written in the form  $\sigma^{-1}(n^* + p^*)$ ;  $p^* t_\lambda = 0$ . Then we have

$$(3.26) \quad 0 = B_b^\mu (V_\mu \sigma t_\lambda) (n^\lambda + p^\lambda) = B_b^\mu V_\mu \sigma + \sigma B_b^\mu (V_\mu t_\lambda) p^\lambda$$

or

$$(3.27) \quad h_{ba} p^a = \partial_b \log \sigma.$$

Hence, if  $h_{ba}$  has rank  $n-1$ , the vector  $p^a$  can be solved

$$(3.28) \quad p^a = h^{ab} \partial_b \log \sigma$$

and, as was to be foreseen, the new normal vector is uniquely determined. But if the rank of  $h_{ba}$  is  $< n-1$ , it may happen that the vector  $\partial_b \log \sigma$  does not lie in the  $b$ -domain of  $h_{ba}$  and in this case there is no normal vector for the new normalization of  $t_\lambda$ .

The quantity

$$(3.29) \quad \tilde{t}_\lambda \stackrel{\text{def}}{=} \frac{1}{(n-1)!} \tilde{e}_{\lambda_1 \dots \lambda_n} B_{a_1 \dots a_n}^{\lambda_1 \dots \lambda_n} \tilde{F}^{a_1 \dots a_n}$$

is a covariant vector  $\Delta$ -density of weight  $-1$  in  $A_n$  and weight  $+1$  in  $X_{n-1}$ . It is a concomitant of the  $X_{n-1}$ , independent of any rigging and its  $(n-1)$ -direction is tangent to the  $X_{n-1}$ . Now if  $h_{cb}$  has rank  $n-1$ , the quantity

$$(3.30) \quad \mathfrak{h} \stackrel{\text{def}}{=} |\text{Det}(h_{cb})|$$

is a scalar density of weight  $+2$  in  $X_{n-1}$ . Hence, if a screwsense is given in  $X_{n-1}$  and a scalar  $\Delta$ -density  $\tilde{q}$  of weight  $+1$  in  $A_n$ , at least at all points of  $X_{n-1}$ , then densities and  $\Delta$ -densities in  $X_{n-1}$  can be

<sup>1)</sup> Cf. SCHOUTEN 1923, 1, p. 173 ff.

identified and a normalization of  $t_\lambda$  can be fixed by the equation

$$(3.31) \quad t_\lambda = \tilde{q} \, \mathfrak{h}^{-\frac{1}{2}} \tilde{t}_\lambda.$$

If the connexion in  $A_n$  happens to be volume preserving, the condition  $\nabla_\lambda \tilde{q} = 0$  determines a field  $\tilde{q}$  and therefore also the field  $t_\lambda$  to within a constant scalar factor, hence

*For an  $X_{n-1}$  in a volume preserving  $A_n$  for which  $h_{cb}$  has rank  $n-1$  at all points, special fields  $t_\lambda$  and  $n^\kappa$  satisfying both normalization conditions are determined to within a constant scalar factor and its reciprocal.*

In an  $E_n$  the equations of GAUSS and CODAZZI take the very simple form

$$(3.32) \quad 2l_{[d}^a h_{c]b} = {}'R_{dcb}^a$$

$$(3.33) \quad {}'\nabla_{[d} h_{c]b} = 0$$

$$(3.34) \quad {}'\nabla_{[d} l_{c]}^a = 0.$$

These equations are the n.a.s. integrability conditions for the construction of an  $E_n$  in which the  $A_{n-1}$  with a given volume preserving connexion  $'T_{cb}^a$  and given fields  $h_{cb}$  (of rank  $n-1$ ) and  $l_c^a$  can be imbedded. Let the coordinate system  $\xi^\kappa$  in the  $E_n$  to be constructed be rectilinear,  $\Gamma_{\mu\lambda}^\kappa = 0$ . Then the unknowns are  $\xi^\kappa$ ,  $t_\lambda$  and  $n^\kappa$  as functions of the  $\eta^a$  and the equations for these unknowns are (cf. 3.10, 11, 12)

$$(3.35) \quad \text{a)} \partial_b \xi^\kappa = B_b^\kappa \quad \text{b)} \quad t_\lambda n^\lambda = 1 \quad \text{c)} \quad (\partial_c t_\lambda) n^\lambda = 0 \quad \text{d)} \quad B_b^\lambda t_\lambda = 0$$

$$(3.36) \quad \text{a)} \quad B_b^\lambda \partial_c t_\lambda = -h_{cb} \quad \text{b)} \quad \partial_c n^\kappa = -B_a^\kappa l_c^a \quad \text{c)} \quad \partial_c B_b^\kappa = {}'T_{cb}^a B_a^\kappa + h_{cb} n^\kappa.$$

Taking into account that the given connexion  $'T_{cb}^a$  is volume preserving we get the following scheme for the integrability conditions of (3.35, 36):

$$(3.37) \quad \left\{ \begin{array}{ll} 3.35 \text{a} & 3.36 \text{c} \\ 3.35 \text{b} & 3.35 \text{c, d}; 3.36 \text{b} \\ 3.35 \text{c} & 3.36 \text{a}; 3.32 \\ 3.35 \text{d} & 3.35 \text{b}; 3.36 \text{a, c} \\ 3.36 \text{a} & 3.33; 3.35 \text{c} \\ 3.36 \text{b} & 3.34; 3.35 \text{c}; 3.36 \text{a} \\ 3.36 \text{c} & 3.32; 3.33. \end{array} \right.$$

In this scheme the integrability conditions of the equation to the left are identically satisfied as a consequence of the equation or equations to the right on the same line. Hence the conditions are all identically satisfied and this implies that there is one and only one solution of (3.35, 36) satisfying the initial conditions for  $\eta^a = \eta^a_0$ ,  $h_{cb} = h_{cb}^0$ ,  $l_c^a = l_c^a_0$ ,  $'T_{cb}^a = {}'T_{cb}^a$ :

$$(3.38) \quad \left\{ \begin{array}{ll} \text{a)} & \xi^* = \frac{\xi^*}{0} \\ \text{b)} & t_\lambda = \frac{t_\lambda}{0} \\ \text{c)} & n^* = \frac{n^*}{0} \\ \text{d)} & B_b^* = \frac{B_b^*}{0} \\ \text{e)} & B_b^\lambda \partial_c t_\lambda = - \frac{h_{cb}}{0} \\ & \\ \text{f)} & \frac{n^\lambda}{0} \partial_c t_\lambda = 0 \\ \text{g)} & \partial_c n^* = - \frac{B_a^*}{0} \frac{l_c^a}{0} \\ \text{h)} & \partial_c B_b^* = \frac{T_{cb}^a}{0} \frac{B_a^*}{0} + \frac{h_{cb}}{0} \frac{n^*}{0} \\ \text{i)} & \frac{t_\lambda}{0} \frac{n^\lambda}{0} = 1 \\ \text{j)} & \frac{B_b^\lambda}{0} \frac{t_\lambda}{0} = 0. \end{array} \right.$$

This means that the imbedding and rigging of an  $\mathfrak{N}(\eta^*)$  of an  $A_{n-1}$  with given fields  $T_{cb}^a$ ,  $h_{cb}$  (of rank  $n-1$ ) and  $l_c^a$ , satisfying (3.32, 33, 34) and  $'R_{abc}^a = 0$ , in an  $E_n$  is determined if we give 1. the point  $\xi^*$  corresponding to  $\eta^*$ ; 2. the two vectors  $t_\lambda$ ,  $n^*$  at  $\xi^*$ , satisfying (3.38i); 3.  $n-1$  linearly independent vectors  $B_b^*$  in the  $E_{n-1}$  of  $t_\lambda$ . If instead we give in an arbitrary way the point  $*\xi^*$  and the vectors  $*t_\lambda$ ,  $*n^*$  and  $*B_b^*$  satisfying  $*t_\lambda *n^\lambda = 1$  and  $*B_b^\lambda *t_\lambda = 0$ , obviously there always exist  $n^2 + n$  constants  $C_\lambda^*$ ,  $c^*$  such that

$$(3.39) \quad *\xi^* = C_\lambda^* \frac{\xi^*}{0} + c^*; \quad *\frac{t_\lambda}{0} = \frac{-1}{C_\lambda^*} \frac{t_\lambda}{0}; \quad *\frac{n^*}{0} = C_\lambda^* \frac{n^*}{0}; \quad *\frac{B_b^*}{0} = C_\lambda^* \frac{B_b^*}{0}.$$

If now  $\xi^*$ ,  $t_\lambda$ ,  $n^*$  are the solutions satisfying the old initial conditions it is clear that  $C_\lambda^* \xi^* + c^*$ ,  $C_\lambda^* t_\lambda$  and  $C_\lambda^* n^*$  are the solutions satisfying the new conditions. Hence the imbedding and rigging of the given  $A_{n-1}$  is determined to within a non homogeneous affine transformation in  $E_n$ .

In an  $X_{n-1}$  in a general  $A_n$  the field  $h_{ba}$  is fixed to within an arbitrary scalar factor. This means that, if  $h_{ba}$  has rank  $n-1$ , a conformal geometry with a fundamental pseudotensor  $[h_{ba}]$  is induced in the  $X_{n-1}$ . It also means that the asymptotic lines of an  $X_{n-1}$  in  $A_n$  are independent of the rigging.

### Exercise.

V 3.1. Prove for a rigged  $X_{n-1}$  in  $A_n$  satisfying (3.4b) and (3.18):

- a) The covariant differential of  $t_\lambda$  in an asymptotic direction contains this direction.
- b) The covariant differential of  $n^*$  in a direction of principal curvature lies in this direction.
- c) For every line of curvature on  $X_{n-1}$  the covariant differential of the bivector  $n^{[x]} d\xi^{[1]}/dt$  in the direction of the curve is simple and contains the direction of  $n^*$ .

#### § 4. The $V_{n-1}$ in $V_n$ .<sup>1), 2)</sup>

Let  $\overset{\circ}{i}_\lambda$  be the tangent unitvector of a real  $X_{n-1}$  in an ordinary  $V_n$ . Then the  $X_{n-1}$  is a  $V_{n-1}$  and the contravariant vector  $\overset{\circ}{i}^\kappa$  determines a rigging. Because of

$$(4.1) \quad \overset{\circ}{i}_\lambda \overset{\circ}{i}^\lambda = 1,$$

$$(4.2) \quad B_c^\mu (\nabla_\mu \overset{\circ}{i}^\lambda) \overset{\circ}{i}_\lambda = 0.$$

$\overset{\circ}{i}_\lambda$  and  $\overset{\circ}{i}^\kappa$  satisfy the normalization conditions of  $t_\lambda$  and  $n^\kappa$  in the preceding section. From now on we write  $n^\kappa$  for  $\overset{\circ}{i}^\kappa$ . Note that the normalization in  $A_n$  is quite different from that in  $V_n$  because in  $V_n$  it depends entirely on the fundamental tensor  $g_{\lambda\kappa}$ . In a  $V_n$  the difference between  $h_{cb}$  and  $l_{cb}$  vanishes because

$$(4.3) \quad h_{cb} = -B_{c b}^{\mu\lambda} \nabla_\mu n_\lambda = -B_{c b}^{\mu\lambda} g_{\lambda\kappa} \nabla_\mu n^\kappa = -'g_{ba} B_{c\kappa}^{\mu a} \nabla_\mu n^\kappa = l_c^{a'} g_{ba}.$$

' $g_{ba}$  is the section of  $g_{\lambda\kappa}$  and hence ' $g_{\lambda\kappa} = g_{\lambda\kappa} - n_\lambda n_\kappa$ . The induced connexion is Riemannian because

$$(4.4) \quad 'V_c 'g_{ba} = B_{c b a}^{\mu\lambda\kappa} \nabla_\mu 'g_{\lambda\kappa} = -B_{c b a}^{\mu\lambda\kappa} \nabla_\mu n_\lambda n_\kappa = 0.$$

The equations of GAUSS and CODAZZI take the form

$$(4.5) \quad \boxed{'K_{dcba} = B_{d c b a}^{\nu\mu\lambda\kappa} K_{\nu\mu\lambda\kappa} - 2h_{[d[b} h_{c]a]}},$$

$$(4.6) \quad \boxed{2 'V_{[d} h_{c]b} = B_{d c b}^{\nu\mu\lambda} K_{\nu\mu\lambda\kappa} n^\kappa}$$

and for  $V_n = R_n$

$$(4.7) \quad \boxed{'K_{dcba} = -2h_{[d[b} h_{c]a]}},$$

$$(4.8) \quad \boxed{'V_{[d} h_{c]b} = 0}.$$

<sup>1)</sup> Cf. for a more elaborate treatment of the properties of a  $V_{n-1}$  in  $V_n$  dealt with in V § 4, 5; EISENHART 1926, 1, Ch. IV; DUSCHEK and MAYER 1930, 1; E II 1938, 2, Ch. II.

<sup>2)</sup> General references on  $V_{n-1}$  in  $V_n$ : BERWALD 1922, 1; 2; WEYL 1922, 3 FIALKOW 1938, 2; 3 ( $R_n$  and  $S_n$ ); GHOSH 1938, 1; E II 1938, 2, p. 57ff.; CARTAN 1939, 1; 2 ( $V_{n-1}$  isopar. in  $S_n$ ); COBURN 1940, 1 (EINSTEIN  $V_{n-1}$ ); COTTON 1940, 1 MUTÔ 1940, 2; MAEDA 1940, 1; URBAN 1947, 1 and 1948, 1 ( $V_1$  on umb.  $V_{n-1}$ ) VERBICKI 1949, 1 (hyperquadrics); GOLAB 1949, 1 and GOLAB and WRÓBEI 1951, 1 (curves in  $V_{n-1}$  in  $R_n$ ).

$h_{ba}$  is often called the *second fundamental tensor* of the  $V_{n-1}$  in  $V_n$ . If a  $V_{n-1}$  is given in which the curvature tensor can be written in the form (4.7), where  $h_{cb}$  is a symmetric tensor satisfying (4.8) it is always possible to construct an  $R_n$  in which a sufficiently small part of this  $V_{n-1}$  can be imbedded. For  $n=3$  this is a fundamental theorem of classical differential geometry. The proof for an arbitrary  $n$  gives another example of the significance of the GAUSS-CODAZZI equations as integrability conditions. The unknowns are now  $\xi^x$  and  $n_\lambda$  and the system ( $x$ ) in  $R_n$  is supposed to be cartesian. Then the equations are

$$(4.9) \quad \begin{cases} \text{a)} & \partial_b \xi^x = B_b^x \\ \text{b)} & n_\lambda n^\lambda = I; \quad n^x \stackrel{\text{def}}{=} g^{x\lambda} n_\lambda \\ \text{c)} & (\partial_c n_\lambda) n^\lambda = 0 \\ \text{d)} & B_b^\lambda n_\lambda = 0 \\ \text{e)} & B_b^{\lambda x} g_{\lambda x} = 'g_{ba}; \quad g_{\lambda x} \stackrel{\text{def}}{=} \delta_{\lambda x} \end{cases}$$

$$(4.10) \quad \begin{cases} \text{a)} & B_b^\lambda \partial_c n_\lambda = -h_{cb} \\ \text{b)} & \partial_c B_b^x = '\{_{cb}^a\} B_a^x + h_{cb} n^x \end{cases}$$

and the integrability conditions are identically satisfied according to the scheme (cf. scheme 3.37)

$$(4.11) \quad \begin{cases} 4.9a & 4.10b \\ 4.9b & 4.9c, d \\ 4.9c & 4.10a, 4.7 \\ 4.9d & 4.9b; 4.10a, b \\ 4.9e & 4.9d; 4.10b \\ 4.10a & 4.8; 4.9c \\ 4.10b & 4.7; 4.8. \end{cases}$$

Hence the imbedding is fixed if in  $R_n$  we give the point  $\xi^x$  corresponding to  $\eta^a$ , at this point a unitvector  $n^x$  and in the  $R_{n-1}$  of  $n_\lambda$  a set of vectors  $B_0^1, \dots, B_{n-1}^x$  having the same lengths and the same angles between them as the basis vectors  $e_1^a, \dots, e_{n-1}^a$  at  $\eta^a$  in  $V_{n-1}$ . This proves that the imbedding in  $R_n$  is fixed to within translations, rotations and reflexions.

If ' $g_{ba}$ ' is given we know ' $K_{dcba}$ ' and according to a wellknown theorem of tensor calculus (cf. Exerc. I 8,9) the symmetric tensor  $h_{cb}$  is determined by (4.7) to within the sign if such an equation holds and the rank of ' $K_{dcba}$ ' is  $> 2$ . Hence a sufficiently small part of a  $V_{n-1}$ ,  $n \geq 4$ , whose curvature tensor can be written in the form (4.7) with a symmetric field  $h_{cb}$  satisfying (4.8), can always be imbedded in an  $R_n$  and if the

rank of ' $K_{abc}$ ' is  $> 2$  the imbedding is determined to within translations, rotations and reflexions. For  $n=3$  the curvature tensor has rank 0 or 2 and can always be written in the form (4.7) but in this case this equation determines  $\text{Det}(h_{ba})$  only and still leaves two degrees of freedom for the symmetric tensor  $h_{ba}$ . A transformation of a  $V_{n-1}$  in  $R_n$  preserving the linear element is called a *bending* and the bending is said to be *genuine* if it is not a combination of a translation and a rotation or reflexotation. Using this term we are now able to state the theorem of SCHUR:

*A  $V_{n-1}$  in  $R_n$  can for  $n=3$  always suffer a genuine bending and for  $n>3$  can do so if and only if  $h_{ba}$  has rank 0, 1 or 2.<sup>1)</sup>*

In order to prove that the condition is sufficient it is necessary to prove that for a rank  $< 3$  the tensor  $h_{ba}$  can always be changed continuously in such a way that the equations of GAUSS and CODAZZI remain valid.

#### Exercise.

V 4.1. Let a  $V_{n-1}$  be given in a  $V_n$ . We choose the coordinate system in such a way that  $\xi^n = 0$  is the equation of the  $V_{n-1}$ , that the  $\xi^\alpha$ ;  $\alpha = 1, \dots, n-1$ , are coordinates in the  $V_{n-1}$  and that  $\xi^n$  is the distance from a point to the  $V_{n-1}$  along a geodesic perpendicular to the  $V_{n-1}$ . Prove that with respect to this special coordinate system

$$\text{V 4.1 } \alpha) \quad h_{\gamma\beta}^* = -\frac{1}{2} \partial_n g_{\gamma\beta}; \quad \beta, \gamma = 1, \dots, n-1.$$

### § 5. Congruences in $V_n$ .<sup>2)</sup>

The symmetric tensor  $h_{\mu\lambda}$  of the preceding section is a special case of a quantity occurring in the theory of congruences. Let  $i^\lambda$  be the unitvector tangent to a real congruence in an ordinary  $V_n$ . Then, because  $(\nabla_\mu i_\lambda) i^\lambda = 0$  the covariant derivative of the field  $i_\lambda$  can always be written in the form

$$(5.1) \quad \nabla_\mu i_\lambda = l_{\mu\lambda} + i_\mu u_\lambda; \quad i^\mu l_{\mu\lambda} = 0; \quad l_{\mu\lambda} i^\lambda = 0,$$

where  $u_\lambda$  is the curvature vector of the congruence (cf. V § 1). Now the n.a.s. condition for  $i_\lambda$  to be  $V_{n-1}$ -forming is (cf. II § 5 or § 7)

$$(5.2) \quad 0 = i_{[\nu} \nabla_\mu i_{\lambda]} = i_{[\nu} l_{\mu\lambda]}$$

<sup>1)</sup> BEEZ 1874, 1; 1875, 1 (in  $R_n$ ); SCHUR 1886, 1; 2; 1887, 1 (in  $R_4$ ); BOMPIANI 1914, 1 (in  $R_n$ ); KILLING 1885, 1; 1893, 1. Cf. for  $V_{n-1}$  in  $S_n$  CARTAN 1916, 1; E II 1938, 2, p. 143ff., also for literature. Special forms of bending of a surface in  $R_3$  leaving the principal axes of  $h_{ba}$  invariant are dealt with by CARTAN 1941, 2; 1942, 1; 1943, 1.

<sup>2)</sup> RICCI 1895, 1; RICCI and LEVI CIVITA 1901, 1; SCHOUTEN-STRUUK 1919, 1; 1921, 2; LEVI CIVITA 1925, 1; EISENHART 1926, 1; E II 1938, 2, p. 27ff.

or

$$(5.3) \quad h_{\mu\lambda} \stackrel{\text{def}}{=} -l_{(\mu\lambda)} = -l_{\mu\lambda}; \quad l_{[\mu\lambda]} = 0.$$

Now for every  $V_{n-1}$  of this set we have

$$(5.4) \quad B_\lambda^\alpha = A_\lambda^\alpha - i_\lambda i^\alpha,$$

and accordingly the symmetric tensor

$$(5.5) \quad h_{\mu\lambda} = -B_{\mu\lambda}^{\sigma\eta} V_\sigma i_\eta$$

is the second fundamental tensor of these  $V_{n-1}$ 's.

Returning to the general case  $l_{[\mu\lambda]} \neq 0$  we may require the principal directions of  $h_{\mu\lambda}$ . One of these is the direction of  $i^\alpha$ , belonging to the eigenvalue zero. The others with unitvectors  $i_1^\alpha, \dots, i_{n-1}^\alpha$  form  $n-1$  congruences perpendicular to  $i^\alpha$ . They are determined uniquely if and only if the eigenvalues of  $h_{\mu\lambda}$  are all different. RICCI<sup>1)</sup> called them the *canonical congruences* of  $i^\alpha$ .

If one of these congruences, for instance  $i_1^\alpha$ , is  $V_{n-1}$ -normal (cf. II § 5, III § 9) we have

$$(5.6) \quad \left\{ \begin{array}{l} (i^\mu V_\mu i_\lambda - i_1^\mu V_\mu i_\lambda)_a = -2i_1^\mu (V_{(\mu} i_{\lambda)})_a + 2i^\mu (V_{[\mu} i_{\lambda]})_a \\ \quad = -2i_1^\mu i_\lambda^\lambda V_{(\mu} i_{\lambda)} = 0; \quad a = 2, \dots, n-1 \end{array} \right.$$

and this proves (cf. II § 5) that

*Every  $V_{n-1}$ -normal canonical congruence of  $i^\alpha$  is together with  $i^\alpha$   $V_2$ -forming.*

In other words: the equations  $i^\mu \partial_\mu f = 0$ ;  $i_1^\mu \partial_\mu f = 0$  form a complete system.

The congruence  $i^\alpha$  is *geodesic* if  $i^\alpha = 0$ . A geodesic congruence need not be  $V_{n-1}$ -normal as is wellknown from congruences of straight lines in ordinary space. But if a congruence is geodesic and normal to *one*  $V_{n-1}$  it is  $V_{n-1}$ -normal throughout.<sup>2)</sup> To prove this we start from  $i^\nu V_\mu i^\alpha = 0$  and  $V_\mu i_\lambda = l_{\mu\lambda}$ , valid everywhere and  $V_\mu i_\lambda = -h_{\mu\lambda}$  valid at all points of a  $V_{n-1}$ . Then we have everywhere (cf. III 4.9a)

$$(5.7) \quad i^\nu V_\nu V_\mu i_\lambda = i^\nu V_\mu V_\nu i_\lambda - i^\nu K_{\nu\mu\lambda} i_\alpha = -(V_\mu i^\nu) V_\nu i_\lambda - i^\nu K_{\nu\mu\lambda} i_\alpha,$$

<sup>1)</sup> RICCI 1895, 1; cf. SCHOUTEN-STRUIK 1919, 1.

<sup>2)</sup> This theorem is due to BELTRAMI 1869, 1. Cf. DARBOUX 1889, 1; STRUIK 1922, 1, p. 51; EISENHART 1926, 1, p. 57; E II 1938, 2, p. 45. The theorem in its simplest form had already been found by GAUSS, 1827, 1, §§ 15, 16.

hence putting  $k_{\mu\lambda} \stackrel{\text{def}}{=} -l_{[\mu\lambda]}$

$$(5.8) \quad \frac{\delta}{ds} k_{\mu\lambda} = k_{[\mu}{}^{\nu} h_{|\nu|\lambda]} + h_{[\mu}{}^{\nu} k_{|\nu|\lambda]} = 2k_{[\mu}{}^{\nu} h_{\lambda]\nu}$$

from which it follows that  $k_{\mu\lambda}$  is zero at all points if it is zero at the points of one  $V_{n-1}$ . This proves that  $i_\lambda$  is a gradient vector  $i_\lambda = \partial_\lambda f$  and from this  $i^\mu \partial_\mu f = df/ds = 1$ . Hence  $f = \text{const.}$  is the equation of the normal  $V_{n-1}$ 's and any two of these  $V_{n-1}$ 's cut off segments of the same length from all curves of the congruence.

A set of  $n$  mutually perpendicular real congruences in an ordinary  $V_n$  is called an *orthogonal net*.<sup>1)</sup> It is said to be an *orthogonal system* if the congruences are *all*  $V_{n-1}$ -normal.<sup>2)</sup> Let  $\overset{n}{i}{}^\mu = j_\mu$ ;  $j = 1, \dots, n$  be the tangent unitvectors. Hence we have for the  $n$ -th congruence of an orthogonal system an equation of the form (cf. 5.1, 3)

$$(5.9) \quad \overset{n}{V}_\mu i_\lambda = -h_{\mu\lambda} + \overset{n}{i}_\mu u_\lambda; \quad \overset{n}{i}^\mu h_{\mu\lambda} = 0; \quad h_{[\mu\lambda]} = 0.$$

The orthogonal net is an orthogonal system if and only if each set of two congruences is  $V_2$ -forming. The n.a.s. condition is (cf. II § 5)

$$(5.10) \quad \overset{n}{i}^\mu \overset{n}{V}_\mu i_j^\infty - \overset{n}{i}^\mu \overset{n}{V}_\mu i_i^\infty = \alpha \overset{n}{i}_i^\infty + \beta \overset{n}{i}_j^\infty$$

or (cf. III § 9)

$$(5.11) \quad \overset{n}{\Gamma}_{[j}{}^h{}_{i]} \stackrel{*}{=} -\overset{n}{i}^\mu \overset{n}{i}^\lambda \overset{n}{V}_{[\mu} i_{\lambda]}^\hbar = 0; \quad h, i, j \neq .$$

But because  $\overset{n}{\Gamma}_{j i}^h \stackrel{*}{=} -\overset{n}{\Gamma}_{j h}^i$  this condition is satisfied if and only if

$$(5.12) \quad -\overset{n}{\Gamma}_{ij}^h \stackrel{*}{=} \overset{n}{i}^\mu \overset{n}{i}^\lambda \overset{n}{V}_\mu i_\lambda^\hbar = 0; \quad h, i, j \neq .$$

This equation expresses the fact that every set of  $n-1$  of the congruences is canonical with respect to the remaining congruence.

Let us now suppose that the principal directions of  $h_{\mu\lambda}$  are uniquely determined and that the congruences  $\overset{n}{i}^\mu_b$ ;  $b = 1, \dots, n-1$  are chosen in these principal directions. Then we have  $h_{\mu\lambda} \stackrel{*}{=} \sum_a h_{aa} \overset{n}{i}_\mu^a \overset{n}{i}_\lambda^a$ . The  $n$  con-

<sup>1)</sup> The case of a  $V_n$  with an indefinite fundamental tensor is dealt with by many authors. Cf. for instance EISENHART 1926, 1; 1949, 1; E I 1935, 1; E II 1938, 2; WONG 1945, 2.

<sup>2)</sup> Cf. also for literature SCHOUTEN and STRUIK 1919, 1; R. K. 1924, 1, p. 190ff.; EISENHART 1923, 2; 1926, 1, p. 117ff.; SCHOUTEN 1927, 3; WALBERER 1934, 1; E II 1938, 2, p. 27ff.; WONG 1945, 2.

gruences form an orthogonal system if and only if

$$(5.13) \quad \begin{cases} \text{a)} & i^{\mu} \underset{n}{\underset{b}{i^{\lambda}}} \nabla_{\mu} i^{\alpha}_{\lambda} = 0 \\ \text{b)} & i^{\mu} \underset{c}{\underset{b}{i^{\lambda}}} \nabla_{\mu} i^{\alpha}_{\lambda} = 0; \quad a, b, c \neq; \quad a, b, c = 1, \dots, n - 1 \end{cases}$$

since the remaining equation  $i^{\mu} \underset{c}{\underset{b}{i^{\lambda}}} \nabla_{\mu} i^{\alpha}_{\lambda} = 0$  of (5.12) is already satisfied because of the special choice of the  $i^{\alpha}$ . Now by differentiation of

$$(5.14) \quad i^{\mu} \underset{c}{\underset{b}{i^{\lambda}}} h_{\mu\lambda} = 0; \quad b \neq c$$

we get

$$(5.15) \quad \begin{cases} \text{a)} & (h_{bb} - h_{cc}) i^{\nu} \underset{n}{\underset{b}{i^{\lambda}}} \nabla_{\nu} i^{\alpha}_{\lambda} + i^{\nu} \underset{n}{\underset{c}{i^{\mu}}} \underset{b}{i^{\lambda}} \nabla_{\nu} h_{\mu\lambda} = 0 \\ \text{b)} & (h_{bb} - h_{cc}) i^{\nu} \underset{d}{\underset{b}{i^{\lambda}}} \nabla_{\nu} i^{\alpha}_{\lambda} + i^{\nu} \underset{d}{\underset{c}{i^{\mu}}} \underset{b}{i^{\lambda}} \nabla_{\nu} h_{\mu\lambda} = 0; \quad b, c, d \neq; \end{cases}$$

and in these equations  $h_{bb} \neq h_{cc}$ . This proves the proposition:

*A real congruence in an ordinary  $V_n$  whose canonical congruences are uniquely determined, belongs to an orthogonal system of  $n$  congruences if and only if*

$$(5.16) \quad \boxed{\begin{cases} \text{a)} & i^{\nu} \underset{n}{\underset{c}{i^{\mu}}} \underset{b}{i^{\lambda}} \nabla_{\nu} h_{\mu\lambda} = 0 \\ \text{b)} & i^{\nu} \underset{d}{\underset{c}{i^{\mu}}} \underset{b}{i^{\lambda}} \nabla_{\nu} h_{\mu\lambda} = 0 \end{cases}; \quad b, c, d = 1, \dots, n - 1; \quad b, c, d \neq.}$$

The geometrical meaning of these conditions is that the covariant differential of  $h_{\mu\lambda}$  for a direction perpendicular to any set of  $m$  principal directions of  $h_{\mu\lambda}$  has a component in the  $R_m$  of these directions whose principal multidirections (cf. I § 9) contain these principal directions.<sup>1)</sup>

If the principal directions of  $h_{\mu\lambda}$  are not uniquely determined results can be obtained but they are rather complicated.<sup>2)</sup> It can be proved in the same way that the principal directions of a symmetric tensorfield  $T_{\mu\lambda}$  in  $V_n$  in the special case that they are uniquely determined are  $V_{n-1}$ -normal if and only if

$$(5.17) \quad i^{\nu} \underset{j}{\underset{i}{i^{\mu}}} \underset{h}{i^{\lambda}} \nabla_{\nu} T_{\mu\lambda} = 0; \quad j, i, h = 1, \dots, n; \quad j, i, h \neq. \quad .^3)$$

<sup>1)</sup> LÉVY 1870, 1 for  $V_2$  in  $R_3$ .

<sup>2)</sup> SCHOUTEN 1927, 3.

<sup>3)</sup> SCHOUTEN 1927, 3, p. 723ff. also for the general case.

The conditions (5.16, 17) can only be used if the principal directions are already determined. TONOLO<sup>1)</sup> gave for  $n=3$  a condition for  $T_{\mu\lambda}$  that only involves  $T_{\mu\lambda}$  and its adjoint (cf. Exerc. I 8,2) and SCHOUTEN<sup>2)</sup> gave the n.a.s. conditions

$$(5.18) \quad \text{a)} \quad T_{[\nu}^{\cdot\infty} V_\mu T_{\lambda]\infty} = 0 \quad \text{b)} \quad \bar{T}_{[\nu}^I V_\mu T_{\lambda]\infty} = 0 \quad \text{c)} \quad \bar{T}_{[\nu}^{\cdot\infty} V_\mu \bar{T}_{\lambda]\infty}^I = 0$$

for the case when  $T_{\mu\lambda}$  has  $n$  different eigenvalues none of which is zero. The proof runs as follows. Multiplying (5.17) with the eigenvalue  $\lambda$  and using the fact that  $\frac{i}{h}$  is an eigenvector we get

$$(5.19) \quad i^\mu i^\lambda i^\nu T_{\nu}^{\cdot\infty} V_\mu T_{\lambda\infty} = 0; \quad h, i, j = 1, \dots, n; \quad h, i, j \neq .$$

Alternation over  $j, i, h$  gives (5.18a). The other equations (5.18) are derived in a similar way. Hence (5.18) is necessary. Conversely, using the anholonomic system  $(h)$ , (5.18) can be replaced by

$$(5.20) \quad \left\{ \begin{array}{l} T_{[k}^h V_j T_{i]h} = 0 \\ \bar{T}_{[k}^I V_j T_{i]h} = 0; \quad \bar{T}_i^2 \stackrel{\text{def}}{=} \bar{T}_j^I \bar{T}_{i]I}^J \\ \bar{T}_{[k}^2 \bar{T}_j^I V_{i]} T_{hI} = 0 \end{array} \right.$$

and if these equations are written out using the special values of the components of  $h_{\mu\lambda}$  with respect to  $(h)$  we get

$$(5.21) \quad V_j T_{ih} = 0; \quad h, i, j \neq .$$

which is equivalent to (5.17).

NIJENHUIS<sup>3)</sup> has given yet another form connected with the question of whether a tensor  $T_{\cdot\lambda}^*$  in  $X_n$  with  $n$  different eigenvalues has covariant eigenvectors which are  $X_{n-1}$ -forming. For this problem without a metric he introduces the new concomitant (cf. Exerc. II 2,3)

$$(5.22) \quad H_{\mu\lambda}^{\cdot\infty} \stackrel{\text{def}}{=} 2 T_{[\mu}^{\cdot\infty} \partial_{|\rho|} T_{\lambda]\infty} - 2 T_{\rho}^{\cdot\infty} \partial_{[\mu} T_{\lambda]\rho}^{\cdot\infty}$$

of the field  $T_{\lambda}\cdot^*$  and gives several forms of n.a.s. conditions. For instance it is n.a.s. that  $H_{\mu\lambda}^{\cdot\infty}$  can be written in the form

$$(5.23) \quad H_{\mu\lambda}^{\cdot\infty} = \hat{p}_{[\mu} A_{\lambda]}^{\cdot\infty} + \hat{p}_{[\mu} T_{\lambda]\infty}^{\cdot\infty} + \cdots + \hat{p}_{[\mu}^{n-2} T_{\lambda]\infty}^{\cdot\infty}$$

<sup>1)</sup> TONOLO 1941, 2; 1949, 1; 2; 3; 1953, 1; cf. TONOLO 1941, 1. A  $V_3$  whose principal directions are  $V_2$ -normal was already discussed by SLEBODZINSKI 1927, 1.

<sup>2)</sup> SCHOUTEN 1951, 2.

<sup>3)</sup> NIJENHUIS 1951, 1.

where  $\overset{\circ}{p}_\lambda, \dots, \overset{n-2}{p}_\lambda$  are suitably chosen and where  $\overset{q}{T}_\lambda^*$  represents the  $q$ -th power of  $T_\lambda^*$ . Going back to the metric case he found other forms for the conditions (5.18). The most elegant form is

$$(5.24) \quad \left\{ \begin{array}{ll} \text{a)} & H_{[\mu\lambda]}^* g_{\nu}\vphantom{\frac{1}{2}} = 0 \\ \text{b)} & H_{[\mu\lambda]}^* T_{\nu}\vphantom{\frac{1}{2}} = 0 \\ \text{c)} & H_{[\mu\lambda]}^* \overset{2}{T}_{\nu}\vphantom{\frac{1}{2}} = 0 \end{array} \right.$$

also suitable if one of the eigenvalues of  $T_{\mu\lambda}$  is zero.

### Exercises.

V 5,1<sup>4</sup>). An ordinary  $V_n$  is an  $R_n$  if and only if there exists an orthogonal system of real geodesic congruences.

V 5,2<sup>2</sup>). A real geodesic congruence with unit tangent vector  $i^*$  is  $V_{n-1}$ -normal if and only if the integral  $\int i_\mu d\xi^\mu$  is zero for every closed curve in some non singular  $V_{n-1}$  that has a point in common with each curve of the congruence.

V 5,3<sup>3</sup>). A  $V_{n-1}$  in  $R_n$  belongs to an orthogonal system in  $R_n$  if and only if there exist  $n$  different principal directions of  $h_{cb}$  that form an orthogonal system in  $V_{n-1}$ .

V 5,4<sup>4</sup>). If the congruences with the unit tangent vectors  $i^*_j$ ;  $j=1, \dots, n$ , in  $V_n$  form an orthogonal system and if the field  $\alpha i_1^* + \beta i_2^*$  is  $V_{n-1}$ -normal, then the field  $-\alpha i_1^* + \beta i_2^*$  is also  $V_{n-1}$ -normal.  $\alpha$  and  $\beta$  need not be constants.

V 5,5<sup>5</sup>). The curvature vector of the congruence of a field  $v^*$  in  $V_n$  that satisfies KILLING's equation  $V_{(\mu} v_{\lambda)} = 0$ , is a gradient vector.

V 5,6<sup>6</sup>). Every congruence in  $V_n$  perpendicular to a field  $v^*$  that satisfies KILLING's equation, has a curvature vector perpendicular to  $v^*$ .

V 5,7<sup>7</sup>). If a field  $v^*$  of  $V_n$  is  $V_{n-1}$ -normal and satisfies KILLING's equation and if the length of  $v^*$  is constant, its congruence is geodesic and normal to a set of  $\infty^1$  geodesic  $V_{n-1}$ 's.

<sup>1)</sup> E II 1938, 2, p. 31.

<sup>2)</sup> SCHOUTEN-STRUIK 1921, 2.

<sup>3)</sup> BRAUNER 1951, 1.

<sup>4)</sup> DEMOULIN 1913, 1 for  $R_3$ .

<sup>5)</sup> KILLING 1892, 1, p. 167; RICCI 1898, 1; 2; 1901, 1, p. 608.

<sup>6)</sup> RICCI 1898, 1; 2.

<sup>7)</sup> STRUIK 1922, 1, p. 157.

### § 6. Properties of curvature of a $V_{n-1}$ in $V_n$ .

Let  $i^x$  be the tangent unitvector of a real congruence of  $V_{n-1}$ . Every curve of this congruence may be considered as a curve of  $V_n$  and as a curve of  $V_{n-1}$ . Writing ' $u^a$ ' for the curvature vector (cf. V § 1) in  $V_{n-1}$  we get

$$(6.1) \quad u^x \stackrel{\text{def}}{=} i^\mu V_\mu i^x = B_a^x i^c V_c i^a + i^\mu (V_\mu i^e) n_e n^x = B_a^x 'u^a + i^c i^b h_{cb} n^x.$$

We call  $u^x$  the *absolute curvature vector* of the curve, ' $u^a$ ' the *relative curvature vector*,

$$(6.2) \quad "u^x \stackrel{\text{def}}{=} i^c i^b h_{cb} n^x = u^x - 'u^{x-1})$$

the *enforced curvature vector* and  $i^c i^b h_{cb}$  the *enforced curvature*. The sign of this latter scalar depends of course on the choice of the sign of  $n^x$ . The vector " $u^x$ " is the same for all curves that have the same tangent at the point considered.<sup>2)</sup> It is equal to  $u^x$  if and only if the tangent of the curve, considered as a curve of the  $V_{n-1}$ , is stationary (cf. VI § 1) at the point and it is equal to  $u^x$  all along the curve if and only if the curve is a geodesic of  $V_{n-1}$ . This proves that the curves of  $V_{n-1}$  through one of its points, looked upon as curves of the  $V_n$ , have extreme values of the enforced curvature if and only if the tangent lies in a principal direction of  $h_{cb}$  at that point. These directions are therefore called the *directions of principal curvature* and the corresponding values of  $i^c i^b h_{cb}$  are called the *principal curvatures* of the  $V_{n-1}$ . The scalar

$$(6.3) \quad h \stackrel{\text{def}}{=} \frac{1}{n-1} 'g^{ba} h_{ba}$$

is called the *mean curvature*. Principal curvatures and mean curvature are differential invariants of the  $V_{n-1}$ .

A curve of  $V_{n-1}$  whose tangent lies everywhere in a direction of principal curvature is said to be a *line of curvature*. Hence the lines of curvature belong to canonical congruences of the congruences normal to  $V_{n-1}$ . As a result of this the  $V_{n-1}$ 's of an orthogonal system intersect in lines of curvature. This is the theorem of DUPIN.

If  $i^c i^b h_{cb} = 0$  at some point, the direction of  $i^a$  is called a *principal tangent* and the curves of  $V_{n-1}$  whose tangents are everywhere principal are called *asymptotic lines*.<sup>3)</sup>

1) If instead of the direction of  $n^x$  the direction of an arbitrary congruence given over  $V_{n-1}$  is taken, we get instead of the relative curvature the *union curvature* of SPRINGER 1950, 1 and instead of the geodesics in  $V_{n-1}$  the *union curves* with respect to this congruence. Cf. YANO 1948, 1 also for a generalization in  $A_n$ ; MISHRA 1952, 1.

2) This is the quintessence of the theorem of MEUSNIER ( $n = 3$ ).

3) KRONECKER 1869, 1; VOSS 1880, 1. Cf. for asymptotic lines of higher order STRUIK 1922, 1, p. 80; E II 1938, 2, p. 73ff.; GOLAB 1949, 1.

If  $h_{cb}$  has rank  $r$ ,  $n-1-r$  principal curvatures are zero and the cone of principal tangents consists of  $\infty'$   $R_{n-r-1}$ 's all containing the  $(n-1-r)$ -direction belonging to these curvatures. If  $h_{cb}=0$  at a point, every curve of  $V_{n-1}$  through this point has an enforced curvature zero at that point. The  $V_{n-1}$  is then called *geodesic at that point*. A *geodesic*  $V_{n-1}$  is a  $V_{n-1}$  that is geodesic at all its points. Every geodesic of a geodesic  $V_{n-1}$  is also a geodesic of  $V_n$  and this property is characteristic for a geodesic  $V_{n-1}$ .

If  $h_{cb}$  equals ' $g_{cb}$ ' to within a scalar factor

$$(6.4) \quad h_{cb} = h' g_{cb}$$

the point is called *umbilical*. At an umbilical point all principal curvatures are equal and all directions are directions of principal curvature.<sup>1)</sup> If all points of  $V_{n-1}$  are umbilical it is called *umbilical*. It follows from (4.6) that for an umbilical  $V_{n-1}$

$$(6.5) \quad 2(\nabla_{[a} h)' g_{c]b} = B_{dc}^{\nu\mu\lambda} K_{\nu\mu\lambda\nu} n^\nu.$$

Hence, if umbilical  $V_{n-1}$ 's are possible through every point of  $V_n$  and with every  $(n-1)$ -direction at that point, all components  $K_{kjih}$  with respect to any orthogonal (in general anholonomic) coordinate system ( $h$ ) with *four* different indices are zero. In the next chapter we shall prove that this condition is also sufficient.<sup>2)</sup> If the  $V_{n-1}$  is not only umbilical but if moreover the mean curvature  $h$  is constant, the left hand side of (6.5) vanishes. Now  $B_{\lambda}^{\nu} = A_{\lambda}^{\nu} - n_{\lambda} n^{\nu}$ , hence by contraction of (6.5)

$$(6.6) \quad 0 = B_{\nu}^{\sigma} K_{\sigma\sigma} n^{\nu}$$

and this proves that the normal of every umbilical  $V_{n-1}$  with constant mean curvature lies at each point in a principal direction of  $V_n$  [cf. Exerc. III 5,3].<sup>3)</sup>  $V_{n-1}$ 's of this kind through every point and with every  $(n-1)$ -direction at that point are only possible in an EINSTEIN space because according to (6.6)  $K_{\mu\lambda}$  must equal  $g_{\mu\lambda}$  to within a scalar factor. But in § 5 of the next chapter it will be proved that this condition is not sufficient but that the space must be an  $S_n$  [cf. III § 5].<sup>4)</sup> As a corollary we get that a geodesic  $V_{n-1}$  is at each point always perpendicular to a principal direction of  $V_n$  and that only in an  $S_n$  geodesic  $V_{n-1}$ 's are

<sup>1)</sup> Cf. on umbilical points T. Y. THOMAS 1938, 2; SASAKI 1939, 1; MUTÔ 1940, 1; WONG 1943, 3 (EINSTEIN space); ADATI 1951, 1; 2.

<sup>2)</sup> SCHOUTEN 1921, 1, p. 86 for  $n > 3$  (cf. footnote 1, p. 309).

<sup>3)</sup> RICCI 1903, 1, p. 415 for  $h=0$ ; RIMINI 1904, 1, p. 35 for  $n=3$ ; STRUIK 1922, 1, p. 143 for the general case.

<sup>4)</sup> SCHOUTEN 1921, 1, p. 87.

possible through every point and with every  $(n-1)$ -direction through this point.<sup>1)</sup>

Some interesting propositions may be derived for  $V_n = S_n$  from the integrability equations (4.5, 6). According to (III 5.31) in an  $S_n$  these equations take the form

$$(6.7) \quad 'K_{dcb}a = -2\kappa'g_{[d[b}'g_{c]a]} - 2h_{[d[b}h_{c]a]}$$

$$(6.8) \quad 'V_{[d}h_{c]b} = 0$$

where  $\kappa$  is the (constant) scalar curvature of the  $S_n$ . If the  $V_{n-1}$  is geodesic it follows from (6.7) that it is an  $S_{n-1}$  with the same constant scalar curvature  $\kappa$ . In that case (6.8) is trivial. If the  $V_{n-1}$  has only umbilical points we have

$$(6.9) \quad 'K_{dcb}a = -2(\kappa + h^2)'g_{[d[b}'g_{c]a]},$$

hence the  $V_{n-1}$  must be an  $S_{n-1}$  with the constant curvature  $\kappa + h^2$  and its mean curvature must be constant. From (6.7) it follows that a  $V_{n-1}$  in  $S_n$  can be an  $S_{n-1}$  if and only if

$$(6.10) \quad h_{[d[b}h_{c]a]} = (\kappa - \kappa)'g_{[d[b}'g_{c]a]}.$$

For  $n \geq 4$  this implies that the  $S_{n-1}$  is umbilical and that  $h_{cb} = \pm'g_{cb} \times \sqrt{\kappa - \kappa}$  but for  $n = 3$  the equation (6.10) only determines the determinant of  $h_{cb}$  and leaves the possibility of bending (cf. V § 4).

From (6.7) we get by contraction

$$(6.11) \quad 'K_{cb} = \kappa(n-2)'g_{cb} - h_{.b}^a h_{ca} + h_{.a}^b h_{cb}$$

from which it follows that the principal directions of  $h_{cb}$  are also principal directions of  $'K_{cb}$ . Hence the directions of principal curvature of a  $V_{n-1}$  in  $S_n$  are principal directions of  $V_{n-1}$ .<sup>2)</sup>

The quadric in the tangent  $R_{n-1}$

$$(6.12) \quad h_{ba}x^bx^a = \pm 1; \quad a, b = 1, \dots, n-1$$

is the *indicatrix* of DUPIN. Let  $(a)$  be an anholonomic orthogonal coordinate system in  $V_{n-1}$  with unit basis vectors  $i_b^a$  and let  $\alpha$  be the angles between an arbitrary congruence  $i^a$  and the congruences  $i_b^a$ . Then the enforced curvature of the congruence  $i^a$  is

$$(6.13) \quad i^c i^b h_{cb} = \sum_b h_{bb} \cos^2 \alpha.$$

<sup>1)</sup> A  $V_n$  in which geodesic  $V_{n-1}$ 's with every  $(n-1)$ -direction are possible through one definite point is called a SCHUR space and the point is called *centre*. DUSCHEK and MAYER gave 1930, 1, p. 167 ff. the form of the linear element in normal coordinates with respect to the centre.

<sup>2)</sup> Cf. COBURN 1940, 1.

This is the *theorem of EULER* ( $n=3$ ). The absolute value of the enforced curvature for the direction of  $i^a$  equals the inverse quadrate of the length of the radiusvector  $x^a$  of DUPIN's indicatrix in that direction.

### Exercises.

V 6,1<sup>1)</sup>. Prove for a real curve in a real  $V_{n-1}$  in an ordinary  $V_n$ :

If the curve is geodesic in  $V_{n-1}$  and a line of curvature, then its second curvature in  $V_n$  is zero.

If the curve is geodesic in  $V_{n-1}$  and if its second curvature in  $V_n$  is zero, then it is a line of curvature.

V 6,2<sup>2)</sup>. In a real  $V_{n-1}$  in an ordinary  $V_n$  there are two real congruences with unit tangent vectors  $i^x$  and  $j^x$ . Prove that

$$V 6,2 \alpha) \quad i^\mu V_\mu j^x = i^\mu V_\mu j^x + i^c j^b h_{cb} n^x.$$

### § 7. The rigged $X_n^m$ in $L_n$ and $X_n$ .<sup>3)</sup>

Let a rigged  $X_n^m$  be given in an  $L_n$  and let the basis vectors  $\overset{a}{e}_b^x$ ;  $a, b, \dots = 1, \dots, m$  span the  $E_m$  of the  $X_n^m$  and the  $\overset{y}{e}_\lambda^x$ ;  $y, \lambda, \dots = m+1, \dots, n$  span the  $E_{n-m}$  of the rigging  $X_n^{m'}$ ;  $m'=n-m$  at each point. Then at each point we have the basis vectors  $\overset{i}{e}_i^x$ ;  $\overset{h}{e}_\lambda^x$ ;  $h, i = 1, \dots, n$  of the anholonomic coordinate system ( $h$ ) and the connecting quantities

$$(7.1) \quad \begin{cases} B_b^x = \overset{a}{e}_b^x \overset{a}{e}_b^x \overset{a}{A}_b^x; & C_\lambda^x = \overset{y}{e}_\lambda^x \overset{y}{e}_\lambda^x \overset{y}{A}_\lambda^x; & B_\lambda^x = \overset{a}{e}_\lambda^x \overset{a}{e}_\lambda^x; & B_a^b \overset{b}{=} \delta_a^b; \\ C_y^x = \overset{x}{e}_y^x \overset{x}{e}_y^x \overset{x}{A}_y^x; & B_\lambda^a = \overset{a}{e}_b^x \overset{b}{e}_\lambda^x \overset{a}{A}_\lambda^x; & C_\lambda^x = \overset{x}{e}_\lambda^x \overset{x}{e}_\lambda^x; & C_y^x \overset{x}{=} \delta_y^x. \end{cases}$$

<sup>1)</sup> E II 1938, 2, p. 72.

<sup>2)</sup> E II 1938, 2, p. 66; cf. GRAUSTEIN 1934, 1; RICCI 1902, 2; Aoust 1864, 1.

<sup>3)</sup> General references on the  $X_n^m$  (cf. also II § 5): SCHOUTEN 1923, 1 ( $X_m$  in  $A_n$ ); 1928, 1; 1929, 3; VRANCEANU 1926, 1; 2; 1927, 1; 2; 1928, 1; 2; 5; 1929, 1; 2; 1930, 1; 1931, 1; 1932, 1; 1934, 1; 1936, 1; 1938, 1; 1942, 2; HORAK 1927, 1; 2; 1928, 1; SYNGE 1928, 1; HLAVATY 1930, 1; 2; 1934, 3; MOISIL 1930, 1; SCHOUTEN and v. KAMPEN 1930, 2; BORTOLOTTI 1931, 1; 2; 1936, 3 (also for literature); 1937, 2 ( $X_n^m$  project.); 1941, 1 (also for literature); DIENES 1932, 1; AGOSTINELLI 1933, 1 ( $V_n$ ); MORINAGA 1934, 1; VANDERSLICE 1934, 1; FABRICIUS-BJERRE 1936, 1 (torsionfree); WAGNER 1936, 1; 1938, 1; 2; 1940, 1 (non linear in  $E_3$ ); 1941, 1 ( $X_n^{m-1}$ ); 1943, 2 (non linear in  $X_n$ ); PAUC 1937, 1; 1938, 1; 2; YANO 1937, 1; 1938, 1; 1939, 1; YANO and PETRESCU 1940, 2; BOMPIANI 1938, 1; MAXIA 1939, 1; 2; 1940, 1; MIKAN 1939, 1; HAIMOVICI 1940, 1; 1946, 3; SU 1943, 1 ( $X_n^{m-1}$  project.); PANTAZZI 1943, 1; 1947, 1; PETRESCU 1943, 1 ( $V_n^{m-1}$ ); 2; 3 ( $V_2^{2p+1}, V_2^{2p+2}$ ); 1944, 1 ( $L_n^m$  in  $E_n$ ); 1945, 1; 1948, 1 ( $V_n^m$ ); WANG 1943, 1; 2; ROZENFEL'D 1947, 2 ( $X_n^m$  in  $E_n$ ); ROGOVOI 1949, 1 and 1950, 1 ( $X_3^2$  in  $E_3$ ); DAVIES 1953, 1; TAKASU 1953, 1 (in the large).

In order to consider questions of curvature we could proceed in the same way as in the case of an  $X_{n-1}$  in  $A_n$  (cf. V § 3) but then the formulae would become crowded with factors  $B$  and  $C$  and would be very unreadable. Therefore we introduce here the so-called  $D$ -symbolism due to VAN DER WAERDEN and BORTOLOTTI.<sup>1)</sup>

If  $p^* = B_a^* p^a$  and  $q_\lambda = B_\lambda^b q_b$  are vectors of  $X_n^m$ ;  $r^* = C_x^* r^x$  and  $s_\lambda = C_\lambda^y s_y$  vectors of  $X_n^{m'}$  and  $v^*$  and  $w_\lambda$  general vectors of  $X_n$ , the working of the operators  $D_\mu$ ,  $D_c$  and  $D_z$  on these quantities and on scalars is defined by the equations

$$(7.2) \quad \begin{cases} D_\mu p = V_\mu p = \partial_\mu p \\ D_b p = B_b^\mu V_\mu p = B_b^\mu \partial_\mu p \\ D_y p = C_y^\mu V_\mu p = C_y^\mu \partial_\mu p \end{cases}$$

$$(7.3) \quad \text{a)} \begin{cases} \alpha) & D_\mu p^* = V_\mu p^* \\ \beta) & D_c p^* = B_c^\mu V_\mu p^* \\ \gamma) & D_z p^* = C_z^\mu V_\mu p^* \end{cases} \quad \text{b)} \begin{cases} \alpha) & D_\mu q_\lambda = V_\mu q_\lambda \\ \beta) & D_c q_\lambda = B_c^\mu V_\mu q_\lambda \\ \gamma) & D_z q_\lambda = C_z^\mu V_\mu q_\lambda \end{cases}$$

$$(7.4) \quad \text{a)} \begin{cases} \alpha) & D_\mu p^a = B_\mu^a V_\mu p^a \\ \beta) & D_c p^a = B_c^\mu V_\mu p^a \\ \gamma) & D_z p^a = C_z^\mu B_\mu^a V_\mu p^a \end{cases} \quad \text{b)} \begin{cases} \alpha) & D_\mu q_b = B_b^\lambda V_\mu q_\lambda \\ \beta) & D_c q_b = B_c^\mu V_\mu q_\lambda \\ \gamma) & D_z q_b = C_z^\mu B_b^\lambda V_\mu q_\lambda \end{cases}$$

$$(7.5) \quad \text{a)} \begin{cases} \alpha) & D_\mu r^* = V_\mu r^* \\ \beta) & D_c r^* = B_c^\mu V_\mu r^* \\ \gamma) & D_z r^* = C_z^\mu V_\mu r^* \end{cases} \quad \text{b)} \begin{cases} \alpha) & D_\mu s_\lambda = V_\mu s_\lambda \\ \beta) & D_c s_\lambda = B_c^\mu V_\mu s_\lambda \\ \gamma) & D_z s_\lambda = C_z^\mu V_\mu s_\lambda \end{cases}$$

$$(7.6) \quad \text{a)} \begin{cases} \alpha) & D_\mu r^* = C_x^* V_\mu r^* \\ \beta) & D_c r^* = B_c^\mu C_x^* V_\mu r^* \\ \gamma) & D_z r^* = C_z^\mu V_\mu r^* \end{cases} \quad \text{b)} \begin{cases} \alpha) & D_\mu s_y = C_y^\lambda V_\mu s_\lambda \\ \beta) & D_c s_y = B_c^\mu C_y^\lambda V_\mu s_\lambda \\ \gamma) & D_z s_y = C_z^\mu C_y^\lambda V_\mu s_\lambda \end{cases}$$

$$(7.7) \quad \text{a)} \begin{cases} \alpha) & D_\mu v^* = V_\mu v^* \\ \beta) & D_c v^* = B_c^\mu V_\mu v^* \\ \gamma) & D_z v^* = C_z^\mu V_\mu v^* \end{cases} \quad \text{b)} \begin{cases} \alpha) & D_\mu w_\lambda = V_\mu w_\lambda \\ \beta) & D_c w_\lambda = B_c^\mu V_\mu w_\lambda \\ \gamma) & D_z w_\lambda = C_z^\mu V_\mu w_\lambda \end{cases}$$

<sup>1)</sup> V. D. WAERDEN 1927, 1; BORTOLOTTI 1927, 3. All formulae containing indices  $a, b, \dots$  or  $x, y, \dots$  should of course be written with  $\overset{*}{=}$  because they are meant to hold only for the special choice of the basis vectors  $\overset{x}{e}_\mu^*$ ,  $\overset{y}{e}_\mu^*$  with respect to the  $X_n^m$  and  $X_n^{m'}$ . Since this is understood we nearly always write in this section  $=$ . Cf. for historical notes concerning the  $D$ -symbolism, SCHOUTEN and v. KAM-PEN 1930, 2, p. 774. Cf. BOMPIANI 1952, 2.

and by the rule of LEIBNIZ. Hence, if a quantity of higher valence is given, for instance  $v_{\cdot\cdot\mu}^{x\lambda}$ , lying with the index  $x$  in  $X_n^m$ , with the index  $\lambda$  in  $X_n^{m'}$ , and with the index  $\mu$  in  $X_n$  but neither in  $X_n^m$  nor in  $X_n^{m'}$ , there exist full sets of components of four kinds, c.q.  $v_{\cdot\cdot\mu}^{x\lambda}$ ,  $v_{\cdot\cdot\mu}^{a\lambda}$ ,  $v_{\cdot\cdot\mu}^{xx}$  and  $v_{\cdot\cdot\mu}^{ax}$ ,<sup>1)</sup> and we have the twelve derivatives

$$(7.8) \quad \left\{ \begin{array}{ll} \text{a)} & \begin{cases} \alpha) D_v v_{\cdot\cdot\mu}^{x\lambda} = V_v v_{\cdot\cdot\mu}^{x\lambda} \\ \beta) D_d v_{\cdot\cdot\mu}^{x\lambda} = B_d^v V_v v_{\cdot\cdot\mu}^{x\lambda} \\ \gamma) D_u v_{\cdot\cdot\mu}^{x\lambda} = C_u^v V_v v_{\cdot\cdot\mu}^{x\lambda} \end{cases} & \text{b)} & \begin{cases} \alpha) D_v v_{\cdot\cdot\mu}^{a\lambda} = B_\alpha^a V_v v_{\cdot\cdot\mu}^{x\lambda} \\ \beta) D_d v_{\cdot\cdot\mu}^{a\lambda} = B_{d\alpha}^a V_v v_{\cdot\cdot\mu}^{x\lambda} \\ \gamma) D_u v_{\cdot\cdot\mu}^{a\lambda} = C_u^a B_\alpha^a V_v v_{\cdot\cdot\mu}^{x\lambda} \end{cases} \\ \text{c)} & \begin{cases} \alpha) D_v v_{\cdot\cdot\mu}^{xx} = C_\lambda^x V_v v_{\cdot\cdot\mu}^{x\lambda} \\ \beta) D_d v_{\cdot\cdot\mu}^{xx} = B_d^x C_\lambda^x V_v v_{\cdot\cdot\mu}^{x\lambda} \\ \gamma) D_u v_{\cdot\cdot\mu}^{xx} = C_u^x V_v v_{\cdot\cdot\mu}^{x\lambda} \end{cases} & \text{d)} & \begin{cases} \alpha) D_v v_{\cdot\cdot\mu}^{ax} = B_\alpha^a C_\lambda^x V_v v_{\cdot\cdot\mu}^{x\lambda} \\ \beta) D_d v_{\cdot\cdot\mu}^{ax} = B_{d\alpha}^a C_\lambda^x V_v v_{\cdot\cdot\mu}^{x\lambda} \\ \gamma) D_u v_{\cdot\cdot\mu}^{ax} = B_u^a C_\alpha^x V_v v_{\cdot\cdot\mu}^{x\lambda} \end{cases} \end{array} \right.$$

From this we see that the following rules hold:

1. With respect to the index of differentiation,  $D_\mu$  corresponds to  $V_\mu$ ;  $D_c$  to  $B_c^\mu V_\mu$  and  $D_z$  to  $C_z^\mu V_\mu$ ;
2. The only possible other indices that can occur in the derivative are those that appear in *possible* components of the quantity to be differentiated. For instance  $D_v v_{\cdot\cdot\mu}^{x\lambda}$  or  $D_v v_{\cdot\cdot z}^{x\lambda}$  can not appear because there are no full sets of components  $v_{\cdot\cdot\mu}^{x\lambda}$  or  $v_{\cdot\cdot z}^{x\lambda}$ .
3. For an index  $a, b, \dots$  or  $x, y, \dots$  in the quantity to be differentiated a transvection with  $B$  or  $C$  respectively occurs and for an index  $\alpha, \lambda, \dots$  there is no such transvection.
4. In order to get the meaning of a formula with  $V$  or  $\delta$  we have only to look at the *skeleton* (cf. I § 11). But the meaning of a formula with  $D$  depends also on the kinds of indices used. For instance  $V_v v_{\cdot\cdot\mu}^{x\lambda}$ ,  $V_k v_{\cdot\cdot j}^{h\lambda}$ ,  $V_\mu v_{\cdot\cdot j}^{h\lambda}$  and  $V_k v_{\cdot\cdot j}^{x\lambda}$  are all components of the same quantity, the covariant derivative of  $v_{\cdot\cdot\mu}^{x\lambda}$ . But  $D_v v_{\cdot\cdot\mu}^{x\lambda}$ ,  $D_d v_{\cdot\cdot\mu}^{x\lambda}$ ,  $D_u v_{\cdot\cdot\mu}^{x\lambda}$  and  $D_u v_{\cdot\cdot\mu}^{ax}$  are four different quantities.

In (7.4a  $\beta, \gamma$ ) and (7.4b  $\beta, \gamma$ ) the covariant derivatives  $V_\mu p^x$  and  $V_\mu q_\lambda$  are split up into parts that may be considered as a kind of covariant derivatives themselves. The same holds for  $V_\mu r^x$  and  $V_\mu s_\lambda$  in (7.6a  $\beta, \gamma$ ) and (7.6b  $\beta, \gamma$ ). The (two) parts of  $V_\mu p^x$ ,  $V_\mu q_\lambda$ ,  $V_\mu r^x$  and  $V_\mu s_\lambda$  not occurring in these formulae are no derivatives of these fields, because for instance  $C_\alpha^x V_\mu p^a = -p^a V_\mu C_\alpha^x$ . This leads to the investigation of

<sup>1)</sup> By a *full set of components* we mean a set that fixes the quantity completely. For instance the components  $v_{\cdot\cdot c}^{x\lambda}$  exist but they fix the quantity  $v_{\cdot\cdot\mu}^{x\lambda}$  only if combined with the  $v_{\cdot\cdot z}^{x\lambda}$ .

the derivative  $\nabla_\mu B_\lambda^x = -\nabla_\mu C_\lambda^x$ . If we introduce the quantities<sup>1)</sup>

$$(7.9) \quad \overset{m}{H}_{\mu\lambda}^{::x} \stackrel{\text{def}}{=} B_{\mu\lambda}^{\tau\sigma} \nabla_\tau B_\sigma^x = -B_{\mu\lambda}^{\tau\sigma} \nabla_\tau C_\sigma^x = B_\mu^\tau C_\sigma^x \nabla_\tau B_\lambda^\sigma = -B_\mu^\tau C_\sigma^x \nabla_\tau C_\lambda^\sigma;$$

$$(7.10) \quad \overset{m}{L}_{\mu\cdot\lambda}^{::x} \stackrel{\text{def}}{=} B_{\mu\cdot\lambda}^{\tau x} \nabla_\tau B_\lambda^0 = -B_{\mu\cdot\lambda}^{\tau x} \nabla_\tau C_\lambda^0 = B_\mu^\tau C_\lambda^0 \nabla_\tau B_\lambda^x = -B_\mu^\tau C_\lambda^0 \nabla_\tau C_\lambda^x;$$

$$(7.11) \quad \overset{m'}{H}_{\mu\cdot\lambda}^{::x} \stackrel{\text{def}}{=} C_{\mu\lambda}^{\tau\sigma} \nabla_\tau C_\sigma^x = -C_{\mu\lambda}^{\tau\sigma} \nabla_\tau B_\sigma^x = C_\mu^\tau B_\sigma^x \nabla_\tau C_\lambda^\sigma = -C_\mu^\tau B_\sigma^x \nabla_\tau B_\lambda^\sigma;$$

$$(7.12) \quad \overset{m'}{L}_{\mu\cdot\lambda}^{::x} \stackrel{\text{def}}{=} C_{\mu\cdot\lambda}^{\tau x} \nabla_\tau C_\lambda^0 = -C_{\mu\cdot\lambda}^{\tau x} \nabla_\tau B_\lambda^0 = C_\mu^\tau B_\lambda^0 \nabla_\tau C_\lambda^x = -C_\mu^\tau B_\lambda^0 \nabla_\tau B_\lambda^x;$$

we see that each of these tensors lies with one index or two indices in  $X_n^m$  and with the remaining index or indices in  $X_n^{m'}$ . The full sets of components

$$(7.13) \quad \overset{m}{H}_{cb}^{::x}; \quad \overset{m}{L}_{c\cdot y}^{::a}; \quad \overset{m'}{H}_{z\cdot y}^{::a}; \quad \overset{m'}{L}_{z\cdot b}^{::x}$$

exist; this follows from (7.9–12). From these full sets we can see which of the indices lie in  $X_n^m$  and which in  $X_n^{m'}$ . The covariant derivative  $\nabla_\mu B_\lambda^x$  can be expressed in terms of these four quantities

$$(7.14) \quad \nabla_\mu B_\lambda^x = \overset{m}{H}_{\mu\lambda}^{::x} + \overset{m}{L}_{\mu\cdot\lambda}^{::x} - \overset{m'}{H}_{\mu\lambda}^{::x} - \overset{m'}{L}_{\mu\cdot\lambda}^{::x}.$$

The relationship  $A_\lambda^x = B_\lambda^x + C_\lambda^x$  shows that a quantity with valence 3 in general splits up into eight parts, but here four of these parts are zero, for instance  $B_{\mu\lambda\eta}^{\sigma\alpha} \nabla_\tau B_\sigma^0$ .

The quantities  $\overset{m}{H}_{\mu\lambda}^{::x}$  and  $\overset{m}{L}_{\mu\cdot\lambda}^{::x}$  are called the *first* and *second curvature tensor of valence 3 of  $X_n^m$* .<sup>2)</sup> In the same way to  $X_n^{m'}$  belong  $\overset{m'}{H}_{\mu\lambda}^{::x}$  and  $\overset{m'}{L}_{\mu\cdot\lambda}^{::x}$ . The following formulae are useful

$$(7.15) \quad \text{a) } \overset{m}{H}_{cb}^{::x} = C_x^y D_c B_b^x = -B_b^x D_c C_x^y; \quad \text{b) } \overset{m}{H}_{c\cdot b}^{::x} = D_c B_b^x; \quad \text{c) } \overset{m}{H}_{c\cdot\lambda}^{::x} = -D_c C_\lambda^x$$

$$(7.16) \quad \text{a) } \overset{m}{L}_{c\cdot y}^{::a} = C_y^\lambda D_c B_\lambda^a = -B_\lambda^a D_c C_y^\lambda; \quad \text{b) } \overset{m}{L}_{c\cdot\lambda}^{::a} = D_c B_\lambda^a; \quad \text{c) } \overset{m}{L}_{c\cdot y}^{::a} = -D_c C_y^a$$

$$(7.17) \quad \text{a) } \overset{m'}{H}_{z\cdot y}^{::a} = B_\lambda^a D_z C_y^a = -C_y^a D_z B_\lambda^a; \quad \text{b) } \overset{m'}{H}_{z\cdot y}^{::x} = D_z C_y^x; \quad \text{c) } \overset{m'}{H}_{z\cdot\lambda}^{::a} = -D_z B_\lambda^a$$

$$(7.18) \quad \text{a) } \overset{m'}{L}_{z\cdot b}^{::x} = B_b^\lambda D_z C_\lambda^x = -C_\lambda^x D_z B_b^\lambda; \quad \text{b) } \overset{m'}{L}_{z\cdot\lambda}^{::x} = D_z C_\lambda^x; \quad \text{c) } \overset{m'}{L}_{z\cdot b}^{::x} = -D_z B_\lambda^x.$$

<sup>1)</sup> SCHOUTEN 1923, 1 for  $X_m$  in  $A_n$ .

<sup>2)</sup> For  $V_n$  the curvature tensor of valence 3 (cf. V § 9) was introduced by SCHOUTEN and STRUIK 1921, 3 and at the same time by BOMPIANI 1921, 1. Its components correspond to the  $Q_{r\cdot ij}$  of VOSS 1880, 1, to the  $b_{\alpha\mid rs}$  in RICCI 1903, 1 and to the  $b_{fg}^{(r)}$  in KÜHNE 1904, 1. Cf. SCHOUTEN and STRUIK 1922, 1; 2; STRUIK 1922, 1; BOMPIANI 1951, 3. Some authors call it the "EULER-SCHOUTEN" tensor which is not correct from a historical point of view.

Using the anholonomic coordinate system ( $h$ ) in  $L_n$  we have the following formulae for the contravariant vectors  $p^x$  and  $r^x$

$$(7.19) \quad \begin{cases} \text{a)} & D_c p^a \stackrel{*}{=} \partial_c p^a + \Gamma_{cb}^a p^b; \quad \partial_c \stackrel{\text{def}}{=} B_c^\mu \partial_\mu \\ \text{b)} & D_z p^a \stackrel{*}{=} \partial_z p^a + \Gamma_{zb}^a p^b; \quad \partial_z \stackrel{\text{def}}{=} C_z^\mu \partial_\mu \\ \text{c)} & D_c r^x \stackrel{*}{=} \partial_c r^x + \Gamma_{cy}^x r^y \\ \text{d)} & D_z r^x \stackrel{*}{=} \partial_z r^x + \Gamma_{zy}^x r^y \end{cases}$$

and corresponding formulae for the covariant vectors  $q_\lambda$ ,  $s_\lambda$ . These equations fix a linear connexion for quantities of  $X_n^m$  and also a linear connexion for quantities of  $X_n^{m'}$ . We express this by writing  $L_n^m$ ,  $L_n^{m'}$  for  $X_n^m$ ,  $X_n^{m'}$  from now on, and also  $A_n^m$ ,  $A_n^{m'}$  if the  $L_n$  happens to be an  $A_n$ . In  $L_n$  we know that (cf. III 9.3)

$$(7.20) \quad \Gamma_{[j i]}^h = S_{j i}^h - Q_{j i}^h,$$

hence the expressions

$$(7.21) \quad \Gamma_{[c b]}^a \stackrel{*}{=} S_{c b}^a - Q_{c b}^a; \quad \Gamma_{[z y]}^x \stackrel{*}{=} S_{z y}^x - Q_{z y}^x$$

depend only on the rigging and on  $S_{\mu \lambda}^x$  and not on the symmetric part  $\Gamma_{(\mu \lambda)}^x$  of the connexion in  $L_n$ . In an  $A_n$  they depend only on the rigging and not on the connexion in  $A_n$ .

From (7.15 b) we get

$$(7.22) \quad \begin{cases} \tilde{H}_{[c b]}^x = -B_{[c b]}^{\mu \lambda} (V_\mu C_\lambda^x) C_x^z = -B_{[c b]}^{\mu \lambda} V_\mu (\overset{\circ}{e}_\lambda e^x) C_x^z \\ = -B_{c b}^{\mu \lambda} (V_{[\mu} \overset{\circ}{e}_{\lambda]})_y e^x = -B_{c b}^{\mu \lambda} (\partial_{[\mu} \overset{\circ}{e}_{\lambda]} - S_{\mu \lambda}^x \overset{\circ}{e}_x)_y e^x \\ = B_{c b}^{j i} C_h^x (-Q_{j i}^h + S_{j i}^h) \\ = -\tilde{Z}_{c b}^x + B_{c b}^{j i} C_h^x S_{j i}^h \stackrel{*}{=} \Gamma_{[c b]}^x; \quad \tilde{Z}_{c b}^x \stackrel{\text{def}}{=} B_{c b}^{j i} C_h^x Q_{j i}^h \stackrel{*}{=} Q_{c b}^x; \end{cases}$$

and this implies that  $\tilde{Z}_{c b}^x$  is a tensor belonging to the given  $L_n^m$  and that it lies with two indices in  $L_n^m$  and with one index in  $L_n^{m'}$ . In II § 9 we have already proved that the  $X_n^m$  is holonomic (that is  $X_m$ -forming) if and only if  $\tilde{Z}_{c b}^x = 0$ . This n.a.s. condition can now be expressed in the form

$$(7.23) \quad \tilde{H}_{[c b]}^x - B_{c b}^{j i} C_h^x S_{j i}^h = 0.$$

In the same way the tensor  $\tilde{Z}_{z y}^{m'} \stackrel{*}{=} \tilde{H}_{[z y]}^{m'}$  belongs to  $L_n^{m'}$  and this  $L_n^{m'}$  is holonomic if and only if

$$(7.24) \quad -\tilde{Z}_{z y}^{m'} \stackrel{\text{def}}{=} \tilde{H}_{[z y]}^{m'} - C_{z y}^{j i} B_h^a S_{j i}^h = 0.$$

In an  $A_n$  the two quantities  $\tilde{Z}_{c b}^x$  and  $\tilde{Z}_{z y}^{m'}$  are identical with  $-\tilde{H}_{[c b]}^x$  and  $-\tilde{H}_{[z y]}^{m'}$ .

Besides the tensors  $\overset{m}{Z}_{cb}^x$  and  $\overset{m'}{Z}_{zy}^a$  four geometric objects can be formed from  $\Omega_{ci}^h$ , viz.  $\overset{*}{\Omega}_{cb}^a \stackrel{\text{def}}{=} \Omega_{cb}^a$  with the transformation (cf. II 9.3)

$$(7.25) \quad \overset{*}{\Omega}_{c'b'}^{a'} \stackrel{*}{=} A_{c'b'a}^{c'b'a'} \Omega_{cb}^a + A_{c'b'}^{c'b} \partial_{[c} A_{b]}^{a'}$$

and  $\overset{*}{\Omega}_{cy}^a \stackrel{\text{def}}{=} \Omega_{cy}^a$  with the transformation

$$(7.26) \quad \overset{*}{\Omega}_{c'y'}^{a'} \stackrel{*}{=} A_{c'y'a}^{c'y'a'} \Omega_{cy}^a - \frac{1}{2} A_{c'y'}^{c'y} \partial_y A_c^{a'}$$

and the two corresponding objects  $\overset{*}{\Omega}_{zy}^x$  and  $\overset{*}{\Omega}_{zb}^x$ . The reason why  $\Omega_{ci}^h$  splits up into six separate geometric objects is of course that here only transformations of (h) are allowed with  $A_y^{a'} = 0$ ;  $A_y^a = 0$ ;  $A_b^{x'} = 0$ ;  $A_b^x = 0$ . From now on we write every one of these separate objects with its own kernel letter.

If only one  $X_m$  is given in  $L_n$ , only the operator  $D_c$  is important as long as only fields in  $X_m$  are considered. But the other operators can be used also if the fields are prolonged in the neighbourhoods of all points of the  $X_m$ . This is often convenient and the results as far as fields of  $X_m$  are concerned are independent of the manner of prolongation (cf. II § 4). The vectors  $e_a^x$  may be chosen in such a way that they are basis vectors of some holonomic coordinate system (a) in  $X_m$ . But this is not necessary and it should only be done if it is convenient for some purpose.

The equations concerning the curvature of the  $L_n^m$  in  $L_n$  can now be obtained in a very simple way using the operators  $D$ . We have first to compute the alternated second derivatives of quantities of  $L_n$ ,  $L_n^m$  and  $L_n^{m'}$ :

$$(7.27A) \quad \left\{ \begin{array}{l} \text{a)} \quad D_{[\nu} D_{\mu]} \phi = - S_{\nu\mu}^{..o} \partial_\rho \phi \\ \text{b)} \quad D_{[\nu} D_{\mu]} v^\kappa = \frac{1}{2} R_{\nu\mu\lambda}^{..i} v^\lambda - S_{\nu\mu}^{..o} V_\rho v^\kappa \\ \text{c)} \quad D_{[\nu} D_{\mu]} w_\lambda = - \frac{1}{2} R_{\nu\mu}^{..i} w_\kappa - S_{\nu\mu}^{..o} V_\rho w_\lambda \\ \text{d)} \quad D_{[\nu} D_{\mu]} \phi^a = \left( \frac{1}{2} B_{b\kappa}^{\lambda a} R_{\nu\mu}^{..i} \lambda^x + L_{[\nu..|x]}^{..a} H_{\mu]b}^{..m} - L_{[\nu..|x]}^{..a} L_{\mu].b}^{..m'} - H_{[\nu..|x]}^{..a} H_{\mu]b}^{..m} + H_{[\nu..|x]}^{..a} L_{\mu].b}^{..m'} \right) \phi^b - S_{\nu\mu}^{..o} D_\rho \phi^a \\ \text{e)} \quad D_{[\nu} D_{\mu]} q_b = - \left( \frac{1}{2} B_{b\kappa}^{\lambda a} R_{\nu\mu}^{..i} \lambda^x + L_{[\nu..|x]}^{..a} H_{\mu]b}^{..m} - L_{[\nu..|x]}^{..a} L_{\mu].b}^{..m'} - H_{[\nu..|x]}^{..a} H_{\mu]b}^{..m} + H_{[\nu..|x]}^{..a} L_{\mu].b}^{..m'} \right) q_a - S_{\nu\mu}^{..o} D_\rho q_b \\ \text{f)} \quad D_{[\nu} D_{\mu]} r^x = \left( \frac{1}{2} C_{y\kappa}^{\lambda x} R_{\nu\mu}^{..i} \lambda^x - L_{[\nu..|y]}^{..a} H_{\mu]a}^{..m} + L_{[\nu..|y]}^{..a} L_{\mu].a}^{..m'} + H_{[\nu..|y]}^{..a} H_{\mu]a}^{..m} - H_{[\nu..|y]}^{..a} L_{\mu].a}^{..m'} \right) r^y - S_{\nu\mu}^{..o} D_\rho r^x \\ \text{g)} \quad D_{[\nu} D_{\mu]} s_y = - \left( \frac{1}{2} C_{y\kappa}^{\lambda x} R_{\nu\mu}^{..i} \lambda^x - L_{[\nu..|y]}^{..a} H_{\mu]a}^{..m} + L_{[\nu..|y]}^{..a} L_{\mu].a}^{..m'} + H_{[\nu..|y]}^{..a} H_{\mu]a}^{..m} - H_{[\nu..|y]}^{..a} L_{\mu].a}^{..m'} \right) s_x - S_{\nu\mu}^{..o} D_\rho s_y. \end{array} \right.$$

$$(7.27B) \left\{ \begin{array}{l} \text{a)} D_{[d} D_{c]} \dot{\rho} = - (B_{dc}^{\nu\mu} S_{\nu\mu}^{;\lambda} - H_{[dc]}^{;\lambda}) \partial_\lambda \dot{\rho} \\ \text{b)} D_{[d} D_{c]} v^\lambda = \frac{1}{2} B_{dc}^{\nu\mu} R_{\nu\mu}^{;\lambda} v^\lambda - (B_{dc}^{\nu\mu} S_{\nu\mu}^{;\varrho} - H_{[dc]}^{;\varrho}) V_\varrho v^\lambda \\ \text{c)} D_{[d} D_{c]} w_\lambda = - \frac{1}{2} B_{dc}^{\nu\mu} R_{\nu\mu}^{;\lambda} w_\lambda - (B_{dc}^{\nu\mu} S_{\nu\mu}^{;\varrho} - H_{[dc]}^{;\varrho}) V_\varrho w_\lambda \\ \text{d)} D_{[d} D_{c]} \dot{\rho}^a = (\frac{1}{2} B_{dc}^{\mu\lambda a} R_{\nu\mu}^{;\lambda} - H_{[dc]}^{;\lambda} L_{c].x}^a) \dot{\rho}^b - \\ \quad - (B_{dc}^{\nu\mu} S_{\nu\mu}^{;\varrho} - H_{[dc]}^{;\varrho}) D_\varrho \dot{\rho}^a \\ \text{e)} D_{[d} D_{c]} q_b = - (\frac{1}{2} B_{dc}^{\mu\lambda a} R_{\nu\mu}^{;\lambda} - H_{[dc]}^{;\lambda} L_{c].x}^a) q_a - \\ \quad - (B_{dc}^{\nu\mu} S_{\nu\mu}^{;\varrho} - H_{[dc]}^{;\varrho}) D_\varrho q_b \\ \text{f)} D_{[d} D_{c]} r^\lambda = (\frac{1}{2} B_{dc}^{\nu\mu} C_{y\lambda}^{\lambda x} R_{\nu\mu}^{;\lambda} + H_{[dc]}^{;\lambda} L_{c].y}^x) r^y - \\ \quad - (B_{dc}^{\nu\mu} S_{\nu\mu}^{;\varrho} - H_{[dc]}^{;\varrho}) D_\varrho r^\lambda \\ \text{g)} D_{[d} D_{c]} s_y = - (\frac{1}{2} B_{dc}^{\nu\mu} C_{y\lambda}^{\lambda x} R_{\nu\mu}^{;\lambda} + H_{[dc]}^{;\lambda} L_{c].y}^x) s_x - \\ \quad - (B_{dc}^{\nu\mu} S_{\nu\mu}^{;\varrho} - H_{[dc]}^{;\varrho}) D_\varrho s_y. \end{array} \right.$$
  

$$(7.27C) \left\{ \begin{array}{l} \text{a)} D_{[u} D_{z]} \dot{\rho} = - (C_{uz}^{\nu\mu} S_{\nu\mu}^{;\lambda} - H_{[uz]}^{;\lambda}) \partial_\lambda \dot{\rho} \\ \text{b)} D_{[u} D_{z]} v^\lambda = \frac{1}{2} C_{uz}^{\nu\mu} R_{\nu\mu}^{;\lambda} v^\lambda - (C_{uz}^{\nu\mu} S_{\nu\mu}^{;\varrho} - H_{[uz]}^{;\varrho}) V_\varrho v^\lambda \\ \text{c)} D_{[u} D_{z]} w_\lambda = - \frac{1}{2} C_{uz}^{\nu\mu} R_{\nu\mu}^{;\lambda} w_\lambda - (C_{uz}^{\nu\mu} S_{\nu\mu}^{;\varrho} - H_{[uz]}^{;\varrho}) V_\varrho w_\lambda \\ \text{d)} D_{[u} D_{z]} \dot{\rho}^a = (\frac{1}{2} C_{uz}^{\nu\mu} B_{bx}^{\lambda a} R_{\nu\mu}^{;\lambda} + H_{[u|x}^{\lambda a} L_{z].b}^x) \dot{\rho}^b - \\ \quad - (C_{uz}^{\nu\mu} S_{\nu\mu}^{;\varrho} - H_{[uz]}^{;\varrho}) D_\varrho \dot{\rho}^a \\ \text{e)} D_{[u} D_{z]} q_b = - (\frac{1}{2} C_{uz}^{\nu\mu} B_{bx}^{\lambda a} R_{\nu\mu}^{;\lambda} + H_{[u|x}^{\lambda a} L_{z].b}^x) q_a - \\ \quad - (C_{uz}^{\nu\mu} S_{\nu\mu}^{;\varrho} - H_{[uz]}^{;\varrho}) D_\varrho q_b \\ \text{f)} D_{[u} D_{z]} r^\lambda = (\frac{1}{2} C_{uz}^{\nu\mu} \lambda_{yx} R_{\nu\mu}^{;\lambda} - H_{[u|y}^{\lambda a} L_{z].a}^x) r^y - \\ \quad - (C_{uz}^{\nu\mu} S_{\nu\mu}^{;\varrho} - H_{[uz]}^{;\varrho}) D_\varrho r^\lambda \\ \text{g)} D_{[u} D_{z]} s_y = - (\frac{1}{2} C_{uz}^{\nu\mu} \lambda_{yx} R_{\nu\mu}^{;\lambda} - H_{[u|y}^{\lambda a} L_{z].a}^x) s_x - \\ \quad - (C_{uz}^{\nu\mu} S_{\nu\mu}^{;\varrho} - H_{[uz]}^{;\varrho}) D_\varrho s_y. \end{array} \right.$$

Using the anholonomic system (h) and the equations (7.20) and

$$(7.28) \quad \partial_{[j} \partial_{i]} \dot{\rho} = - \Omega_{ji}^h \partial_h \dot{\rho} \quad (\text{cf. II 9.4})$$

we find another expression for  $D_{[d} D_{c]} \dot{\rho}^a$

$$(7.29) \left\{ \begin{array}{l} D_{[d} D_{c]} \dot{\rho}^a = H_{[dc]}^{;\lambda} D_\lambda \dot{\rho}^a + \\ \quad + \{\partial_{[d} \Gamma_{c]b}^a + \Gamma_{[d|e}^a \Gamma_{c]b}^e + \Omega_{dc}^i \Gamma_{ib}^a\} \dot{\rho}^b - S_{\mu\lambda}^{;\varrho} B_{d\lambda}^{\mu\lambda} D_\varrho \dot{\rho}^a. \end{array} \right.$$

Hence, if we write

$$(7.30) \quad \overset{m}{r}_{dc}{}^a \stackrel{\text{def}}{=} 2\partial_{[d}\Gamma_{c]b}^a + 2\Gamma_{[d|e]}^a\Gamma_{c]b}^e + 2\Omega_{dc}^j\Gamma_{jb}^a$$

it follows from (7.29) that

$$(7.31) \quad 2D_{[d}D_{c]}p^a = \overset{m}{r}_{dc}{}^a p^b - 2Z_{dc}^m D_y p^a - 2S_{dc}^e D_e p^a$$

and from (7.27 B d) that

$$(7.32) \quad \boxed{\overset{m}{r}_{dc}{}^a = B_{dc}^{\nu\mu\lambda} R_{\nu\mu\lambda}^a - 2H_{[d|b|}^{\dot{x}} L_{c]\cdot x}^{\dot{a}}.}$$

In the same way we get for  $L_n^{m'}$

$$(7.33) \quad 2D_{[u}D_{z]}r^x = \overset{m'}{r}_{uz}{}^x r^y - 2Z_{uz}^{m'} D_a r^x - 2S_{uz}^y D_y r^x$$

and

$$(7.34) \quad \boxed{\overset{m'}{r}_{uz}{}^x = C_{uzy}^{\nu\mu\lambda} R_{\nu\mu\lambda}^x - 2H_{[u|y|}^{\dot{a}} L_{z]\cdot a}^{\dot{x}}.}$$

(7.32) and (7.34) are a kind of generalized equations of GAUSS for  $L_n^m$  and  $L_n^{m'}$  in  $L_n$ . But  $\overset{m}{r}$  and  $\overset{m'}{r}$  can not be considered as the true curvature tensors of the induced connexion in  $L_n^m$  and  $L_n^{m'}$ . In fact there is something queer in (7.31). The left hand side depends only on the rigging and the  $\Gamma_{cb}^a$ , that is on the displacement of vectors of the  $L_n^m$  in directions of the  $L_n^m$ . But none of the terms of the right hand side has this property. Now because of (7.26) the equations

$$(7.35)^1) \quad \begin{cases} \text{a)} & \mathfrak{D}_z p^a \stackrel{\text{def}}{=} \partial_z p^a + 2\overset{m}{\Omega}_{bz}^a p^b \\ \text{b)} & \mathfrak{D}_z q_b \stackrel{\text{def}}{=} \partial_z q_b - 2\overset{m}{\Omega}_{bz}^a q_a \end{cases}$$

define a covariant derivative of  $p^a$  and  $q_b$  depending only on the rigging of the  $L_n^m$  and not on any connexion in  $L_n$ . As is to be expected this derivative is intimately connected with the LIE derivative. In fact, if  $v^x$  is a vector field of  $L_n^{m'}$  it is easy to prove that

$$(7.36) \quad v^y \mathfrak{D}_y p^a = B_x^a \underset{v}{\xi} p^x. {}^2)$$

This means that the covariant differential  $(d\xi)^y \mathfrak{D}_y p^a$  for a direction lying in  $L_n^{m'}$  is zero if the  $L_n^m$ -part of the LIE differential vanishes or in other words if the field  $p^x$  dragged along over  $d\xi^x$  differs from the original value at  $\xi^x + d\xi^x$  only by a vector in  $L_n^{m'}$ .

<sup>1)</sup> For (7.35c, d) see p. 261.

<sup>2)</sup> Cf. VII § 3.

An operator  $\mathfrak{D}_b$  operating on  $r^x$  and  $s_y$  may be defined in the same way:

$$(7.35) \quad \begin{cases} \text{c)} & \mathfrak{D}_c r^x = \partial_c r^x + 2 \overset{m'}{\Omega}_{yc}^x r^y \\ \text{d)} & \mathfrak{D}_c s_y = \partial_c s_y - 2 \overset{m'}{\Omega}_{yc}^x s_x. \end{cases}$$

Note that the operation of  $\mathfrak{D}_y$  on quantities of  $L_n^{m'}$  and of  $\mathfrak{D}_b$  on quantities of  $L_n^m$  is not yet defined. The relation between  $\mathfrak{D}_y$  and  $D_y$  is the identity

$$(7.37) \quad \mathfrak{D}_z p^a = D_z p^a + \overset{m}{L}_{cz}^a p^c + 2 S_{cz}^a p^c$$

and if this is substituted in (7.31) we get

$$(7.38) \quad 2 D_{[d} D_{c]} p^a = - 2 \overset{m}{Z}_{dc}^y \mathfrak{D}_y p^a - 2 S_{dc}^e D_e p^a + \overset{m}{R}_{dc}{}^b{}^a p^b$$

where

$$(7.39) \quad \overset{m}{R}_{dc}{}^b{}^a \stackrel{\text{def}}{=} \overset{m}{r}_{dc}{}^b{}^a + 2 \overset{m}{Z}_{dc}^y (L_{b.y}^a + 2 S_{b.y}^a)$$

is a quantity depending only on the displacement of vectors of  $L_n^m$  in directions of  $L_n^m$  and on the rigging and not extra on any connexion in  $L_n$ . This quantity is the *true* curvature tensor of the induced connexion in the  $L_n^m$  and there is just such a quantity  $\overset{m'}{R}_{uz}{}^x$  for the  $L_n^{m'}$ .<sup>1)</sup> If the  $L_n^m$  is holonomic,  $\overset{m}{R}$  is identical with  $\overset{m}{r}$  as follows from (7.39). The *true* generalizations of the equations of GAUSS for  $L_n^m$  and  $L_n^{m'}$  in  $L_n$  are (cf. 3.15)

$$(7.40) \quad \boxed{\begin{aligned} \overset{m}{R}_{dc}{}^b{}^a &= B_{dc}^{\nu\mu\lambda a} R_{\nu\mu\lambda}{}^x - 2 H_{[d|\dot{b}|}^{\dot{x}} L_{c].x}^a + \\ &\quad + 2 Z_{dc}^y (L_{b.y}^a + 2 S_{b.y}^a) \end{aligned}} \quad (\text{GAUSS})$$

$$(7.41) \quad \boxed{\begin{aligned} \overset{m'}{R}_{uz}{}^x &= C_{uzy\kappa}^{\nu\mu\lambda x} R_{\nu\mu\lambda}{}^x - 2 H_{[\dot{u}|\dot{y}|}^{\dot{x}} L_{z].a}^a + \\ &\quad + 2 Z_{uz}^b (L_{y.b}^x + 2 S_{y.b}^x) \end{aligned}} \quad (\text{GAUSS}).$$

The following expression can be deduced for  $D_{[d} D_{c]} r^x$

$$(7.42) \quad 2 D_{[d} D_{c]} r^x = \overset{m'm'}{R}_{dc}{}^y{}^x r^y - 2 \overset{m}{Z}_{dc}^y D_y r^x - 2 S_{dc}^e D_e r^x$$

where

$$(7.43) \quad \boxed{R_{dc}{}^y{}^x \stackrel{\text{def}}{=} B_{dc}^{\nu\mu} C_{y\kappa}^{\lambda x} R_{\nu\mu\lambda}{}^x + 2 H_{[d|\dot{b}|}^{\dot{x}} L_{c].y}^b} \quad (\text{RICCI})^2)$$

<sup>1)</sup> SCHOUTEN 1929, 3, p. 163f. also for historical notes; SCHOUTEN and v. KAMPEN 1930, 2, p. 777; E II 1938, 2, p. 161. The case of  $V_n^m$  in  $V_n$  was treated earlier by VRANCEANU 1926, 1; 2; 1928, 1; 3.

<sup>2)</sup> For  $V_m$  in  $V_n$  FABRICIUS-BJERRE 1934, 1; 1936, 1 called this tensor the  $\overset{m'm'}{R}_{dc}{}^y{}^x$  "torsion" of the  $V_m$  and discussed the case  $R_{dc}{}^y{}^x = 0$ .

and similarly

$$(7.44) \quad 2D_{[u} D_{z]} p^a = R_{uzb}^{m'm} p^b - 2Z_{uz}^{m'} D_b p^a - 2S_{uz}^{m'} D_y p^a$$

where

$$(7.45) \quad \boxed{R_{uzb}^{m'm} \stackrel{\text{def}}{=} C_{uz}^{\nu\mu} B_{\nu\lambda}^{\lambda a} R_{\nu\mu\lambda}^{m'x} + 2H_{[u]y}^{m'} L_{z]}^y} \quad (\text{RICCI}).$$

(7.43, 45) are the generalized equations of RICCI for  $L_n^m$  and  $L_n^{m'}$  in  $L_n$ .

It is easily proved that the rule of LEIBNIZ holds for the operators  $D_{[d} D_{c]}$  and  $D_{[u} D_{z]}$  (cf. III § 4). Hence for a product  $v^\kappa q_b$  we get from (7.27B b, e)

$$(7.46) \quad \left\{ \begin{array}{l} D_{[d} D_{c]} v^\kappa q_b = -\frac{1}{2} r_{dc}^{m'} v^\kappa q_a + \frac{1}{2} B_{dc}^{\nu\mu} R_{\nu\mu\lambda}^{m'x} v^\lambda q_b + \\ + H_{[dc]}^m D_y v^\kappa q_b - B_{dc}^{\nu\mu} S_{\nu\mu}^{m'} D_\varrho v^\kappa q_b \end{array} \right.$$

and thus

$$(7.47) \quad \left\{ \begin{array}{l} D_{[d} H_{c]}^m v^\kappa = D_{[d} D_{c]} B_b^\kappa = -\frac{1}{2} r_{dc}^{m'} B_a^\kappa + \frac{1}{2} B_{dc}^{\nu\mu} R_{\nu\mu\lambda}^{m'x} B_b^\lambda + \\ + H_{[dc]}^m D_y B_b^\kappa - B_{dc}^{\nu\mu} S_{\nu\mu}^{m'} D_\varrho B_b^\kappa. \end{array} \right.$$

Transvection of this equation with  $B_x^a$  leads back to (7.32) but a new identity arises from transvection with  $C_x^x$  (cf. 3.19):

$$(7.48) \quad \boxed{2D_{[d} H_{c]}^m v^\kappa = B_{dc}^{\mu\lambda} C_x^\nu R_{\nu\mu\lambda}^{m'x} - 2H_{[dc]}^m L_y^x - 2S_{dc}^{m'} H_{ab}^{m'x} + 2S_{\nu\mu}^{m'} B_{dc}^{\nu\mu} C_\varrho^x L_y^x} \quad (\text{CODAZZI})$$

and in the same way we get for  $L_n^{m'}$ :

$$(7.49) \quad \boxed{2D_{[u} H_{z]}^{m'} v^a = C_{uzy}^{\nu\mu\lambda} B_\nu^a R_{\nu\mu\lambda}^{m'x} - 2H_{[uz]}^{m'} L_b^a - 2S_{uz}^{m'} H_{xy}^{m'a} + 2S_{\nu\mu}^{m'} C_{uz}^{\nu\mu} B_\varrho^a L_b^a} \quad (\text{CODAZZI}).$$

Differentiation of  $L_c^{m'\lambda}$  gives, using (7.27A c, 27B c, d)

$$(7.50) \quad \left\{ \begin{array}{l} D_{[d} L_{c]}^{m'} = D_{[d} D_{c]} B_\lambda^a = -\frac{1}{2} r_{dc}^{m'} B_\lambda^b - \frac{1}{2} B_{dc}^{\nu\mu} R_{\nu\mu\lambda}^{m'x} B_\lambda^a + \\ + H_{[dc]}^{m'} D_x B_\lambda^a - B_{dc}^{\nu\mu} S_{\nu\mu}^{m'} D_\varrho B_\lambda^a. \end{array} \right.$$

Transvection with  $B_b^\lambda$  leads back to (7.32) but by transvection with  $C_y^\lambda$  we get the new identity (cf. 3.20)

$$(7.51) \quad \boxed{2D_{[d} L_{c]}^{m'} = -B_{dc}^{\mu\alpha} C_\varrho^\lambda R_{\nu\mu\lambda}^{m'x} + 2Z_{dc}^{m'} H_{xy}^{m'a} - 2S_{dc}^{m'} L_b^a} \quad (\text{CODAZZI})$$

and in the same way for  $L_n^{m'}$

$$(7.52) \quad 2D_{[u} L_{z]}^{m'} = -C_{uzx}^{\nu\mu x} B_b^\lambda R_{\nu\mu\lambda}^{z''} + 2Z_{uz}^{m'} H_{ab}^m - 2S_{uz}^{y'} L_y^{m'} \quad (\text{CO-DAZZI}).$$

(7.48, 49, 51, 52) are the generalized equations of CODAZZI for  $L_n^m$  and  $L_n^{m'}$  in  $L_n$ .<sup>1)</sup>

A curve of  $L_n^m$ , that is a curve whose direction at every point lies in  $L_n^m$ , is called a geodesic of  $L_n^m$  if the covariant differential of its tangent vector  $d\xi^x/dt$  along the curve for the in  $L_n^m$  induced connexion has the direction of the tangent. Accordingly the n.a.s. condition that every geodesic of  $L_n^m$  is at the same time a geodesic of  $L_n$  is that the vector

$$(7.53) \quad \frac{d\xi^\mu}{dt} \left( V_\mu \frac{d\xi^x}{dt} - V_\mu \frac{d\xi^x}{dt} \right) = \frac{d\xi^\mu}{dt} \left( V_\mu \frac{d\xi^\lambda}{dt} \right) C_\lambda^x = \frac{d\xi^\mu}{dt} \frac{d\xi^\lambda}{dt} H_{\mu\lambda}^m$$

has the direction of  $d\xi^x/dt$  for every choice of  $d\xi^x/dt$  in the tangent  $E_m$ .

But from the definition of  $H_{\mu\lambda}^m$  we see that this is only possible if  $H_{(\mu\lambda)}^m = 0$ . If  $H_{(\mu\lambda)}^m = 0$  at a point, the  $L_n^m$  is said to be *geodesic at this point*. It is called *geodesic* if it is geodesic at all points of  $L_n$ . A geodesic  $A_m$  in  $E_n$  is of course always an  $E_m$ . But a geodesic  $A_n^m$  in  $E_n$  need not have a curvature zero. Here is a very simple example in an  $R_3$ . The null system of a system of forces in  $R_3$  fixes at every point the 2-direction of the vector

$$(7.54) \quad p_\lambda = a_\lambda + r^x f_{x\lambda}$$

where  $r^x$  is the radiusvector,  $a_\lambda$  a constant vector and  $f_{x\lambda}$  a constant bivector. If  $p_\lambda = p i_\lambda$  where  $i_\lambda$  is a unitvector, we have

$$(7.55) \quad \text{a) } \overset{2}{H}_{\mu\lambda}^{x''} = -\frac{1}{p} f_{\mu\lambda} i^x; \quad \text{b) } \overset{2}{L}_{\mu\lambda}^{x''} = \frac{1}{p} f_{\mu\lambda} i_\lambda; \quad f_{\mu\lambda} = B_{\mu\lambda}^{\tau\sigma} f_{\tau\sigma}$$

and accordingly (cf. 7.32, 39)

$$(7.56) \quad \begin{cases} \text{a) } \overset{2}{r}_{dcb}^{x''a} = \frac{2}{p^2} f_{[d|b]} f_{c]}^a, \\ \text{b) } \overset{2}{R}_{dcb}^{x''a} = \frac{2}{p^2} f_{[d|b]} f_{c]}^a + \frac{2}{p^2} f_{d c} f_{b}^a. \end{cases}$$

The straight lines of the null system are also geodesics in the  $L_3^2$  but neither  $\overset{2}{r}$  nor  $\overset{2}{R}$  vanishes.

If a rigged  $X_n^m$  is given in  $X_n$ , a connexion for quantities of the  $X_n^m$  only, can be fixed by giving the  $T_{cb}^a$  as functions of the coordinates  $\xi^x$ .

<sup>1)</sup> The GAUSS-CODAZZI equations for  $V_m$  in  $V_n$  were first given by Voss in 1880, 1, the RICCI equations for  $V_m$  in  $R_n$  by RICCI 1888, 1. KÜHNE 1903, 1 had already all equations for  $V_m$  in  $V_n$ . Then followed SCHOUTEN 1929, 3 for  $L_n^m$  in  $L_n$ , SCHOUTEN and v. KAMPEN 1930, 2 for  $V_n^m$  in  $V_n$  and DIENES 1932, 1 for  $L_n^m$  in  $L_n$ . Cf. also for literature STRUIK 1922, 1, p. 136; EISENHART 1926, 1, p. 192; JÄRNEFELT 1928, 1; MAYER 1928, 1; 1935, 2.

Then  $D_c p^a$  (and  $D_c q_b$ ) can be defined just as in (7.19a) but with  $'\Gamma_{cb}^a$  instead of  $\Gamma_{cb}^a$  and besides these derivatives we have also the derivatives  $\mathfrak{D}_z p^a$  and  $\mathfrak{D}_z q_b$  defined already in (7.35a, b). These derivatives *together* fix a covariant differentiation for quantities of  $X_n^m$  in every direction:

$$(7.57) \quad \begin{cases} \text{a)} & \mathfrak{D}_c p^a \stackrel{\text{def}}{=} D_c p^a \stackrel{\text{def}}{=} \partial_c p^a + '\Gamma_{cb}^a p^b; \quad \mathfrak{D}_z p^a \stackrel{\text{def}}{=} \partial_z p^a + 2\Omega_{yz}^a p^b \\ \text{b)} & \mathfrak{D}_c q_b \stackrel{\text{def}}{=} D_c q_b \stackrel{\text{def}}{=} \partial_c q_b - '\Gamma_{cb}^a q_a; \quad \mathfrak{D}_z q_b \stackrel{\text{def}}{=} \partial_z q_b - 2\Omega_{yz}^a q_a. \end{cases}$$

In the same way, if the  $"\Gamma_{zy}^x$  were given as functions of the  $\xi^x$ , the derivatives (cf. 7.35 c, d)

$$(7.57) \quad \begin{cases} \text{c)} & \mathfrak{D}_z r^x \stackrel{\text{def}}{=} D_z r^x \stackrel{\text{def}}{=} \partial_z r^x + "I_{zy}^x r^y; \quad \mathfrak{D}_c r^x \stackrel{\text{def}}{=} \partial_c r^x + 2\Omega_{yc}^x r^y \\ \text{d)} & \mathfrak{D}_z s_y \stackrel{\text{def}}{=} D_z s_y \stackrel{\text{def}}{=} \partial_z s_y - "I_{zy}^x s_x; \quad \mathfrak{D}_c s_y \stackrel{\text{def}}{=} \partial_c s_y - 2\Omega_{yc}^x s_x \end{cases}$$

*together* would fix a covariant differentiation for quantities of  $X_n^{m'}$  in every direction. The quantities  $H_{\mu\lambda}^{m'}$  and  $L_{\mu\lambda}^{m'}$  do not exist now but  $H_{[cb]}^{m'}$  can be defined by (cf. 7.22)

$$(7.58) \quad H_{[cb]}^{m'} \stackrel{\text{def}}{=} -Z_{cb}^{m'}$$

without the term with  $S_{ji}^{ih}$  because in  $X_n$  there is no connexion and no  $S_{ji}^{ih}$ . For this connexion repeated differentiation leads to [cf. (7.38)]

$$(7.59) \quad 2\mathfrak{D}_{[d}\mathfrak{D}_{c]}p^a = -2Z_{dc}^{m'}\mathfrak{D}_y p^a + *R_{dc}^{m'}p^b - 2('I_{[dc]}^e + '\Omega_{dc}^e)D_e p^a$$

with

$$(7.60) \quad *R_{dc}^{m'} \stackrel{\text{def}}{=} 2\partial_{[d}''\Gamma_{c]b}^a + 2'\Gamma_{[d|e]}'\Gamma_{c]b}^e + 2'\Omega_{de}'\Gamma_{eb}^a + 4Z_{dc}^{m'}\Omega_{by}^a$$

and

$$(7.61) \quad (\mathfrak{D}_u \mathfrak{D}_c - \mathfrak{D}_c \mathfrak{D}_u) p^a = *R_{uc}^{m'} p^b$$

with

$$(7.62) \quad \begin{cases} *R_{uc}^{m'm'} \stackrel{\text{def}}{=} \partial_u ''\Gamma_{cb}^a + 2\partial_c \Omega_{ub}^a + 2'\Gamma_{cb}^e \Omega_{eu}^a + 2'\Gamma_{eb}^a \Omega_{uc}^e + \\ + 2'\Gamma_{ce}^a \Omega_{ub}^e + 4\Omega_{uc}^y \Omega_{by}^a. \end{cases}$$

A rigged  $X_n^m$  in  $X_n$  provided in this way with a linear connexion is called an  $L_n^m$  in  $X_n$  and in the special case when  $'\Gamma_{[cb]}^a + '\Omega_{cb}^a = 0$  it is also called an  $A_n^m$  in  $X_n$ .

### Exercises.

V 7.1. If a rigged  $X_n^m$  is given in  $L_n$  and if we take for the connexion in the  $X_n^m$  the connexion induced from the  $L_n$ , prove that  $*R_{dc}^{m'} \stackrel{\text{def}}{=}$  as defined in (V 7.60) with respect to this induced connexion is identical with  $R_{dc}^{m'} \stackrel{\text{def}}{=}$  as defined in (V 7.39).

V 7.2. Prove that for a rigged  $L_n^m$  in  $L_n$

$$V\ 7.2\ \alpha) \quad L_{c,y}^{;a} \stackrel{*}{=} -\Gamma_{cy}^a.$$

V 7.3<sup>1)</sup>. Prove that for a rigged  $L_n^m$  in  $L_n$

$$V\ 7.3\ \alpha) \quad a) -H_{\mu\lambda}^{;a} + L_{\mu\cdot\lambda}^{;a} = D_\mu B_\lambda^a; \quad b) -H_{\mu\lambda}^{;x} + L_{\mu\cdot\lambda}^{;x} = D_\mu C_\lambda^x.$$

V 7.4<sup>1)</sup>. For every vector field  $v^x$  of a rigged  $L_n^m$  in  $L_n$  the following identities hold

$$V\ 7.4\ \alpha) \quad \begin{cases} a) & C_x^x V_\mu v^x = (H_{\mu\lambda}^{;x} - L_{\mu\cdot\lambda}^{;x}) v^\lambda \\ b) & B_c^\mu C_x^x V_\mu v^x = H_{cb}^{;x} v^b \\ c) & C_{y\cdot x}^{\mu x} V_\mu v^x = -L_{y\cdot b}^{;x} v^b. \end{cases}$$

V 7.5<sup>2)</sup>. Prove for a rigged  $L_m$  in  $L_n$  that  $\tilde{H}_{[cb]}^{;x}$  is always zero if the connexion in  $L_n$  is semi-symmetric.

### § 8. The rigged $X_m$ in $A_n$ .<sup>3)</sup>

In  $A_n S_{\mu\lambda}^{;x}$  is zero and therefore an  $X_n^m$  in  $A_n$  is holonomic, that is, it consists of  $\infty^{n-m} X_m$ 's if and only if (cf. II § 9, V 7.22)  $\tilde{H}_{[cb]}^{;x} = -\tilde{Z}_{cb}^{;x} = 0$ . In this case the  $e_b^x$  in  $A_n$  can be chosen in such a way that they are contravariant basis vectors of a holonomic coordinate system (a) on every  $X_m$ . If this has been done we have

$$(8.1) \quad 'Q_{cb}^a \stackrel{*}{=} e_c^\mu e_b^\lambda \partial_{[\mu} \tilde{e}_{\lambda]}^a = e_c^e e_b^d \partial_{[e} \tilde{e}_{d]}^a = 0.$$

Of course the  $X_n^m$  need not be holonomic.

Now we consider the case when only one rigged  $X_m$  is given in  $A_n$ . Then this  $X_m$  is an  $A_m$  (cf. V § 7). Let the coordinate system (a) in  $A_m$  be holonomic and let the  $e_y^x$  be given over  $A_m$  in an arbitrary way in the  $m'$ -direction of the rigging. Then the  $e_i^x$ ,  $\tilde{e}_\lambda^h$  and the  $A_i^x$ ,  $A_i^h$ ,  $h, i = 1, \dots, n$  are known over  $A_m$  and the same holds for the  $\Gamma_{cb}^a$ ,  $\Gamma_{cb}^x$ ,  $\Gamma_{cy}^a$  and  $\Gamma_{cy}^x$ :

$$(8.2) \quad \Gamma_{ci}^h = A_{cix}^{\mu\lambda h} \Gamma_{\mu\lambda}^x - A_i^\lambda \partial_c A_\lambda^h.$$

The only derivatives that are interesting are those in a direction of  $A_m$ . This means that only  $D_c$  will be used and that only  $\tilde{H}_{cb}^{;x}$  and  $\tilde{L}_{c\cdot y}^{;a}$ , defined by (7.15, 16), will occur. The equations (7.31), (7.32), (7.42)

<sup>1)</sup> E II 1938, 2, p. 157.

<sup>2)</sup> E I 1935, 1, p. 97.

<sup>3)</sup> Cf. footnote 2 on p. 232.

and (7.43) take the form

$$(8.3) \quad 2D_{[d} D_{c]} p^a = 'R_{dc}{}^a p^b$$

$$(8.4) \quad 'R_{dc}{}^a = B_{dc}{}^{\mu\lambda} R_{\nu\mu}{}^x - 2H_{[d|b|}{}^x L_{c].x}^m \quad (\text{GAUSS})$$

$$(8.5) \quad 2D_{[d} D_{c]} r^x = R_{dc}{}^y {}^{mm'} r^y$$

$$(8.6) \quad R_{dc}{}^y {}^x = B_{dc}{}^{\nu\mu} C_{y\nu}^{\lambda x} R_{\nu\mu}{}^x + 2H_{[d|b|}{}^x L_{c].y}^m \quad (\text{RICCI})$$

where  $'R_{dc}{}^a$  is the curvature tensor of the  $A_m$  which equals  $R_{dc}{}^a$  and also  $r_{dc}{}^a$  because of (7.30) and (7.39). These equations are only valid at points of the  $A_m$ . The equations (7.48) and (7.51) take the form

$$\left. \begin{aligned} (8.7) \quad 2D_{[d} H_{c]}{}^x &= B_{dc}{}^{\nu\mu\lambda} C_{\nu\mu}^x R_{\lambda}{}^z \\ (8.8) \quad 2D_{[d} L_{c]}{}^a &= -B_{dc}{}^{\nu\mu\lambda} C_{\nu\mu}^{\lambda} R_{\lambda}{}^a \end{aligned} \right\} \quad (\text{CODAZZI})$$

and also are only valid at points of the  $A_m$ .

All these equations can be expressed in terms of the tangent vectors  $\dot{e}_{\lambda}$ , the rigging vectors  $e_y^x$  and their derivatives. It follows from (7.9, 10) that (cf. 3.11, 12)

$$(8.9) \quad H_{\mu}{}^x = -B_{\mu}{}^{\sigma} V_{\tau} \dot{e}_{\sigma}{}^x = \dot{h}_{\mu}{}_{\lambda}{}^x e^{\lambda}; \quad \dot{h}_{\mu}{}_{\lambda} \stackrel{\text{def}}{=} -B_{\mu}{}^{\tau\sigma} V_{\tau} \dot{e}_{\sigma}$$

$$(8.10) \quad L_{\mu}{}^x = -B_{\mu}{}^{\tau\sigma} V_{\tau} \dot{e}_{\lambda}{}^x = l_{\mu}{}^x \dot{e}_{\lambda}; \quad l_{\mu}{}^x \stackrel{\text{def}}{=} -B_{\mu}{}^{\tau\sigma} V_{\tau} e_{\sigma}^{\lambda}$$

hence the equations (7.40, 43) of GAUSS and RICCI can be written in the form

$$(8.11) \quad 'R_{dc}{}^a = B_{dc}{}^{\mu\lambda} R_{\nu\mu}{}^x - 2\dot{h}_{[d|b|}{}^x l_{c].x}^m \quad (\text{cf. 3.15}) \quad (\text{GAUSS})$$

$$(8.12) \quad R_{dc}{}^y {}^x = B_{dc}{}^{\nu\mu} C_{y\nu}^{\lambda x} R_{\nu\mu}{}^x + 2\dot{h}_{[d|b|}{}^x l_{c].y}^m \quad (\text{RICCI}).$$

If we introduce the  $(m')^2$  vectors of  $A_m$

$$(8.13) \quad \dot{v}_b \stackrel{\text{def}}{=} B_b^{\mu} (V_{\mu} \dot{e}_{\lambda})_y{}^{\lambda} = -B_b^{\mu} (V_{\mu} e^{\lambda})_y{}^{\lambda} \dot{e}_{\lambda} = -I_{b}{}^x$$

we have

$$(8.14) \quad 2'V_{[d} \dot{v}_{c]} = -R_{dc}{}^y {}^x + 2\dot{v}_{[d} \dot{v}_{c]}^x$$

and accordingly the equations of Ricci can be written in the form

$$(8.15) \quad 2'V_{[d} \tilde{v}_{c]} = -B_{d\mu}^{\nu} R_{\nu\lambda}^{\mu} \tilde{e}_y^{\lambda} \tilde{e}_{\lambda} - 2\tilde{h}_{[d|b]}^{\lambda} l_{\lambda}^b + 2\tilde{v}_{[d}^{\lambda} \tilde{v}_{c]}^{\lambda} \quad (\text{RICCI}).$$

If (8.13) is substituted in (8.7, 8) we get another form for the equations of Codazzi

$$(8.16) \quad 2'V_{[d} \tilde{h}_{c]}^{\lambda} b = B_{d\mu}^{\nu\lambda} R_{\nu\lambda}^{\mu} \tilde{e}_y^{\lambda} + 2\tilde{v}_{[d}^{\lambda} \tilde{h}_{c]}^{\lambda} b \quad (\text{CODAZZI})$$

$$(8.17) \quad 2'V_{[d} l_{c]}^{\mu} = -B_{d\mu}^{\nu\lambda} R_{\nu\lambda}^{\mu} \tilde{e}_y^{\lambda} - 2\tilde{v}_{[d}^{\lambda} l_{c]}^{\lambda} \quad (\text{CODAZZI}).$$

The equations (8.11, 15, 16, 17) are especially interesting for the case of an  $A_m$  in  $E_n$ . Then they take the simple form

$$(8.18) \quad 'R_{dcb}^{\cdot\cdot\cdot a} = -2\tilde{h}_{[d|b]}^{\lambda} l_{\lambda}^a \quad (\text{GAUSS})$$

$$(8.19) \quad 2'V_{[d} \tilde{v}_{c]} = -2\tilde{h}_{[d|b]}^{\lambda} l_{\lambda}^b + 2\tilde{v}_{[d}^{\lambda} \tilde{v}_{c]}^{\lambda} \quad (\text{RICCI})$$

$$(8.20) \quad 'V_{[d} \tilde{h}_{c]}^{\lambda} b = \tilde{v}_{[d}^{\lambda} \tilde{h}_{c]}^{\lambda} b \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad (\text{CODAZZI}).$$

We prove that an  $A_m$  can be imbedded in an  $E_n$  if and only if there exist, in this  $A_m$ ,  $2m' + m'^2$  fields  $\tilde{h}_{cb}$ ,  $l_c^a$ ,  $\tilde{v}_b$  satisfying (8.18–21) and  $\tilde{h}_{[cb]} = 0$ . Let  $(x)$  be a rectilinear coordinate system in the  $E_n$  to be constructed. Then we have from (8.9, 10, 13)

$$(8.22) \quad \tilde{h}_{cb}^{\lambda} \stackrel{*}{=} (\partial_c C_{\lambda}^x) B_{\lambda}^b = -(\partial_c B_b^{\lambda}) C_{\lambda}^x$$

$$(8.23) \quad l_c^a \stackrel{*}{=} (\partial_c C_y^x) B_{\lambda}^a = -(\partial_c B_{\lambda}^a) C_y^x$$

$$(8.24) \quad \tilde{v}_c^{\lambda} \stackrel{*}{=} (\partial_c C_{\lambda}^x) C_y^{\lambda} = -(\partial_c C_y^{\lambda}) C_{\lambda}^x$$

and from  $'V_c w_b \stackrel{*}{=} B_{cb}^{\mu\lambda} \partial_{\mu} w_{\lambda}$

$$(8.25) \quad I_{cb}^a \stackrel{*}{=} (\partial_c B_b^{\lambda}) B_{\lambda}^a = -(\partial_c B_{\lambda}^a) B_b^{\lambda}.$$

From these equations we get the differential equations for the unknowns  $B_b^x$ ,  $C_y^x$

$$(8.26) \quad \partial_c B_b^x \stackrel{*}{=} I_{cb}^a B_a^x - \tilde{h}_{cb}^{\lambda} C_{\lambda}^x$$

$$(8.27) \quad \partial_c C_y^x \stackrel{*}{=} l_c^a B_a^x - \tilde{v}_c^{\lambda} C_{\lambda}^x$$

and similar equations for  $B_\lambda^a, C_\lambda^x$  that do not need to be considered. The integrability conditions of (8.26, 27) are

$$(8.28) \quad \left\{ \begin{array}{l} 0 \stackrel{*}{=} (\partial_{[d} \Gamma_{c]b}^a) B_a^x + \Gamma_{[c|b]}^a \partial_{d]} B_a^x - (\partial_{[d} \tilde{h}_{c]b}) C_x^x - \tilde{h}_{[c|b]} \partial_{d]} C_x^x \\ = \frac{1}{2} ('R_{d}{}^{c}{}_{b}{}^{a} + 2 \tilde{h}_{[d|b]}^x l_{c]}^a) B_a^x - ('V_{[d} \tilde{h}_{c]b}^x - \tilde{v}_{[d} \tilde{h}_{c]b}^y) C_x^x \end{array} \right.$$

$$(8.29) \quad \left\{ \begin{array}{l} 0 \stackrel{*}{=} (\partial_{[d} l_{c]}^a) B_a^x + l_{[c|}^a \partial_{d]} B_a^x - (\partial_{[d} \tilde{v}_{c]}^x) C_x^x - \tilde{v}_{[c|}^x \partial_{d]} C_x^x \\ = ('V_{[d} l_{c]}^a + \tilde{v}_{[d} l_{c]}^a) B_a^x - (\partial_{[d} \tilde{v}_{c]}^x + \tilde{h}_{[d|b]}^x l_{c]}^b + \tilde{v}_{[c|}^z \tilde{v}_{d]}^x) C_x^x \end{array} \right.$$

and these are satisfied identically because of (8.18–21). Hence there exist solutions  $B_b^x, C_y^x$  that take arbitrarily given values  $B_b^x, C_y^x$  at some point  $\eta^a$  of  $A_m$ . When the  $B_b^x$  have been found in this way the equations

$$(8.30) \quad B_b^x = \partial_b \xi^x$$

are completely integrable because of the symmetry in  $bc$  of  $\Gamma_{c}^a$  and  $\tilde{h}_{c}^x$  in (8.26). The  $\xi^x$  are then the rectilinear coordinates in the  $E_n$  that were to be determined.

In these considerations the number  $n$  and the fields  $\tilde{h}_{c}^x, \tilde{l}_{c}^x$  and  $\tilde{v}_{c}^x$  are given beforehand. But there is another much deeper problem. We may require a number  $n$  such that every  $A_m$  can always be imbedded (in the neighbourhood of one of its points) in an  $E_n$ . By counting the number of equations and of variables for the case of a  $V_m$  in  $R_n$  SCHLÄFLI 1871 suggested that the value  $n$  would be  $\binom{m+1}{2}$ . The rigorous proof of SCHLÄFLI's supposition was given by JANET 1926 and CARTAN 1927.<sup>1)</sup> The number  $n - m$  is called the *class* of the  $V_m$ .<sup>2)</sup> T. Y.

<sup>1)</sup> SCHLÄFLI 1871, 1; JANET 1926, 1; CARTAN 1927, 4; BURSTIN 1931, 1; 1932, 1. Cf. also WEISE 1934, 1; TOMPKINS 1941, 1. LAPTEV proved 1943, 1; 1945, 1 that every  $A_n$  can be imbedded in an  $E_N$ ;  $N \geq \frac{1}{2}(n^2 + 2n - 1)$ . CHERN proved 1937, 1 that every projectively connected  $n$ -dimensional space can be imbedded in an ordinary  $N$ -dimensional projective space with  $N \geq \frac{1}{2}(n^2 - 1) + n$  dimensions. Cf. for the projective case KANITANI 1947, 1; 1948, 1; 1949, 1; 1950, 2 and for the conformal case RYŽKOV 1950, 1; ICHINOHE 1951, 1. A general theory for imbedding in an  $R_N$  was given by ROZENFEL'D 1945, 1. Cf. for imbedding in the large for instance ALLENDOERFER 1937, 1; 1939, 1.

<sup>2)</sup> There is much literature on subspaces of special classes and their properties. We mention here only SCHOUTEN 1918, 2; VRANCEANU 1930, 2; WEISE 1934, 1; T. Y. THOMAS 1936, 4 (class 1, here is the definition of "type"); ALLENDOERFER 1937, 1; 2 (class 1, EINSTEIN spaces); 1939, 1 (class  $> 1$ , rigidity); BURSTIN 1937, 1; EISENHART 1937, 1 (class  $> 1$ ); E II 1938, 2, p. 141; KOCZIAN 1939, 1 (class 1); LAURA 1940, 1 (class 1); ROSENSON 1940, 1 (class 1); 1941, 1 (class 1); 1943, 1 (class 1); PALATINI 1941, 1 (class 1); MASINI VENTURELLI 1943, 1; 2 (class 1); SCHWARTZ 1946, 1 (second curvature); BLUM 1946, 2; 1947, 2; 3; MATSUMOTO

THOMAS<sup>1)</sup> has proved that for  $n - m \geq 4$  the equations of CODAZZI are consequences of the equations of GAUSS.<sup>2)</sup>

### Exercise.

V 8.1. If  $A_r$  is groupspace of an  $r$ -parameter group and if  $X_p$  is geodesic in  $A_r$ , prove that

$$(V 8.1 \alpha) \quad B_{\nu \mu \lambda}^{d c b} C_a^\nu c_d^{\dot{c}} c_e^{\dot{b}} = 0$$

(cf. IV 1.29 and IV 6.11 and use CODAZZI).

### § 9. The $V_n^m$ and $V_m$ in $V_n$ and $X_n$ .<sup>3)</sup>

We consider a real  $X_n^m$  in an ordinary  $V_n$ . Such an  $X_n^m$  is always rigged by the  $m'$ -direction perpendicular to the tangent  $m$ -direction and these  $m'$ -directions form an  $X_n^{m'}$  rigged by the  $X_n^m$ . The fundamental tensor of the  $X_n^m$  is, taking the anholonomic system  $(h)$  as in V § 7 and § 8,

$$(9.1) \quad {}'g_{ba} = B_{ba}^{\lambda \kappa} g_{\lambda \kappa}$$

and from (7.9, 12) we get

$$(9.2) \quad D_c {}'g_{ba} = B_{cb}^{\mu \lambda \kappa} V_\mu B_{\lambda \kappa}^{\sigma \rho} g_{\sigma \rho} = 2 B_{cb}^{\mu \lambda \kappa} {}'g_{\sigma (\kappa} V_{|\mu|} B_{\lambda)}^\sigma = 2 B_{(a}^\kappa H_{|c|}^m {}'g_{\sigma \kappa} = 0$$

$$(9.3) \quad D_y {}'g_{ba} = C_y^\mu B_{ba}^{\lambda \kappa} V_\mu B_{\lambda \kappa}^{\sigma \rho} g_{\sigma \rho} = - 2 B_{(a}^\kappa L_{|y|}^{m'} {}'g_{\sigma \kappa} = 0.$$

1950, 1; 2; 3 (class 2); 1951, 1 (class 2); 2 (affine, class 1); 1952, 1 (project., class 1); VERBICKII 1950, 1 (class 2); BRAUNER 1951, 1 (class 1, orth. syst.).

1) T. Y. THOMAS 1936, 4.

2) The GAUSS-CODAZZI-RICCI equations are not always independent. Cf. MAYER 1935, 2; ALLENDÖRFER 1939, 1; SCHWARTZ 1941, 1; BLUM 1946, 2; 1947, 2.

3) General references on imbedding in  $V_n$ ,  $R_n$  and  $S_n$  (cf. VI § 5 for conformal imbedding): CARTAN 1919, 1; 1920, 3; 1939, 1; 2 ( $R_n$  and  $S_n$ ); SCHOUTEN and STRUIK 1921, 3; 1922, 1; 2; R. K. 1924, 1, Ch. V; BOMPIANI 1921, 1; 1924, 1 (geod.  $V_m$  in  $V_n$ ); 1950, 1; 2 ( $V_2$  in  $R_n$ ); 1951, 1; 2 (higher order curvature); 3 ( $V_n$ ); KÄMMERER 1922, 1; v. d. WAERDEN 1927, 1; BORTOLOTTI 1927, 3; HAYDEN 1932, 1; MAYER 1933, 1 ( $V_n$ ); 1935, 2 ( $R_n$ ); FABRICIUS-BJERRE 1934, 1 ( $S_n$ ); 1936, 1 ( $R_n$ ); KAPLAN 1934, 1 ( $R_6$ ); 2 ( $R_{n+1}$ ); 1936, 1 ( $R_6$ ); PEREPELKINE 1934, 1 ( $R_n$ ); 1935, 1 ( $V_n$ ); 1936, 1 ( $R_n$ ); DUSCHEK 1935, 1 ( $V_m$  in  $R_n$ , GAUSS-BONNET); DAVIES 1935, 1 ( $V_n$ ); FUCHS 1935, 1 (geod. in  $V_4$ ); ANDERSON and INGOLD 1936, 1 ( $R_n$ ); TOMPKINS 1939, 1 (flat isometric in  $R_n$ ); 1941, 1 ( $R_4$ ); T. Y. THOMAS 1939, 3 ( $R_n$ ); COBURN 1939, 1 ( $S_n$ ); 1940, 1 ( $S_n$ ); WONG 1940, 1 ( $V_n$ ); LENSE 1940, 1 ( $R_n$ ); 1944, 1 ( $R_n$ ); FINSLER 1940, 1; GHOSH 1940, 1 ( $R_n$ ); 1941, 1 ( $R_n$ ); WAGNER 1942, 2 (anhol. in  $S_n$ ); CHERN 1944, 1 ( $R_n$ ); WONG 1945, 1; CASTOLDI 1946, 1 ( $V_n$ ); 1949, 1 ( $V_n$ ); LIBER 1947, 1 ( $S_m$  in  $S_n$ ); PETRESCU 1948, 1 (anhol. in  $V_n$ ); HLAVATY 1949, 1; 1952, 1 ( $W_n$ ); YANENKO 1949, 1; 1953, 1 ( $R_n$ ); LOPSCHITZ 1950, 1 ( $E_{m+1}$ ); 2 (centr.  $E_{m+2}$ ); ABRAMOV 1950, 1; 1951, 1; 2 (all topol. inv.  $V_n$ ); OTSUKI 1950, 2 and 1951, 1 (conformal).

The difference between  $\overset{m}{H}_{\mu\lambda}^{\cdot\cdot\cdot\cdot\cdot}$  and  $\overset{m}{L}_{\mu\lambda}^{\cdot\cdot\cdot\cdot\cdot}$  vanishes because we can show from (7.9, 10) that

$$(9.4) \quad \begin{cases} \overset{m}{L}_{\mu\lambda}^{\cdot\cdot\cdot\cdot\cdot} = \overset{m}{L}_{\mu\cdot\sigma}^{\cdot\cdot\cdot\cdot\cdot} g^{\sigma\kappa} g_{\lambda\rho} = B_{\mu\nu}^{\tau\sigma} (V_{\tau} B_{\sigma}^{\nu}) g^{\sigma\kappa} g_{\lambda\rho} = B_{\mu}^{\tau} (V_{\tau} g^{\nu\kappa})' g_{\lambda\rho} \\ = B_{\mu\lambda}^{\tau\sigma} V_{\tau} g^{\nu\kappa} g_{\sigma\rho} = B_{\mu\lambda}^{\tau\sigma} V_{\tau} B_{\sigma}^{\nu} = \overset{m}{H}_{\mu\lambda}^{\cdot\cdot\cdot\cdot\cdot} \end{cases}$$

and it can be proved in the same way that  $\overset{m'}{L}_{\mu\lambda}^{\cdot\cdot\cdot\cdot\cdot} = \overset{m'}{H}_{\mu\lambda}^{\cdot\cdot\cdot\cdot\cdot}$ .

For  $\mathfrak{D}_y$  we get from (7.37) and (9.3)

$$(9.5) \quad \mathfrak{D}_y' g_{ba} = D_y' g_{ba} - \overset{m}{L}_{b\cdot y}^c g_{ca} - \overset{m}{L}_{a\cdot y}^c g_{bc} = -2 \overset{m}{H}_{(ba)y}$$

and from (7.35 b) we get another form

$$(9.6) \quad \mathfrak{D}_y' g_{ba} = \partial_y' g_{ba} - 4 \overset{m}{\Omega}_{(b|y|}^c g_{a)c}.$$

If  $(h)$  is an anholonomic orthogonal system with unit basis vectors we have  $\partial_y' g_{ba} \neq 0$  and consequently in this special case

$$(9.7) \quad \mathfrak{D}_y' g_{ba} \neq 4 \overset{m}{\Omega}_{(b|y|}^c g_{a)c}.$$

Similar formulae hold for the  $X_n^m$ . We call an  $X_n^m$  rigged in this way a  $V_n^m$  in  $V_n$ .<sup>1)</sup>

Now the question arises whether a displacement of quantities of  $X_n^m$  in  $X_n$  can be fixed by giving only  $'g_{ba}$  and a rigging. The parameters  $'\Gamma_{cb}^a$  of this displacement can be found from (9.2)

$$(9.8) \quad 0 = D_c' g_{ba} = \partial_c' g_{ba} - '\Gamma_{cb}^d g_{da} - '\Gamma_{ca}^d g_{bd}$$

and (7.20)

$$(9.9) \quad '\Gamma_{[cb]}^a = -'\Omega_{cb}^a.$$

We may proceed as in V§ 7, putting  $\mathfrak{D}_c p^a = D_c p^a$  and  $\mathfrak{D}_x p^a = \partial_x p^a + 2 \overset{m}{\Omega}_{bx}^a p^b$ . Then we get for the parameters of this covariant differentiation

$$(9.10) \quad *'\Gamma_{cb}^a = '\Gamma_{cb}^a; \quad *'\Gamma_{yb}^a = -2 \overset{m}{\Omega}_{yb}^a$$

hence

$$(9.11) \quad \mathfrak{D}_y' g_{ba} = \partial_y' g_{ba} + 4 \overset{m}{\Omega}_{(b|y|}^c g_{a)c}$$

in accordance with (9.6).

A rigged  $X_n^m$  in  $X_n$  provided in this way with a metric linear connexion is called a  $V_n^m$  in  $X_n$ . Note that in this case  $\overset{m}{H}_{cb}^{\cdot\cdot\cdot\cdot\cdot}$  does not exist because there is no connexion in  $X_n$  but that  $\overset{m}{H}_{(ba)y}^{\cdot\cdot\cdot\cdot\cdot}$  is already known because of (9.5). For repeated differentiation see V § 7.

<sup>1)</sup> Cf. SCHOUTEN 1929, 3, p. 164, 167 ff.; BORTOLOTTI 1930, 2; MOISIL 1940, 1; VRANCEANU 1942, 1.

We will now deduce some of the most important properties of a  $V_m$  in  $V_n$ .<sup>1)</sup> If  $i^\alpha$  is the tangent unitvector of a curve of  $V_m$  we have

$$(9.12) \quad i^\mu 'V_\mu i^\alpha = i^\mu (V_\mu i^\theta) B_\theta^\alpha = i^\mu V_\mu i^\alpha + i^\mu i^\lambda V_\mu C_\lambda^\alpha = i^\mu V_\mu i^\alpha - i^\mu i^\lambda H_{\mu\lambda}^\alpha$$

writing  $H_{\mu\lambda}^\alpha$  instead of  $\overset{m}{H}_{\mu\lambda}^\alpha$ , or

$$(9.13) \quad u^\alpha = 'i^\alpha + ''u^\alpha$$

where  $u^\alpha$  is the *absolute curvature vector*, ' $i^\alpha$ ' the *relative curvature vector* and '' $u^\alpha$  the *enforced curvature vector* (cf. V § 6). If '' $u^\alpha = 0$ , the direction of  $i^\alpha$  is called a *principal tangent* and a curve whose tangent is principal at all points is said to be an *asymptotic line*.<sup>2)</sup> Now let  $i^a$  be the unit basis vectors of an orthogonal anholonomic system (a) of  $V_m$ . Then we have

$$(9.14) \quad i^\alpha = \sum_b i^a \cos_a \alpha$$

where the  $\alpha$  are the angles between  $i^\alpha$  and the basis vectors. Hence (cf. 6.13)

$$(9.15) \quad ''u^\alpha = \sum_{abc} i^a i^b \cos_a \alpha \cos_b \alpha H_{\mu\lambda}^\alpha$$

and this proves that, at every point, the vector '' $u^\alpha$  lies in an  $R_{m_1}$ ;  $m_1 \leq n-m$ ,  $m_1 \leq \frac{1}{2}m(m+1)$  in the tangent  $R_n$ , and perpendicular to the tangent  $R_m$ . The  $R_m$  and  $R_{m_1}$  determine an  $R_{m+m_1}$  called the *first curvature region* of the  $V_m$  at the point considered.<sup>3)</sup> If  $m_1 = \frac{1}{2}m(m+1)$  the vector '' $u^\alpha$  lies on a cone in the local  $R_{m_1}$  and its endpoint describes a  $V_{m-1}$  of degree  $2^{m-1}$  which lies in an  $R_{m_1-1}$ . This  $V_{m-1}$  is the *curvature figure* (Krümmungsgebilde) and its centre of gravity has the radius-vector in tangent space

$$(9.16a) \quad M^\alpha = \frac{1}{m} 'g^{ab} H_{ab}^\alpha.$$

In general the  $V_{m-1}$  does not contain the origin, hence principal tangents only occur in special cases. There is a one to one correspondence between the  $\infty^{m-1}$  directions in the  $V_m$  at the point considered and the points of the curvature figure. For  $m_1 < \frac{1}{2}m(m+1)$  there are several subcases.<sup>4)</sup>

<sup>1)</sup> Cf. for a more detailed treatment EISENHART 1926, 1, Ch. IV, V; DUSCHEK and MAYER 1930, 1, Ch. VII; MAYER 1935, 2; E II 1938, 2, Ch. III, IV.

<sup>2)</sup> SCHOUTEN and STRUIK 1922, 1; 2; HAYDEN 1930, 2; E II 1938, 2, p. 89.

<sup>3)</sup> An osculating space of another kind was introduced by BERZOLARI 1897, 1 making use of an idea of KILLING 1885, 1. Cf. E II 1938, 2, p. 90ff.; BOMPIANI 1950, 3.

<sup>4)</sup> Cf. for an elaborate treatment that deals also with other geometric figures connected with the curvature and that contains many examples E II 1938, 2, p. 94ff.

If  $m=1, 2$  or  $3$  at some point of the  $V_m$  this point is called *axial*, *planar* and *spatial* respectively.<sup>1)</sup>

If the enforced curvature vector is the same for every direction of the  $V_m$  at a point, this point is said to be *umbilical*.<sup>2)</sup> N.a.s. conditions are

$$(9.16b) \quad 0 = M_{cb}^{xx} \stackrel{\text{def}}{=} H_{cb}^{xx} - \frac{1}{m} g_{cb}^{} g^{aa} H_{da}^{xx}.$$

The  $V_m$  is called *umbilical* [cf. V § 6]<sup>3)</sup> if all its points are umbilical. Every  $V_1$  is umbilical because  $H_{cb}^{xx} = {}_1^x j_c j_b j^x$ . We prove that the section of an umbilical  $V_p$  and an umbilical  $V_q$  is always umbilical. This section is a  $V_s$ ;  $s \geq p + q - n$ . At a point of this section, to every direction of  $V_p$  there belongs a curvature vector and the projection of this vector on the normal  $R_{n-p}$  is the enforced curvature vector that is the same for all these directions. The same holds for the directions of  $V_q$ . Now let us consider the directions of  $V_s$  and the projections of the curvature vectors belonging to them on the normal  $R_{n-s}$ . If such a projection is projected on the normal  $R_{n-p}$  and on the normal  $R_{n-q}$  (both lying in  $R_{n-s}$ ) we get the enforced curvature vectors belonging to the direction of  $V_s$ , considered as direction of  $V_p$  and of  $V_q$  respectively. Hence the enforced curvature vector of a direction of  $V_s$  is a vector of  $R_{n-s}$  from which we know that its projection on  $R_{n-p}$  and on  $R_{n-q}$  is the same for all directions of  $V_s$ .  $R_{n-p}$  and  $R_{n-q}$  have an  $R_{n-p-q+s}$  in common and  $n-p-q+s \geq 0$ . Hence the enforced curvature vector of a direction of  $V_s$  is wholly determined by these two projections and this proves that it is the same for all directions of  $V_s$ .

$M^x$  is the *mean curvature vector* of the  $V_m$ . For  $m=1$  it is identical with the curvature vector of the curve. A  $V_m$  in  $V_n$  is called *minimal* if for every part of it bounded by a closed  $V_{m-1}$ , the  $m$ -dimensional volume has an extreme value for variations that do not change the boundary. Now let  $(a)$  be a *holonomic* coordinate system of  $V_m$  with the coordinates  $\eta^a$ . Then  $d\eta^1 \dots d\eta^m$  is the component of a contravariant  $m$ -vector represented by an infinitesimal volume element. This component is a scalar density of  $V_m$  of weight  $-1$ . If  $(a')$  is an anholonomic orthogonal coordinate system in  $V_m$  with unit basis vectors, the volume of such an element is  $|(d\eta)^1 \dots (d\eta)^m|$ . Now  $'g \stackrel{\text{def}}{=} |\text{Det}(g_{ba})|$  is a scalar density of  $V_m$  of weight  $+2$  and its value with respect to  $(a')$  is  $+1$ . Hence the volume is

$$(9.17) \quad d\tau_m \stackrel{\text{def}}{=} |(d\eta)^1 \dots (d\eta)^m| = |d\eta^1 \dots d\eta^m|'g^{\frac{1}{2}}.$$

1) Cf. for  $V_m$ 's in  $S_n$  with planar points COBURN 1939, 1.

2) Cf. for  $V_m$ 's admitting umbilical  $V_m$ 's ADATI 1951, 1; 2.

3) Cf. E II 1938, 2, Ch. III; PEREPELKINE 1936, 1; WONG 1943, 1; 3; 1946, 1.

If the part of the  $V_m$  considered undergoes the finite point transformation  $\overset{t\eta}{T}$  belonging to the one-parameter group generated by the infinitesimal point transformation  $\xi^\kappa \rightarrow \xi^\kappa + v^\kappa dt$  (cf. II § 10) the change of the volume can be computed most easily by performing afterwards the inverse transformation  $\overset{-t\eta}{T}$  and dragging along the coordinate systems ( $\kappa$ ) and ( $a$ ) and the field  $g_{\lambda\kappa}$  with this latter transformation. Then the  $V_m$  returns to its original position and the changed volume we acquire is the volume in this original position with respect to the field  $\overset{m}{g}_{\lambda\kappa}$  dragged along (cf. II § 10). This is the *method of transposition* that will be used often hereafter. Now we have (cf. III 5.48)

$$(9.18) \quad \overset{m}{\mathcal{L}} g_{\lambda\kappa} = v^\mu V_\mu g_{\lambda\kappa} + g_{\mu\kappa} V_\lambda v^\mu + g_{\lambda\mu} V_\kappa v^\mu = 2 V_{(\lambda} v_{\kappa)}$$

$$(9.19) \quad \overset{2}{\mathcal{L}} g_{\lambda\kappa} = 2 v^\mu V_\mu V_{(\lambda} v_{\kappa)} + 2 (V_{(\mu} v_{\kappa)}) V_\lambda v^\mu + 2 (V_{(\lambda} v_{\mu)}) V_\kappa v^\mu$$

and hence, according to (II 10.38)

$$(9.20) \quad \left\{ \begin{array}{l} \overset{m}{g}_{\lambda\kappa} = g_{\lambda\kappa} - 2t V_{(\lambda} v_{\kappa)} + t^2 v^\mu V_\mu V_{(\lambda} v_{\kappa)} + \\ \qquad \qquad \qquad + t^2 (V_{(\mu} v_{\kappa)}) V_\lambda v^\mu + t^2 (V_{(\lambda} v_{\mu)}) V_\kappa v^\mu - \dots \end{array} \right.$$

Because both ( $\kappa$ ) and ( $a$ ) are dragged along,  $B_b^\kappa$  is not changed and consequently

$$(9.21) \quad \left\{ \begin{array}{l} {}' \overset{m}{g}_{b\alpha} = {}' g_{b\alpha} - 2t B_{b\alpha}^{\lambda\kappa} V_{(\lambda} v_{\kappa)} + t^2 B_{b\alpha}^{\lambda\kappa} v^\mu V_\mu V_{(\lambda} v_{\kappa)} + \\ \qquad \qquad \qquad + t^2 B_{b\alpha}^{\lambda\kappa} (V_{(\mu} v_{\kappa)}) V_\lambda v^\mu + t^2 B_{b\alpha}^{\lambda\kappa} (V_{(\lambda} v_{\mu)}) V_\kappa v^\mu - \dots \end{array} \right.$$

Hence for that variation of the field  $'g_{b\alpha}$  at the point considered, which is a consequence of the dragging along, we get

$$(9.22) \quad \frac{d}{dt} {}' g_{b\alpha} = - 2 B_{b\alpha}^{\lambda\kappa} V_{(\lambda} v_{\kappa)} \quad \text{for} \quad t = 0$$

and accordingly (cf. Exerc. I, 1,1)

$$(9.23) \quad \left\{ \begin{array}{l} \frac{d}{dt} {}' g^{\frac{1}{2}} = \frac{1}{2} {}' g^{\frac{1}{2}} \frac{d}{dt} \log {}' g = \frac{1}{2} {}' g^{\frac{1}{2}} {}' g^{ba} \frac{d}{dt} {}' g_{ba} \\ \qquad \qquad \qquad = - {}' g^{\frac{1}{2}} {}' g^{ba} B_{b\alpha}^{\lambda\kappa} V_\lambda v_\kappa = - {}' g^{\frac{1}{2}} B_\alpha^\lambda V_\lambda v^\kappa. \end{array} \right.$$

Now  $v^\kappa$  can always be split up into  $'v^\kappa$  perpendicular to  $V_m$  and  $''v^\kappa$  in  $V_m$ . Then we get

$$(9.24) \quad \left\{ \begin{array}{l} \frac{d}{dt} d\tau_m = - |d\eta^1 \dots d\eta^m| {}' g^{\frac{1}{2}} ('V_b ''v^b - {}' g^{cb} {}' v_\kappa D_c B_b^\kappa) \\ \qquad \qquad \qquad = - d\tau_m ('V_b ''v^b - m {}' v_\mu M^\mu) \end{array} \right.$$

and accordingly

$$(9.25) \quad \frac{d}{dt} \int_{\tau_m} d\tau_m = - \int_{\tau_m} 'V_b ''v^b d\tau_m + m \int_{\tau_m} 'v_\mu M^\mu d\tau_m.$$

The first integral at the right hand side vanishes if  $''v^x = 0$  on the boundary (cf. II § 8). Hence the variation of the volume is zero for all infinitesimal point transformations  $v^x dt$  for which the  $V_m$ -part of  $v^x$  vanishes at the boundary if and only if the mean curvature vector is zero at all points of the region considered.<sup>1)</sup> For  $m=1$  this proves that the variation of the length of a part of a curve in  $V_n$  vanishes for all infinitesimal transformations for which the component in the tangent vanishes at the endpoints if and only if the curve is a geodesic. We can express this in a less rigorous way: geodesics are not only the straightest but also the shortest (longest) curves. Whether the volume really has an extreme value or not, can only be investigated by means of the higher terms of the series (9.20).<sup>2)</sup>

### Exercises.

V 9,1<sup>3)</sup>. In a real  $V_m$  in an ordinary  $V_n$  there are two congruences with unit tangent vectors  $i^x$  and  $j^x$ . Prove that

$$i^\mu V_\mu j^x = i^\mu 'V_\mu j^x + i^c j^b H_{cb}^x$$

(cf. (9.12) and Exerc. V 6,2.)

V 9,2<sup>4)</sup>. The vectorial mean-value of the enforced curvature vectors with respect to  $m$  mutually perpendicular directions of a real  $V_m$  in an ordinary  $V_n$  is independent of the choice of this set of directions and equals the mean curvature vector.

V 9,3<sup>5)</sup>. A set of  $\infty^1$  real  $V_{n-1}$ 's in an ordinary  $V_n$  contains only minimal  $V_{n-1}$ 's if and only if the orthogonal trajectories make correspond the points of the  $V_{n-1}$ 's in such a way that the  $(n-1)$ -dimensional volume is invariant.

V 9,4 (cf. Exerc. VI 5,2)<sup>6)</sup>. According to HAANTJES and WRONA the scalar curvature of a non singular  $m$ -direction at a point of a  $V_n$  is the scalar curvature of a geodesic  $V_m$  with this  $m$ -direction at this

<sup>1)</sup> LIPSCHITZ 1874, 2; BOMPIANI 1921, 1, p. 1141; STRUIK 1922, 1, p. 98; SCHOUTEN and v. KAMPEN 1933, 1, p. 19 also for  $V_n^m$ ; E II 1938, 2, p. 165 ff.

<sup>2)</sup> Minimal surfaces in an  $R_3$  with an indefinite fundamental tensor are dealt with by many authors. We mention here only the elder investigators LIE 1879, 1; BIANCHI 1888, 1; CAYLEY 1890, 1 and WOODS 1895, 1.

<sup>3)</sup> E II 1938, 2, p. 85.

<sup>4)</sup> E II 1938, 2, p. 87.

<sup>5)</sup> BIANCHI 1902, 2, II, p. 577 for  $V_3$  in  $R_4$ ; BOMPIANI 1921, 1, p. 1141 for  $V_{n-1}$  in  $V_n$ .

<sup>6)</sup> HAANTJES and WRONA 1939, 1.

point. If now  $n=2m$ , the  $V_{2m}$  is an EINSTEIN space if and only if at every point two mutually perpendicular  $m$ -directions have always the same curvature.

V 9,5<sup>1)</sup>). Prove that a geodesic  $V_m$  in an  $S_n(R_n)$  is an  $S_m(R_m)$ .

### § 10. Higher curvatures of a $V_m$ in $V_n$ .

If  $i^\alpha, j^\alpha$  and  $k^\alpha$  are the tangent unitvectors of three real congruences in a real  $V_m$  in an ordinary  $V_n$  we have (cf. Exerc. V 9,1)

$$(10.1) \quad \left\{ \begin{array}{l} j^\mu \nabla_\mu i^\alpha = j^\mu \nabla_\mu i^\alpha + j^\mu (\nabla_\mu i^\alpha) C_\alpha = j^\mu \nabla_\mu i^\alpha + j^\mu i^\lambda H_{\mu\lambda}^\alpha \\ \quad = j^\mu \nabla_\mu i^\alpha + j^\nu i^\beta D_\nu B_\nu^\alpha \end{array} \right.$$

$$(10.2) \quad \left\{ \begin{array}{l} k^\nu \nabla_\nu j^\mu \nabla_\mu i^\alpha = k^\nu \nabla_\nu j^\mu \nabla_\mu i^\alpha + k^\nu (\nabla_\nu j^\mu \nabla_\mu i^\alpha) C_\alpha = k^\nu \nabla_\nu j^\mu i^\lambda H_{\mu\lambda}^\alpha \\ \quad = k^\nu \nabla_\nu j^\mu \nabla_\mu i^\alpha + (k^\mu j^\nu \nabla_\nu i^\lambda + i^\lambda k^\nu \nabla_\nu j^\mu + j^\mu k^\nu \nabla_\nu i^\lambda) H_{\mu\lambda}^\alpha + \\ \quad \quad + k^\mu j^\nu i^\beta D_\nu B_\nu^\alpha \end{array} \right.$$

and in the same way the  $p$ -th covariant derivative of  $i^\alpha$  with respect to  $p$  directions of  $V_m$  can be proved to be a vector lying in the  $\alpha$ -region of the set  $B_\nu^\alpha, D_{b_1} B_\nu^\alpha, \dots, D_{b_p} \dots D_{b_k} B_\nu^\alpha$ . This region is called the  $p$ -th curvature region of  $V_m$  at the point considered.<sup>2)</sup> Calling its dimension  $m+m_1+\dots+m_p$  for all values of  $p$  for which  $m_p \neq 0$  we have the inequalities

$$(10.3) \quad m_p \leq \frac{1}{2} m^p (m+1); \quad \Sigma m_p \leq n - m. \quad ^3)$$

The  $V_m$  is called *asymptotic of order p* if the  $p$ -th (and not the  $(p+1)$ -th) curvature region is contained in the tangent  $R_m$ . Then every curve of  $V_m$  is an *asymptotic line of order p*, i.e. its osculating  $R_{p+1}$  is contained in the tangent  $R_m$ .<sup>4)</sup>

Since the  $V_m$  is real, none of the curvature regions has an extraordinary position with respect to the nullcone. There is a number  $k$  such that for each value of  $l \geq 1$  the  $(k+l)$ -th curvature region is contained in the  $k$ -th curvature region. Then these regions determine uniquely an  $R_{m_0} = R_m, R_{m_1}, \dots, R_{m_k}$  perpendicular to each other. For the space containing all directions perpendicular to  $R_{m_0}, \dots, R_{m_k}$  we write  $R_{m_{k+1}}$ ;  $m_{k+1} = n - m - m_1 - \dots - m_k$ . The  $x$ -th curvature region  $x \leq k$  is spanned by  $R_{m_0}, \dots, R_{m_x}$ . If the  $V_n$  is an  $R_n$  it follows immediately from the foregoing that the  $V_m$  is imbedded in a flat  $R_{n-m_{k+1}}$  in  $R_n$ .

<sup>1)</sup> E II 1938, 2, p. 133. Cf. p. 150 for generalizations.

<sup>2)</sup> Higher curvatures of curves in  $V_m$  in  $V_n$  are considered by DEL PEZZO 1886, 1; HAYDEN 1930, 2; 1934, 1; PEREPELKINE 1936, 1; ANDERSON and INGOLD 1936, 1 ( $R_n$ ).

<sup>3)</sup> An improvement was given by SCHWARTZ 1946, 1; cf. ALLENDOERFER 1939, 1.

<sup>4)</sup> Cf. p. 250 footnote 3).

Now we choose an anholonomic coordinate system with unit basis vectors such that for each value of  $x; x=0, \dots, k+1$  the vectors  $i_x^\nu; p_x^\nu, q_x, \dots = \tilde{1}, \dots, \tilde{m}_x$  span  $R_{m_x}$ . Then if a quantity lies with one index in  $R_{m_x}$  this index can be greek or  $p_x, q_x, \dots$ . If we write  $\tilde{g}_{q_x p_x}$  and  $\tilde{B}_{q_x}^{p_x}$  for the fundamental tensor and the unity tensor in  $R_{m_x}$  we have

$$(10.4) \quad 'g_{q_0 p_0} = \frac{\partial}{\partial q_0 p_0}; \quad B_{q_0}^{p_0} = \tilde{B}_{q_0}^{p_0}; \quad \tilde{g}_{q_x p_x} = \tilde{B}_{q_x}^{\lambda} \tilde{g}_{\lambda x}$$

$$(10.5) \quad \tilde{B}_{q_x}^{\nu} \tilde{B}_{\nu}^{p_y} = \begin{cases} \tilde{B}_{q_x}^{p_x} & \text{for } x=y \\ 0 & \text{for } x \neq y \end{cases}; \quad x, y = 0, \dots, k+1$$

and the  $D$ -symbolism can be extended with the operators  $D_{r_x}$ :

$$(10.6) \quad D_{r_x} p \stackrel{\text{def}}{=} \tilde{B}_{r_x}^{\mu} V_{\mu} p$$

$$(10.7) \quad D_{r_x} v^\nu \stackrel{\text{def}}{=} \tilde{B}_{r_x}^{\mu} V_{\mu} v^\nu; \quad D_{r_x} w_\lambda \stackrel{\text{def}}{=} \tilde{B}_{r_x}^{\mu} V_{\mu} w_\lambda$$

$$(10.8) \quad \begin{cases} \text{a)} \quad D_\mu r^{p_x} \stackrel{\text{def}}{=} \tilde{B}_{\mu}^{p_x} V_\mu r^x; & D_\mu s_{q_x} \stackrel{\text{def}}{=} \tilde{B}_{\mu}^{\lambda} V_\mu s_\lambda \\ \text{b)} \quad D_{r_x} r^{p_y} \stackrel{\text{def}}{=} \tilde{B}_{r_x}^{\mu} \tilde{B}_{\mu}^{p_y} V_\mu r^y; & D_{r_x} s_{q_y} \stackrel{\text{def}}{=} \tilde{B}_{r_x}^{\mu} \tilde{B}_{\mu}^{\lambda} V_\mu s_\lambda \end{cases}$$

The curvature tensor  $H_{\mu\lambda}^{\nu}$  appears in the form

$$(10.9) \quad \tilde{H}_{r_0 q_0}^{\nu} = D_{r_0} \tilde{B}_{q_0}^{\nu}; \quad \tilde{H}_{r_0 \cdot \lambda}^{\nu} = D_{r_0} \tilde{B}_{\lambda}^{\nu}$$

and is called now the *first curvature tensor of valence 3*. Its  $\nu$ -region coincides with  $R_{m_1}$ . The *second curvature tensor of valence 3* is defined by

$$(10.10) \quad \tilde{H}_{r_0 q_1}^{\nu} \stackrel{\text{def}}{=} D_{r_0} \tilde{B}_{q_1}^{\nu} + \tilde{H}_{r_0 \cdot q_1}^{\nu}.$$

Because

$$(10.11) \quad \tilde{B}_{\nu}^{p_0} \tilde{H}_{r_0 q_1}^{\nu} = - \tilde{H}_{r_0 \cdot \nu}^{p_0} \tilde{B}_{q_1}^{\nu} + \tilde{B}_{\nu}^{p_0} \tilde{H}_{r_0 \cdot q_1}^{\nu} = 0$$

and

$$(10.12) \quad \tilde{B}_{\nu}^{p_1} \tilde{H}_{r_0 q_1}^{\nu} = \tilde{B}_{\nu}^{p_1} D_{r_0} \tilde{B}_{q_1}^{\nu} = \tilde{B}_{\nu}^{p_1 \lambda} D_{r_0} \tilde{B}_{\lambda}^{\nu} = \tilde{B}_{\nu}^{p_1} D_{r_0} \tilde{B}_{q_1}^{\nu} - \tilde{B}_{\lambda}^{p_1} D_{r_0} \tilde{B}_{q_1}^{\lambda} = 0;$$

the  $\nu$ -region of  $\tilde{H}_{r_0 q_1}^{\nu}$  is  $m_2$ -dimensional and perpendicular to  $R_{m_0}$  and  $R_{m_1}$  and therefore coincides with  $R_{m_2}$ . It can be proved in the same way that the  $\nu$ -region of

$$(10.13) \quad \tilde{H}_{r_0 q_x}^{\nu} \stackrel{\text{def}}{=} D_{r_0} \tilde{B}_{q_x}^{\nu} + \tilde{H}_{r_0 \cdot q_x}^{\nu}; \quad x = 1, \dots, k;$$

coincides with  $R_{m_{x+1}}$  and that this quantity, the  $(x+1)$ -th *curvature tensor of valence 3* of the  $V_m$  vanishes for  $x \geq k$ . The formulae

$$(10.14a) \quad \begin{cases} D_{r_0} \tilde{B}_{q_x}^{\nu} = - \tilde{H}_{r_0 \cdot q_x}^{\nu} + \tilde{H}_{r_0 q_x}^{\nu} \\ \tilde{H}_{r_0 \cdot q_0}^{\nu} = 0; \quad \tilde{H}_{r_0 q_1}^{\nu} = 0; \quad \tilde{H}_{r_0 q_{k+1}}^{\nu} = 0; \quad x = 0, \dots, k; \end{cases}$$

are a generalization for  $V_m$  in  $V_n$ <sup>1)</sup> of the FRENET formulae (cf. V § 1) and become these formulae for  $m=1$ . They can also be written in the form

$$(10.14\text{b}) \quad \begin{cases} D_{r_0} \overset{x}{B}_{\lambda}^{px} = -\overset{x}{H}_{r_0 \lambda}^{px} + \overset{x+1}{H}_{r_0 \lambda}^{px} \\ \overset{\theta}{H}_{r_0 \lambda}^{px} = 0; \quad \overset{k+1}{H}_{r_0 \lambda}^{px} = 0; \quad \overset{k+2}{H}_{r_0 \lambda}^{px} = 0; \quad x = 0, 1, \dots, k. \end{cases}$$

Note that  $\overset{x}{H}_{\mu \lambda}^{px}$  lies with the second index in  $R_{m_{x-1}}$  and with the third index in  $R_{mx}$ .

The equations (10.14a or b) can not be used to find out whether a given  $V_m$  can be imbedded in an  $R_n$  or  $S_n$  because these equations contain connecting quantities. The method that will be followed here, using equations that only contain quantities of the  $V_m$  is due to BURSTIN and MAYER and is developed here in the invariant form given by SCHOUTEN and v. KAMPEN.<sup>2)</sup>

The following quantities are formed from the  $\overset{x}{H}_{r_0 \lambda}^{px}$

$$(10.15) \quad \begin{cases} \text{a)} \quad \overset{1}{J}_{c b}^{xx} \stackrel{\text{def}}{=} \overset{1}{H}_{cb}^{xx} \\ \text{b)} \quad \overset{x}{J}_{b_{x+1} \dots b_1}^{xx} \stackrel{\text{def}}{=} \overset{x}{H}_{b_{x+1} \lambda}^{xx} \overset{x-1}{J}_{b_x \dots b_1}^{\lambda}; \quad x = 2, \dots, k \end{cases}$$

where the indices  $b, c, b_1, \dots, b_{x+1}$  belong to a coordinate system (a) in  $V_m$ . Then

$$(10.16) \quad \overset{x}{L}_{b_{x+1} \dots b_1 d_{x+1} \dots d_1} \stackrel{\text{def}}{=} \overset{x}{J}_{b_{x+1} \dots b_1}^{xx} \overset{x}{J}_{d_{x+1} \dots d_1}^{\lambda} g_{x \lambda}; \quad x = 1, \dots, k$$

are  $k$  quantities of  $V_m$ . It can be proved that a  $V_m$  in  $S_n$  is determined by the quantities  $\overset{x}{L}$  to within rotations and reflexions. But the quantities  $\overset{x}{L}$  can not be given at random<sup>3)</sup>, because they must satisfy certain conditions. A method for building a possible set of quantities  $\overset{x}{L}$  was given by SCHOUTEN and STRUIK.<sup>4)</sup>

Of course the theory of imbedding involving only the first curvature, developed in V § 8 for the rigged  $A_m$  in  $A_n$  can also be used for  $V_m$  in  $V_n$ . If for this purpose we take quite another anholonomic system (h), namely the system (a) orthogonal and anholonomic with unit basis vectors  $i_b^x; a, b = 1, \dots, m$ , in  $V_m$  and for  $i_y^x; x, y = m+1, \dots, n$

<sup>1)</sup> SCHOUTEN and v. KAMPEN 1931, 2, p. 151; E II 1938, 2, p. 120; cf. also WONG 1940, 2; IWAMOTO 1944, 1; BOMPIANI 1951, 2.

<sup>2)</sup> BURSTIN and MAYER 1926, 1; MAYER 1928, 1; 1931, 1; 1935, 2; BURSTIN 1929, 1; DUSCHEK and MAYER 1930, 1, II, p. 201—232; SCHOUTEN and v. KAMPEN 1931, 2, p. 153.

<sup>3)</sup> MAYER 1928, 1; 1935, 2.

<sup>4)</sup> E II 1938, 2, p. 125ff., also for literature.

unitvectors mutually perpendicular and perpendicular to  $V_m$ , the difference between  $\overset{m}{H}, \overset{m}{h}$  and  $\overset{m}{L}, \overset{m}{l}$  as defined in V §§ 7, 8 vanishes and we get immediately from (8.9, 13)

$$(10.17) \quad \text{a) } \overset{x}{h}_{cb} \stackrel{\text{def}}{=} -B_{c b}^{\mu \lambda} \nabla_{\mu} \overset{x}{i}_{\lambda}; \quad \text{b) } \overset{x}{v}_b \stackrel{\text{def}}{=} B_b^{\mu} (\nabla_{\mu} \overset{x}{i}_{\lambda}) \overset{x}{i}_{\lambda}; \quad x, y = m+1, \dots, n$$

and from (8.11, 15, 16)

$$(10.18) \quad 'K_{dcba} = B_{dcba}^{\mu \lambda \kappa} K_{\nu \mu \lambda \kappa} - 2 \sum_x \overset{x}{h}_{[d[b} \overset{x}{h}_{c]a]} \quad (\text{GAUSS})$$

$$(10.19) \quad 2'V_{[d} \overset{x}{v}_{y]c} = -B_{dc}^{\nu \mu} K_{\nu \mu \lambda \kappa} \overset{x}{i}_{\lambda} \overset{x}{i}_{\kappa} - 2 \overset{x}{h}_{[d[b} \overset{y}{h}_{c]b} + 2 \overset{x}{v}_{[d} \overset{z}{v}_{y]c]} \quad (\text{RICCI})^1$$

$$(10.20) \quad 2'V_{[d} \overset{x}{h}_{c]b} = B_{dc}^{\nu \mu \lambda} K_{\nu \mu \lambda \kappa} \overset{x}{i}_{\kappa} + 2 \overset{x}{v}_{[d} \overset{y}{h}_{c]b} \quad (\text{CODAZZI}).$$

For  $R_n$  the terms with  $K_{\nu \mu \lambda \kappa}^x$  drop out and for  $S_n$  the quantity  $K_{\nu \mu \lambda \kappa}$  has to be replaced by  $-2\eta g_{[\nu} [\lambda} g_{\mu] \kappa]}$ . Using the method of V § 8 it can be proved that a  $V_m$  in  $R_n$  or  $S_n$  is determined to within translations, rotations and reflexions by  $n-m$  symmetric tensor fields  $\overset{x}{h}_{cb}$  and  $\binom{n-m}{2}$  vector fields  $\overset{x}{v}_b$  provided that these quantities satisfy the equations (10.18 to 20).<sup>2)</sup>

Another form of the GAUSS-CODAZZI-RICCI equations, involving also higher curvatures can be derived in the following way.<sup>3)</sup> Let the unit basis vectors  $\overset{q}{i}_x$ ;  $x = 0, \dots, k+1$  once more be chosen in  $R_{mx}$ . Then we have for any vector of  $R_{mx}$

$$(10.21) \quad \left\{ \begin{array}{l} D_{[s_0} D_{r_0]} w_{qx} = D_{[s_0} D_{r_0]} w_{\overset{p}{i}_{qx}} \\ \quad = w_{\overset{p}{i}_x} (-\partial_{[s_0} \Gamma_{r_0] q_x}^{\overset{p}{i}_x} + \Gamma_{[s_0 r_0]}^{t_0} \Gamma_{t_0 q_x}^{\overset{p}{i}_x} + \Gamma_{[s_0 q_x]}^{t_x} \Gamma_{r_0] t_x}^{\overset{p}{i}_x}) \\ \quad = -\frac{1}{2} K_{s_0 r_0 q_x}^{\overset{x}{i} \cdot \overset{p}{i}_x} w_{\overset{p}{i}_x}; \quad x = 0, \dots, k+1; \end{array} \right.$$

where (cf. III 9.3, 20)

$$(10.22) \quad \left\{ \begin{array}{l} \overset{x}{K}_{s_0 r_0 q_x}^{\overset{x}{i} \cdot \overset{p}{i}_x} \stackrel{\text{def}}{=} 2(\partial_{[s_0} \Gamma_{r_0] q_x}^{\overset{p}{i}_x} - \Gamma_{[s_0 q_x]}^{t_x} \Gamma_{r_0] t_x}^{\overset{p}{i}_x} + \Omega_{s_0 r_0}^{t_0} \Gamma_{t_0 q_x}^{\overset{p}{i}_x}); \\ \quad x = 0, \dots, k+1. \end{array} \right.$$

<sup>1)</sup> RICCI 1888, 1; KÜHNE gave 1903, 1 the whole set (10.18–20) for  $V_m$  in  $V_n$ . SCHOUTEN and v. KAMPEN gave 1930, 2 the corresponding equations for  $V_m^n$  in  $V_n$ . An analogous set for  $L_n^m$  in  $L_n$  was given by DIENES 1932, 1. Cf. also JÄRNEFELT 1928, 1; MAYER 1928, 1; 1935, 2; E II 1938, 2, p. 131.

<sup>2)</sup> E II 1938, 2, p. 121ff. Cf. EISENHART 1926, 1, p. 192; LENSE 1940, 1; KAPLAN 1941, 1; YANO and MUTO 1942, 1 and YANO 1942, 2 (conformal).

<sup>3)</sup> SCHOUTEN and v. KAMPEN 1931, 2, § 3; E II 1938, 2, p. 121ff.

Of course  $K_{s_0 r_0 q_0}^{\theta}$  is identical with the curvature tensor of the  $V_m$ , viz.  $'K_{s_0 r_0 q_0}^{\theta}$ .

Differentiation of the FRENET formulae (10.14a) leads to

$$(10.23) \quad D_{[s_0] D_{r_0]} \tilde{B}_{q_x}^x = -D_{[s_0] \tilde{H}_{r_0}^x \cdot q_x} + D_{[s_0] \tilde{H}_{r_0}^x} \tilde{B}_{q_x}^{x+1}; \quad x=0, 1, \dots, k+1$$

or in another form

$$(10.24) \quad \tilde{B}_{s_0 r_0}^{\theta} \tilde{B}_{q_x}^x \tilde{B}_{\mu}^{\lambda} K_{\nu \mu \lambda}^{\cdot \cdot \cdot x} - \tilde{B}_{p_x}^x \tilde{B}_{s_0 r_0 q_x}^{\theta} K_{\nu \mu \lambda}^{\cdot \cdot \cdot p_x} = -2D_{[s_0] \tilde{H}_{r_0}^x \cdot q_x} + 2D_{[s_0] \tilde{H}_{r_0}^x} \tilde{B}_{q_x}^{x+1}.$$

According to (10.14) the right hand side of this equation lies with the index  $x$  in the space spanned by  $R_{m_{x-2}}$ ,  $R_{m_{x-1}}$ ,  $R_{m_x}$ ,  $R_{m_{x+1}}$  and  $R_{m_{x+2}}$ . Hence it can be split up into the following four equations<sup>1)</sup>

$$(10.25) \quad \left\{ \begin{array}{l} \text{a)} \quad \tilde{B}_{s_0 r_0}^{\theta} \tilde{B}_{\mu}^{x-g} \tilde{B}_{q_x}^x K_{\nu \mu \lambda}^{\cdot \cdot \cdot x} = 0; \quad x=g, \dots, k+1; \quad g \geq 3 \\ \text{b)} \quad \tilde{B}_{s_0 r_0}^{\theta} \tilde{B}_{\mu}^{x-2} \tilde{B}_{q_x}^x K_{\nu \mu \lambda}^{\cdot \cdot \cdot x} = 2\tilde{H}_{[s_0 \cdot |x|] \tilde{H}_{r_0}^x}^{\theta p_{x-2}} \tilde{B}_{q_x}^x; \quad x=2, \dots, k+1 \\ \text{c)} \quad \tilde{B}_{s_0 r_0}^{\theta} \tilde{B}_{\mu}^{x-1} \tilde{B}_{q_x}^x K_{\nu \mu \lambda}^{\cdot \cdot \cdot x} = -2D_{[s_0] \tilde{H}_{r_0}^x} \tilde{B}_{q_x}^{p_{x-1}}; \quad x=1, \dots, k+1 \\ \text{d)} \quad \tilde{B}_{s_0 r_0}^{\theta} \tilde{B}_{p_x}^x \tilde{B}_{q_x}^x K_{\nu \mu \lambda}^{\cdot \cdot \cdot x} = \tilde{K}_{s_0 r_0 q_x}^{\cdot \cdot \cdot p_x} - 2\tilde{H}_{[s_0 \cdot |x|] \tilde{H}_{r_0}^x}^{\theta p_x} \tilde{H}_{r_0}^x \tilde{B}_{q_x}^x - \\ \quad \quad \quad - 2\tilde{H}_{[s_0 \cdot |x|] \tilde{H}_{r_0}^x}^{\theta p_x} \tilde{B}_{q_x}^{x+1}; \quad x=0, \dots, k+1. \end{array} \right.$$

(10.25) are the GAUSS-CODAZZI-RICCI equations of a  $V_m$  in  $V_n$  for higher curvatures. For  $V_m$  in  $R_n$  only three equations remain

$$(10.26) \quad \left\{ \begin{array}{l} \text{a)} \quad \tilde{H}_{[s_0 \cdot |x-1|] \tilde{H}_{r_0}^x}^{\theta p_{x-2}} \tilde{B}_{q_x}^{x-1} = 0; \quad x=2, \dots, k \\ \text{b)} \quad D_{[s_0] \tilde{H}_{r_0}^x} \tilde{B}_{q_x}^{p_{x-1}} = 0; \quad x=1, \dots, k \\ \text{c)} \quad \tilde{K}_{s_0 r_0 q_x}^{\cdot \cdot \cdot p_x} = 2\tilde{H}_{[s_0 |x-1|] \tilde{H}_{r_0}^x}^{\theta p_x} \tilde{B}_{q_x}^{x-1} + 2\tilde{H}_{[s_0 \cdot |x+1|] \tilde{H}_{r_0}^x}^{\theta p_x} \tilde{B}_{q_x}^{x+1}; \\ \quad \quad \quad x=0, \dots, k+1. \end{array} \right.$$

In the equations (cf. 10.14a) in an  $R_n$  with *rectilinear* coordinates  $\xi^x$

$$(10.27) \quad \left\{ \begin{array}{l} \text{a)} \quad \partial_{q_0} \xi^x = \tilde{B}_{q_0}^x; \quad \partial_{q_0} \stackrel{\theta}{=} \tilde{B}_{q_0}^b \partial_b \\ \text{b)} \quad D_{r_0} \tilde{B}_{q_x}^x = -\tilde{H}_{r_0}^x \tilde{B}_{q_x}^{x-1} \tilde{B}_{p_{x-1}}^{x-1} + \tilde{H}_{r_0}^x \tilde{B}_{q_x}^{x+1} \tilde{B}_{p_{x+1}}^{x+1}; \\ \quad \quad \quad x=0, \dots, k+1; \end{array} \right.$$

<sup>1)</sup> The first equation is due to SCHWARTZ 1941, 1; cf. CUTLER 1931, 1; 2.

the  $\xi^x$ ,  $\overset{0}{B}_{q_0}^x, \dots, \overset{k+1}{B}_{q_{k+1}}^x$  may be considered as the unknowns and the  $\overset{1}{\Gamma}_{r_0 q_0}^{p_0}, \dots, \overset{l}{\Gamma}_{r_0 q_{k+1}}^{p_{k+1}}, \overset{1}{H}_{r_0 \cdot q_1}^{1 \cdot p_0}, \dots, \overset{k}{H}_{r_0 \cdot q_k}^{k \cdot p_{k-1}}$  as given functions of the coordinates  $\eta^a$  of  $V_m$ . Then (10.26) and  $H_{[s_0 q_0]}^{l \cdot p_1} = 0$  are the integrability conditions.<sup>1)</sup> If these conditions are satisfied identically by the given functions, (10.27) always has a solution that takes initial values  $\overset{0}{\xi^x}, \overset{0}{B}_{q_0}^x, \dots, \overset{k+1}{B}_{q_{k+1}}^x$  at an arbitrary point  $\eta^a$  of  $V_m$ . But these initial values must satisfy the conditions

$$(10.28) \quad \begin{cases} a) & g_{\lambda x} \overset{x}{B}_{q_x}^{\lambda} \overset{x}{B}_{p_x}^{\lambda} = \delta_{q_x p_x} \\ b) & g_{\lambda x} \overset{x}{B}_{q_x}^{\lambda} \overset{y}{B}_{p_y}^{\lambda} = 0; \quad x, y = 0, \dots, k+1; \quad x \neq y \end{cases}$$

where the  $g_{\lambda x}$  are some given constants with  $g_{\lambda x} = g_{x \lambda}$  and  $\text{Det}(g_{\lambda x}) \neq 0$ . Then similar equations are satisfied by  $\overset{0}{B}_{q_0}^x, \dots, \overset{k+1}{B}_{q_{k+1}}^x$  at all points in an  $\mathfrak{N}(\eta^a)$ . In the  $R_n$  of the  $\xi^x$  with the fundamental tensor  $g_{\lambda x}$  the first  $m+m_1+\dots+m_k$  vectors span a flat  $R_{m+\dots+m_k}$  in which the  $V_m$  is imbedded. Hence:

If in a  $V_m$  the  $\overset{1}{\Gamma}_{r_0 q_0}^{p_0}$  are the parameters of the connexion with respect to an anholonomic coordinate system with unit basis vectors and if in this  $V_m$  the  $\overset{1}{\Gamma}_{r_0 q_1}^{p_1}, \dots, \overset{l}{\Gamma}_{r_0 q_{k+1}}^{p_{k+1}}$  and the  $\overset{1}{H}_{r_0 \cdot q_1}^{1 \cdot p_0}, \dots, \overset{k}{H}_{r_0 \cdot q_k}^{k \cdot p_{k-1}}$ ;  $p_x, q_x = \overset{x}{1}, \dots, \overset{x}{m_x}$ ;  $x = 1, \dots, k+1$  are given functions of the coordinates  $\eta^a$  of the  $V_m$  satisfying the conditions  $H_{[r_0 q_0]}^{l \cdot p_1} = 0$  and (10.26), in which the operator  $D_{s_0}$  is defined by

$$(10.29) \quad \begin{cases} a) & D_{s_0} v^{p_x} \stackrel{\text{def}}{=} \partial_{s_0} v^{p_x} + \overset{1}{\Gamma}_{s_0 q_x}^{p_x} v^{q_x}; \quad x = 0, \dots, k+1 \\ b) & D_{s_0} w_{q_x} \stackrel{\text{def}}{=} \partial_{s_0} w_{q_x} - \overset{1}{\Gamma}_{s_0 q_x}^{p_x} w_{p_x} \end{cases}$$

and  $K_{s_0 r_0 q_x}^{l \cdot p_x}$  by (10.22), then an  $R_n$ ;  $n = m+m_1+\dots+m_{k+1}$  can be constructed and in this  $R_n$  a flat  $R_{m+\dots+m_k}$ , in which the  $V_m$  is imbedded in such a way that its  $x$ -th curvature region is an  $R_{m+m_1+\dots+m_x}$ ;  $x = 1, \dots, k$ , can also be constructed.

If for a  $V_m$  in  $R_n$  we have  $m_l = 1$ ,  $l \geq 1$  and if  $\overset{l}{n}^x$  is a unitvector in  $R_{m_l}$  the quantity  $\overset{l}{H}_{r_0 p_{l-1}}^{l \cdot p_x}$  can be written in the form

$$(10.30) \quad \overset{l}{H}_{r_0 p_{l-1}}^{l \cdot p_x} = \overset{l}{H}_{r_0 p_{l-1}}^{l \cdot p_x} \overset{l}{n}_{\lambda}^{l+1} H_{r_0 \cdot q_{l+1}}^{l \cdot p_x}.$$

Then (10.26a) gives for  $x = l+1$

$$(10.31) \quad \overset{l}{H}_{[s_0 \cdot p_{l-1}]}^{l \cdot p_x} \overset{l}{n}_{\lambda}^{l+1} H_{r_0 \cdot q_{l+1}}^{l \cdot p_x} = 0.$$

<sup>1)</sup> SCHOUTEN and v. KAMPEN 1931, 2, § 4.

For  $l=k$  this is trivial but for  $l < k$  (10.31) expresses that for every value of  $p_{l-1}$  and  $q_{l+1}$  the vectors  $\overset{l}{H}_{s_0 p_{l-1}}$  and  $\overset{l}{n}_\lambda \overset{l+1}{H}_{r_0 q_{l+1}}$  all have the same direction. Hence for  $m=1$  and  $1 \leq l < k$  the curvature tensors of valence 3 from  $\overset{l}{H}_{r_0 \lambda}^*$  to  $\overset{k}{H}_{r_0 \lambda}^*$  can be written as follows

$$(10.32) \quad \begin{cases} \text{a)} & \overset{l}{H}_{r_0 \lambda}^* = H_{r_0} u_\lambda \overset{l}{n}^* \\ \text{b)} & \overset{l+1}{H}_{r_0 \lambda}^* = {}^\alpha \overset{l+1}{H}_{r_0} \overset{l}{n}_\lambda \overset{l+1}{n}^* \\ & \vdots \\ \text{c)} & \overset{k}{H}_{r_0 \lambda}^* = {}^\alpha \overset{k}{H}_{r_0} \overset{k-1}{n}_\lambda \overset{k}{n}^* \end{cases}$$

in which  $H_{r_0}$  is a definite vector of  $V_m$  and  $u_\lambda$  a definite unitvector of  $R_{m_{l-1}}$  and where  $\overset{l}{n}^*, \dots, \overset{k}{n}^*$  are unitvectors in the one dimensional spaces  $R_{m_l}, \dots, R_{m_k}$ . For  $k=l$  the vector  $H_{r_0}$  does not exist and instead of (10.32) we have (10.30) only.<sup>1)</sup> Note that  $D_{r_0} \overset{x}{n}_{q_x} = 0$ ;  $x=l, \dots, k$ .

From (10.26b) and (10.32) we get for  $1 \leq l < k$  and  $x=l, \dots, k$

$$(10.33) \quad \begin{cases} \text{a)} & (D_{[s_0} H_{r_0]} u^{p_{l-1}} \overset{l}{n}_{q_l} + H_{[r_0} (D_{s_0]} u^{p_{l-1}}) \overset{l}{n}_{q_l}) = 0 \\ \text{b)} & (D_{[s_0} {}^\alpha \overset{l+1}{H}_{r_0]} \overset{l}{n}^{p_l} \overset{l+1}{n}_{q_{l+1}}) = 0 \\ & \vdots \\ \text{c)} & (D_{[s_0} {}^\alpha \overset{k}{H}_{r_0]} \overset{k-1}{n}^{p_{k-1}} \overset{k}{n}_{q_k}) = 0 \end{cases}$$

from which we see (by transvection of (10.33a) with  $u_{p_{l-1}}$ ) that  $H_{r_0}$  is a gradientvector,  $H_{r_0} = \partial_{r_0} H$ , and that the coefficients  ${}^\alpha, \dots, {}^\alpha$  can all be written as functions of  $H$  only. The equation  $H = \text{const.}$  represents a normal system of  $V_{m-1}$ 's in  $V_m$ . If  $v^{q_0}$  is a vector in these  $V_{m-1}$ 's we have  $v^{q_0} H_{r_0} = 0$  and from (10.33a) it follows that

$$(10.34) \quad v^{q_0} D_{q_0} u^{p_{l-1}} = 0;$$

but this equation is only part of a more important result that can be derived from the equations of FRENET.

In fact, according to (10.14a), (10.32) and (10.34) we have

$$(10.35) \quad \begin{cases} v^{q_0} D_{q_0} u^* = v^{q_0} u^{p_{l-1}} D_{q_0} \overset{l-1}{B}_{p_{l-1}}^* \\ = - v^{q_0} u^{p_{l-1}} \overset{l-1}{H}_{q_0 p_{l-1}}^* + v^{q_0} u^{p_{l-1}} H_{q_0} u_{p_{l-1}} \overset{l}{n}^*. \end{cases}$$

<sup>1)</sup> EISENHART 1937, 1. Although in most parts of this paper EISENHART only considers the case  $l=2$ , his point of view is more general because his fundamental tensor in  $V_n$  is also taken to be indefinite. This gives some very interesting results that could not be included here.

But (10.26a) gives, for  $x=l$ ,

$$(10.36) \quad H_{[s_0 \cdot t_{l-2} \cdot] \lambda}^{\cdot l-1} H_{r_0} u^\lambda n_{q_l}^l = 0,$$

hence

$$(10.37) \quad v^{q_0} D_{q_0} u^\lambda = 0$$

and this proves that for  $m_l=1$  and  $1 \leq l < k$  the field  $u^\lambda$  is covariant constant over each of the  $V_{m-1}$ 's of the normal system.<sup>1)</sup>

Returning now to the equations of FRENET (10.14) for  $x=l, \dots, k$  we get according to (10.32) for  $m_l=1$ ;  $1 \leq l < k$

$$(10.38) \quad \left\{ \begin{array}{ll} \text{a)} & (D_{r_0} n_{q_l}^l) n_{q_l}^l = -H_{r_0} u^\lambda n_{q_l}^l + {}^l \alpha H_{r_0} n_{q_l}^l n_{q_l}^{l+1} \\ \text{b)} & (D_{r_0} n_{q_{l+1}}^{l+1}) n_{q_{l+1}}^{l+1} = -{}^l \alpha H_{r_0} n_{q_{l+1}}^{l+1} n_{q_{l+1}}^{l+2} + {}^{l+2} \alpha H_{r_0} n_{q_{l+1}}^{l+1} n_{q_{l+1}}^{l+2} \\ \vdots & \\ \text{c)} & (D_{r_0} n_{q_k}^k) n_{q_k}^k = -{}^k \alpha H_{r_0} n_{q_k}^{k-1} n_{q_k}^k \end{array} \right.$$

or

$$(10.39) \quad \left\{ \begin{array}{ll} \text{a)} & D_{r_0} n_{q_l}^l = -H_{r_0} (u^\lambda - {}^l \alpha n_{q_l}^{l+1}) \\ \text{b)} & D_{r_0} n_{q_{l+1}}^{l+1} = -H_{r_0} ({}^l \alpha n_{q_{l+1}}^{l+1} - {}^{l+2} \alpha n_{q_{l+1}}^{l+2}) \\ \vdots & \\ \text{c)} & D_{r_0} n_{q_k}^k = -H_{r_0} {}^k \alpha n_{q_k}^{k-1}, \end{array} \right.$$

hence

$$(10.40) \quad v^{r_0} D_{r_0} \ddot{n}^\lambda = 0; \quad x = l, \dots, k;$$

and this proves that for  $m_l=1$ ,  $1 \leq l < k$  the fields  $\dot{n}^\lambda, \dots, \ddot{n}^\lambda$  also are covariant constant over each of the  $V_{m-1}$ 's of the normal system  $H=\text{const.}$ <sup>1)</sup>

Substituting (10.32) in (10.26c) we get for  $m_l=1$ ,  $1 \leq l < k$

$$(10.41) \quad K_{s_0 r_0 q_x}^x \cdot \cdot p_x = 0; \quad x = l, \dots, k$$

and for  $m_l=1$ ,  $1 \leq l = k$

$$(10.42) \quad \left\{ \begin{array}{ll} \text{a)} & K_{s_0 r_0 q_k}^k \cdot \cdot p_k = 2 H_{[s_0 \cdot t_{k-1}]}^k H_{r_0}^k n_{q_k}^k n_{q_k}^k = 0 \\ \text{b)} & K_{s_0 r_0 q_{k-1}}^{k-1} \cdot \cdot p_{k-1} = 2 H_{[s_0 \cdot t_{k-2}]}^{k-1} H_{r_0}^{k-1} n_{q_{k-1}}^{k-1} n_{q_{k-1}}^k + 2 H_{[s_0 \cdot t_{k-1}]}^k H_{r_0}^k n_{q_{k-1}}^k \end{array} \right.$$

We see from (10.26c, 32a, 42b) that for  $l=1$

$$(10.43) \quad K_{s_0 r_0 q_0}^0 \cdot \cdot p_0 = \begin{cases} 2 H_{[s_0 \cdot t_0]}^1 H_{r_0}^1 n_{q_0}^1 & \text{for } k=1 \\ 0 & \text{for } k>1. \end{cases}$$

<sup>1)</sup> EISENHART 1937, 1 for  $l=2$ .

Hence if  $m_1 = 1$  the  $V_m$  lies in a flat  $R_{m+1}$  in  $R_n$  or it is a bent  $R_m$ . This implies that if a  $V_m$  is imbedded in an  $R_n$  such that  $m_1 = 1$ , it can always be imbedded in an  $R_{m+1}$ .<sup>1)</sup> We prove the following more general theorem:

*A  $V_m$  in  $R_n$  with  $m_l = 1$ ;  $2 \leq l < k$  can always be imbedded in an  $R_{m+m_1+\dots+m_{l-1}}$  such that its  $x$ -th curvature region is an  $R_{m+m_1+\dots+m_x}$ ;  $x = 1, \dots, l-1$ .*<sup>2)</sup>

If we take (10.32) into account we have from (10.14a) for  $2 \leq l < k$

$$(10.44) \quad \left\{ \begin{array}{ll} \text{a)} & B_{q_0}^{\theta} = \partial_{q_0} \xi^x \\ \text{b)} & D_{r_0} B_{q_0}^{\theta} = H_{r_0 q_0}^{l-1} \cdot {}^{p_1} B_{p_1}^{\theta} \\ \text{c)} & D_{r_0} \dot{B}_{q_x}^{\theta} = - \dot{H}_{r_0}^{x \cdot p_{x-1}} {}_{q_x} B_{p_{x-1}}^{x-1} + H_{r_0 q_x}^{x+1} \cdot {}^{p_{x+1}} B_{p_{x+1}}^{x+1} \quad x=1, \dots, l-2 \\ \text{d)} & D_{r_0} \dot{B}_{q_{l-1}}^{\theta} = - \dot{H}_{r_0}^{l-1} \cdot {}^{p_{l-2}} {}_{q_{l-1}} B_{p_{l-2}}^{l-2} + H_{r_0} u_{q_{l-1}} {}^l n^{\theta} \\ \text{e)} & D_{r_0} {}^l n^{\theta} = - H_{r_0} u^{p_{l-1}} {}^{l-1} B_{p_{l-1}}^{\theta} + {}^{l+1} \alpha H_{r_0} {}^{l+1} n^{\theta} \\ \text{f)} & D_{r_0} {}^y n^{\theta} = - {}^y \alpha H_{r_0} {}^{y-1} n^{\theta} + {}^{y+1} \alpha H_{r_0} {}^{y+1} n^{\theta}; \quad y=l+1, \dots, k-1 \\ \text{g)} & D_{r_0} {}^k n^{\theta} = - {}^k \alpha H_{r_0} {}^{k-1} n^{\theta} \\ \text{h)} & D_{r_0} {}^{k+1} B_{q_{k+1}}^{\theta} = 0 \end{array} \right.$$

and from (10.26) the integrability conditions of (10.44)

$$(10.45) \quad \left\{ \begin{array}{ll} \text{a)} & \dot{H}_{[s_0 \cdot |t_x|}^{x \cdot p_{x-1}} H_{r_0] \cdot q_{x+1}}^{x+1} t_x = 0; \quad x=1, \dots, l-2 \\ \text{b)} & \dot{H}_{[s_0 \cdot |t_{l-1}|}^{l-1} H_{r_0]} u^{t_{l-1}} = 0; \quad u^{p_{l-1}} u_{p_{l-1}} = 1 \\ \text{c)} & D_{[s_0} \dot{H}_{r_0]}^{z \cdot p_{z-1}} {}_{q_z} = 0; \quad z=1, \dots, l-1 \\ \text{d)} & D_{[s_0} H_{r_0]} = 0 \\ \text{e)} & H_{[r_0} \partial_{s_0]} {}^{l+1} \alpha = 0; \dots; \quad H_{[r_0} \partial_{s_0]} {}^k \alpha = 0 \\ \text{f)} & K_{s_0 r_0 q_0}^{\theta} \cdot {}^l p_0 = 2 \dot{H}_{[s_0 \cdot |t_1|}^l H_{r_0] \cdot q_0}^{l-1} t_1 \\ \text{g)} & \dot{K}_{s_0 r_0 q_x}^{\theta} \cdot {}^l p_x = 2 \dot{H}_{[s_0 \cdot |t_{x-1}|}^x p_x H_{r_0] \cdot q_x}^{x \cdot t_{x-1}} + 2 \dot{H}_{[s_0 \cdot |t_{x+1}|}^{x+1} H_{r_0] \cdot q_x}^{x+1} t_{x+1}; \\ & \quad x=1, \dots, l-2 \\ \text{h)} & \dot{K}_{s_0 r_0 q_{l-1}}^{\theta} \cdot {}^{l-1} p_{l-1} = 2 \dot{H}_{[s_0 \cdot |t_{l-2}|}^{l-1} p_{l-1} H_{r_0] \cdot q_{l-1}}^{l-1} t_{l-2} \\ \text{i)} & \dot{K}_{s_0 r_0 q_u}^{\theta} \cdot {}^l p_u = 0; \quad u=l, \dots, k+1 \\ \text{j)} & \dot{H}_{[r_0 q_0]}^l p_1 = 0. \end{array} \right.$$

<sup>1)</sup> EISENHART 1937, 1.

<sup>2)</sup> EISENHART 1937, 1 for  $l=2$ .

In (10.44) the variables  $\xi^\alpha, \overset{\theta}{B}_{q_0}^\alpha, \dots, \overset{l-1}{B}_{q_{l-1}}^\alpha, \overset{l}{n}^\alpha, \dots, \overset{k}{n}^\alpha, \overset{k+1}{B}_{q_{k+1}}^\alpha$  are the unknowns and the  $\overset{p_0}{I}_{r_0 q_0}, \dots, \overset{p_{l-1}}{I}_{r_0 q_{l-1}}, \overset{l}{H}_{r_0 q_0}^{\cdot p_1}, \dots, \overset{l-1}{H}_{r_0 q_{l-2}}^{\cdot p_{l-1}}, H_{r_0}, u^{p_{l-1}}, \overset{l+1}{\alpha}, \dots, \overset{k}{\alpha}$  are known functions of the coordinates in  $V_m$ , satisfying the conditions (10.45 a-h, j). Now this same  $V_m$  can be imbedded in a flat  $R_{m+m_1+\dots+m_{l-1}}$  in  $R_n$  with curvature regions of dimension  $m+m_1, m+m_1+m_2, \dots, m+\dots+m_{l-1}$  if and only if equations of the form

$$(10.46) \quad \left\{ \begin{array}{ll} \text{a)} & 'B_{q_0}^\alpha = \partial_{q_0} \xi^\alpha \\ \text{b)} & D_{r_0} 'B_{q_0}^\alpha = 'H_{r_0 q_0}^{\cdot p_1} 'B_{p_1}^\alpha \\ \text{c)} & D_{r_0} 'B_{q_x}^\alpha = - 'H_{r_0 \cdot q_x}^{\cdot p_{x-1}} 'B_{p_{x-1}}^{x-1} + 'H_{r_0 q_x}^{\cdot p_{x+1}} 'B_{p_{x+1}}^{x+1}; \\ & \quad x = 1, \dots, l-2 \\ \text{d)} & D_{r_0} 'B_{q_{l-1}}^{l-1} = - 'H_{r_0 \cdot q_{l-1}}^{l-1} 'B_{p_{l-2}}^{l-2} \\ \text{e)} & D_{r_0} 'B_{q_l}^l = 0 \end{array} \right.$$

have solutions. Now the integrability conditions of these equations are

$$(10.47) \quad \left\{ \begin{array}{ll} \text{a)} & 'H_{[s_0 \cdot |t_x]}^{\cdot p_{x-1}} 'H_{r_0 \cdot q_{x+1}}^{x+1} t_x = 0; \quad x = 1, \dots, l-2 \\ \text{b)} & D_{[s_0} 'H_{r_0] \cdot q_x}^{\cdot p_{x-1}} = 0; \quad z = 1, \dots, l-1 \\ \text{c)} & K_{s_0 r_0 q_0}^{\theta} = 2 'H_{[s_0 \cdot |t_1]}^l 'H_{r_0] q_0}^l t_1 \\ \text{d)} & 'K_{s_0 r_0 q_x}^{\cdot p_x} = 2 'H_{[s_0 \cdot |t_{x-1}]}^x 'H_{r_0] \cdot q_x}^{x-1} + 2 'H_{[s_0 \cdot |t_{x+1}]}^{x+1} 'H_{r_0] q_x}^{x+1} t_{x+1}; \\ & \quad x = 1, \dots, l-2 \\ \text{e)} & 'K_{s_0 r_0 q_{l-1}}^{\cdot p_{l-1}} = 2 'H_{[s_0 \cdot |t_{l-2}]}^{l-1} 'H_{r_0] q_{l-1}}^{l-1} t_{l-2} \\ \text{f)} & 'K_{s_0 r_0 q_l}^{\cdot p_l} = 0 \end{array} \right.$$

and these equations are identical with (10.45 a, c, f, g, h, i) if we take

$$(10.48) \quad \left\{ \begin{array}{ll} 'H_{s_0 \cdot q_x}^{\cdot p_{x-1}} = H_{s_0 \cdot q_x}^{\cdot p_{x-1}}; & x = 1, \dots, l-1 \\ 'I_{r_0 q_x}^{\cdot p_x} = I_{r_0 q_x}^{\cdot p_x}; & x = 1, \dots, l-1 \\ H_{r_0} = 0^1 & \end{array} \right.$$

and zero for all  $'I_{r_0 q_l}^{\cdot p_l}$ , where the indices  $p_l, q_l$  now belong to a region of dimension  $m_l + \dots + m_{k+1}$ .

<sup>1)</sup> It is not necessary to take  $H_{r_0} = 0$ .

A theory of the imbedding of a  $W_m$  in  $W_n$ <sup>1)</sup> and of an  $A_m$  in  $A_n$ <sup>2)</sup> involving higher curvatures was developed by HLAVATY. Problems of higher curvature for  $V_{m'}$  in  $V_m$  in  $V_n$  were dealt with by LONGO 1948<sup>3)</sup>. Umbilical subspaces in umbilical subspaces were considered by YANO<sup>4)</sup>.

### § 11. Product spaces.<sup>5)</sup>

If two riemannian spaces  $V_m$  and  $V_{n-m}$  are given with coordinates  $\xi^\alpha$ ;  $\alpha, \beta, \gamma = 1, \dots, m$ , and  $\xi^\xi$ ;  $\xi, \eta, \zeta = m+1, \dots, n$ , and fundamental tensors  $g_{\beta\alpha}$  and  $g_{\eta\xi}$ , the  $V_n$  with the coordinates  $\xi^\alpha$ ;  $\alpha, \lambda, \mu = 1, \dots, n$ , and the fundamental tensor  $g_{\lambda\alpha}$  for which  $g_{\eta\alpha} = 0$ ;  $g_{\beta\xi} = 0$ , is called the *product* of  $V_m$  and  $V_{n-m}$ . A  $V_n$  that is a product space is said to be *decomposable* (or *separable*). That means that  $V_n$  is decomposable if and only if there can be found a coordinate system ( $\alpha$ ) such that  $g_{\eta\alpha} = 0$ ;  $g_{\beta\xi} = 0$  and the  $g_{\beta\alpha}$  and the  $g_{\eta\xi}$  are functions of the  $\xi^\alpha$  and the  $\xi^\xi$  respectively.

If two spaces  $L_m$  and  $L_{n-m}$  are given with coordinates  $\xi^\alpha$  and  $\xi^\xi$  and the connexions  $\Gamma_{\gamma\beta}^\alpha$  and  $\Gamma_{\xi\eta}^\xi$ , the  $L_n$  with the coordinates  $\xi^\alpha$  and the connexion  $\Gamma_{\mu\lambda}^\alpha$  for which all components with two indices of different kind vanish, is called the *product* of  $L_m$  and  $L_{n-m}$ . An  $L_n$  that is a product space is also called *decomposable*. Hence an  $L_n$  is decomposable if and only if there exists a coordinate system ( $\alpha$ ) such that all parameters of the connexion with two indices of different kind vanish and that the others only depend on the set of coordinates to which they belong. Obviously an  $L_n$  that is decomposable according to the second definition and happens to be a  $V_n$  is also decomposable according to the first definition.

An object in a decomposable  $L_n$  is called *breakable* if its components with respect to a special coordinate system as defined above are always zero when they have indices from both ranges. A breakable object is said to be *decomposable* if and only if the components belonging to the subspace  $L_m$  ( $L_{n-m}$ ) depend on the  $\xi^\alpha$  ( $\xi^\xi$ ) only. The objects  $g_{\lambda\alpha}$  and  $\Gamma_{\mu\lambda}^\alpha$  in a decomposable  $V_n$  or  $L_n$  are examples of decomposable objects.

A decomposable  $L_n$  contains a set of  $\infty^{n-m}$   $X_m$ 's and a set of  $\infty^m$   $X_{n-m}$ 's with the equations

$$(11.1) \quad \text{a) } \xi^\xi \neq 0; \quad \xi = m+1, \dots, n \quad \text{and} \quad \text{b) } \xi^\alpha \neq 0; \quad \alpha = 1, \dots, m$$

<sup>1)</sup> HLAVATY 1949, 1; 1952, 1.

<sup>2)</sup> HLAVATY 1949, 2; 3; 4.

<sup>3)</sup> LONGO 1948, 1.

<sup>4)</sup> YANO 1940, 3.

<sup>5)</sup> General references: EISENHART 1923, 1; LEVY 1925, 2; DUSCHEK and MAYER 1930, 1; T. Y. THOMAS 1939, 1; 2; FICKEN 1939, 1; LICHNEROWICZ 1944, 1.

such that nowhere an  $X_m$  has a direction in common with an  $X_{n-m}$ . Each of these sets is geodesic and parallel in  $L_n$ . In fact, if  $v^\alpha; v^\xi = 0$  is a field in the  $X_m$ 's, we have for each displacement in these spaces ( $d\xi^\xi = 0$ )

$$(11.2) \quad \delta v^\xi * d\xi^\gamma \Gamma_{\gamma\beta}^\xi v^\beta = 0$$

and for each displacement in the other spaces ( $d\xi^\alpha = 0$ )

$$(11.3) \quad \delta v^\xi * d\xi^\zeta \Gamma_{\zeta\beta}^\xi v^\beta = 0.$$

Conversely an  $L_n$  is decomposable if there exist a set of  $\infty^{n-m}$  geodesic and parallel  $X_m$ 's and a set of  $\infty^m$  geodesic and parallel  $X_{n-m}$ 's having nowhere a direction in common. In an ordinary  $V_n$  it is sufficient that there is a set of  $\infty^{n-m}$  real geodesic and parallel  $V_m$ 's because in this case the set of orthogonal  $V_{n-m}$ 's exists and is geodesic and parallel. But if the fundamental tensor of  $V_n$  is indefinite it may happen that the tangent  $m$ -direction and the  $(n-m)$ -directions perpendicular to it have a direction in common.

If a  $V_n$  is decomposable,  $g_{\lambda\alpha}$  can be split up into two symmetric tensors

$$(11.4) \quad \begin{cases} g_{\lambda\alpha} = 'g_{\lambda\alpha} + ''g_{\lambda\alpha}; & 'g_{\beta\xi} * 0; & 'g_{\eta\alpha} * 0; & 'g_{\eta\xi} * 0 \\ & ''g_{\beta\xi} * 0; & ''g_{\eta\alpha} * 0; & ''g_{\beta\alpha} * 0 \end{cases}$$

and each of these tensors is covariant constant.

$$(11.5) \quad \text{a)} \nabla_\mu 'g_{\lambda\alpha} = 0; \quad \text{b)} \nabla_\mu ''g_{\lambda\alpha} = 0.$$

Moreover the  $'g_{\beta\alpha}$  ( $''g_{\eta\xi}$ ) depend on the  $\xi^\alpha$  ( $\xi^\xi$ ) only. But this implies that also the symmetric tensor  $'c'g_{\lambda\alpha} + ''c''g_{\lambda\alpha}$  with two constants  $'c$  and  $''c$ , is covariant constant. Conversely, if there exists in an ordinary  $V_n$  a real covariant constant symmetric tensor  $h_{\lambda\alpha}$  that is not a multiple of  $g_{\lambda\alpha}$ , the principal multidirections of  $h_{\lambda\alpha}$  form parallel fields. Hence:

*An ordinary  $V_n$  is decomposable if and only if there exists a real covariant constant symmetric tensor field  $h_{\lambda\alpha}$  that is not a multiple of  $g_{\lambda\alpha}$ .<sup>1)</sup>*

In a decomposable  $L_n$  any figure can be "projected" by the  $X_{n-m}$ 's on one of the  $X_m$ 's and by the  $X_m$ 's on one of the  $X_{n-m}$ 's. These projections have many interesting properties. For instance it can be proved that a curve in a decomposable  $A_n$  is a geodesic if and only if its projections are both geodesics.

There are several generalizations possible. WONG<sup>2)</sup> considered a  $V_n$  whose linear element can be written in the form

$$(11.6) \quad ds^2 = g_{\beta\alpha} d\xi^\beta d\xi^\alpha + g_{\eta\xi} d\xi^\eta d\xi^\xi$$

<sup>1)</sup> T. Y. THOMAS 1939, 1; 2; LICHNEROWICZ 1944, 1.

<sup>2)</sup> WONG 1943, 1.

where the  $g_{\beta\alpha}$  and  $g_{\eta\xi}$  are general functions of the  $\xi^\alpha$  and  $\xi^\xi$  both. Such a  $V_n$  need not be decomposable. In this case the  $V_m$ 's :  $\xi^\xi = 0$  and the  $V_{n-m}$ 's :  $\xi^\alpha = 0$  are called complementary subspaces.

If in (11.6)  $g_{\beta\alpha}$  and  $g_{\eta\xi}$  have the form

$$(11.7) \quad g_{\beta\alpha} \stackrel{*}{=} \varrho' g_{\beta\alpha}; \quad g_{\eta\xi} \stackrel{*}{=} \sigma'' g_{\eta\xi}$$

where the ' $g_{\beta\alpha}$  (" $g_{\eta\xi}$ ) depend on the  $\xi^\alpha$  ( $\xi^\xi$ ) only but  $\varrho$  and  $\sigma$  are general functions of the  $\xi^\alpha$  and  $\xi^\xi$  both, the  $V_n$  is said to be *conformally decomposable (separable)*.<sup>1)</sup>

Many recent investigations deal with the geometry in the large of decomposable and indecomposable compact spaces.

### Exercise.

V 11,1<sup>2)</sup>). Every symmetric  $V_n$  that is not decomposable is an EINSTEIN space.

## VI. Projective and conformal transformations of connexions.<sup>3)</sup>

### § 1. Projective transformations of a symmetric connexion.

In Ch. III § 7 we proved that the transformations of the  $\Gamma_{\mu\lambda}^\alpha$  in  $A_n$  which preserve the geodesics have the form

$$(1.1) \quad \Gamma_{\mu\lambda}^\alpha = \Gamma_{\mu\lambda}^\alpha + p_\mu A_\lambda^\alpha + p_\lambda A_\mu^\alpha \quad \text{4)}$$

where  $p_\lambda$  is an arbitrary vector. We call this transformation *projective* and  $p_\lambda$  the *vector of the transformation*. From (1.1) and (III 2.3, 7) we get

$$(1.2) \quad \left\{ \begin{array}{l} \text{a)} \quad 'V_\mu v^\alpha = V_\mu v^\alpha + A_\mu^\alpha p_\lambda v^\lambda + v^\alpha p_\mu \\ \text{b)} \quad 'V_\mu w_\lambda = V_\mu w_\lambda - 2w_{(\mu} p_{\lambda)}. \end{array} \right.$$

<sup>1)</sup> YANO 1940, 1; WONG 1943, 2.

<sup>2)</sup> LICHNEROWICZ 1944, 1.

<sup>3)</sup> In this chapter a  $V_n$  is always ordinary and imbedded spaces are always real if another supposition is not made explicitly.

<sup>4)</sup> A transformation of the  $\Gamma_{\mu\lambda}^\alpha$  in  $L_n$  of the form  $\Gamma_{\mu\lambda}^\alpha \rightarrow \Gamma_{\mu\lambda}^\alpha + p_\mu A_\lambda^\alpha$  preserves not only the geodesics but also parallelism of directions (cf. VI § 9). The simplest object that can be formed of a connexion given to within transformations of this kind is  $\Gamma_{\mu\lambda}^\alpha - \frac{1}{n} \Gamma_\mu A_\lambda^\alpha$  as was proved by J. M. THOMAS 1926, 1. Cf. FRIESECKE 1925, 1; BORTOLOTTI 1930, 3, p. 75 ff.; 1931, 4; SCHEIBE 1952, 1 and footnotes 5, p. 322 and 6, p. 334.

If (1.1) is substituted in (III 4.2) it follows that

$$(1.3) \quad 'R_{\nu\mu\lambda}^{\cdot\cdot\cdot\cdot\lambda} = R_{\nu\mu\lambda}^{\cdot\cdot\cdot\cdot\lambda} - 2\dot{p}_{[\nu\mu]} A_{\lambda}^{\lambda} + 2A_{[\nu}^{\lambda} \dot{p}_{\mu]\lambda}; \quad \dot{p}_{\mu\lambda} \stackrel{\text{def}}{=} -V_{\mu} \dot{p}_{\lambda} + \dot{p}_{\mu} p_{\lambda}$$

and accordingly

$$(1.4) \quad \begin{cases} \text{a)} & 'R_{\mu\lambda} = R_{\mu\lambda} + n \dot{p}_{\mu\lambda} - \dot{p}_{\lambda\mu} \\ \text{b)} & 'V_{\nu\mu} = -2'R_{[\nu\mu]} = V_{\nu\mu} - 2(n+1) \dot{p}_{[\nu\mu]} = V_{\nu\mu} + 2(n+1) V_{[\nu} \dot{p}_{\mu]} \end{cases}$$

(1.4b) implies that a volume preserving connexion (Exerc. III 4.5) is transformed into a connexion with the same property if and only if  $\dot{p}_{\lambda}$  is a gradient. The transformation is then called *restricted projective*.<sup>1)</sup> If we wish to make ' $V_{\nu\mu}$  zero, the integrability conditions of (1.4b) are  $V_{[\omega} V_{\nu\mu]} = 0$ . But these are identically satisfied (cf. III 5.24), hence *every connexion can be transformed into a volume preserving connexion by a projective transformation*.<sup>2)</sup>

An  $A_n$  and its connexion are said to be *projectively euclidean*<sup>3)</sup> and the  $A_n$  is called a  $D_n$  if it can be transformed projectively into an  $E_n$ .<sup>4)</sup> The n.a.s. condition is that there exists a vector field  $P_{\lambda}$  such that

$$(1.5) \quad \begin{cases} \text{a)} & R_{\nu\mu\lambda}^{\cdot\cdot\cdot\cdot\lambda} = 2P_{[\nu\mu]} A_{\lambda}^{\lambda} - 2A_{[\nu}^{\lambda} P_{\mu]\lambda} \\ \text{b)} & P_{\mu\lambda} \stackrel{\text{def}}{=} -V_{\mu} P_{\lambda} + P_{\mu} P_{\lambda}. \end{cases}$$

Now in an  $A_n$  the curvature tensor satisfies only the first and the second identity (III 5.1, 4) and therefore the number of its independent components is  $n^2 \binom{n}{2} - n \binom{n}{3} = \frac{1}{3} n^2 (n^2 - 1)$ . This number is  $> n^2$  for  $n > 2$ , hence (1.5a) can always be satisfied for  $n = 2$ , but for  $n > 2$  only if  $R_{\nu\mu\lambda}^{\cdot\cdot\cdot\cdot\lambda}$  satisfies certain algebraic conditions. In this case we get from (1.4a) for  $\dot{p}_{\mu\lambda} = P_{\mu\lambda}$  and ' $R_{\mu\lambda} = 0$

$$(1.6) \quad \begin{cases} \text{a)} & P_{\mu\lambda} = -\frac{n}{n^2 - 1} R_{\mu\lambda} - \frac{1}{n^2 - 1} R_{\lambda\mu} \\ & = -\frac{1}{n-1} R_{\mu\lambda} - \frac{1}{n^2 - 1} V_{\mu\lambda} = -\frac{1}{n-1} R_{(\mu\lambda)} - \frac{1}{n+1} R_{[\mu\lambda]} \\ \text{b)} & P_{[\mu\lambda]} = \frac{1}{2(n+1)} V_{\mu\lambda}. \end{cases}$$

<sup>1)</sup> The term was introduced by COBURN 1941, 1.

<sup>2)</sup> Cf. EISENHART 1922, 2, p. 236.

<sup>3)</sup> Many authors use the term projectively flat. But an  $A_{n-1}$  in  $E_n$  can be euclidean, that is an  $E_{n-1}$ , without being flat in the ordinary sense, for instance a cone has this property. There is no objection to the term *locally projectively flat*.

<sup>4)</sup> In BORTOLOTTI 1931, 4 a  $D_n$  is an  $L_n$  whose connexion is given to within a transformation of the form  $\Gamma_{\mu\lambda}^{\lambda} \rightarrow \Gamma_{\mu\lambda}^{\lambda} + \dot{p}_{\mu} A_{\lambda}^{\lambda}$ . Cf. footnote 4 on page 287.

The first integrability condition of (1.5b) is

$$(1.7) \quad R_{\nu\mu}^{\cdot\cdot\lambda} P_\lambda = 2V_{[\nu} P_{\mu]\lambda} - 2(V_{[\nu} P_{\mu]}) P_\lambda - 2P_{[\mu} V_{\nu]} P_\lambda$$

or, according to (1.5a, b)

$$(1.8) \quad V_{[\nu} P_{\mu]\lambda} = 0.$$

Now, if the identity of BIANCHI (III 5.21) is applied to (1.5a) we get

$$(1.9) \quad 0 = V_{[\omega} P_{\nu\mu]} A_{\lambda}^{\omega} - A_{[\nu}^{\omega} V_{\mu]} P_{\lambda}$$

and from this equation by contraction over  $\omega\lambda$

$$(1.10) \quad V_{[\omega} P_{\nu\mu]} = 0$$

and by contraction over  $\nu\lambda$

$$(1.11) \quad V_{[\omega} P_{\lambda\mu]} = \frac{1}{3}(n-2)V_{[\omega} P_{\mu]\lambda}.$$

Hence, for  $n > 2$  the integrability condition of (1.5b) is a consequence of (1.5a). This proves the theorem<sup>1)</sup>:

An  $A_n$  is projectively euclidean if and only if

$$(1.12) \quad P_{\nu\mu}^{\cdot\cdot\lambda} \stackrel{\text{def}}{=} R_{\nu\mu}^{\cdot\cdot\lambda} - 2P_{[\nu\mu]} A_{\lambda}^{\omega} + 2A_{[\nu}^{\omega} P_{\mu]\lambda} = 0$$

where

$$(1.13) \quad P_{\mu\lambda} \stackrel{\text{def}}{=} -\frac{n}{n^2-1}R_{\mu\lambda} - \frac{1}{n^2-1}R_{\lambda\mu}$$

and if for  $n=2$  moreover

$$(1.14) \quad V_{[\nu} P_{\mu]\lambda} = 0.$$

For  $n=2$ , (1.12) is identically satisfied and for  $n > 2$  (1.14) is a consequence of (1.12).

$P_{\nu\mu}^{\cdot\cdot\lambda}$  is called the *projective curvature tensor*.<sup>2)</sup> From (1.13) and (1.4a) it follows that

$$(1.15) \quad {}'P_{\mu\lambda} = P_{\mu\lambda} - p_{\mu\lambda}$$

and by substituting this in (1.12) we see that  $P_{\nu\mu}^{\cdot\cdot\lambda}$  is invariant for projective transformations of the connexion. For  $n=2$  it vanishes identically and for  $n > 2$  it satisfies the identities

$$(1.16) \quad \begin{cases} \text{a)} & P_{(\nu\mu)\lambda}^{\cdot\cdot\lambda} = 0; \\ \text{c)} & P_{\nu\mu\lambda}^{\cdot\cdot\nu} = 0; \\ \text{b)} & P_{[\nu\mu\lambda]}^{\cdot\cdot\lambda} = 0; \\ \text{d)} & P_{\nu\mu\lambda}^{\cdot\cdot\lambda} = 0. \end{cases}$$

<sup>1)</sup> WEYL 1921, 2, p. 105. In EISENHART 1927, 1, p. 97 there is a mistake in the condition for a  $V_n$  of being a  $D_n$ . It is probably caused by a wrong interpretation of WEYL's term "scalar curvature". The same mistake occurs in LOVELL 1934, 1. Cf. footnote 4, Ch. III, p. 148. For  $n=2$  there is a mistake in SCHOUTEN 1953, 1, p. 74.

<sup>2)</sup> Cf. for the semi-symmetric case SCHOUTEN 1925, 2.

As a corollary we get from (1.5a) that for a projectively euclidean connexion the curvature tensor has at the most  $n^2$  independent components in the general case and  $\frac{1}{2}n(n+1)$  if the connexion is volume preserving.

From (1.12) we see that in a  $D_n$  all the components of  $R_{\nu\mu\lambda}^\kappa$  vanish for which  $\kappa$  is not equal to at least one of the indices  $\nu, \mu$  or  $\lambda$ . It can be proved that for  $n > 2$  this condition, if valid for every coordinate system, is also sufficient:<sup>1)</sup>

*An  $A_n$ ,  $n > 2$ , is a  $D_n$  if and only if the equation*

$$(1.17) \quad R_{\nu\mu\lambda}^\kappa = 0 \quad \text{for } \kappa \neq \nu, \mu, \lambda$$

*holds for every coordinate system at every point.*

Another n.a.s. condition can be derived as follows. The  $X_{n-1}$  with the equation  $f(\xi^\kappa) = c$  is geodesic (V § 7) if and only if  $v^\mu v^\lambda V_\mu V_\lambda f = 0$  is a consequence of  $v^\mu V_\mu f = 0$  and  $f = c$ . Hence a n.a.s. condition is that (cf. Exerc. I 3,2)

$$(1.18) \quad V_\mu w_\lambda = 2q_{(\mu} w_{\lambda)}; \quad w_\lambda \stackrel{\text{def}}{=} V_\lambda f,$$

at all points of the  $X_{n-1}$ . Now the first integrability conditions of this equation are

$$(1.19) \quad \left\{ \begin{array}{l} -R_{\nu\mu\lambda}^\kappa w_\kappa = 2(V_{[\nu} q_{\mu]}) w_\lambda + 2q_{[\mu} V_{\nu]} w_\lambda + 2w_{[\mu} V_{\nu]} q_\lambda \\ \quad = 2w_\lambda V_{[\nu} q_{\mu]} + 2w_{[\nu} (q_{\mu]} q_\lambda - V_{\mu]} q_\lambda). \end{array} \right.$$

If (1.18) is totally integrable, that is if (1.19) can be identically satisfied for every value of  $w_\lambda$ , then for every point of  $A_n$  and every  $(n-1)$ -direction through this point, there exists a geodesic  $X_{n-1}$  having at this point this given  $(n-1)$ -direction. Now taking  $w_\lambda = \dot{e}_\lambda^i$  it follows from (1.19) that  $R_{\nu\mu\lambda}^\kappa \dot{e}_\lambda^i = 0$  for  $\lambda, \mu, \nu \neq 1$  and this holds also mutatis mutandis for  $w_\lambda = \dot{e}_\lambda^2, \dots, w_\lambda = \dot{e}_\lambda^n$  and for every choice of the coordinate system. Hence, for  $n > 2$ , the  $A_n$  must be a  $D_n$  and because in an  $E_n$  and therefore also in a  $D_n$  there exist geodesic  $X_{n-1}$ 's through every point in every  $(n-1)$ -direction, the condition is also necessary:<sup>2)</sup>

*An  $A_n$ ,  $n > 2$ , is projectively euclidean if and only if there exist through every point geodesic  $X_{n-1}$ 's with every  $(n-1)$ -direction at that point.*

In the case when the  $A_n$  is already an  $E_n$  we may require all projective transformations of the  $I_{\mu\lambda}^\kappa$  for which the  $E_n$  remains an  $E_n$ . From (1.4, 5) it follows that  $p_{\mu\lambda} = 0$  and that accordingly  $p_\lambda$  is a solution of the equation

$$(1.20) \quad \partial_\mu p_\lambda - \partial_\mu p_\lambda = 0$$

<sup>1)</sup> E II 1938, 2, p. 181.

<sup>2)</sup> E II 1938, 2, p. 182.

if the coordinate system is chosen such that  $I_{\mu\lambda}^* = 0$ . From (1.20) it follows that

$$(1.21) \quad d\dot{p}_\lambda = \dot{p}_\lambda \dot{p}_\mu d\xi^\mu$$

and this means that  $\dot{p}_\lambda$  is a parallel field (cf. III § 1). Taking the rectilinear coordinate system such that  $\dot{p}_\lambda = \dot{p} \dot{e}_\lambda^I$  we get for  $\dot{p}$  the equations

$$(1.22) \quad \partial_1 \dot{p} = \dot{p} \dot{p}; \quad \partial_\beta \dot{p} = 0; \quad \beta = 2, \dots, n$$

leading to the general solution for  $\dot{p}_\lambda$

$$(1.23) \quad \dot{p}_\lambda = \dot{p} \dot{e}_\lambda^I = -(\xi^I + C)^{-1} \dot{e}_\lambda^I; \quad C = \text{const.}$$

For every choice of  $\dot{e}_\lambda^I$  and  $C$  there is one solution. If for any definitely chosen solution we lay the first  $E_{n-1}$  of the  $\dot{p}_\lambda$  belonging to the point  $\xi^*$  through this point, the second  $E_{n-1}$  goes through the point  $\xi^I = -C$  on the 1-axis.<sup>1)</sup> Hence this second  $E_{n-1}$  is common to the  $\dot{p}_\lambda$ 's of the whole field. In fact it is the improper  $E_{n-1}$  for the transformed connexion. To prove this we take the affine parameter  $z$  on the geodesic 1-axis for the given connexion equal to  $\xi^I$ . Then, according to (III 7.11) we get for the affine parameter belonging to the transformed connexion

$$(1.24) \quad \left\{ \begin{array}{l} z = C_1 \int e^{+2 \int \dot{p}_\mu d\xi^\mu} dz + C_2 \\ = C_1 \int e^{-2 \log(\xi^I + C)} d\xi^I + C_2 \\ = -C_1 (\xi^I + C)^{-1} + C_2 \end{array} \right.$$

which takes the value  $\infty$  for  $\xi^I = -C$ . Hence

*If the connexion of an  $E_n$  is transformed projectively so that the  $E_n$  remains an  $E_n$ , the vector field  $\dot{p}_\lambda$  consists of parallel covariant vectors which all have the same second  $E_{n-1}$  and this  $E_{n-1}$  is the improper  $E_{n-1}$  of the transformed  $E_n$ .<sup>2)</sup>*

### Exercises.

VI 1,1<sup>3)</sup>. Prove that  $V_{v\mu} = 0$  in a  $D_n$  in which a covariant constant field  $v^*$  exists.

VI 1,2. An  $A_n$  is a  $E_n$  if and only if for every value of  $\overset{\theta}{p}_\lambda$  at an arbitrary point  $\overset{\theta}{\xi}^*$  there exists a projective transformation that leaves the curvature tensor invariant and whose vector takes the value  $\overset{\theta}{p}_\lambda$  at  $\overset{\theta}{\xi}^*$ .

<sup>1)</sup> Cf. for the geometric representation of a covariant vector I § 2.

<sup>2)</sup> R. K. 1924, 1, p. 133; E II 1938, 2, p. 184.

<sup>3)</sup> LEVINE 1949, 1. The term parallel is used there in the sense of covariant constant. Cf. III § 2, VI § 9. Cf. LEVINE 1950, 1 for the  $C_n$ .

VI 1,3<sup>1)</sup>. The projective transformations with vectors  $p_\lambda$ ,  $2p_\lambda$  and  $3p_\lambda$  transform  $R_{\nu\mu\lambda}^\zeta$  into  $\bar{R}_{\nu\mu\lambda}^1$ ,  $\bar{R}_{\nu\mu\lambda}^2$  and  $\bar{R}_{\nu\mu\lambda}^3$ . Prove that

$$\text{VI 1,3 } \alpha) \quad R_{\nu\mu\lambda}^\zeta - 3\bar{R}_{\nu\mu\lambda}^1 + 3\bar{R}_{\nu\mu\lambda}^2 - \bar{R}_{\nu\mu\lambda}^3 = 0.$$

VI 1,4<sup>2)</sup>. Prove the identity for  $A_n$ ,  $n > 2$

$$\text{VI 1,4 } \alpha) \quad V_{[\omega} P_{\nu\mu]}^\zeta - \frac{1}{n-2} A_{[\omega}^\zeta V_{|\sigma]} P_{\nu\mu]}^\sigma = 0.$$

## § 2. Projective transformation of the connexion in a $V_n$ .<sup>3)</sup>

Every  $S_n$  is a  $D_n$  because in consequence of (cf. III § 5)

$$(2.1) \quad K_{\nu\mu\lambda}^\zeta = 2\zeta A_{[\nu}^\zeta g_{\mu]\lambda}$$

the equations (1.12) are identically satisfied for  $P_{\mu\lambda} = -\zeta g_{\mu\lambda}$  and  $P_\lambda$  is a solution of the equations

$$(2.2) \quad V_\mu P_\lambda - P_\mu P_\lambda = \zeta g_{\mu\lambda}$$

whose integrability conditions are identically satisfied. Because two  $D_n$ 's can always be mapped on each other so that geodesics remain geodesics, the same holds for two  $S_n$ 's and for an  $S_n$  and an  $R_n$ .<sup>4)</sup>

If, in a  $V_n$ , a new (symmetric) fundamental tensor ' $g_{\lambda\zeta}$ ' is introduced, a new connexion arises. As follows from (1.1) this change of connexion preserves the geodesics if and only if a vector  $p_\lambda$  exists such that

$$(2.3) \quad V_\mu 'g_{\lambda\zeta} = 2p_\mu 'g_{\lambda\zeta} + p_\lambda 'g_{\mu\zeta} + p_\zeta 'g_{\lambda\mu}.$$

By transvection with ' $\bar{g}^{\lambda\zeta}$ ' we get (cf. III 3.12)

$$(2.4) \quad \begin{cases} 2(n+1)p_\mu = \bar{g}^{\lambda\zeta} V_\mu 'g_{\lambda\zeta} = \bar{g}^{\lambda\zeta} \partial_\mu 'g_{\lambda\zeta} - 2\Gamma_\mu = \partial_\mu \log \frac{'g}{g}; \\ 'g \stackrel{\text{def}}{=} |\text{Det}('g_{\lambda\zeta})|. \end{cases}$$

As a corollary we get that  $p_\lambda$  is a gradient vector, which could have been foreseen because the connexion remains riemannian and therefore remains volume preserving. From (2.3) and (2.4) it follows that the transformation of the connexion is projective if and only if

$$(2.5) \quad 2(n+1)V_\mu 'g_{\lambda\zeta} = 'g_{\lambda\zeta} \partial_\mu \log \frac{'g}{g} + 3 'g_{(\lambda\zeta} \partial_{\mu)} \log \frac{'g}{g}.$$

<sup>1)</sup> BERWALD gave a similar proposition for conformal transformations in  $V_2$  (Arch. Math. u. Phys. 1918, 27, p. 81; Jahr. D. M. V. 1923, 22, p. 48).

<sup>2)</sup> R. K. 1924, 1, p. 132; E II 1938, 2, p. 182.

<sup>3)</sup> Cf. E II 1938, 2, p. 185 ff. also for  $S_n$ .

<sup>4)</sup> Note that we do not consider spaces in the large.

The differential equation of the geodesics of an  $A_n$

$$(2.6) \quad \frac{\delta}{dz} \frac{d\xi^\lambda}{ds} = 0$$

is said to have a *first integral of degree p*<sup>1)</sup> if a tensor field  $q_{\lambda_1 \dots \lambda_p}$  exists such that

$$(2.7) \quad q_{\lambda_1 \dots \lambda_p} \frac{d\xi^{\lambda_1}}{ds} \dots \frac{d\xi^{\lambda_p}}{ds} = \text{const.}$$

along every geodesic, and this condition is satisfied if and only if

$$(2.8) \quad V_{(\mu} q_{\lambda_1 \dots \lambda_p)} = 0.$$

Now it follows from (2.5) that

$$(2.9) \quad V_{(\mu} \mu^2 'g_{\lambda)\nu} = 0; \quad \mu^{-(n+1)} = \frac{'g}{g}$$

and this proves that in a  $V_n$  which can be transformed projectively into another  $V_n$ , the equation of the geodesics admits besides  $g_{\lambda\nu} \frac{d\xi^\lambda}{ds} \frac{d\xi^\nu}{ds} = \text{const.}$  another first integral  $\mu^2 'g_{\lambda\nu} \frac{d\xi^\lambda}{ds} \frac{d\xi^\nu}{ds} = \text{const.}$

If the principal directions of ' $g_{\lambda\nu}$ ' are uniquely determined, it follows from (2.3) and (V 5.17) that they are all  $V_{n-1}$ -normal. The integrability conditions of (2.3) are

$$(2.10) \quad 2K_{\nu\mu}^{\lambda} \dot{g}_{\lambda\nu} + 4p_{[\nu} (\lambda 'g_{\mu]\lambda}) = 0$$

because  $p_\lambda$  is a gradientvector. This equation could also be derived from (1.3) by substituting  $K_{\nu\mu}^{\lambda} \dot{g}_{\lambda\nu}$  and ' $K_{\nu\mu}^{\lambda} \dot{g}_{\lambda\nu}$ ' for  $R_{\nu\mu}^{\lambda} \dot{g}_{\lambda\nu}$  and ' $R_{\nu\mu}^{\lambda} \dot{g}_{\lambda\nu}$ ' because  $p_{[\nu\mu]} = 0$  and ' $K_{\nu\mu}^{\lambda} \dot{g}_{\lambda\nu}$ ' = 0.

If the  $V_n$  is an  $S_n$  (2.10) takes the form (cf. III § 5)

$$(2.11) \quad 0 = + 4\kappa g_{[\nu} (\lambda A_{\mu]}^\nu 'g_{\lambda}\nu) + 4p_{[\nu} (\lambda 'g_{\mu]\lambda}) = 4(p_{[\nu} (\lambda + \kappa g_{[\nu} (\lambda) 'g_{\mu]\lambda})$$

but this equation expresses the fact that  $p_{\nu\lambda} + \kappa g_{\nu\lambda}$  and ' $g_{\nu\lambda}$ ' differ only by a scalar factor. Let this factor be  $\alpha$

$$(2.12) \quad p_{\nu\lambda} + \kappa g_{\nu\lambda} = \alpha 'g_{\nu\lambda}$$

then we get from (1.3)

$$(2.13) \quad 'K_{\nu\mu}^{\lambda} \dot{g}_{\lambda\nu} = -2\kappa g_{[\nu} (\lambda A_{\mu]}^\nu - 2\kappa A_{[\nu}^\lambda g_{\mu]\lambda} + 2\alpha A_{[\nu}^\lambda 'g_{\mu]\lambda} = 2\alpha A_{[\nu}^\lambda 'g_{\mu]\lambda}$$

and this proves that for  $n > 2$  after the transformation once more we have an  $S_n$  but with the scalar curvature  $+\alpha$ . Hence

<sup>1)</sup> Cf. EISENHART 1924, 1; 1927, 1, p. 83–86; LIBER 1941, 1.

If the connexion of an  $S_n$ ,  $n > 2$ , is transformed projectively and if the transform happens to be the connexion of a  $V_n$ , this  $V_n$  is an  $S_n$ .<sup>1)</sup>

As a corollary we get that a  $V_n$  is projectively euclidean if and only if it is an  $S_n$ .<sup>2)</sup>.

We return now to the general  $V_n$ . By transvecting (2.10) with  $g^{\mu\lambda}$  we get

$$(2.14) \quad K_{v}^{\cdot\theta} g_{\theta\lambda} + K_{v\lambda}^{\cdot\theta} g_{\theta\sigma} = p_v^{\cdot\theta} g_{\theta\lambda} - p_{\theta\sigma} g^{\theta\sigma} g_{\theta\lambda} + p_{v\lambda}^{\cdot\theta} g_{\theta\sigma} g^{\theta\sigma} - p_{\theta\lambda}^{\cdot\theta} g_{\theta\sigma}$$

hence from (2.14) and (2.10) transvected with  $g^{\lambda\mu}$

$$(2.15) \quad K_{v}^{\cdot\theta} g_{[\theta]\lambda} = 0.$$

But this equation expresses the fact that the symmetric tensors ' $g_{\lambda\mu}$ ' and ' $K_{\lambda\mu}$ ' have at least one set of  $n$  mutually perpendicular principal directions in common (cf. Exerc. I 9,3). Transvection of (2.10) with  $g^{\lambda\mu}$  leads to

$$(2.16) \quad p_{[v}^{\cdot\lambda} g_{\mu]\lambda} = 0$$

and this expresses the fact that also the symmetric tensors ' $g_{\lambda\mu}$ ' and ' $p_{\lambda\mu}$ ' have at least one set of  $n$  mutually perpendicular principal directions in common. But (2.10) contains still more information about these principal directions. If a local coordinate system ( $h$ ) with unit basis vectors is laid in principal directions common to ' $g_{\lambda\mu}$ ' and ' $p_{\lambda\mu}$ ', (2.10) can be written as

$$(2.17) \quad K_{kjih} ('g_{hh} - 'g_{ii}) \stackrel{*}{=} p_{ki} 'g_{jh} - p_{ji} 'g_{kh} + p_{kh} 'g_{ji} - p_{jh} 'g_{ki}.$$

For  $k=j$  both sides vanish and the same holds for  $i=h$ . For  $k, j, i \neq h$  we get<sup>3)</sup>

$$(2.18a) \quad K_{kjih} \stackrel{*}{=} 0$$

provided that  $h$  and  $i$  do not belong to the same principal multidirection. For  $k=i$ ,  $h=j$  we get

$$(2.18b) \quad K_{iji} ('g_{jj} - 'g_{ii}) \stackrel{*}{=} p_{ii} 'g_{jj} - p_{jj} 'g_{ii}.$$

Hence, if ' $g_{ii} = 'g_{jj}$ ', it follows that  $p_{ii} = p_{jj}$  and this means that every principal multidirection of ' $g_{\lambda\mu}$ ' is contained in a principal multidirection of ' $p_{\lambda\mu}$ '. But this has as a consequence that every principal direction of ' $g_{\lambda\mu}$ ' is also a principal direction of ' $p_{\lambda\mu}$ '. Hence the mutually perpendicular principal directions common to ' $g_{\lambda\mu}$ ' and ' $K_{\lambda\mu}$ ' are also principal directions of ' $p_{\lambda\mu}$ ', and thus from (1.4a) in the form

$$(2.19) \quad 'K_{\mu\lambda} = K_{\mu\lambda} + (n-1) p_{\mu\lambda}$$

<sup>1)</sup> BELTRAMI 1868, 1; cf. EISENHART 1926, 1, p. 134.

<sup>2)</sup> WEYL 1921, 2, p. 110.

<sup>3)</sup> FUBINI 1905, 1, p. 306.

it follows that they are also principal directions of ' $K_{\lambda \nu}$ '. This proves the theorem

The symmetric tensors ' $g_{\lambda \nu}$ ', ' $\rho_{\lambda \nu}$ ', ' $K_{\lambda \nu}$ ' and ' $K'_{\lambda \nu}$  have always at least one set of  $n$  mutually perpendicular principal directions in common.<sup>1)</sup>

We now consider the special case where the principal directions of ' $g_{\lambda \nu}$ ' are uniquely determined. Taking the anholonomic coordinate system ( $h$ ) with unit basis vectors  $i^x_i$ ,  $i^h_\lambda$  in these directions, (2.3) takes the form

$$(2.20) \quad \left\{ \begin{array}{l} \partial_j 'g_{ih} - 2\Gamma_{j(i}^l 'g_{h)l} \stackrel{*}{=} -'g_{ih} \partial_j \log \rho - \frac{1}{2}'g_{hj} \partial_i \log \rho - \frac{1}{2}'g_{ij} \partial_h \log \rho; \\ \rho_i \stackrel{\text{def}}{=} -\frac{1}{2} \partial_i \log \rho \end{array} \right.$$

and from this we derive (cf. III 9.11) for all possible assumptions with respect to  $h$ ,  $i$  and  $j$

$$(2.21) \quad \left\{ \begin{array}{ll} \text{a)} & \Gamma_{jih} ('g_{hh} - 'g_{ii}) \stackrel{*}{=} 0; & (h, i, j \neq) \\ \text{b)} & \partial_j 'g_{ii} \stackrel{*}{=} -'g_{ii} \partial_j \log \rho; & (h = i \neq j) \\ \text{c)} & \Gamma_{iih} ('g_{ii} - 'g_{hh}) \stackrel{*}{=} -\frac{1}{2}'g_{ii} \partial_h \log \rho; & (h \neq i = j) \\ \text{d)} & \partial_i 'g_{ii} \stackrel{*}{=} -2'g_{ii} \partial_i \log \rho; & (h = i = j). \end{array} \right.$$

From (2.21 a) and (V 5.12) it follows once more that the  $n$  congruences  $i^x_i$  are  $V_{n-1}$ -normal. Accordingly the coordinate system ( $\nu$ ) can be chosen such that its contravariant basis vectors are tangent to those congruences. This means that the linear element can be written in the form

$$(2.22) \quad ds^2 = \sum_h \overset{h}{H} \overset{h}{H} \overset{h}{e}_\mu^h \overset{h}{e}_\lambda^h d\xi^\mu d\xi^\lambda; \quad \overset{h}{e}_\lambda^h \stackrel{\text{def}}{=} \delta_\nu^h \overset{x}{e}_\lambda^x$$

and

$$(2.23) \quad \overset{h}{i}_\lambda = \overset{h}{H} \overset{h}{e}_\lambda.$$

Then we have for  $h \neq i$

$$(2.24) \quad \left\{ \begin{array}{l} \Gamma_{ii}^h \stackrel{*}{=} -V_i \overset{h}{i}_i \stackrel{*}{=} -i^h_i i^\lambda V_\mu \overset{h}{i}_\lambda = i^h_i i^\lambda V_\mu i_\lambda^i = i^\mu i^\lambda V_\mu H^i e_\lambda \\ = H i^\mu i^\lambda V_\mu e_\lambda = H i^\mu i^\lambda V_\mu H^{-1} i_\lambda^i \stackrel{*}{=} H \partial_h H^{-1} \end{array} \right.$$

and accordingly from (2.21 b, c, d), writing  $\varrho_i$  for the ' $g_{ii}$ ' from (2.21)

$$(2.25) \quad \left\{ \begin{array}{ll} \text{a)} & \partial_j \rho \varrho_i \stackrel{*}{=} 0; & i \neq j \\ \text{b)} & (\varrho_h - \varrho_i) \partial_h \log \overset{i}{H} \stackrel{*}{=} \frac{1}{2} \partial_h \varrho_i; & h \neq i \\ \text{c)} & \partial_i \rho \varrho_i + \varrho_i \partial_i \rho \stackrel{*}{=} 0 \end{array} \right.$$

<sup>1)</sup> R. K. 1924, 1, p. 206.

in which  $\partial_j$  can be interpreted as  $A_j^\mu \partial_\mu$  but also as  $\delta_j^\mu \partial_\mu$ , because  $A_j^\mu \stackrel{j}{\equiv} H^{-1} \delta^\mu$  (no summation over  $j$ ). The integrability conditions of (2.25) are the well known equations of LAMÉ<sup>1)</sup>

$$(2.26) \quad \partial_j \partial_i H^{\frac{h}{j}} \stackrel{i}{\equiv} H^{-1} (\partial_j H^{\frac{i}{j}}) \partial_i H^{\frac{h}{j}} + H^{-1} (\partial_i H^{\frac{j}{i}}) (\partial_j H^{\frac{h}{j}}); \quad h, i, j \neq; \\ \text{(no summation over } i \text{ and } j\text{)}$$

which are equivalent to

$$(2.27) \quad K_{kji} \stackrel{k}{\equiv} 0; \quad k, j, i \neq$$

and this is a special case of (2.18a).

### Exercises.

VI 2,1<sup>2)</sup>. A  $V_n$  can be transformed projectively into a  $V_n$  if and only if there exist a gradient vector  $p_\lambda \neq 0$  and a symmetric tensor ' $g_{\lambda\kappa}$ ' of rank  $n$  satisfying (2.3).

VI 2,2<sup>3)</sup>. In a  $V_n$  a covariant constant symmetric tensor field ' $g_{\lambda\kappa}$ ' of rank  $n$  can exist if and only if

a) the differential equation of the geodesics admits a first integral ' $g_{\lambda\kappa} \frac{d\xi^\lambda}{dz} \frac{d\xi^\kappa}{dz} = \text{const.}$ ';

b) the connexion of the  $V_n$  can be transformed projectively into the riemannian connexion belonging to ' $g_{\lambda\kappa}$ '.

### § 3. Imbedded spaces in $A_n$ under projective transformations of the connexion.<sup>4)</sup>

Let  $\eta^a$ ;  $a = 1, \dots, m$  be a set of coordinates in an  $X_m$  imbedded and rigged in  $A_n$  and let  $\Gamma'{}^a_{cb}$  be the parameters of the connexion induced in  $X_m$  (cf. V § 3). Then we have for each vector field  $v^a$  of  $X_m$

$$(3.1) \quad V'_c v^a = \partial_c v^a + \Gamma'{}^a_{cb} v^b = B'{}^a_{c\kappa} (\partial_\mu v^\kappa + \Gamma^\kappa_{\mu\lambda} v^\lambda)$$

<sup>1)</sup> Cf. BIANCHI-LUKAT 1899, 1, p. 485. Cf. for the integration of (2.25) LEVI CIVITA 1896, 1; FUBINI 1905, 1; WRIGHT 1908, 1, p. 80ff.

<sup>2)</sup> E II 1938, 2, p. 186.

<sup>3)</sup> LEVY 1925, 2; EISENHART 1927, 1, p. 80; E II 1938, 2, p. 193. Cf. CARTAN 1927, 3 for another property of  $V_n$ 's of this kind.

<sup>4)</sup> General references on projective imbedding: SCHOUTEN and HAANTJES 1936, 1; BORTOLOTTI 1941, 2; BOMPIANI 1943, 1; KIMPARA 1943, 1; KANITANI 1943, 1; 2; 1947, 1; 1948, 1; 1949, 1; 1950, 2; NORDEN 1945, 3; 1947, 1; 1948, 1; 1949, 1; BOL 1950, 1.

hence (cf. V 3.10) <sup>1)</sup>

$$(3.2) \quad \Gamma'{}^a_{cb} = B^{\mu\lambda a}_{cb\nu} \Gamma^\nu_{\mu\lambda} - B^{\mu\lambda}_{cb} \partial_\mu B^\nu_\lambda.$$

If now the connexion in  $A_n$  suffers the projective transformation (1.1) we get for the new connexion in  $X_m$

$$(3.3) \quad \begin{cases} \Gamma'{}^a_{cb} = B^{\mu\lambda a}_{cb\nu} (\Gamma^\nu_{\mu\lambda} + p_\mu A^\nu_\lambda + p_\lambda A^\nu_\mu) - B^{\mu\lambda}_{cb} \partial_\mu B^\nu_\lambda \\ = \Gamma'{}^a_{cb} + p'_c B^\nu_b + p'_b B^\nu_c; \quad p'_b \stackrel{\text{def}}{=} B^\lambda_b p_\lambda \end{cases}$$

and this proves that *the induced connexion is also transformed projectively*. From this it follows at once that a geodesic  $X_m$  in  $A_n$  remains geodesic. This is also one of the consequences of the fact that the quantities  $\overset{m}{H}_{\mu\lambda}{}^\nu$  and  $\overset{m'}{H}_{\mu\lambda}{}^\nu$  defined in (V 7.9, 11) are invariant for projective transformations of the connexion in  $A_n$ . The quantities  $\overset{m}{L}_{\mu\cdot\lambda}{}^\nu$  and  $\overset{m'}{L}_{\mu\cdot\lambda}{}^\nu$  are in general not invariant, for instance

$$(3.4) \quad \begin{cases} \overset{m}{L}_{\mu\cdot\lambda}{}^\nu = \overset{m}{L}_{\mu\cdot\lambda}{}^\nu + B_{\mu\sigma}^\nu (-p_\tau A_\lambda^\sigma B_\sigma^\nu - p_\lambda A_\tau^\sigma B_\sigma^\nu + p_\tau A_\sigma^\nu B_\lambda^\sigma + p_\sigma A_\tau^\nu B_\lambda^\sigma) \\ = \overset{m}{L}_{\mu\cdot\lambda}{}^\nu - B_\mu^\nu p_\lambda''; \quad p_\lambda'' \stackrel{\text{def}}{=} C_\lambda^\nu p_\nu. \end{cases}$$

If for an  $X_{n-1}$  in  $A_n$  the rigging is chosen in the way described in V § 3, it depends on the connexion in  $A_n$ . Because it was assumed in that section that the connexion was volume preserving we need only consider here those projective transformations for which  $p_\lambda$  is a gradient,  $p_\lambda = \partial_\lambda p$ . In V § 3 the vector  $t_\lambda$  was fixed by (V 3.31) and in that formula  $\tilde{q}$  was a covariant constant scalar density field of weight + 1. Let now  $'\tilde{q} = \varrho \tilde{q}$  be covariant constant for the transformed connexion

$$(3.5) \quad 'V_\lambda '\tilde{q} = ('V_\lambda \varrho) \tilde{q} + \varrho 'V_\lambda \tilde{q} = \tilde{q} V_\lambda \varrho - (n+1) \varrho p_\lambda \tilde{q},$$

then

$$(3.6) \quad V_\lambda \log \varrho = (n+1) p_\lambda$$

and according to (V 3.31), if  $'t_\lambda = \sigma t_\lambda$  is the new normalized tangent vector, we have

$$(3.7) \quad \sigma t_\lambda = '\tilde{q} \sigma^{-\frac{n}{2}} h^{-\frac{1}{2}} t_\lambda = \varrho \sigma^{-\frac{n}{2}} t_\lambda$$

or

$$(3.8) \quad (n+2) \log \sigma = 2 \log \varrho$$

at all points of  $X_{n-1}$ , hence

$$(3.9) \quad \partial_b \log \sigma = 2 \frac{n+1}{n+2} p'_b; \quad p'_b \stackrel{\text{def}}{=} B^\lambda_b p_\lambda.$$

---

<sup>1)</sup> We use an accent on the right here to denote the induced connexion in order to preserve the accent on the left for the transformed objects.

For the new vector ' $n^*$ ' we find from (V 3.28)

$$(3.10) \quad 'n^* = \sigma^{-1} (n^* + B_b^* h^{ab} \partial_a \log \sigma) = \sigma^{-1} \left( n^* + 2 \frac{n+1}{n+2} B_b^* h^{ab} 'p_a \right).$$

According to (V 3.11, 12) we get for the new values ' $h_{cb}$ ' and ' $'l_c'$ '

$$(3.11) \quad 'h_{cb} = - B_{cb}^* V_\mu 't_\lambda = \sigma h_{cb} - 2 B_{cb}^* p_\mu t_\lambda = \sigma h_{cb}$$

$$(3.12) \quad \begin{cases} 'l_c^a = - B_{ca}^* V_\mu 'n^* \\ \quad = \sigma^{-1} \left( l_c^a - B_c^a p_\lambda n^\lambda + \frac{n}{n+2} v^a p'_c - B_c^a p'_b v^b - B_{ca}^* V_\mu v^* \right) \end{cases}$$

where

$$(3.13) \quad v^a \stackrel{\text{def}}{=} 2 \frac{n+1}{n+2} h^{ab} p'_b.$$

From (3.11) we see that the asymptotic lines are invariant for the transformations under consideration (cf. V § 3). The lines of curvature have not this independence as can be seen from (3.12).

If a curve is given in  $A_n$  with an arbitrary parameter  $t$ , the covariant derivative  $\frac{\delta}{dt} \frac{d\xi^*}{dt}$  depends on the choice of this parameter. If  $z = z(t)$  is introduced as a new parameter, we get

$$(3.14) \quad \frac{\delta}{dt} \frac{d\xi^*}{dt} = z' \frac{\delta}{dz} z' \frac{d\xi^*}{dz} = z'^2 \frac{\delta^2 \xi^*}{dz^2} + z'' \frac{d\xi^*}{dz}; \quad z' \stackrel{\text{def}}{=} \frac{dz}{dt}; \quad z'' \stackrel{\text{def}}{=} \frac{d^2 z}{dt^2}$$

hence the 2-direction spanned by  $d\xi^*/dt$  and  $\delta^2 \xi^*/dt^2$  is independent of the choice of the parameter. Now if we write

$$(3.15) \quad 't^{\alpha\lambda} \stackrel{\text{def}}{=} \frac{d\xi^*}{dt} \frac{\delta^2 \xi^{\lambda}}{dt^2},$$

this quantity transforms into

$$(3.16) \quad 't^{\alpha\lambda} \stackrel{\text{def}}{=} \frac{d\xi^*}{dz} \frac{\delta^2 \xi^{\lambda}}{dz^2} = z'^{-3} t^{\alpha\lambda}.$$

A curve in  $A_n$  will be called *quasi-plane*<sup>1)</sup> if ' $t^{\alpha\lambda}$ ' is parallel along the curve:

$$(3.17) \quad \frac{\delta}{dt} t^{\alpha\lambda} = \alpha t^{\alpha\lambda}.$$

In this case we have

$$(3.18) \quad z'^{-1} \frac{\delta}{dz} z'^3 't^{\alpha\lambda} = 3z'' 't^{\alpha\lambda} + z'^2 \frac{\delta}{dz} 't^{\alpha\lambda} = \alpha z' 't^{\alpha\lambda}$$

and this proves that

$$(3.19) \quad \frac{\delta}{dz} 't^{\alpha\lambda} = 0$$

---

<sup>1)</sup> YANO and TAKANO 1944, 1; 2; cf. YANO, TAKANO and TOMONAGA 1948, 2; YANO 1949, 2.

provided that  $z$  is a solution of the equation

$$(3.20) \quad \frac{z''}{z'} = \frac{1}{3} \alpha.$$

The general solution of (3.20) has the form

$$(3.21) \quad z = C_1 \int e^{\int \frac{1}{3} \alpha dt} dt + C_2;$$

where  $C_1$  and  $C_2$  are constants and  $C_1 \neq 0$ .  $z$  is called the *affine parameter* of the quasi-plane curve. It is fixed to within an affine transformation.  $C_2$  fixes the nullpoint of the scale and  $C_1$  fixes a kind of gauge. If  $z$  is chosen in this way we call  $n^\kappa = \delta^2 \xi^\kappa / dz^2$  the *affine normal* of the curve. It is fixed to within a *constant* factor. According to (3.19) we have along the curve

$$(3.22) \quad \frac{\delta^3 \xi^\kappa}{dz^3} + k(z) \frac{d \xi^\kappa}{dz} = 0$$

where  $k$  is a function of  $z$ . From (3.17) we get

$$(3.23) \quad \frac{\delta^3 \xi^\kappa}{\delta t^3} - \alpha(t) \frac{\delta^2 \xi^\kappa}{\delta t^2} - \beta(t) \frac{d \xi^\kappa}{dt} = 0$$

and after the introduction of  $z$

$$(3.24) \quad z'^3 \frac{\delta^3 \xi^\kappa}{dz^3} + (z''' - \alpha z'' - \beta z') \frac{d \xi^\kappa}{dz} = 0$$

from which

$$(3.25) \quad k = z'^{-3} (z''' - \alpha z'' - \beta z') = C_1^{-2} e^{-\frac{1}{3} \int \alpha dt} \left( \frac{1}{3} \alpha' - \frac{2}{9} \alpha^2 - \beta \right).$$

If  $k = \text{const.}$ , the curve is called an *affine conic*. Hence the n.a.s. conditions for a quasi-plane curve to be an affine conic are

$$(3.26) \quad \frac{1}{3} \alpha'' - \frac{2}{3} \alpha \alpha' + \frac{4}{27} \alpha^3 + \frac{2}{3} \alpha \beta - \beta' = 0.$$

If the connexion is transformed projectively we have for any vector  $v^\kappa$  along the curve (cf. 1.1)

$$(3.27) \quad \frac{d}{dt} v^\kappa = \frac{\delta}{dt} v^\kappa + p_\mu \frac{d \xi^\mu}{dt} v^\kappa + p_\mu v^\mu \frac{d \xi^\kappa}{dt}$$

hence, for any functions ' $\alpha(t)$ ' and ' $\beta(t)$ '

$$(3.28) \quad \begin{cases} \frac{d^3 \xi^\kappa}{dt^3} - \alpha \frac{d^2 \xi^\kappa}{dt^2} - \beta \frac{d \xi^\kappa}{dt} \\ = \frac{\delta^3 \xi^\kappa}{\delta t^3} - (\alpha - 3p) \frac{\delta^2 \xi^\kappa}{\delta t^2} - (\beta + 3p' - 2\alpha p - 2p^2 - q) \frac{d \xi^\kappa}{dt}; \\ p \stackrel{\text{def}}{=} \frac{d \xi^\mu}{dt} p_\mu; \quad q \stackrel{\text{def}}{=} \frac{d \xi^\mu}{dt} \frac{\delta p_\mu}{dt}. \end{cases}$$

Equating this to zero we get on account of (3.23)

$$(3.29) \quad \begin{cases} {}' \alpha = \alpha + 3p \\ {}' \beta = \beta + 3p' - q - 2p^2 - 2\alpha p. \end{cases}$$

From (3.23) and (3.29) it follows that (cf. 1.1)

$$(3.30) \quad \begin{cases} \frac{1}{3} {}' \alpha'' - \frac{2}{3} {}' \alpha \cdot {}' \alpha' + \frac{4}{27} {}' \alpha^3 + \frac{2}{3} {}' \alpha \cdot {}' \beta - {}' \beta' = \frac{1}{3} \alpha'' - \frac{2}{3} \alpha \alpha' + \\ + \frac{4}{27} \alpha^3 + \frac{2}{3} \alpha \beta - \beta' + 2 \frac{d\xi^\mu}{dt} \frac{\delta^2 \xi^\lambda}{dt^2} p_{\mu\lambda} - \frac{4}{3} \alpha \Pi + \Pi' \\ \Pi \stackrel{\text{def}}{=} p_{\mu\lambda} \frac{d\xi^\mu}{dt} \frac{d\xi^\lambda}{dt} \quad (\text{cf. 1.3}). \end{cases}$$

Hence  $k$  remains a constant for all conics if and only if  $p_{\mu\lambda} = 0$ . This proves the theorem<sup>1)</sup> (cf. 1.3):

*A projective transformation (1.1) of the connexion in  $A_n$  changes all affine conics into affine conics if and only if  $p_{\mu\lambda} = 0$ , that is for  $n \geq 2$ , if and only if the transformation leaves the curvature tensor invariant.*

That means that an affine conic is in general not invariant for all projective transformations.<sup>2)</sup>

### Exercises.

VI 3,1<sup>3)</sup>). If a connexion in  $L_n$  suffers a transformation that preserves parallelism of directions, the transformation of the connexion induced in a rigged  $X_n$  has the same property.

VI 3,2<sup>4)</sup>). Among all parameters  $\Gamma_{\mu\lambda}^x$  in  $L_n$  with the same parallelism of directions there is at most one set that is symmetric in  $\mu\lambda$ .

## § 4. Projective connexions.<sup>5)</sup>

T. Y. THOMAS was the first to remark that the expressions

$$(4.1) \quad \overset{\circ}{\Gamma}_{\mu\lambda}^x \stackrel{\text{def}}{=} \Gamma_{\mu\lambda}^x - \frac{2}{n+1} A_{\mu}^x \Gamma_{\lambda}$$

<sup>1)</sup> YANO and TAKANO 1944, 2, p. 424.

<sup>2)</sup> HOMBURG and MIKAMI 1941, 1 and MIKAMI 1941, 2 defined projective conics. Cf. HOMBURG and MIKAMI 1942, 1; YANO and TAKANO 1944, 1 and 1949, 3 also for literature. It can be proved that an affine conic is transformed into a projective conic by every projective transformation of the connexion.

<sup>3)</sup> BORTOLOTTI 1931, 2, p. 19ff.

<sup>4)</sup> BORTOLOTTI 1931, 4.

<sup>5)</sup> References. General: VEBLEN 1922, 1; 2; 1928, 1; 1929, 1; T. Y. THOMAS 1922, 1; 1925, 2; 3; 1926, 2; 1927, 3; 1934, 2, Ch. III; VEBLEN and T. Y. THOMAS 1923, 1; J. M. THOMAS 1925, 2; VEBLEN and J. M. THOMAS 1926, 1; WEYL 1929, 2; WEYL and ROBERTSON 1929, 1; WHITEHEAD 1929, 1; GOLAB 1930, 1; SCHOUTEN

are invariant for the projective transformations (1.1) of a symmetric connexion. They are called the *projective parameters* of T. Y. THOMAS.<sup>1)</sup> Their transformation is

$$(4.2) \quad \overset{\circ}{\Gamma}_{\mu'\lambda'}^{\kappa'} = A_{\mu'\lambda'}^{\mu\lambda} \overset{\circ}{\Gamma}_{\mu\lambda}^{\kappa} + A_{\kappa'}^{\kappa} \partial_{\mu'} A_{\lambda}^{\lambda} + \frac{2}{n+1} A_{(\mu'}^{\kappa'} \partial_{\lambda)} \log \Delta.$$

Hence the  $\overset{\circ}{\Gamma}_{\mu\lambda}^{\kappa}$  are components of a geometric object of a more complicated kind than the  $\Gamma_{\mu\lambda}^{\kappa}$ . Therefore the  $\overset{\circ}{\Gamma}_{\mu\lambda}^{\kappa}$  do not fix one linear connexion but only an infinity of symmetric linear connexions that transform into each other by projective transformations and that all belong to the same system of geodesics.

If we first prefer one coordinate system and then consider coordinate transformations with  $\Delta = \text{const.}$  only, the  $\overset{\circ}{\Gamma}_{\mu\lambda}^{\kappa}$  transform in the same way as the  $\Gamma_{\mu\lambda}^{\kappa}$  and in this case the  $\overset{\circ}{\Gamma}_{\mu\lambda}^{\kappa}$  represent the only connexion that has the same geodesics as the connexion  $\Gamma_{\mu\lambda}^{\kappa}$  and whose parameters give zero if transvected with  $A_{\kappa}^{\lambda}$ .

The  $\overset{\circ}{\Gamma}_{\mu\lambda}^{\kappa}$  can be used to investigate an affine geometry that is given to within projective transformations. On every geodesic a projective parameter can be defined that is fixed to within an affine or a projective transformation with constant coefficients. The equation of the geodesics can be written in the form (cf. III 7.2)

$$(4.3) \quad \frac{d^2\xi^{\kappa}}{dt^2} + \overset{\circ}{\Gamma}_{\mu\lambda}^{\kappa} \frac{d\xi^{\mu}}{dt} \frac{d\xi^{\lambda}}{dt} = \alpha(t) \frac{d\xi^{\kappa}}{dt} - \frac{2}{n+1} A_{\mu}^{\kappa} \Gamma_{\lambda} \frac{d\xi^{\mu}}{dt} \frac{d\xi^{\lambda}}{dt}$$

with an arbitrary parameter  $t$ . If a new parameter  $\zeta$  is introduced and if we write  $\zeta' \stackrel{\text{def}}{=} d\zeta/dt$ ;  $\zeta'' \stackrel{\text{def}}{=} d^2\zeta/dt^2$ , (4.3) takes the form

$$(4.4) \quad \frac{d^2\xi^{\kappa}}{d\zeta^2} + \overset{\circ}{\Gamma}_{\mu\lambda}^{\kappa} \frac{d\xi^{\mu}}{d\zeta} \frac{d\xi^{\lambda}}{d\zeta} = \frac{1}{\zeta'} \alpha \frac{d\xi^{\kappa}}{d\zeta} - \frac{2}{n+1} \Gamma_{\lambda} \frac{d\xi^{\lambda}}{d\zeta} \frac{d\xi^{\kappa}}{d\zeta} - \frac{\zeta''}{(\zeta')^2} \frac{d\xi^{\kappa}}{d\zeta}$$

and the right hand side of this equation is zero if and only if  $\zeta$  is a solution of

$$(4.5) \quad \frac{\zeta''}{\zeta'} = \alpha - \frac{2}{n+1} \Gamma_{\lambda} \frac{d\xi^{\lambda}}{dt}$$

and GOLAB 1930, 3; EISENHART 1930, 1; BORTOLOTTI 1931, 6; 1932, 1; 1933, 2; v. DANTZIG 1932, 1; 2; 3; 1934, 1; HLAVATY 1933, 1; SCHOUTEN and v. DANTZIG 1933, 2; CARTAN 1934, 1; CARTAN 1935, 1; 1937, 1; 3; BERWALD 1936, 1; SCHOUTEN and HAANTJES 1936, 1; BORTOLOTTI and HLAVATY 1936, 2 (literature); HAANTJES 1937, 1; KANITANI 1941, 1; YANO 1942, 4; CHI TA YEN 1948, 1. *Imbedding:* BORTOLOTTI 1941, 2 (literature); BOMPIANI 1943, 1; KIMPARA 1943, 1; KANITANI 1943, 1; 2; 1947, 1; 2; 1948, 1; 1949, 1; 1950, 2; NORDEN 1945, 3; 1947, 1; 1948, 1; 1949, 1; BOL 1950, 1. *Mapping:* EISENHART and KNEBELMAN 1927, 2; WHITEHEAD 1929, 1 ( $D_n$ ); KANITANI 1947, 2; CZECH 1949, 1; 1950, 1; 2.

<sup>1)</sup> T. Y. THOMAS 1925, 3. The special case  $\Delta = +1$  was called by him *equi-projective*.

hence

$$(4.6) \quad \zeta = C_1 \int e^{\int \alpha dt} e^{-\frac{2}{n+1} \int \Gamma_\lambda d\xi^\lambda} dt + C_2.$$

Now it follows from (1.1) and (III 7.2) that  $\alpha dt - \frac{2}{n+1} \Gamma_\lambda d\xi^\lambda$  is invariant for projective transformations of  $\overset{\circ}{I}_{\mu\lambda}^\kappa$ , hence  $\zeta$  is a projective invariant. It is the *projective parameter* on a geodesic introduced by T. Y. THOMAS.<sup>1)</sup> If  $\overset{\circ}{I}_{\mu\lambda}^\kappa$  is given,  $\zeta$  is determined on each geodesic to within an affine transformation. It is remarkable that  $\zeta$  is not a scalar. If the coordinates are transformed we get

$$(4.7) \quad e^{\frac{2}{n+1} \int \Gamma_{\lambda'} d\xi^{\lambda'}} = e^{\frac{2}{n+1} \int \Gamma_\lambda d\xi^\lambda} e^{-\frac{2}{n+1} \int d \log A}$$

hence

$$(4.8) \quad d\xi^{(\kappa)} = A^{\frac{2}{n+1}} d\xi^{(\kappa)}$$

and this proves that  $d\xi$  is a scalar  $A$ -density of weight  $-\frac{2}{n+1}$ . If a projective parameter on a curve is fixed in any way this parameter can be used to build up the projective differential geometry of the curve, generalized FRENET formulae etc.<sup>2)</sup>

But the  $\overset{\circ}{I}_{\mu\lambda}^\kappa$  can also be used in a more general way. The most simple method is to construct a scalar density that is a concomitant of the  $\overset{\circ}{I}_{\mu\lambda}^\kappa$ .<sup>3)</sup> Such a density is invariant for projective transformations of the connexion. It fixes a set of coordinate systems with respect to which it has the component  $+1$ . These coordinate systems transform into each other with  $A=+1$  and accordingly the  $\overset{\circ}{I}_{\mu\lambda}^\kappa$  now fix a linear connexion that is projectively invariant. The disadvantage of the method is not that it would be difficult to form a projectively invariant density, in fact there are many of them, but there is no one specially preferred and they all depend on higher derivatives. CARTAN<sup>4)</sup> proposed in 1924 to use a flat projective space instead of an affine space as local space. SCHOUTEN<sup>5)</sup> proved in the same year that this amounts to introducing a linear connexion for local projective spaces and that it is intimately connected with the linear connexions of local

<sup>1)</sup> T. Y. THOMAS 1925, 3; cf. VEBLEN and J. M. THOMAS 1926, 1.

<sup>2)</sup> WHITEHEAD 1931, 2 has introduced a scalar projective parameter determined to within a "projective" transformation and BORTOLOTTI 1932, 1 has used another one. Cf. HLAVATY 1931, 1; BERWALD 1936, 1; HAANTJES 1937, 1; SAKI 1937, 1; YANO 1937, 2; 3; 1944, 3; 4; E II 1938, 2, p. 194.

<sup>3)</sup> WEITZENBÖCK 1932, 1.

<sup>4)</sup> CARTAN 1924, 1.

<sup>5)</sup> SCHOUTEN 1924, 2.

$E_N$ 's;  $N \neq n$  already introduced by KÖNIG in 1920.<sup>1)</sup> EISENHART, VEBLEN and T. Y. THOMAS first considered the generalized projective geometry as "geometry of paths"<sup>2)</sup>,<sup>3)</sup> but soon after the introduction of the projective parameter of T. Y. THOMAS several authors developed theories that could all be considered in some way as theories of linear connexion of local  $E_{n+1}$ 's with a restricted transformation group.<sup>4)</sup> Another line of thought found its origin in the idea of v. DANTZIG<sup>5)</sup> of using in  $X_n$  a set of  $n+1$  homogeneous coordinates subjected to homogeneous transformations of degree one. Then local space is automatically a flat projective space and for the vectors in this space (called *projectors*) a linear displacement can be determined uniquely.<sup>6)</sup>,<sup>7)</sup>

### Exercise.

VI 4,1. For every choice of an object  $\overset{*}{\Gamma}_\lambda$  with the transformation

$$\text{VI 4,1 } \alpha) \quad \overset{*}{\Gamma}_{\lambda'} = A_{\lambda'}^\lambda \overset{*}{\Gamma}_\lambda - \partial_{\lambda'} \log A$$

the  $\overset{*}{\Gamma}_{\mu\lambda}^*$  defined by

$$\text{VI 4,1 } \beta) \quad \overset{*}{\Gamma}_{\mu\lambda}^* = \overset{*}{\Gamma}_{\mu\lambda}^{\rho} + \frac{2}{n+1} A_{(\mu}^* \overset{*}{\Gamma}_{\lambda)}^*$$

transform like the parameters of a symmetric linear connexion and  $\overset{*}{\Gamma}_{\mu\lambda}^* = \overset{*}{\Gamma}_{\mu}^*$  (cf. Exerc. VI 7,1).

<sup>1)</sup> KÖNIG 1920, 1.

<sup>2)</sup> EISENHART and VEBLEN 1922, 3; VEBLEN 1922, 1; 1928, 1; VEBLEN and T. Y. THOMAS 1923, 1.

<sup>3)</sup> DOUGLAS gave two generalizations. In 1928, 1 he considered systems of curves such that one and only one curve passes through every point in every direction. A general system of this kind can be defined as solutions of a differential equation of the second order. This theory is intimately connected with FINSLER geometry. Cf. T. Y. THOMAS 1934, 2, p. 7, also for literature. In 1931, 1 DOUGLAS considered systems of  $X_K$ 's ( $K$ -spreads) such that one and only one passes through every point in every  $K$ -direction. There are many recent investigations on the geometry in the space of  $K$ -spreads especially by chinese and japanese authors. These theories are beyond the realm of this book.

<sup>4)</sup> T. Y. THOMAS 1925, 3; 1926, 2; 1934, 2; WEYL 1929, 2; WEYL and ROBERTSON 1929, 1; GOLAB 1930, 1; SCHOUTEN and GOLAB 1930, 3; BORTOLOTTI 1932, 1 (literature); HLAVATY 1933, 1 (literature); BERWALD 1936, 1.

<sup>5)</sup> v. DANTZIG 1932, 1; 2; 3.

<sup>6)</sup> v. DANTZIG 1932, 3; SCHOUTEN and v. DANTZIG 1932, 1; 2; SCHOUTEN and HAANTJES 1936, 1 (literature).

<sup>7)</sup> Cf. for the relations between the different points of view also CARTAN 1924, 1; SCHOUTEN 1924, 2; 3; WEYL 1929, 2; WEYL and ROBERTSON 1929, 1; v. DANTZIG 1932, 1; 2; 1934, 1; YANO 1937, 3; 1938, 4; 5; 6; 1940, 4; 1941, 6; 1942, 4; 1944, 3; 8; 9; 10; 11; 1945, 4; 1947, 1; HLAVATY 1939, 1; YANO and SASAKI 1949, 4.

### § 5. Conformal transformation of a connexion in $V_n$ .

If the fundamental tensor  $g_{\lambda\kappa}$  in  $V_n$  is transformed into

$$(5.1) \quad 'g_{\lambda\kappa} = \sigma g_{\lambda\kappa}; \quad 'g^{\lambda\kappa} = \sigma^{-1} g^{\lambda\kappa} {}^1)$$

where  $\sigma$  is an arbitrary scalar, the parameters  $I_{\mu\lambda}^\kappa$  transform into

$$(5.2) \quad \left\{ \begin{array}{l} 'I_{\mu\lambda}^\kappa = \frac{1}{2} 'g^{\lambda\sigma} (\partial_\mu 'g_{\lambda\sigma} + \partial_\lambda 'g_{\mu\sigma} - \partial_\sigma 'g_{\mu\lambda}) \\ \quad = I_{\mu\lambda}^\kappa + \frac{1}{2} (A_\lambda^\kappa s_\mu + A_\mu^\kappa s_\lambda - g^{\lambda\sigma} g_{\mu\lambda} s_\sigma); \\ s_\lambda \stackrel{\text{def}}{=} \partial_\lambda \log \sigma. {}^2) \end{array} \right.$$

We call this transformation *conformal*. From (5.2) and (III 2.3, 7) we get

$$(5.3) \quad \left\{ \begin{array}{l} \text{a)} \quad 'V_\mu v^\kappa = V_\mu v^\kappa + \frac{1}{2} s_\mu v^\kappa + \frac{1}{2} s_\lambda v^\lambda A_\mu^\kappa - \frac{1}{2} g^{\lambda\sigma} s_\sigma g_{\mu\lambda} v^\lambda \\ \text{b)} \quad 'V_\mu w_\lambda = V_\mu w_\lambda - w_{(\mu} s_{\lambda)} + \frac{1}{2} g_{\mu\lambda} s_\sigma g^{\sigma\tau} w_\tau. \end{array} \right.$$

In dealing with these transformations the process of raising and lowering should be used very carefully because there are now two fundamental tensors  $g_{\lambda\kappa}$  and  $'g_{\lambda\kappa}$  and the process can only be defined with respect to one of them. From (5.3) it follows that

$$(5.4) \quad 'V_\mu v^\mu = V_\mu v^\mu + \frac{1}{2} n s_\mu v^\mu.$$

If a transformation of the  $I_{\mu\lambda}^\kappa$  in  $V_n$  is at the same time projective and conformal we have from (1.1) and (5.2)

$$(5.5) \quad 2p_\mu A_\lambda^\kappa + 2p_\lambda A_\mu^\kappa = A_\lambda^\kappa s_\mu + A_\mu^\kappa s_\lambda - g^{\lambda\sigma} g_{\mu\lambda} s_\sigma$$

from which by transvection with  $g_{\mu\lambda}$  and by contraction with respect to  $\lambda$  two equations can be derived which are consistent if and only if  $s_\lambda = 0$ . Hence

*A riemannian connexion is wholly determined if its geodesics are given and the fundamental tensor to within a scalar factor. Then the fundamental tensor is also fixed to within a constant scalar factor* <sup>3)</sup> *or in other words*

*The metric of a  $V_n$  is determined to within a constant scalar factor by the projective and the conformal properties of the  $V_n$  together.*

<sup>1)</sup> Since raising and lowering of indices is performed by means of  $g^{\lambda\kappa}$  and  $g_{\lambda\kappa}$ ,  $'g^{\lambda\kappa}$  can not be written as  $'g^{\lambda\kappa}$ .

<sup>2)</sup> FUBINI 1909, 1, p. 144; SCHOUTEN 1918, 1; WEYL 1918, 1.

<sup>3)</sup> WEYL 1921, 2, p. 100.

If (5.2) is substituted in (III 4.2) it follows that

$$(5.6) \quad 'K_{\nu\mu\lambda}^{\cdot\cdot\cdot} = K_{\nu\mu\lambda}^{\cdot\cdot\cdot} + g_{[\nu[\lambda} s_{\mu]\sigma]} g^{\sigma\lambda}$$

where

$$(5.7) \quad s_{\mu\lambda} = s_{\lambda\mu} \stackrel{\text{def}}{=} 2V_{\mu} s_{\lambda} - s_{\mu} s_{\lambda} + \frac{1}{2} g_{\mu\lambda} s_{\sigma} s^{\sigma}$$

and accordingly

$$(5.8) \quad 'K_{\mu\lambda} = K_{\mu\lambda} - \frac{1}{4}(n-2)s_{\mu\lambda} - \frac{1}{4}g_{\mu\lambda}s_{\sigma\tau}g^{\sigma\tau}$$

$$(5.9) \quad 'K = \sigma^{-1}K - \frac{1}{2}(n-1)\sigma^{-1}s_{\mu\lambda}g^{\mu\lambda}$$

$$(5.10) \quad 'G_{\mu\lambda} = G_{\mu\lambda} - \frac{1}{4}(n-2)s_{\mu\lambda} + \frac{1}{4}(n-2)g_{\mu\lambda}s_{\sigma\tau}g^{\sigma\tau}.$$

A  $V_n$  and its connexion are said to be *conformally euclidean* and the  $V_n$  is called a  $C_n$  if it can be transformed conformally into an  $R_n$ .<sup>2)</sup> N.a.s. condition is for  $n > 2$  that there exists a vector field  $L_{\lambda}$  such that

$$(5.11) \quad \begin{cases} \text{a)} & K_{\nu\mu\lambda\kappa} = \frac{4}{n-2}g_{[\nu[\lambda} L_{\mu]\kappa]}; \quad L_{[\mu\lambda]} = 0 \\ \text{b)} & -\frac{4}{n-2}L_{\mu\lambda} = 2V_{\mu}L_{\lambda} - L_{\mu}L_{\lambda} + \frac{1}{2}g_{\mu\lambda}L_{\sigma}L^{\sigma} \end{cases}$$

(the factor  $\frac{4}{n-2}$  is introduced only for convenience). The number of the independent components of  $K_{\nu\mu\lambda\kappa}$  is  $\frac{1}{2}n^2(n^2-1)$  and this is  $> \binom{n+1}{2}$  for  $n > 3$ , hence (5.11 a) can always be satisfied for  $n \leq 3$  but for  $n > 3$  only if  $K_{\nu\mu\lambda\kappa}$  satisfies certain algebraical conditions. In this case we get from (5.8) for  $s_{\mu\lambda} = -\frac{4}{n-2}L_{\mu\lambda}$  and  $'K_{\mu\lambda} = 0$

$$(5.12) \quad L_{\mu\lambda} = -K_{\mu\lambda} + \frac{1}{2(n-1)}Kg_{\mu\lambda}.$$

The first integrability condition of (5.11 b) is

$$(5.13) \quad \frac{n-2}{4}K_{\nu\mu\lambda}^{\cdot\cdot\cdot}L_{\kappa} = +V_{[\nu}L_{\mu]\lambda} + g_{[\nu[\lambda}L_{\mu]\kappa]}L^{\kappa}$$

from which we get by substituting (5.11 a)

$$(5.14) \quad V_{[\nu}L_{\mu]\lambda} = 0.$$

<sup>1)</sup> Note that  $\sigma^{-1}'K_{\nu\mu\lambda\kappa} = \sigma^{-1}g_{\kappa\theta}'K_{\nu\mu\lambda}^{\cdot\cdot\theta} = K_{\nu\mu\lambda\kappa} + g_{[\nu[\lambda}s_{\mu]\kappa]}g^{\theta\lambda}$ ; cf. STRUIK 1922, 1, p. 151; FINZI 1923, 1; SCHOUTEN and STRUIK 1921, 4; R. K. 1924, 1, p. 168, footnote <sup>3)</sup>.

<sup>2)</sup> Cf. for the geometry of a  $C_n$ : HAANTJES and WRONA 1939, 1; HAANTJES 1940, 1; 1941, 1; 1942, 1; 2; 3; 1943, 1; YANO 1945, 4; 5; 6; 1946, 2; 3; WRONA 1948, 1; 2; 1949, 1; 1950, 1; LEVINE 1950, 1 (parallel fields); MATSUMOTO 1951, 3 (class one).

For an  $S_n$ ,  $n \geq 2$  we have (cf. III 5.31) a solution of (5.11a)

$$(5.15) \quad L_{\mu\lambda} = -\frac{1}{2}(n-2)\varkappa g_{\mu\lambda}$$

and for  $n > 2$  this is the only solution. Because  $\varkappa$  is constant (5.14) is satisfied and this implies that every  $S_n$  is a  $C_n$ . For a  $V_2$  which is not an  $S_2$  (5.15) is also a solution of (5.11a) but it does not satisfy (5.14) because  $\varkappa$  is no longer constant. But in this case (5.11a) has an infinite number of solutions and it is well known from elementary differential geometry that every  $V_2$  can be conformally mapped on an  $R_2$ .<sup>1)</sup> For a  $V_n$  the identity of BIANCHI (III 5.21) can be applied to (5.11a). Then we get

$$(5.16) \quad (n-3)V_{[\nu}L_{\mu]\lambda} = 0.$$

Hence for  $n > 3$  the integrability condition of (5.11b) is a consequence of (5.11a). That proves<sup>2)</sup>

A  $V_2$  is always a  $C_2$ . A  $V_n$ ,  $n > 2$ , is conformally euclidean if and only if

$$(5.17) \quad C_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda} \stackrel{\text{def}}{=} K_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda} - \frac{4}{n-2}g_{[\nu[\lambda}L_{\mu]\rho]}g^{\rho\lambda} = 0$$

where

$$(5.18) \quad L_{\mu\lambda} \stackrel{\text{def}}{=} -K_{\mu\lambda} + \frac{1}{2(n-1)}Kg_{\mu\lambda}$$

and if for  $n=3$  moreover

$$(5.19) \quad V_{[\nu}L_{\mu]\lambda} = 0.$$

For  $n=3$ , (5.17) is identically satisfied and for  $n > 3$ , (5.19) is a consequence of (5.17).

$C_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda}$  is called the *conformal curvature tensor*. From (5.8, 12) it follows that

$$(5.20) \quad 'L_{\mu\lambda} = L_{\mu\lambda} + \frac{n-2}{4}s_{\mu\lambda}$$

and by substituting this in (5.17) it follows that  $C_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda}$  (but not  $C_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda}$ ) is invariant for conformal transformations of the connexion.<sup>3)</sup> For  $n \leq 3$  it vanishes identically and for  $n > 3$  it satisfies the identities

$$(5.21) \quad \begin{cases} \text{a)} & C_{(\nu\mu)\lambda}^{\cdot\cdot\cdot\lambda} = 0; \\ \text{c)} & C_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda} = 0; \end{cases} \quad \begin{cases} \text{b)} & C_{[\nu\mu]\lambda}^{\cdot\cdot\cdot\lambda} = 0; \\ \text{d)} & C_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda} = 0; \\ \text{e)} & C_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda} = -C_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda}. \end{cases}$$

<sup>1)</sup> Cf. for instance WEATHERBURN 1927, 1, p. 168; HAACK 1948, 1.

<sup>2)</sup> SCHOUTEN 1921, 1, p. 82ff.; COTTON 1899, 1, p. 412 and FINZI 1902, 1 for  $n=3$ . Cf. FINZI 1921, 1. WEYL proved 1918, 1, p. 404 that  $C_{\nu\mu\lambda}^{\cdot\cdot\cdot\lambda}$  is zero in a  $C_n$ . FINZI proved 1921, 1 that (5.17) and (5.19) together are n.a.s. for the  $V_n$  to be a  $C_n$ . Cf. also FINZI 1922, 1; CARTAN 1922, 3 for  $n=4$  and LAGRANGE 1923, 1, p. 43 for  $n \neq 3$ . DOUGLAS gave a synthetic criterion 1925, 1.

<sup>3)</sup> Cf. for the semi-symmetric case SCHOUTEN 1925, 2.

As a corollary we get that for a conformally euclidean connexion the curvature tensor has at most  $\frac{1}{2}n(n+1)$  independent components.

From (5.11a) we see that for a  $C_n$  all components of  $K_{\nu\mu\lambda\kappa}$  with respect to an (holonomic or anholonomic) orthogonal coordinate system, with four different indices vanish. It can be proved<sup>1)</sup> that for  $n > 3$  this condition is also sufficient:

*A  $V_n$ ,  $n > 3$ , is a  $C_n$  if and only if with respect to every holonomic or anholonomic orthogonal coordinate system ( $h$ )*

$$(5.22) \quad K_{k_1 i h} \stackrel{*}{=} 0; \quad h, i, j, k \neq$$

*at every point.<sup>2)</sup>*

Another form of the n.a.s. conditions is found as follows. We consider here only an ordinary  $V_n$  and real subspaces. Let a  $V_n$  be such that it contains orthogonal systems (cf. definition V § 5) in every direction, i.e. that for every set of  $n$  directions at any arbitrarily chosen point there exists at least one orthogonal system having just these directions at that point. Then writing  $i^{\alpha}_b$ ;  $b = 1, \dots, n-1$  and  $i^{\alpha}_n$  for the unitvectors in the directions of this system we have (cf. V 5.16b)

$$(5.23) \quad 0 = 2 \sum_d i^{\alpha}_d i^{\mu}_c i^{\lambda}_b V_{[\nu} h_{\mu]\lambda} = i^{\alpha}_d i^{\mu}_c K_{\nu\mu\lambda\kappa} i^{\lambda}_b i^{\alpha}_n \stackrel{*}{=} K_{d c b n}; \quad b, c, d \neq$$

from which we see that all orthogonal components of  $K_{\nu\mu\lambda\kappa}$  with four different indices vanish and that accordingly for  $n > 3$  the  $V_n$  is a  $C_n$ . Now an  $R_n$  contains of course orthogonal systems in every direction and orthogonal systems remain orthogonal under conformal transformations. Hence

*A  $V_n$ ,  $n > 3$  is a  $C_n$  if and only if for every choice of a point and an  $(n-1)$ -direction at this point, it contains at least one  $V_{n-1}$  belonging to an orthogonal system and having just this  $(n-1)$ -direction at this point.<sup>3)</sup>*

In V § 6 it was proved that if in a  $V_n$  there exist umbilical  $V_{n-1}$ 's through every point and with every  $(n-1)$ -direction at this point, all orthogonal components  $K_{k_1 i h}$  with four different indices vanish. But we just proved that for  $n > 3$  this means that the  $V_n$  is a  $C_n$ . Now let an anholonomic coordinate system with mutually perpendicular unit-vectors  $i^{\alpha}_j$  be chosen such that  $i^{\alpha}_n$  is normal to a set of  $\infty^1$   $V_{n-1}$ 's. Then we have for the second fundamental tensor  $h_{cb}$  of these  $V_{n-1}$ 's

$$(5.24) \quad h_{cb} = -B_{cb}^{\mu\lambda} V_{\mu} i_{\lambda}; \quad b, c = 1, \dots, n-1$$

<sup>1)</sup> SCHOUTEN 1921, 1, p. 84; cf. EISENHART 1926, 1, p. 122ff.

<sup>2)</sup> Cf. Exerc. III 5.5.

<sup>3)</sup> SCHOUTEN 1927, 3, p. 719; E II 1938, 2, p. 204 also for  $n = 3$  and for literature.

and after the transformation  $g_{\lambda\mu} \rightarrow \sigma g_{\lambda\mu}$  according to (5.3 b), denoting the components of the new  $h_{cb}$  with respect to the old unitvectors  $\overset{\infty}{i}_b$  by ' $h_{cb}$

$$(5.25) \quad \left\{ \begin{array}{l} 'h_{cb} = -B_{cb}^{\mu\lambda} V_\mu \sigma^\frac{1}{n} i_\lambda = -\sigma^\frac{1}{n} B_{cb}^{\mu\lambda} (V_\mu \overset{\infty}{i}_\lambda - \overset{\infty}{i}_{(\mu} s_{\lambda)} + \frac{1}{2} g_{\mu\lambda} s_\sigma \overset{\infty}{i}_\tau g^{\sigma\tau}) \\ \qquad \qquad \qquad = \sigma^\frac{1}{n} (h_{cb} - \frac{1}{2} s_\sigma \overset{\infty}{i}_\tau g^{\sigma\tau} g_{cb}). \end{array} \right.$$

This proves that the directions of principal curvature are conformally invariant and that all umbilical points remain umbilical points after the conformal transformation. Now in an  $R_n$ , umbilical  $V_{n-1}$ 's are possible through every point with every  $(n-1)$ -direction at that point and this implies that the same holds in a  $C_n$ . So we have proved for  $n > 3$  that the existence of these  $V_{n-1}$ 's is a necessary and sufficient condition for the  $V_n$  to be a  $C_n$ . STELLMACHER<sup>1)</sup> has proved that the condition is also n.a.s. for  $n = 3$ . We prove this in the following way:

Let  $\eta^\alpha$ ;  $\alpha = 1, 2$  be coordinates in an umbilical  $V_2$  in  $V_3$ . Then we have on the one hand from (V 5.9)

$$(5.26) \quad 2 B_{\delta\gamma\beta}^{\nu\mu\lambda} V_{[\nu} h_{\mu]\lambda} = -2 B_{\delta\gamma\beta}^{\nu\mu\lambda} V_{[\nu} V_{\mu]} \overset{\infty}{i}_\lambda = B_{\delta\gamma\beta}^{\nu\mu\lambda} K_{\nu\mu\lambda} \overset{\infty}{i}_3$$

if  $\overset{\infty}{i}_3$  is a congruence normal to the  $V_2$ . On the other hand we have because all points are umbilical

$$(5.27) \quad 2 B_{\delta\gamma\beta}^{\nu\mu\lambda} V_{[\nu} h_{\mu]\lambda} = 2 B_{\delta\gamma\beta}^{\nu\mu\lambda} V_{[\nu} h (g_{\mu]\lambda} - \overset{\infty}{i}_{[\mu} \overset{\infty}{i}_{\lambda)}) = 2 'g_{\beta[\gamma} \partial_{\delta]} h.$$

But in a  $V_3$  the equation (5.17) is identically satisfied, hence from (5.26, 27)

$$(5.28) \quad B_{\delta\gamma\beta}^{\nu\mu\lambda} (g_{\nu\lambda} L_{\mu\nu} - g_{\mu\lambda} L_{\nu\mu}) \overset{\infty}{i}_3 = 2 'g_{\beta[\gamma} \partial_{\delta]} h$$

or

$$(5.29) \quad 'g_{\beta[\delta} B_{\gamma]}^\mu L_{\mu\nu} \overset{\infty}{i}_3 = 'g_{\beta[\gamma} \partial_{\delta]} h$$

and this implies that  $B_\beta^\lambda L_{\lambda\nu} \overset{\infty}{i}_3$  is a gradient vector in  $V_2$ :

$$(5.30) \quad B_\beta^\lambda L_{\lambda\nu} \overset{\infty}{i}_3 = -\partial_\beta h.$$

The rotation of this vector is zero and equal to the  $V_2$ -part of the rotation of  $L_{\lambda\nu} \overset{\infty}{i}_3$

$$(5.31) \quad \left\{ \begin{array}{l} 0 = B_{\gamma\beta}^{\mu\lambda} V_{[\mu} L_{\lambda]\nu} \overset{\infty}{i}_3 = B_{\gamma\beta}^{\mu\lambda} \overset{\infty}{i}_3^\nu V_{[\mu} L_{\lambda]\nu} - B_{\gamma\beta}^{\mu\lambda} L_{\nu[\lambda} h_{\mu]\nu} \\ \qquad \qquad \qquad = B_{\gamma\beta}^{\mu\lambda} \overset{\infty}{i}_3^\nu V_{[\mu} L_{\lambda]\nu} - h B_{\gamma\beta}^{\mu\lambda} L_{\nu[\lambda} g_{\mu]\nu} = B_{\gamma\beta}^{\mu\lambda} \overset{\infty}{i}_3^\nu V_{[\mu} L_{\lambda]\nu}. \end{array} \right.$$

<sup>1)</sup> STELLMACHER 1951, 1.

Hence, if in  $V_3$  there exist umbilical  $V_2$ 's through every point with every 2-direction, all orthogonal components of  $\nabla_{[\mu} L_{\lambda]} \times$  with three different indices vanish. From this we see that from the quantity (cf. I 9.20)

$$(5.32) \quad N_{\nu \times} \stackrel{\text{def}}{=} I_{\nu}^{\mu \lambda} \nabla_{[\mu} L_{\lambda]} \times$$

all orthogonal components with two equal indices vanish. But this is only possible with respect to all orthogonal systems if  $N_{\nu \times} = N_{[\nu \times]}$ . Returning now to the identity of BIANCHI (III 5.21) we have according to (5.17)

$$(5.33) \quad 0 = g_{\lambda [\nu} \nabla_{\omega} L_{\mu] \times} - g_{\times [\nu} \nabla_{\omega} L_{\mu] \lambda}$$

or, transvecting with  $I^{\nu \omega \mu}$

$$(5.34) \quad N_{\lambda \times} = N_{[\lambda \times]} = 0$$

which implies that in a  $V_3$  of the kind considered  $\nabla_{[\mu} L_{\lambda]} \times = 0$ .

Gathering results we have<sup>1)</sup>

A  $V_n$ ,  $n > 2$ , is conformally euclidean if and only if there exist umbilical  $V_{n-1}$ 's through every point with every  $(n-1)$ -direction at that point.

If a  $V_m$  in  $V_n$  can be transformed into a geodesic  $V_m$  by a conformal transformation of the connexion in  $V_n$  it is called conformally geodesic. If  $m=1$  we have for the tangent unitvector ( $\sigma > 0$ )

$$(5.35) \quad 'j^{\times} = \sigma^{-\frac{1}{2}} j^{\times}; \quad 'j_{\lambda} = \sigma^{\frac{1}{2}} j_{\lambda}$$

hence (cf. V § 1)

$$(5.36) \quad 'u^{\times} = 'j^{\mu} 'V_{\mu} 'j^{\times} = \sigma^{-1} (u^{\times} - \frac{1}{2} z^{\times}),$$

where  $u^{\times}$  is the curvature vector and  $z^{\times}$  is defined by

$$(5.37) \quad z^{\times} \stackrel{\text{def}}{=} s^{\times} - s_{\lambda} j^{\lambda} j^{\times}$$

is the projection of  $s^{\times}$  on the  $R_{n-1}$  perpendicular to  $j^{\times}$ . If the transformed curve is geodesic it follows that  $u^{\times}$  is the projection of a gradient vector. Conversely, if we know that  $u^{\times}$  is the projection of a vector  $g^{\times \lambda} \partial_{\lambda} \log \sigma$ , the transformation  $g_{\lambda \times} \rightarrow \sigma g_{\lambda \times}$  transforms the curve into a geodesic.

<sup>1)</sup> For  $n > 3$  this was proved by SCHOUTEN 1921, 1, p. 86. But for  $n=3$  it was stated there erroneously that every  $V_3$  contains umbilical  $V_2$ 's through every point in every 2-direction. This wrong statement occurred also in the first edition of this book R. K. 1924, 1, p. 180. But it was not republished in E II 1938, 2 § 19 because at that time we smelled a rat without knowing exactly its lare. STELLMACHER gave the solution in a letter of 19. 11. '49 to the author and published this with many other results concerning conformal geometry in 1951, 1; cf. SAKAI 1939, 1; 1940, 1.

Hence<sup>1)</sup>

*Every single curve is conformally geodesic. A real congruence of curves in an ordinary  $V_n$  is conformally geodesic if and only if the curvature vector is the projection of a gradientvector.*

Orthogonality being invariant for conformal transformations, the theorem of BELTRAMI (cf. V § 5) leads to the theorem of THOMSON and TAIT<sup>2)</sup>

*If a real conformally geodesic congruence in an ordinary  $V_n$  is normal to one  $V_{n-1}$  it is  $V_{n-1}$ -normal throughout.*

But such a congruence is not in any way special because it can be proved that every  $V_{n-1}$ -normal congruence is conformally geodesic.<sup>3)</sup>

Because of

$$(5.38) \quad v^\mu 'V_\mu v^x = v^\mu V_\mu v^x + v^\mu s_\mu v^x - \frac{1}{2} v^\mu v_\mu s^x$$

*every null geodesic (geodesic whose tangent lies everywhere in a null direction) remains a null geodesic after conformal transformation of  $g_{\lambda x}$ .*

For  $m > 1$  we get [cf. (5.37) and V § 9]<sup>4)</sup>

$$(5.39) \quad \left\{ \begin{array}{l} 'H_{cb}^{xx} = - B_{cb}^{\mu\lambda} ('V_\mu 'i_\lambda)_x^x = H_{cb}^{xx} - \frac{1}{2} 'g_{cb} z^x; \\ b, c = 1, \dots, m; \quad x = m+1, \dots, n; \end{array} \right.$$

hence

*An umbilical point of a  $V_m$  in  $V_n$  remains umbilical after conformal transformation of  $g_{\lambda x}$ .*

This proves that a conformally geodesic  $V_m$  is for  $m > 1$  the same as an umbilical  $V_m$ . But this does not imply that a system of  $\infty^{n-m}$  umbilical  $V_m$ 's;  $m = 1, \dots, n-1$ , can be transformed at the same time by one conformal transformation of the  $g_{\lambda x}$  into  $\infty^{n-m}$  geodesic  $V_m$ 's. According to (5.39) this is possible if and only if the mean curvature vector (cf. V § 9) is the projection of a gradientvector.<sup>5)</sup>

According to (5.39) the tensor  $M_{cb}^{xx}$  defined in (V 9.16b) is conformal invariant. The same holds for the vectors  $\tilde{i}_b^x$  defined in (V 8.13).

<sup>1)</sup> SCHOUTEN 1928, 2.

<sup>2)</sup> THOMSON and TAIT 1879, 1, p. 353 for  $n = 3$  and SCHOUTEN and STRUIK 1921, 2 for general values of  $n$ .

<sup>3)</sup> SCHOUTEN and STRUIK 1921, 2; 1922, 4; E II 1938, 2, p. 51; SCHOUTEN 1928, 2. Cf. for families of curves that may be considered as conformal geodesics PAINLEVÉ 1894, 1; KASNER 1910, 1; 1913, 1; DOUGLAS 1924, 1; SCHOUTEN 1928, 2; BLASCHKE 1928, 1; FIALKOW 1939, 2; 1940, 1; 1942, 2; YANO 1950, 1.

<sup>4)</sup> SCHOUTEN and STRUIK 1923, 4; R. K. 1924, 1, p. 202.

<sup>5)</sup> E II 1938, 2, p. 211.

For a  $V_m$  they take the form

$$(5.40) \quad \ddot{\tilde{v}}_b = B_b^\mu (\nabla_\mu \dot{\tilde{i}}_\lambda) \dot{\tilde{i}}^\lambda,$$

hence

$$(5.41) \quad ' \ddot{\tilde{v}}_b = B_b^\mu (' \nabla_\mu ' \dot{\tilde{i}}_\lambda) ' \dot{\tilde{i}}^\lambda = \ddot{\tilde{v}}_b \cdot 1)$$

In (V § 8) it was proved that an  $A_m$  can be imbedded in an  $E_n$  if there exist fields  $\ddot{\tilde{h}}_{cb}$ ,  $\dot{l}_c^a$  and  $\ddot{\tilde{v}}_b$  satisfying the equations (V 8.18–21) of GAUSS, CODAZZI and RICCI. For a  $V_m$  in  $R_n$  the fields  $\ddot{\tilde{h}}_{cb}$  and  $\dot{l}_c^a$  coincide and accordingly a  $V_m$  can be imbedded in an  $R_n$  if there exist fields  $\ddot{\tilde{h}}_{cb}$  and  $\ddot{\tilde{v}}_b$  satisfying the GAUSS-CODAZZI-RICCI equations. YANO and MUTO<sup>2)</sup> proved that a  $V_m$  with the fields  $M_{cb(x)}$  and  $\ddot{\tilde{v}}_b$  can be conformally imbedded in a  $C_n$  provided that these fields satisfy three equations that may be considered as the conformal generalizations of the GAUSS-CODAZZI-RICCI equations. These equations were derived before by YANO.<sup>3)</sup>

In V § 6 we have found that an umbilical  $V_{n-1}$  with constant mean curvature ( $h = \text{const.}$ ) is at every point perpendicular to a principal direction of  $V_n$  and that accordingly  $V_{n-1}$ 's of this kind are only possible through every point and with every  $(n-1)$ -direction at that point if the  $V_n$  is an EINSTEIN space,  $K_{\mu\lambda} \propto g_{\mu\lambda}$ . But then also  $L_{\mu\lambda}$  equals  $g_{\mu\lambda}$  to within a scalar factor and according to (5.17) the  $V_n$  is an  $S_n$ <sup>4)</sup> (cf. V § 6):

*A  $V_n$ ,  $n > 2$ , is an  $S_n$  if and only if there exist umbilical  $V_{n-1}$ 's with constant mean curvature through every point with every  $(n-1)$ -direction at this point.*

A geodesic  $V_{n-1}$  is a special case of an umbilical  $V_{n-1}$  ( $h = 0$ ), hence

*A  $V_n$ ,  $n > 2$ , is an  $S_n$  if and only if there exist geodesic  $V_{n-1}$ 's through every point with every  $(n-1)$ -direction at this point.*

According to (5.7, 12) an  $R_n$  is transformed conformally into another  $R_n$  if  $s_\lambda = \partial_\lambda \log \sigma$  satisfies the equations

$$(5.42) \quad 2\nabla_\mu s_\lambda - s_\mu s_\lambda + \frac{1}{2} g_{\mu\lambda} s_\sigma s_\tau g^{\sigma\tau} = 0.$$

This equation is totally integrable<sup>5)</sup>.

<sup>1)</sup> SASAKI 1936, 1.

<sup>2)</sup> YANO and MUTO 1942, 1; 1946, 1.

<sup>3)</sup> YANO 1939, 2; 3; 1940, 3; 1943, 2.

<sup>4)</sup> SCHOUTEN 1921, 1, p. 87.

<sup>5)</sup> Cf. for  $R_n$  transformed conformally into  $R_n$ , E II 1938, 2, p. 205 ff.

LOUVILLE proved the theorem

*For  $n > 2$  every conformal mapping of an  $R_n$  on itself is always a similarity transformation with respect to a point or a combination of an inversion at a point and a similarity transformation with respect to that point*

for  $n = 3$  and LIE gave the generalization for  $n > 3$ .<sup>1)</sup> The conformal transformations of an  $R_2$  that leads to another  $R_2$  are well known from the theory of functions.<sup>2)</sup>

### Exercises.

VI 5,1<sup>3)</sup>. A  $V_2$  is transformed conformally into an  $R_2$ . If  $'g_{\lambda\mu} = \sigma g_{\lambda\mu}$  and  $s_\lambda = \partial_\lambda \log \sigma$ , prove that

$$\text{VI 5,1 } \alpha) \quad \nabla_\lambda s^\lambda = K.$$

VI 5,2 [cf. Exerc. V 9,4]<sup>4)</sup>. A  $V_{2m}$  is a  $C_{2m}$  if and only if the sum of the scalar curvatures of 2 arbitrary mutually perpendicular  $m$ -directions is always independent of the choice of these  $m$ -directions.

VI 5,3<sup>5)</sup>. A normal system of  $\infty^{n-m} V_m$ 's in a  $V_n$  can be transformed conformally into a system of  $\infty^{n-m}$  minimal  $V_m$ 's if and only if the mean curvature vector is the projection of a gradient vector on the local  $R_{n-m}$  perpendicular to the  $V_m$ .

VI 5,4<sup>6)</sup>. If a congruence in  $V_m$  in  $V_n$  is given, there is always a conformal transformation in  $V_n$  that transforms the congruence into a congruence of asymptotic lines of the  $V_m$ .

## § 6. Conformal transformations of the connexion in an EINSTEIN space.<sup>7)</sup>

A  $V_n$ ,  $n > 2$ , is transformed into an EINSTEIN space by a conformal transformation of the connexion if and only if

$$(6.1) \quad 'K_{\mu\lambda} = \frac{'K}{n} \sigma g_{\mu\lambda}; \quad 'K = \sigma^{-1} K - \frac{1}{2} (n-1) \sigma^{-1} s_\mu s^\mu g^{\mu\lambda} = \text{const.}$$

<sup>1)</sup> LIE 1872, 1; BIANCHI 1902, 2, p. 375. The proof is not difficult but rather tiresome and uninteresting. A generalization was given by HAANTJES 1937, 2. Cf. E II 1938, 2, p. 209.

<sup>2)</sup> Cf. E II 1938, 2, p. 210.

<sup>3)</sup> E II 1938, 2, p. 205.

<sup>4)</sup> HAANTJES and WRONA 1939, 1.

<sup>5)</sup> SCHOUTEN and STRUIK 1923, 4; E II 1938, 2, p. 212.

<sup>6)</sup> SCHOUTEN and STRUIK 1923, 4; R. K. 1924, 1, p. 202.

<sup>7)</sup> General references: BRINKMAN 1923, 1; 1924, 1; 1925, 1; HAANTJES and WRONA 1939, 1; FIALKOW 1939, 1; 1942, 1; SASAKI 1942, 1; YANO 1943, 1; WONG 1943, 2; 3; WRONA 1948, 2; KUIPER 1950, 1; 2; 1951, 1.

On account of (5.12) this is equivalent to

$$(6.2) \quad {}'L_{\mu\lambda} = -\frac{n-2}{2n(n-1)} {}'K \sigma g_{\mu\lambda},$$

hence, according to (5.20)

$$(6.3) \quad s_{\mu\lambda} = -\frac{4}{n-2} L_{\mu\lambda} - \frac{2}{n(n-1)} {}'K \sigma g_{\mu\lambda}.$$

This gives for  $\sigma$  the differential equation (cf. 5.7)

$$(6.4) \quad 2\nabla_\mu s_\lambda - s_\mu s_\lambda + \frac{1}{2} g_{\mu\lambda} s_\sigma s^\sigma = -\frac{4}{n-2} L_{\mu\lambda} - \frac{2}{n(n-1)} {}'K \sigma g_{\mu\lambda}$$

with the first integrability conditions (cf. 5.13)

$$(6.5) \quad \boxed{\nabla_{[\nu} L_{\mu]\lambda} - \frac{n-2}{4} C_{\nu\mu\lambda}{}^\kappa s_\kappa = 0.}$$

These conditions can also be derived immediately from (6.2):

$$(6.6) \quad {}'\nabla_{[\nu} {}'L_{\mu]\lambda} = 0$$

because the left hand sides of (6.5) and (6.6) are equal.<sup>1)</sup>

Now from (III 5.27) it can be derived that<sup>2)</sup>

$$(6.7) \quad \nabla_\kappa C_{\nu\mu\lambda}{}^\kappa = -\frac{2(n-3)}{n-2} \nabla_{[\nu} L_{\mu]\lambda}$$

and from (6.5) and (6.7) after some calculation

$$(6.8) \quad \boxed{\nabla^\mu \nabla_{[\nu} L_{\mu]\lambda} + \frac{1}{2} L^{\mu\sigma} C_{\nu\mu\lambda\sigma} = -\frac{n-4}{2} s^\sigma \nabla_{[\lambda} L_{\sigma]\nu}.} ; \quad \nabla^\mu \stackrel{\text{def}}{=} g^{\mu\nu} \nabla_\nu$$

This equation is a part of the second integrability condition of (6.4). It is remarkable that the right hand side is zero for  $n=4$ . That implies that for  $n=4$  the left hand side must be a conformal concomitant to within a scalar factor. In fact it can be proved that for  $n=4$  the symmetric tensor density of weight  $+\frac{1}{2}$

$$(6.9) \quad \mathfrak{C}_{\nu\lambda} \stackrel{\text{def}}{=} g^{\frac{1}{4}} (\nabla^\mu \nabla_{[\nu} L_{\mu]\lambda} + \frac{1}{2} L^{\mu\sigma} C_{\nu\mu\lambda\sigma})$$

<sup>1)</sup> E II 1938, 2, p. 203.

<sup>2)</sup> E II 1938, 2, p. 204.

is invariant for conformal transformations of the connexion.<sup>1)</sup> For  $n=4$  the vanishing of  $\mathbb{C}_{\nu\lambda}$  is a necessary though not a sufficient condition for the  $V_4$  to be conformally transformable into an EINSTEIN  $V_4$ .

Returning to the first integrability conditions (6.5) for  $n>2$  we first remark that (6.4) is totally integrable if and only if  $C_{\nu\mu\lambda}^{\alpha\beta\gamma}$  and  $\bar{V}_{[\nu}L_{\mu]\lambda}$  vanish, that is if the  $V_n$  is conformally euclidean. But in that case the EINSTEIN space is conformally euclidean as well, and from (5.17) it follows that a conformally euclidean EINSTEIN space is always an  $S_n$ .<sup>2)</sup> Now let us suppose that the  $V_n$  is a *special* EINSTEIN space,  $L_{\mu\lambda}=0$ . Then (6.5) reduces to

$$(6.10) \quad K_{\nu\mu\lambda}^{\alpha\beta\gamma} s_\alpha = 0; \quad s_\alpha \neq 0$$

because in this case  $C_{\nu\mu\lambda}^{\alpha\beta\gamma}$  is equal to  $K_{\nu\mu\lambda}^{\alpha\beta\gamma}$  according to its definition. Now (6.10) implies that  $K_{\nu\mu\lambda\alpha}$  can not have the  $\alpha$ -rank  $n$ . But the rank of  $K_{\nu\mu\lambda\alpha}$  is the same with respect to all indices, hence at every point  $K_{\nu\mu\lambda\alpha}$  lies in an  $R_{n-1}$  perpendicular to  $s^\alpha$ . For  $n=4$  this has an important consequence. According to its symmetry properties  $K_{\nu\mu\lambda\alpha}$  can always be written in the form ' $a_{[\nu[\lambda} Q_{\mu]\alpha]}$ ' where ' $a_{\lambda\alpha}$ ' is the fundamental tensor of the  $R_3$  perpendicular to  $s^\alpha$ , ' $a_{\lambda\alpha} = a_{\lambda\alpha} - \varrho s_\lambda s_\alpha$ ';  $\varrho s_\lambda s^\lambda = 1$ . But because  $K_{\nu\mu\lambda}^{\alpha\beta\gamma} = C_{\nu\mu\lambda}^{\alpha\beta\gamma}$ , the RICCI tensor  $K_{\mu\lambda}$  must vanish and this is only possible if  $Q_{\lambda\alpha} = 0$ . This proves the theorem of BRINKMAN:<sup>3)</sup>

*A conformally euclidean special EINSTEIN space is always an  $R_n$ . If a special EINSTEIN space is not conformally euclidean then for  $n=4$  it is impossible to map it conformally on another special EINSTEIN space<sup>4)</sup>.*

This is very important for general relativity. Time-space is a special EINSTEIN space in all regions without matter. In these regions the metric can only be investigated by means of light signals and these signals fix only the null geodesics. That would imply that in empty time-space a metric could only be determined to within conformal transformations. But according to the theorem of BRINKMAN the condition  $G_{\mu\lambda}=0$  (usually derived from a variational principle) is sufficient to fix the metric<sup>5)</sup>.

<sup>1)</sup> This quantity was discovered by SCHOUTEN and HAANTJES 1936, 2. It played an important role in the imbedding of an EINSTEIN  $V_4$  in a five-dimensional space with a general projective geometry and it was rediscovered by HESSELBACH 1949, 1. He proved that  $\bar{V}_\mu \mathbb{C}_{\nu\lambda}^{\mu} = 0$ .

<sup>2)</sup> SCHOUTEN and STRUIK 1921, 4.

<sup>3)</sup> BRINKMAN 1923, 1; 1924, 1; 1925, 1 also for many other properties of conformal mapping of EINSTEIN spaces; cf. E II 1938, 2, p. 213; KUIPER 1950, 1; 2 also for literature on EINSTEIN spaces.

<sup>4)</sup> It is also impossible to map it conformally on a non special EINSTEIN space.

<sup>5)</sup> This remark is due to KASNER 1921, 1; 1922, 1 who proved the theorem of BRINKMAN for a special case.

### § 7. Conformal connexions.<sup>1)</sup>

If the (symmetric) fundamental tensor  $g_{\lambda\kappa}$  is given to within a scalar factor the (symmetric) tensor density of weight  $-\frac{2}{n}$

$$(7.1) \quad \mathfrak{G}_{\lambda\kappa} \stackrel{\text{def}}{=} g^{-\frac{1}{n}} g_{\lambda\kappa}$$

is uniquely determined. Its determinant (or in the indefinite case the absolute value) is +1. This has as a consequence that on each curve a *conformal parameter*  $\zeta$  is defined whose differential

$$(7.2) \quad d\overset{(x)}{\zeta} = (\mathfrak{G}_{\lambda\kappa} d\xi^\lambda d\xi^\kappa)^{\frac{1}{2}}$$

is a scalar density of weight  $-\frac{1}{n}$ .<sup>2)</sup> If the process that leads from the  $g_{\lambda\kappa}$  to the  $\overset{c}{I}_{\mu\lambda}^{\kappa}$  belonging to  $g_{\lambda\kappa}$ , is applied to the  $\mathfrak{G}_{\lambda\kappa}$  we get

$$(7.3) \quad \begin{cases} \overset{c}{I}_{\mu\lambda}^{\kappa} \stackrel{\text{def}}{=} \frac{1}{2} \overset{-1}{\mathfrak{G}}{}^{\kappa\sigma} (\partial_\mu \mathfrak{G}_{\lambda\sigma} + \partial_\lambda \mathfrak{G}_{\mu\sigma} - \partial_\sigma \mathfrak{G}_{\mu\lambda}) \\ = I_{\mu\lambda}^{\kappa} - \frac{2}{n} I_{(\mu} A_{\lambda)}^{\kappa} + \frac{1}{n} g_{\mu\lambda} g^{\kappa\sigma} I_\sigma. \end{cases}$$

These are the *conformal parameters* of J. M. THOMAS<sup>3)</sup>, which are invariant for conformal transformations of  $g_{\lambda\kappa}$ . Their transformation is

$$(7.4) \quad \begin{cases} \overset{c}{I}_{\mu'\lambda'}^{\kappa'} = A_{\mu'\lambda'\kappa'}^{\mu\lambda\kappa} \overset{c}{I}_{\mu\lambda}^{\kappa} + A_{\kappa'}^\kappa \partial_{\mu'} A_{\lambda'}^\mu + \frac{2}{n} A_{(\mu'}^\kappa \partial_{\lambda')}^\mu \log \Delta - \\ - \frac{1}{n} \mathfrak{G}_{\mu'\lambda'} \overset{-1}{\mathfrak{G}}{}^{\kappa'q'} \partial_{q'} \log \Delta. \end{cases}$$

<sup>1)</sup> References. General: CARTAN 1923, 3; 1937, 3; SCHOUTEN and STRUIK 1923, 4; T. Y. THOMAS 1926, 3; 1932, 1; 1934, 2, Ch. IV; VEBLEN 1928, 3; 1930, 1; 1935, 1; WHITEHEAD 1929, 1; VANDERSLICE 1934, 2; HLAVATY 1935, 2; 1936, 1; SCHOUTEN and HAANTJES 1936, 2; YANO 1938, 2; 1939, 4; 5; SASAKI 1940, 2; 1941, 2; 1948, 1; SCHIROKOW 1940, 1; VRANCEANU 1940, 1; 1943, 1; YANO and MUTO 1941, 1; 2; SCHMIDT 1949, 1. Curves: AGOSTINELLI 1933, 3; HLAVATY 1935, 2, III; MODESTITT 1938, 1; YANO 1938, 3; 1944, 4; 1950, 1; FIALKOW 1939, 2; 1940, 1; 1942, 2; SASAKI 1939, 2; MUTO 1939, 1; 1942, 1; HAANTJES 1941, 1; 1942, 1; LAGRANGE 1941, 1; 2; YANO and MUTO 1941, 1; 2; 3; 4. Imbedding: BRINKMAN 1923, 2; PEREPELKINE 1935, 2; HLAVATY 1935, 2, II; SASAKI 1936, 1; 1939, 3; 1940, 2; 3; HAIMOVICI 1938, 1; FIALKOW 1939, 1; 1942, 1; 1944, 1; YANO 1939, 2; 3 (GAUSS and CODAZZI); 1940, 3; 1942, 2; 1943, 2; 4; MUTO 1939, 2; 1940, 3; 1942, 1; YANO and MUTO 1940, 9; 1941, 5; 1942, 1; 1946, 1; HAANTJES 1942, 2; 3; 1943, 1; BECKENBACH and BING 1943, 1; NORDEN 1945, 3; PETRESCU 1946, 1; 1948, 1; OTSUKI 1950, 2; VERBICKII 1951, 1; 2 (class); ICHILIONE 1951, 1; AKIVIS 1952, 1. Product spaces: YANO 1940, 1; WONG 1943, 2. Mapping: GOLAB 1935, 1; HAANTJES 1937, 2; ZAREMBA 1937, 1; COBURN 1941, 2.

<sup>2)</sup> T. Y. THOMAS 1926, 3.

<sup>3)</sup> J. M. THOMAS 1925, 1; 1926, 3.

Hence the  $\overset{c}{I}_{\mu\lambda}^{\kappa}$  are not the components of a geometric object because their transformation formula contains not only  $\overset{c}{I}_{\mu\lambda}^{\kappa}$  and  $A_{\lambda}^{\kappa}$  but also  $\mathfrak{G}_{\mu\lambda}$  (or  $g_{\mu\lambda}$ , because  $\mathfrak{G}_{\mu\lambda}\mathfrak{G}^{\kappa\rho} = g_{\mu\lambda}g^{\kappa\rho}$ ). But  $\overset{c}{I}_{\mu\lambda}^{\kappa}$  together with  $\mathfrak{G}_{\mu\lambda}$  form a geometric object. This geometric object does not fix one metric and symmetric connexion but an infinity of such connexions transforming into each other by conformal transformations. If we first prefer one coordinate system and then consider coordinate systems with  $A = \text{const.}$  only, the  $\overset{c}{I}_{\mu\lambda}^{\kappa}$  transform in the same way as the  $I_{\mu\lambda}^{\kappa}$  and in this case the  $\overset{c}{I}_{\mu\lambda}^{\kappa}$  represent the only symmetric connexion that leaves  $\mathfrak{G}_{\lambda\kappa}$  invariant and whose parameters give zero if transvected with  $A_{\lambda}^{\kappa}$ .

The  $\overset{c}{I}_{\mu\lambda}^{\kappa}$  can be used in several ways to fix some kind of connexion. The simplest method is, as in the projective case, to construct an invariant scalar density.<sup>1)</sup> In 1924 CARTAN<sup>2)</sup> proposed to use a flat projective  $(n+1)$ -dimensional space with a quadric instead of an affine space as local space. SCHOUTEN<sup>3)</sup> proved in the same year that this amounts to introducing a linear connexion for local conformal spaces and that it is intimately connected with the linear connexions of local  $E_N$ 's;  $N \neq n$  already introduced by KÖNIG in 1920.<sup>4)</sup> T. Y. THOMAS<sup>5)</sup> succeeded in constructing from the  $\overset{c}{I}_{\mu\lambda}^{\kappa}$  a connexion for certain quantities in an  $X_{n+1}$  with a restricted group of coordinate transformations.

It is well known that the conformal geometry of an  $R_n$  is identical with the projective geometry on a quadric in a flat projective  $(n+1)$ -dimensional space. This can be proved for instance by introducing  $n+2$  polyspherical coordinates.<sup>6)</sup> This can be generalized for the conformal geometry of a  $V_n$ . SCHOUTEN and HAANTJES<sup>7)</sup> have proved that for  $n$  odd an  $X_n$  with a general conformal geometry can be imbedded in an  $X_{n+1}$  with a general projective connexion, and that *this connexion is uniquely determined*. This means that  $n$ -dimensional general conformal geometry can for  $n$  odd be treated with  $n+2$  homogeneous coordinates that satisfy one condition. For  $n$  even the theorem holds only if special conditions are satisfied. For  $n=4$  the only condition is that  $\mathfrak{C}_{\lambda\kappa}=0$  (cf. 6.9).

<sup>1)</sup> COTTON 1899, 1 for  $n=3$ ; HLAVATY 1935, 2; LEVINE 1935, 1.

<sup>2)</sup> CARTAN 1924, 1; cf. 1923, 3.

<sup>3)</sup> SCHOUTEN 1924, 2; 3.

<sup>4)</sup> KÖNIG 1920, 1.

<sup>5)</sup> T. Y. THOMAS 1932, 1; 1934, 2 also for literature; cf. VEBLEN 1928, 2; 3; 1935, 1; VANDERSLICE 1934, 2; HLAVATY 1935, 2; 1936, 1; YANO 1938, 2; 1940, 4.

<sup>6)</sup> Cf. for instance WOODS 1922, 1, p. 251; KLEIN 1926, 1, p. 193; SCHOUTEN and HAANTJES 1936, 2.

<sup>7)</sup> SCHOUTEN and HAANTJES 1935, 3; 1936, 2.

**Exercise.**

VI 7,1. For every choice of an object  $\overset{*}{\Gamma}_\lambda$  with the transformation

$$\text{VI 7,1 } \alpha) \quad \overset{*}{\Gamma}_{\lambda'} = A_\lambda^\lambda \overset{*}{\Gamma}_\lambda - \partial_{\lambda'} \log A$$

the  $\overset{*}{\Gamma}_{\mu\lambda}^x$  defined by

$$\text{VI 7,1 } \beta) \quad \overset{*}{\Gamma}_{\mu\lambda}^x = \overset{*}{\Gamma}_{\mu\lambda}^x + \frac{2}{n} \overset{*}{\Gamma}_{(\mu} A_{\lambda)}^x - \frac{1}{n} g_{\mu\lambda} g^{x\eta} \overset{*}{\Gamma}_\eta^x$$

transform like the parameters of a linear connexion leaving  $\mathfrak{G}_{\lambda x}$  invariant and  $\overset{*}{\Gamma}_{\mu\lambda}^x = \overset{*}{\Gamma}_{\mu}^x$  (cf. Exerc. VI 4,1).

## § 8. Subprojective connexions.

An  $A_n$  is said to be *k-fold projective*<sup>1)</sup> if there exists a coordinate system with respect to which every geodesic can be given by means of  $k$  linear equations and  $n-k-1$  equations that need not be linear. Hence a  $D_n$  is  $(n-1)$ -fold projective. For  $k=n-2$  it may happen that there exists a coordinate system such that every geodesic is given with respect to this system by  $n-2$  homogeneous linear equations and one other equation that need not be linear. Then the  $A_n$  is called *subprojective*<sup>2)</sup> and the origin of the special coordinate system a *pole*.<sup>3)</sup> At any point the direction of the geodesic through this point and a pole is called a *pole direction*. The special coordinate systems with respect to which  $n-2$  equations of any geodesic are homogeneous and linear transform into each other by the transformations of the group

$$(8.1) \quad \xi^{h'} \overset{*}{=} \varrho(\xi^i) P_{;h}^{h'} \xi^h; \quad P_{;h}^{h'} = \text{const.}$$

if they have the same origin, as was proved by RACHEVSKI and will be proved here later on.<sup>4)</sup>

$n-2$  homogeneous linear equations in one of these special coordinate systems fix an  $X_2$  that is totally geodesic in  $A_n$  because every geodesic having two points in common with such an  $X_2$  lies wholly in it. Hence through the origin totally geodesic  $X_{n-1}$ 's can be laid with every  $(n-1)$ -direction and every one of them is given by one homogeneous linear equation. In the following we do not consider the case  $n=2$  because according to the definition every 1-fold projective  $A_2$  is a  $D_2$  and every  $A_2$  is subprojective.

<sup>1)</sup> KAGAN 1930, 1; 1933, 1; RACHEVSKI 1930, 1; 1933, 1; SCHAPIRO 1930, 1; 1933, 1.

<sup>2)</sup> KAGAN 1930, 1; 1935, 1.

<sup>3)</sup> The case  $k < n-2$  was recently discussed by VRANCEANU 1947, 2; 1950, 1; 2. A linear element was given for a  $k$ -fold projective  $V_n$ .

<sup>4)</sup> RASCHEVSKI 1933, 1, p. 128. Cf. for this and some of the following results also E II 1938, 2, p. 215 ff.

Now let  $(h)$  be such a special coordinate system and let  $v^h$  be an arbitrary vector field with *constant* components  $v^h$ . The equations

$$(8.2) \quad \xi^h \underset{1}{=} c^h t + v^h \underset{2}{}, \quad c^h = \text{const}; \quad v^h = \text{const}.$$

represent the parameter form of a geodesic  $X_2$  at every point of which the vector  $v^h$  has a tangent direction. But because the  $X_2$  is geodesic this implies that

$$(8.3) \quad v^j \nabla_j v^h \underset{*}{=} I_{ji}^h v^j v^i \underset{*}{=} \lambda v^h + \mu \xi^h$$

for every choice of the direction of  $v^h$ . Because the field  $v^h$  is arbitrary, we may now look upon the  $v^h$  and  $\xi^h$  in (8.3) as independent variables. Then by differentiation with respect to  $v^h$  we get

$$(8.4) \quad 2I_{ji}^h \underset{*}{=} \frac{\partial^2 \lambda}{\partial v^i \partial v^j} v^h + \frac{\partial \lambda}{\partial v^j} A_i^h + \frac{\partial \lambda}{\partial v^i} A_j^h + \xi^h \frac{\partial^2 \mu}{\partial v^i \partial v^j},$$

which is only possible if  $\frac{\partial^2 \lambda}{\partial v^i \partial v^j} = 0$  because the  $I_{ji}^h$  depend on the  $\xi^h$  only and not on the  $v^h$ . Hence  $\frac{\partial \lambda}{\partial v^j}$  and  $\frac{\partial^2 \mu}{\partial v^i \partial v^j}$  do not depend on the  $v^h$  and we get for  $I_{ji}^h$  an expression of the form

$$(8.5) \quad I_{ji}^h \underset{*}{=} 2\varphi_{(j} A_{i)}^h + \varphi_{[j} \xi^{h]}; \quad \varphi_{[j} \underset{*}{=} 0$$

valid with respect to all special coordinate systems.<sup>1)</sup> If we apply the coordinate transformation (8.1) we have

$$(8.6) \quad \begin{cases} \text{a)} & A_i^h \underset{*}{=} \varrho P_{\cdot i}^h + \xi^h \partial_i \log \varrho \\ \text{b)} & \partial_j A_i^h \underset{*}{=} (\partial_j \varrho) P_{\cdot i}^h + \varrho P_{\cdot j}^h \partial_i \log \varrho + \xi^h (\partial_j \log \varrho) \partial_i \log \varrho + \xi^h \partial_j \partial_i \log \varrho \end{cases}$$

and

$$(8.7) \quad I_{j'i'}^h = 2\varphi_{(j'} A_{i')}^h + \varphi_{j'i'} \xi^h$$

with<sup>2)</sup>

$$(8.8) \quad \begin{cases} \text{a)} & \varphi_{i'} \underset{*}{=} A_{i'}^i \varphi_i - \partial_{i'} \log \varrho \\ \text{b)} & \varphi_{j'i'} \underset{*}{=} (1 + \xi^h \partial_h \log \varrho) (A_{j'i'}^h \varphi_{j'} - \varrho^{-1} \partial_{j'} \partial_{i'} \varrho). \end{cases}$$

From these equations we see that indeed a transformation of the form (8.1) transforms a special coordinate system into another special system with the same pole. The other part of RACHEVSKI's theorem follows from the condition that the transformation must transform every homogeneous linear equation into an equation with the same property. The  $\varphi_i$  and  $\varphi_{j'i'}$  do not behave like a vector field or a tensor field respectively and each of them forms a geometric object for transformations of the

<sup>1)</sup> KAGAN 1933, 1, p. 41.

<sup>2)</sup> Cf. E II 1938, 2, p. 219.

form (8.4) only. More general transformations disturb the form of (8.5). It is often convenient to consider a vector field and a symmetric tensor field defined by  $u_i \stackrel{*}{=} \varphi_i$  and  $u_{j;i} \stackrel{*}{=} \varphi_{ji}$ , but these fields depend on the choice of  $(h)$  and they change if another special coordinate system is introduced instead of  $(h)$ . From (8.18a) we see that  $\varphi_i$  can be made zero if and only if the field  $u_i$  is a gradient field. That implies that  $2\partial_{[j}\varphi_{i]}$  is an invariant bivector and a concomitant of the connexion. Another consequence of (8.7) is that a *subprojective connexion remains subprojective for every projective transformation* (1.1). From a geometrical point of view this is quite obvious because a projective transformation of the connexion does not change the geodesics.

The form (8.5) leads to the following expression for the curvature tensor

$$(8.9) \quad R_{k;j;i}^h = 2A_{[k}^h T_{j]i} - 2A_{i}^h T_{[k}j] + 2U_{k;j;i} x^{h-1},$$

where  $U_{k;j;i}$ ,  $T_{j;i}$  and  $x^h$  are tensor fields and a vector field, defined by

$$(8.10) \quad U_{k;j;i} \stackrel{*}{\underset{\text{def}}{=}} \partial_{[k} \varphi_{j]}_i + \xi^l \varphi_{l[k} \varphi_{j]}_i,$$

$$(8.11) \quad T_{j;i} \stackrel{*}{\underset{\text{def}}{=}} -\partial_j \varphi_i + \varphi_{j;i} (1 + \xi^l \varphi_l) + \varphi_j \varphi_i,$$

$$(8.12) \quad x^h \stackrel{*}{\underset{\text{def}}{=}} \xi^h.$$

For  $R_{j;i}$ ,  $V_{k;j}$ ,  $P_{j;i}$  and  $P_{k;j;i}^h$  (cf. VI § 1) we have

$$(8.13) \quad R_{j;i} \stackrel{*}{=} (n-1) T_{j;i} - 2\partial_{[j} \varphi_{i]} + 2x^h U_{h;j;i},$$

$$(8.14) \quad V_{j;i} \stackrel{*}{=} 2(n+1) \partial_{[j} \varphi_{i]} + 2U_{j;i} x^h,$$

$$(8.15) \quad P_{j;i} = -T_{j;i} + U_{j;i},$$

$$(8.16) \quad P_{k;j;i}^h = 2U_{k;j;i} x^h + 2A_{[k}^h U_{j]i} - 2A_{i}^h U_{[k}j],$$

where

$$(8.17) \quad U_{j;i} \stackrel{\text{def}}{=} -\frac{2}{n-1} x^l U_{l;j;i} - \frac{2}{n^2-1} U_{j;il} x^l,$$

$$(8.18) \quad U_{[j;i]} = \frac{1}{n+1} U_{j;i} x^l.$$

From (8.16, 17) it is seen that for  $n > 2$  the connexion is projectively euclidean if  $U_{k;j;i} = 0$ . In order to prove that this condition is also necessary we consider an expression of the form<sup>2)</sup>

$$(8.19) \quad \left\{ \begin{array}{l} A_{[k}^h B_{j]}_i + A_i^h C_{kj} + \sum_i^h Z_{ikj} + \cdots + \sum_p^h Z_{ikj}; \\ C_{(k}j)} = 0; \quad Z_{i(k}j)} = 0; \dots; Z_{i(k}j)} = 0 \end{array} \right.$$

<sup>1)</sup> In this equation the sign \* can be dropped because the equation has the invariant form.

<sup>2)</sup> SCHOUTEN 1953, 1.

with  $\overset{z^h}{\underset{i}{z^h}}, \dots, \overset{z^h}{\underset{p}{z^h}}$  linearly independent. For  $B_{ji} \neq 0, C_{kj} \neq 0$  the  $h$ -rank of the first term is  $\geq n-1$  and the second term has  $h$ -rank  $n^1)$ . In order to find the  $h$ -rank of the first two terms together we try to find a solution  $w_i$  of

$$(8.20) \quad w_{[k} B_{j]i} + w_i C_{kj} \stackrel{*}{=} 0.$$

Such a solution is only possible if  $B_{ji}$  has the form  $p_j r_i + r_j q_i$  and if  $C_{kj} = p_{[k} r_{j]}^2)$ . For  $p_i \neq q_i$  the solution is  $w_i \propto r_i$ . Hence the  $h$ -rank of the first two terms together is  $n-1$  in this exceptional case and  $n$  in the ordinary case. The  $h$ -rank of the last  $p$  terms of (8.19) together is  $\leq p$ , hence

*Auxiliary theorem:*

For  $n > p+2$  an expression of the form (8.19) is zero if and only if  $B_{kj}, C_{kj}$  and  $Z_{ikj}, \dots, Z_{ijk}$  are all zero. For  $n = p+1$  the expression can possibly also be zero if  $B_{ji}$  has the form  $p_j r_i + r_j q_i$  and if  $C_{kj} = p_{[k} r_{j]}$ .

If an expression can be written in the form (8.19) and also in the form

$$(8.21) \quad \begin{cases} A_{[k}^h 'B_{j]i} + A_i^h 'C_{kj} + \overset{z^h}{\underset{i}{Z_{ikj}}} + \dots + \overset{z^h}{\underset{q}{Z_{ijk}}} \\ 'C_{(kj)} = 0; \quad \overset{z}{\underset{i}{Z_{i(kj)}}} = 0; \quad \dots; \quad \overset{z}{\underset{q}{Z_{i(kj)}}} = 0 \end{cases}$$

and if there are  $p+q-s$  linearly independent ones among the  $z^h$  and  $'z^h$ , it follows that

$$(8.22) \quad 'B_{ji} = B_{ji}; \quad 'C_{kj} = C_{kj}$$

for  $n > p+q-s+2$ . This is also true for  $n = p+q-s+1$  but it need not be true if  $B_{ji} - 'B_{ji}$  has the form  $p_j r_i + r_j q_i$  and if  $C_{kj} - 'C_{kj} = p_{[k} r_{j]}$ .

Now let us suppose that there exists another decomposition of  $P_{kji}^h$

$$(8.23) \quad P_{kji}^h \stackrel{*}{=} 2Z_{kji} s^h + 2A_{[k}^h Z_{j]i} - 2A_i^h Z_{[kj]} \quad Z_{(kj)i} = 0$$

then we have  $p=1, q=1$  and  $s=1$  or  $s=0$ . Then the exceptional case can for  $n > 2$  only occur for  $s=0, n=3$  (necessary conditions for this case are that  $U_{ji} - Z_{ji}$  has the form  $p_j r_i + r_j q_i$  and that  $2p_{[k} r_{j]} = q_{[k} r_{j]}$ ). This proves that, for  $n > 2$ ,  $U_{kj}$  and  $U_{kji}$  are uniquely determined by  $P_{kji}^h$  provided that the direction of  $x^h$  is known and that, for  $n > 3$ , the direction of  $x^h$  is uniquely determined by  $P_{kji}^h$  provided that  $P_{kji}^h \neq 0$ . For  $n > 2$ ,  $U_{kji} = 0$  if  $P_{kji}^h = 0$ .

Gathering results we have

1) The expressions  $B_{ij}, C_{kj}$  etc. need not be tensors. Notwithstanding it is possible to speak of a  $h$ -rank as long as the coordinate system is not transformed.

2) In SCHOUTEN 1953, 1 only the case  $q_i = 0$  was considered. I owe this correction to a personal communication of J. HAANTJES.

A subprojective connexion is for  $n > 2$  projectively euclidean if and only if  $U_{kji} = 0$ .<sup>1)</sup> If it is not projectively euclidean the pole direction at every point is for  $n > 3$  uniquely determined and can be computed from  $P_{kji}^h$ .

### § 9. ADATI's problem<sup>2)</sup>.

ADATI considered the question whether a connexion that is not projectively euclidean and that takes the form

$$(9.1) \quad I_{j'i}^h * 2w_{(j} A_{i)}^h + w_{ji} v^h; \quad w_{[j} i] = 0;$$

with respect to some coordinate system ( $h$ ) is subprojective if some suitable conditions are introduced for the field  $v^h$ . In (9.1)  $w_i$  and the product  $w_{ji} v^h$  are supposed to be given as functions of the coordinates. Of course it can not be said of  $w_j, v^h$  and  $w_{ji}$  that they transform in the way of vectors or of a tensor because a coordinate transformation generally disturbs the form of (9.1). But it is allowed and also convenient to consider a vector field and a symmetric tensor field that have just the components  $w_j, v^h$  and  $w_{ji}$  with respect to ( $h$ ). Of course these fields belong then to the coordinate system ( $h$ ) used in (9.1) and to those systems that arise from ( $h$ ) by a homogeneous linear transformation with *constant* coefficients.

N.a.s. condition for the connexion (9.1) to be subprojective is that there exists a coordinate system ( $h'$ ) such that (cf. 8.5)

$$(9.2) \quad I_{j'i'}^{h'} * 2\varphi_{(j'} A_{i')}^{h'} + \varphi_{j'i'} \xi^{h'}.$$

From (9.1) we get for the curvature tensor

$$(9.3) \quad \left\{ \begin{array}{l} R_{kji}^h * 2A_i^h \partial_{[k} w_{j]} + 2A_{[k}^h (-\partial_{j]} w_i + w_{j]} w_i + w_{j]i} w_i v^l) + \\ + 2v^h (\partial_{[k} w_{j]} i + 2w_{[k} l] w_{j]i} v^l) + 2w_{i[j} \partial_{k]} v^h \end{array} \right.$$

and this expression could be equated to (8.9) because (8.9) is an invariant equation valid with respect to every coordinate system. Of course ( $h'$ ) is now the system for which (9.2) holds and for  $x^h$  we have now

$$(9.4) \quad x^h = A_h^{h'} x^{h'} * \xi^{h'}; \quad x^h \neq \xi^h.$$

But comparing (9.3) and (8.9) we see that simple results can only be obtained if  $\partial_j v^h$  has the form

$$(9.5) \quad \partial_j v^h * \alpha' A_j^h + \gamma_j v^h.$$

<sup>1)</sup> ADATI 1951, 8, p. 136.

<sup>2)</sup> ADATI 1951, 3; 4; 5; 6; 7; 8; SCHOUTEN 1953, 1.

This is equivalent to the invariant condition introduced by ADATI for the vector field  $v^\kappa$

$$(9.6) \quad \nabla_\mu v^\kappa = \alpha A_\mu^\kappa + \beta_\mu v^\kappa.$$

A vector field in  $L_n$  satisfying an equation of the form (9.6) is called *torse forming* because in an  $E_n$  the lines of the vectors of such a field at the points of any curve form a torse.<sup>1)</sup> By multiplying  $v^\kappa$  with a scalar  $\sigma$  another torse forming field arises and if  $\alpha \neq 0$ ,  $\sigma$  can always be chosen such that the new factor  $\alpha$  equals +1.

Important special cases are:

the *concircular* field<sup>2)</sup>:  $\beta_\mu = \text{gradient}$ ;  $\alpha$  arbitrary. If  $v^\kappa$  is concircular,  $\sigma v^\kappa$  has the same property and  $\beta_\mu = \partial_\mu \beta$  can always be reduced to zero by taking  $\sigma = C e^{-\beta}$ ;  $C = \text{const.}$ ;

the *special concircular* field:  $\beta_\mu = 0$ ;  $\alpha$  arbitrary. The property is only invariant if  $\sigma = \text{const.}$ ;

the *concurrent* field<sup>3)</sup>:  $\beta_\mu = 0$ ;  $\alpha = \text{const.}$  By multiplying  $v^\kappa$  with  $\alpha^{-1}$  the constant  $\alpha$  is reduced to +1;

the *recurrent* or *parallel* field (cf. III § 2)<sup>4)</sup>:  $\alpha = 0$ ;  $\beta_\mu$  arbitrary;

the *covariant constant* field (cf. III § 2):  $\alpha = 0$ ;  $\beta_\mu = 0$ .

It is easily proved that the only transformations of the  $I_{\mu\lambda}^\kappa$  that leave every torse forming field torse forming are the transformations of the form (cf. Exerc. VI 9,1)

$$(9.7) \quad 'I_{\mu\lambda}^\kappa = I_{\mu\lambda}^\kappa + p_\mu A_\lambda^\kappa + q_\lambda A_\mu^\kappa. \quad ^5)$$

These are at the same time those transformations in  $L_n$  that preserve all geodesics and transform every semi-symmetric connexion into a connexion with the same property. Here is a survey of the invariance of the six vector fields just defined under the transformations (9.7)

<sup>1)</sup> If for a field  $v^\kappa$  in an  $A_n$  or  $L_n$  satisfying (9.5) a CARTAN displacement (cf. III § 2) is effected along a curve the images of the lines of the vectors in the moving  $E_n$  form a torse. Fields of this kind were introduced by YANO 1944, 5. The property of being torse forming is in fact not a property of the vector field but of its field of directions. Obviously the field of the pole directions of a subprojective connexion is torse forming.

<sup>2)</sup> YANO 1940, 5; 6; 7; 8; 1943, 3; ADATI 1951, 3, p. 161.

<sup>3)</sup> Concurrent fields were first introduced by MYLLER 1924, 1; 1928, 1 for  $V_m$  in  $R_n$ , cf. BORTOLOTTI 1931, 5; HAIMOVICI 1938, 1; SCHIROKOW 1939, 1; YANO 1943, 3.

<sup>4)</sup> Cf. for instance EISENHART 1922, 4; 1925, 1; 1926, 1, p. 67ff.; 1938, 1; LOPSCHITZ 1936, 1; YANO 1943, 3; LEVINE 1948, 1; 1949, 1; RUSE 1949, 1; 1950, 1; WALKER 1949, 1; 1950, 1; 3; 4; PATTERSON 1951, 1.

<sup>5)</sup> Transformations of this kind were considered by HLAVATY 1927, 3; 1928, 2; 1933, 1; SCHOUTEN and GOLAB 1930, 3; HOMBU and OKADA 1941, 2; YANO 1944, 12; MIKAMI 1949, 1.

|                      | $\alpha$  | $\beta_\mu$ | Transformation                                                                           |
|----------------------|-----------|-------------|------------------------------------------------------------------------------------------|
| Torse forming        | arbitrary | arbitrary   | $\alpha \rightarrow \alpha + q_\mu v^\mu; \quad \beta_\mu \rightarrow \beta_\mu + p_\mu$ |
| concircular          | arbitrary | gradient    | invariant for $p_\mu = \text{gradient}$                                                  |
| spec. concircular    | arbitrary | 0           | invariant for $p_\mu = 0$                                                                |
| concurrent           | constant  | 0           | invariant for $p_\mu = 0; \quad q_\mu = 0$                                               |
| recurrent (parallel) | 0         | arbitrary   | invariant for $q_\mu = 0$ <sup>1)</sup>                                                  |
| cov. constant        | 0         | 0           | invariant for $p_\mu = 0; \quad q_\mu = 0$                                               |

Torse forming fields are not possible in every  $A_n$ .

From (9.1, 5, 6) we get for  $n > 2$

$$(9.8) \quad \alpha' * \alpha - w_i v^i; \quad \gamma_j * \beta_j - w_j - w_{ji} v^i.$$

If (9.5) is satisfied, its integrability conditions

$$(9.9) \quad (\partial_{[k} \alpha' - \alpha' \gamma_{[k}) A_{j]}^h + \partial_{[k} \gamma_{j]} v^h * 0$$

must also hold. But if in the auxiliary theorem (cf. VI §8)  $i$  is strangled and  $C_{kj}$  dropped it follows that for  $n > 2$  (9.9) is equivalent to

$$(9.10) \quad \begin{cases} \text{a)} & \partial_j \alpha' - \alpha' \gamma_j * 0 \\ \text{b)} & \partial_{[j} \gamma_{i]} * 0. \end{cases}$$

(9.10b) is a consequence of (9.10a), and (9.10a) can also be written in the form

$$(9.11) \quad \partial_j \alpha - \alpha \beta_j * - v^i (\alpha w_{ji} - \partial_j w_i + w_j w_i).$$

If (9.5) is substituted in (9.3) it follows that  $R_{kji}^{*};^h$  has the special form (8.9). If then the auxiliary theorem is applied to (8.9) and (9.3) it follows for  $n > 3$  that

$$(9.12) \quad \begin{cases} \text{a)} & T_{ji} * \alpha w_{ji} - \partial_j w_i + w_j w_i \\ \text{b)} & T_{[j i]} = - \partial_{[j} w_{i]} \\ \text{c)} & v^h * \varrho x^h \\ \text{d)} & U_{kji} * \varrho (\partial_{[k} w_{j]}_i + \beta_{[k} w_{j]}_i - w_{[k} w_{j]}_i). \end{cases}$$

Consequently (9.11) takes for  $n > 3$  the form

$$(9.13) \quad \partial_j \alpha - \alpha \beta_j * - v^i T_{ii}.^2)$$

<sup>1)</sup> Cf. footnote 4 on p. 287.

<sup>2)</sup> In order to derive this equation we have only used (9.1) and (9.5) and these conditions are sufficient to ensure that  $R_{kji}^{*};^h$  has the special form (8.9) without it being known whether the connexion is subprojective or not.

The  $I_{ji}^h$  transform into the  $I_{j'i'}^{h'}$  as follows

$$(9.14) \quad I_{j'i'}^{h'} = A_{j'i'}^{j'i} I_{ji}^h - A_{j'i'}^{j'i} \partial_j A_i^{h'}.$$

Hence, in order that  $I_{j'i'}^{h'}$  takes the form (9.2) it is necessary and sufficient that  $\partial_j A_i^{h'}$  takes the form

$$(9.15) \quad \partial_j A_i^{h'} \stackrel{*}{=} \psi_j A_i^{h'} + \psi_i A_j^{h'} + \psi_{ji} v^{h'}; \quad \psi_{[j} i] = 0.$$

Now we have from (9.4, 5, 12c, 15)

$$(9.16) \quad \left\{ \begin{array}{l} A_j^{h'} \stackrel{*}{=} \partial_j \xi^{h'} \stackrel{*}{=} \partial_j \varrho^{-1} A_i^{h'} v^i \stackrel{*}{=} A_j^{h'} \varrho^{-1} (\alpha' + \psi_i v^i) + \\ \qquad \qquad \qquad + \varrho^{-1} v^{h'} (\psi_j + \psi_{ji} v^i + \gamma_j - \partial_j \log \varrho) \end{array} \right.$$

hence for  $n > 3$

$$(9.17) \quad \alpha' \stackrel{*}{=} \varrho - \psi_i v^i$$

$$(9.18) \quad \gamma_j \stackrel{*}{=} -\psi_j - \psi_{ji} v^i + \partial_j \log \varrho.$$

The integrability conditions of (9.15) are

$$(9.19) \quad \left\{ \begin{array}{l} 0 \stackrel{*}{=} A_i^{h'} \partial_{[k} \psi_{j]} + A_{[k}^{h'} (\psi_{j]} \psi_i - \partial_{j]} \psi_i + \psi'_{j]i}) + \\ \qquad \qquad \qquad + \varrho^{-1} v^{h'} (\psi'_{i[k} \psi_{j]} + \partial_{[k} \psi'_{j]i}); \quad \psi'_{ji} \stackrel{*}{=}_{\det} \varrho \psi_{ji}; \end{array} \right.$$

and according to the auxiliary theorem these are for  $n > 3$  equivalent to

$$(9.20) \quad \left\{ \begin{array}{l} \text{a)} \quad \partial_j \psi_i - \psi_j \psi_i - \psi'_{ji} \stackrel{*}{=} 0 \\ \text{b)} \quad \partial_{[k} \psi'_{j]i} - \psi'_{[k} \psi_{j]i} \stackrel{*}{=} 0. \end{array} \right.$$

Now the integrability condition of (9.20a) is (9.20b) and the integrability condition of (9.20b) is identically satisfied. Moreover if (9.17, 18) is substituted in the integrability conditions (9.10) these equations are identically satisfied on account of (9.17, 18, 20). Hence the system of differential equations (9.4, 12c, 15, 17, 18, 20a, 20b) for the unknowns  $\xi^{h'}$ ,  $\varrho$ ,  $A_i^{h'}$ ,  $\psi_j$ ,  $\psi_{ji}$  is totally integrable. Every solution of (9.20a) if substituted in (9.17) gives a value of  $\varrho$  that satisfies (9.18) on account of (9.20a). That proves the theorem<sup>1)</sup>

*If the connexion of a non projectively euclidean  $A_n$  can be written in the form*

$$(9.21) \quad I_{ji}^h \stackrel{*}{=} 2w_{(j} A_{i)}^h + w_{ji} v^h; \quad w_{[j} i] = 0$$

<sup>1)</sup> SCHOUTEN 1953, 1; ADATI proved this theorem for  $V_n$  but for  $A_n$  he introduced (9.13), equivalent to (9.10a) as an additional condition and he did not observe that this condition is in fact the integrability condition of (9.5) or (9.6) and that it can be derived using (9.1) and (9.5) only [see footnote 2 on p. 323]. Cf. ADATI 1951, 7, p. 127 where he gives the final form of his theorem for  $A_n$ .

with respect to some coordinate system  $(h)$  and if the vector field which has the components  $v^h$  with respect to  $(h)$  is torse forming the connexion is subprojective. The pole direction is the direction of  $v^h$ .<sup>1)</sup>

The system (9.20a) has some very simple solutions. There are two cases:

1.  $\alpha' \neq 0$ . Because the product  $w_{ji} v^h$  is given, we may fix the unknown factor in  $v^h$  in such a way that  $\alpha' = 1$ . Then

$$(9.22) \quad \partial_j v^h * A_j^h + \gamma_j v^h.$$

But the integrability conditions of this equation are

$$(9.23) \quad (\partial_{[k} \gamma_{j]}) v^h + A_{[k}^h \gamma_{j]} * 0$$

and these can only be satisfied if  $\gamma_j = 0$ . Then from (9.22) we get

$$(9.24) \quad \partial_j v^h * \partial_j \xi^h$$

hence

$$(9.25) \quad v^h * \xi^h + c^h; \quad c^h = \text{const.}$$

Now if we take

$$(9.26) \quad \xi^h * \delta_h^h (\xi^h + c^h)$$

we have  $A_h^h * \delta_h^h$  and accordingly

$$(9.27) \quad \Gamma_{j'i'}^h * 2w_{(j'} A_{i')}^h + w_{j'i'} \xi^h$$

as was demanded.

2.  $\alpha' = 0$ . In this case we have

$$(9.28) \quad \partial_j v^h * \gamma_j v^h$$

for any choice of the scalar factor in  $v^h$ . The integrability conditions are

$$(9.29) \quad \partial_{[k} \gamma_{j]} * 0,$$

hence by changing the scalar factor in  $v^h$  we can always obtain

$$(9.30) \quad \partial_j v^h * 0; \quad v^h * C^h = \text{const.}$$

Now we take for the  $\psi_i$   $n$  arbitrary constants  $c_i$  such that  $c_i C^i = 1$  and for  $\psi'_i$  we take  $-c_j c_i$ . Then (9.20a, 20b) are satisfied and from (9.17) we get  $\varrho = 1$ , satisfying also (9.18). That leaves for the  $A_i^h$  the equations (9.15) in the form

$$(9.31) \quad \partial_j A_i^h * 2c_{(j} A_{i)}^h - c_j c_i C^h A_h^h$$

---

<sup>1)</sup> The condition for  $v^h$  is sufficient but not necessary. Necessary and sufficient conditions are not yet found. The theorem holds also for  $n = 3$  but for that case it is not proved that the direction of  $v^h$  is the only pole-direction at every point.

with constants  $c_i$  and  $C^h$ . Because the  $c_i$  and  $C^h$  are constants the coordinate system  $(h)$  can be transformed by a homogeneous linear transformation with *constant* coefficients such that after the transformation  $c_i$  coincides with  $\bar{e}_i^l$  and  $C^h$  with  $\bar{e}^h$  and this transformation does not disturb the form of (9.4) and (9.30).

Then (9.31) takes the form

$$(9.32) \quad \begin{cases} a) \quad \partial_1 A_1^{h'} = A_1^{h'} \\ b) \quad \partial_b A_1^{h'} = \partial_1 A_b^{h'} = A_b^{h'}; \quad b, c = 2, \dots, n \\ c) \quad \partial_c A_b^{h'} \equiv 0 \end{cases}$$

from which by integration

$$(9.33) \quad \begin{cases} a) \quad \partial_1 \xi^{h'} = A_1^{h'} = (C_b^{h'} \xi^b + C_1^{h'}) e^{\xi^1}; \quad C_h^{h'} = \text{const.}; \quad b = 2, \dots, n \\ b) \quad \partial_b \xi^{h'} = A_b^{h'} \equiv C_b^{h'} e^{\xi^1} \end{cases}$$

with constants  $C_h^{h'}$  that can be chosen arbitrarily provided that  $\text{Det}(C_h^{h'}) \neq 0$ . By a second integration we get

$$(9.34) \quad \xi^{h'} \equiv (C_b^{h'} \xi^b + C_1^{h'}) e^{\xi^1} = \bar{e}^{h'} = v^{h'}$$

and this leads to the following expression for the  $I_{j'i'}^{h'}$

$$(9.35) \quad I_{j'i'}^{h'} \equiv (w_{j'} + \bar{e}_{j'}) A_{i'}^{h'} + (w_{i'} + \bar{e}_{i'}) A_{j'}^{h'} + (w_{j'i'} - \bar{e}_{j'} \bar{e}_{i'}) \xi^{h'}$$

as was demanded. (9.35) is independent of the choice of the  $C_h^{h'}$ . In fact a change of these constants is equivalent to a homogeneous linear transformation of the  $\xi^{h'}$  with constant coefficients and such a transformation does not disturb the form of (9.35).

From (9.34) we see that the case  $\alpha' = 0$  is not at all abnormal and that it only arises because the coordinates are chosen such that they get infinite at the pole. VRANCEANU<sup>1)</sup> used 1947 this special choice of the coordinates in order to simplify his calculations.

### Exercises.

VI 9,1<sup>2)</sup>). The transformations  $I_{\mu\lambda}^{\alpha} \rightarrow I_{\mu\lambda}^{\alpha} + P_{\mu\lambda}^{\alpha}$  in  $L_n$  transforms every torse forming vector field into a field with the same property if and only if  $P_{\mu\lambda}^{\alpha}$  has the form

$$\text{VI 9,1 } \alpha) \quad P_{\mu\lambda}^{\alpha} = p_{\mu} A_{\lambda}^{\alpha} + A_{\mu}^{\alpha} q_{\lambda}.$$

VI 9,2. The transformation of Exerc. VI 9,1 leaves the tensor  $R_{\nu\mu\lambda}^{\alpha} + A_{\lambda}^{\alpha} R_{\nu\mu}$  invariant for  $q_{\lambda} = 0$ .

<sup>1)</sup> VRANCEANU 1947, 2.

<sup>2)</sup> Cf. footnote 5 on p. 322.

VI 9,3<sup>1)</sup>. The object

$$\text{VI 9,3 } \alpha) \quad A_{\mu\lambda}^{\kappa} = \Gamma_{\mu\lambda}^{\kappa} + \frac{1}{n^2 - 1} \{ A_{\mu}^{\kappa} (\Gamma_{\lambda\theta}^{\theta} - n \Gamma_{\theta\lambda}^{\theta}) + A_{\lambda}^{\kappa} (\Gamma_{\theta\mu}^{\theta} - n \Gamma_{\mu\theta}^{\theta}) \}.$$

is invariant for the transformations of Exerc. VI 9,1.

## § 10. Subprojective transformations of a connexion in $A_n$ .

A transformation of a connexion of the form

$$(10.1) \quad 'T_{j|i}^h = \Gamma_{j|i}^h + 2p_{(j} A_{i)}^h + q_{[j|i]} v^h; \quad q_{[j|i]} = 0$$

with arbitrary vectors  $p_i$  and  $v^h$  and an arbitrary symmetric tensor  $q_{ji}$  is called a *subprojective transformation belonging to the direction*  $[v^h]$ .<sup>2)</sup> All subprojective transformations belonging to the same direction  $[v^h]$  form a group.

The curves satisfying equations of the form

$$(10.2) \quad \frac{d^2 \xi^h}{dt^2} + \Gamma_{j|i}^h \frac{d\xi^j}{dt} \frac{d\xi^i}{dt} = \kappa \frac{d\xi^h}{dt} + \lambda v^h$$

are called *subgeodesics with respect to the field*  $[v^h]$ .<sup>3)</sup> The subprojective transformations belonging to  $[v^h]$  leave the form of these equations invariant. If  $[v^h]$  is torse forming we know that subprojective connexions exist with the pole-direction  $[v^h]$  and in this case the subprojective transformations belonging to  $[v^h]$  are those transformations that transform these subprojective connexions into each other. The set of all subprojective connexions belonging to  $[v^h]$  and the set of all streamlines of  $[v^h]$  determine each other uniquely. Hence a transformation  $\xi^h \rightarrow \xi^h + u^h dt$  leaves this set of connexions invariant if and only if it leaves the streamlines invariant, that is if  $\frac{d}{dt} v^h \propto v^h$ . This can be proved also from (II 10.34). If this equation is written in the form<sup>4)</sup>

$$(10.3) \quad \frac{d}{dt} \Gamma_{j|i}^h = u^l \partial_l \Gamma_{j|i}^h - \Gamma_{j|i}^l \partial_l u^h + \Gamma_{l|i}^h \partial_j u^l + \Gamma_{j|i}^h \partial_i u^l + \partial_j \partial_i u^h$$

it must be proved that  $\frac{d}{dt} \Gamma_{j|i}^h$  has the form  $\omega_j A_i^h + \omega_i A_j^h + \omega_{ji} v^h$ <sup>5)</sup> if and only if  $\frac{d}{dt} v^h \propto v^h$ . Now if (9.1) is substituted in (10.3) and if all terms

<sup>1)</sup> HLAVATY 1927, 3, p. 85.

<sup>2)</sup> YANO 1944, 6; ADATI 1951, 6, p. 106.

<sup>3)</sup> Equations of the form (10.1) and (10.2) were first considered by v. DANTZIG 1932, 3; cf. SCHOUTEN and HAANTJES 1936, 1; HAANTJES 1937, 1; YANO 1944, 3; 6, he introduced the name "subpaths".

<sup>4)</sup> Cf. E I 1935, 1, p. 142.

<sup>5)</sup> YANO 1944, 6 has formulated this condition but he seems to have overlooked that it is satisfied if and only if the streamlines are invariant.

are dropped that have already the right form, the only remaining terms are

$$(10.4) \quad w_{ji} \alpha' u^h - w_{ji} v^l \partial_l u^h = w_{ji} \underline{v}^h - w_{ji} u^l \gamma_l v^h; \quad (\text{cf. 9.5})$$

and these take the right form if and only if  $\underline{v}^h \propto v^h$ .

### Exercise.

VI 10,1<sup>1)</sup>. Every subprojective  $A_n$  can be transformed into an  $E_n$  by a subprojective transformation of its connexion.

## § 11. The subprojective $V_n$ .

If the subprojective  $A_n$  is a  $V_n$ , (8.9) takes the form

$$(11.1) \quad K_{kjih} \stackrel{*}{=} 2g_{[k|h]} T_{j]i} - 2g_{ih} T_{[kj]} + 2U_{kji} x_h$$

and this expression must be alternating in  $ih$ :

$$(11.2) \quad g_{[k(h} T_{j]i)} - g_{ih} T_{[kj]} + U_{kji(i} x_h) \stackrel{*}{=} 0.$$

For  $n > 2$  we may take two vectors  $y^h, z^h$ , perpendicular to  $x^h$  and to each other. Then by transvection of (11.2) with  $y^h z^i$  we get

$$(11.3) \quad y_{[k} T_{j]i} z^i + z_{[k} T_{j]i} y^i \stackrel{*}{=} 0$$

and this means that  $T_{ji} y^i \propto y_j$  for every vector  $y^h$  perpendicular to  $x^h$ . But this is only possible if  $T_{ji}$  has the form

$$(11.4) \quad T_{ji} \stackrel{*}{=} 2\mu g_{ji} + l_j x_i$$

with suitably chosen  $\mu$  and  $l_j$ . If (11.4) is substituted in (11.1) we get

$$(11.5) \quad K_{kjih} \stackrel{*}{=} 4\mu g_{[k|h} g_{j]i} + 2g_{[k|h} l_{j]} x_i - 2g_{ih} l_{[k} x_{j]} + 2U_{kji} x_h.$$

As this expression must be alternating in  $ih$  we get

$$(11.6) \quad g_{[k(h} l_{j]} x_i) - g_{ih} l_{[k} x_{j]} + U_{kji(i} x_h) \stackrel{*}{=} 0$$

and by transvection with  $y^i y^h$  it follows that  $l_{[k} x_{j]} = 0$ . Hence  $T_{ji}$  is symmetric

$$(11.7) \quad T_{ji} \stackrel{*}{=} 2\mu g_{ji} + \lambda x_j x_i$$

and this implies that  $\varphi_j$  is a gradient and that there exist special co-ordinate systems for which  $\varphi_j$  vanishes (cf. VI § 8). If (11.7) is substituted in (11.2) we get

$$(11.8) \quad \text{a)} \quad U_{kjh} = \lambda x_{[k} g_{j]h}; \quad \text{b)} \quad U_{kjh} x^h = 0$$

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<sup>1)</sup> YANO 1944, 6, p. 104.

and consequently

$$(11.9) \quad \begin{cases} \text{a)} & K_{kjih} = \frac{4}{n-2} g_{[k[i} L_{j]h]} \\ \text{b)} & L_{jh} \stackrel{*}{=} (n-2)(\mu g_{jh} + \lambda x_j x_h). \end{cases}$$

For  $n > 3$ , (11.9) is the n.a.s. condition for the  $V_n$  to be conformally euclidean. For  $n = 3$ ,  $K_{kjih}$  can always be written as the double alternating product of  $g_{ki}$  with another symmetric tensor. But in that case  $\nabla_{[k} L_{j]h} = 0$  is the n.a.s. condition (cf. VI § 5). Now we can prove that this latter condition [that is a consequence of (11.9) for  $n > 3$ ] is always satisfied for a subprojective  $V_n$ .<sup>1)</sup> We take the coordinate system such that  $\varphi_i = 0$ . Then  $\alpha = 1$  and from (8.5, 10, 11, 12) it follows that

$$(11.10) \quad T_{ji} \stackrel{*}{=} \varphi_{ji} \stackrel{*}{=} 2\mu g_{ji} + \lambda x_j x_i,$$

$$(11.11) \quad I_{ji}^h \stackrel{*}{=} \varphi_{ji} x^h,$$

$$(11.12) \quad U_{kji} \stackrel{*}{=} \nabla_{[k} \varphi_{j]i}.$$

But on account of (11.8) this leads to

$$(11.13) \quad \nabla_i \mu = \lambda x_i; \quad \nabla_i \lambda = \zeta x_i$$

where  $\zeta$  is a scalar. If (11.13) is substituted in (11.9), differentiating gives  $\nabla_{[k} L_{j]h} = 0$ . Moreover RACHEVSKI<sup>1)</sup> proved that the condition is also sufficient and can be put in the following form

*A  $V_n$ ,  $n > 2$  is subprojective if and only if it is a  $C_n$  satisfying the special condition*

$$(11.14) \quad L_{jh} = \sigma g_{jh} + \tau (\nabla_j \sigma) V_h \sigma$$

where  $\sigma$  is an arbitrary scalar and  $\tau$  a function of  $\sigma$  only.

From

$$(11.15) \quad \nabla_j x^h \stackrel{*}{=} A_j^h + \varphi_{ji} x^i x^h \stackrel{*}{=} A_j^h + (2\mu + \lambda x_i x^i) x_j x^h$$

and (11.13) it follows immediately that  $\beta_j$  is a gradient. Hence  $x^h$  is not only torse forming but also concircular.<sup>2)</sup> ADATI gave also the forms of  $L_{jh}$  for the cases that  $x^h$  is parallel or the unitvector of  $x^h$  concurrent and he gave also other forms of the linear element of subprojective  $V_n$ 's for different cases.<sup>3)</sup>

<sup>1)</sup> RACHEVSKI 1933, 1, p. 137; cf. E II 1938, 2, p. 220ff.; cf. ADATI 1951, 5, p. 343ff.

<sup>2)</sup> ADATI 1951, 3, p. 172.

<sup>3)</sup> ADATI 1951, 4; 5, p. 333, 357f.

### Exercises.

VI 11,1<sup>1)</sup>. If in a subprojective  $V_n$ ;  $n > 2$ , the coefficient  $\mu$  in (11.4) is a constant, the  $V_n$  is an  $S_n$  (use 11.9 and 11.13).

VI 11,2<sup>2)</sup>. The  $C_n$  with the linear element

$$ds^2 \stackrel{*}{=} f(p) \sum_{\lambda} d\xi^{\lambda} d\xi^{\lambda}; \quad p \stackrel{*}{=} \sum_{\mu} \xi^{\mu} \xi^{\mu}$$

is subprojective.

## § 12. Concircular transformations of a $V_n$ .

From (5.36) we get for the curvature vector  $u^{\alpha} = j^{\mu} V_{\mu} j^{\alpha}$  of a curve with tangent unitvector  $j^{\alpha}$  under conformal transformations of  $g_{\lambda\alpha}$

$$(12.1) \quad 'u^{\alpha} = \sigma^{-1} (u^{\alpha} - \frac{1}{2} z^{\alpha}); \quad z^{\alpha} \stackrel{\text{def}}{=} s^{\alpha} - j^{\mu} s_{\mu} j^{\alpha}; \quad s_{\lambda} \stackrel{\text{def}}{=} \partial_{\lambda} \log \sigma.$$

Now we consider a *geodesic circle*<sup>3)</sup>, that is a curve whose curvature vector has constant length and whose second curvature is zero

$$(12.2) \quad \text{a)} \quad \frac{d}{ds} u^{\alpha} u^{\lambda} g_{\lambda\alpha} = 0; \quad \text{b)} \quad j^{[\alpha} \frac{\delta}{ds} u^{\lambda]} = 0.$$

Then we have after the transformation

$$(12.3) \quad \frac{d}{ds} ('u^{\alpha} 'u^{\lambda} g_{\lambda\alpha}) = \sigma j^{\mu} 'u^{\lambda} \left( -V_{\mu} s_{\lambda} + \frac{1}{2} s_{\mu} s_{\lambda} \right)$$

and this proves that a conformal transformation transforms every geodesic circle into a geodesic circle if and only if an equation holds of the form<sup>4)</sup>

$$(12.4) \quad 2V_{\mu} s_{\lambda} - s_{\mu} s_{\lambda} = 4\psi g_{\mu\lambda} \quad ^5)$$

or (cf. 5.7)

$$(12.5) \quad s_{\mu\lambda} = 4\varphi g_{\mu\lambda}; \quad \varphi = \psi + \frac{1}{8} s_{\lambda} s^{\lambda}.$$

Conformal transformations satisfying this condition are called *concircular*. They leave invariant the equation of all geodesic circles, that

<sup>1)</sup> SCHAPIRO 1933, 1, p. 112; E II 1938, 2, p. 222 and 225.

<sup>2)</sup> Cf. BUCHHOLZ 1899, 1; KAGAN 1933, 1, p. 94; E II 1938, 2, p. 225.

<sup>3)</sup> FIALKOW 1939, 2. YANO introduced 1938, 3 other generalized circles and proved 1940, 6 that these curves are geodesic circles if their tangent direction is everywhere a principal direction of  $V_n$ .

<sup>4)</sup> BRINKMAN 1925, 1; FIALKOW 1939, 2, p. 461; cf. also FIALKOW 1940, 1; 1942, 2; YANO 1940, 5; 6; 7; 8; YANO and ADATI 1944, 7.

<sup>5)</sup> The factor 4 is introduced only in order to get the same  $\varphi$  and  $\psi$  as in the paper of YANO. His  $\varrho_{\mu\lambda}$  is equal to  $\frac{1}{4} s_{\mu\lambda}$ .

can be proved to be<sup>1)</sup>

$$(12.6) \quad \frac{\delta^3 \xi^\lambda}{ds^3} + g_{\mu\lambda} \frac{\delta^2 \xi^\mu}{ds^2} \frac{\delta^2 \xi^\lambda}{ds^2} \frac{d\xi^\lambda}{ds} = 0.$$

Transvection of (12.4) with  $s^\mu$  leads to

$$(12.7) \quad 2s^\mu V_\mu s^\lambda = \dot{s}^\mu s_\mu s^\lambda + 4\psi s^\lambda$$

hence the streamlines of  $s^\lambda$  are geodesics and the congruence of these geodesics is normal to the  $V_{n-1}$ 's tangent to the gradient  $s_\lambda$ . The integrability conditions of (12.4) are

$$(12.8) \quad -K_{\nu\mu\lambda} s_\lambda - (2\psi s_{[\mu} - 4V_{[\mu}\psi) g_{\nu]\lambda} = 0$$

from which by transvection with  $g^{\mu\lambda}$

$$(12.9) \quad K_{\nu}{}^\lambda s_\lambda = (n-1)(2\psi s_\nu - 4V_\nu\psi).$$

But by transvection of (12.8) with  $s^\lambda$  it follows that  $V_\nu\psi$  is a multiple of  $s_\nu$ . Hence  $s^\lambda$  lies in a principal direction (cf. III § 5) of  $V_n$ <sup>2)</sup> and the function  $\psi$  is constant on each of the  $V_{n-1}$ 's orthogonal to the congruence  $s^\lambda$ .

If  $i^\lambda$  is the unitvector of  $s^\lambda$  and  $s^\lambda = \zeta i^\lambda$  we have from (12.4)

$$(12.10) \quad 2(V_\mu \zeta) i_\lambda + 2\zeta V_\mu i_\lambda - \zeta^2 i_\mu i_\lambda = 4\psi g_{\mu\lambda}$$

and

$$(12.11) \quad 2V_\mu \zeta = \zeta^2 i_\mu + 4\psi i_\mu.$$

But this implies that  $V_\mu i_\lambda$  has the form

$$(12.12) \quad V_\mu i_\lambda = -h(g_{\mu\lambda} - i_\mu i_\lambda).$$

Hence the second fundamental tensor of the  $V_{n-1}$ 's normal to  $i^\lambda$  is

$$(12.13) \quad h_{cb} = -B_{c b}^{\mu\lambda} V_\mu i_\lambda = h' g_{cb}$$

and this proves that these  $V_{n-1}$ 's are umbilical. From the CODAZZI equation (V 6.5)

$$(12.14) \quad 2(V_{[d} h)' g_{c]b} = B_{d c b}^{\nu\mu\lambda} K_{\nu\mu\lambda\nu} i^\nu$$

we get by transvection with  $'g^{cb}$

$$(12.15) \quad (n-2)V_d h = B_d^\nu (g^{\mu\lambda} - i^\mu i^\lambda) K_{\nu\mu\lambda\nu} i^\nu = B_d^\nu K_{\nu\nu} i^\nu$$

<sup>1)</sup> YANO 1940, 5, p. 195. He also considers curves that can be transformed conformally into a geodesic circle and calls them conformally geodesic circles. But because every single curve is conformally geodesic (cf. VI § 5), it is also a conformally geodesic circle and the term is useful only in connexion with systems of curves.

<sup>2)</sup> YANO 1940, 6, p. 356.

but, because  $i^*$  lies in a principal direction of  $K_{\mu\nu}$  this proves that  $h$  is constant over  $V_{n-1}$ , i.e. that the  $V_{n-1}$ 's have each constant mean curvature. So we see that the  $V_{n-1}$ 's normal to a geodesic and normal congruence are always umbilical if the tangent direction of the congruence is torse forming<sup>1)</sup> and that the constancy of the mean curvature is due to the fact that in the special case under consideration the tangent lies in a principal direction of the  $V_n$ .

Now let us suppose that there exists in a  $V_n$  a congruence with unit vector  $i^*$  satisfying (12.12) with  $\nabla_b h = 0$ . Then we prove that there exists a scalar  $\eta$  such that the field defined by  $s_\lambda = \eta i_\lambda$  satisfies an equation of the form (12.4). From

$$(12.16) \quad 2(\nabla_\mu \eta) i_\lambda - 2\eta h g_{\mu\lambda} + 2\eta h i_\mu i_\lambda - \eta^2 i_\mu i_\lambda = 4\psi g_{\mu\lambda}$$

follows

$$(12.17) \quad \begin{cases} \text{a)} & 2\partial_\mu \eta = (\eta^2 - 2\eta h) i_\mu \\ \text{b)} & \eta h = -2\psi. \end{cases}$$

The integrability condition of (12.17a) is

$$(12.18) \quad (2\eta \partial_{[\nu} \eta - 2\eta \partial_{[\nu} h - 2h \partial_{[\nu} \eta) i_{\mu]} = 0$$

and because  $i_{[\mu} \partial_{\nu]} h = 0$ , this implies that  $\eta$  is a function of  $u$  only, if we write  $i_\mu = \partial_\mu u$ . Now  $h$  is also a function of  $u$  only, hence if  $\eta \stackrel{\text{def}}{=} f(u)$  and  $f'(u) \stackrel{\text{def}}{=} \frac{df}{du}$  the differential equation for  $\eta$  takes the form

$$(12.19) \quad 2f' = f^2 - 2h f$$

and this is a RICCATI equation. Gathering results we have<sup>2)</sup>

*A  $V_n$  admits a concircular transformation if and only if there exists at least one geodesic congruence, normal to a set of umbilical  $V_{n-1}$ 's with constant mean curvature.*

The concircular transformations form a subgroup of all conformal transformations. If a  $V_m$  in  $V_n$  is given, every concircular transformation  $g_{\mu\lambda} \rightarrow \sigma g_{\mu\lambda}$  in  $V_n$  induces a conformal transformation  $'g_{cb} \rightarrow \sigma' g_{cb}$  in  $V_m$  and we may ask<sup>3)</sup> whether the induced transformation is also concircular. Now if  $'s_b \stackrel{\text{def}}{=} B_b^\lambda s_\lambda$  we have

$$(12.20) \quad \left\{ \begin{array}{l} 2'\nabla_c 's_b = 2'\nabla_c B_b^\lambda s_\lambda = 2B_{cb}^\lambda \nabla_\mu s_\lambda + 2s_\lambda D_c B_b^\lambda \\ \qquad \qquad \qquad = 's_c 's_b + 4\psi 'g_{cb} + 2H_{cb}^{..}\lambda s_\lambda, \end{array} \right.$$

<sup>1)</sup> YANO 1944, 5.

<sup>2)</sup> YANO 1940, 6, p. 359; cf. also FIALKOW 1939, 2. The proof given here is much simpler than YANO's proof that is based on a special form of the linear element and makes use of a function suggested by A. KAWAGUCHI.

<sup>3)</sup> YANO 1940, 8, p. 506.

hence the induced transformation is concircular if and only if  $H_{cb}^{\cdot\lambda}$  is a multiple of ' $g_{cb}$ ' or, in another form, if and only if<sup>1)</sup> (cf. V 9.16b)

$$(12.21) \quad M_{cb}^{\cdot\lambda} s_\lambda = 0.$$

As a corollary we get that a concircular transformation induces a concircular transformation in every umbilical  $V_m$ . This implies that *every geodesic and every geodesic circle in an umbilical  $V_m$  is a geodesic circle in  $V_n$* .<sup>2)</sup>

If a field  $u^\alpha$  is subjected to a parallel displacement over  $v^\alpha dt$  we have for the field value of the new field at  $\xi^\alpha + v^\alpha dt$

$$(12.22) \quad u^\alpha - v^\mu \Gamma_{\mu\lambda}^\alpha u^\lambda dt.$$

In order that this value has the same direction as the field dragged along over  $v^\alpha dt$

$$(12.23) \quad u^\alpha + u^\mu \partial_\mu v^\alpha dt$$

it is necessary and sufficient that

$$(12.24) \quad u^\mu \nabla_\mu v^\alpha = \alpha u^\alpha$$

where  $\alpha$  is a function of the  $\xi^\alpha$ . Hence, a point transformation  $v^\alpha dt$  leaves invariant the tangent direction of every curve if and only if  $v^\alpha$  is a special concircular field (cf. VI § 9):

$$(12.25) \quad \nabla_\mu v^\alpha = \alpha A_\mu^\alpha.$$

Because  $\nabla_\mu v_\lambda = \alpha g_{\mu\lambda}$ , the vector  $v_\lambda$  is a gradient. If we write  $v_\lambda = \partial_\lambda \varrho^{-1}$  it follows that

$$(12.26) \quad \nabla_\mu \nabla_\lambda \log \varrho - (\nabla_\mu \log \varrho) \nabla_\lambda \log \varrho = -\alpha \varrho g_{\mu\lambda}$$

and this proves that the  $V_n$  admits a concircular transformation  $g_{\mu\lambda} \rightarrow \varrho^2 g_{\mu\lambda}$  with  $\psi = -\alpha \varrho$  (cf. 12.4). Conversely, if we have a  $V_n$  with a field  $s_\lambda$  satisfying (12.4) and if  $s_\lambda = \partial_\lambda \log \sigma$ , the field  $v^\alpha = g^{\lambda\alpha} \partial_\lambda \sigma^{-\frac{1}{2}}$  is a special concircular field. Hence<sup>3)</sup>

*A  $V_n$  admits a concircular transformation if and only if it admits at least one special concircular vector field and this is the case if and only if it admits at least one infinitesimal point transformation that displaces the tangent directions of every curve parallel.*

<sup>1)</sup> See footnote 3, page 332.

<sup>2)</sup> YANO 1943, 3, p. 193.

<sup>3)</sup> YANO and ADATI 1944, 7. This paper deals also with the case where  $v^\alpha$  is a concurrent field,  $\alpha = \text{const.}$

FRENET formulae for curves in concircular geometry can be developed by means of the vector (cf. 12.6)

$$(12.27) \quad V^* \stackrel{\text{def}}{=} \frac{\delta^3 \xi^*}{ds^3} + g_{\mu\lambda} \frac{\delta^2 \xi^\mu}{ds^2} \frac{\delta^2 \xi^\lambda}{ds^2} \frac{d\xi^*}{ds}$$

that transforms into  $\sigma^{-\frac{2}{n}} V^*$  for a conformal transformation of  $g_{\mu\lambda}$  if and only if this transformation is concircular.<sup>1)</sup>

From (12.5) and (5.6) it follows that for a concircular transformation

$$(12.28) \quad K'_{\nu\mu\lambda} = K_{\nu\mu\lambda} + 4\varphi g_{[\nu} g_{\mu]\sigma} g^{\sigma*}.$$

Hence, a  $V_n$  that can be transformed into an  $R_n$  by a concircular transformation must be an  $S_n$ . Moreover it follows from (12.28) that a concircular transformation transforms every EINSTEIN space into an EINSTEIN space.<sup>2)</sup> Another consequence of (12.28) is that<sup>3)</sup><sup>4)</sup>

$$(12.29) \quad Z_{\nu\mu\lambda}^* \stackrel{\text{def}}{=} K_{\nu\mu\lambda} - \frac{2}{n(n-1)} K A_{[\nu}^* g_{\mu]\lambda}$$

is invariant for concircular transformations. YANO<sup>3)</sup> proved that every conformal transformation that leaves  $Z_{\nu\mu\lambda}^*$  invariant is concircular.  $Z_{\nu\mu\lambda}^*$  is called the *concircular curvature tensor*. In the same paper YANO proved that its vanishing is for  $n > 2$  n.a.s. for the  $V_n$  to be transformable into an  $R_n$  by a concircular transformation. The  $V_n$  could be called *concircular euclidean* in this case but because for  $n > 2$  there is no difference between an  $S_n$  and a concircular euclidean space it is better not to introduce a new name. For  $Z_{\nu\mu\lambda}^*$  the following identities hold

$$(12.30) \quad \begin{cases} \text{a) } Z_{\nu\mu\lambda}^* = K_{\mu\lambda} - \frac{1}{2} K K_{\mu\lambda} \\ \text{b) } g^{\mu\lambda} Z_{\nu\mu\lambda}^* = 0. \end{cases}$$

In the last of YANO's papers on concircular transformations and in a paper of 1943<sup>5)</sup> he studied conformal and concircular properties of EINSTEIN spaces. Recently TACHIBANA<sup>6)</sup> dealt with the relations between concircular transformations and the investigations of SASAKI<sup>7)</sup> on the holonomy group of conformal spaces.

### Exercise.

VI 12,<sup>18)</sup>. If every geodesic circle of a  $V_{n-1}$  in  $V_n$  is a geodesic circle of  $V_n$ , the  $V_{n-1}$  is umbilical and its mean curvature is constant.

<sup>1)</sup> YANO 1940, 4; 7.

<sup>2)</sup> YANO 1940, 5, p. 200.

<sup>3)</sup> YANO 1940, 5.

<sup>4)</sup> TACHIBANA 1951, 2, p. 150.

<sup>5)</sup> YANO 1942, 3; 1943, 1.

<sup>6)</sup> TACHIBANA 1951, 2.

<sup>7)</sup> SASAKI 1943, 1; 2; 3.

<sup>8)</sup> YANO 1940, 7; 1943, 3.

## VII. Variations and deformations.

### § 1. General deformation problems.<sup>1)</sup>

In this chapter several problems concerning variations and deformations are dealt with. The general form of such a problem is as follows: certain geometrical objects suffer a finite or infinitesimal transformation and the behaviour of some other objects depending on them is required. In deformation problems we deal with the special case where the variation is due to displacements of some kind, for instance a dragging along or a parallel displacement. The case occurring most frequently is that some objects are left at rest, others are dragged along and others are displaced parallel. The theory of variation and deformation is very important because a great number of problems in differential geometry can be treated in a very elegant way by using the methods of this theory.

If a geometric object field is defined in an  $\mathfrak{N}(\xi^x)$  we have at  $\xi^x + v^x dt$ .<sup>2)</sup>

1. the *natural value* of the field or the value of the field *at rest*, for instance for a contravariant vector

$$(1.1) \quad p^x + d\overset{m}{p}^x = p^x + v^\mu \partial_\mu p^x dt.$$

2. the value arising from *dragging along* the field over  $v^x dt$ , for instance (cf. II § 10) for vectors

$$(1.2) \quad \begin{cases} \text{a)} & \overset{m}{p}^x + d\overset{m}{p}^x = p^x + \overset{m}{p}^\mu \partial_\mu v^x dt \\ \text{b)} & q_\lambda + d\overset{m}{q}_\lambda = q_\lambda - q_\mu \partial_\lambda v^\mu dt. \end{cases}$$

If a parallel displacement in the direction of  $v^x$  is defined for the quantities considered we also have

3. the value arising from parallel displacement of the field over  $v^x dt$ , for instance (cf. III § 2)

$$(1.3) \quad \begin{cases} \text{a)} & \overset{*}{p}^x + d\overset{*}{p}^x = p^x - \Gamma_{\mu\lambda}^x p^\lambda v^\mu dt \\ \text{b)} & q_\lambda + \overset{*}{d}q_\lambda = q_\lambda + \Gamma_{\mu\lambda}^x q_\mu v^\mu dt. \end{cases}$$

<sup>1)</sup> Cf. Mc CONNELL 1929, 1 (curves in  $V_n$ ); HAYDEN 1931, 1; SCHOUTEN and v. KAMPEN 1933, 1; DAVIES 1933, 1 (in  $U_n$ ); 1937, 1; 1938, 1; DIENES 1933, 3 (in  $L_n$ ); E I 1935, 1, § 12; E II 1938, 2, § 16; COBURN 1940, 2 (on SCHOUTEN and v. KAMPEN); SHABBAR 1942, 1 (finite deform.); YANO 1945, 7; 8; 9; 10 (on COBURN); 1949, 2 (literature); TONOLO 1950, 1 (in  $V_3$  and  $S_3$ ).

<sup>2)</sup> We follow here the line of thought developed in SCHOUTEN and v. KAMPEN 1933, 1 and E I 1935, 1, p. 140ff.

The three differentials arising in this way are all non invariant<sup>1)</sup> but their differences are invariant:

1. the *covariant differential*  $\delta$  (cf. III § 2), for instance for vectors

$$(1.4) \quad \begin{cases} \text{a)} & \delta p^x = d \overset{*}{p}{}^x - d \overset{*}{\dot{p}}{}^x = v^\mu V_\mu \overset{*}{p}{}^x dt \\ \text{b)} & \delta q_\lambda = d q_\lambda - d \overset{*}{q}_\lambda = v^\mu V_\mu q_\lambda dt. \end{cases}$$

2. the *LIE differential*  $\frac{\xi}{v} dt$  (cf. II § 10 and III 5.48), for instance

$$(1.5) \quad \begin{cases} \text{a)} & \frac{\xi}{v} p^x dt = d \overset{*}{p}{}^x - d \overset{m}{\dot{p}}{}^x = (v^\mu \partial_\mu p^x - \overset{*}{p}{}^\mu \partial_\mu v^x) dt \\ & = v^\mu V_\mu \overset{*}{p}{}^x dt - \overset{*}{p}{}^\mu v_\mu^x dt. \\ \text{b)} & \frac{\xi}{v} q_\lambda dt = d q_\lambda - d \overset{m}{\dot{q}}_\lambda = (v^\mu \partial_\mu q_\lambda + q_\mu \partial_\lambda v^\mu) dt \\ & = v^\mu V_\mu q_\lambda dt + q_\mu v_\lambda^\mu dt. \\ & v_\lambda^x \stackrel{\text{def}}{=} \partial_\lambda v^x + I_{\mu\lambda}^{x*} v^\mu. \end{cases}$$

3. the *apparent differential*<sup>2)</sup>  $\frac{D}{s} dt$ , for instance

$$(1.6) \quad \begin{cases} \text{a)} & \frac{D}{s} p^x dt = d \overset{*}{p}{}^x - d \overset{m}{\dot{p}}{}^x = - \overset{*}{p}{}^\mu v_\mu^x dt \\ \text{b)} & \frac{D}{s} q_\lambda dt = d \overset{*}{q}_\lambda - d \overset{m}{\dot{q}}_\lambda = + q_\mu v_\lambda^\mu dt. \end{cases}$$

Between these three differentials applied to quantities we have the relation

$$(1.7) \quad \delta - \frac{\xi}{v} dt + \frac{D}{s} dt = 0.$$

The geometrical interpretation is as follows. Let the field  $v^x$  be given in an  $\mathfrak{N}(\xi^x)$  and let the field of a quantity  $\Omega$  (indices suppressed) undergo on the one hand a parallel displacement over  $v^x dt$ . Then the value  $\Omega$  at  $\xi^x$  changes into  $\Omega - \delta\Omega$ . On the other hand let the field  $\Omega$  be dragged along over  $v^x dt$ . Then the value at  $\xi^x$  changes into  $\Omega - \frac{\xi}{v} \Omega dt$ . The difference between the two new values at  $\xi^x$  is  $- \frac{D}{s} \Omega dt$ , hence we get exactly the field value  $\Omega + \frac{D}{s} \Omega dt$  at  $\xi^x$  if the field  $\Omega$  is first dragged along over  $-v^x dt$  and afterwards displaced parallel over  $+v^x dt$ . This illustrates the fact that  $\frac{D}{s} \Omega dt$  is really not a true differential of the field  $\Omega$  but only a linear transformation of the components of  $\Omega$  at  $\xi^x$

<sup>1)</sup> If by applying a differential operator to the components of a geometric object of a certain kind the components of another geometric object of the same kind arise, we call the operator *invariant with respect to objects of this kind*.

<sup>2)</sup> "Scheinbares Differential" in SCHOUTEN and v. KAMPEN 1933, 1 and E I 1935, 1, p. 142.

depending on the values of  $\Omega$  and of the field  $v_\lambda^\kappa$  at the point  $\xi^\kappa$  only, and not on the values of these fields at neighbouring points.

From (II 10.34) or (III 5.47) we get

$$(1.8) \quad \frac{\delta}{v} \Gamma_{\mu\lambda}^\kappa = V_\mu V_\lambda v^\kappa + 2V_\mu S_\varrho v_\lambda^\kappa v^\varrho + v^\nu R_{\nu\mu\lambda}^\kappa = V_\mu v_\lambda^\kappa + v^\nu R_{\nu\mu\lambda}^\kappa.$$

In (III 5.50) we obtained the first of the following three identities

$$(1.9) \quad \begin{cases} \text{a)} & \left( \frac{\delta}{v} V_\mu - V_\mu \frac{\delta}{v} \right) p^\kappa = \left( \frac{\delta}{v} \Gamma_{\mu\lambda}^\kappa \right) p^\lambda = (V_\mu v_\lambda^\kappa + v^\nu R_{\nu\mu\lambda}^\kappa) p^\lambda \\ \text{b)} & \left( \frac{\delta}{v} V_\mu - V_\mu \frac{\delta}{v} \right) q_\lambda = - \left( \frac{\delta}{v} \Gamma_{\mu\lambda}^\kappa \right) q_\kappa = - (V_\mu v_\lambda^\kappa + v^\nu R_{\nu\mu\lambda}^\kappa) q_\kappa \\ \text{c)} & \left( \frac{\delta}{v} V_\mu - V_\mu \frac{\delta}{v} \right) q = - w \left( \frac{\delta}{v} \Gamma_\mu \right) q = - w (V_\mu v_\lambda^\lambda + v^\nu V_{\nu\mu}) q \end{cases}$$

for the vectors  $p^\kappa$ ,  $q_\lambda$  and the density  $q$  of weight  $w$ , from which the last two can be derived easily from the first. From (III 4.9a, b) we get for these same quantities

$$(1.10) \quad \begin{cases} \text{a)} & (\delta V_\mu - V_\mu \delta) p^\kappa = (v^\nu R_{\nu\mu\lambda}^\kappa p^\lambda - v_\mu^\varrho V_\varrho p^\kappa) dt \\ \text{b)} & (\delta V_\mu - V_\mu \delta) q_\lambda = - (v^\nu R_{\nu\mu\lambda}^\kappa q_\kappa + v_\mu^\varrho V_\varrho q_\lambda) dt \\ \text{c)} & (\delta V_\mu - V_\mu \delta) q = - w (v^\nu V_{\nu\mu} q + v_\mu^\varrho V_\varrho q) dt \end{cases}$$

and from these equations and (1.7) it follows that<sup>1)</sup>

$$(1.11) \quad \begin{cases} \text{a)} & \left( D_s V_\mu - V_\mu D_s \right) p^\kappa = p^\lambda V_\mu v_\lambda^\kappa + v_\mu^\varrho V_\varrho p^\kappa \\ \text{b)} & \left( D_s V_\mu - V_\mu D_s \right) q_\lambda = - q_\kappa V_\mu v_\lambda^\kappa + v_\mu^\varrho V_\varrho q_\lambda \\ \text{c)} & \left( D_s V_\mu - V_\mu D_s \right) q = - w q V_\mu v_\lambda^\lambda + w v_\mu^\varrho V_\varrho q. \end{cases}$$

All these identities are important because they give us an opportunity of changing the order of the operators  $\frac{\delta}{v}$ ,  $\delta$  or  $D_s$  and the operator  $V_\mu$  in any formula.

If the operators  $\delta$ ,  $\frac{\delta}{v} dt$  and  $D_s dt$  are applied to a number of objects, other objects, depending on these will in general suffer variations of a more complicated kind, that could not arise by the mere dragging along or parallel displacement of these latter objects. Let such a variation transform an object field  $\Omega$  (indices suppressed) with fieldvalues  $\frac{\Omega}{v}$  at  $\xi^\kappa$  into a field with fieldvalues  $\frac{\Omega}{v} + \frac{d}{v}\Omega$  at  $\xi^\kappa + v^\kappa dt$ . Then  $\frac{d}{v}$  is not an invariant operator but the following three invariant operators can be defined

<sup>1)</sup> SCHOUTEN and v. KAMPEN 1933, 1, p. 9.

1. The *natural variation*  $\overset{n}{D}\Omega dt$ , that is the difference between the new value at  $\overset{\theta}{\xi} + v^\kappa dt$  and the old value at that same point

$$(1.12) \quad \overset{n}{D} \stackrel{\text{def}}{=} \frac{\overset{v}{d}}{dt} - \frac{d}{dt} = \frac{\overset{v}{d}}{dt} - v^\mu \partial_\mu.$$

*The natural variation does in general not depend on a connexion.*

2. The *absolute variation*  $\overset{a}{D}\Omega dt$ , that is the difference between the new value at  $\overset{\theta}{\xi} + v^\kappa dt$  and the value arising at that point from dragging along from  $\overset{\theta}{\xi}$  over  $v^\kappa dt$

$$(1.13) \quad \overset{a}{D} \stackrel{\text{def}}{=} \frac{\overset{v}{d}}{dt} - \frac{\overset{m}{d}}{dt}.$$

*The absolute variation does in general not depend on a connexion.*

3. The *geodesic variation*  $\overset{g}{D}\Omega dt$ , that is the difference between the new value at  $\overset{\theta}{\xi} + v^\kappa dt$  and the value arising at that point from parallel displacement from  $\overset{\theta}{\xi}$  to  $\overset{\theta}{\xi} + v^\kappa dt$

$$(1.14) \quad \overset{g}{D} \stackrel{\text{def}}{=} \frac{\overset{v}{d}}{dt} - \frac{\overset{*}{d}}{dt}.$$

*Of course these operators do not always exist.* Between them and the operators  $v^\mu \overset{v}{V}_\mu$ ,  $\overset{f}{D}$  and  $\overset{s}{D}$  the following relations hold

$$(1.15) \quad \left\{ \begin{array}{lcl} \overset{n}{D} & = & \overset{a}{D} - \overset{f}{D} = \overset{g}{D} - v^\mu \overset{v}{V}_\mu \\ \overset{a}{D} & = & \overset{n}{D} + \overset{f}{D} = \overset{g}{D} + \overset{s}{D} \\ \overset{g}{D} & = & \overset{n}{D} + v^\mu \overset{v}{V}_\mu = \overset{a}{D} - \overset{s}{D} \end{array} \right.$$

If  $\Phi$  is a field (indices suppressed) in an  $\mathfrak{N}(\overset{\theta}{\xi})$  we also have the following geometrical interpretation. After the variation the field value at  $\overset{\theta}{\xi}$  is  $\overset{\theta}{\Phi} - v^\mu \partial_\mu \overset{\theta}{\Phi} dt + \overset{v}{d}\overset{\theta}{\Phi} = \overset{\theta}{\Phi} + \overset{n}{D}\overset{\theta}{\Phi} dt$ . So the natural variation  $\overset{n}{D}\overset{\theta}{\Phi} dt$  is just the change of the field at  $\overset{\theta}{\xi}$  that we observe looking at the field during the variation. The changes  $\overset{g}{D}\overset{\theta}{\Phi} dt$  and  $\overset{a}{D}\overset{\theta}{\Phi} dt$  appear if we compare the new fieldvalues at  $\overset{\theta}{\xi}$  not with the original fieldvalues at that point but with the values arising at  $\overset{\theta}{\xi}$  after a parallel displacement or a dragging along respectively of the field over  $v^\kappa dt$ .

Accordingly we have for a field *left at rest*:

$$(1.16) \quad \overset{n}{D} = 0; \quad \overset{a}{D} = \overset{f}{D}; \quad \overset{g}{D} = v^\mu \overset{v}{V}_\mu,$$

for a *field dragged along*

$$(1.17) \quad \overset{\text{v}}{D} = -\overset{\text{v}}{L}; \quad \overset{\text{s}}{D} = 0; \quad \overset{\text{s}}{D} = -\overset{\text{s}}{D}$$

and for a field *displaced parallel*

$$(1.18) \quad \overset{\text{v}}{D} = -v^\mu V_\mu; \quad \overset{\text{s}}{D} = +\overset{\text{s}}{D}; \quad \overset{\text{s}}{D} = 0.$$

As a first example of a deformation problem we try to compute the variation of the expression  $u^\mu V_\mu P_{\cdot\lambda}^x$  if 1) the vector field  $u^x$  is left unchanged, 2) the tensor field  $P_{\cdot\lambda}^x$  is displaced parallel over  $v^x dt$  and 3) the object field  $I_{\mu\lambda}^x$  is dragged along over  $v^x dt$ . Then we have

$$(1.19) \quad \begin{cases} \text{a)} & \overset{\text{v}}{D} u^x = 0; \\ \text{b)} & \overset{\text{v}}{D} P_{\cdot\lambda}^x = -v^\mu V_\mu P_{\cdot\lambda}^x; \\ \text{c)} & \overset{\text{v}}{D} I_{\mu\lambda}^x = -\overset{\text{v}}{L} I_{\mu\lambda}^x = -V_\mu v_\lambda^{x\sigma} - v^\rho R_{\nu\mu\lambda}^{x\sigma}, \end{cases}$$

hence after some calculation

$$(1.20) \quad \begin{cases} \overset{\text{v}}{D} u^\mu V_\mu P_{\cdot\lambda}^x = u^\mu V_\mu \overset{\text{v}}{D} P_{\cdot\lambda}^x - u^\mu (\overset{\text{v}}{D} I_{\mu\lambda}^x) P_{\cdot\sigma}^x + u^\mu (\overset{\text{v}}{D} I_{\mu\sigma}^x) P_{\cdot\lambda}^x \\ = -v^\nu V_\nu u^\mu V_\mu P_{\cdot\lambda}^x + (\overset{\text{v}}{L} u^\mu) V_\mu P_{\cdot\lambda}^x + \\ + (u^\mu V_\mu v_\lambda^{\sigma\sigma}) P_{\cdot\sigma}^x - (u^\mu V_\mu v_\sigma^{\cdot x}) P_{\cdot\lambda}^x; \end{cases}$$

and, according to (1.15)

$$(1.21) \quad \overset{\text{g}}{D} u^\mu V_\mu P_{\cdot\lambda}^x = (\overset{\text{v}}{L} u^\mu) V_\mu P_{\cdot\lambda}^x + (u^\mu V_\mu v_\lambda^{\sigma\sigma}) P_{\cdot\sigma}^x - (u^\mu V_\mu v_\sigma^{\cdot x}) P_{\cdot\lambda}^x.$$

$$(1.22) \quad \begin{cases} \overset{\text{a}}{D} u^\mu V_\mu P_{\cdot\lambda}^x = \overset{\text{g}}{D} u^\mu V_\mu P_{\cdot\lambda}^x + \overset{\text{s}}{D} u^\mu V_\mu P_{\cdot\lambda}^x \\ = (\overset{\text{v}}{L} u^\mu) V_\mu P_{\cdot\lambda}^x + u^\mu V_\mu (v_\lambda^{\sigma\sigma} P_{\cdot\sigma}^x - v_\sigma^{\cdot x} F_{\cdot\lambda}^x). \end{cases}$$

Formula (1.22) can also be found by starting from

$$(1.23) \quad \begin{cases} \overset{\text{a}}{D} u^x = \overset{\text{v}}{L} u^x \\ \overset{\text{a}}{D} P_{\cdot\lambda}^x = \overset{\text{s}}{D} P_{\cdot\lambda}^x \\ \overset{\text{a}}{D} I_{\mu\lambda}^x = 0. \end{cases}$$

The formulae (1.15–18) are very convenient for computational purposes because they make it possible to choose always the easiest operators. But it should be remembered that  $v^\mu V_\mu$ ,  $\overset{\text{s}}{D}$  and  $\overset{\text{g}}{D}$  can only be applied if a parallel displacement is given and then only to quantities.

A more important application, connected with SCHLESINGER's product integrals, is the following. Let  $C$  be a closed curve  $\xi^* = f^*(t)$  in  $L_n$  through  $\underset{\theta}{\xi}^*$  and let  $t$  be a parameter on this curve,  $t_0 \leq t \leq t_1$ , such that  $f^*(t_0) = f^*(t_1) = \underset{\theta}{\xi}^*$ . Now let the vectors in the tangent  $E_n$  at  $\underset{\theta}{\xi}^*$  be displaced parallel along  $C$  till they return to  $\underset{\theta}{\xi}^*$ . Then they have suffered a linear homogeneous transformation, depending on the form of  $C$  and on the connexion. This transformation may be written in the form

$$(1.24) \quad u^* = \underset{*}{\left[ \begin{smallmatrix} t_1 & * \\ t_0 & \lambda \end{smallmatrix} \right]} u^\lambda,$$

in which the kernel of the tensor of the transformation is symbolized by  $\underset{t_0}{\left[ \begin{smallmatrix} t_1 \\ t_0 \end{smallmatrix} \right]}^*$ . This transformation can be expressed by means of a product integral

$$(1.25) \quad \underset{t_0}{\left[ \begin{smallmatrix} t_1 & * \\ t_0 & \lambda \end{smallmatrix} \right]} = \int_{t_0}^{t_1} \left( A_\lambda^* - I_{\mu\lambda}^* \frac{d\xi^\mu}{dt} dt \right).$$

A product integral is defined in matrix notation by

$$(1.26) \quad \int_{t_0}^{t_n} (I + P dt) \stackrel{\text{def}}{=} \lim_{\substack{n \rightarrow \infty \\ t_k - t_{k-1} \rightarrow 0}} \prod_{k=1}^{k=n} \{ I + P(t_k) (t_k - t_{k-1}) \}.$$

Product integrals have many properties that can be found in the papers of SCHLESINGER and other authors.<sup>1)</sup> Here we need only the generalization of LEIBNIZ' rule for the variation<sup>2)</sup>:

$$(1.27) \quad d \int_{t_0}^{t_1} (I + P dt) = \int_{t_0}^{t_1} \left\{ \int_{t_0}^t (I + P du) \right\} dP(t) \left\{ \int_t^{t_1} (I + P du) \right\} dt$$

and the differentiation rule

$$(1.28) \quad \frac{d}{dt} \int_{t_0}^t (I + P du) = \left\{ \int_{t_0}^t (I + P du) \right\} P(t)$$

that can be derived easily.

Starting from a vector  $\underset{\theta}{\xi}^*$  at  $\underset{\theta}{\xi}^*$ , at an arbitrary point  $\xi^*$  of  $C$  we get

$$(1.29) \quad u^* = \underset{*}{\left[ \begin{smallmatrix} t & * \\ t_0 & \lambda \end{smallmatrix} \right]} u^\lambda; \quad t_0 \leq t \leq t_1.$$

Here the tensor  $\underset{t_0}{\left[ \begin{smallmatrix} t & * \\ t_0 & \lambda \end{smallmatrix} \right]}^*$  is a connecting quantity lying with the lower index in the tangent  $E_n$  of  $\underset{\theta}{\xi}^*$  and with the upper one in the tangent  $E_n$

<sup>1)</sup> SCHLESINGER 1928, 1; 1931, 1; 2; 1932, 1; cf. VOLTERRA 1887, 1; MORINAGA 1934, 2; E I 1935, 1, p. 134; NIJENHUIS 1952, 1, Ch. II § 13.

<sup>2)</sup> Cf. MICHAL 1945, 1; JOHNSON 1948, 1.

of  $\xi^\mu$ . Because  $u^\mu$  is covariant constant along  $C$  we have along this curve according to (1.28)

$$(1.30) \quad \theta = \frac{\delta}{dt} [t_0]_\lambda^\mu = \frac{d}{dt} [t_0]_\lambda^\mu + \frac{d\xi^\mu}{dt} \Gamma_{\mu\sigma}^\sigma [t_0]_\lambda^\mu$$

and in the same way

$$(1.31) \quad \theta = \frac{\delta}{dt} [t_1]_\lambda^\mu = \frac{d}{dt} [t_1]_\lambda^\mu - \frac{d\xi^\mu}{dt} \Gamma_{\mu\lambda}^\sigma [t_1]_\sigma^\mu.$$

Now let  $C$  be subjected to the variation  $\varepsilon v^\mu$ . Then a point  $\xi^\mu$  of  $C$  is transformed into  $\xi^\mu + \varepsilon v^\mu$  and the variation  $\overset{\nu}{d} [t_0]_\lambda^\mu$  must be computed from the values  $\Gamma_{\mu\lambda}^\mu + \varepsilon v^\nu \partial_\nu \Gamma_{\mu\lambda}^\mu$  at the points of the displaced curve.<sup>1)</sup> Instead we transpose the problem in the way described in V § 9 by dragging back the curve over  $-\varepsilon v^\mu$  to its original position and at the same time dragging along the field  $\Gamma_{\mu\lambda}^\mu$  over  $-\varepsilon v^\mu$ . If now we compute the natural variation of  $[t_0]_\lambda^\mu$  for the transposed problem this variation is the difference between the new value and the original value both at  $\overset{\theta}{\xi}^\mu$ . But this new value arises from the new value of the original (not transposed) problem at  $\overset{\theta}{\xi}^\mu + \varepsilon v^\mu$  by dragging along over  $-\varepsilon v^\mu$ . This means that *the natural variation for the transposed problem equals the absolute variation for the original problem*.

Hence, if we denote the natural variation for the transposed problem by ' $\overset{\theta}{D}$ ', we have ' $\overset{\theta}{D} = \overset{\theta}{D}$ '. Now we have according to (1.25, 27)

$$(1.32) \quad \left\{ \begin{array}{l} [t_0]_\lambda^\mu + \varepsilon' \overset{\theta}{D} [t_0]_\lambda^\mu = \int_{t_0}^{t_1} \left\{ A_\lambda^\mu - (\Gamma_{\mu\lambda}^\mu + \varepsilon \overset{\nu}{\mathcal{L}} \Gamma_{\mu\lambda}^\nu) \frac{d\xi^\mu}{dt} dt \right\} \\ = [t_1]_\lambda^\mu - \varepsilon \int_{t_0}^{t_1} [t_0]_\lambda^\sigma \frac{d\xi^\mu}{dt} (\overset{\nu}{\mathcal{L}} \Gamma_{\mu\sigma}^\nu) [t_1]_\sigma^\mu dt, \end{array} \right.$$

hence, using (1.8)

$$(1.33) \quad \overset{\theta}{D} [t_0]_\lambda^\mu = - \int_{t_0}^{t_1} [t_0]_\lambda^\sigma \frac{d\xi^\mu}{dt} (\nabla_\mu v_\sigma^\mu + v^\nu R_{\nu\mu\sigma}^\mu) [t_1]_\sigma^\mu dt.$$

According to (1.30) we get, integrating by parts

$$(1.34) \quad \int_{t_0}^{t_1} [t_0]_\lambda^\sigma \delta v_\sigma^\mu [t_1]_\mu^\mu = \left( [t_0]_\lambda^\sigma v_\sigma^\mu [t_1]_\mu^\mu \right)_{t_0}^{t_1} = [t_1]_\lambda^\sigma v_\sigma^\mu - v_\lambda^\mu [t_0]_\mu^\mu;$$

hence, returning to the original problem

$$(1.35) \quad \overset{\theta}{D} [t_0]_\lambda^\mu = - \int_{t_0}^{t_1} [t_0]_\lambda^\sigma v^\nu \frac{d\xi^\mu}{dt} R_{\nu\mu\sigma}^\mu [t_1]_\sigma^\mu dt - [t_1]_\lambda^\sigma v_\sigma^\mu + v_\lambda^\mu [t_0]_\mu^\mu.$$

<sup>1)</sup> Cf. JOHNSON 1948, 1, who also considers finite shifts of  $C$ .

The natural variation of  $[t_1]_\lambda^\kappa$  for the original problem does not exist because  $[t_0]_\lambda^\kappa$  is not a field defined in an  $\mathfrak{N}(\xi^\kappa)$  and accordingly has no field value at  $\xi^\kappa + \epsilon v^\kappa$ . But the geodesic variation  $\overset{\text{g}}{D}[t_1]_\lambda^\kappa$  exists because it is the difference between the new value at  $\xi^\kappa + \epsilon v^\kappa$  and the value arising from the parallel displacement of  $[t_0]_\lambda^\kappa$  from  $\xi^\kappa$  to  $\xi^\kappa + \epsilon v^\kappa$ . From (1.15) and (1.35) we get

$$(1.36) \quad \overset{\text{g}}{D}[t_1]_\lambda^\kappa = - \int_{t_0}^{t_1} [t]_{t_0}^\sigma v^\nu \frac{d\xi^\mu}{dt} R_{\nu\mu\sigma}^{\lambda} [t]_q^\kappa dt;$$

and this equation does not contain any derivatives of the field  $v^\kappa$ . The case where  $v^\kappa = 0$  at  $\xi^\kappa$  is very important. Since  $\xi^\kappa$  is then fixed,  $\overset{*}{d}$  is zero at that point and  $\overset{\text{g}}{D}[t_0]^\kappa$  equals the  $\overset{n}{D}[t_0]^\kappa$  that exists in this special case and represents the variation at  $\xi^\kappa$  that really appears. The formula (1.36) will be used for the discussion of the holonomy group of a connexion in VII § 4.

As another interesting application we consider the generalization of BERTRAND curves for  $V_n$ . In  $R_3$  a curve is called a BERTRAND curve if there exists another curve such that the first normals of both curves are the same.<sup>1)</sup> Let a curve in  $R_n$  be given by

$$(1.37) \quad \xi^\kappa = f^\kappa(s)$$

as in V § 1 with  $s$  as parameter of length but with the restriction that  $(\kappa)$  is a rectilinear coordinate system. Then a point on the first normal of a point  $f^\kappa(s)$  has the coordinates

$$(1.38) \quad \varphi^\kappa(s) = f^\kappa(s) + \psi(s) \overset{2}{j}^\kappa(s).$$

We try to choose  $\psi(s)$  in such a way that the curve  $\xi^\kappa = \varphi^\kappa(s)$  has the same first normals as the curve (1.37). According to the FRENET formulae (V 1.10) we have

$$(1.39) \quad \frac{d\varphi^\kappa}{ds} = \frac{df^\kappa}{ds} + \frac{d\psi}{ds} \overset{2}{j}^\kappa + \psi \frac{d\overset{2}{j}^\kappa}{ds} = (1 - k\psi) \overset{2}{j}^\kappa + \frac{d\psi}{ds} \overset{2}{j}^\kappa + k\psi \overset{3}{j}^\kappa$$

and because  $d\varphi^\kappa/ds$  must be perpendicular to  $\overset{2}{j}^\kappa$ , this means that  $\psi$  is constant:  $\psi = c$ . Now  $s$  need not be a length on the second curve, hence, if  $'s$  is the parameter of length on this second curve we have for its tangent unitvector

$$(1.40) \quad \overset{1}{k}^\kappa = \frac{ds}{d's} \frac{d\varphi^\kappa}{ds}; \quad \frac{ds}{d's} \neq 0; \quad \frac{d's}{ds} \neq 0.$$

<sup>1)</sup> Cf. for instance WEATHERBURN 1927, 1, p. 34ff.; HAACK 1948, 1, p. 31f.

From this it follows that the vector

$$(1.44) \quad \frac{d^2 s}{d's^2} \frac{d\varphi^\kappa}{ds} + \left( \frac{ds}{d's} \right)^2 \frac{d^2 \varphi^\kappa}{ds^2}$$

has the direction of  $\dot{\gamma}^\kappa$ . But this is only possible if

$$(1.42) \quad \begin{cases} a) & (1 - \frac{c}{1} \frac{k}{1}) \frac{d^2 s}{d's^2} - \frac{c}{1} \frac{d k}{ds} \left( \frac{ds}{d's} \right)^2 = 0 \\ b) & \frac{c}{1} \frac{k}{2} \frac{d^2 s}{d's^2} + \frac{c}{1} \frac{d k}{ds} \left( \frac{ds}{d's} \right)^2 = 0 \\ c) & \frac{c}{1} \frac{k}{2} \frac{k}{3} \left( \frac{ds}{d's} \right)^2 = 0. \end{cases}$$

There are two trivial solutions:  $c_1 = 0$  and  $k_2 = 0$  (hence  $k_3 = 0$ ). If we exclude these, (1.42) leads to the n.a.s. conditions

$$(1.43) \quad \begin{cases} a) & \frac{c}{1} \frac{k}{1} + \frac{c}{2} \frac{k}{2} = 1; \quad \frac{c}{1} \text{ and } \frac{c}{2} \text{ constants} \\ b) & k_3 = 0 \end{cases}$$

for a curve to be a BERTRAND curve in  $R_n$ . If these conditions are satisfied it is easily proved that

$$(1.44) \quad \frac{d's}{ds} = k_2 \sqrt{\frac{c^2}{1} + \frac{c^2}{2}}$$

and that  $\frac{c_1}{c_2}$  is the tangent of the (constant) angle between the tangents of the curves at corresponding points.  $\frac{c}{1}$  is the (constant) distance between corresponding points and from this we see that to a BERTRAND curve there belongs in general one and only one corresponding curve. But if  $k_1$  and  $k_2$  are constants, there is a second curve for every choice of  $c$  as is well known from the theory of circular helices in  $R_3$ .

Now this problem can be generalized for curves in  $V_n$  by requiring that a curve passes into another curve by an infinitesimal deformation in the direction of its first normal and such that the first normals of both curves are parallel at corresponding points.<sup>1)</sup> The curve may be given by (1.37) but  $(\kappa)$  is no longer a rectilinear system. Instead of (1.38) we get

$$(1.45) \quad \varphi^\kappa(s) = f^\kappa(s) + \varepsilon \dot{\gamma}_2^\kappa(s)$$

and it can be proved as before that  $\varepsilon$  is an infinitesimal constant.<sup>2)</sup> But from here we must proceed in quite another way because the covariant derivatives are no longer ordinary derivatives and the  $\Gamma_{\mu\lambda}^\kappa$

<sup>1)</sup> PEARS 1935, 1; YANO, TAKANO and TOMONAGA 1948, 2. The latter paper contains many theorems on deformation of curves.

<sup>2)</sup> It is convenient to use FERMI coordinates, cf. III § 8.

do not have the same values at corresponding points of the two curves. Again we transpose the problem, dragging back the second curve and the field  $g_{\lambda\kappa}$  over  $-\varepsilon j_2^\kappa$ . We note that although the curve dragged back coincides with the original curve,  $s$  need no longer be a length parameter on it.

For the fields  $g_{\lambda\kappa}$  and  $I_{\mu\lambda}^\kappa$  dragged back over  $-\varepsilon v^\kappa$  we have according to (1.17)

$$(1.46) \quad 'D g_{\mu\lambda} = \frac{d}{ds} g_{\mu\lambda} = 2 V_{(\mu} v_{\lambda)}$$

$$(1.47) \quad 'D I_{\mu\lambda}^\kappa = \frac{d}{ds} I_{\mu\lambda}^\kappa = V_\mu V_\lambda v^\kappa + v^\nu K_{\nu\mu\lambda}^\kappa$$

but, because  $j_2^\kappa$  is defined over  $C$  only, we must put  $v^\kappa = j_2^\kappa$  on this curve and prolong this field over a neighbourhood of  $C$  in order that the derivatives in (1.46, 47) get a meaning. Of course all results will be independent of the manner of prolongation.

From (1.46) it follows that

$$(1.48) \quad 'D (ds)^2 = 2 d\xi^\mu d\xi^\lambda V_\mu v_\lambda = 2 (ds)^2 j^\mu j^\lambda V_\mu j_\lambda = -2 k_1 (ds)^2$$

hence

$$(1.49) \quad \begin{cases} \text{a)} \quad \frac{'D ds}{ds} = -k_1; \\ \text{b)} \quad 'D j^\kappa = D \frac{d\xi^\kappa}{ds} = -\frac{d\xi^\kappa}{(ds)^2} D ds = \frac{d\xi^\kappa}{(ds)^2} k_1 ds = k_1 j^\kappa. \end{cases}$$

For the first curvature vector we get

$$(1.50) \quad \begin{cases} 'D j^\mu V_\mu j^\kappa = ('D j^\mu) V_\mu j^\kappa + j^\mu V_\mu 'D j^\kappa + j^\mu j^\lambda 'D I_{\mu\lambda}^\kappa \\ \quad = k_1 j^\mu V_\mu j^\kappa + j^\mu (\nabla_\mu k_1) j^\kappa + j^\mu k_1 V_\mu j^\kappa + \\ \quad \quad \quad + j^\mu j^\lambda V_\mu V_\lambda v^\kappa + j^\nu j^\mu j^\lambda K_{\nu\mu\lambda}^\kappa \\ \quad = 2 k_1 j^\mu V_\mu j^\kappa + j^\kappa \frac{d}{ds} k_1 + \frac{\delta^2}{ds^2} j_2^\kappa - k_1 j^\mu V_\mu v^\kappa + j^\nu j^\mu j^\lambda K_{\nu\mu\lambda}^\kappa; \end{cases}$$

hence, returning to the original untransposed problem

$$(1.51) \quad (\overset{a}{D} k_2) j_2^\kappa + k_1 \overset{a}{D} j_2^\kappa = 2 k_1^2 j_2^\kappa + j_2^\kappa \frac{d}{ds} k_1 + \frac{\delta^2}{ds^2} j_2^\kappa - k_1 j^\mu V_\mu v^\kappa + j^\nu j^\mu j^\lambda K_{\nu\mu\lambda}^\kappa;$$

or, according to (1.15) (cf. V 1.10)

$$(1.52) \quad \begin{cases} k_1 \overset{g}{D} j_2^\kappa = (2 k_1^2 - \overset{a}{D} k_1) j_2^\kappa + j_2^\kappa \frac{d}{ds} k_1 + \frac{\delta^2}{ds^2} j_2^\kappa + j_2^\nu j^\mu j^\lambda K_{\nu\mu\lambda}^\kappa \\ \quad = (k_1^2 - \overset{a}{D} k_1 - k_2^2) j_2^\kappa + j_3^\kappa \frac{d}{ds} k_2 + k_2 k_3 j_4^\kappa + j_2^\nu j^\mu j^\lambda K_{\nu\mu\lambda}^\kappa. \end{cases}$$

Now  $\overset{g}{D}_2 j^{\infty} = 0$  according to our suppositions, hence, taking an anholonomic coordinate system ( $h$ ) whose unit basis vectors coincide with the  $j_2^{\infty}, j_3^{\infty}, \dots, j_n^{\infty}$  at all points of  $C$  we have

$$(1.53) \quad \left\{ \begin{array}{l} \text{a) } \overset{a}{D}_1 k_1 = k_1^2 - k_2^2 + K_{2112} \\ \text{b) } \frac{d}{ds} k_2 = -K_{2113} \\ \text{c) } k_2 k_3 = -K_{2114} \\ \text{d) } 0 = K_{211p}; \quad p = 5, \dots, n. \end{array} \right.$$

This is in accordance with the results (1.42) derived for  $R_n$  ( $\epsilon$  infinitesimal). The problem of two curves in a general  $V_n$  with common normal geodesics having on both curves the direction of the first normal can not be tackled in this way. In an  $R_n$  the components of  $K_{\nu\mu\lambda\nu}$  occurring in (1.53 b-d) are zero, hence either  $k_3 = 0$  and  $k_2 = \text{const.}$  or  $k_2 = k_3 = 0$ .

The direction of  $j^{\infty}$  can be compared with the tangent direction at the corresponding point of the other curve if this tangent is displaced back parallel. This displaced tangent has the direction of the unit-vector  $j^{\infty} + \epsilon \overset{g}{D}_2 j^{\infty}$ . This gives for the angle  $d\alpha$  between both directions according to (1.15) and (1.49b)

$$(1.54) \quad \left\{ \begin{array}{l} (d\alpha)^2 = \epsilon^2 (\overset{g}{D}_2 j^{\infty}) (\overset{g}{D}_2 j^{\lambda}) g_{\lambda\nu} \\ \quad = \epsilon^2 (\overset{a}{D}_1 k_2 + j^{\mu} V_{\mu} j^{\infty}) (\overset{a}{D}_1 k^{\lambda} + j^{\mu} V_{\mu} j^{\lambda}) g_{\lambda\nu} = \epsilon^2 \frac{k^2}{2} j_3^{\infty} j^{\lambda} g_{\lambda\nu} = \epsilon^2 k^2. \end{array} \right.$$

Hence the angle  $d\alpha$  is constant if  $k$  is constant. This is always the case in an  $R_n$ .

### Exercises.

VII 1,1. Prove that the rule of LEIBNIZ holds for the operators  $O_1 O_2 - O_2 O_1$  if it holds for both operators  $O_1$  and  $O_2$  and if in the formulae considered all operations can be effected.

VII 1,2. If the variations of a quantity  $\Phi$  (indices suppressed) and of the connexion  $I_{\mu\lambda}^{\infty}$  are given, the quantity  $(\overset{n}{D} V_{\mu} - V_{\mu} \overset{n}{D}) \Phi$  is known. Prove that this quantity does not change if the variation of  $\Phi$  is changed, as long as the variation of  $I_{\mu\lambda}^{\infty}$  remains unchanged. Prove the same for the operators  $\overset{g}{D} V_{\mu} - V_{\mu} \overset{g}{D}$  and  $\overset{a}{D} V_{\mu} - V_{\mu} \overset{a}{D}$ .<sup>1)</sup>

<sup>1)</sup> SCHOUTEN and v. KAMPEN 1933, 1, p. 8ff.

VII 1,3. Prove that

$$\text{VII 1,3 } \alpha) \quad \overset{\circ}{D} V_\mu - V_\mu \overset{\circ}{D} = 0$$

if the connexion is not changed.<sup>1)</sup>

## § 2. Groups of “motions” in $V_n$ and $L_n$ .<sup>2)</sup>

A deformation in  $V_n$  is called a *motion* if it leaves the metric invariant. Hence an infinitesimal deformation  $\xi^\alpha \rightarrow \xi^\alpha + v^\alpha dt$  is a motion if and only if  $\frac{\delta}{v} g_{\lambda\alpha} = 0$ . This idea can be generalized. A deformation in  $U_n$  is a motion if  $\frac{\delta}{v} g_{\lambda\alpha} = 0$ ;  $\frac{\delta}{v} S_{\mu\lambda}^{\alpha} = 0$ . In  $L_n$  we may call a deformation an *affine motion* if the connexion is left invariant. Hence an infinitesimal affine motion in  $L_n$  is characterized by  $\frac{\delta}{v} \Gamma_{\mu\lambda}^\alpha = 0$ . Further generalizations are the *conformal motions* in  $V_n$  characterized by  $\frac{\delta}{v} g_{\lambda\alpha} = \alpha g_{\lambda\alpha}$ , the *homothetic motion* in  $V_n$ , being a conformal motion for which  $\alpha$  is a constant and the *projective motion* in  $A_n$  characterized by  $\frac{\delta}{v} \Gamma_{\mu\lambda}^\alpha = \varphi_\mu A_\lambda^\alpha + \varphi_\lambda A_\mu^\alpha$ .

The conformal and projective motions are also characterized by  $\frac{\delta}{v} \overset{c}{I}_{\mu\lambda}^\alpha = 0$  and  $\frac{\delta}{v} \overset{p}{I}_{\mu\lambda}^\alpha = 0$  where  $\overset{c}{I}_{\mu\lambda}^\alpha$  and  $\overset{p}{I}_{\mu\lambda}^\alpha$  are the conformal and projective parameters respectively of J. M. THOMAS (cf. VI § 7) and of T. Y. THOMAS (cf. VI § 4). Hence every motion is characterized by the *absolute* invariance (cf. II § 10) of one or more fields.<sup>3)</sup>

The differential equations of the affine motions in  $L_n$  are according to (1.8)

$$(2.1) \quad \begin{cases} \text{a)} & V_\mu v_\lambda^{\alpha} = -v^\nu R_{\nu\mu\lambda}^\alpha \\ \text{b)} & V_\lambda v^\alpha = v_\lambda^{\alpha} + 2S_{\mu\lambda}^{\alpha} v^\mu \end{cases}$$

with the unknowns  $v^\alpha$  and  $v_\lambda^{\alpha}$ . The first integrability conditions (cf. II § 6, III § 6) can be derived by differentiation, alternation and using

<sup>1)</sup> SCHOUTEN and v. KAMPEN 1933 1. p. 8 ff.

<sup>2)</sup> SLEBODZINSKI 1932, 1. Cf. EISENHART 1935, 1; 1936, 1 (simply trans. gr. in  $V_n$ ); LEVINE 1937, 1 and 1939, 1 (in  $C_n$ ); 1950, 2 (in  $A_2$  and project.); 1951, 1 (in  $W_2$ ); 1951, 2 (in  $A_2$ ); KNEBELMAN 1945, 1 (in  $A_n$  and  $V_n$ ); YANO 1946, 5; YANO and TAKANO 1946, 6 and YANO 1946, 7; 1947, 3 (in  $L_n$ ); 1949, 1; 1951, 2 (homoth. in  $V_n$ ); YANO and TOMONAGA 1946, 8; 9 (in  $L_n$ ); EGOROV 1947, 1 (in  $A_n$ ); 1948, 1 (project.); 1949, 1 (in  $V_n$ ); 1949, 2 (in  $L_n$ ); 1950, 1 (in  $L_n$ ); SU 1949, 1 (conform); SHANKS 1950, 1 (homothet. in  $V_n$ ); YANO and IMAI 1950, 2; MUTO 1951, 1 (simply trans. gr. in  $V_n$ ).

<sup>3)</sup> Many results on these different “motions” are to be found in YANO’s booklet 1949, 1. Cf. also KOSAMBI 1951, 1; 1952, 1.

the second identity and the identity of BIANCHI. This gives

$$(2.2) \quad \left\{ \begin{array}{l} \text{a)} \quad \frac{\mathfrak{L}}{v} R_{\nu\mu\lambda}^{\cdot\cdot\cdot\cdot} = v^\omega V_\omega R_{\nu\mu\lambda}^{\cdot\cdot\cdot\cdot} + v_\nu^\sigma R_{\sigma\mu\lambda}^{\cdot\cdot\cdot\cdot} + v_\mu^\sigma R_{\nu\sigma\lambda}^{\cdot\cdot\cdot\cdot} + \\ \qquad \qquad \qquad + v_\lambda^\sigma R_{\nu\mu\sigma}^{\cdot\cdot\cdot\cdot} - v_\nu^{\sigma\mu} R_{\nu\mu\lambda}^{\cdot\cdot\cdot\cdot} = 0 \\ \text{b)} \quad \frac{\mathfrak{L}}{v} S_{\mu\lambda}^{\cdot\cdot\cdot\cdot} = v^\nu V_\nu S_{\mu\lambda}^{\cdot\cdot\cdot\cdot} + v_\mu^\sigma S_{\sigma\lambda}^{\cdot\cdot\cdot\cdot} + v_\lambda^\sigma S_{\mu\sigma}^{\cdot\cdot\cdot\cdot} - v_\nu^{\sigma\mu} S_{\mu\lambda}^{\cdot\cdot\cdot\cdot} = 0. \end{array} \right.$$

The second and following integrability conditions are found by covariant differentiation of (2.2). As  $V_\mu$  and  $\frac{\mathfrak{L}}{v}$  can be interchanged on account of (1.9) and  $\frac{\mathfrak{L}}{v} \Gamma_{\mu\lambda}^{\cdot\cdot} = 0$ , we get

$$(2.3) \quad \left\{ \begin{array}{l} \text{a)} \quad \frac{\mathfrak{L}}{v} V_{\omega_k \dots \omega_1} R_{\nu\mu\lambda}^{\cdot\cdot\cdot\cdot} = 0; \\ \text{b)} \quad \frac{\mathfrak{L}}{v} V_{\omega_k \dots \omega_1} S_{\mu\lambda}^{\cdot\cdot\cdot\cdot} = 0; \quad k = 0, 1, 2, 3, \dots \end{array} \right.$$

These equations express that all covariant derivatives of the fields  $R_{\nu\mu\lambda}^{\cdot\cdot\cdot\cdot}$  and  $S_{\mu\lambda}^{\cdot\cdot\cdot\cdot}$  are absolutely invariant for the field  $v^\nu$  (cf. II § 10). If the set (2.3) for  $k=m+1$  depends algebraically on the sets for  $k=0, \dots, m$ , the process stops because all other sets depend for  $k > m$  on the foregoing ones (cf. II § 6). The equations (2.1) have solutions if and only if at some point  $\xi^\nu$  there exist values  $v_0^\nu, v_0^\lambda$  satisfying the algebraic equations (2.3). Then these values can be taken as the initial values at  $\xi^\nu$  and there is one and only one solution of (2.1) taking these values at this point. As the number of independent sets of values  $v_0^\nu, v_0^\lambda$  satisfying (2.3) is finite, the  $v^\nu$  generate a set of one-parameter groups of transformations, depending on a finite number of parameters. Let  $u^\nu$  and  $v^\nu$  be two solutions:

$$(2.4) \quad \frac{\mathfrak{L}}{u} \Gamma_{\mu\lambda}^{\cdot\cdot} = 0; \quad \frac{\mathfrak{L}}{v} \Gamma_{\mu\lambda}^{\cdot\cdot} = 0$$

then we have (cf. Exerc. II 10,9)

$$(2.5) \quad 0 = \left( \frac{\mathfrak{L}}{u} \frac{\mathfrak{L}}{v} - \frac{\mathfrak{L}}{v} \frac{\mathfrak{L}}{u} \right) \Gamma_{\mu\lambda}^{\cdot\cdot} = \frac{\mathfrak{L}}{w} \Gamma_{\mu\lambda}^{\cdot\cdot}; \quad w^\nu \stackrel{\text{def}}{=} \frac{\mathfrak{L}}{u} v^\nu;$$

and this proves that  $\frac{\mathfrak{L}}{u} v^\nu$  is also a solution. This shows that the one-parameter groups together form a group.

The other cases mentioned above can be treated in the same way. The equations of motion can always be found by equating the LIE derivative of the fundamental geometric object or objects to zero. The integrability conditions of these equations only express that the LIE derivatives of all differential concomitants vanish.<sup>1)</sup>

<sup>1)</sup> KOSAMBI 1951, 1 has hinted at the possibility of a theorem of this kind, valid for every geometry whose fundamental geometric objects possess a LIE derivative. But a proof of such a general theorem has not yet been given. We only know that it is found to be true in all cases so far investigated. The same holds for the supposition that  $\frac{\mathfrak{L}}{v}$  commutes with all invariant differential operators if and only if  $v^\nu$  defines a motion.

The maximal number of parameters of a group of motions in a  $V_n$  that is not an  $S_n$  was proved to be  $\frac{1}{2}n(n+1)-1$  for  $n > 2$  by FUBINI.<sup>1)</sup> EGOROV proved<sup>2)</sup> that for a  $V_n$  that is not an EINSTEIN  $V_n$  the maximum is  $\frac{1}{2}n(n-1)+1$  and that it is  $\frac{1}{2}n(n-1)+2$  for a  $V_n$  that is not an  $S_n$ . VRANCEANU reduced this latter number in 1951<sup>3)</sup> to  $\frac{1}{2}(n-1)(n-2)+5$ . In 1947 EGOROV proved<sup>4)</sup> that for an  $L_n$  that is not an  $A_n$  the maximum is  $n^2$  and that an  $A_n$  with an  $n^2$ -parameter group of motions is necessarily projective euclidean (cf. VI §1). The maximum has been established by EGOROV and VRANCEANU for a great number of other cases.<sup>5)</sup> YANO proved 1953<sup>6)</sup> that the number of parameters for  $n > 4$ ,  $n \neq 8$  is  $\binom{n}{2} + 1$  if and only if the  $V_n$  is either the product space of an  $R_1$  and an  $S_{n-1}$  or an  $S_n$  with negative curvature.

For the motions in a  $V_n$  we have because of  $\mathcal{L}_v g_{\lambda\kappa} = 0$ ;  $\mathcal{L}_v g^{\kappa\lambda} = 0$  and the commutativity of  $\mathcal{L}_v$  and  $\partial_\mu$

$$(2.6) \quad \begin{cases} a) \quad \mathcal{L}_v \{_{\mu\lambda}^{\kappa}\} = \frac{1}{2} g^{\kappa\nu} (\partial_\mu \mathcal{L}_v g_{\lambda\nu} + \partial_\lambda \mathcal{L}_v g_{\mu\nu} - \partial_\nu \mathcal{L}_v g_{\mu\lambda}) = 0; \\ \quad \quad \quad \text{(also valid with } V \text{ instead of } \partial) \\ b) \quad V_v \mathcal{L}_v g_{\mu\lambda} = (\mathcal{L}_v \{_{\nu\mu}^{\sigma}\}) g_{\sigma\lambda} - (\mathcal{L}_v \{_{\nu\lambda}^{\sigma}\}) g_{\mu\sigma} = 0. \end{cases}$$

From (2.6) we see that the group of motions in  $V_n$  is a subgroup of the group of affine motions and that the former group is determined by those solutions of (2.1) with  $R_{\nu\mu\lambda}{}^\kappa = K_{\nu\mu\lambda}{}^\kappa$ ,  $S_{\mu\lambda}{}^\kappa = 0$  which also satisfy the equations

$$(2.7) \quad \mathcal{L}_v g_{\mu\lambda} = 2V_{(\mu} v_{\lambda)} = 2v_{(\mu\lambda)} = 0.$$

Now from (2.1a) it follows that  $V_\mu v_{(\lambda\kappa)} = 0$  because  $K_{\nu\mu\lambda\kappa}$  is alternating in  $\lambda\kappa$ , hence, if  $v_{(\lambda\kappa)}$  is zero at one point of  $V_n$ , this quantity vanishes at all points. Therefore an affine motion in  $V_n$  is an ordinary motion in that space if and only if  $v_{(\lambda\kappa)} = 0$  at one point. Then  $v_{(\lambda\kappa)}$  is also zero (cf. Exerc. V 5, 6, 7) at all other points. The equation  $v_{(\lambda\kappa)} = 0$  is known as KILLING's equation.<sup>7)</sup>

<sup>1)</sup> FUBINI 1903, 1; 1904, 1.

<sup>2)</sup> EGOROV 1949, 1.

<sup>3)</sup> VRANCEANU 1951, 2; 1953, 1.

<sup>4)</sup> EGOROV 1947, 1.

<sup>5)</sup> EGOROV 1948, 1; 1949, 1; 2; 1950, 1; 1951, 1; 1952, 1; 2; VRANCEANU 1947, 1, 1949, 1; 2; 1951, 1; 2; 3; this last publication gives a good summary of results in french.

<sup>6)</sup> YANO 1953, 1.

<sup>7)</sup> KILLING 1892, 1; cf. BIANCHI 1918, 1; EISENHART 1926, 1 and for the more general case where  $S_{\mu\lambda\kappa} = S_{[\mu\lambda\kappa]} \neq 0$  HAYDEN 1934, 2; E II 1938, 2, p. 163.

We prove here only a few theorems on motions in  $V_n$ .<sup>1)</sup> From  $\frac{d}{v} g_{\lambda\kappa} = 0$  and  $\frac{d}{v} v^\kappa = 0$  it follows:

*Along the streamlines of a one-parameter group of motions in  $V_n$  the length of the displacement vector is constant.*

A *translation* is a motion whose streamlines are geodesics. Because of

$$(2.8) \quad V_\mu v_\lambda v^\lambda = 2v^\lambda V_\mu v_\lambda = -2v^\lambda V_\lambda v_\mu$$

we have:

*A motion in  $V_n$  is a translation if and only if the length of the displacement vector is constant.<sup>2)</sup>*

We consider two  $V_n$ 's that can be mapped on each other so that geodesics correspond to geodesics. Then we wish to prove that the groups of motions in both  $V_n$ 's have the same number of parameters. This is not trivial because the notion of a motion in  $V_n$  is based on the metric and the mapping is projective and will in general disturb the metric. Let corresponding points have the same coordinates  $\xi^\kappa$  and let  $g_{\lambda\kappa}, \{{}^\kappa_{\mu\lambda}\}, V_\mu$  belong to one  $V_n$  and ' $g_{\lambda\kappa}$ ', ' $\{{}^\kappa_{\mu\lambda}\}$ ', ' $V_\mu$ ' to the other  $V_n$ , called ' $V_n$ ' hereafter. Because the mapping is projective we have (cf. III § 7)

$$(2.9) \quad \{{}^\kappa_{\mu\lambda}\} = \{{}^\kappa_{\mu\lambda}\} + 2p_{(\mu} A^\kappa_{\lambda)}$$

and (cf. III 3.19)

$$(2.10) \quad \partial_\mu \log 'g = \partial_\mu \log g + 2(n+1)p_\mu$$

which proves that  $p_\lambda$  is a gradientvector

$$(2.11) \quad p_\lambda = \frac{1}{2} \partial_\lambda \log \varphi; \quad \varphi \stackrel{\text{def}}{=} \left( \frac{'g}{g} \right)^{\frac{1}{n+1}}.$$

Now let  $d\xi^\kappa = v^\kappa dt$  be an infinitesimal motion in  $V_n$ . Then we consider in ' $V_n$ ' the transformation  $d\xi^\kappa = 'v^\kappa dt$  with

$$(2.12) \quad 'v_\lambda \stackrel{\text{def}}{=} \varphi v_\lambda; \quad 'v^\kappa = 'g^{\kappa\lambda} 'v_\lambda = \varphi 'g^{\kappa\mu} g_{\mu\lambda} v^\lambda.$$

For this transformation we get

$$(2.13) \quad \left\{ \begin{array}{l} \frac{d}{v} 'g_{\lambda\kappa} = 2'V_{(\lambda} 'v_{\kappa)} = 2v_{(\lambda} \partial_{\lambda)} \varphi + 2\varphi 'V_{(\lambda} v_{\kappa)} \\ \quad = 2v_{(\lambda} \partial_{\lambda)} \varphi + 2\varphi V_{(\lambda} v_{\kappa)} - 4\varphi A^\mu_{(\lambda} p_{\mu)} v_\mu \\ \quad = 2\varphi V_{(\lambda} v_{\kappa)} = \varphi \frac{d}{v} g_{\lambda\kappa} = 0; \end{array} \right.$$

<sup>1)</sup> Cf. for more facts and more literature e.g. EISENHART 1926, 1 or 1949, 1, Ch. VI; 1933, 1, p. 208ff.; YANO 1949, 1, Ch. IV.

<sup>2)</sup> EISENHART 1933, 1, p. 212.

and this proves:

*If two  $V_n$ 's can be mapped on each other projectively, their groups of motions have the same number of parameters.<sup>1)</sup>*

Of course the groups need not be isomorphic because for two motions  $u^\alpha$  and  $v^\alpha$  in  $V_n$  we have in general

$$(2.14) \quad \mathcal{L}_v u^\alpha \neq \varphi' g^{\alpha\mu} g_{\mu\lambda} \mathcal{L}_v u^\lambda.$$

It is well known that the group of all motions in an  $S_n$  has exactly  $\frac{1}{2}n(n+1)$  parameters. If we know that the group of all motions in a  $V_n$  has exactly  $\frac{1}{2}n(n+1)$  parameters,  $v^\alpha$  and  $v_{[\lambda\alpha]}$  can be chosen arbitrarily at one point and the fields  $v^\alpha$ ,  $v_\lambda^\alpha$  can then be computed at all points. Hence the equations

$$(2.15) \quad \mathcal{L}_v V_{\omega_k \dots \omega_1} K_{\nu\mu\lambda\alpha} = 0; \quad k = 0, 1, 2, 3, \dots$$

must be satisfied for all values of  $v_{\lambda\alpha}$  satisfying  $v_{[\lambda\alpha]} = 0$  and they should not contain  $v^\alpha$ . This implies that in (2.2a) the first term in the right hand side must vanish

$$(2.16) \quad V_\omega K_{\nu\mu\lambda\alpha} = 0$$

and that accordingly this equation can be written as

$$(2.17) \quad v^{\tau\sigma} (g_{\nu[\tau} K_{\sigma]\mu\lambda\alpha} + g_{\mu[\tau} K_{|\nu|\sigma]\lambda\alpha} + g_{\lambda[\tau} K_{|\nu|\mu|\sigma]\alpha} + g_{\alpha[\tau} K_{|\nu|\mu|\lambda|\sigma]}) = 0$$

valid for all values of  $v^{[\tau\sigma]}$ . But this is only possible if the expression between brackets is zero, hence, transvecting with  $g^{\tau\alpha}$

$$(2.18) \quad (n-1) K_{\nu\mu\lambda\alpha} + 2g_{[\nu|\lambda|} K_{\mu]\alpha} = 0$$

from which it follows for  $n > 2$  immediately, after transvecting with  $g^{\nu\alpha}$ , that the  $V_n$  is an  $S_n$ . For  $n=2$  the constancy of  $K$  does not follow from (2.18) but in this case it follows from (2.16). Hence<sup>2)</sup>

*The group of all motions of a  $V_n$  has  $\frac{1}{2}n(n+1)$  parameters if and only if the  $V_n$  is an  $S_n$ .*

### Exercises.

VII 2,1<sup>3)</sup>. An  $L_n$  with an integrable non-symmetric connexion is group space of a transitive  $n$ -parameter group if and only if  $\nabla_\nu S_{\mu\lambda}^\alpha = 0$ .

VII 2,2<sup>3)</sup>. An  $A_n$  has the (0)-connexion of an  $n$ -parameter group space if and only if there is a tensor field  $S_{\mu\lambda}^\alpha$  alternating in  $\mu\lambda$  such that

$$\text{VII 2,2 } \alpha) \quad R_{\nu\mu\lambda}^\alpha = S_{\nu\mu}^\alpha S_{\sigma\lambda}^\sigma; \quad \nabla_\nu S_{\mu\lambda}^\alpha = 0.$$

<sup>1)</sup> KNEBELMAN 1930, 1; YANO 1949, 1.

<sup>2)</sup> FUBINI 1903, 1; 1904, 1; EISENHART 1933, 1, p. 215; YANO 1949, 1, p. 36.

<sup>3)</sup> EISENHART 1933, 1, p. 197; YANO 1949, 1, p. 21.

VII 2,3<sup>1)</sup>. Let  $\xi^* \rightarrow \xi^* + v^* dt$  be a motion in an  $L_n$  with an integrable connexion and let  $e^*_i$  be  $n$  covariant constant vectorfields. Prove that the vectors  $\sum_v e^*_v$  are linear combinations of the  $e^*_i$  with *constant* coefficients.

VII 2,4<sup>2)</sup>. The fields  $v^*$  and  $\varrho v^*$  determine both a motion in  $A_n$  if and only if

$$\text{VII 2,4 } \alpha) \quad v^* V_\mu V_\lambda \varrho + 2(V_{(\lambda} \varrho)V_{\mu)} v^* = 0.$$

VII 2,5<sup>3)</sup>. An  $L_n$  admits a one-parameter group of motions if and only if there exists a coordinate system such that either the  $I_{\mu\lambda}^*$  are independent of  $\xi^1$  or that the  $I_{\mu\lambda}^*$  are homogeneous of degree  $-1$  with respect to the coordinates.

VII 2,6 [cf. 1.8 or III 5.47].<sup>4)</sup> The field  $v_\lambda^*$  in  $L_n$  is covariant constant along every streamline of  $v^*$ . Streamlines, along which  $v_\lambda^* = 0$ , are geodesics but this condition is only sufficient and not necessary. A necessary and sufficient condition is that  $v^*$  lies at all points of the curve in an eigendirection of  $v_\lambda^*$ .

VII 2,7. The motion in  $V_n$  defined by the field  $v^*$  is a translation if and only if the angle between  $v^*$  and any geodesic is constant along this geodesic.<sup>5)</sup>

VII 2,8 (cf. IV § 8). The translations in the group space of a semi-simple group are the transformations of the first and second parameter-group.<sup>6)</sup>

VII 2,9. Let  $v^*$  be a displacement vector of a motion in an  $S_3$ . The equations of motion can be written in the form (cf. I 9.20)

$$\text{VII 2,9 } \alpha) \quad V_\mu v_\lambda = 2I_{\mu\lambda\nu} u^\nu; \quad V_\mu u_\lambda = 2\nu I_{\mu\lambda\nu} v^\nu; \quad u^\nu \stackrel{\text{def}}{=} I^{\nu\mu\lambda} V_\mu v_\lambda.$$

Prove that this system is totally integrable and that  $u^\nu$  is the displacement vector of another motion.<sup>7)</sup>

<sup>1)</sup> ROBERTSON 1932, 1; YANO 1949, 1, p. 15.

<sup>2)</sup> YANO 1949, 1, p. 17.

<sup>3)</sup> EISENHART 1933, 1, p. 232; YANO 1949, 1, p. 18. The booklet of YANO contains in Ch. III a great number of interesting propositions on affine motions in  $L_n$  and  $A_n$ . Note that  $A_n$  is used there in the sense of  $L_n$  and that an  $A_n$  is called an  $A_n$  with a symmetric linear connexion.

<sup>4)</sup> A. NIJENHUIS, personal communication.

<sup>5)</sup> EISENHART 1933, 1, p. 212; YANO 1949, 1, p. 32. As NIJENHUIS pointed out in 1952, 1, theses No. 4, the proof at these places of the lemma that the trajectories of two translations meet always under a constant angle is not sufficient and it seems probable that the lemma is not true.

<sup>6)</sup> CARTAN and SCHOUTEN 1926, 3, p. 811; EISENHART 1933, 1, p. 213; YANO 1949, 1, p. 33.

<sup>7)</sup> TONOLO 1950, 1.

### § 3. Deformation of subspaces.<sup>1)</sup>

The deformation of a subspace can often be dealt with, using the method of transposition already explained in V § 9 and VII § 1. Here we give a detailed treatment of a  $V_m$  in  $V_n$  subjected to the infinitesimal transformation  $\xi^x \rightarrow \xi^x + v^x dt$ . Let a holonomic coordinate system ( $a$ );  $a = 1, \dots, m$  in  $V_m$  be completed by basis vectors  $e_y^x$ ;  $y = m+1, \dots, n$  perpendicular to  $V_m$  so that there results an anholonomic system ( $h$ );  $h = 1, \dots, n$ , defined over  $V_m$ . We suppose that the coordinate system ( $a$ ) in  $V_m$  with its basis vectors  $e_b^a$ , and also the basis vectors  $\tilde{e}_\lambda$ , are dragged along over  $v^x dt$ . Further we suppose that the  $e_y^x$  and  $\tilde{e}_\lambda$  remain perpendicular to the  $e_b^a$  and  $\tilde{e}_\lambda$  respectively and that the equations  $\tilde{e}_\lambda e^\lambda = \delta_y^x$ ;  $\tilde{e}_\lambda e^\lambda = \delta_b^a$  remain valid. This implies that in general the  $e_y^x$  and  $\tilde{e}_\lambda$  are not dragged along, because the deformation may disturb orthogonality. In order to transpose the problem, the  $V_m$  and all basis vectors  $e_i^x$ ,  $\tilde{e}_\lambda$  are dragged back over  $-v^x dt$  and at the same time the field  $g_{\lambda x}$  is dragged along over  $-v^x dt$ . Then the  $e_b^a$  and  $\tilde{e}_\lambda$  return to their original positions but the fields  $g_{\lambda x}$ ,  $e_y^x$  and  $\tilde{e}_\lambda$  over  $V_m$  will have been changed. As we have seen before, the absolute differentials  $\overset{\alpha}{D} dt$  of all quantities connected with  $V_m$  in the original problem equal the natural differentials  $\overset{\alpha}{D} dt$  in the transposed problem. Hence

$$(3.1) \quad \overset{\alpha}{D} g_{\lambda x} = \overset{\alpha}{D} g_{\lambda x} = \overset{\alpha}{D} g_{\lambda x} = 2 V_{(\lambda} v_{\mu)} = 2 v_{(\lambda x)}; \quad v_{\lambda}^x = V_\lambda v^x$$

$$(3.2) \quad \overset{\alpha}{D} B_b^x = \overset{\alpha}{D} B_b^x = 0$$

$$(3.3) \quad \overset{\alpha}{D} C_\lambda^x = \overset{\alpha}{D} C_\lambda^x = 0$$

$$(3.4) \quad \left\{ \begin{array}{l} \overset{\alpha}{D} g_{cb} = \overset{\alpha}{D} g_{cb} = B_{cb}^{\mu\lambda} \overset{\alpha}{D} g_{\mu\lambda} = 2 B_{cb}^{\mu\lambda} v_{(\mu\lambda)} = 2 B_{(b}^{\lambda} D_c) v_\lambda \\ \text{(cf. V § 7).} \end{array} \right.$$

<sup>1)</sup> General references: CARTAN 1916, 1 (in  $R_n$ ); 1917, 1 (in  $C_n$ ,  $n \geq 5$ ); 1920, 1 (project.); 2; 1927, 5 (second var.); LEVI CIVITA 1926, 1 (second var.); SYNELL 1926, 1; 2 and 1934, 1 (second var.); SCHOUTEN 1927, 2 ( $V_m$  in  $V_n$ ); Mc CONNELL 1929, 1 (curves in  $V_n$ ); HLAVATY 1929, 3 (curves in  $U_n$ ); 1950, 1; 1952, 5 (families of  $V_m$  in  $V_n$ ); BURSTIN 1931, 1; 2; HAYDEN 1931, 1 (curves in  $V_n$ ); 1934, 2 ( $X_m$  in  $U_n$ ); 1936, 1 ( $L_m$  in  $L_n$ ); DIENES 1933, 4 and 1934, 2 ( $X_m$  in  $L_n$ ); BOMPIANI 1935, 1; LAND 1936, 1; DAVIES 1936, 1 ( $V_m$  in  $V_n$ ); 1938, 1 (curves in  $V_n$ ); 1938, 2 ( $V_n^m$ ); 1942, 1 (second var.  $V_m$  in  $V_n$ ); YANO and MUTO 1936, 2 (second var.); DIENES and DAVIES 1937, 1 ( $X_m$  in  $V_n$  and  $A_n$ , literature); E II 1938, 2, p. 164ff.; YANO and ADATI 1944, 7 (par. tang. and concirc.); YANO 1945, 9; 10; 1949, 2 (literature); YANO, TAKANO and TOMONAGA 1946, 10 and 1948, 2 (curves in  $L_n$ ); CASTOLDI 1948, 2 (finite deform. of  $V_m$  in  $V_n$ ); YANENKO 1949, 2 ( $V_m$  in  $V_n$ ).

If  $v^*$  is split up into ' $v^*$ ' in  $V_m$  and '' $v^*$  perpendicular to  $V_m$ , (3.4) leads to (cf. V 7.15)

$$(3.5) \quad \begin{aligned} \text{a)} \quad & \overset{\alpha}{D}' g_{cb} = 2 D_{(c} 'v_{b)} - 2 ''v_x H_{c\dot{b}}{}^x \\ \text{b)} \quad & \overset{\alpha}{D}' g^{ab} = - 2 'g^{c(a} D_c 'v^{b)} + 2 ''v_x H^{ab}{}_x \end{aligned}$$

From (3.1-4) and  $B_\lambda^a = g_{\lambda\kappa} B_b^x g^{b\kappa}$  we get

$$(3.6) \quad \boxed{\overset{\alpha}{D} B_\lambda^a = 2 'g^{ab} B_b^x C_\lambda^\mu v_{(\mu\kappa)}}$$

and accordingly

$$(3.7) \quad \boxed{\overset{\alpha}{D} B_\lambda^x = - \overset{\alpha}{D} C_\lambda^x = 2 'g^{x\sigma} C_\lambda^\mu v_{(\mu\sigma)}}.$$

Because of  $\overset{\alpha}{D} g_{\lambda\kappa} = \frac{f}{v} g_{\lambda\kappa}$  we have (cf. 1.8)

$$(3.8) \quad \overset{\alpha}{D} I_{\mu\lambda}^x = \frac{f}{v} I_{\mu\lambda}^x = \nabla_\mu v_{\lambda}{}^x + v^\sigma K_{\sigma\mu}{}^x$$

and from this we get for  $\overset{\alpha}{D}' T_{cb}^a$  (cf. 3.2, 6 and V 3.10)

$$(3.9) \quad \left\{ \begin{aligned} \overset{\alpha}{D}' T_{cb}^a &= \overset{\alpha}{D} (B_{cb}^{\mu\lambda} I_{\mu\lambda}^x + B_\lambda^a \partial_c B_b^x) \\ &= B_{cb}^{\mu\lambda} I_{\mu\lambda}^x \overset{\alpha}{D} B_\lambda^a + B_{cb}^{\mu\lambda} \overset{\alpha}{D} I_{\mu\lambda}^x + (\overset{\alpha}{D} B_\lambda^a) \partial_c B_b^x \\ &= B_{cb}^{\mu\lambda} \frac{f}{v} I_{\mu\lambda}^x + (\overset{\alpha}{D} B_\lambda^a) (D_c B_b^x + 'T_{cb}^d B_d^x) \end{aligned} \right.$$

hence

$$(3.10) \quad \boxed{\overset{\alpha}{D}' T_{cb}^a = B_{cb}^{\mu\lambda} \frac{f}{v} I_{\mu\lambda}^x + H_{cb}{}^x \overset{\alpha}{D} B_\lambda^a} .^1)$$

It can be proved in the same way that

$$(3.11) \quad \boxed{\overset{\alpha}{D} H_{c\dot{b}}{}^x = C_x^z B_{c\dot{b}}^{\mu\lambda} \frac{f}{v} I_{\mu\lambda}^x} .$$

By means of (3.8) and the CODAZZI equations (V 8.7) this equation can be transformed into

$$(3.12) \quad \left\{ \begin{aligned} \overset{\alpha}{D} H_{c\dot{b}}{}^x &= 'v^d D_d H_{c\dot{b}}{}^x + 2 H_{d(c}{}^x D_{b)} 'v^d - H_{c\dot{b}}{}^y ''v_y{}^x + \\ &\quad + ''v^y (C_{y\kappa}^x B_{c\dot{b}}^{\mu\lambda} K_{\nu\mu}{}^x - H_{c\dot{b}}{}^z H_{b\cdot y}{}^d) + D_c D_b ''v^x; \\ ''v_y{}^x &\stackrel{\text{def}}{=} C_{y\kappa}^x v_\lambda{}^x. \end{aligned} \right.$$

In the exercises (VII 3.1; 2; 3) we give some other formulae concerning the  $V_m$  in  $V_n$ .<sup>2)</sup>

<sup>1)</sup> DAVIES 1936, 1; cf. BORTOLOTTI 1928, 2; 3 (literature); HAYDEN 1934, 2.

<sup>2)</sup> Cf. for finite deformations of  $V_m$  in  $V_n$  CASTOLDI 1948, 2.

The  $V_n^m$  in  $V_n$ , the rigged  $L_m$  in  $L_n$  and the rigged  $L_n^m$  in  $L_n$  can be dealt with in the same way.<sup>1)</sup>

The method of transposition cannot be used for a subspace imbedded in an  $X_n$  because in the  $X_n$  there is no field that could be dragged back. But a more general method that can also be used in this case, was developed by NIJENHUIS.<sup>2)</sup> Let a rigged  $X_n^m$  in  $X_n$  be subjected to the infinitesimal transformation  $\xi^x \rightarrow \xi^x + v^x dt$  defined over  $X_n$  and let there be introduced an anholonomic coordinate system  $(h)$  such that before the transformation the  $e_b^x$ ;  $b = 1, \dots, m$  lie in  $X_n^m$  and the  $e_y^x$ ;  $y = m+1, \dots, n$  in the rigging. Note that the tangent  $E_m$  and  $E_{m'}$ ;  $m' = n - m$ , at  $\xi^x$  after having been dragged along over  $v^x dt$  do not coincide with the tangent  $E_m$  and  $E_{m'}$  at  $\xi^x + v^x dt$  before the transformation. This implies, that for a vector  $p^a$  of  $X_n^m$ , having components  $p^a = B_b^x p^b$  with respect to  $(x)$ , a LIE derivative is as yet not defined because after the transformation the vector is no longer a vector in the tangent  $E_m$ . We now *define* the LIE derivatives for the fields  $p^a, q_b$  as the  $X_n^m$ -parts of the LIE derivatives of the fields  $p^x, q_\lambda$  considered as fields of  $X_n$  (cf. V 7.36):

$$(3.13) \quad \begin{cases} a) & \frac{\mathfrak{L}}{v} p^a \stackrel{\text{def}}{=} B_x^a \frac{\mathfrak{L}}{v} p^x \\ b) & \frac{\mathfrak{L}}{v} q_b \stackrel{\text{def}}{=} B_b^1 \frac{\mathfrak{L}}{v} q_\lambda. \end{cases}$$

The definition (3.13) implies that the LEIBNIZ rule remains valid at least for quantities. We *assume* that it holds always. If the  $X_n^m$  is holonomic (i.e. consisting of  $\infty^{n-m} X_m$ 's) and if  $v^x$  happens to be a vector of  $X_n^m$ , the vector  $\frac{\mathfrak{L}}{v} p^a$  defined in the ordinary way with coordinates  $\eta^a$  in the  $X_m$ 's, that is  $v^b \partial_b p^a - p^b \partial_b v^a$ , is the  $X_m$ -part of  $\frac{\mathfrak{L}}{v} p^x$ . This proves that  $\frac{\mathfrak{L}}{v} p^a$  as defined in (3.13a) has the ordinary meaning in this special case. By the definition-formulae (3.13) the operator  $\frac{\mathfrak{L}}{v}$  gets the properties of a v. d. WAERDEN-BORTOLOTTI operator (cf. V § 7). From (3.13) we get immediately (cf. II 9.10)

$$(3.14) \quad \begin{cases} \frac{\mathfrak{L}}{v} p^a = B_x^a (v^\mu \partial_\mu B_b^x p^b - p^b B_b^\mu \partial_\mu v^x) \\ \quad \equiv v^\mu \partial_\mu p^a - (\partial_b v^a + 2v^\mu Q_{\mu b}^a B_b^\lambda) p^b; \end{cases}$$

hence (cf. V 7.25, 26, 35)

$$(3.15) \quad \begin{cases} a) & \frac{\mathfrak{L}}{v} p^a = 'v^b \partial_b p^a - (\partial_b 'v^a + 2'v^c Q_{cb}^a) p^b \\ b) & \frac{\mathfrak{L}}{v} p^a = ''v^y (\partial_y p^a + 2Q_{cy}^a p^c) = ''v^y \mathfrak{D}_y p^a \\ c) & \frac{\mathfrak{L}}{v} p^a = \frac{\mathfrak{L}}{v} p^a + \frac{\mathfrak{L}}{v} p^a = \frac{\mathfrak{L}}{v} p^a + ''v^y \mathfrak{D}_y p^a \end{cases}$$

<sup>1)</sup> SCHOUTEN and v. KAMPEN 1933, 1; E II 1938, 2, p. 165 ff.

<sup>2)</sup> Personal communication.

where  $\mathfrak{D}_y$  is the invariant operator defined in V § 7 that is independent of any connexion in  $X_n$ . Note the resemblance of (3.15 a) to the expression for the LIE derivative in anholonomic coordinates in exerc. II 10,6. In the same way we can derive

$$(3.16) \quad \left\{ \begin{array}{l} \text{a)} \quad \mathcal{L}_{\dot{v}} q_b = 'v^c \partial_c q_b + (\partial_b 'v^a + 2'v^c \Omega_{cb}^a) q_a \\ \text{b)} \quad \mathcal{L}_{\dot{v}} q_b = ''v^y (\partial_y q_b - 2\overset{m}{\Omega}_{by}^a q_a) = ''v^y \mathfrak{D}_y q_b \\ \text{c)} \quad \mathcal{L}_{\dot{v}} q_b \stackrel{*}{=} v^\mu \partial_\mu q_b + (\partial_b 'v^a + 2v^\mu \Omega_{\mu\lambda}^a B_b^\lambda) q_a = \mathcal{L}_{\dot{v}} q_b + \mathcal{L}_{\dot{v}} q_b \\ \qquad \qquad \qquad = \mathcal{L}_{\dot{v}} q_b + ''v^y \mathfrak{D}_y q_b. \end{array} \right.$$

If we define

$$(3.17) \quad \left\{ \begin{array}{l} \text{a)} \quad \mathcal{L}_{\dot{v}} r^x \stackrel{\text{def}}{=} C_x^x \mathcal{L}_{\dot{v}} r^x \\ \text{b)} \quad \mathcal{L}_{\dot{v}} s_y \stackrel{\text{def}}{=} C_y^x \mathcal{L}_{\dot{v}} s_x \end{array} \right.$$

for the fields  $r^x$  and  $s_y$  in the rigging (cf. 3.13) we get in the same way

$$(3.18) \quad \left\{ \begin{array}{l} \text{a)} \quad \mathcal{L}_{\dot{v}} r^x = 'v^b \mathfrak{D}_b r^x; \\ \text{b)} \quad \mathcal{L}_{\dot{v}} r^x = ''v^y \partial_y r^x - (\partial_y ''v^x + 2''v^z ''\Omega_{zy}^x) r^y \\ \text{c)} \quad \mathfrak{D}_b r^x = \partial_b r^x + 2\overset{m}{\Omega}_{yb}^x r^y \quad (\text{cf. V 7.35 c)}) \\ \text{d)} \quad \mathcal{L}_{\dot{v}} r^x \stackrel{*}{=} v^\mu \partial_\mu r^x - (\partial_y ''v^x + 2v^\mu \Omega_{\mu\lambda}^x C_y^\lambda) r^y = \mathcal{L}_{\dot{v}} r^x + \mathcal{L}_{\dot{v}} r^x \end{array} \right.$$

$$(3.19) \quad \left\{ \begin{array}{l} \text{a)} \quad \mathcal{L}_{\dot{v}} s_y = 'v^b \mathfrak{D}_b s_y \\ \text{b)} \quad \mathcal{L}_{\dot{v}} s_y = ''v^x \partial_z s_y + (\partial_y ''v^x + 2''v^z ''\Omega_{zy}^x) s_x \\ \text{c)} \quad \mathfrak{D}_b s_y = \partial_b s_y - 2\overset{m}{\Omega}_{yb}^x s_x \quad (\text{cf. V 7.35 d)}) \\ \text{d)} \quad \mathcal{L}_{\dot{v}} s_y \stackrel{*}{=} v^\mu \partial_\mu s_y + (\partial_y ''v^x + 2v^\mu \Omega_{\mu\lambda}^x C_y^\lambda) s_x = \mathcal{L}_{\dot{v}} s_y + \mathcal{L}_{\dot{v}} s_y. \end{array} \right.$$

All these formulae can also be derived easily from (Exerc. II 10,6). It is now easy to derive the general formula for a quantity lying with some indices in  $X_n$ , with other indices in  $X_n^m$  and with the remaining ones in the rigging, for instance

$$(3.20) \quad \left\{ \begin{array}{l} \mathcal{L}_{\dot{v}} P_{..y}^{xa} \stackrel{*}{=} v^\mu \partial_\mu P_{..y}^{xa} - P_{..y}^{\mu a} \partial_\mu v^x - P_{..y}^{xb} (\partial_b 'v^a + 2v^\mu \Omega_{\mu\lambda}^a B_b^\lambda) + \\ \qquad \qquad \qquad + P_{..x}^{xa} (\partial_y ''v^x + 2v^\mu \Omega_{\mu\lambda}^x C_y^\lambda). \end{array} \right.$$

For a density  $\overset{(a)}{\mathfrak{p}}$  or  $\overset{(x)}{\mathfrak{p}}$  of  $X_n^m$  or  $X_n^{m'}$  with weight ' $w$ ' or '' $w$  respectively we find (cf. Exerc. II 10, 6 δ)

$$(3.21) \quad \left\{ \begin{array}{l} \text{a)} \quad \mathcal{L}_{\dot{v}} \overset{(a)}{\mathfrak{p}} \stackrel{*}{=} v^\mu \partial_\mu \overset{(a)}{\mathfrak{p}} + 'w \overset{(a)}{\mathfrak{p}} (\partial_b 'v^b + 2v^\mu \Omega_{\mu\lambda}^b B_b^\lambda) \\ \text{b)} \quad \mathcal{L}_{\dot{v}} \overset{(x)}{\mathfrak{p}} \stackrel{*}{=} v^\mu \partial_\mu \overset{(x)}{\mathfrak{p}} + ''w \overset{(x)}{\mathfrak{p}} (\partial_y ''v^y + 2v^\mu \Omega_{\mu\lambda}^y C_y^\lambda). \end{array} \right.$$

For any quantity  $\Phi(\Psi)$  (indices suppressed) of  $X_n^m$  ( $X_n^{m'}$ ) we have (cf. V § 7)

$$(3.22) \quad \frac{\delta}{v} \Phi = "v^y \mathfrak{D}_y \Phi; \quad \frac{\delta}{v} \Psi = 'v^b \mathfrak{D}_b \Psi.$$

Of course  $\mathfrak{D}_b$  ( $\mathfrak{D}_y$ ) can not yet be applied to a quantity of  $X_n^m$  ( $X_n^{m'}$ ) because so far no connexion has been given (cf. V § 7).

An  $X_n^m$  is invariant for a transformation  $v^x dt$  if and only if every vector  $p^a$  of  $X_n^m$  remains a vector of  $X_n^m$ , that is if

$$(3.23) \quad \frac{\delta}{v} p^x = \frac{\delta}{v} B_a^x p^a = B_a^x \frac{\delta}{v} p^a + p^a \frac{\delta}{v} B_a^x$$

is a vector of  $X_n^m$ . Now the connecting quantity depending on  $v^x$

$$(3.24) \quad \boxed{H_b^x \stackrel{\text{def}}{=} \frac{\delta}{v} B_b^x}$$

lies with the index  $x$  in  $X_n^{m'}$  because

$$(3.25) \quad B_\kappa^a H_b^x = B_{\kappa b}^a \frac{\delta}{v} B_\lambda^x = - B_{\kappa b}^a \frac{\delta}{v} C_\lambda^x = - B_{\kappa b}^a \frac{\delta}{v} C_y^x C_\lambda^y = 0.$$

Hence the necessary and sufficient condition for the invariance of  $X_n^m$  is that the LIE derivative of  $B_b^x$  is zero:  $H_b^x = 0$ .

In the same way the connecting quantity

$$(3.26) \quad \boxed{L_y^x \stackrel{\text{def}}{=} \frac{\delta}{v} C_y^x}$$

lies with the index  $x$  in  $X_n^m$  and it is zero if and only if the rigging  $X_n^{m'}$  is invariant. From (3.24, 26) it follows easily that

$$(3.27) \quad \boxed{\begin{aligned} \text{a)} \quad & \frac{\delta}{v} B_\lambda^a = - L_\lambda^a; & \text{b)} \quad & \frac{\delta}{v} C_\lambda^x = - H_\lambda^x \end{aligned}}$$

and

$$(3.28) \quad \frac{\delta}{v} B_\lambda^x = - \frac{\delta}{v} C_\lambda^x = H_\lambda^x - L_\lambda^x.$$

For  $H_b^x$  and  $L_y^a$  it is easily proved that

$$(3.29) \quad \left\{ \begin{array}{l} \text{a)} \quad H_b^x = C_\kappa^x (v^\mu \partial_\mu B_b^x - B_b^\mu \partial_\mu v^\kappa) \\ \text{b)} \quad L_y^a = B_\kappa^a (v^\mu \partial_\mu C_y^x - C_y^\mu \partial_\mu v^\kappa) \end{array} \right.$$

and these equations express that  $H_b^x$  depends on the  $X_n^m$  only and not on the rigging, and in the same way  $L_y^a$  depends on the  $X_n^{m'}$  only.

If  $v^x$  lies in  $X_n^m$ ,  $v^x = 'v^x$ , we get (cf. V 7.22)

$$(3.30) \quad \left\{ \begin{array}{l} H_b^x = C_\kappa^x ('v^c \partial_c B_b^x - (\partial_b B_c^x) 'v^c - B_c^x \partial_b 'v^c) \\ = 2 C_\kappa^x 'v^c \partial_{[c} B_{b]}^x \stackrel{*}{=} - 2 'v^c Q_{cb}^x \stackrel{*}{=} - 2 'v^c Z_{cb}^x \end{array} \right.$$

and from this identity we see that an  $X_n^m$  is invariant for all tangent deformations if and only if it is holonomic.

If  $v^x$  lies in  $X_n^{m'}$ ,  $v^x = {}''v^x$ , we get from (3.29) and (V 7.35c)

$$(3.31) \quad \begin{aligned} {}''H_b^x &= C_x^x({}'v^x \partial_x B_b^x - (\partial_b C_x^x) {}''v^x - C_x^x \partial_b {}''v^x) \\ &= -\partial_b {}''v^x - 2 {}''v^x \Omega_{xb}^{m'} = -\mathfrak{D}_b {}''v^x \end{aligned}$$

hence, from (3.30, 31) and the analogous reasoning for  $X_n^{m'}$

$$(3.32) \quad \begin{cases} \text{a)} & H_v^x = -\mathfrak{D}_b {}''v^x - 2'v^c Z_{cb}^{m'} \\ \text{b)} & L_y^a = -\mathfrak{D}_y {}'v^a - 2 {}''v^x Z_{xy}^{m'} \end{cases}$$

If  $u^x$  and  $v^x$  are both vectors of  $X_n^m$ :  $u^x = {}'u^x$ ;  $v^x = {}'v^x$ , we get from (3.30)

$$(3.33) \quad \underline{\underline{L}}_u {}'v^x = \underline{\underline{L}}_u B_b^x {}'v^b = B_b^x \underline{\underline{L}}_u {}'v^b + {}'v^b H_u^x = B_b^x \underline{\underline{L}}_u {}'v^b - 2'{}u^c {}'v^b Z_{cb}^{m'},$$

hence  $\underline{\underline{L}}_u {}'v^x$  lies in  $X_n^m$  for every choice of  $'u^x$  and  $'v^x$  if and only if  $X_n^m$  is holonomic.

In Ch. II § 10 we found (Exerc. II 10,9)

$$(3.34) \quad \underline{\underline{L}}_{[u v]} = \underline{\underline{L}}_{(u, v)}; \quad (u, v)^x \stackrel{\text{def}}{=} \underline{\underline{L}}_u v^x = -\underline{\underline{L}}_v u^x$$

for all quantities of  $X_n$ . But this simple rule does not hold for the operators  $\underline{\underline{L}}_u$  and  $\underline{\underline{L}}_v$  as defined in (3.13, 17), if applied to quantities of  $X_n^m$  or  $X_n^{m'}$ . Instead we get from (3.28)

$$(3.35) \quad \boxed{\begin{array}{l} \text{a)} \quad 2\underline{\underline{L}}_{[u v]} \underline{\underline{L}}_v p^a = \underline{\underline{L}}_{(u, v)} p^a + 2H_b^x L_x^a p^b \\ \text{b)} \quad 2\underline{\underline{L}}_{[u v]} q_b = \underline{\underline{L}}_{(u, v)} q_b - 2H_b^x L_x^a q_a \\ \text{c)} \quad 2\underline{\underline{L}}_{[u v]} r^x = \underline{\underline{L}}_{(u, v)} r^x + 2L_y^a H_a^x r^y \\ \text{d)} \quad 2\underline{\underline{L}}_{[u v]} s_y = \underline{\underline{L}}_{(u, v)} s_y - 2L_y^a H_a^x s_x \end{array}}$$

hence, by applying (3.34, 35) to  $B_b^x$  and  $C_y^x$

$$(3.36) \quad \begin{cases} \text{a)} & 2\underline{\underline{L}}_{[u v]} B_b^x = H_b^x - 2H_b^x L_x^x \\ \text{b)} & 2\underline{\underline{L}}_{[u v]} C_y^x = L_y^x - 2L_y^a H_a^x \end{cases}$$

and by transvection of these equations with  $C_x^*$  and  $B_a^*$  respectively

$$(3.37) \quad \begin{cases} \text{a)} & H_b^x = 2 C_x^* \underset{[u \ v]}{\mathfrak{L}} B_b^x = 2 \underset{[u \ v]}{\mathfrak{L}} H_b^x \\ \text{b)} & L_y^a = 2 B_a^* \underset{[u \ v]}{\mathfrak{L}} C_y^* = 2 \underset{[u \ v]}{\mathfrak{L}} L_y^a. \end{cases}$$

There is also a commutation formula for the operators  $\underset{v}{\mathfrak{L}}$  and  $\mathfrak{D}_y$  (or  $\mathfrak{D}_b$ ) that can be derived from (3.15, 16, 32, 35)

$$(3.38) \quad \begin{cases} \text{a)} & (\mathfrak{D}_y \underset{v}{\mathfrak{L}} - \underset{v}{\mathfrak{L}} \mathfrak{D}_y) p^a = \underset{p}{\mathfrak{L}} L_y^a + 2 Z_{y \ x}^{m'} H_v^x p^b \\ \text{b)} & (\mathfrak{D}_b \underset{v}{\mathfrak{L}} - \underset{v}{\mathfrak{L}} \mathfrak{D}_b) r^x = \underset{r}{\mathfrak{L}} H_b^x + 2 Z_{b \ a}^{m'} L_y^a r^y. \end{cases}$$

So far no connexion or metric has been introduced either in  $X_n$  or in  $X_n^m$ . By the introduction of a connexion or a metric in the rigged  $X_n^m$  only, as was described in V § 7 and V § 9 respectively we get the rigged  $L_n^m$  and  $V_n^m$  in  $X_n$ . Introduction of a connexion or a metric in  $X_n$  leads to the rigged  $L_n^m$  in  $L_n$  and the  $V_n^m$  in  $V_n$  respectively. For all these special cases the formulae derived here for the rigged  $X_n^m$  in  $X_n$  remain valid, only new additional invariant operators and formulae arise.

Here we deal only with the rigged  $L_n^m$  in  $X_n$  arising by the introduction of a connexion  $T_{cb}^a$  in the  $X_n^m$ . Then, as we saw in V § 7 the operator  $\mathfrak{D}_b$  can be extended to quantities of  $X_n^m$  (cf. V 7, 35, 57)

$$(3.39) \quad \begin{cases} \text{a)} & \mathfrak{D}_c p^a = D_c p^a = \partial_c p^a + T_{cb}^a p^b \\ \text{b)} & \mathfrak{D}_c q_b = D_c q_b = \partial_c q_b - T_{cb}^a q_a \end{cases}$$

and the operator  $\mathfrak{D}_\mu$  can be defined as

$$(3.40) \quad \mathfrak{D}_\mu \stackrel{\text{def}}{=} B_\mu^b \mathfrak{D}_b + C_\mu^y \mathfrak{D}_y.$$

Then we get from (3.15, 16) (cf. III 9.3 and III 5.48)

$$(3.41) \quad \begin{cases} \text{a)} & \underset{v}{\mathfrak{L}} p^a = v^\mu \mathfrak{D}_\mu p^a - 'v_b^a p^b; \quad 'v_b^a = D_b 'v^a - 2 S_{b \ c}^{a \ b} v^c \\ \text{b)} & \underset{v}{\mathfrak{L}} q_b = v^\mu \mathfrak{D}_\mu q_b + 'v_b^a q_a; \quad 'S_{c \ b}^{a \ b} = 'T_{[c \ b]}^a + 'Q_{c \ b}^a. \end{cases}$$

For  $\underset{v}{\mathfrak{L}} r^x$ ;  $\underset{v}{\mathfrak{L}} s_y$ , there are no formulae of this kind as long as there is no connexion defined in  $X_n^m$ .

We wish to find  $\underset{v}{\mathfrak{L}} T_{cb}^a$ . Although  $\underset{v}{\mathfrak{L}}$  and  $\partial_\mu$  commute,  $\underset{v}{\mathfrak{L}}$  and  $\partial_c$  have this property if and only if the  $X_n^m$  is invariant for the transformation  $v^x dt$ , because, using the rule of LEIBNIZ (cf. p. 354) we have

$$(3.42) \quad \underset{v}{\mathfrak{L}} \partial_c p^a = \underset{v}{\mathfrak{L}} B_c^\mu \partial_\mu p^a = (\underset{v}{\mathfrak{L}} B_c^\mu) \partial_\mu p^a + B_c^\mu \underset{v}{\mathfrak{L}} \partial_\mu p^a = H_c^\mu \partial_\mu p^a + \partial_c \underset{v}{\mathfrak{L}} p^a.$$

Hence, if the  $X_n^m$  is holonomic and if  $v^*$  lies in  $X_n^m$ , it follows from (3.30) that  $\frac{\xi}{v}$  and  $\partial_c$  are always commuting whether (c) is holonomic or anholonomic, as was to be expected (cf. II 10.17).

Applying (3.42) to  $\frac{\xi}{v} \mathfrak{D}_c p^a$  we get

$$(3.43) \quad \left\{ \begin{array}{l} \frac{\xi}{v} \mathfrak{D}_c p^a = \frac{\xi}{v} (\partial_c p^a + 'I_{cb}^a p^b) \\ \qquad = H_c^{\mu} \partial_{\mu} p^a + \partial_c \frac{\xi}{v} p^a + 'I_{cb}^a \frac{\xi}{v} p^b + p^b \frac{\xi}{v} 'I_{cb}^a \end{array} \right.$$

hence (cf. V 7.35a)

$$(3.44) \quad p^b \frac{\xi}{v} 'I_{cb}^a = (\frac{\xi}{v} \mathfrak{D}_c - \mathfrak{D}_c \frac{\xi}{v}) p^a - H_c^{\mu} \mathfrak{D}_{\mu} p^a + 2 H_c^{\mu} \frac{\xi}{v} \Omega_{b\mu}^a p^b.$$

Now it can be proved from (V 7.59–62) and (3.32a) that

$$(3.45) \quad \left\{ \begin{array}{l} \left( \frac{\xi}{v} \mathfrak{D}_c - \mathfrak{D}_c \frac{\xi}{v} \right) p^a - H_c^{\mu} \mathfrak{D}_{\mu} p^a \\ \qquad = ('v^d * R_{a\mu}^{\mu} + ''v^y * R_{y\mu}^{\mu} + \mathfrak{D}_c 'v_b^a) p^b, \end{array} \right.$$

hence

$$(3.46) \quad \boxed{\frac{\xi}{v} 'I_{cb}^a = 'v^d * R_{a\mu}^{\mu} + ''v^y * R_{y\mu}^{\mu} + \mathfrak{D}_c 'v_b^a + 2 H_c^{\mu} \frac{\xi}{v} \Omega_{b\mu}^a}.$$

In order to find  $\frac{\xi}{v} \Omega_{cy}^a$  we start from  $\frac{\xi}{v} \partial_y p^a$ :

$$(3.47) \quad \frac{\xi}{v} \partial_y p^a = \frac{\xi}{v} C_y^{\mu} \partial_{\mu} p^a = L_y^{\mu} \partial_{\mu} p^a + \partial_y \frac{\xi}{v} p^a$$

and apply this result to  $\frac{\xi}{v} \mathfrak{D}_y p^a$  (cf. V 7.35a):

$$(3.48) \quad \frac{\xi}{v} \mathfrak{D}_y p^a = L_y^{\mu} \partial_{\mu} p^a + \mathfrak{D}_y \frac{\xi}{v} p^a + 2 p^c \frac{\xi}{v} \Omega_{cy}^a$$

or

$$(3.49) \quad (\frac{\xi}{v} \mathfrak{D}_y - \mathfrak{D}_y \frac{\xi}{v}) p^a = L_y^{\mu} \partial_{\mu} p^a - L_y^{\mu} \mathfrak{D}_c p^a + 2 p^c \frac{\xi}{v} \Omega_{cy}^a.$$

Now from (3.15, 19) (cf. 3.45 and V 7.24, 58–61) it can be proved that

$$(3.50) \quad \left\{ \begin{array}{l} \left( \frac{\xi}{v} \mathfrak{D}_y - \mathfrak{D}_y \frac{\xi}{v} \right) p^a - L_y^{\mu} \mathfrak{D}_c p^a \\ \qquad = (- \mathfrak{D}_b L_y^{\mu} + 2' S_{bc}^{\mu} L_y^{\mu} - 2 Z_{yx}^{\mu} H_b^x) p^b; \end{array} \right.$$

hence

$$(3.51) \quad 2 \frac{\xi}{v} \Omega_{by}^a = - \mathfrak{D}_b L_y^{\mu} + L_y^{\mu} ('I_{cb}^a - 2' S_{cb}^a) - 2 Z_{yx}^{\mu} H_b^x$$

or

$$(3.52) \quad \boxed{\frac{\xi}{v} \Omega_{by}^a = - \frac{1}{2} \partial_b L_y^{\mu} + \Omega_{yb}^{\mu} L_y^{\mu} + 'I_{cb}^a L_y^{\mu} - Z_{yx}^{\mu} H_b^x}.$$

Note that in the anholonomic case neither  $\xi_v' T_{cb}^a$  nor  $\xi_v \Omega_{b,y}^m$ , is a quantity, as could have been foreseen, and that  $\xi_v \Omega_{b,y}^m$  does not depend on the connexion in  $L_n^m$ .

If an  $L_n^m$  is rigged in an  $L_n$ , there is induced a connexion in  $L_n^m$  and we have seen in (Exerc. V 7,1) that in this case  $*R_{d;cb}^{m;a}$  as defined in (V 7.60) with respect to this induced connexion is identical with  $R_{d;cb}^{m;a}$  as defined in (V 7.39).  $\xi_v' T_{cb}^a$  can now also be computed by means of (cf. III 9.2; V 3.10)

$$(3.53) \quad \xi_v' T_{cb}^a = B_{cbx}^{\mu\lambda a} I_{\mu\lambda}^x + B_x^a \partial_c B_b^x$$

using (3.24, 26, 27; III 5.47) with the result

$$(3.54) \quad \boxed{\begin{aligned} \xi_v' T_{cb}^a &\equiv B_{cbx}^{\mu\lambda a} (\nabla_\mu v_\lambda^x + v^\sigma R_{\sigma\mu\lambda}^x) - \\ &- H_b^y L_{c,y}^m - L_z^a H_{cb}^m + H_c^y I_{yb}^a \end{aligned}}.$$

From (3.46) and (3.54) we get after some calculations the following relation between  $*R_{y;cb}^{m;a}$  and  $R_{\nu\mu\lambda}^x$ ,

$$(3.55) \quad \boxed{\begin{aligned} *R_{y;cb}^{m;a} &\equiv C_y^* B_{cbx}^{\mu\lambda a} R_{\nu\mu\lambda}^x - \\ &- \mathfrak{D}_c (L_b^a{}_y + 2S_{by}^a) - H_{cb}^m H_{yz}^m + L_{c,x}^m L_{y,b}^m \end{aligned}}.$$

### Exercises.

VII 3,1. Prove that for a  $V_m$  in  $V_n$

$$\text{VII 3,1 a)} \quad D C_y^x = -'g^{xc} (D_c ''v_y + H_{cdy} 'v^d) - B_\lambda^x C_y^\mu \nabla_\mu v^\lambda.$$

VII 3,2. Prove that for a  $V_m$  in  $V_n$

$$\text{VII 3,2 a)} \quad \left\{ \begin{aligned} D' T_{cb}^a &= 'v^d K_{d;cb}^a + D_c D_b 'v^a + 2D_{[c} H_{b]a}^x ''v^x - \\ &- 'g^{ad} D_d H_{cbx} ''v^x. \end{aligned} \right.$$

VII 3,3. Prove that for a  $V_m$  in  $V_n$

$$\text{VII 3,3 a)} \quad \left\{ \begin{aligned} D' K_{d;cb}^a &= \xi_v' K_{d;cb}^a + 4D_{[d} D_{[b} H_{c]a]}^x ''v^x - \\ &- 2'K_{d;c[b}^e H_{a]ex} ''v^x. \end{aligned} \right.$$

VII 3,4. Prove that for an  $X_n^m$  in  $X_n$

$$\text{VII 3,4 a)} \quad \frac{\partial}{\partial} Z_{cb}^m = \partial_{[c} H_{b]}^m + H_{a}^m \partial_{cb}^a + 2 \Omega_{y[c}^m H_{b]}^y.$$

VII 3,5. Prove that for an  $X_n^m$  in  $L_n$

$$\text{VII 3,5 a)} \quad \frac{\partial}{\partial} H_{cb}^m = D_c H_b^m + B_{cb}^{\mu\lambda} C_{\mu}^x \frac{\partial}{\partial} I_{\lambda}^x - H_c^y L_{y,b}^m.$$

VII 3,6<sup>1)</sup>. If the first and second fundamental tensor of a  $V_{n-1}$  in  $R_n$  admit the same infinitesimal transformation in  $V_{n-1}$  there exists at least one relation between the principal curvatures (cf. V § 6).

VII 3,7. (3.52) can also be derived by taking the LIE derivative of

$$\text{VII 3,7 a)} \quad \frac{\partial}{\partial} B_{b,y}^a \equiv B_b^{\mu} C_{\mu}^y \partial_{[\mu} B_{\lambda]}^a \quad (\text{cf. V § 7}).$$

VII 3,8. Prove that (3.54) is in accordance with (3.46) if for the  $T_{cb}^a$  in (3.46) we take the parameters of the induced connexion (use 3.38 and V 7.37).

#### § 4. The holonomy group of an $L_n$ .

Let  $C$  be a closed curve, in an  $\mathfrak{N}(\xi)$ , through  $\xi$  that can be contracted into a point and let  $u$  be a vector at  $\xi$ . Then the components of  $u$  after having been displaced parallel along the curve are (cf. VII § 1)

$$(4.1) \quad u^* = \overset{C}{T}_{\cdot\lambda} u^{\lambda}$$

where  $\overset{C}{T}_{\cdot\lambda}$  is a tensor depending on the choice of  $C$  but not on the choice of  $u$  at  $\xi$ . The homogeneous linear transformations (4.1) for all choices of  $C$  form a group, the *holonomy group* of the  $L_n$ . If instead of parallel displacements we use CARTAN displacements (cf. III § 2) we get a non-homogeneous linear transformation of the points of the tangent  $E_n$  forming the *non-homogeneous holonomy group* of the  $L_n$ <sup>2)</sup>

<sup>1)</sup> TONOLO 1943, 1.

<sup>2)</sup> The homogeneous holonomy group in  $V_n$  was already used by SCHOUTEN 1918, 2 who remarked that the dimension of the  $R_N$  in which the  $V_n$  can be imbedded is  $n$  more than the number of parameters of this group. But the systematical treatment of the holonomy groups is due to CARTAN, 1924, 2; 1925, 1; 1926, 5. He introduced the name "groupe d'holonomie" though the group vanishes if the  $E_n$  moves "holonomically", that is if it always returns to its original position. Moreover the word holonomic or anholonomic gives the false impression that the group has something to do with an anholonomic coordinate system or with an  $X_n^m$  in  $L_n$ . The term "circuit group" would certainly be more suitable but the name given by CARTAN is so widely spread in literature that it seems better to leave it at that and to warn only against misinterpretation. Some american authors recently used the term "holonomic group" but this gives the false impression that the group itself has some property of holonomy. Of course holonomy groups can also be defined for the more general linear connexions mentioned in III § 2, footnote 1), p. 123 and especially for conformal and projective connexions, cf. CARTAN 1937, 1, p. 278 ff.

If  $C'$  is a closed curve in  $\mathfrak{N}(\xi^*)$  through another point  $\xi^*_1$  and if  $\overset{C'}{T}$  (matrix notation) is the transformation belonging to it, we may form a curve through  $\xi^*_0$  by going first from  $\xi^*_0$  to  $\xi^*_1$  along some curve  $C''$  in  $\mathfrak{N}(\xi^*)$ , then along  $C'$  and finally back to  $\xi^*_0$  along  $C''$ . If then  $U$  is the transformation along  $C''$  from  $\xi^*_0$  to  $\xi^*_1$ , we get for the whole transformation  $U^{-1}\overset{C'}{T}U$  and this must be a transformation of the holonomy group defined at  $\xi^*_0$ . Hence the groups defined at  $\xi^*_0$  and  $\xi^*_1$  are isomorphic and this means that there is in fact only one holonomy group for all points of  $L_n$ . In fact, if  $\overset{C'}{T}$  and  $U^{-1}\overset{C'}{T}U$  are described with respect to two local coordinate systems that can be transformed into each other by parallel displacement along  $C''$ , the numbers in these descriptions are identical.

If for  $C$  we take the boundary of an infinitesimal facet  $d f^{\mu}$  at  $\xi^*_0$ , the vector  $u^*$  is (cf. III § 4)

$$(4.2) \quad u^* = u^* - \frac{1}{2} R_{\nu\mu\lambda}^* u^\lambda d f^{\mu}.$$

For every choice of the facet this is an infinitesimal transformation of the holonomy group. But this does not imply that these at most  $\binom{n}{2}$  linearly independent infinitesimal transformations generate this group. For the non-homogeneous holonomy group we must use the transformations of the vectors together with the transformation of the point that originally coincided with the contactpoint and that after the displacement has the radiusvector (cf. III § 2)

$$(4.3) \quad - S_{\nu\mu}^* d f^{\mu}.$$

In order to find all infinitesimal transformations of the homogeneous holonomy group we may use the results of VII § 1.<sup>1)</sup> With the notations introduced in that section we have for a curve arising from  $C$  by a displacement  $\varepsilon v^*(t)$  with  $v^*(t_0) = v^*(t_1) = 0$

$$(4.4) \quad u^* = ([t_1]_\lambda^* + \varepsilon \overset{g}{D} [t_1]_\lambda^*) u_\theta^*$$

and accordingly the infinitesimal transformation belonging to these two curves is given by the tensor (cf. 1.36)

$$(4.5) \quad A_\lambda^* + \varepsilon [t_1]_\sigma^* \overset{g}{D} [t_1]_\lambda^* = A_\lambda^* - \varepsilon \int_{t_0}^{t_1} [t]_\lambda^* v^\nu \frac{d\xi^\mu}{dt} R_{\nu\mu\tau}^* [t]_\sigma^* dt.$$

<sup>1)</sup> There are many other methods. Cf. for instance KANITANI 1952, 1 where the transformations are developed in series.

Now we may take all curves through  $\xi^\alpha$  and all deformations  $\varepsilon v^\alpha$ , hence every tensor of the form

$$(4.6) \quad C_{\dot{\lambda}}^\alpha = [t_0]_{\dot{\lambda}}^\alpha p^\nu \frac{d\xi^\mu}{dt} R_{\nu\mu\dot{\lambda}}^\alpha$$

is a *generator* of the group (cf. IV § 3), i.e.  $A_{\dot{\lambda}}^\alpha + C_{\dot{\lambda}}^\alpha dt$  is an infinitesimal transformation of the group, and the same holds for all derivatives of  $C_{\dot{\lambda}}^\alpha$  with respect to  $t$ . In (4.6)  $d\xi^\mu/dt$  and  $R_{\nu\mu\dot{\lambda}}^\alpha$  are values at the point  $\xi^\alpha(t)$  and  $p^\nu$  is an arbitrary vector at that point. Using (1.30, 31) we find for these derivatives

$$(4.7) \quad \begin{cases} \frac{d}{dt} C_{\dot{\lambda}}^\alpha = [t_0]_{\dot{\lambda}}^\alpha \left\{ \frac{d\xi^\omega}{dt} V_\omega p^\nu \frac{d\xi^\mu}{dt} R_{\nu\mu\dot{\lambda}}^\alpha \right\} [t_0]_\sigma^\alpha \\ \vdots \\ \frac{d^k}{dt^k} C_{\dot{\lambda}}^\alpha = [t_0]_{\dot{\lambda}}^\alpha \left\{ \left( \frac{d\xi^\omega}{dt} V_\omega \right)^k p^\nu \frac{d\xi^\mu}{dt} R_{\nu\mu\dot{\lambda}}^\alpha \right\} [t_0]_\sigma^\alpha \end{cases}$$

and in particular for  $t=t_0$

$$(4.8) \quad \begin{cases} C_{\dot{\lambda}}^\alpha = p^\nu \frac{d\xi^\mu}{dt} R_{\nu\mu\dot{\lambda}}^\alpha \\ \frac{d}{dt} C_{\dot{\lambda}}^\alpha = \frac{d\xi^\omega}{dt} V_\omega p^\nu \frac{d\xi^\mu}{dt} R_{\nu\mu\dot{\lambda}}^\alpha \\ \vdots \\ \frac{d^k}{dt^k} C_{\dot{\lambda}}^\alpha = \left( \frac{d\xi^\omega}{dt} V_\omega \right)^k p^\nu \frac{d\xi^\mu}{dt} R_{\nu\mu\dot{\lambda}}^\alpha. \end{cases}$$

Conversely *all* generators depend linearly on the expressions (4.8) because the expressions (4.7) can be expanded in powers of  $t-t_0$  and the coefficients in these series are the expressions (4.8) multiplied by constants. Hence the generators of the group are the quantities

$$(4.9) \quad p^\nu q^\mu R_{\nu\mu\dot{\lambda}}^\alpha; \quad q^\omega V_\omega p^\nu q^\mu R_{\nu\mu\dot{\lambda}}^\alpha; \dots; (q^\omega V_\omega)^k p^\nu q^\mu R_{\nu\mu\dot{\lambda}}^\alpha; \dots$$

for all fields  $q^\alpha$  and all fields  $p^\alpha$  with  $p^\alpha=0$  at  $\xi^\alpha$ . That proves that the generators lie in the  $\dot{\lambda}$ -domain of the quantities

$$(4.11) \quad R_{\nu\mu\dot{\lambda}}^\alpha; \quad V_\omega R_{\nu\mu\dot{\lambda}}^\alpha; \dots; \quad V_{\omega_k \dots \omega_1} R_{\nu\mu\dot{\lambda}}^\alpha; \dots$$

Moreover, as was proved by NIJENHUIS<sup>1)</sup> both domains coincide. Hence

*The generators of the holonomy group of an  $L_n$  span the  $\dot{\lambda}$ -domain of the curvature tensor  $R_{\nu\mu\dot{\lambda}}^\alpha$  and its covariant derivatives.*

Let  $r \leq n^2$  be the rank of this  $\dot{\lambda}$ -domain and let the domain be spanned by the  $r$  linearly independent tensors  $C_{(b)\dot{\lambda}}^\alpha$ ;  $b=\dot{i}, \dots, \dot{r}$ . Here  $b$  is a dead index but it may be considered as a living one belonging to an auxiliary  $E$ , in which every tensor of the domain is represented by a contravariant vector.<sup>1)</sup> Then  $C_{b\dot{\lambda}}^\alpha$  is a connecting quantity

<sup>1)</sup> NIJENHUIS 1952, 1, Ch. II § 13.

between this  $E_r$  and the  $\lambda$ -domain. The infinitesimal transformations of the holonomy group are

$$(4.12) \quad X_b = C_{b\lambda}^{\alpha} x^{\lambda} \frac{\partial}{\partial x^{\alpha}}$$

where the  $x^{\alpha}$  are rectilinear coordinates in the tangent  $E_n$  and from this we see that the  $E_r$  is identical with the tangent  $E_r$  at the point of group space corresponding to the identical transformation (cf. IV § 2). Now  $R_{\nu\mu\lambda}^{\alpha}$  and its covariant derivatives can be written in the form

$$(4.13) \quad \begin{cases} \text{a)} & R_{\nu\mu\lambda}^{\alpha} = B_{\nu\mu}^{\alpha} C_{b\lambda}^{\alpha} \\ \vdots & \\ \text{b)} & V_{\omega_k \dots \omega_1} R_{\nu\mu\lambda}^{\alpha} = B_{\omega_k \dots \omega_1 \nu\mu}^{\alpha} C_{b\lambda}^{\alpha}; \quad k = 1, 2, \dots \end{cases}$$

with uniquely determined coefficients  $B$  and  $B$ .

To every point of  $L_n$  there now belongs also besides the ordinary tangent  $E_n$  a tangent  $E_r$  independent of this  $E_n$ , and covariant differentiation can be extended to the quantities of these  $E_r$ 's by introducing the condition that the covariant derivative of the connecting quantity  $C_{b\lambda}^{\alpha}$  be zero for displacements in  $L_n$ .<sup>1)</sup> In fact, if we put

$$(4.14) \quad \theta = V_{\mu} C_{b\lambda}^{\alpha} = \partial_{\mu} C_{b\lambda}^{\alpha} - I_{\mu b}^a C_{a\lambda}^{\alpha} - I_{\mu\lambda}^a C_{b\alpha}^{\alpha} + I_{\mu\sigma}^{\alpha} C_{b\lambda}^{\sigma}$$

the  $I_{\mu b}^a$  are determined uniquely because  $C_{b\mu}^{\alpha}$  has the maximum  $b$ -rank. If the  $I_{\mu b}^a$  are found it follows from (4.13) that

$$(4.15) \quad \begin{cases} V_{\omega} B_{\nu\mu}^{\alpha} = B_{\omega\nu\mu}^{\alpha} \\ \vdots \\ V_{\omega} B_{\omega_k \dots \omega_1 \nu\mu}^{\alpha} = B_{\omega\omega_k \dots \omega_1 \nu\mu}^{\alpha}; \quad k = 1, 2, \dots \end{cases}$$

It is easily proved that the coefficients  $I_{\mu b}^a$  transform in the following way if the coordinates in  $L_n$  and the (rectilinear) coordinates in  $E_r$  are transformed:

$$(4.16) \quad I_{\mu b'}^{a'} = A_{\mu}^{\mu} A_{b'}^b A_{a'}^a I_{\mu b}^a + A_{a'}^a \partial_{\mu} A_{b'}^b.$$

For the part  $R_{\nu\mu b}^{\alpha}$  of the extended curvature tensor (defined in the ordinary way) concerning the  $E_r$  we find

$$(4.17) \quad R_{\nu\mu b}^{\alpha} = 2 \partial_{[\nu} I_{\mu]b}^a + 2 I_{[\nu|c]}^a I_{\mu]b}^c.$$

Because the covariant derivative of  $C_{b\lambda}^{\alpha}$  vanishes, the following identity holds

$$(4.18) \quad \theta = 2 V_{[\nu} V_{\mu]} C_{b\lambda}^{\alpha} = - R_{\nu\mu b}^{\alpha} C_{a\lambda}^{\alpha} - R_{\nu\mu\lambda}^{\alpha} C_{b\alpha}^{\alpha} + R_{\nu\mu\lambda}^{\alpha} C_{b\lambda}^{\alpha},$$

<sup>1)</sup> Cf. SCHOUTEN 1926, 1.

hence, according to (4.13)

$$(4.19) \quad R_{\nu\mu b}^{\cdot\cdot\cdot a} C_{a\lambda}^{\cdot\cdot\cdot x} = B_{\nu\mu}^{\cdot\cdot\cdot c} (C_{c\lambda}^{\cdot\cdot\cdot\sigma} C_{b\sigma}^{\cdot\cdot\cdot x} - C_{b\lambda}^{\cdot\cdot\cdot\sigma} C_{c\sigma}^{\cdot\cdot\cdot x}).$$

Since  $C_{b\lambda}^{\cdot\cdot\cdot x}$  has maximum  $b$ -rank, this implies that there exist uniquely determined coefficients  $c_{cb}^{\cdot\cdot\cdot a}$  such that

$$(4.20) \quad \begin{cases} a) & C_{c\lambda}^{\cdot\cdot\cdot\sigma} C_{b\sigma}^{\cdot\cdot\cdot x} - C_{b\lambda}^{\cdot\cdot\cdot\sigma} C_{c\sigma}^{\cdot\cdot\cdot x} = c_{cb}^{\cdot\cdot\cdot a} C_{a\lambda}^{\cdot\cdot\cdot x} \\ b) & R_{\nu\mu b}^{\cdot\cdot\cdot a} = B_{\nu\mu}^{\cdot\cdot\cdot c} c_{cb}^{\cdot\cdot\cdot a}. \end{cases}$$

It follows from (4.12, 20a) that

$$(4.21) \quad (X_c X_b) = 2 C_{[c|\lambda|}^{\cdot\cdot\cdot\sigma} x^\lambda \frac{\partial}{\partial x^\sigma} C_{b]\mu}^{\cdot\cdot\cdot\tau} x^\mu \frac{\partial}{\partial x^\tau} = c_{cb}^{\cdot\cdot\cdot a} X_a$$

and this proves that the  $c_{cb}^{\cdot\cdot\cdot a}$  are at each point of  $L_n$  the structural constants of the holonomy group at that point. In fact they satisfy the identities (cf. IV § 2)

$$(4.22) \quad a) \quad c_{(cb)}^{\cdot\cdot\cdot a} = 0 \quad b) \quad c_{[a;c}^{\cdot\cdot\cdot e} c_{b]e}^{\cdot\cdot\cdot a} = 0$$

as can be checked easily by means of (4.20a). By covariant differentiation of (4.20a) we get because of (4.14)

$$(4.23) \quad \nabla_\omega c_{cb}^{\cdot\cdot\cdot a} = 0.$$

The  $c_{cb}^{\cdot\cdot\cdot a}$  are not in general constants throughout  $L_n$  as follows from the fact that the coordinates in the tangent  $E_r$ 's are chosen quite arbitrarily at every point. But if, starting from  $\xi^x$  we displace the local coordinate system in  $E_r$  parallel from  $\xi^x$  along some curve to an arbitrary point  $\xi^x$ , the  $c_{cb}^{\cdot\cdot\cdot a}$  remain constant. If the system returns to  $\xi^x$  by some way or other, always moving parallel, the local coordinate system has suffered in the end a linear homogeneous transformation but the initial and the final values of  $c_{cb}^{\cdot\cdot\cdot a}$  are the same. Hence the coordinate systems in the tangent  $E_r$ 's can always be chosen in such a way that the  $c_{cb}^{\cdot\cdot\cdot a}$  are constants over  $L_n$ , but it is not necessary to do so.

Now let  $C$  be a closed curve through  $\xi^x$  and let the tangent  $E_n$  and the tangent  $E_r$  at  $\xi^x$  be displaced parallel along  $C$  until they are back at  $\xi^x$ . Let a general vector  $u^x$  at  $\xi^x$  be changed by this process into

$$(4.24) \quad 'u^x = T_{\lambda}^{\cdot\cdot\cdot x} u^\lambda;$$

then  $T$  is a general transformation of the holonomy group and every transformation of this group can be obtained in this way. If  $C_\lambda^{\cdot\cdot\cdot x}$  is any

generator of the group at  $\xi_\theta^x$ , this quantity changes into

$$(4.25) \quad 'C_{\lambda}^{;x} = \bar{T}_{\lambda}^{; \sigma} C_{\sigma}^{; \theta} T_{\theta}^{; x},$$

and this equation thus represents a transformation of the linear adjoint group, that is the adjoint group applied to the infinitesimal transformations of the holonomy group only (cf. IV § 2). Now for every choice of  $C_{\lambda}^{;x}$  we always have  $C_{\lambda}^{;x} = e^b C_{b\lambda}^{;x}$  with uniquely determined constants  $e^b$  and the infinitesimal transformation represented by  $C_{\lambda}^{;x}$  is at the same time represented by the vector  $e^b$  in the tangent  $E_r$ . The quantity  $C_{b\lambda}^{;x}$  is covariant constant, i.e. it remains unchanged if it is displaced parallel along  $C$ . Hence we may write  $'C_{\lambda}^{;x} = e^b C_{b\lambda}^{;x}$  and then we have

$$(4.26) \quad 'e^b C_{b\lambda}^{;x} = \bar{T}_{\lambda}^{; \sigma} e^b C_{\sigma}^{; \theta} T_{\theta}^{; x}.$$

The  $b$ -rank of  $C_{b\lambda}^{;x}$  has the highest possible value  $r$  and therefore at least one quantity  $\bar{C}_{\cdot\cdot\lambda}^{;x}$  must exist such that

$$(4.27) \quad C_{b\theta}^{; \sigma} \bar{C}_{\cdot\cdot\lambda}^{; \sigma} = A_b^{\alpha}.$$

By means of this quantity we get from (4.26)

$$(4.28) \quad 'e^a = \bar{T}_{\lambda}^{; \sigma} C_{b\sigma}^{; \theta} T_{\theta}^{; x} \bar{C}_{\cdot\cdot\lambda}^{; x} e^b$$

and this equation expresses how the vector  $e^a$  is transformed if the infinitesimal transformations of the holonomy group suffer a transformation  $T$  of the linear adjoint group.

In the same way as above it could now be proved that the generators of the linear adjoint group span the  $\theta$ -domain of the set

$$(4.29) \quad R_{v\mu b}^{; ; a}; \quad \nabla_{\omega} R_{v\mu b}^{; ; a}; \dots; \nabla_{\omega_k \dots \omega_1} R_{v\mu b}^{; ; a}; \dots$$

and, according to (4.14, 15, 20b) also the  $\theta$ -domain of the set

$$(4.30) \quad B_{v\mu}^{; ; c} c_{cb}^{; a}; \quad \dot{B}_{\omega v\mu}^{; ; c} c_{cb}^{; a}; \dots; \dot{B}_{\omega_k \dots \omega_1 v\mu}^{; ; c} c_{cb}^{; a}; \dots$$

But because the set of all  $B$ 's has the maximum  $c$ -rank  $r$ , this proves that the generators of the linear adjoint group span the  $\theta$ -domain of  $c_{cb}^{; a}$  exactly, a result well known from IV § 2.

The holonomy group of an  $L_n$  is intimately connected with covariant constant fields of quantities. We prove the theorem<sup>1)</sup>:

<sup>1)</sup> WAGNER 1943, 1; 1946, 2. This paper deals with general geometric objects. The proof is from NIJENHUIS 1952, 1, Ch. II § 13.

*A covariant constant field of a quantity with a given kind of transformation in an  $L_n$  is possible if and only if there exists a quantity of this kind which is invariant for the holonomy group.*

The proof is given here for a tensor  $P_{\cdot\lambda}^*$  of valence two. The generalization is obvious. The equation

$$(4.31) \quad \nabla_\mu P_{\cdot\lambda}^* = 0$$

has the first integrability condition

$$(4.32) \quad R_{\nu\mu\sigma}^{*\cdot\lambda} P_{\cdot\lambda}^\sigma - R_{\nu\mu\lambda}^{*\cdot\sigma} P_{\cdot\sigma}^* = 0$$

and the following integrability conditions are obtained by successive covariant differentiation of (4.32) (cf. III § 6):

$$(4.33) \quad (\nabla_{\omega_k \dots \omega_1} R_{\nu\mu\sigma}^{*\cdot\lambda}) P_{\cdot\lambda}^\sigma - (\nabla_{\omega_k \dots \omega_1} R_{\nu\mu\lambda}^{*\cdot\sigma}) P_{\cdot\sigma}^* = 0; \quad k = 1, 2 \dots .$$

Because of (4.13) the conditions (4.32, 33) are equivalent to

$$(4.34) \quad C_{b\sigma}^{*\cdot\lambda} P_{\cdot\lambda}^\sigma - C_{b\lambda}^{*\cdot\sigma} P_{\cdot\sigma}^* = 0$$

and this equation expresses the invariance of  $P_{\cdot\lambda}^*$  for the holonomy group.<sup>1)</sup>

If the  $L_n$  is a  $V_n$  special results can be obtained. Instead of (4.13 a) we get

$$(4.35) \quad K_{\nu\mu\lambda\kappa} = B_{\nu\mu}^{*\cdot b} C_{b\lambda\kappa};$$

hence, according to the fourth identity (cf. III § 5)

$$(4.36) \quad B_{\nu\mu}^{*\cdot b} C_{b\lambda\kappa} = B_{\lambda\kappa}^{*\cdot b} C_{b\nu\mu}.$$

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<sup>1)</sup> There are a great number of investigations on holonomy groups that leave certain geometric objects invariant. We give here some references. *Symmetric tensor of valence two in  $V_n$ :* LEVY 1925, 2; SAMUEL 1945, 1; LICHNEROWICZ 1950, 1; *Bivector in  $V_n$ :* IWAMOTO 1950, 1; LICHNEROWICZ 1950, 2. *Multivector in  $V_n$ :* LICHNEROWICZ 1951, 1; 2; IWAMOTO 1951, 1;  *$E_p$ -field in  $A_n$  and  $V_n$ :* ABE 1944, 1; 2; RUSE 1949, 1; 1950, 1; WALKER 1949, 1; 1950, 3; 4; YANO and SASAKI 1948, 3; WILLMORE 1951, 1; *Pointfield in  $V_n$ :* TACHIBANA 1949, 1; *Hyperquadric in a normal projective connexion:* SASAKI and YANO 1949, 1; ÔTSUKI 1950, 1; 1952, 1; *Point or hypersphere in a normal conformal connexion:* SASAKI 1943, 1; 2; 3; YANO and SASAKI 1947, 2. Connected with these investigations are a number of papers on classification of connexions by means of the holonomy group: SASAKI 1942, 1 (conform.); 1943, 1; 2; 3 (conform.); 1951, 1 (one-parameter groups); 1951, 2 (global); YANO 1943, 5 (conform.); YANO and SASAKI 1944, 13 (conform.); 1946, 4 (conform.); 1947, 2 (conform.); 1948, 3; WAGNER 1949, 2 ( $L_n$ ); ÔTSUKI 1949, 1; 2 ( $V_4$ ); 1950, 2 (conform.); 1950, 3 ( $W_4$ ); 1951, 1 (conform.); LIBER 1949, 1 (one-parameter groups); DEBEVER 1950, 1 (conform.); KANITANI 1950, 1; 1952, 1 (projective); KUIPER 1951, 2; TAKIZAWA 1951, 1; HASHIMOTO 1951, 1; KURITA 1952, 1; WAKAKUWA 1952, 1; NIJENHUIS 1953, 1. Further literature is given in these papers.

But this implies that the  $\nu\mu$ -domain of  $B_{\nu\mu}^{;a}$  and of  $C_{b\nu\mu}$  are identical. Hence there must exist equations of the form

$$(4.37) \quad B_{\nu\mu}^{;a} = a^{ab} C_{b\nu\mu}$$

where  $a^{ab}$  is a uniquely determined symmetric tensor that can be computed from (4.35). Since the  $b$ -rank of  $C_{b\nu\mu}$  is  $r$ , the rank of  $a^{ab}$  is equal to the  $a$ -rank of  $B_{\nu\mu}^{;a}$ , and to the  $\lambda\kappa$ -rank of  $K_{\nu\mu\lambda\kappa}$ .

*The rank of the symmetric tensor  $a^{ab}$  has the maximum value  $r$  if and only if the  $\lambda\kappa$ -rank of  $K_{\nu\mu\lambda\kappa}$  equals  $r$ , or in other words, if the  $\lambda\kappa$ -domain of  $V_\omega K_{\nu\mu\lambda\kappa}$  lies in the  $\lambda\kappa$ -domain of  $K_{\nu\mu\lambda\kappa}$ .<sup>1)</sup>*

From (4.37) and (4.13) it follows that

$$(4.38) \quad K_{\nu\mu\lambda\kappa} = a^{ab} C_{a\nu\mu} C_{b\lambda\kappa}$$

and

$$(4.39) \quad \overset{b}{B}_{\omega_k \dots \omega_1 \nu \mu}^{; ; a} = C_{b\nu\mu} V_{\omega_k \dots \omega_1} a^{ab}$$

and from this latter equation we see that the  $a$ -rank of the set consisting of  $a^{ab}$  and all its covariant derivatives equals  $r$ .

As an example we consider a  $V_n$  with recurrent curvature tensor (cf. VIII § 9). This is a  $V_n$  with a vector field  $k_\omega$  such that

$$(4.40) \quad V_\omega K_{\nu\mu\lambda\kappa} = k_\omega K_{\nu\mu\lambda\kappa}; \quad k_\omega \neq 0.$$

From (4.38, 40) it follows that

$$(4.41) \quad V_\omega a^{ab} = k_\omega a^{ab}$$

and from BIANCHI's identity, because  $a^{ab}$  has rank  $r$

$$(4.42) \quad C_{b[\nu\mu} k_{\omega]} = 0.$$

But this equation expresses that for every choice of  $b$  the bivector  $C_b$  is simple and has a 2-direction containing the direction of  $k_\omega$ . This implies that  $C_{b\nu\mu}$  can be written in the form

$$(4.43) \quad C_{b\nu\mu} = 2u_{b[\nu} k_{\mu]}$$

in which  $u_{b\nu}$  is uniquely determined to within an additive term of the form  $u_b k_\nu$  with an arbitrary vector  $u_b$ . Now we may derive from (4.40)

$$(4.44) \quad 2V_{[\sigma\sigma]} K_{\nu\mu\lambda\kappa} = 2(V_{[\sigma} k_{\sigma]}) K_{\nu\mu\lambda\kappa}$$

but on the other hand we have (cf. III 4.9 a)

$$(4.45) \quad 2V_{[\sigma\sigma]} K_{\nu\mu\lambda\kappa} = -2K_{\sigma\sigma}^{;[\nu} K_{|\tau|\mu]\lambda\kappa} - 2K_{\sigma\sigma}^{;[\lambda} K_{|\nu\mu\tau|\kappa]}.$$

<sup>1)</sup> NIJENHUIS, personal communication.

Hence  $\nabla_{[\sigma} k_{\rho]}$  lies in the  $\sigma\rho$ -domain of  $K_{\sigma\epsilon\lambda\nu}$ , and thus in the  $\sigma\rho$ -domain of  $C_{b\sigma\rho}$  also, and this implies that there is an equation of the form

$$(4.46) \quad \nabla_{[\sigma} k_{\rho]} = 'k^b C_{b\sigma\rho} = 2'k^b u_{b[\sigma} k_{\rho]}.$$

This proves that  $k_\lambda$  is a gradient to within a scalar factor (cf. II § 7). Substituting (4.46) and (4.38) in (4.44, 45) we get

$$(4.47) \quad \left\{ \begin{array}{l} 'k^a C_{a\sigma\rho} a^{c\mu} C_{c\nu\mu} C_{d\lambda\nu} = -a^{ab} C_{a\sigma\rho} C_{b[\nu}{}^\tau a^{c\mu} C_{c\tau]\mu} C_{d\lambda\nu} - \\ \quad - a^{ab} C_{a\sigma\rho} C_{b[\lambda}{}^\tau a^{c\mu} C_{c\nu\mu} C_{d\tau]\nu} \end{array} \right.$$

from which it follows immediately by writing out

$$(4.48) \quad 'k^{(a} a^{c)d)} C_{a\sigma\rho} C_{c\nu\mu} C_{d\lambda\nu} = 0$$

which is only possible if

$$(4.49) \quad 'k^{(a} a^{c)d)} = 0$$

that is, if  $'k^a$  is zero. This proves that  $k_\lambda$  is a gradient vector.<sup>1)</sup> If we write  $k_\lambda = -\nabla_\lambda \log \sigma$  we get from (4.41)

$$(4.50) \quad \nabla_\mu \sigma a^{ab} = 0.$$

The  $C_{b\lambda}{}^\nu$ ;  $b = \dot{1}, \dots, \dot{r}$  are the generators of the holonomy group, hence, from (4.20a) and (4.43)

$$(4.51) \quad \left\{ \begin{array}{l} (u_{c\lambda} k^\sigma - u_{c}{}^\sigma k_\lambda) (u_{b\sigma} k^\nu - u_b{}^\nu k_\sigma) - \\ \quad - (u_{b\lambda} k^\sigma - u_b{}^\sigma k_\lambda) (u_{c\sigma} k^\nu - u_c{}^\nu k_\sigma) = c_{cb}{}^a (u_{a\lambda} k^\nu - u_a{}^\nu k_\lambda). \end{array} \right.$$

Now the  $r$  vectors  $u_{b\lambda}$ ;  $b = \dot{1}, \dots, \dot{r}$  and  $k_\lambda$  are linearly independent because  $C_{b\mu\nu}$  has  $b$ -rank  $r$  (cf. 4.43). Hence (4.51) is equivalent to

$$(4.52) \quad \left\{ \begin{array}{l} a) \quad k_\sigma k^\sigma u_{[c|\lambda]} u_{b]}{}^\nu = 0 \\ b) \quad 2u_{[c}{}^\sigma u_{b]}{}^\nu k_\lambda k_\sigma = \quad c_{cb}{}^a u_a{}^\nu k_\lambda. \end{array} \right.$$

If  $k_\sigma k^\sigma \neq 0$  this implies that

$$(4.53) \quad u_{[c|\lambda]} u_{b]}{}^\nu = 0; \quad k_\sigma k^\sigma \neq 0$$

and if this substituted in (4.52b) we get  $c_{cb}{}^a = 0$  because  $u_b{}^\nu$  has  $b$ -rank  $r$ . Now for  $r > 1$ , (4.53) would imply that  $u_1{}^\lambda$  and  $u_2{}^\lambda$  were linearly dependent, hence  $r = 1$ . This proves<sup>2)</sup>

<sup>1)</sup> WALKER 1950, 1, the proof given here is from NIJENHUIS (personal communication). Cf. RUSE 1946, 3; 1948, 2; 1949, 2. If the scalar curvature is not zero the proposition can also be proved by transvecting (4.40) with  $g^{\nu\lambda} g^{\mu\lambda}$ .

<sup>2)</sup> NIJENHUIS, personal communication.

*The holonomy group of a  $V_n$  with a recurrent curvature tensor is a one-parameter group if  $k^x$  is not a nullvector.*

Now let us suppose that  $k^x$  is not a nullvector and that the 2-direction of the simple bivector  $C_b^{;\alpha\lambda}$ ;  $b=1$  is not tangent to the nullcone. Then the holonomy group consists of all rotations in the tangent  $R_n$  around this 2-direction. This means that the 2-direction and the  $(n-2)$ -direction perpendicular to it are covariant constant because both are invariant for the holonomy group. But this implies that the  $V_n$  is a product  $V_2 \times V_{n-2}$  (cf. V § 11). The  $V_2$ 's are geodesic and parallel and their tangent  $R_2$  is spanned by  $u_b^{\alpha}$  and  $k^x$  (cf. 4.43). The  $V_{n-2}$ 's are also geodesic and parallel and their curvature tensor is the  $V_{n-2}^{;\alpha\beta}$ -part of the curvature tensor in  $V_n$  (cf. V § 10, GAUSS). But according to (4.38) this part is zero. Hence

*If in a  $V_n$  with a recurrent curvature tensor the vector  $k_\lambda$  is not a nullvector and if the simple bivector of the rotations of the one-parameter holonomy group is not tangent to the nullcone, the  $V_n$  is a product  $V_2 \times R_{n-2}$ .<sup>1)</sup>*

Note that accordingly every ordinary  $V_n$  with a recurrent curvature tensor is a product  $V_2 \times R_{n-2}$ .<sup>2)</sup>

### Exercise.

VII 4,1<sup>3)</sup>. The non-homogeneous holonomy group of a  $V_n$  leaves a point invariant if and only if there exists a concurrent contravariant vector field (cf. VI § 9).

It leaves a direction invariant if and only if there exists a covariant constant contravariant vector field.

## § 5. Affine motions and the holonomy group in a symmetric $A_n$ .

In a symmetric  $A_n$  (cf. III § 7) the covariant derivative of  $R_{\nu\mu\lambda}^{;\alpha}$  vanishes and consequently the holonomy group is generated by the  $\lambda$ -domain of  $R_{\nu\mu\lambda}^{;\alpha}$  only. This means that in the factorization (4.13 a)

$$(5.1) \quad R_{\nu\mu\lambda}^{;\alpha} = B_{\nu\mu}^{;\alpha} C_{\alpha\lambda}^{;\alpha}$$

the  $a$ -rank of  $B_{\nu\mu}^{;\alpha}$  and the  $a$ -rank of  $C_{\alpha\lambda}^{;\alpha}$  both are maximum and equal to the number  $r$  of parameters of the holonomy group. Hence the quan-

<sup>1)</sup> WALKER 1950, 1, p. 46. In this paper the author gave a detailed discussion of all other cases. He proved that the  $V_n$  can also be a product  $V_3 \times R_{n-3}$  or  $V_4 \times R_{n-4}$  where  $V_3$  and  $V_4$  have also recurrent curvature tensors. The proof given here is from NIJENHUIS (personal communication).

<sup>2)</sup> We mention some further recent publications on holonomy groups: LAPTEV 1950, 1; SASAKI 1951, 2 (symmetric spaces); KANITANI 1952, 1; BOREL and LICHNEROWICZ 1952, 2 ( $V_n$ ).

<sup>3)</sup> YANO 1943, 3.

tities  $\dot{B}$  in (4.13 b) vanish. The equations (4.13 a), (4.14) and (4.16–23) remain valid but instead of (4.15) we have

$$(5.2) \quad \nabla_\omega B_{\nu\mu}^{\cdot\lambda} = 0$$

and there is another identity

$$(5.3) \quad C_{c\nu}^{\cdot\lambda} B_{\mu\lambda}^{\cdot a} - C_{c\mu}^{\cdot\lambda} B_{\nu\lambda}^{\cdot a} = c_{cb}^{\cdot a} B_{\nu\mu}^{\cdot b}$$

that can be derived from  $\nabla_{[\omega} \nabla_{\nu]} B_{\mu\lambda}^{\cdot a} = 0$  because the  $a$ -rank of  $B_{\nu\mu}^{\cdot a}$  is  $r$ .

The equations for the group of affine motions are now [cf. (2.1)]

$$(5.4) \quad \begin{array}{ll} \text{a)} & \boxed{\nabla_\mu v_\lambda^{\cdot x} = -v^\nu R_{\nu\mu\lambda}^{\cdot x}} \\ \text{b)} & \boxed{\nabla_\lambda v^x = v_\lambda^{\cdot x}} \end{array}$$

and the integrability conditions [cf. (2.2, 3)] reduce to

$$(5.5) \quad v_\nu^{\cdot\sigma} R_{\sigma\mu\lambda}^{\cdot x} + v_\mu^{\cdot\sigma} R_{\nu\sigma\lambda}^{\cdot x} + v_\lambda^{\cdot\sigma} R_{\nu\mu\sigma}^{\cdot x} - v_\rho^{\cdot x} R_{\nu\mu\lambda}^{\cdot\rho} = 0.$$

This equation expresses the fact that in the tangent  $E_n$  the tensor  $R_{\nu\mu\lambda}^{\cdot x}$  is invariant for the infinitesimal transformation  $x^x \rightarrow x^x + v_\lambda^{\cdot x} x^\lambda dt$ . If (5.1) is substituted in (5.5) we get

$$(5.6) \quad (v_\nu^{\cdot\sigma} B_{\sigma\mu}^{\cdot a} + v_\mu^{\cdot\sigma} B_{\nu\sigma}^{\cdot a}) C_{a\lambda}^{\cdot x} + B_{\nu\mu}^{\cdot a} (v_\lambda^{\cdot\sigma} C_{a\sigma}^{\cdot x} - v_\rho^{\cdot x} C_{a\lambda}^{\cdot\rho}) = 0.$$

But because  $C_{a\lambda}^{\cdot x}$  and  $B_{\nu\mu}^{\cdot a}$  both have maximum  $a$ -rank, (5.6) can only be true if there exist uniquely determined coefficients  $v_b^{\cdot a}$  such that

$$(5.7) \quad \begin{array}{ll} \text{a)} & \boxed{v_\mu^{\cdot\eta} B_{\eta\lambda}^{\cdot a} + v_\lambda^{\cdot\eta} B_{\mu\eta}^{\cdot a} - v_b^{\cdot a} B_{\mu\lambda}^{\cdot b} = 0} \\ \text{b)} & \boxed{v_c^{\cdot a} C_{a\lambda}^{\cdot x} + v_\lambda^{\cdot\eta} C_{c\eta}^{\cdot x} - v_\sigma^{\cdot x} C_{c\lambda}^{\cdot\sigma} = 0.} \end{array}$$

The  $v_b^{\cdot a}$  are the components of a tensor in the auxiliary tangent  $E_r$ 's introduced in VII § 4 and (5.7) expresses the fact that in the tangent  $E_{n+r}$  spanned by the tangent  $E_n$  and the tangent  $E_r$ , the quantities  $B_{\mu\lambda}^{\cdot a}$  and  $C_{a\lambda}^{\cdot x}$  are invariant for the infinitesimal transformation

$$(5.8) \quad x^x \rightarrow x^x + v_\lambda^{\cdot x} x^\lambda dt; \quad x^a \rightarrow x^a + v_b^{\cdot a} x^b dt$$

in the tangent  $E_{n+r}$ . Moreover (5.7) is another form of the integrability conditions (5.5). As to  $C_{a\lambda}^{\cdot x}$  this means that the holonomy group is invariant for all affine motions. Of course this is also true in a general  $A_n$  because a motion does not change the geometry in the space and consequently can not change the holonomy group. If  $p^a$  is arbitrary,

$\dot{p}^a C_{\dot{a}\dot{\lambda}}^{\dot{x}}$  is an arbitrary generator of the group. Taking the LIE derivative of this quantity we get according to (5.7b)

$$(5.9) \quad \mathcal{L}_v \dot{p}^a C_{\dot{a}\dot{\lambda}}^{\dot{x}} = C_{\dot{a}\dot{\lambda}}^{\dot{x}} (v^\mu V_\mu \dot{p}^a - \dot{p}^c v_c^a)$$

and this proves in fact that the totality of all infinitesimal transformations of the holonomy group is invariant for affine motions.

If  $V_\mu$  is applied to (5.7b) it follows, using (5.4a) and (4.18), that

$$(5.10) \quad \boxed{V_\mu v_b^a = -v^\nu R_{\nu\mu b}^{a\dot{c}} c_{\dot{c}\dot{b}}^a}.$$

From (5.7) and (4.20a) it can be deduced that

$$(5.11) \quad \boxed{v_c^d c_{\dot{d}\dot{b}}^a + v_b^d c_{\dot{c}\dot{d}}^a - v_d^a c_{\dot{c}\dot{b}}^d = 0}$$

and this equation expresses the fact that in the tangent  $E$ , the quantity  $c_{\dot{c}\dot{b}}^a$  is invariant for the infinitesimal transformation  $x^a \rightarrow x^a + v_b^a x^b dt$ .

The equations (5.4, 10) and (5.7) are equivalent to (5.4, 5) and thus (cf. VII § 2) form a totally integrable system with the unknowns  $v^x$ ,  $v_{\dot{\lambda}}^x$  and  $v_b^a$ . In order to prove that the solutions of this system generate a group we can use the same way as was followed in VII § 2. Thus we extend the LIE operator  $\mathcal{L}_v$  to the  $X_{n+r}$ , formed by the  $\infty^n$  tangent  $E$ 's. Of course this can be done in several ways, but if we put the condition that  $\mathcal{L}_v C_{\dot{c}\dot{\lambda}}^{\dot{x}} = 0$  we get uniquely for a vector of  $E$ , [for instance for the contravariant vector  $\dot{p}^a$  from (5.9)]

$$(5.12) \quad \begin{cases} a) & \mathcal{L}_v \dot{p}^a = v^\mu V_\mu \dot{p}^a - v_b^a \dot{p}^b = v^\mu \partial_\mu \dot{p}^a + v^\mu \Gamma_{\mu b}^a \dot{p}^b - v_b^a \dot{p}^b \\ b) & \mathcal{L}_v q_b = v^\mu V_\mu q_b + v_b^a q_a = v^\mu \partial_\mu q_b - v^\mu \Gamma_{\mu b}^a q_a + v_b^a q_a. \end{cases}$$

It is interesting to look at these equations from the point of view of the  $X_{n+r}$ . The general formula for the LIE-derivative of a vector  $u^A$ ;  $A = 1, \dots, n, 1, \dots, r$  in this space with respect to the field  $v^A$ , resulting from  $v^x$  and  $v^a$ , is

$$(5.13) \quad \begin{cases} a) & \mathcal{L}_v u^x = v^\mu \partial_\mu u^x + v^c \partial_c u^x - u^\mu \partial_\mu v^x - u^c \partial_c v^x \\ b) & \mathcal{L}_v u^a = v^\mu \partial_\mu u^a + v^c \partial_c u^a - u^\mu \partial_\mu v^a - u^c \partial_c v^a. \end{cases}$$

For a field  $u^A$  with  $u^a = 0$ , only dependent on the  $\xi^x$ , this gives the ordinary expression for  $\mathcal{L}_v u^x$ . But for a field with  $u^x = 0$  and  $u^a$  depending on

the  $\xi^x$  only, we get

$$(5.14) \quad \begin{cases} \text{a)} & \frac{\mathcal{L}}{v} u^x = 0 \\ \text{b)} & \frac{\mathcal{L}}{v} u^a = v^\mu \partial_\mu u^a - u^c \partial_c v^a \end{cases}$$

because  $v^x$  depends on the  $\xi^x$  only. Now (5.14b) is in accordance with (5.12a) if

$$(5.15) \quad v_b^*{}^a = \partial_b v^a + v^\mu I_{\mu b}^a.$$

This means that the LIE derivative  $\frac{\mathcal{L}}{v} p^a$  in (5.12a) is the same as would have been found if the vector field  $v^x$  were extended with a field  $v^a$ , dependent on the  $\xi^x$  and the  $x^a$ , to  $v^x, v^a$ , in such a way that  $\partial_b v^a$  were satisfying (5.15). This implies that

$$(5.16) \quad \frac{\mathcal{L}}{v} V_\mu - V_\mu \frac{\mathcal{L}}{v} = 0$$

for all quantities of  $X_{n+}$ , but this equation can also be derived directly from (4.14, 22) and (5.1, 7, 11). It is very remarkable that  $v^a$  itself does not occur in (5.14b) but only its derivative  $\partial_b v^a$ . Hence, if  $v^a$  is an arbitrary field not depending on the  $x^a$ , the field  $v^a = v^a + (v_b^*{}^a - v^\mu I_{\mu b}^a) x^b$  satisfies the condition (5.15).

The group property can now be proved by showing that  $w^x \stackrel{\text{def}}{=} \frac{\mathcal{L}}{u} v^x$  is a solution of (5.4, 7, 10, 11) if  $u^x$  and  $v^x$  are solutions. This is done by substituting in these equations

$$(5.17) \quad \begin{cases} \text{a)} & w^x \stackrel{\text{def}}{=} \frac{\mathcal{L}}{u} v^x = u^\mu V_\mu v^x - v^\mu V_\mu u^x = u^\mu v_\mu^x - v^\mu u_\mu^x \\ \text{b)} & w_\lambda^x \stackrel{\text{def}}{=} V_\lambda w^x = V_\lambda \frac{\mathcal{L}}{u} v^x = \frac{\mathcal{L}}{u} v_\lambda^x = u^\nu v^\mu R_{\nu\mu}^x + v_\sigma^x u_\lambda^\sigma - u_\sigma^x v_\lambda^\sigma \\ \text{c)} & w_b^a \stackrel{\text{def}}{=} \frac{\mathcal{L}}{u} v_b^a = u^\nu v^\mu R_{\nu\mu b}^a + v_c^a u_b^c - u_c^a v_b^c. \end{cases}$$

The equations (5.17) give the composition of two infinitesimal affine motions. If we drop the curvature tensor in (5.17b, c) these equations give the well known composition of affine motions in  $E_n$  where  $u^x dt$ ,  $v^x dt$  and  $w^x dt$  are the displacements and  $u_\lambda^x dt$ ,  $v_\lambda^x dt$  and  $w_\lambda^x dt$  the affine transformations of the tangent  $E_n$ 's.<sup>1)</sup> From (5.17a) it follows that the affine motions leaving one arbitrary point  $\xi^x$  invariant, form a group. This group is called following CARTAN the *group of isotropy with respect to  $\xi^x$* .

We prove now that the *group of affine motions in a symmetric  $A_n$  is transitive and that it has at least  $n+r$  parameters, where  $r$  is the number of parameters of the holonomy group*.

<sup>1)</sup> Some authors call them "affine rotations".

The equations (5.4, 7, 10, 11) are totally integrable, hence there exists a solution for every set of initial values  $v_{\theta}^{\alpha}$ ,  $v_{\lambda}^{\alpha}$  and  $v_b^{\alpha}$  at  $\xi^{\alpha}$ , that satisfies (5.7, 11). But because  $v^{\alpha}$  does not occur in these conditions,  $v_{\theta}^{\alpha}$  can be chosen arbitrarily and this proves that the group is transitive. Now according to (4.20a, 22b; 5.3) the set defined by

$$(5.18) \quad v_{\lambda}^{\alpha} \stackrel{\text{def}}{=} v^c C_{c\lambda}^{\alpha}; \quad v_b^{\alpha} \stackrel{\text{def}}{=} v^c c_{cb}^{ca};$$

satisfies (5.7, 11) for every choice of the constants  $v^{\alpha}$ . This proves that the group has at least  $n+r$  parameters. The group with the parameters  $v_{\theta}^{\alpha}$ ,  $v^{\alpha}$  defined by (5.18) is an  $(n+r)$ -parameter subgroup of the group of affine motions. First we remark that (5.18) implies that equations of the same form

$$(5.19) \quad \text{a)} \quad v_{\lambda}^{\alpha} = v^c C_{c\lambda}^{\alpha}; \quad \text{b)} \quad v_b^{\alpha} = v^c c_{cb}^{ca}$$

with suitable values of  $v^c$ , hold at every point of  $A_n$ . In fact, according to (5.1, 4, 10) and (4.14, 20b) we have

$$(5.20) \quad \text{a)} \quad C_{c\lambda}^{\alpha} \nabla_{\mu} v^c = -v^{\nu} B_{\nu\mu}^{ca} C_{c\lambda}^{\alpha}; \quad \text{b)} \quad c_{cb}^{ca} \nabla_{\mu} v^c = -v^{\nu} B_{\nu\mu}^{ca} c_{cb}^{ca}$$

or

$$(5.21) \quad \nabla_{\mu} v^a = -v^{\nu} B_{\nu\mu}^{ca}$$

and the integrability conditions of this equation are identically satisfied because according to (4.20b) and (5.3)

$$(5.22) \quad \left\{ \begin{array}{l} 2 \nabla_{[\nu} \nabla_{\mu]} v^a = -2(\nabla_{[\nu} v^{\omega}) B_{|\omega|\mu]}^{ca} = -2v^c C_{c[\nu}^{\omega} B_{|\omega|\mu]}^{ca} \\ = v^c c_{cb}^{ca} B_{\nu\mu}^{cb} = v^b c_{bc}^{ca} B_{\nu\mu}^{bc} = R_{\nu\mu}^{ca} v^b. \end{array} \right.$$

That proves that there is one and only one field  $v^a$  that satisfies (5.19, 21) and takes at  $\xi^{\alpha}$  the values  $v_{\theta}^{\alpha}$  satisfying (5.18).

Now we prove that the subgroup defined by (5.18) is an *invariant* subgroup of the group of affine motions (cf. VII § 2). Therefore we substitute (5.19) in (5.4, 10) and (5.7, 11). Then (5.7, 11) are satisfied identically as a consequence of (4.20a, 22b; 5.3), and (5.4, 10) reduces to the set

$$(5.23) \quad \left\{ \begin{array}{l} \nabla_{\lambda} v^{\alpha} = v^c C_{c\lambda}^{\alpha} \\ \nabla_{\mu} v^a = -v^{\nu} B_{\nu\mu}^{ca} \end{array} \right.$$

which is of course totally integrable. If now  $u^{\alpha}$  is the vector of an affine motion not necessarily belonging to the subgroup it follows

from (5.17, 23) that  $\underline{\underline{v}}^{\alpha}$ ,  $\underline{\underline{v}}^{\mu}$  also satisfy (5.23) and this means that the subgroup is invariant.

**Exercise.**

VII 5.1. Prove (5.22) by writing out  $2V_{[\nu}V_{\mu]}v_{\lambda}^{\alpha}$  in two different ways and using (4.18) (use the rule of LEIBNIZ for  $V_{[\nu}V_{\mu]}$ ).

## § 6. CARTAN's method applied to the holonomy group and the symmetric $A_n$ .

In order to give further examples of CARTAN's symbolical method we apply this method here independently of VII § 4, 5 to the homogeneous holonomy group. The equations of VII § 4, 5 will not be used in this section but results will be compared.

Starting from any point  $\xi^{\alpha}$  of an  $L_n$  we fix an anholonomic coordinate system  $(h)$  at all points of an  $\mathfrak{N}(\xi^{\alpha})$  by parallel displacement of some given local coordinate system in the tangent  $E_n$  of  $\xi^{\alpha}$  along any definitely chosen curves to all points of  $\mathfrak{N}(\xi^{\alpha})$ . If this local system at  $\xi^{\alpha}$  suffers a linear homogeneous transformation, the local coordinate systems of  $(h)$  at all points are transformed in the same way and all these transformations are even numerically the same if they are expressed at all points with respect to  $(h)$ . When this has been done we consider now at every point of  $\mathfrak{N}(\xi^{\alpha})$  not only the local coordinate systems of  $(h)$  but all coordinate systems derivable from them by means of all transformations of the holonomy group. The whole set of all local coordinate systems obtained in this way will be called the *allowable local coordinate systems*.

If from this set we choose in an arbitrary way one local coordinate system at every point of  $\mathfrak{N}(\xi^{\alpha})$  we get an *allowable anholonomic coordinate system* in that region.

It is clear that the set is independent of the choice of the curves used above. If the holonomy group is an  $r$ -parameter group, the local coordinate systems depend at each point on  $r$  parameters. If  $\xi_1^{\alpha}$  and  $\xi_2^{\alpha}$  are two points of  $\mathfrak{N}(\xi^{\alpha})$  and if the tangent  $E_n$  at  $\xi_1^{\alpha}$  is displaced parallel from  $\xi_1^{\alpha}$  to  $\xi_2^{\alpha}$  along some arbitrary curve, the local coordinates of the points of  $E_n$  at  $\xi_1^{\alpha}$  with respect to some allowable local coordinate system and the local coordinates of the displaced points in the  $E_n$  at  $\xi_2^{\alpha}$  with respect to some other allowable local coordinate system always transform into each other by a transformation of the holonomy group.

Now let  $x^h$  be the radiusvector of a point of the tangent  $E_n$  of a point  $\xi^x$  in  $\mathfrak{N}(\xi^x)$  with respect to some allowable local coordinate system. The general infinitesimal transformation of the  $r$ -parameter holonomy group has the form (cf. IV § 3)

$$(6.1) \quad dx^h = E_b^h(x^i)(d\eta)^b; \quad h = 1, \dots, n; \quad b = i, \dots, r.$$

But because this group contains linear homogeneous transformations only,  $dx^h$  must be linear homogeneous in  $x^h$ , hence there exist equations of the form

$$(6.2) \quad dx^h = C_{bi}^h x^i (d\eta)^b$$

in which the  $C_{bi}^h$  are constants.

If a vector  $v^x$  is displaced parallel from  $\xi^x$  to  $\xi^x + d\xi^x$  we have for the anholonomic components  $v^h$  (cf. III 2.10; 9.1; 10.1)

$$(6.3) \quad dv^h = -\Gamma_{\mu i}^h v^i d\xi^\mu$$

and because of the choice of the allowable local coordinate systems this transformation is a transformation of the holonomy group in the variables  $v^h$ . Hence there must be a relation of the form

$$(6.4) \quad I_{\mu i}^h = \frac{1}{2} C_{ai}^h \theta_\mu^a(\xi^x)$$

in which the  $\theta_\mu^a$  are not in general constants.

The infinitesimal transformations of the holonomy group are

$$(6.5) \quad X_b = C_{bi}^h x^i \frac{\partial}{\partial x^h}$$

and as we know that  $(X_c X_b) = c_{cb}^a X_a$ , it follows by substitution of (6.5) in this equation that (cf. 4.20a)

$$(6.6) \quad C_{ci}^j C_{bj}^h - C_{bi}^j C_{cj}^h = c_{cb}^a C_{ai}^h.$$

Now we substitute (6.4, 6) in the structural equations (III 10.4, 5) of an  $A_n$ . Then  $S_{\mu i}^x = 0$  and we get

$$(6.7) \quad \left\{ \begin{array}{ll} \text{a)} & \partial_{[\mu} A_{\lambda]}^h = \frac{1}{2} C_{ai}^h A_{[\mu}^i \theta_{\lambda]}^a \\ \text{C} & \text{b)} \quad [dA^h] = \frac{1}{2} C_{ai}^h [A^i \theta^a] \end{array} \right.$$

and

$$(6.8) \quad \left\{ \begin{array}{ll} \text{a)} & C_{ai}^h \partial_{[\nu} \theta_{\mu]}^a - \frac{1}{2} c_{cb}^a C_{ai}^h \theta_{[\nu}^c \theta_{\mu]}^b = R_{\nu \mu i}^{h a} \\ \text{C} & \text{b)} \quad C_{ai}^h [d \theta^a] - \frac{1}{2} c_{cb}^a C_{ai}^h [\theta^c \theta^b] = R_i^h \end{array} \right.$$

or [cf. (4.13)]

$$(6.9) \quad \left\{ \begin{array}{ll} \text{a)} & R_{\nu\mu i}^{\cdot\cdot h} = C_{ai}^{\cdot\cdot h} B_{\nu\mu}^{\cdot\cdot a} \\ \text{C} & \text{b)} \quad R_i^{\cdot\cdot h} = C_{ai}^{\cdot\cdot h} B^a \end{array} \right.$$

where

$$(6.10) \quad \left\{ \begin{array}{ll} \text{a)} & B_{\nu\mu}^{\cdot\cdot a} \stackrel{\text{def}}{=} \partial_{[\nu} \theta_{\mu]}^a - \frac{1}{2} c_{cb}^{\cdot\cdot a} \theta_{[\nu}^c \theta_{\mu]}^b \\ \text{C} & \text{b)} \quad B^a \stackrel{\text{def}}{=} [d\theta^a] - \frac{1}{2} c_{cb}^{\cdot\cdot a} [\theta^c \theta^b]. \end{array} \right.$$

As an example we consider first the case of a *symmetric*  $A_n$  (cf. III § 7, VII § 5). Then we have  $V_\omega R_{\nu\mu i}^{\cdot\cdot h} = 0$  and the components  $R_{kji}^{\cdot\cdot h}$  have the same values at *all* points with respect to *all* allowable coordinate systems  $(h)$ .

The infinitesimal transformation of the radiusvector  $x^h$  in the tangent  $E_n$  (cf. III 4.1)

$$(6.11) \quad x^h \rightarrow x^h - \frac{1}{2} R_{kji}^{\cdot\cdot h} x^i d\gamma^{kj}$$

is the transformation of the holonomy group belonging to the facet  $d\gamma^{kj}$ . Hence there must exist relations of the form (cf. 4.13a)

$$(6.12) \quad R_{kji}^{\cdot\cdot h} = C_{ai}^{\cdot\cdot h} B_{kj}^{\cdot\cdot a}$$

in which now *not only the*  $C_{ai}^{\cdot\cdot h}$  *but also the coefficients*  $B_{kj}^{\cdot\cdot a}$  *are constants.* Substituting (6.12) in (6.8a) we get

$$(6.13) \quad C_{ai}^{\cdot\cdot h} (\partial_{[\nu} \theta_{\mu]}^a - \frac{1}{2} c_{cb}^{\cdot\cdot a} \theta_{[\nu}^c \theta_{\mu]}^b - B_{kj}^{\cdot\cdot a} A_{[\nu}^k A_{\mu]}^j) = 0,$$

and this is equivalent to

$$(6.14) \quad \left\{ \begin{array}{ll} \text{a)} & \partial_{[\nu} \theta_{\mu]}^a = \frac{1}{2} c_{cb}^{\cdot\cdot a} \theta_{[\nu}^c \theta_{\mu]}^b + B_{kj}^{\cdot\cdot a} A_{[\nu}^k A_{\mu]}^j \\ \text{C} & \text{b)} \quad [d\theta^a] = \frac{1}{2} c_{cb}^{\cdot\cdot a} [\theta^c \theta^b] + B_{kj}^{\cdot\cdot a} [A^k A^j] \end{array} \right.$$

because the  $a$ -rank of  $C_{ai}^{\cdot\cdot h}$  is  $r$ .

We return to the case of a *general*  $A_n$ . The allowable local coordinate systems in  $A_n$  may be considered as depending on the general coordinates  $\eta^\alpha; \alpha = \dot{1}, \dots, \dot{r}$  in the group space  $X_r$  of the holonomy group. To this end we have only to fix one allowable local coordinate system at every point of  $\mathfrak{N}(\xi^*)$  and to give these systems the coordinates  $\eta^\alpha$  belonging to the identical transformation. Of course an allowable anholonomic system  $(h)$  can then be fixed by giving the  $\eta^\alpha$  as functions of  $\xi^*$ . In the same way as in III § 10 the  $A_\lambda^h$  may be considered as the section of the  $A_B^h; B = 1, \dots, n, \dot{1}, \dots, \dot{r}$  in the  $X_{n+r}$  of  $\xi^*, \eta^\alpha$  with  $A_\beta^h = 0$ . But in contradistinction to III § 10 we have now  $r \leq n^2$  and in the  $X_r$  we have not only the holonomic coordinates  $\eta^\alpha$  but among others also the

anholonomic system  $(a)$ . Hence in  $X_{n+r}$  we have besides the holonomic system  $(x, \alpha); 1, \dots, n, \dot{1}, \dots, \dot{r}$  the three anholonomic systems  $(x, a); 1, \dots, n, \dot{1}, \dots, \dot{r}$  and  $(h, \alpha); 1, \dots, n, \dot{1}, \dots, \dot{r}$  and  $(h, a); 1, \dots, n, \dot{1}, \dots, \dot{r}$ .

According to (6.2, 5) the infinitesimal transformation of  $A_\lambda^h$  by the holonomy group is given by

$$(6.15) \quad d\eta^\beta \partial_\beta A_\lambda^h = C_{bi}^{ch} A_\lambda^i (d\eta)^b$$

and from this equation we get

$$(6.16) \quad A_i^h \partial_\gamma A_\lambda^h = A_\gamma^a C_{ai}^{ch}.$$

Hence, if we write

$$(6.17) \quad \theta_\gamma^a \stackrel{\text{def}}{=} -A_\gamma^a$$

we get besides (6.7a) the equations

$$(6.18) \quad \begin{cases} \partial_{[\gamma} A_{\lambda]}^h = \frac{1}{2} C_{ai}^{ch} A_{[\gamma}^i \theta_{\lambda]}^a \\ \partial_{[\gamma} A_{\beta]}^h = \frac{1}{2} C_{ai}^{ch} A_{[\gamma}^i \theta_{\beta]}^a; \end{cases} \quad (\text{Note that } A_\beta^h = 0)$$

that can be gathered together with (6.7a) into the equations

$$(6.19) \quad \begin{cases} \text{a)} \quad \partial_{[C} A_{B]}^h = \frac{1}{2} C_{ai}^{ch} A_{[C}^i \theta_{B]}^a; \quad B, C = 1, \dots, n, \dot{1}, \dots, \dot{r}; \\ \text{b)} \quad [dA^h] = \frac{1}{2} C_{ai}^{ch} [A^i \theta^a]; \quad h, i = 1, \dots, n; \quad a = \dot{1}, \dots, \dot{r}. \end{cases}$$

Note that (6.19b) has exactly the same form as (6.7b), though (6.19b) contains much more.

Now we will prove that the equations (6.14) for the symmetric  $A_n$  can be generalized in the same way to

$$(6.20) \quad \begin{cases} \text{a)} \quad \partial_{[C} \theta_{B]}^a = \frac{1}{2} c_{cb}^{ca} \theta_{[C}^b \theta_{B]}^a + B_{kj}^{ia} A_{[C}^k A_{B]}^j \\ \text{b)} \quad [d\theta^a] = \frac{1}{2} c_{cb}^{ca} [\theta^c \theta^b] + B_{kj}^{ia} [A^k A^j]. \end{cases}$$

We note that (6.20b) has the same form as (6.14b) but contains much more. We look at the  $\Gamma_{\mu i}^h$  with respect to a definitely chosen allowable anholonomic coordinate system  $(h)$ . If a transformation of the holonomy group is applied simultaneously to all local coordinate systems of  $(h)$  we get a new allowable anholonomic coordinate system  $(h')$  and the  $A_{\mu i}^h, A_{\mu i}^{h'}$  are constants. That means that in this case the transformation  $\Gamma_{\mu i}^h \rightarrow \Gamma_{\mu i}^{h'}$  is linear homogeneous (cf. 4.16). For an infinitesimal transformation of this kind it follows from (6.15) that

$$(6.21) \quad d\Gamma_{\mu i}^h = (C_{bj}^{ch} \Gamma_{\mu i}^j - C_{bi}^{ch} \Gamma_{\mu j}^i) (d\eta)^b$$

hence, according to (6.4, 6, 17)

$$(6.22) \quad \left\{ \begin{array}{l} \partial_\delta \Gamma_{\mu i}^h = C_{\dot{\delta} j}^h \Gamma_{\mu i}^j - C_{\dot{\delta} i}^j \Gamma_{\mu j}^h \\ = \frac{1}{2} (C_{\dot{\delta} j}^h C_{\dot{a} i}^j - C_{\dot{\delta} i}^j C_{\dot{a} j}^h) \theta_\mu^a = -\frac{1}{2} c_{\dot{c} \dot{b}}^a C_{\dot{a} i}^h \theta_\mu^c \theta_\lambda^b. \end{array} \right.$$

But on the other hand we have from (6.4)

$$(6.23) \quad \partial_\delta \Gamma_{\mu i}^h = \frac{1}{2} C_{\dot{a} i}^h \partial_\delta \theta_\mu^a$$

and because the  $a$ -rank of  $C_{\dot{a} i}^h$  is  $r$ , (6.22, 23) have as a consequence

$$(6.24) \quad \partial_\gamma \theta_\lambda^a = c_{\dot{c} \dot{b}}^a \theta_\gamma^c \theta_\lambda^b.$$

As  $\theta_\gamma^a$  depends here on the  $\eta^\alpha$  only, we have  $\partial_\lambda \theta_\gamma^a = 0$  and

$$(6.25) \quad \partial_{[\gamma} \theta_{\lambda]}^a = \frac{1}{2} c_{\dot{c} \dot{b}}^a \theta_{[\gamma}^c \theta_{\lambda]}^b + B_{k j}^a A_{[\gamma}^k A_{\lambda]}^j$$

where the last term could be added because  $A_{\gamma}^k = 0$ . Now we have already two parts of (6.20) viz. (6.14) and (6.25). The last part is [cf. (6.17)]

$$(6.26) \quad \partial_{[\gamma} A_{\beta]}^a = -\frac{1}{2} c_{\dot{c} \dot{b}}^a A_{[\gamma}^c A_{\beta]}^b$$

but this equation is equivalent to CARTAN's structural equations of the parameter groups of the holonomy group (IV 1,19), already proved in IV § 1.

In the  $X_{n+r}$ , of  $\xi^\alpha, \eta^\alpha$  the  $A_B^h$ ;  $h=1, \dots, n$  and  $\theta_B^a$ ;  $a=\dot{i}, \dots, \dot{r}$  are  $n+r$  covariant vector fields. They are linearly independent because an equation of the form

$$(6.27) \quad \alpha_h A_B^h + \alpha_a \theta_B^a = 0$$

would be equivalent to

$$(6.28) \quad \left\{ \begin{array}{l} \text{a)} \quad \alpha_h A_\lambda^h + \alpha_a \theta_\lambda^a = 0 \\ \text{b)} \quad \alpha_a \theta_\beta^a = 0 \end{array} \right.$$

and these equations can only be true for  $\alpha_a = 0$ ;  $\alpha_h = 0$  because of (6.17). Hence the  $n+r$  covariant vectors determine an anholonomic coordinate system in the  $X_{n+r}$ , and the intermediate components  $A_B^X$  are

$$(6.29) \quad \left\{ \begin{array}{l} A_B^h; \quad A_B^a \stackrel{\text{def}}{=} -\theta_B^a; \quad X = 1, \dots, n; \quad \dot{i}, \dots, \dot{r}; \\ h = 1, \dots, n; \quad a = \dot{i}, \dots, \dot{r}; \\ B = 1, \dots, n; \quad \dot{i}, \dots, \dot{r}. \end{array} \right.$$

With this notation the equations (6.19, 20) of the symmetric  $A_n$  can be written together

$$(6.30) \quad \left\{ \begin{array}{l} \text{a)} \quad \partial_{[C} A_B^X] = \frac{1}{2} C_{ZY}^X A_{[C}^Z A_{B]}^Y \\ \text{b)} \quad [dA^X] = \frac{1}{2} C_{ZY}^X [A^Z A^Y]; \quad X, Y, Z = 1, \dots, n, \dot{i}, \dots, \dot{r} \end{array} \right.$$

with constant coefficients  $C_{ZY}^X$

$$(6.31) \quad \left\{ \begin{array}{l} C_{cb}^a \stackrel{\text{def}}{=} c_{cb}^a; \quad C_{cb}^h \stackrel{\text{def}}{=} 0; \quad a, b, c = 1, \dots, r \\ C_{ic}^a = C_{ci}^a \stackrel{\text{def}}{=} 0; \quad C_{ic}^h \stackrel{\text{def}}{=} -C_{ci}^h; \quad h, i, j = 1, \dots, n \\ C_{ji}^a \stackrel{\text{def}}{=} -B_{ji}^{;a}; \quad C_{ji}^h \stackrel{\text{def}}{=} 0. \end{array} \right.$$

But as was proved in IV § 2 this is only possible if

$$(6.32) \quad C_{[ZY}^U C_{X]U}^V = 0.$$

Now if these latter equations are written out making use of (6.31), we see that they are in fact satisfied as a consequence of (III 5.4), (IV 1.28b), (6.6, 12) and  $V_\omega R_{\nu\mu}^{;i} \dot{\xi}^\nu = 0$ .

As we have seen in IV § 2, (6.32) has as a consequence that (6.30) are the structural equations after CARTAN of an  $(n+r)$ -parameter group of point transformations in  $X_{n+r}$ . The structural equations of the same group after LIE are

$$(6.33) \quad (A_Z A_Y) = -C_{ZY}^X A_X$$

or, written out,

$$(6.34) \quad \left\{ \begin{array}{l} (A_c A_b) = c_{cb}^a A_a; \quad A_b = A_b^\mu \partial_\mu + A_b^\gamma \partial_\gamma; \\ (A_c A_i) = -C_{ci}^h A_h; \quad A_i = A_i^\mu \partial_\mu + A_i^\gamma \partial_\gamma; \\ (A_j A_i) = B_{ji}^{;a} A_a = -C_{ji}^a A_a. \end{array} \right.$$

This proves that if a system of constants  $c_{cb}^a$ ,  $C_{ci}^h$ ,  $B_{ji}^{;a}$  is given, satisfying (6.31, 32) there always exists a symmetric  $A_n$  whose holonomy group depends on  $r$  parameters and has the  $c_{cb}^a$  for structural constants (cf. IV § 2). In this  $A_n$  there exists an anholonomic coordinate system  $(h)$  such that  $I_{\mu i}^h = -\frac{1}{2} A_\mu^a C_{ai}^h$ , and  $R_{kij}^{;i}{}^h = C_{ai}^h B_{kj}^{;a}$ .<sup>1)</sup>

As was proved in IV § 2 there exists in  $X_{n+r}$  an  $(n+r)$ -parameter group leaving the  $n+r$  covariant vector fields  $A_B^h$  (with  $A_\beta^h = 0$ ) and  $A_B^a$  invariant. Hence, if  $\xi^\alpha \rightarrow \xi^\alpha + v^\alpha dt$  combined with  $\eta^\alpha \rightarrow \eta^\alpha + v^\alpha dt$  is an infinitesimal transformation of this group, the LIE derivative of these  $n+r$  fields must vanish:

$$(6.35) \quad \left\{ \begin{array}{l} \text{a)} \quad v^\mu \partial_\mu A_\lambda^h + v^\gamma \partial_\gamma A_\lambda^h + A_\mu^h \partial_\lambda v^\mu + (A_\gamma^h \partial_\lambda v^\gamma) = 0 \\ \text{b)} \quad (v^\mu \partial_\mu A_\beta^h) + (v^\gamma \partial_\gamma A_\beta^h) + A_\mu^h \partial_\beta v^\mu + (A_\gamma^h \partial_\beta v^\gamma) = 0 \\ \text{c)} \quad v^\mu \partial_\mu A_\lambda^a + (v^\gamma \partial_\gamma A_\lambda^a) + A_\mu^a \partial_\lambda v^\mu + A_\gamma^a \partial_\lambda v^\gamma = 0 \\ \text{d)} \quad (v^\mu \partial_\mu A_\beta^a) + (v^\gamma \partial_\gamma A_\beta^a) + A_\mu^a \partial_\beta v^\mu + A_\gamma^a \partial_\beta v^\gamma = 0 \end{array} \right.$$

<sup>1)</sup> This theorem was proved by E. CARTAN in 1926, 1, p. 264 ff. for the case of a  $V_n$ .

(the terms in brackets vanish already because 1°.  $A_\beta^h = 0$ ; 2°. the  $A_\lambda^a$  are independent of the  $\xi^\alpha$  and 3°. the  $A_\lambda^a$  are independent of the  $\eta^\alpha$ ). From (6.35b) it follows that the  $v^\alpha$  are functions of the  $\xi^\alpha$  only. Hence  $v^\alpha dt$  represents an ordinary infinitesimal transformation of the  $X_n$ . With this transformation the fields  $\overset{h}{e}_\lambda$  are dragged along and accordingly the changes of these fields are at every point of  $X_n$  equal to their negative LIE differentials. Apart from this transformation,  $v^\alpha dt$  represents an extra infinitesimal transformation of these local vectors  $\overset{h}{e}_\lambda$  and because in total the local fields  $\overset{h}{e}_\lambda$  must be invariant, these two infinitesimal transformations must annihilate each other. This is just what the equation (6.35a) expresses. Moreover the  $A_\lambda^a$  are invariant, hence the  $\Gamma_{\mu i}^h = -\frac{1}{2} A_\mu^a C_{ai}^h$  are invariant, and this means that the connexion in  $A_n$  is invariant. This proves that there exists in a symmetric  $A_n$  an  $(n+r)$ -parameter group of point transformations, which leaves the connexion invariant, and that this group has the  $-C_{ZY}^X$  for structural constants.<sup>1)</sup> If from this group we take the subgroup that leaves invariant a point  $\underset{\theta}{\xi}^\alpha$ , this subgroup transforms the vectors in the tangent  $E_n$  of  $\underset{\theta}{\xi}^\alpha$  in the same way as the holonomy group does.

## VIII. Miscellaneous examples.

### § 1. The harmonic $V_n$ .

Let  $x^\alpha$  be rectilinear coordinates in an ordinary  $R_n$ . Then the distance  $s$  of an arbitrary point  $x^\alpha$  from the origin is

$$(1.1) \quad s = (g_{\mu\lambda} x^\mu x^\lambda)^{\frac{1}{2}}$$

and by differentiation we get

$$(1.2) \quad \nabla_\nu s^p = p s^{p-2} g_{\mu\nu} x^\mu$$

$$(1.3) \quad \nabla^\nu \nabla_\nu s^p = p(n+p-2)s^{p-2}; \quad \nabla^\nu \stackrel{\text{def}}{=} g^{\nu\mu} \nabla_\mu.$$

Hence  $\nabla^\nu \nabla_\nu s$  is a function of  $s$  only, and  $s^{2-n}$  is for  $n \neq 2$  a solution of the equation of LAPLACE

$$(1.4) \quad \nabla^\nu \nabla_\nu V = 0.$$

In a  $V_n$  (1.4) is an invariant equation but we can not be certain that solutions always exist. The  $V_n$  is called *harmonic*<sup>3)</sup> at the point  $\underset{\theta}{\xi}^\alpha$

<sup>1)</sup> This was proved by E. CARTAN 1926, 1 for the case of a  $V_n$ .

<sup>2)</sup> From now we write in this section  $d\sigma^2$  instead of  $ds^2$  for the linear element.

<sup>3)</sup> COPSON and RUSE 1940, 1, p. 117. Cf. for the affine generalization RUSE 1952, 1; PATTERSON 1952, 1.

if (1.4) has at least one solution  $\varphi(s)$  which is a function of  $s$  only.  $s$  is here the distance of the point  $\xi^*$  with respect to the point  $\underset{\theta}{\xi}^*$ . If this solution exists we have

$$(1.5) \quad \left\{ \begin{array}{l} 0 = \nabla^\nu \nabla_\nu \varphi(s) = \varphi' \nabla^\nu \nabla_\nu s + \varphi''(\nabla^\nu s) \nabla_\nu s; \\ \varphi' \stackrel{\text{def}}{=} \frac{d\varphi}{ds}; \quad \varphi'' \stackrel{\text{def}}{=} \frac{d^2\varphi}{ds^2} \end{array} \right.$$

hence

$$(1.6) \quad \nabla^\nu \nabla_\nu s = (-\varphi''(\nabla^\nu s) \nabla_\nu s)/\varphi' = -\varphi''/\varphi'$$

and this is a function of  $s$  only. Conversely, if it is known that  $\nabla^\nu \nabla_\nu s$  is a function of  $s$  only:

$$(1.7) \quad \nabla^\nu \nabla_\nu s = \chi(s)$$

and if we write  $V(s)$  for a still unknown solution of (1.4), we have

$$(1.8) \quad \nabla^\nu \nabla_\nu V = V''(\nabla^\nu s) \nabla_\nu s + V' \chi(s) = 0$$

from which  $V$  can be found by quadratures as a function of  $s$ . Hence

*A  $V_n$  is harmonic at a point  $\underset{\theta}{\xi}^*$  if and only if the distance  $s$  with respect to  $\underset{\theta}{\xi}^*$  satisfies an equation of the form (1.7).<sup>1)</sup>*

It is convenient to introduce the function  $\Omega \stackrel{\text{def}}{=} \frac{1}{2}s^2$ <sup>2)</sup> and to write

$$(1.9) \quad f(\Omega) \stackrel{\text{def}}{=} \nabla^\nu \nabla_\nu \Omega = s \chi(s) + 1 = (2\Omega)^{\frac{1}{2}} \chi\{(2\Omega)^{\frac{1}{2}}\} + 1.$$

According to COPSON and RUSE a  $V_n$  that is harmonic at all points of the region considered is called *completely harmonic*. If it is harmonic at one point only, the term *centrally harmonic* will be used. In a completely harmonic  $V_n$  the functions  $\chi(s)$  and  $f(\Omega)$  are each the same for all points, as was proved by LICHNEROWICZ.<sup>3)</sup>

RUSE<sup>4)</sup> has proved that an elementary solution of (1.4) is given by the integral

$$(1.10) \quad \varphi(s) = A \int_a^s \varrho^{-1} s^{1-n} ds + B$$

where  $A, B$  and  $a > 0$  are constants,  $s$  is the distance from  $\xi^*$  to  $\underset{\theta}{\xi}^*$  and where the integration has to be effected along the geodesic through

<sup>1)</sup> COPSON and RUSE 1940, 1, p. 121.

<sup>2)</sup>  $\Omega$  is a function of  $\xi^*$  and  $\underset{\theta}{\xi}^*$ , and is often called the *characteristic function* of the  $V_n$ . Its covariant derivatives for fixed  $\xi^*$  were derived by SYNGE 1930, 1.

Cf. COPSON and RUSE 1940, 1; LICHNEROWICZ 1944, 1, p. 158.

<sup>3)</sup> LICHNEROWICZ 1944, 1, p. 150; cf. WALKER 1945, 1.

<sup>4)</sup> RUSE 1930, 2; 3; 1939, 1 for  $S_n$ ; cf. COPSON and RUSE 1940, 1.

$\xi^*$  and  $\xi^*$ .  $\varrho$  is RUSE's invariant

$$(1.11) \quad \varrho \stackrel{\text{def}}{=} g^{\frac{1}{2}}(\xi^*) g^{\frac{1}{2}}\left(\frac{\xi^*}{\varrho}\right) / \text{Det} \left( \frac{\partial^2 \Omega}{\partial \xi^\mu \partial \xi^\lambda} \right);$$

it is a function of  $s$  only, if and only if the  $V_n$  is centrally harmonic at  $\xi^*$ .

Its development in a power series of  $s$  and its geometric interpretation were given by WALKER.<sup>1)</sup> The solution (1.10) behaves like  $s^{2-n}$  at  $\xi^*$  and on all null geodesics through this point, and it passes into  $s^{2-n}$  if the  $V_n$  is an  $R_n$ .<sup>2)</sup>

A completely harmonic  $V_n$  with  $\varrho = 1$  is called *simply harmonic*.<sup>3)</sup> WALKER proved that this condition is equivalent to  $\chi(s) = (n-1) s^{-1}$ .<sup>4)</sup>

If the  $V_n$  is centrally harmonic at  $\xi^*$  and if the geodesic joining  $\xi^*$  and  $\xi^*$  has a unit tangent vector at  $\xi^*$  with components  $t^{h'}$ ;  $h' = 1', \dots, n'$  with respect to some fixed local orthogonal coordinate system  $(h')$  in the tangent  $R_n$  of  $\xi^*$ , we may introduce the coordinates

$$(1.12) \quad \xi^{a'} \stackrel{\text{def}}{=} t^{a'}; \quad \xi^{n'} \stackrel{\text{def}}{=} s; \quad a' = 1', \dots, (n-1)'.$$

Then the linear element takes the form

$$(1.13) \quad d\sigma^2 \stackrel{*}{=} d\xi^{n'} d\xi^{n'} + g_{a'b'} d\xi^{a'} d\xi^{b'}$$

and for  $\chi(s)$  we get

$$(1.14) \quad \begin{cases} \chi(s) = V^\mu V_\mu s \stackrel{*}{=} (\partial_{h'} g^{i'h'}) \partial_{i'} s + g^{-\frac{1}{2}} \partial^{h'} g^{\frac{1}{2}} \partial_{h'} s \\ \qquad \qquad \qquad = g^{-\frac{1}{2}} \partial^{n'} g^{\frac{1}{2}} \partial_{n'} s = \frac{1}{2} \frac{\partial}{\partial s} \log g, \end{cases}$$

from which by integration

$$(1.15) \quad \frac{1}{2} \log g \stackrel{(h')}{=} \int \chi(s) ds + \Phi(\xi^{a'}); \quad a' = 1', \dots, (n-1)'.$$

Instead of  $(h')$  we now introduce the coordinate system  $(h)$ ;  $h = 1, \dots, n$ , defined by

$$(1.16) \quad \xi^a \stackrel{\text{def}}{=} \delta_{a'}^a s t^{a'}; \quad \xi^n \stackrel{\text{def}}{=} s t^n; \quad a = 1, \dots, n-1$$

<sup>1)</sup> WALKER 1942, 1, p. 16.

<sup>2)</sup> LICHNEROWICZ 1944, 1, p. 148f. Mean value theorems for harmonic functions in a completely harmonic  $V_n$  were dealt with by WILLMORE 1950, 1. The more general equation  $V^\nu V_\nu = F(V)$  was discussed by NORDON 1944, 1.

<sup>3)</sup> WALKER 1942, 1; cf. PATTERSON 1951, 2; 1952, 1.

<sup>4)</sup> WALKER 1945, 1.

where  $t^m$  is a known function of the  $t^{a'}$ . Then

$$(1.17) \quad \log \text{Det}(A_h^h) = (n - 1) \log s + \log t^m(\xi^{a'})$$

and

$$(1.18) \quad \frac{1}{2} \log \overset{(h)}{\mathfrak{g}} = \int \chi(s) ds - (n - 1) \log s + \Phi(\xi^{a'}) - \log t^m(\xi^{a'}).$$

But, for  $s = 0$ ,  $\overset{(h)}{\mathfrak{g}}$  is independent of the  $\xi^{a'}$  and has the value +1. Moreover in III § 7 we have seen that the transformation of two normal coordinate systems with respect to the same point is linear homogeneous with constant coefficients. Hence

*In a centrally harmonic  $V_n$  with centrum  $\overset{(h)}{\xi^x}$  the components  $\overset{(h)}{\mathfrak{g}}$  of  $\mathfrak{g}$  with respect to any arbitrary normal coordinate system  $(h)$  belonging to  $\overset{(h)}{\xi^x}$  is a function of  $s$  only and satisfies the equation*

$$(1.19) \quad \left\{ \begin{array}{l} \frac{1}{2} \frac{d}{ds} \log \overset{(h)}{\mathfrak{g}} = \chi(s) - \frac{n-1}{s} \\ \qquad \qquad \qquad = V^\mu \nabla_\mu s - \frac{n-1}{s} \end{array} \right.$$

From (1.9) and (1.19) we get

$$(1.20) \quad V^\nu V_\nu \Omega = (n - 1) + \frac{1}{2} s \frac{d \log \overset{(h)}{\mathfrak{g}}}{ds} + (V^\nu s) V_\nu s^2.$$

According to (III 3.19) it follows from (1.19) that for a positive definite fundamental tensor

$$(1.21) \quad \left\{ \begin{array}{l} I_{ji}^i = \frac{1}{2} \partial_j \log \overset{(h)}{\mathfrak{g}} = \left( \chi(s) - \frac{n-1}{s} \right) \partial_j s \\ \qquad \qquad \qquad = \frac{s \chi(s) - n + 1}{s^2} \partial_j \Omega = \frac{f(\Omega) - n}{2 \Omega} \partial_j \Omega \end{array} \right.$$

From the series (III 7.28) for  $I_{ji}^h$  we get for  $I_{ji}^i$

$$(1.22) \quad I_{ji}^i = \xi^k N_{kji}^4(\xi) + \frac{1}{2!} \xi^{k_1} \xi^{k_2} N_{k_1 k_2 j i}^5(\xi) + \dots$$

Now in (1.21),  $f(\Omega)$  can be developed in the series

$$(1.23) \quad f(\Omega) = f(0) + \Omega f'(0) + \frac{1}{2} \Omega^2 f''(0) + \dots; \quad f(0) = n.$$

<sup>1)</sup> LICHNEROWICZ 1944, 1, p. 155; the converse is also true. The proof given there is not quite satisfactory because the result seems to depend on the choice of the normal system, which in fact it does not.

<sup>2)</sup> LICHNEROWICZ 1944, 1, p. 150.

<sup>3)</sup> LICHNEROWICZ 1944, 1, p. 156.

Making use of the following identities in normal coordinates

$$(1.24) \quad \partial_i s \stackrel{*}{=} g_{ij} \frac{\xi^j}{s}; \quad \partial_i \Omega \stackrel{*}{=} g_{ij} \xi^j;$$

$$(1.25) \quad (\partial_j \partial_i \Omega)_0 \stackrel{*}{=} (g_{ji})_0$$

we get by equating corresponding terms in the development of both sides of (1.21)

$$(1.26) \quad \left\{ \begin{array}{l} \text{a)} \quad N_{k_1 k_2 k_3}^{4; i} \left( \frac{\xi}{\theta} \right) = \frac{1}{2} f'(0) g_{k_1 k_2} \left( \frac{\xi}{\theta} \right) \\ \text{b)} \quad N_{k_1 k_2 k_3 k_4}^{5; i} \left( \frac{\xi}{\theta} \right) = 0 \\ \text{c)} \quad N_{k_1 k_2 k_3 k_4 k_5}^{6; i} \left( \frac{\xi}{\theta} \right) = \frac{3}{4} f''(0) g_{j(k_1} \left( \frac{\xi}{\theta} \right) g_{k_2 k_3)} \left( \frac{\xi}{\theta} \right) \\ \text{d)} \quad N_{k_1 k_2 k_3 k_4 k_5 k_6}^{7; i} \left( \frac{\xi}{\theta} \right) = 0 \\ \vdots \end{array} \right.$$

known as the *equations of COPSON and RUSE*.<sup>1)</sup>

Using the relations (III 7.36, 40; Exerc. III 7.5) between the normal tensors and the curvature tensor and its covariant derivatives, and the identities III (5.27, 30), the equations (1.26a, b) appear to be equivalent to

$$(1.27) \quad \left\{ \begin{array}{l} \text{a)} \quad \frac{1}{3} K_{\nu \mu \lambda}^{;\lambda} \left( \frac{\xi}{\theta} \right) + \frac{1}{3} K_{\nu \lambda \mu}^{;\lambda} \left( \frac{\xi}{\theta} \right) = -\frac{1}{3} K_{\nu \mu} \left( \frac{\xi}{\theta} \right) = \frac{1}{2} f'(0) g_{\nu \mu} \left( \frac{\xi}{\theta} \right) \\ \text{b)} \quad V_{(\omega} K_{\nu \mu)} \left( \frac{\xi}{\theta} \right) = 0; \end{array} \right.$$

hence (cf. III § 5)

$$(1.28) \quad f'(0) = -\frac{2}{3n} K \left( \frac{\xi}{\theta} \right) = -\frac{2}{3} (n-1) \kappa \left( \frac{\xi}{\theta} \right).$$

If the  $V_n$  is completely harmonic, (1.27a, b) hold for every choice of  $\xi^\mu$ . Then (1.27a) expresses that the  $V_n$  is an EINSTEIN space.<sup>2)</sup> (1.27b) is a consequence of (1.27a) for  $n > 2$  and expresses for  $n = 2$  that the  $V_2$  is an  $S_2$ . As every EINSTEIN  $V_3$  is an  $S_3$  this proves that a completely harmonic  $V_n$  is always an  $S_n$  if  $n = 2$  or 3.<sup>2)</sup> Of course every  $S_n$  is completely harmonic because for every choice of  $\xi^\mu$ ,  $V^\mu V_\mu$  is a function of  $s$  only. In the beginning of the investigations it seemed

<sup>1)</sup> COPSON and RUSE 1940, 1, p. 130; cf. LICHNEROWICZ 1944, 1, p. 156; RUSE 1945, 1. COPSON and RUSE went via the covariant derivatives of  $\Omega$  in order to make use of SYNGE's relations in SYNGE 1930, 1.

<sup>2)</sup> COPSON and RUSE 1940, 1, p. 132. In the same paper it was proved that every SCHUR space [cf. V, p. 244, footnote 2)] is centrally harmonic with respect to the origin but that for  $n > 2$  not every centrally harmonic space is a SCHUR space.

quite probable that every completely harmonic  $V_n$  should be an  $S_n$ . Every conformally euclidean EINSTEIN space is for  $n > 2$  an  $S_n$  (cf. VI § 6) hence every completely harmonic space that is conformally euclidean is an  $S_n$ . But as WALKER proved by a counterexample<sup>1)</sup> a space can for  $n > 3$  be completely harmonic without being an  $S_n$ .

(1.26b) contains derivatives of the components of the curvature tensor. But for  $n = 2$  it follows by differentiation of (1.26a) that (1.26b) is identically satisfied. In an analogous way the second derivatives of the curvature tensor that occur in (1.26c), after expressing  $\overset{6}{N}$  in terms of  $\overset{4}{K}$  and  $\overset{6}{K}$  (cf. III § 7), can be eliminated after differentiating the foregoing formulae of (1.26). Then we get after some calculation<sup>2)</sup>

$$(1.29) \quad K_{\nu(\mu\lambda|\kappa)} K_{\cdot\sigma\varrho}^{\nu\cdot\cdot\cdot\kappa} = -45f''(0) g_{(\mu\lambda} g_{\sigma\varrho)}.$$

LICHNEROWICZ and WALKER<sup>3)</sup> used this equation to prove that a completely harmonic  $V_n$  is always an  $S_n$  if its index (cf. I § 8) is 1 or  $n-1$ . We give here a survey of results:

*A completely harmonic  $V_n$  is an  $S_n$  if one of the following conditions is satisfied*

- a)  $n = 2$  or 3;
- b) the  $V_n$  is conformally euclidean;
- c) the index is 1 or  $n-1$ .

*If the  $V_n$  is simply harmonic it is an  $R_n$  if either one of these three conditions is satisfied or the condition<sup>4)</sup>*

- d) the index is 0 or  $n$ .<sup>5)</sup>

*For any  $n \geq 4$  there is a completely (simply) harmonic  $V_n$  which is not an  $S_n$  ( $R_n$ ), with any index not satisfying c (c or d).<sup>6)</sup>*

<sup>1)</sup> WALKER 1945, 1; 2.

<sup>2)</sup> LICHNEROWICZ 1944, 1, p. 159. RUSE 1945, 1, p. 156, gave the geometric interpretation of (1.29) for  $n=4$  and he determined all algebraically possible  $V_4$ 's satisfying (1.27) and (1.29). Cf. LICHNEROWICZ 1953, 2 also for literature on the harmonic  $V_n$ .

<sup>3)</sup> WALKER 1942, 1 for  $n=4$ ; LICHNEROWICZ and WALKER 1945, 1 for the general case.

<sup>4)</sup> WALKER 1945, 1.

<sup>5)</sup> This is in accordance with T. Y. THOMAS and TITT 1939, 4, who proved that for the definite case  $s^{2-n}$  for  $n > 2$  and  $\log s$  for  $n = 2$  can only be solutions of LAPLACE's equation if the  $V_n$  is an  $R_n$ . Cf. WALKER 1942, 1, p. 26; LICHNEROWICZ 1944, 1, p. 160.

<sup>6)</sup> WALKER 1945, 1; 2.

WALKER gave the following examples.<sup>1)</sup> The  $V_4$  with the linear element with index 2

$$(1.30) \quad \left\{ \begin{array}{l} d\sigma^2 = C(\xi^2 d\xi^1 - \xi^1 d\xi^2)^2 + 2d\xi^1 d\xi^3 + 2d\xi^2 d\xi^4; \\ C = \text{const.} \neq 0 \end{array} \right.$$

is simply harmonic. It has a curvature tensor with the only non-zero component  $K_{1212}$ . Hence  $K_{\mu\lambda} = 0$ , that means that the  $V_4$  is a special EINSTEIN space. But it can not be an  $S_4$  because an  $S_4$  with  $K_{\mu\lambda} = 0$  would be an  $R_4$ . The  $V_4$  is symmetric because  $\nabla_\omega K_{\nu\mu\lambda\kappa} = 0$ .

The  $V_4$  with the linear element with index 2

$$(1.31) \quad \left\{ \begin{array}{l} d\sigma^2 = 2f^{-2}(d\xi^1 d\xi^2 + d\xi^3 d\xi^4) + \\ \quad + 2Cf^{-2}(\xi^4 d\xi^1 - \xi^1 d\xi^4)(\xi^3 d\xi^2 - \xi^2 d\xi^3) \\ f = 1 + C(\xi^1 \xi^2 + \xi^3 \xi^4); \quad f > 0; \quad C = \text{const.} \end{array} \right.$$

is completely but not simply harmonic. The curvature tensor satisfies the equation

$$(1.32) \quad \left\{ \begin{array}{l} K_{\nu\mu\lambda\kappa} = 4Cg_{[\nu[\lambda} g_{\mu]\kappa]} + S_{\nu\mu\lambda\kappa}; \\ S_{\nu\mu\lambda\kappa} = -S_{\mu\nu\lambda\kappa} = -S_{\nu\mu\kappa\lambda} = S_{\lambda\kappa\nu\mu}; \quad S_{\nu[\mu\lambda\kappa]} = 0; \end{array} \right.$$

where  $S_{\nu\mu\lambda\kappa}$  is a tensor with the only non-zero components

$$(1.33) \quad S_{1234} = Cf^{-3}; \quad S_{1342} = Cf^{-3}; \quad S_{1423} = -2Cf^{-3}.$$

The  $V_4$  is symmetric but no  $S_4$ . A space with the same properties but a definite fundamental tensor can be derived from (1.31) by a non-real coordinate transformation.

In a later paper<sup>2)</sup> WALKER proved that a completely harmonic  $V_4$  is always symmetric if its fundamental tensor is definite.

In the same paper he proved that the equations of a completely harmonic  $V_4$  with a fundamental tensor of index two, only containing  $g_{\lambda\kappa}$ ,  $K_{\nu\mu\lambda\kappa}$  and  $\nabla_\omega K_{\nu\mu\lambda\kappa}$  [viz. the equations of COPSON and RUSE (1.27, 29)] admit solutions that do neither satisfy  $\nabla_\omega K_{\nu\mu\lambda\kappa} = 0$  nor  $\nabla_\omega K_{\nu\mu\lambda\kappa} = k_\omega K_{\nu\mu\lambda\kappa}$ ;  $k_\omega \neq 0$ . But, as WALKER remarks on p. 22 this does not yet prove that there exists a harmonic  $V_n$  that is neither symmetric nor of recurrent curvature.

LICHNEROWICZ<sup>3)</sup> proved from (1.27) and (1.29) that in a completely harmonic  $V_n$  the following inequality is valid

$$(1.34) \quad f''(0) \leq -2\theta(n-1)f'(0)$$

<sup>1)</sup> WALKER 1945, 1; 2; 1946, 1; cf. RUSE 1946, 3.

<sup>2)</sup> WALKER 1949, 2.

<sup>3)</sup> LICHNEROWICZ 1944, 1, p. 161ff; 2.

and that the sign = holds if and only if the  $V_n$  is an  $S_n$ . Moreover he proved in the same paper that a completely harmonic  $V_n$  is an  $R_n$  if there exists a number  $k > 1$  such that

$$(1.35) \quad f'^2(0) \geq -20k(n-1)f''(0).$$

This is a generalization of the result of THOMAS and TITT mentioned before. The proof of (1.34, 35) is not difficult but rather elaborate.

In 1938<sup>1)</sup> FIALKOW proved that an EINSTEIN  $V_n$  in an  $S_{n+1}$  with scalar curvature  $\kappa$  is either an  $S_n$  or a product (cf. V § 11) of an  $S_p$  and an  $S_{n-p}$  with scalar curvatures  $\frac{n-2}{p-1}\kappa$  and  $\frac{n-2}{n-p-1}\kappa$  respectively. LICHNEROWICZ proved that a non-flat completely harmonic  $V_n$  can never be a product space and that according to the theorem of FIALKOW every completely harmonic  $V_n$  in  $S_{n+1}$  must be an  $S_n$ .<sup>2)</sup>

Global properties of harmonic spaces were discussed by WILLMORE.<sup>3)</sup>

RENAUDIE<sup>4)</sup> dealt with properties of the group of isotropy (cf. VII § 5) in harmonic spaces.

### Exercises.

VIII 1,1<sup>5)</sup>. Every completely harmonic  $V_{n-1}$  in  $S_n$  is an  $S_{n-1}$ .

VIII 1,2<sup>6)</sup>. The  $V_4$  with the linear element

$$d\sigma^2 = C(\xi^2 d\xi^1 - \xi^1 d\xi^2)^2 + 2d\xi^1 d\xi^3 + 2d\xi^2 d\xi^4; \quad C = \text{const.} \neq 0$$

is simply harmonic.

VIII 1,3<sup>6)</sup>. In a centrally harmonic  $V_n$ ,  $\log \varrho$  is written as a power series in  $s$ :

$$\text{VIII 1,3 } \alpha) \quad \log \varrho = \frac{1}{2!} w_1 s + \frac{1}{3!} w_2 s^2 + \dots$$

Prove that

$$\text{VIII 1,3 } \beta) \quad w_1 = 0; \quad w_2 = \frac{1}{s^2} K_{\mu\lambda} x^\mu x^\lambda$$

## § 2. Connexions for hybrid quantities.

In I § 10 we defined hybrid quantities in  $E_n$ . In order to consider fields of such quantities in  $X_n$  it is necessary to make an agreement about functional dependency. Ordinary quantities were called analytic

<sup>1)</sup> FIALKOW 1938, 2, p. 782 also for literature.

<sup>2)</sup> LICHNEROWICZ 1944, 1, p. 163ff; 1944, 3.

<sup>3)</sup> WILLMORE 1953, 1.

<sup>4)</sup> RENAUDIE 1954, 1.

<sup>5)</sup> LICHNEROWICZ 1944, 3.

<sup>6)</sup> WALKER 1945, 1.

if their components with respect to some allowable coordinate system ( $\alpha$ ) were analytic functions of the  $\xi^\alpha$  (cf. II § 2). It seems convenient to call a quantity of the second kind (cf. I § 10) analytic if its components with respect to ( $\alpha$ ) are analytic functions of the  $\xi^\alpha$ . Then a quantity of the first or second kind is analytic if its complex conjugate is analytic. Accordingly a scalar is called analytic if it is an analytic function either of the  $\xi^\alpha$  or of the  $\xi^{\bar{\alpha}}$ . Now it would be impossible to define analyticity of hybrid quantities in the same way, because such a definition would not be invariant for allowable coordinate transformations. Therefore we introduce for all fields the term *semi-analytic*. A field is called semi-analytic if its components with respect to some allowable coordinate system ( $\alpha$ ) are analytic in the  $2n$  variables  $\xi^\alpha$  and  $\xi^{\bar{\alpha}}$ .<sup>1)</sup> Only in this case they are also analytic in the  $2n$  variables  $\eta^\alpha, \zeta^\alpha$  defined by  $\eta^\alpha = \xi^\alpha + i\zeta^\alpha; \xi^{\bar{\alpha}} = \eta^\alpha - i\zeta^\alpha$ . Obviously the definition of semi-analyticity has the desired invariance. In the following all fields are supposed to be semi-analytic (and perhaps but not necessarily analytic in  $\xi^\alpha$  or in  $\xi^{\bar{\alpha}}$ ) and  $\partial_{\bar{\mu}}$  stands for  $\partial/\partial\xi^{\bar{\mu}}$ . Note that the  $\xi^\alpha$  always remain *analytic* functions of the  $\xi^\alpha$ .<sup>2)</sup>

In order to establish a linear connexion<sup>3)</sup> for all quantities, hybrid quantities included, we start with the four conditions C I, II, III, V of Ch. III § 2. But instead of C IV we take the condition

C IV' *The covariant differential of a quantity  $\Psi$  (indices suppressed) is linear homogeneous in  $d\xi^\alpha$  and  $d\xi^{\bar{\alpha}}$ :*

$$(2.1) \quad \delta\Psi = V_\mu \Psi d\xi^\mu + V_{\bar{\mu}} \Psi d\xi^{\bar{\mu}};$$

and as a sixth condition we introduce

C VI *The covariant differentials of complex conjugate quantities are complex conjugate.*

From these conditions we get in the same way as in III § 2 the most general form of the covariant differentials of contra- and covariant vectors of both kinds

$$(2.2) \quad \begin{cases} a) & \delta v^\alpha = dv^\alpha + \Gamma_{\mu\lambda}^\alpha v^\lambda d\xi^\mu + \Gamma_{\bar{\mu}\lambda}^\alpha v^\lambda d\xi^{\bar{\mu}} \\ b) & \delta w_\lambda = dw_\lambda - \Gamma_{\mu\lambda}^\alpha w_\alpha d\xi^\mu - \Gamma_{\bar{\mu}\lambda}^\alpha w_\alpha d\xi^{\bar{\mu}} \end{cases}$$

$$(2.3) \quad \begin{cases} a) & \delta v^{\bar{\alpha}} = dv^{\bar{\alpha}} + \Gamma_{\mu\bar{\lambda}}^{\bar{\alpha}} v^{\bar{\lambda}} d\xi^{\bar{\mu}} + \Gamma_{\bar{\mu}\bar{\lambda}}^{\bar{\alpha}} v^{\bar{\lambda}} d\xi^\mu \\ b) & \delta w_{\bar{\lambda}} = dw_{\bar{\lambda}} - \Gamma_{\mu\bar{\lambda}}^{\bar{\alpha}} w_{\bar{\alpha}} d\xi^{\bar{\mu}} - \Gamma_{\bar{\mu}\bar{\lambda}}^{\bar{\alpha}} w_{\bar{\alpha}} d\xi^\mu \end{cases}$$

<sup>1)</sup> SCHOUTEN 1929, 1.

<sup>2)</sup> GHOSH 1950, 1 considers the more general case where the  $\xi^\alpha$  are semi-analytic.

<sup>3)</sup> SCHOUTEN and v. DANTZIG 1929, 5; 1930, 1; SCHOUTEN 1929, 1.

from which the covariant differentials of all other quantities can be derived. In these formulae  $\Gamma_{\mu\lambda}^{\nu}$  and  $\Gamma_{\bar{\mu}\bar{\lambda}}^{\nu}$  are arbitrary but semi-analytic and  $\Gamma_{\bar{\mu}\lambda}^{\bar{\nu}}$  and  $\Gamma_{\mu\bar{\lambda}}^{\bar{\nu}}$  are their complex conjugates. In the following we often suppress formulae that are the complex conjugates of other formulae already written out and write "conj." instead. The parameters of the connexion transform in the following way

$$(2.4) \quad \begin{cases} \text{a)} & \Gamma_{\mu'\lambda'}^{\nu'} = A_{\nu\mu'\lambda'}^{\nu}\Gamma_{\mu\lambda}^{\nu} + A_{\nu}^{\nu'} \partial_{\mu'} A_{\lambda}^{\nu}; \quad \text{conj.} \\ \text{b)} & \Gamma_{\bar{\mu}'\lambda'}^{\nu'} = A_{\nu\lambda'\bar{\mu}'}^{\nu}\Gamma_{\bar{\mu}\lambda}^{\nu}; \quad \text{conj.} \end{cases}$$

from which we see that the  $\Gamma_{\mu\lambda}^{\nu}$  transform in the ordinary way (cf. III 2.5) but that the  $\Gamma_{\bar{\mu}\lambda}^{\nu}$  are components of a hybrid tensor.

It is sometimes convenient to introduce an auxiliary  $X_{2n}$  whose coordinates are the  $\xi^x$  and  $\xi^{\bar{x}}$  considered as  $2n$  independent variables that can each take all complex values. Of course the  $\eta^x, \zeta^x$  could also be used as (non allowable) coordinates. In this  $X_{2n}$  not all analytic transformations of the  $\xi^x, \xi^{\bar{x}}$  are allowable but only those that can be written in the form

$$(2.5) \quad \xi^x = f^x(\xi^{\bar{x}}); \quad \text{conj.}$$

But this implies that the  $X_{2n}$  contains two invariant sets of  $\infty^n X_n$ 's with the equations  $\xi^{\bar{x}} = \text{const.}$  and  $\xi^x = \text{const.}$  respectively.<sup>1)</sup> In the  $X_n$ 's of the first (second) invariant set the  $\xi^x, (\xi^{\bar{x}})$  can be used as coordinates. Through every point of the  $X_{2n}$  there goes one  $X_n$  of the first and one  $X_n$  of the second set and two  $X_n$ 's of different sets intersect exactly in one point. From this it follows that the points of each  $X_n$  of one set are in one to one correspondence with the points of every other  $X_n$  of the same set and that corresponding points lie in the same  $X_n$  of the other set. A figure in one  $X_n$  and the corresponding figure in another  $X_n$  of the same set are said to be *equipollent*.

The tangent space of the  $X_{2n}$  is an  $E_{2n}$  and in this  $E_{2n}$  the tangent  $n$ -directions of the two invariant  $X_n$ 's through the point considered form with their parallels the two sets of invariant  $E_n$ 's defined in I § 10. A general contravariant vector in  $X_{2n}$  has  $2n$  components  $v^x$  and  $v^{\bar{x}}$  and here the  $v^{\bar{x}}$  are in general not the complex conjugates of the  $v^x$ . The  $n$ -dimensional subspace of the  $X_{2n}$  with the equations

$$(2.6) \quad \xi^{\bar{x}} = \bar{\xi}^x$$

is invariant and is called the *principal*  $X_n$  of the  $X_{2n}$ . It may be considered as the image of the original  $X_n$ . If the vector  $v^x, v^{\bar{x}}$  belongs to

<sup>1)</sup> In SCHOUTEN and v. DANTZIG 1930, 1, p. 327 we called these  $X_n$ 's "isotropic". Here the word isotropic has another meaning, cf. I § 9 and VIII § 3.

a point of the principal  $X_n$  it lies in this  $X_n$  if and only if  $v^{\bar{x}} = \bar{v}^x$ . In this way the original  $X_n$  appears as a subspace of the auxiliary  $X_{2n}$  and the quantities of the original  $X_n$ , hybrid quantities included, are those quantities of the  $X_{2n}$  at points of the principal  $X_n$  that satisfy the condition, that dropping a bar where there is one and inserting a bar where there is none changes every component into its complex conjugate.

Apart from the connexion in the original  $X_n$  defined in (2.2, 3) we now consider the most general linear connexion in the  $X_{2n}$ . This connexion has the form (cf. III § 2)

$$(2.7) \quad \left\{ \begin{array}{l} \text{a)} \quad \delta v^x = dv^x + (\Gamma_{\mu\lambda}^x v^\lambda + \Gamma_{\mu\bar{\lambda}}^x v^{\bar{\lambda}}) d\xi^\mu + (\Gamma_{\bar{\mu}\lambda}^x v^\lambda + \Gamma_{\bar{\mu}\bar{\lambda}}^x v^{\bar{\lambda}}) d\xi^{\bar{\mu}} \\ \text{b)} \quad \delta v^{\bar{x}} = dv^{\bar{x}} + (\Gamma_{\mu\lambda}^{\bar{x}} v^\lambda + \Gamma_{\mu\bar{\lambda}}^{\bar{x}} v^{\bar{\lambda}}) d\xi^\mu + (\Gamma_{\bar{\mu}\lambda}^{\bar{x}} v^\lambda + \Gamma_{\bar{\mu}\bar{\lambda}}^{\bar{x}} v^{\bar{\lambda}}) d\xi^{\bar{\mu}} \end{array} \right.$$

where the  $v^x, v^{\bar{x}}$  are  $2n$  arbitrary components of a general contravariant vector of  $X_{2n}$  and where the  $\Gamma$  are  $8n^3$  independent parameters not yet in any way connected with each other or with the  $\Gamma$  in (2.2, 3). Because the two sets of  $X_n$ 's are invariant we may now *first* introduce the invariant condition that their tangent  $n$ -directions are parallel for *every* parallel displacement in  $X_{2n}$ . This gives immediately

$$(2.8) \quad \Gamma_{\bar{\mu}\lambda}^x = 0; \quad \Gamma_{\mu\bar{\lambda}}^x = 0; \quad \Gamma_{\mu\lambda}^{\bar{x}} = 0; \quad \Gamma_{\bar{\mu}\bar{\lambda}}^{\bar{x}} = 0.$$

The geometrical meaning is that the  $X_n$ 's of either set are geodesic and parallel. *Secondly* we introduce the invariant condition that the principal  $X_n$  is geodesic. Because for every displacement in this  $X_n$  we have  $d\xi^{\bar{x}} = \bar{d\xi^x}$ , the n.a.s. conditions are

$$(2.9) \quad \left\{ \begin{array}{l} \text{a)} \quad \Gamma_{\bar{\mu}\bar{\lambda}}^{\bar{x}} = \bar{\Gamma}_{\mu\lambda}^x \\ \text{b)} \quad \Gamma_{\mu\bar{\lambda}}^{\bar{x}} = \bar{\Gamma}_{\bar{\mu}\lambda}^x \end{array} \right\} \quad \text{for} \quad \xi^{\bar{x}} = \bar{\xi^x}.$$

Hence the linear connexion in  $X_{2n}$  for which the  $X_n$ 's of either invariant set are geodesic and parallel and for which the principal  $X_n$  is geodesic, is at the same time a connexion in the original  $X_n$  of the kind (2.2, 3), that is a connexion satisfying the conditions C I, C II, C III, C IV', C V and C VI.<sup>1)</sup>

The covariant derivatives belonging to this connexion are

$$(2.10) \quad \left\{ \begin{array}{l} \text{a)} \quad V_\mu v^x = \partial_\mu v^x + \Gamma_{\mu\lambda}^x v^\lambda; \quad \text{conj.} \\ \text{b)} \quad V_\mu w_\lambda = \partial_\mu w_\lambda - \Gamma_{\mu\lambda}^x w_x; \quad \text{conj.} \end{array} \right.$$

$$(2.11) \quad \left\{ \begin{array}{l} \text{a)} \quad V_{\bar{\mu}} v^x = \partial_{\bar{\mu}} v^x + \Gamma_{\bar{\mu}\lambda}^x v^\lambda; \quad \text{conj.} \\ \text{b)} \quad V_{\bar{\mu}} w_\lambda = \partial_{\bar{\mu}} w_\lambda - \Gamma_{\bar{\mu}\lambda}^x w_x; \quad \text{conj.} \end{array} \right.$$

<sup>1)</sup> Cf. footnote 3) on p. 389.

They can be looked upon as parts of the covariant derivatives of the vectors  $v^x, v^{\bar{x}}$  and  $w_{\lambda}, w_{\bar{\lambda}}$  in the auxiliary  $X_{2n}$ .

The curvature tensors belonging to the connexion (2.2, 3) may be considered as the only non-vanishing parts of the curvature tensor in  $X_{2n}$ . There are eight of them:

$$(2.12) \quad \left\{ \begin{array}{ll} \text{a)} & R_{v\mu\lambda}^{;;x} = 2\partial_{[\nu} I_{\mu]\lambda}^x + 2I_{[\nu|\rho]}^x I_{\mu]\lambda}^{\rho}; \quad \text{conj.} \\ \text{b)} & R_{\bar{v}\mu\lambda}^{;;x} = \partial_{[\bar{\nu}} I_{\mu]\lambda}^x - \partial_{\mu} I_{\bar{\nu}\lambda}^x + I_{\bar{\nu}\rho}^x I_{\mu\lambda}^{\rho} - I_{\mu\rho}^x I_{\bar{\nu}\lambda}^{\rho}; \quad \text{conj.} \\ \text{c)} & R_{v\bar{\mu}\lambda}^{;;x} = -R_{\bar{\mu}v\lambda}^{;;x}; \quad \text{conj.} \quad (\text{isomer of (b)}) \\ \text{d)} & R_{\bar{v}\bar{\mu}\lambda}^{;;x} = 2\partial_{[\bar{\nu}} I_{\bar{\mu}]\lambda}^x + 2I_{[\bar{\nu}|\rho]}^x I_{\bar{\mu}]\lambda}^{\rho}; \quad \text{conj.} \end{array} \right.$$

the other eight,  $R_{v\mu\lambda}^{;;\bar{x}}, R_{\bar{v}\mu\lambda}^{;;\bar{x}}, R_{v\bar{\mu}\lambda}^{;;\bar{x}}, R_{\bar{v}\bar{\mu}\lambda}^{;;\bar{x}}$ , conj. being zero. The tensors of asymmetry can in the same way be looked upon as the only non-vanishing parts of the tensor of asymmetry in  $X_{2n}$ . There are six of them:

$$(2.13) \quad \left\{ \begin{array}{ll} \text{a)} & S_{\mu\lambda}^{;;x} = I_{[\mu\lambda]}^x; \quad \text{conj.} \\ \text{b)} & S_{\bar{\mu}\lambda}^{;;x} = \frac{1}{2} I_{\bar{\mu}\lambda}^x; \quad \text{conj.} \\ \text{c)} & S_{\mu\bar{\lambda}}^{;;x} = -S_{\bar{\lambda}\mu}^{;;x}; \quad \text{conj.} \quad (\text{isomer of (b)}) \end{array} \right.$$

the other two,  $S_{\mu\lambda}^{;;\bar{x}}$ , conj. being zero.

A contravariant vector of an  $X_n$  of the first invariant set is characterized by  $v^x \neq 0; v^{\bar{x}} = 0$ . If this vector is displaced parallel in a direction lying in an  $X_n$  of the second invariant set, we have  $d\xi^x = 0, d\xi^{\bar{x}} \neq 0$ , hence

$$(2.14) \quad dv^x = d\xi^{\bar{\mu}} I_{\bar{\mu}\lambda}^x v^{\lambda}.$$

But this proves that the transformation is an equipollence for every choice of  $v^x$ , if and only if  $I_{\bar{\mu}\lambda}^x = 0$ . Hence, if we introduce as a *third* invariant condition that the parallel displacement of a vector of an  $X_n$  of one invariant set in a direction lying in an  $X_n$  of the other set is always an equipollence, this leads to the equations

$$(2.15) \quad I_{\bar{\mu}\lambda}^x = 0; \quad I_{\mu\bar{\lambda}}^{\bar{x}} = 0.$$

Because  $I_{\bar{\mu}\lambda}^x$  and  $I_{\mu\bar{\lambda}}^{\bar{x}}$  are hybrid tensors as we have seen, the condition (2.15) is also invariant in the original  $X_n$ . Another geometrical meaning of (2.15) is that it is the n.a.s. condition for the existence of infinitesimal parallelograms in  $X_{2n}$  with two parallel sides in an arbitrary direction of an  $X_n$  of one invariant set and the other two in an arbitrary direction of an  $X_n$  of the other set. This is quite clear because  $I_{\lambda\bar{\mu}}^x$  and  $I_{\bar{\lambda}\mu}^{\bar{x}}$  are

already zero according to (2.8), and (2.15) is therefore equivalent to the vanishing of the components  $I_{[\bar{\mu}\lambda]}^\kappa, I_{[\mu\bar{\lambda}]}^\kappa$  of the tensor of asymmetry in  $X_{2n}$ . So (2.15) is in fact a kind of condition of symmetry of the connexion in  $X_{2n}$  but only halfway.

If the third condition is also satisfied we have<sup>1)</sup>

$$(2.16) \quad \left\{ \begin{array}{ll} \text{a)} & \nabla_\mu v^\kappa = \partial_\mu v^\kappa + I_{\mu\lambda}^\kappa v^\lambda; \quad \nabla_{\bar{\mu}} v^\kappa = \partial_{\bar{\mu}} v^\kappa; \quad \text{conj.} \\ \text{b)} & \nabla_\mu w_\lambda = \partial_\mu w_\lambda - I_{\mu\lambda}^\kappa w_\kappa; \quad \nabla_{\bar{\mu}} w_\lambda = \partial_{\bar{\mu}} w_\lambda; \quad \text{conj.} \end{array} \right.$$

and this connexion differs from an ordinary connexion in  $L_n$  as dealt with in III § 2 only by its special equations for fields that are only semi-analytic.

From (2.16) it can be derived in the ordinary way via the  $n$ -vectors that for a density with the weights  $w$  and  $w'$  there is for  $\nabla_\mu$  a term  $-w I_{\mu\tau}^\kappa$  and for  $\nabla_{\bar{\mu}}$  a term with  $-w' I_{\bar{\mu}\tau}^\kappa$ . An  $X_n$  with the connexion (2.16) is called an  $\tilde{L}_n$ . It is the most general linear connexion for hybrid quantities that satisfies all invariant conditions and in this way the  $\tilde{L}_n$  is analogous to the  $L_n$  in III § 2. The auxiliary  $X_{2n}$  of an  $\tilde{L}_n$  is an  $L_{2n}$ .

In an  $\tilde{L}_n$  only the curvature tensors  $R_{\nu\mu\lambda}^{\cdot\kappa}$ ; conj.,  $R_{\bar{\nu}\mu\lambda}^{\cdot\kappa}$ ; conj. and  $R_{\nu\bar{\mu}\lambda}^{\cdot\kappa}$ ; conj. remain and (2.12 b) takes the simpler form

$$(2.17) \quad R_{\nu\mu\lambda}^{\cdot\kappa} = \partial_\nu I_{\mu\lambda}^\kappa; \quad \text{conj.}$$

From the tensors of asymmetry only  $S_{\mu\lambda}^{\cdot\kappa}$ ; conj. remain.

In an  $\tilde{L}_n$  the quantities  $R_{\nu\mu\lambda}^{\cdot\kappa}$  and  $S_{\mu\lambda}^{\cdot\kappa}$  satisfy the identities (III 5.1), (III 5.2) and (III 5.19) and similar identities hold for their conjugates. The quantities  $R_{\nu\mu\lambda}^{\cdot\kappa}, R_{\bar{\nu}\mu\lambda}^{\cdot\kappa}, R_{\nu\bar{\mu}\lambda}^{\cdot\kappa}, S_{\mu\lambda}^{\cdot\kappa}$  and their conjugates satisfy the following identities that can be easily derived from (III 5.1, 2, 19) if these equations are applied to the  $X_{2n}$ :

$$(2.18) \quad R_{\bar{\nu}\mu\lambda}^{\cdot\kappa} = -R_{\mu\bar{\nu}\lambda}^{\cdot\kappa}; \quad \text{conj.} \quad (\text{first identity, cf. III 5.1})$$

$$(2.19) \quad \left\{ \begin{array}{ll} R_{\bar{\nu}[\mu\lambda]}^{\cdot\kappa} = \nabla_{\bar{\nu}} S_{\mu\lambda}^{\cdot\kappa} = \partial_{\bar{\nu}} S_{\mu\lambda}^{\cdot\kappa}; & \text{conj.} \\ & (\text{second identity, cf. III 5.2}) \end{array} \right.$$

$$(2.20) \quad \left\{ \begin{array}{ll} \text{a)} & 2\nabla_{[\omega} R_{\nu]\bar{\mu}\lambda}^{\cdot\kappa} + \partial_{\bar{\mu}} R_{\nu\cdot\lambda}^{\cdot\kappa} = 2S_{\omega\cdot\lambda}^{\cdot\kappa} R_{\bar{\mu}\cdot\lambda}^{\cdot\kappa}; \quad \text{conj.} \\ \text{b)} & \nabla_{[\bar{\omega}} R_{\nu]\mu\lambda}^{\cdot\kappa} = S_{\bar{\omega}\nu}^{\cdot\kappa} R_{\mu\lambda}^{\cdot\kappa}; \quad \text{conj.} \end{array} \right. \quad (\text{identity of BIANCHI, cf. III 5.19})$$

<sup>1)</sup> SCHOUTEN 1929, 1, p. 460.

The RICCI tensor  $R$  and the bivector  $V$  in  $L_{2n}$  have the non vanishing components

$$(2.21) \quad \left\{ \begin{array}{ll} \text{a)} & R_{\mu\lambda} = R_{\nu\mu}^{\cdot\cdot\lambda} + R_{\bar{\nu}\mu}^{\cdot\cdot\lambda} = R_{\nu\mu}^{\cdot\cdot\lambda}; \quad \text{conj.} \\ \text{b)} & R_{\bar{\nu}\lambda} = R_{\bar{\nu}\bar{\mu}}^{\cdot\cdot\lambda} + R_{\bar{\mu}\bar{\nu}}^{\cdot\cdot\lambda} = R_{\bar{\mu}\bar{\nu}}^{\cdot\cdot\lambda}; \quad \text{conj.} \\ \text{c)} & V_{\nu\mu} = R_{\nu\mu}^{\cdot\cdot\lambda} + R_{\bar{\nu}\mu}^{\cdot\cdot\lambda} = R_{\nu\mu}^{\cdot\cdot\lambda}; \quad \text{conj.} \\ \text{d)} & V_{\bar{\nu}\mu} = R_{\bar{\nu}\mu}^{\cdot\cdot\lambda} + R_{\bar{\nu}\bar{\mu}}^{\cdot\cdot\lambda} = V'_{\bar{\nu}\mu} - V'_{\mu\bar{\nu}}; \quad \text{conj.}; \\ & V'_{\bar{\nu}\mu} \stackrel{\text{def}}{=} R_{\bar{\nu}\mu}^{\cdot\cdot\lambda} = \partial_{\bar{\nu}} I_{\mu\lambda}^{\lambda} \end{array} \right.$$

but from (2.18) it does not follow that  $V'_{\bar{\nu}\mu} = -V'_{\mu\bar{\nu}}$ , because  $V'_{\mu\bar{\nu}}$  is according to our definitions the complex conjugate of  $V'_{\bar{\nu}\mu}$ , and thus equal to  $R_{\mu\bar{\nu}}^{\cdot\cdot\lambda}$  and not equal to  $R_{\mu\bar{\nu}}^{\cdot\cdot\lambda}$ . From (2.19) we get the equation

$$(2.22) \quad R_{\bar{\nu}\lambda} + V'_{\bar{\nu}\lambda} = 2\partial_{\bar{\nu}} S_{\lambda\mu}^{\cdot\cdot\mu} = (n-1)\partial_{\bar{\nu}} S_{\lambda},$$

which will be used hereafter. From (2.21) we see that  $V_{\bar{\nu}\mu}$  (and not  $V'_{\bar{\nu}\mu}$ ) is always a hybrid bivector and that  $V'_{\bar{\nu}\mu}$  is hermitian if and only if  $V_{\bar{\nu}\mu} = 0$ . From (2.20) we derive by contraction over  $\kappa\lambda$

$$(2.23) \quad \boxed{\begin{array}{ll} \text{a)} & 2V_{[\omega} V'_{|\bar{\mu}| \nu]} - \partial_{\bar{\mu}} V_{\omega\nu} = -2S_{\omega\nu}^{\cdot\cdot\mu} V'_{\bar{\mu}\mu}; \quad \text{conj.} \\ \text{b)} & V_{[\bar{\omega}} V'_{\bar{\nu}]\mu} = -S_{\bar{\omega}\bar{\nu}}^{\cdot\cdot\bar{\mu}} V'_{\bar{\mu}\mu}; \quad \text{conj.} \end{array}}$$

from which, by means of (2.21 d)

$$(2.24) \quad 2V_{[\bar{\omega}} V'_{\bar{\nu}]\mu} + \partial_{\mu} V_{\bar{\omega}\bar{\nu}} = -2S_{\bar{\omega}\bar{\nu}}^{\cdot\cdot\bar{\mu}} V'_{\bar{\mu}\mu}; \quad \text{conj.}$$

in accordance with (III 5.22). By contraction over  $\kappa\nu$  and  $\kappa\mu$  we get from (2.20a) and (2.20b)

$$(2.25) \quad \boxed{V_{\omega} R_{\bar{\mu}\lambda} + V_{\nu} R_{\bar{\mu}\bar{\omega}}^{\cdot\cdot\lambda} - \partial_{\bar{\mu}} R_{\omega\lambda} = 2S_{\omega\nu}^{\cdot\cdot\mu} R_{\bar{\mu}\bar{\mu}}^{\cdot\cdot\lambda}; \quad \text{conj.}}$$

$$(2.26) \quad \boxed{V_{[\bar{\nu}} R_{\bar{\omega}]\lambda} = -S_{\bar{\nu}\bar{\omega}}^{\cdot\cdot\bar{\mu}} R_{\bar{\mu}\lambda}; \quad \text{conj.}}$$

that could also be derived from (III 5.25). (2.23 b) and (2.26) are in accordance with (2.22).

An  $\tilde{L}_n$  with a symmetric connexion is called an  $\tilde{A}_n$ . It is analogous to the  $A_n$  of III § 2. The auxiliary  $X_{2n}$  of an  $\tilde{A}_n$  is an  $A_{2n}$ . In an  $\tilde{A}_n$  we have instead of (2.19, 20, 22, 23, 24, 25, 26)

$$(2.27) \quad R_{\bar{\nu}[\mu}^{\cdot\cdot\lambda}\zeta = 0; \quad \text{conj.}$$

$$(2.28) \quad \left\{ \begin{array}{ll} \text{a)} & 2V_{[\omega} R_{\nu]\bar{\mu}}^{\cdot\cdot\lambda}\zeta + \partial_{\bar{\mu}} R_{\omega\nu}^{\cdot\cdot\lambda}\zeta = 0; \quad \text{conj.} \\ \text{b)} & V_{[\bar{\omega}} R_{\bar{\nu}]\mu}^{\cdot\cdot\lambda}\zeta = 0; \quad \text{conj.} \end{array} \right.$$

$$(2.29) \quad R_{\bar{v}\lambda} = -V'_{\bar{v}\lambda} = -\partial_{\bar{v}} I^{\mu}_{\lambda\mu}; \quad \text{conj.}$$

$$(2.30) \quad \begin{cases} \text{a)} & 2V_{[\omega} V'_{|\bar{\mu}|}{}_{\nu]} - \partial_{\bar{\mu}} V_{\omega\nu} = -2V_{[\omega} R_{|\bar{\mu}|}{}_{\nu]} + 2\partial_{\bar{\mu}} R_{[\omega}{}_{\nu]} = 0; \quad \text{conj.} \\ \text{b)} & V_{[\bar{\omega}} V'_{\bar{v}]\mu} = -V_{[\bar{\omega}} R_{\bar{v}]\mu} = 0; \quad \text{conj.} \end{cases}$$

$$(2.31) \quad 2V_{[\bar{\omega}} V_{\bar{v}]\mu} + \partial_{\mu} V_{\bar{\omega}\bar{v}} = 0; \quad \text{conj.}$$

$$(2.32) \quad V_{\omega} R_{\bar{\mu}\lambda} + V_{\nu} R_{\bar{\mu}\omega}{}^{\nu} - \partial_{\bar{\mu}} R_{\omega\lambda} = 0; \quad \text{conj.}$$

$$(2.33) \quad V_{[\bar{v}} R_{\bar{\mu}]\lambda} = 0; \quad \text{conj.}$$

Of course the equations (III 5.2-4) remain valid, hence we have in an  $\tilde{L}_n$ :

$$(2.34) \quad R_{[\nu\mu\lambda]}{}^{\kappa} = 2V_{[\nu} S_{\mu\lambda]}{}^{\kappa} - 4S_{[\nu\mu}{}^{\rho} S_{\lambda]}{}_{\rho}{}^{\kappa}; \quad \text{conj.,}$$

in a semi-symmetric  $\tilde{L}_n$ :

$$(2.35) \quad R_{[\nu\mu\lambda]}{}^{\kappa} = 2A_{[\nu} V_{\mu} S_{\lambda]}{}^{\kappa}; \quad \text{conj.,}$$

and in an  $\tilde{A}_n$ :

$$(2.36) \quad R_{[\nu\mu\lambda]}{}^{\kappa} = 0; \quad \text{conj.}$$

### Exercises.

VIII 2,1<sup>1</sup>). Let  $v^*$  be a semi-analytic field and  $v^{\bar{x}}$  its complex conjugate. This field is analytic if and only if the field  $v^*, 0 (0, v^{\bar{x}})$  in  $X_{2n}$  is constant over each  $X_n$  of the second (first) invariant set.

A semi-analytic scalar field is analytic in  $\xi^*$  ( $\xi^{\bar{x}}$ ) if and only if it is constant in every  $X_n$  of the second (first) invariant set. It is the product of a scalar analytic in  $\xi^*$  with a scalar analytic in  $\xi^{\bar{x}}$  if and only if

$$\text{VIII 2,1 } \alpha) \quad \varphi \partial_{\bar{v}} \partial_{\mu} \varphi = (\partial_{\bar{v}} \varphi) \partial_{\mu} \varphi.$$

VIII 2,2<sup>2</sup>). An analytic transformation of  $X_{2n}$  is an analytic transformation of  $X_n$  if and only if it leaves invariant both invariant sets of  $X_n$ 's and the principal  $X_n$ .

## § 3. Unitary connexions.

If a hermitian tensor field  $a_{\lambda\bar{x}}$  of rank  $n$  is given in  $X_n$ , we may ask for a connexion of the kind (2.2, 3) for which

$$(3.1) \quad \nabla_{\mu} a_{\lambda\bar{x}} = 0; \quad \text{conj.}$$

or

$$(3.2) \quad \partial_{\mu} a_{\lambda\bar{x}} - \Gamma_{\mu\lambda}^{\rho} a_{\rho\bar{x}} - \Gamma_{\mu\bar{x}}^{\bar{\sigma}} a_{\lambda\bar{\sigma}} = 0; \quad \text{conj.}$$

<sup>1</sup>) SCHOUTEN and v. DANTZIG 1930, 1, p. 328 f.; E II 1938, 2, p. 236.

<sup>2</sup>) SCHOUTEN and v. DANTZIG 1930, 1, p. 329.

From (3.2) we get immediately

$$(3.3) \quad \Gamma_{\mu\lambda}^{\kappa} = a^{\bar{q}\kappa} \partial_{\mu} a_{\lambda\bar{q}} - \Gamma_{\mu\bar{q}}^{\bar{q}} a_{\lambda\bar{q}} a^{\bar{q}\kappa}; \quad \text{conj.}$$

which equation expresses  $\Gamma_{\mu\lambda}^{\kappa}$  in terms of  $a_{\lambda\bar{q}}$  and its first derivatives and a hybrid tensor  $\Gamma_{\mu\bar{q}}^{\bar{q}}$  that could be chosen arbitrarily and that is zero if we impose the condition that the  $X_n$  be an  $\tilde{L}_n$ . Then we get<sup>1)</sup>

$$(3.4) \quad \begin{cases} \text{a)} & \Gamma_{\mu\lambda}^{\kappa} = a^{\bar{q}\kappa} \partial_{\mu} a_{\lambda\bar{q}}; \quad \text{conj.} \\ \text{b)} & \Gamma_{\mu\lambda}^{\kappa} = 0; \quad \text{conj.} \end{cases}$$

Hence:

*A hermitian tensor field  $a_{\lambda\bar{q}}$  of rank  $n$  in an  $X_n$  determines one and only one  $\tilde{L}_n$ -connexion for which  $a_{\lambda\bar{q}}$  is covariant constant.*

Such a connexion is called *unitary* and an  $X_n$  with a unitary connexion is called a  $\tilde{U}_n$ .<sup>2)</sup>  $a_{\lambda\bar{q}}$  is called the *fundamental tensor* of the  $\tilde{U}_n$ . It is analogous to the  $U_n$  of III § 3.

In the auxiliary  $X_{2n}$  of  $\xi^{\kappa}$  and  $\xi^{\bar{\kappa}}$  we now use indices  $A, B, \dots$  taking the values  $1, \dots, n, \bar{1}, \dots, \bar{n}$ . Then the  $a_{\lambda\bar{q}}$  are the components of a symmetric tensor  $a_{BA}$  with  $a_{\lambda\kappa} = 0, a_{\bar{\lambda}\bar{q}} = 0$ . Hence the invariant  $X_n$ 's are isotropic (cf. I § 9) with respect to this fundamental tensor. The most general linear connexion in  $X_{2n}$  leaving  $a_{BA}$  invariant is (cf. III 3.5)

$$(3.5) \quad \Gamma_{CB}^A = \frac{1}{2} a^{AD} (\partial_C a_{DB} + \partial_B a_{DC} - \partial_D a_{CB}) + S_{C\dot{B}}^{\dot{A}} - S_{B\cdot C}^{\cdot A} - S_{C\cdot B}^{\cdot A}$$

where  $S_{C\dot{B}}^{\dot{A}}$  is an arbitrary tensor alternating in  $CB$ , or

$$(3.6) \quad \begin{cases} \text{a)} & \Gamma_{\mu\lambda}^{\kappa} = a^{\bar{q}\kappa} \partial_{(\mu} a_{\lambda)\bar{q}} + S_{\mu\dot{\lambda}}^{\dot{\kappa}} - 2S_{(\mu\cdot\lambda)}^{\cdot\kappa} \\ \text{b)} & \Gamma_{\mu\lambda}^{\kappa} = a^{\bar{q}\kappa} \partial_{[\mu} a_{\bar{q}]\lambda} + S_{\dot{\mu}\dot{\lambda}}^{\cdot\kappa} - S_{\dot{\mu}\cdot\lambda}^{\cdot\kappa} \\ \text{c)} & \Gamma_{\mu\lambda}^{\kappa} = a^{\bar{q}\kappa} \partial_{[\lambda} a_{\bar{q}] \mu} + S_{\mu\dot{\lambda}}^{\dot{\kappa}} - S_{\dot{\lambda}\cdot\mu}^{\cdot\kappa} - S_{\mu\cdot\lambda}^{\cdot\kappa} \\ \text{d)} & \Gamma_{\mu\lambda}^{\kappa} = S_{\mu\dot{\lambda}}^{\dot{\kappa}} - S_{\dot{\lambda}\cdot\mu}^{\cdot\kappa} - S_{\mu\cdot\lambda}^{\cdot\kappa} \end{cases}$$

and four expressions for  $\Gamma_{\bar{\mu}\bar{\lambda}}^{\bar{\kappa}}, \Gamma_{\bar{\mu}\lambda}^{\bar{\kappa}}, \Gamma_{\bar{\mu}\lambda}^{\kappa}$  and  $\Gamma_{\mu\bar{\lambda}}^{\bar{\kappa}}$ . Now these latter four expressions must be the complex conjugates of the expressions (3.6) in consequence of the condition that the principal  $X_n$  in the  $X_{2n}$  be geodesic. But this is possible if and only if  $S_{\mu\dot{\lambda}}^{\dot{\kappa}}, S_{\dot{\mu}\dot{\lambda}}^{\cdot\kappa}$  and  $S_{\mu\cdot\lambda}^{\cdot\kappa}$  are the complex conjugates of  $S_{\mu\dot{\lambda}}^{\dot{\kappa}}, S_{\dot{\mu}\dot{\lambda}}^{\cdot\kappa}$  and  $S_{\mu\cdot\lambda}^{\cdot\kappa}$  respectively. If now we introduce the condition (2.8) expressing that the invariant  $X_n$ 's of either set are geodesic and parallel and the condition (2.15) for the equipollence of

<sup>1)</sup> Cf. footnote 3) on p. 389.

<sup>2)</sup> In 1930, 3 we wrote  $U_n$  and in E II 1938, 2 we used the term *unitary*  $K_n$  instead of  $\tilde{U}_n$ . The notations  $\tilde{L}_n, \tilde{A}_n, \tilde{U}_n, \tilde{V}_n, \tilde{S}_n$  and  $\tilde{C}_n$  used now are in strict analogy to the  $L_n, A_n, U_n, V_n, S_n$  and  $C_n$  of Ch. III and VI.

parallel invariant  $X_n$ 's we get

$$(3.7) \quad S_{\mu\lambda}^{\cdot\bar{x}} = 0; \quad S_{\bar{\mu}\lambda}^{\cdot x} = 0; \quad \text{conj.}$$

and

$$(3.8) \quad S_{\bar{\mu}\bar{x}\lambda} = \partial_{[\bar{\mu}} a_{\bar{x}]}\lambda; \quad \text{conj.}$$

or

$$(3.9) \quad S_{\mu\lambda}^{\cdot\bar{x}} = a^{\bar{q}\bar{x}} \partial_{[\mu} a_{\lambda]}\bar{q}; \quad \text{conj.}$$

hence

$$(3.10) \quad \begin{cases} \text{a)} & I_{\mu\lambda}^x = a^{\bar{q}\bar{x}} \partial_{\mu} a_{\lambda}\bar{q}; \quad \text{conj.} \\ \text{b)} & I_{\bar{\mu}\lambda}^x = 0; \quad \text{conj.} \end{cases}$$

in accordance with (3.4). This proves that the connexion in a  $\tilde{U}_n$  can be derived as a connexion in  $X_{2n}$  with the symmetric fundamental tensor  $a_{BA}$  with  $a_{\lambda x} = 0$ ,  $a_{\bar{\lambda}\bar{x}} = 0$ , satisfying instead of the condition of symmetry other conditions concerning the invariant  $X_n$ 's. Hence the auxiliary  $X_{2n}$  of a  $\tilde{U}_n$  is a  $U_{2n}$ .<sup>1)</sup> In general the connexion of  $\tilde{U}_n$  is not symmetric. If it is symmetric we call the  $\tilde{U}_n$  a  $\tilde{V}_n$ . The  $\tilde{V}_n$  is analogous to the  $V_n$  of III § 3 and its auxiliary  $X_{2n}$  is a  $V_{2n}$ . We prove that  $S_{\mu\lambda}^{\cdot\bar{x}} = 0$ ; conj. if and only if there exists a semi-analytic scalar field  $\varphi$  such that

$$(3.11) \quad a_{\bar{x}x} = \partial_{\bar{x}} \partial_x \varphi^2)$$

We have only to prove that (3.11) holds if  $S_{\mu\lambda}^{\cdot\bar{x}} = 0$ ; conj. because the inverse is trivial.<sup>3)</sup> In order to prove this we remark that the equation

$$(3.12) \quad \partial_{\bar{x}} \varphi_x = a_{\bar{x}x}$$

with the unknowns  $\varphi_x$  is totally integrable because of (3.2) and  $S_{\bar{\mu}\lambda}^{\cdot\bar{x}} = 0$ ; conj. if the  $\xi^x$  are considered as parameters in (3.12). Hence, if  $\varphi_x$  is a solution of (3.12) the general solution has the form

$$(3.13) \quad \varphi_x = \varphi_x^0(\xi^0, \xi^{\bar{0}}) + f_x(\xi^0)$$

<sup>1)</sup> Cf. ROZENFEL'D 1949, 1.

<sup>2)</sup> SCHOUTEN and v. DANTZIG 1930, 1 for a special case and KÄHLER 1933, 1 for the general case; cf. LEE 1942, 1; SASAKI 1949, 2; IWAMOTO 1950, 1, p. 125.

<sup>3)</sup> The symmetric unitary connexion or  $\tilde{V}_n$  appeared first in SCHOUTEN and v. DANTZIG 1930, 1, p. 333. In 1933, 1 BERGMAN gave an application of this connexion to the theory of functions of two complex variables by means of his "kernel function". This application has been dealt with since in many publications of BERGMAN and others. In 1933, 1 KÄHLER independently found the same connexion and formulated the condition of symmetry by means of CARTAN's alternating differential forms. A  $\tilde{V}_n$  is sometimes called "KÄHLER space" in the literature. Cf. for instance ECKMANN and GUGGENHEIMER 1949, 1; 2. HODGE 1951, 1; BOCHNER 1953, 1; GUGGENHEIMER 1953, 1. There are a great number of publications of these and other authors dealing with global properties of the  $\tilde{V}_n$ .

where the  $f_\lambda$  are arbitrary functions. This proves that  $\varphi$  is a solution of

$$(3.14) \quad \partial_\lambda \varphi = \varphi_\lambda^\theta + f_\lambda(\xi^\alpha).$$

The integrability conditions of (3.14) are

$$(3.15) \quad \partial_{[\mu} \varphi_{\lambda]}^\theta + \partial_{[\mu} f_{\lambda]} = 0; \quad \text{conj.}$$

From (3.9, 12) and the vanishing of  $S_{\mu\lambda}^\alpha$  it follows that

$$(3.16) \quad \partial_{\bar{\nu}} \partial_{[\mu} \varphi_{\lambda]}^\theta = \partial_{[\mu} \partial_{|\bar{\nu}|} \varphi_{\lambda]}^\theta = \partial_{[\mu} a_{\lambda]} \bar{\nu} = 0$$

and this proves that  $\partial_{[\mu} \varphi_{\lambda]}$  depends on the  $\xi^\alpha$  only and that accordingly  $f_\lambda$  can always be chosen such that (3.15) is satisfied identically.

If  $\varphi$  is a solution of (3.11),  $\bar{\varphi}$  is also a solution and this proves that if (3.11) has solutions, there always exists a *real* solution.

In a  $\tilde{U}_n$  we have

$$(3.17) \quad \partial_{[\nu} \Gamma_{\mu]\lambda}^\alpha + \Gamma_{[\nu|\sigma|}^\alpha \Gamma_{\mu]}^\sigma \lambda = \partial_{[\nu} a^{\bar{q}\alpha} \partial_{\mu]} a_{\lambda\bar{q}} + (a^{\bar{q}\alpha} \partial_{[\nu} a_{|\sigma\bar{q}|}) \partial_{\mu]} a_{\lambda\bar{q}} = 0$$

hence

$$(3.18) \quad R_{\nu\mu\lambda}^\alpha = 0; \quad \text{conj.}$$

and the only remaining curvature tensors are  $R_{\bar{\nu}\mu\lambda}^\alpha$ ; conj. and  $R_{\bar{\nu}\mu\lambda}^\alpha$ ; conj. As in Riemannian geometry it is convenient to lower the last index of  $R_{\bar{\nu}\mu\lambda}^\alpha$ . Then we get from (2.17) and (3.4a)

$$(3.19) \quad \left\{ \begin{array}{l} \text{a)} \quad R_{\bar{\nu}\mu\lambda\bar{q}} = \partial_{\bar{\nu}} \partial_\mu a_{\lambda\bar{q}} - a^{\bar{q}\sigma} (\partial_{\bar{\nu}} a_{\sigma\bar{q}}) \partial_\mu a_{\lambda\bar{q}}; \quad \text{conj.} \\ \text{b)} \quad R_{\bar{\nu}\lambda} = -\partial_{\bar{\nu}} a^{\bar{q}\mu} \partial_\mu a_{\lambda\bar{q}} \\ \quad = -\partial_{\bar{\nu}} \partial_\lambda \log a + (n-1) \partial_{\bar{\nu}} S_\lambda; \quad \text{conj.} \\ \text{c)} \quad V'_{\bar{\nu}\mu} = \partial_{\bar{\nu}} \Gamma_{\mu\lambda}^\lambda = \partial_{\bar{\nu}} \partial_\mu \log a = V'_{\mu\bar{\nu}}; \quad a \stackrel{\text{def}}{=} |\text{Det}(a_{\lambda\bar{q}})| \\ \text{d)} \quad V_{\bar{\nu}\mu} = 0; \quad \text{conj.} \end{array} \right.$$

The vanishing of  $V_{\bar{\nu}\mu}$  is a consequence of the fact that the auxiliary  $X_{2n}$  is a  $U_{2n}$ . But as we have already seen this vanishing implies that  $V'_{\bar{\nu}\mu}$  is hermitian and this is in accordance with (3.19c).

The following identities hold in a  $\tilde{U}_n$ <sup>1)</sup>

$$(3.20) \quad R_{\bar{\nu}\mu\lambda\bar{q}} = -R_{\mu\bar{\nu}\lambda\bar{q}}; \quad \text{conj.}$$

(first identity, cf. III 5.1; VIII 2.18)

$$(3.21) \quad R_{\bar{\nu}[\mu\lambda]\bar{q}} = V_{\bar{\nu}} S_{\mu\lambda\bar{q}}; \quad \text{conj.}$$

(second identity, cf. III 5.2; VIII 2.19)

<sup>1)</sup> E II 1938, 2, p. 234–239.

$$(3.22) \quad R_{\bar{\nu}\mu\lambda\bar{\kappa}} = -R_{\bar{\nu}\mu\bar{\kappa}\lambda}; \quad \text{conj.} \\ (\text{third identity, cf. III 5.13})$$

$$(3.23) \quad R_{\bar{\nu}\mu\lambda\bar{\kappa}} - R_{\lambda\bar{\kappa}\bar{\nu}\mu} = 2V_{\bar{\nu}}S_{\mu\lambda\bar{\kappa}} - 2V_{\lambda}S_{\bar{\kappa}\bar{\nu}\mu}; \quad \text{conj.} \\ (\text{fourth identity, cf. III 5.15 IV})$$

$$(3.24) \quad V_{[\omega}R_{\nu]\bar{\mu}\lambda\bar{\kappa}} = S_{\omega\nu}{}^{\epsilon}R_{\bar{\mu}\epsilon\lambda\bar{\kappa}}; \quad \text{conj.} \\ (\text{identity of BIANCHI, cf. III 5.19, VIII 2.20})^1)$$

In consequence of (3.22) the equations (2.20a, b) lead only to one equation (3.24). The same holds for the equations (2.23a, b) which lead to one equation

$$(3.25) \quad V_{[\omega}V'_{\nu]\bar{\mu}\nu]} = -S_{\omega\nu}{}^{\epsilon}V'_{\bar{\mu}\epsilon}; \quad \text{conj.}$$

In (2.25) the term  $-\partial_{\bar{\mu}}R_{\omega\lambda}$  drops out, (2.26) remains valid without simplifications and (2.34, 35) take the form

$$(3.26) \quad 2V_{[\nu}S_{\mu\lambda]}{}^{\times} - 4S_{[\nu}{}^{\epsilon}S_{\lambda]}{}^{\epsilon} = 0; \quad \text{conj.}$$

for the general  $\tilde{U}_n$  and

$$(3.27) \quad V_{[\mu}S_{\lambda]} = 0; \quad \text{conj.}$$

for the semi-symmetric  $\tilde{U}_n$ ,  $n > 2$ . Hence *in a semi-symmetric*  $\tilde{U}_n$ ,  $n > 2$ ,  $S_{\lambda}$  *always is a gradient vector*.

In a  $\tilde{V}_n$  we have instead of (3.19b) [(cf. III 5.8) but consider the difference between  $V_{\bar{\nu}\lambda}$  and  $V'_{\bar{\nu}\lambda}$ ]

$$(3.28) \quad R_{\bar{\nu}\lambda} = -\partial_{\bar{\nu}}\partial_{\lambda}\log a = R_{\lambda\bar{\nu}} = -V'_{\bar{\nu}\lambda}; \quad \text{conj.}$$

hence  $R_{\bar{\nu}\lambda}$  is now also hermitian.<sup>2)</sup> The identities (3.20–25) take the simpler form

$$(3.29) \quad \begin{cases} \text{a)} \quad R_{\bar{\nu}\mu\lambda\bar{\kappa}} = -R_{\mu\bar{\nu}\lambda\bar{\kappa}}; \quad \text{conj.} & (1^\circ \text{id.}; \text{cf. III 5.15 I}) \\ \text{b)} \quad R_{\bar{\nu}\mu\lambda\bar{\kappa}} = R_{\bar{\nu}\lambda\mu\bar{\kappa}}; \quad \text{conj.} & (2^\circ \text{id.}; \text{cf. III 5.15 II}) \\ \text{c)} \quad R_{\bar{\nu}\mu\lambda\bar{\kappa}} = -R_{\bar{\nu}\mu\bar{\kappa}\lambda}; \quad \text{conj.} & (3^\circ \text{id.}; \text{cf. III 5.15 III}) \\ \text{d)} \quad R_{\bar{\nu}\mu\lambda\bar{\kappa}} = R_{\lambda\bar{\kappa}\bar{\nu}\mu}; \quad \text{conj.} & (4^\circ \text{id.}; \text{cf. III 5.15 IV}) \end{cases}$$

$$(3.30) \quad V_{[\omega}R_{\nu]\bar{\mu}\lambda\bar{\kappa}} = 0; \quad \text{conj.}$$

$$(3.31) \quad V_{[\omega}V'_{\nu]\bar{\mu}\nu]} = 0; \quad \text{conj.}$$

<sup>1)</sup> HOMBU gave 1935, 1 a complete list of all identities containing first derivatives of  $R$  and  $S$ .

<sup>2)</sup> SCHOUTEN and v. DANTZIG 1930, 1, p. 337; KÄHLER 1933, 1; FUCHS 1937, 1.

| The complex conjugates of all equations are valid | $\tilde{L}_n$                                                                                                                                                                                                             | $\tilde{A}_n$                                                                                                                                                                    |
|---------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Definition                                        | $\Gamma_{\bar{\mu}\lambda}^{\kappa} = 0; \quad \Gamma_{\mu\bar{\lambda}}^{\kappa} = 0; \quad \Gamma_{\mu\bar{\lambda}}^{\bar{\kappa}} = 0$                                                                                | $S_{\mu\bar{\lambda}}^{\cdot\cdot\kappa} = 0;$                                                                                                                                   |
| Curvature quantities                              | —                                                                                                                                                                                                                         | —                                                                                                                                                                                |
|                                                   | (2.12 a) $R_{\nu\mu\bar{\lambda}}^{\cdot\cdot\kappa} = 2\partial_{[\nu}\Gamma_{\mu]}^{\kappa}\bar{\lambda} + 2\Gamma_{[\nu \varrho]}^{\kappa}R_{\mu]\varrho}^{\varrho}$                                                   | see $\tilde{L}_n$                                                                                                                                                                |
|                                                   | (2.17) $R_{\bar{\nu}\mu\bar{\lambda}}^{\cdot\cdot\kappa} = \partial_{\bar{\nu}}\Gamma_{\mu\bar{\lambda}}^{\kappa}$                                                                                                        | see $\tilde{L}_n$                                                                                                                                                                |
|                                                   | (2.21 b) $R_{\bar{\nu}\lambda} = -R_{\bar{\nu}\mu\bar{\lambda}}^{\cdot\cdot\mu} = -\partial_{\bar{\nu}}\Gamma_{\mu\bar{\lambda}}^{\mu}$                                                                                   | (2.29) $R_{\bar{\nu}\lambda} = -\partial_{\bar{\nu}}\Gamma_{\lambda\mu}^{\mu}$                                                                                                   |
|                                                   | (2.21 c) $V'_{\bar{\nu}\mu} = R_{\bar{\nu}\mu\bar{\lambda}}^{\cdot\cdot\lambda} = \partial_{\bar{\nu}}\Gamma_{\mu\bar{\lambda}}^{\lambda}$                                                                                | see $\tilde{L}_n$                                                                                                                                                                |
| Identities                                        | (2.21 d) $V_{\bar{\nu}\mu} = V'_{\bar{\nu}\mu} - V'_{\mu\bar{\nu}}$                                                                                                                                                       | see $\tilde{L}_n$                                                                                                                                                                |
|                                                   | (2.34) $R_{[\nu\mu\bar{\lambda}]}^{\cdot\cdot\kappa} = 2V_{[\nu}S_{\mu\bar{\lambda}]}^{\cdot\cdot\kappa} - 4S_{[\nu\mu}^{\cdot\cdot\sigma}S_{\bar{\lambda}]}^{\cdot\cdot\kappa}$                                          | (2.36) $R_{[\nu\mu\bar{\lambda}]}^{\cdot\cdot\kappa} = 0$                                                                                                                        |
|                                                   | (2.35) $R_{[\nu\mu\bar{\lambda}]}^{\cdot\cdot\kappa} = 2A_{[\nu}^{\kappa}V_{\mu}S_{\bar{\lambda}]}^{\cdot\cdot\kappa}$ for $S_{\mu\bar{\lambda}}^{\cdot\cdot\kappa} = S_{[\mu}A_{\bar{\lambda}]}^{\kappa}$                | see $\tilde{L}_n$                                                                                                                                                                |
|                                                   | (2.18) $R_{\bar{\nu}\mu\bar{\lambda}}^{\cdot\cdot\kappa} = -R_{\mu\bar{\nu}\bar{\lambda}}^{\cdot\cdot\kappa}$                                                                                                             | (2.27) $R_{\bar{\nu}[\mu\bar{\lambda}]}^{\cdot\cdot\kappa} = 0$                                                                                                                  |
|                                                   | (2.19) $R_{\bar{\nu}[\mu\bar{\lambda}]}^{\cdot\cdot\kappa} = \partial_{\bar{\nu}}S_{\mu\bar{\lambda}}^{\cdot\cdot\kappa}$                                                                                                 | —                                                                                                                                                                                |
| Identities with differentiated $R$                | —                                                                                                                                                                                                                         | —                                                                                                                                                                                |
|                                                   | —                                                                                                                                                                                                                         | —                                                                                                                                                                                |
|                                                   | (2.22) $R_{\bar{\nu}\lambda} + V'_{\bar{\nu}\lambda} = (n-1)\partial_{\bar{\nu}}S_{\lambda}$                                                                                                                              | (2.29) $R_{\bar{\nu}\lambda} + V'_{\bar{\nu}\lambda} = 0$                                                                                                                        |
|                                                   | (cf. III) $V_{[\omega}R_{\nu\mu]}^{\cdot\cdot\kappa} = 2S_{[\omega\nu}^{\cdot\cdot\sigma}R_{\mu]}^{\cdot\cdot\kappa}$                                                                                                     | (cf. III) $V_{[\omega}R_{\nu\mu]}^{\cdot\cdot\kappa} = 0$                                                                                                                        |
|                                                   | (5.19) $V_{[\omega}R_{\nu\mu]}^{\cdot\cdot\kappa} = 2S_{[\omega\nu}^{\cdot\cdot\sigma}R_{\mu]}^{\cdot\cdot\kappa}$                                                                                                        | (5.21) $V_{[\omega}R_{\nu\mu]}^{\cdot\cdot\kappa} = 0$                                                                                                                           |
|                                                   | (2.20 a) $2V_{[\omega}R_{\nu\mu]}^{\cdot\cdot\kappa} + V_{\bar{\mu}}R_{\omega\nu\bar{\lambda}}^{\cdot\cdot\kappa} = 2S_{\omega\nu}^{\cdot\cdot\sigma}R_{\bar{\mu}\bar{\lambda}}^{\cdot\cdot\kappa}$                       | (2.28 a) $2V_{[\omega}R_{\nu\mu]}^{\cdot\cdot\kappa} + V_{\bar{\mu}}R_{\omega\nu\bar{\lambda}}^{\cdot\cdot\kappa} = 0$                                                           |
|                                                   | (2.20 b) $V_{[\bar{\omega}}R_{\bar{\nu}\mu\bar{\lambda}]}^{\cdot\cdot\kappa} = S_{\bar{\omega}\bar{\nu}}^{\cdot\cdot\bar{\sigma}}R_{\mu\bar{\lambda}}^{\cdot\cdot\kappa}$                                                 | (2.28 b) $V_{[\bar{\omega}}R_{\bar{\nu}\mu\bar{\lambda}]}^{\cdot\cdot\kappa} = 0$                                                                                                |
|                                                   | (2.23 a) $2V_{[\omega}V'_{ \bar{\mu} ,\nu]} - \partial_{\bar{\mu}}V_{\omega\nu} = -2S_{\omega\nu}^{\cdot\cdot\sigma}V'_{\bar{\mu}\sigma}$                                                                                 | (2.30 a) $\begin{cases} 2V_{[\omega}V'_{ \bar{\mu} ,\nu]} - V_{\bar{\mu}}V_{\omega\nu} = -2V_{[\omega}R_{ \bar{\mu} ,\nu]} + \\ + 2V_{\bar{\mu}}R_{[\omega\nu]} = 0 \end{cases}$ |
|                                                   | (2.23 b) $V_{[\bar{\omega}}V'_{\bar{\nu}]\mu} = -S_{\bar{\omega}\bar{\nu}}^{\cdot\cdot\bar{\sigma}}V'_{\bar{\sigma}\mu}$                                                                                                  | (2.30 b) $V_{[\bar{\omega}}V'_{\bar{\nu}]\mu} = -V_{[\bar{\omega}}R_{\bar{\nu}]\mu} = 0$                                                                                         |
|                                                   | (2.24) $2V_{[\bar{\omega}}V_{\bar{\nu}]\mu} + \partial_{\mu}V_{\bar{\omega}\bar{\nu}} = -2S_{\bar{\omega}\bar{\nu}}^{\cdot\cdot\bar{\sigma}}V_{\bar{\sigma}\mu}$                                                          | (2.31) $2V_{[\bar{\omega}}V_{\bar{\nu}]\mu} + V_{\mu}V_{\bar{\omega}\bar{\nu}} = 0$                                                                                              |
|                                                   | (2.25) $V_{\omega}R_{\bar{\mu}\lambda} + V_{\nu}R_{\bar{\mu}\omega\bar{\lambda}}^{\cdot\cdot\bar{\nu}} - V_{\bar{\mu}}R_{\omega\lambda} = 2S_{\omega\nu}^{\cdot\cdot\sigma}R_{\bar{\mu}\bar{\lambda}}^{\cdot\cdot\sigma}$ | (2.32) $V_{\omega}R_{\bar{\mu}\lambda} + V_{\nu}R_{\bar{\mu}\omega\bar{\lambda}}^{\cdot\cdot\bar{\nu}} - V_{\bar{\mu}}R_{\lambda\omega} = 0$                                     |
|                                                   | (2.26) $V_{[\bar{\nu}}R_{\bar{\omega}]\lambda} = -S_{\bar{\nu}\bar{\omega}}^{\cdot\cdot\bar{\sigma}}R_{\bar{\sigma}\lambda}$                                                                                              | (2.33) $V_{[\bar{\nu}}R_{\bar{\mu}]\lambda} = 0$                                                                                                                                 |
|                                                   | —                                                                                                                                                                                                                         | —                                                                                                                                                                                |
|                                                   | —                                                                                                                                                                                                                         | —                                                                                                                                                                                |

| $\tilde{U}_n$                                                                                                                                                                            | $\tilde{V}_n$                                                                                                                      |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------|
| $\nabla_\mu a_{\lambda\bar{\lambda}} = 0; \quad S_{\mu\lambda}^{\cdot\cdot\kappa} + 0$                                                                                                   | $\nabla_\mu a_{\lambda\bar{\lambda}} = 0; \quad S_{\mu\lambda}^{\cdot\cdot\kappa} = 0$                                             |
| (3.4a) $I_{\mu\lambda}^\kappa = a^{\bar{\alpha}\kappa} \partial_\mu a_{\lambda\bar{\alpha}}$                                                                                             | see $\tilde{U}_n$                                                                                                                  |
| (3.18) $R_{\nu\mu\lambda}^{\cdot\cdot\kappa} = 0$                                                                                                                                        | see $\tilde{U}_n$                                                                                                                  |
| (3.19a) $R_{\bar{\nu}\mu\lambda}^{\cdot\cdot\kappa} = \partial_{\bar{\nu}} a^{\bar{\alpha}\kappa} \partial_\mu a_{\lambda\bar{\alpha}}$                                                  | see $\tilde{U}_n$                                                                                                                  |
| (3.19b) $R_{\bar{\nu}\lambda} = -\partial_{\bar{\nu}} \partial_\lambda \log \alpha + 2\partial_{\bar{\nu}} S_{\lambda\mu}^{\cdot\cdot\mu}$                                               | (3.28) $R_{\bar{\nu}\lambda} = -\partial_{\bar{\nu}} \partial_\lambda \log \alpha = R_{\lambda\bar{\nu}} = -V'_{\bar{\nu}\lambda}$ |
| (3.19c) $V'_{\bar{\nu}\mu} = \partial_{\bar{\nu}} \partial_\mu \log \alpha = V'_{\mu\bar{\nu}}$                                                                                          | see $\tilde{U}_n$                                                                                                                  |
| (3.19d) $V_{\bar{\nu}\mu} = 0$                                                                                                                                                           | see $\tilde{U}_n$                                                                                                                  |
| (3.26) $2\nabla_{[\nu} S_{\mu\lambda]}^{\cdot\cdot\kappa} - 4S_{[\nu\mu}^{\cdot\cdot\ell} S_{\lambda]\ell}^{\cdot\cdot\kappa} = 0$                                                       | —                                                                                                                                  |
| (3.27) $\nabla_{[\mu} S_{\lambda]} = 0$ for $n > 2$ if $S_{\mu\lambda}^{\cdot\cdot\kappa} = S_{[\mu} A_{\lambda]}^\kappa$                                                                | —                                                                                                                                  |
| (3.20) $R_{\bar{\nu}\mu\lambda\bar{\lambda}} = -R_{\mu\bar{\nu}\lambda\bar{\lambda}}$                                                                                                    | (3.29a) $R_{\bar{\nu}\mu\lambda\bar{\lambda}} = -R_{\mu\bar{\nu}\lambda\bar{\lambda}}$                                             |
| (3.21) $R_{\bar{\nu}[\mu\lambda]\bar{\lambda}} = \nabla_{\bar{\nu}} S_{\mu\lambda\bar{\lambda}}$                                                                                         | (3.29b) $R_{\bar{\nu}\mu\lambda\bar{\lambda}} = R_{\bar{\nu}\lambda\mu\bar{\lambda}}$                                              |
| (3.22) $R_{\bar{\nu}\mu\lambda\bar{\lambda}} = -R_{\bar{\nu}\mu\bar{\lambda}\lambda}$                                                                                                    | (3.29c) $R_{\bar{\nu}\mu\lambda\bar{\lambda}} = -R_{\bar{\nu}\mu\bar{\lambda}\lambda}$                                             |
| (3.23) $R_{\bar{\nu}\mu\lambda\bar{\lambda}} - R_{\lambda\bar{\lambda}\bar{\nu}\mu} = 2\nabla_{\bar{\nu}} S_{\mu\lambda\bar{\lambda}} - 2\nabla_{\lambda} S_{\bar{\lambda}\bar{\nu}\mu}$ | (3.29d) $R_{\bar{\nu}\mu\lambda\bar{\lambda}} = R_{\lambda\bar{\lambda}\bar{\nu}\mu}$                                              |
| see $\tilde{L}_n$                                                                                                                                                                        | see $\tilde{A}_n$                                                                                                                  |
| —                                                                                                                                                                                        | —                                                                                                                                  |
| (3.24) $\nabla_{[\omega} R_{\nu]\bar{\mu}\lambda\bar{\lambda}} = S_{\omega\nu}^{\cdot\cdot\ell} R_{\bar{\mu}\ell\lambda\bar{\lambda}}$                                                   | (3.30) $\nabla_{[\omega} R_{\nu]\bar{\mu}\lambda\bar{\lambda}} = 0$                                                                |
|                                                                                                                                                                                          | —                                                                                                                                  |
| (3.25) $\nabla_{[\omega} V'_{ \bar{\mu}  \nu]} = -S_{\omega\nu}^{\cdot\cdot\ell} V'_{ \bar{\mu}  \ell}$                                                                                  | (3.31) $\nabla_{[\omega} V'_{ \bar{\mu}  \nu]} = 0$                                                                                |
|                                                                                                                                                                                          | —                                                                                                                                  |
| —                                                                                                                                                                                        | —                                                                                                                                  |
| $\nabla_\omega R_{\bar{\mu}\lambda} + \nabla_\nu R_{\bar{\mu}\omega\lambda} = 2S_{\omega\nu}^{\cdot\cdot\ell} R_{\bar{\mu}\ell\lambda}^{\cdot\cdot\gamma}$                               | $\nabla_\omega R_{\bar{\mu}\lambda} + \nabla_\nu R_{\bar{\mu}\omega\lambda}^{\cdot\cdot\gamma} = 0$                                |
| see $\tilde{L}_n$                                                                                                                                                                        | see $\tilde{A}_n$                                                                                                                  |
| —                                                                                                                                                                                        | —                                                                                                                                  |
| (3.32) $\nabla_\mu G_\nu^{\star\mu} = 0; \quad G_\nu^{\star\mu} \stackrel{\text{def}}{=} R_\nu^{\star\mu} - R A_\nu^{\star\mu}$                                                          | $R \stackrel{\text{def}}{=} R_{\bar{\nu}\lambda} a^{\bar{\nu}\lambda}$                                                             |
|                                                                                                                                                                                          |                                                                                                                                    |

In (2.32) the term  $-\partial_{\bar{\mu}} R_{\omega\lambda}$  drops out but (2.33) remains valid without simplifications. In a  $\tilde{U}_n$  a second contraction is possible. For a  $\tilde{V}_n$  this leads in (2.32) to  $\theta = 0$  but from (2.33) it follows that

$$(3.32) \quad \boxed{V_{\mu} G_{\nu}^{\mu} = 0; \text{ conj.} \quad ; \quad G_{\nu}^{\mu} \stackrel{\text{def}}{=} R_{\nu}^{\mu} - RA_{\nu}^{\mu}; \text{ conj.} \quad R \stackrel{\text{def}}{=} R_{\bar{\nu}\lambda} a^{\bar{\nu}\lambda}}$$

which could be compared with (III 5.29).

On pp. 400, 401 there is a list of the most important formulae for  $\tilde{L}_n$ ,  $\tilde{A}_n$ ,  $\tilde{U}_n$  and  $\tilde{V}_n$ .

If  $u_i^{\kappa}$  are  $n$  fields of unitvectors in a  $\tilde{U}_n$  that are mutually (unitary) perpendicular, if the reciprocal set (cf. I § 2) is  $\hat{u}_{\lambda}$  and if the fundamental tensor  $a_{\lambda\bar{\lambda}}$  is positive definite (cf. I § 10), this tensor can be written in the form

$$(3.33) \quad a_{\lambda\bar{\lambda}} = \sum_h \hat{u}_{\lambda}^h \hat{u}_{\bar{\lambda}}^h$$

but the fields  $\hat{u}_{\lambda}$  need not to be analytic. In fact, it is not necessary that there exist analytic fields  $\hat{u}_{\lambda}^h$  satisfying (3.33). We prove that such fields exist if and only if

$$(3.34) \quad \partial_{\bar{\nu}} I_{\mu\lambda}^{\kappa} = 0^1)$$

or in an equivalent form

$$(3.35) \quad \partial_{\bar{\nu}} \partial_{\mu} a_{\lambda\bar{\lambda}} = (\partial_{\bar{\nu}} a_{\lambda\bar{\lambda}}) (\partial_{\mu} a_{\sigma\bar{\sigma}}) a^{\bar{\sigma}\sigma}.$$

(3.34) expresses the fact that the  $I_{\mu\lambda}^{\kappa}$  are analytic in  $\xi^{\kappa}$ . The condition is necessary. This follows immediately by differentiation of (3.33) if it is supposed that the  $\hat{u}_{\lambda}^h$  are analytic. In order to prove that it is also sufficient we consider the system of differential equations

$$(3.36) \quad \begin{cases} \text{a)} & V_{\mu} \hat{u}_{\lambda}^h = \partial_{\mu} \hat{u}_{\lambda}^h - \hat{u}_{\rho}^h a^{\bar{\sigma}\rho} \partial_{\mu} a_{\lambda\bar{\sigma}} = 0 \\ \text{b)} & V_{\bar{\mu}} \hat{u}_{\lambda}^h = \partial_{\bar{\mu}} \hat{u}_{\lambda}^h = 0. \end{cases}$$

The integrability conditions are

$$(3.37) \quad \begin{cases} \text{a)} & \hat{u}_{\tau}^h a^{\bar{\sigma}\tau} (\partial_{[\nu} a_{|\rho\bar{\omega}]} a^{\bar{\sigma}\rho} \partial_{\mu]} a_{\lambda\bar{\sigma}} + \hat{u}_{\rho}^h (\partial_{[\nu} a^{\bar{\sigma}\rho}) \partial_{\mu]} a_{\lambda\bar{\sigma}} = 0 \\ \text{b)} & \hat{u}_{\rho}^h (\partial_{\bar{\nu}} a^{\bar{\sigma}\rho}) \partial_{\mu} a_{\lambda\bar{\sigma}} + \hat{u}_{\rho}^h a^{\bar{\sigma}\rho} \partial_{\bar{\nu}} \partial_{\mu} a_{\lambda\bar{\sigma}} = 0 \end{cases}$$

(3.37a) is satisfied identically and (3.37b) is a consequence of (3.35). Hence, if (3.35) is satisfied, (3.36) is totally integrable and this means that for every choice of a vector  $\hat{u}_{\lambda}^h$  at some point, there exists a covariant constant field having at this point exactly the value  $\hat{u}_{\lambda}^h$ . Thus, if we

<sup>1)</sup> SCHOUTEN and v. DANTZIG 1930, 1; cf. E II 1938, 2, p. 236.

take  $n$  mutually perpendicular unitvectors at some point we get  $n$  fields of unitvectors that are mutually perpendicular at every point. The equation (3.34) implies that every analytic field remains analytic if it is displaced parallel. If we call such a connexion *analytic*, then (3.34) is the n.a.s. condition for the connexion to be analytic.

From (2.17) and (3.34) it follows that the last curvature tensors still remaining vanish and that the connexion is therefore *integrable* (cf. III § 4). Gathering results we have<sup>1)</sup>:

*In a  $\tilde{U}_n$  the following three conditions are equivalent:*

- a) the connexion is integrable:  $R_{\bar{\nu}\mu\lambda}^{\alpha} = 0$ ; conj.
- b) the connexion is analytic:  $\partial_{\bar{\nu}} R_{\mu\lambda}^{\alpha} = 0$ ; conj.
- c) there exist sets of mutually perpendicular fields of unitvectors that are analytic.

If the connexion is integrable and moreover symmetric, we have from (3.9) and (3.36a)

$$(3.38) \quad \partial_{[\mu} \overset{\text{h}}{u}_{\lambda]} = 0; \quad \partial_{\bar{\mu}} \overset{\text{h}}{u}_{\lambda} = 0$$

and this means that the  $\overset{\text{h}}{u}_{\lambda}$  are analytic gradientvectors, depending on the  $\xi^{\alpha}$  only. Hence in this case the scalars belonging to those fields are analytic in  $\xi^{\alpha}$  and if they are taken as coordinates, the linear element takes the form

$$(3.39) \quad ds^2 \underset{\alpha}{=} \sum d\xi^{\alpha} d\xi^{\alpha}$$

because the  $\overset{\text{h}}{u}_{\lambda}$  are mutually perpendicular unitvectors. From this form every linear coordinate transformation with constant coefficients leads to a form

$$(3.40) \quad ds^2 \underset{\alpha\bar{\alpha}}{=} a_{\alpha\bar{\alpha}} d\xi^{\alpha} d\xi^{\bar{\alpha}}; \quad a_{\alpha\bar{\alpha}} \underset{\alpha}{=} \text{const.}$$

Hence (cf. I § 10)<sup>2)</sup>

*If the connexion of a  $\tilde{V}_n$  is analytic it is also integrable and the  $\tilde{V}_n$  is an  $\tilde{R}_n$ .*

### Exercises.

VIII 3,1. Prove the third identity in  $\tilde{U}_n$

$$\text{VIII 3,1 } \alpha) \quad R_{\bar{\nu}\mu\lambda\bar{\lambda}} = -R_{\bar{\nu}\mu\bar{\lambda}\lambda}; \quad \text{conj.}$$

VIII 3,2. Prove that  $R \stackrel{\text{def}}{=} R_{\bar{\mu}\lambda} a^{\bar{\mu}\lambda}$  is always real in a  $\tilde{V}_n$  but not always in an  $\tilde{U}_n$ .

<sup>1)</sup> SCHOUTEN and v. DANTZIG 1930, 1, p. 333, cf. E II 1938, 2, p. 237.

<sup>2)</sup> SCHOUTEN and v. DANTZIG 1931, 3; E II 1938, 2, p. 237f.

VIII 3,3<sup>1</sup>). In a  $\tilde{V}_n$  with  $a_{\lambda\bar{\lambda}} = \partial_\lambda \partial_{\bar{\lambda}} \varphi$  is given that  $e^\varphi f(\xi^*) g(\xi^{\bar{\lambda}})$  is an ordinary density of weight +2. Prove that there exists a scalar field  $\psi$  such that

$$\text{VIII 3,3 } \alpha) \quad a_{\lambda\bar{\lambda}} - V'_{\lambda\bar{\lambda}} = \partial_\lambda \partial_{\bar{\lambda}} \log \psi.$$

VIII 3,4. Let  $L_{\mu\bar{\lambda}}$  be defined in a  $\tilde{V}_n$  by

$$\text{VIII 3,4 } \alpha) \quad L_{\mu\bar{\lambda}} \stackrel{\text{def}}{=} -R_{\mu\bar{\lambda}} + \frac{2}{n+1} R a_{\mu\bar{\lambda}}.$$

Prove that

$$\text{VIII 3,4)} \quad \begin{cases} \beta) & L_{\mu\bar{\lambda}} = L_{\bar{\lambda}\mu} \\ \gamma) & L \stackrel{\text{def}}{=} L_{\mu}^{\cdot\mu} = L_{\bar{\mu}}^{\cdot\bar{\mu}} = \text{real} \\ \delta) & \nabla_{\nu} L_{\mu}^{\cdot\nu} = \nabla_{\mu} L^{\cdot\nu} \quad (\text{use 3.32}). \end{cases}$$

VIII 3,5. Let  $F_{\bar{\nu}\mu\bar{\lambda}}^*$  be defined in a  $\tilde{V}_n$  by

$$\text{VIII 3,5 } \alpha) \quad F_{\bar{\nu}\mu\bar{\lambda}}^* \stackrel{\text{def}}{=} R_{\bar{\nu}\mu\bar{\lambda}}^* + \frac{2}{n-1} a_{\bar{\nu}\mu} L_{\bar{\lambda}}^*,$$

where  $L_{\mu\bar{\lambda}}$  is defined in (Exerc. VIII 3,4 $\alpha$ ). Prove that

$$\text{VIII 3,5 } \beta) \quad \nabla_{\nu} F_{\bar{\nu}\mu\bar{\lambda}}^* = \nabla_{\bar{\lambda}} L_{\bar{\nu}\mu}.$$

#### § 4. The $\tilde{V}_n$ of constant curvature.<sup>2)</sup>

For a symmetric unitary connexion,  $R_{\bar{\nu}\mu\bar{\lambda}\bar{\lambda}}$  is symmetric in  $\mu\bar{\lambda}$  (cf. 3.21, 29b). Now if

$$(4.1) \quad R_{\bar{\nu}\mu\bar{\lambda}\bar{\lambda}} = \alpha a_{\bar{\nu}(\mu} a_{|\bar{\lambda}| \bar{\lambda})} = \alpha a_{(\bar{\nu}(\mu} a_{\bar{\lambda}) \bar{\lambda})}$$

and therefore

$$(4.2) \quad R_{\bar{\nu}\lambda} = -\frac{1}{2}(n+1)\alpha a_{\bar{\nu}\lambda}$$

the  $\tilde{V}_n$  is said to be of *constant curvature* and is called an  $\tilde{S}_n$ .  $\alpha$  is always *constant* and *real*.<sup>3)</sup> The scalar curvature (cf. III § 5) is also *constant* and *real* because

$$(4.3) \quad \frac{1}{n(n-1)} R = -\frac{1}{2} \frac{n+1}{n-1} \alpha; \quad R \stackrel{\text{def}}{=} R_{\bar{\nu}\lambda} a^{\bar{\nu}\lambda}.$$

<sup>1)</sup> BERGMAN 1933, 1, p. 310.

<sup>2)</sup> SCHOUTEN and v. DANTZIG 1931, 3; FUCHS 1937, 1 also for a  $\tilde{V}_n$  with constant  $R$ ; BOCHNER 1947, 1; E II 1938, 2, p. 239.

<sup>3)</sup> SATO 1950, 1.

If the connexion of a  $\tilde{V}_n$  is transformed projectively (cf. VI § 1) we get a symmetric (but not necessarily unitary) connexion

$$(4.4) \quad \begin{cases} \text{a)} & 'T_{\mu\lambda}^* = T_{\mu\lambda}^* + 2p_{(\mu} A_{\lambda)}^*; \quad \text{conj.} \\ \text{b)} & 'T_{\bar{\mu}\lambda}^* = T_{\bar{\mu}\lambda}^* = 0 \end{cases}$$

and the new curvature tensors are

$$(4.5) \quad \begin{cases} \text{a)} & 'R_{\nu\mu\lambda}^* = -2p_{[\nu\mu]} A_{\lambda}^* + 2A_{[\nu}^* p_{\mu]\lambda}; \quad \text{conj.} \\ \text{b)} & 'R_{\bar{\nu}\mu\lambda}^* = \alpha a_{\bar{\nu}(\mu} A_{\lambda)}^* + 2(V_{\bar{\nu}} p_{(\mu}) A_{\lambda)}^*; \quad \text{conj.} \\ \text{c)} & 'R_{\mu\bar{\nu}\lambda}^* = -'R_{\bar{\nu}\mu\lambda}^*; \quad \text{conj.} \quad p_{\mu\lambda} \stackrel{\text{def}}{=} V_{\mu} p_{\lambda} + p_{\mu} p_{\lambda}; \quad \text{conj.} \end{cases}$$

In order to make the new connexion integrable it is n.a.s. that for  $n \geq 2$ :

$$(4.6) \quad \begin{cases} \text{a)} & V_{\mu} p_{\lambda} = p_{\mu} p_{\lambda}; \quad \text{conj.} \\ \text{b)} & V_{\bar{\mu}} p_{\lambda} = \partial_{\bar{\mu}} p_{\lambda} = -\frac{1}{2}\alpha a_{\bar{\mu}\lambda}; \quad \text{conj.} \end{cases}$$

but this implies that  $p_{\lambda}$  is a gradientvector:

$$(4.7) \quad p_{\lambda} = \partial_{\lambda} \log \mu; \quad \text{conj.}$$

The integrability conditions of (4.6) are identically satisfied in consequence of (4.1) and the constancy of  $\alpha$ . That proves

*An  $\tilde{S}_n$  can always be transformed into an  $\tilde{R}_n$  by a projective transformation of the connexion.*

In this  $\tilde{R}_n$  there exist coordinate systems with respect to which  $'T_{\mu\lambda}^* = 0$ . If we use one of these coordinate systems in the  $\tilde{S}_n$  we get

$$(4.8) \quad \begin{cases} \text{a)} & T_{\mu\lambda}^* \stackrel{*}{=} -2A_{(\lambda}^* \partial_{\mu)} \log \mu; \quad \text{conj.} \\ \text{b)} & T_{\bar{\mu}\lambda}^* \stackrel{*}{=} 0; \quad \text{conj.} \end{cases}$$

where  $\mu$  is a solution of (4.7). For this coordinate system we get

$$(4.9) \quad \begin{cases} \text{a)} & \partial_{\mu} a_{\bar{\lambda}\nu}^* \stackrel{*}{=} -2a_{\bar{\lambda}(\nu} p_{\mu)}; \quad \text{conj.} \\ \text{b)} & \partial_{\mu} a^{\bar{\lambda}\bar{\lambda}} \stackrel{*}{=} a^{\bar{\lambda}\bar{\lambda}} p_{\mu} + A_{\mu}^* p_{\bar{\lambda}}^*; \quad \text{conj.} \\ \text{c)} & \partial_{\mu} p_{\lambda} \stackrel{*}{=} -p_{\mu} p_{\lambda}; \quad \text{conj.} \\ \text{d)} & \partial_{\mu} p^{\bar{\lambda}} \stackrel{*}{=} p^{\bar{\lambda}} p_{\mu}; \quad \text{conj.} \\ \text{e)} & \partial_{\mu} p_{\bar{\lambda}} \stackrel{*}{=} -\frac{1}{2}\alpha a_{\bar{\lambda}\mu}; \quad \text{conj.} \\ \text{f)} & \partial_{\mu} p^{\bar{\lambda}} \stackrel{*}{=} p^{\bar{\lambda}} p_{\mu} - \frac{1}{2}\alpha A_{\mu}^* \left(1 - \frac{2}{\alpha} p^{\bar{\lambda}} p_{\bar{\lambda}}\right); \quad \text{conj.} \end{cases}$$

Now it can easily be verified that

$$(4.10) \quad \mu = 1 - \frac{2}{\alpha} p_{\lambda} p_{\bar{\lambda}} a^{\lambda\bar{\lambda}}$$

is a solution of (4.7) because  $\alpha$  is real. From (4.10) we see that  $\mu$  is also real. If we introduce the vector

$$(4.11) \quad z^* \stackrel{\text{def}}{=} -\frac{2}{\mu\alpha} p^*$$

we get from (4.9c, d) and (4.11)

$$(4.12) \quad \partial_\mu z^* \stackrel{*}{=} A_\mu^*; \quad \partial_{\bar{\mu}} z^* \stackrel{*}{=} 0.$$

Hence the  $z^*$  are analytic and by integration of (4.12) it follows that

$$(4.13) \quad \xi^* \stackrel{*}{=} z^* + c^*; \quad c^* \stackrel{*}{=} \text{const.}$$

But this means that the coordinate system can be chosen such that

$$(4.14) \quad \xi^* \stackrel{*}{=} z^* = -\frac{2}{\mu\alpha} p^*.$$

From (4.6b) and (4.9a, c) it follows that

$$(4.15) \quad b_{\lambda\bar{x}} \stackrel{\text{def}}{=} \frac{\alpha\mu}{2} a_{\lambda\bar{x}} - \mu p_\lambda p_{\bar{x}}$$

is a hermitian tensor with *constant* components:

$$(4.16) \quad \partial_\mu b_{\lambda\bar{x}} \stackrel{*}{=} 0; \quad \partial_{\bar{\mu}} b_{\lambda\bar{x}} \stackrel{*}{=} 0,$$

hence:

*In an  $\tilde{S}_n$  the coordinate system can always be chosen such that in the region where the equation*

$$(4.17) \quad \mu \stackrel{*}{=} 1 - \frac{\alpha\mu^2}{2} a_{\lambda\bar{x}} \xi^\lambda \xi^{\bar{x}}$$

*has a real solution (that is where  $1 + 2\alpha a_{\lambda\bar{x}} \xi^\lambda \xi^{\bar{x}} \geq 0$ ) the fundamental tensor can be written in the form*

$$(4.18) \quad a_{\lambda\bar{x}} \stackrel{*}{=} \frac{2}{\alpha\mu} b_{\lambda\bar{x}} + \frac{\alpha\mu^2}{2} a_{\lambda\bar{q}} a_{\sigma\bar{x}} \xi^\sigma \xi^{\bar{q}}.$$

*where  $b_{\lambda\bar{x}}$  is a hermitian tensor with constant components.*

But this form of  $a_{\lambda\bar{x}}$  gives just the linear element of the hermitian non-euclidean geometry that was investigated by FUBINI and STUDY.<sup>1)</sup> Hence hermitian non-euclidean geometry in the sense of these authors is realized in an  $\tilde{S}_n$ , in the same way as ordinary non-euclidean geometry is realized in an  $S_n$ .

1) FUBINI 1903, 2; STUDY 1905, 1; cf. COOLIDGE 1924, 1; CARTAN 1931, 1; E II 1938, 2, p. 243; BOCHNER 1947, 1, p. 185ff.

**Exercise.**

VIII 4,1<sup>1)</sup>. Let the gradient vector  $p_\lambda$  be the vector of the transformation (4.4a) that transforms an  $\tilde{S}_n$  into an  $\tilde{R}_n$ .  $\mu$  is defined by (4.10). Prove that (use 4.9 and 4.15)

$$\text{VIII 4,1 } \alpha) \quad a_{\lambda \bar{x}} = -2 \partial_\lambda \partial_{\bar{x}} \frac{\log \mu}{\alpha}$$

and

$$\text{VIII 4,1 } \beta) \quad \mu \equiv 1 - b_{\lambda \bar{x}} \xi^\lambda \xi^{\bar{x}}$$

with respect to the coordinate system (4.14).

**§ 5. Imbedding in an  $\tilde{L}_n$ .**

Let an  $X_m$  in  $\tilde{L}_n$  be given by its parameter form

$$(5.1) \quad \xi^x = f^x(\eta^a); \quad a = 1, \dots, m; \quad \text{conj.}$$

where the functions  $f^x$  are analytic in the domain considered.<sup>2)</sup> If we define the connecting quantity  $B_b^x$  in the ordinary way

$$(5.2) \quad B_b^x \stackrel{\text{def}}{=} \partial_b \xi^x; \quad \text{conj.}$$

the analyticity of  $f^x$  ensures that the  $B_b^x$  are analytic:

$$(5.3) \quad \partial_{\bar{c}} B_b^x = 0; \quad \text{conj.}$$

We suppose that the  $X_m$  is rigged and that the vectors  $e_y^x$ ;  $y = m+1, \dots, n$  span the  $E_{n-m}$  of the rigging. These vectors need not be analytic. Besides the  $e_y^x$  we have the vectors of  $\tilde{L}_n$  derived from the basis vectors  $e_a^b$  in  $X_m$

$$(5.4) \quad e_a^x \stackrel{\text{def}}{=} B_b^x e_a^b; \quad \text{conj.}$$

These vectors are analytic and they span the tangent  $E_m$  of  $X_m$ . All these vectors are given at points of  $X_m$  only. From the vectors  $e_b^x$ ,  $e_y^x$ ; conj. we form the reciprocal system  $\tilde{e}_\lambda^x$ ;  $\tilde{e}_{\bar{\lambda}}^x$ ; conj. in the ordinary way. These covariant vectors need not be analytic. Now the formulae (V 7.1) and their conjugates hold over  $X_m$ . From them we need especially the definition of the second connecting quantity  $C_\lambda^x$ :

$$(5.5) \quad C_\lambda^x \stackrel{\text{def}}{=} e_\lambda^x \tilde{e}_\lambda^y; \quad \text{conj.}$$

In contradistinction to  $B_b^x$  this quantity need not be analytic.

<sup>1)</sup> Cf. FUCHS 1937, 1; BOCHNER 1947, 1.

<sup>2)</sup> COBURN also considered  $X_m$ 's in  $\tilde{U}_n$  defined by semi-analytic functions, 1942, 1.

Now let  $\phi$  be a semi-analytic scalar field of  $X_m$ . Then we have the derivatives  $\partial_b \phi$  and  $\partial_{\bar{b}} \phi$  and if  $\phi$  is prolonged somehow in a neighbourhood of  $X_m$  we have also  $\partial_\mu \phi$  and  $\partial_{\bar{\mu}} \phi$  over  $X_m$ . The relations

$$(5.6) \quad \partial_b \phi = B_b^\mu \partial_\mu \phi; \quad \text{conj.}$$

hold between them. If  $v^a$  is a vector field of  $X_m$ , the components of this field considered as a field of  $\tilde{L}_n$  over  $X_m$  are  $v^a = B_b^a v^b$ . Hence a connexion in  $X_m$  can be defined by the equations

$$(5.7) \quad \begin{cases} \text{a)} & 'V_c v^a \stackrel{\text{def}}{=} B_{c\bar{x}}^{\mu a} V_\mu v^x; \quad \text{conj.} \\ \text{b)} & 'V_{\bar{c}} v^a \stackrel{\text{def}}{=} B_{\bar{c}\bar{x}}^{\bar{\mu} a} B_x^\mu V_{\bar{\mu}} v^x; \quad \text{conj.} \end{cases}$$

where  $B_x^a$  is defined as in (V 7.1). From (5.7) and the analyticity of  $B_b^a$  we get the following equations for the parameters of this connexion:

$$(5.8) \quad \begin{cases} \text{a)} & 'T_{cb}^a = B_{c\bar{b}\bar{x}}^{\mu\lambda a} \Gamma_{\mu\lambda}^x - B_{c\lambda}^{\mu a} \partial_\mu B_b^\lambda; \quad \text{conj.} \\ \text{b)} & 'T_{\bar{c}\bar{b}}^a = B_{\bar{c}\bar{b}\bar{x}}^{\bar{\mu}\lambda a} \Gamma_{\bar{\mu}\lambda}^x - B_{\bar{c}\lambda}^{\bar{\mu} a} \partial_{\bar{\mu}} B_b^\lambda = 0; \quad \text{conj.} \end{cases}$$

and this proves that *the  $X_m$  with this induced connexion is an  $\tilde{L}_m$* . From (5.8a) we see that if the connexion in  $\tilde{L}_n$  is analytic,  $\partial_{\bar{\mu}} \Gamma_{\mu\lambda}^x = 0$ , it need not necessarily follow that the connexion in  $\tilde{L}_m$  is also analytic, because  $B_x^a$  need not be analytic. Alternation of (5.8a) gives

$$(5.8) \quad \text{c)} \quad 'S_{c\bar{b}}^{;\bar{x}} = B_{c\bar{b}\bar{x}}^{\mu\lambda a} S_{\mu\lambda}^{;\bar{x}} - B_x^a \partial_{[\bar{c}} B_{\bar{b}]^x} = B_{c\bar{b}\bar{x}}^{\mu\lambda a} S_{\mu\lambda}^{;\bar{x}}; \quad \text{conj.}$$

and this means that *every  $X_m$  imbedded and rigged in an  $\tilde{A}_n$  is itself an  $\tilde{A}_m$  whatever the rigging may be*.

Instead of two curvature tensors with valence 3 as in (V 7.15, 16) here we get four with their complex conjugates

$$(5.9) \quad \text{a)} \quad H_{c\bar{b}}^{;\bar{x}} \stackrel{\text{def}}{=} B_{c\bar{b}}^{\mu\lambda} V_\mu B_\lambda^x \quad \text{b)} \quad h_{\bar{c}\bar{b}}^{;\bar{x}} \stackrel{\text{def}}{=} B_{\bar{c}\bar{b}}^{\bar{\mu}\lambda} V_{\bar{\mu}} B_\lambda^x; \quad \text{conj.}$$

$$(5.10) \quad \text{a)} \quad L_{\bar{c}\cdot\lambda}^{;\bar{a}} \stackrel{\text{def}}{=} B_{\bar{c}\bar{x}}^{\bar{\mu}\bar{a}} V_{\bar{\mu}} B_\lambda^x \quad \text{b)} \quad l_{c\cdot\lambda}^{;\bar{a}} \stackrel{\text{def}}{=} B_{c\bar{x}}^{\mu a} V_\mu B_\lambda^x; \quad \text{conj.}$$

but from these  $h_{\bar{c}\bar{b}}^{;\bar{x}}$  and its complex conjugate are zero. In fact, according to the analyticity of  $B_b^x$  we have

$$(5.11) \quad h_{\bar{c}\bar{b}}^{;\bar{x}} = B_{\bar{c}\bar{b}}^{\bar{\mu}\bar{\lambda}} \partial_{\bar{\mu}} B_\lambda^x = - B_{\bar{c}}^{\bar{\mu}} \partial_{\bar{\mu}} B_b^x = 0; \quad \text{conj.}$$

From (5.9a) it follows that

$$(5.12) \quad \begin{cases} H_{[c\bar{b}]}^{;\bar{x}} = - B_{c\bar{b}}^{\mu\lambda} \partial_{[\mu} C_{\lambda]}^x + B_{c\bar{b}}^{\mu\lambda} \Gamma_{[\mu\lambda]}^x C_{\lambda]}^x - B_{c\bar{b}}^{\mu\lambda} \Gamma_{[\mu|\sigma]}^x C_{\lambda]}^\sigma \\ = C_{\lambda]}^\sigma \partial_{[\mu} B_{\lambda]}^{\mu\lambda} + B_{c\bar{b}}^{\mu\lambda} S_{\mu\lambda}^{;\bar{e}} C_{\bar{e}}^x = B_{c\bar{b}}^{\mu\lambda} S_{\mu\lambda}^{;\bar{e}} C_{\bar{e}}^x; \end{cases} \quad \text{conj.}$$

and this proves that  $H_{[c\bar{b}]}^{;\bar{x}}$  is zero for every choice of the  $X_m$  and its rigging if the  $\tilde{L}_n$  is an  $\tilde{A}_n$  (cf. V 7.23).

An  $\tilde{L}_m$  in  $\tilde{L}_n$  will be called *geodesic* if every vector of  $\tilde{L}_m$  remains in  $\tilde{L}_m$  if it is displaced parallel in a direction of  $\tilde{L}_m$ . N.a.s. conditions are that for every field  $v^x = B_b^x v^b$  the following equations should hold

$$(5.13) \quad \text{a)} \quad v^\mu (V_\mu v^\lambda) C_\lambda^x = 0; \quad \text{conj.}; \quad \text{b)} \quad v^{\bar{\mu}} (V_{\bar{\mu}} v^\lambda) C_\lambda^x = 0; \quad \text{conj.}$$

But these conditions are equivalent to

$$5.14) \quad \text{a)} \quad H_{(cb)}^x = -B_{(cb)}^{\mu\lambda} V_\mu C_\lambda^x = 0; \quad \text{conj.} \quad \text{b)} \quad v^{\bar{c}} v^b B_{cb}^{\bar{\mu}\lambda} V_{\bar{\mu}} C_\lambda^x = 0; \quad \text{conj.}^1)$$

From these equations the first is equivalent to  $H_{cb}^x = 0$  if the  $\tilde{L}_n$  is an  $\tilde{A}_n$  and the second is satisfied identically because  $h_{cb}^{x\bar{x}} = 0$  (cf. 5.11). It is easily proved that the transformation of the connexion in  $\tilde{L}_n$

$$(5.15) \quad I_{\mu\lambda}^x = I_{\mu\lambda}^x + P_{[\mu}^{\cdot\bar{\lambda}} \dot{v}^x; \quad P_{[\mu}^{\cdot\bar{\lambda}} \dot{v}^x] = 0; \quad \text{conj.}$$

leaves invariant the set of all  $\tilde{L}_m$ 's which are geodesic according to this condition, if and only if  $P_{\mu\lambda}^{\cdot\bar{\lambda}}$  has the form  $2p_{(\mu} A_{\lambda)}^x$ , that is, if the transformation is projective (cf. VI § 1) <sup>2)</sup>.

For an  $\tilde{L}_m$  in  $\tilde{L}_n$  a *D*-symbolism may be introduced in the same way as for an  $L_m$  in  $L_n$  (cf. V § 7). Because the fields considered are all defined only over the  $\tilde{L}_m$  here we need only  $D_\mu$  and  $D_c$  and their conjugates. Using these symbols, the tensors  $H_{cb}^x$ ,  $L_{\bar{c}\cdot\lambda}^a$  and  $l_{c\cdot\lambda}^a$  get the form (cf. V 7.15 b, 16b)

$$(5.16) \quad \text{a)} \quad H_{cb}^x = D_c B_b^x; \quad \text{conj.} \quad \text{b)} \quad 0 = D_{\bar{c}} B_b^x; \quad \text{conj.}$$

$$(5.17) \quad \text{a)} \quad L_{\bar{c}\cdot\lambda}^a = D_{\bar{c}} B_\lambda^a; \quad \text{conj.} \quad \text{b)} \quad l_{c\cdot\lambda}^a = D_c B_\lambda^a; \quad \text{conj.}$$

We see from this that the greek index in these three formulae always lies in the rigging and that (cf. V 7.15 a, 16a)

$$(5.18) \quad H_{cb}^x = C_x^z D_c B_b^x = -B_b^x D_c C_x^z; \quad \text{conj.}$$

$$(5.19) \quad \left\{ \begin{array}{ll} \text{a)} & L_{\bar{c}\cdot y}^a = C_y^{\lambda} D_{\bar{c}} B_\lambda^a = -B_\lambda^a D_{\bar{c}} C_y^{\lambda}; \\ \text{b)} & l_{c\cdot y}^a = C_y^{\lambda} D_c B_\lambda^a = -B_\lambda^a D_c C_y^{\lambda}; \end{array} \right. \quad \text{conj.}$$

<sup>1)</sup> If instead of (5.13) we take  $(v^\mu V_\mu v^\lambda + v^{\bar{\mu}} V_{\bar{\mu}} v^\lambda) C_\lambda^x = 0$ ; conj. as the definition of the geodesic  $\tilde{L}_m$ , we get the same result (5.14) because  $h_{cb}^{x\bar{x}} = 0$ . But this latter definition represents just the condition for the  $L_{2m}$  corresponding to the  $\tilde{L}_m$  to be geodesic in the auxiliary  $L_{2n}$ .

<sup>2)</sup> Here is a difference with BOCHNER 1947, 1. He considers all geodesic subspaces in the auxiliary  $L_{2n}$  and requires that they all remain geodesic. Then it can be proved that this stronger condition leads to  $P_{\mu\lambda}^{\cdot\bar{\lambda}} = 0$ . But from our point of view the  $L_{2n}$  is only an auxiliary space in which we are only interested in what happens to subspaces in the principal  $X_n$ , and this leads to  $P_{\mu\lambda}^{\cdot\bar{\lambda}} = 2p_{(\mu} A_{\lambda)}^x$ .

If  $v^a$  is a field of  $\tilde{L}_m$  we have (cf. III 4.9b; V 8.4)

$$(5.20) \quad \left\{ \begin{array}{l} {}'R_{\dot{d}\dot{c}\dot{b}}{}^a v^b - 2 {}'S_{\dot{d}\dot{c}}{}^e {}'V_e v^a = 2 {}'V_{[d} {}'V_{c]} v^a = 2 D_{[d} D_{c]} v^a = 2 D_{[d} B_{c]}^{\mu a} {}'V_\mu v^b \\ = 2 H_{[\dot{d}\dot{c}]}{}^\mu B_{\dot{x}}^\alpha {}'V_\mu v^x + 2 B_{[c}^\mu l_{d]}{}^a {}_y H_{\dot{u}\dot{b}}{}^y v^b + \\ + B_{\dot{d}\dot{c}\dot{x}}^{\nu a} R_{\nu\dot{u}\dot{b}}{}^x v^a - 2 B_{\dot{d}\dot{c}\dot{x}}^{\nu a} S_{\nu\dot{u}}{}^e {}'V_\mu v^x, \end{array} \right.$$

hence

$$(5.21) \quad \left\{ \begin{array}{l} {}'R_{\dot{d}\dot{c}\dot{b}}{}^a + 2 B_{[c}^\mu l_{d]}{}^a {}_y H_{\dot{u}\dot{b}}{}^y + B_{\dot{d}\dot{c}\dot{x}}^{\mu a} R_{\nu\dot{u}\dot{b}}{}^x \\ = 2 S_{\nu\dot{u}\dot{b}}{}^e B_{\dot{d}\dot{c}\dot{x}}^{\mu\lambda} {}'V_\nu v^x + 2 B_{\dot{d}\dot{c}}^{\mu\lambda} S_{\mu\dot{u}}{}^e C_{\dot{v}}^\nu B_{\dot{x}}^\alpha {}'V_\nu v^x - \\ - 2 B_{\dot{d}\dot{c}\dot{x}}^{\nu\mu a} S_{\nu\dot{u}}{}^e {}'V_\mu v^x = 0 \end{array} \right.$$

or (cf. V 7.32; 8.4)

$$(5.22) \quad {}'R_{\dot{d}\dot{c}\dot{b}}{}^a = B_{\dot{d}\dot{c}\dot{b}\dot{x}}^{\nu\mu\lambda a} R_{\nu\dot{u}\dot{b}}{}^x + 2 l_{[\dot{d}\dot{a}}{}^a {}_{|\dot{y}]} H_{\dot{c}\dot{b}}{}^y; \quad \text{conj.} \quad (\text{GAUSS I})$$

In the same way we get (cf. V 7.32; 8.4)

$$(5.23) \quad {}'R_{\dot{d}\dot{c}\dot{b}}{}^a = B_{\dot{d}\dot{c}\dot{b}\dot{x}}^{\bar{\nu}\mu\lambda a} R_{\bar{\nu}\dot{u}\dot{b}}{}^x + L_{\dot{d}\dot{a}\dot{x}}{}^a H_{\dot{c}\dot{b}}{}^x; \quad \text{conj.} \quad (\text{GAUSS II})$$

So we have in  $\tilde{L}_n$  two GAUSS equations and their conjugates.<sup>1)</sup>

Proceeding in the same way we get after some calculations five CODAZZI equations (cf. V 7.48, 51; 8.7, 8):

$$(5.24) \quad 2 D_{[d} H_{c]\dot{b}}{}^x = - 2 {}'S_{\dot{d}\dot{c}}{}^a H_{\dot{a}\dot{b}}{}^x + B_{\dot{d}\dot{c}\dot{b}}^{\nu\mu\lambda} C_{\dot{x}}^\nu R_{\nu\dot{u}\dot{b}}{}^x; \quad \text{conj.} \quad (\text{CODAZZI I})$$

$$(5.25) \quad D_{\dot{d}} H_{\dot{c}\dot{b}}{}^x = C_{\dot{x}}^\nu B_{\dot{c}\dot{b}\dot{d}}^{\mu\lambda\bar{\nu}} R_{\bar{\nu}\dot{u}\dot{b}}{}^x; \quad \text{conj.} \quad (\text{CODAZZI II})$$

$$(5.26) \quad 2 D_{[d} l_{c]\dot{y}}{}^a = - B_{\dot{d}\dot{c}\dot{x}}^{\nu\mu a} C_y^\lambda R_{\nu\dot{u}\dot{b}}{}^x - 2 {}'S_{\dot{d}\dot{c}}{}^e l_{e\dot{y}}{}^a; \quad \text{conj.} \quad (\text{CODAZZI III})$$

$$(5.27) \quad D_{[d} L_{c]\dot{y}}{}^a = - {}'S_{\dot{d}\dot{c}}{}^b L_{\dot{b}\dot{y}}{}^a; \quad \text{conj.} \quad (\text{CODAZZI IV})$$

$$(5.28) \quad - D_{\dot{d}} l_{c\dot{y}}{}^a + D_c L_{\dot{d}\dot{y}}{}^a = C_y^\lambda B_{\dot{c}\dot{x}\dot{d}}^{\mu a\bar{\nu}} R_{\bar{\nu}\dot{u}\dot{b}}{}^x; \quad \text{conj.} \quad (\text{CODAZZI V})$$

with their conjugates and two RICCI equations (cf. V 7.43; 8.6):

$$(5.29) \quad \overset{mm'}{R_{\dot{d}\dot{c}\dot{y}}}{}^x = B_{\dot{d}\dot{c}}^{\nu\mu} C_{\dot{x}\dot{y}}^\lambda R_{\nu\dot{u}\dot{b}}{}^x + 2 H_{[\dot{d}\dot{b}]}{}^x l_{c]\dot{y}}{}^b; \quad \text{conj.} \quad (\text{RICCI I})$$

$$(5.30) \quad \overset{mm'}{R_{\dot{d}\dot{c}\dot{y}}}{}^x = B_{\dot{d}\dot{c}}^{\bar{\nu}\mu} C_{\dot{y}\dot{x}}^\lambda R_{\bar{\nu}\dot{u}\dot{b}}{}^x - H_{\dot{c}\dot{b}}{}^x L_{\dot{d}\dot{b}\dot{y}}{}^y; \quad \text{conj.} \quad (\text{RICCI II})$$

<sup>1)</sup> Cf. HOMBURG 1935, 1.

with their conjugates. Here  $R_{\dot{d}c\dot{y}}^{\mu\mu'}$  and  $R_{\dot{d}c\dot{y}}^{\nu\nu'}$  are the quantities defined by (cf. V 7.42; 8.5)

$$(5.31) \quad 2D_{[d}D_{c]}r^x = R_{\dot{d}c\dot{y}}^{\mu\mu'}r^\nu - 2'S_{\dot{d}c}^\nu D_\nu r^\mu$$

$$(5.32) \quad 2D_{[d}D_{c]}r^x = R_{\dot{d}c\dot{y}}^{\nu\nu'}r^\mu$$

with an arbitrary vector  $r^x$  in the  $(n-m)$ -direction of the rigging. Only the three CODAZZI equations (5.24, 26, 27) take a simpler form if the  $\tilde{L}_n$  is an  $\tilde{A}_n$ .

In a  $\tilde{U}_n$  the induced fundamental tensor is

$$(5.33) \quad 'a_{b\bar{a}} = B_{b\bar{a}}^{\lambda\bar{\lambda}} a_{\lambda\bar{\lambda}}; \quad \text{conj.}$$

and the imbedding is fixed by  $B_b^x$ . Concomitants of  $B_b^x$  and  $a_{\lambda\bar{\lambda}}$  are

$$(5.34) \quad B_\lambda^a = 'a^{\bar{b}a} B_{\bar{b}}^{\bar{\lambda}} a_{\lambda\bar{\lambda}}; \quad \text{conj.}$$

and

$$(5.35) \quad B_\lambda^x = B_{\lambda a}^{ax} = 'a^{\bar{b}a} B_{\bar{b}}^{\bar{x}} a_{\lambda\bar{\lambda}} B_a^x = a^{\bar{\lambda}x} B_{\bar{\lambda}}^{\bar{x}} a_{\lambda\bar{\lambda}}; \quad \text{conj.}$$

We easily derive from these equations that in a  $\tilde{U}_n$

$$(5.36) \quad D_c 'a_{b\bar{a}} = 0; \quad \text{conj.}$$

and

$$(5.37) \quad \begin{cases} \text{a)} & L_{\bar{c}\bar{\lambda}}^{\mu\bar{\mu}} = H_{\bar{c}\bar{\lambda}}^{\bar{\sigma}\bar{\sigma}} a_{\lambda\bar{\sigma}} a^{\bar{\sigma}\bar{\mu}}; \quad \text{conj.} \\ \text{b)} & l_{\bar{c}\bar{\lambda}}^{\mu\bar{\mu}} = h_{\bar{c}\bar{\lambda}}^{\bar{\sigma}\bar{\sigma}} a^{\bar{\sigma}\bar{\mu}} a_{\lambda\bar{\sigma}} = 0; \quad \text{conj.} \end{cases}$$

From (5.36) we see that the imbedded  $X_m$  is a  $\tilde{U}_m$  with  $'a_{b\bar{a}}$  as fundamental tensor. (5.37) expresses the facts that the quantities  $l_{\bar{c}\bar{\lambda}}^{\mu\bar{\mu}}$  and  $h_{\bar{c}\bar{\lambda}}^{\mu\bar{\mu}}$  both vanish and that  $H_{\bar{c}\bar{\lambda}}^{\mu\bar{\mu}}$  and  $L_{\bar{c}\bar{\lambda}}^{\mu\bar{\mu}}$  are different components of one and the same quantity.

The first equation of GAUSS (5.22) drops out because both sides are zero and the second (5.23) takes the form

$$(5.38) \quad \boxed{'R_{\dot{d}c\dot{b}}^{\mu\lambda} = B_{\dot{d}\dot{c}\dot{b}\mu}^{\nu\lambda} R_{\dot{v}\dot{\mu}\dot{\lambda}}^{\nu\lambda} + H_{\dot{d}}^{\mu\bar{\mu}} H_{\dot{c}\dot{b}\bar{\mu}}^{\nu\lambda}; \quad \text{conj.}} \quad (\text{GAUSS}).$$

The third equation of CODAZZI (5.26) drops out, the first (5.24) is equivalent to the fourth (5.27):

$$(5.39) \quad \boxed{D_{[d}H_{c]\bar{b}}^{\mu\bar{\mu}} = -'S_{\dot{d}\dot{c}}^{\mu\bar{\mu}} H_{\dot{a}\dot{b}}^{\nu\bar{\nu}}; \quad \text{conj.}} \quad (\text{CODAZZI I})$$

and the second (5.25) is equivalent to the fifth (5.28):

$$(5.40) \quad D_d H_{cb}^{\cdot x} = C_x^z B_{c b d}^{\mu \lambda \bar{\nu}} R_{\bar{\nu} \mu \lambda}^{\cdot z}; \quad \text{conj.} \quad (\text{CODAZZI II}).$$

The equations of RICCI take the form

$$(5.41) \quad \left\{ \begin{array}{l} \boxed{R_{d c y}^{m m'} = 0; \quad \text{conj.}} \quad (\text{RICCI I}) \\ \boxed{R_{d c y}^{m m'} = B_{dc}^{\bar{\nu} \mu} C_{y x}^{\lambda x} R_{\bar{\nu} \mu \lambda}^{\cdot z} - H_{cb}^{\cdot x} H_{d \cdot y}^{\cdot b}; \quad \text{conj.}} \quad (\text{RICCI II}). \end{array} \right.$$

Among all these equations only the CODAZZI equation (5.39) contains ' $S_{dc}^{\cdot x}$ ' and it is only this equation that is simplified if the  $\tilde{U}_n$  is a  $\tilde{V}_n$ .<sup>1)</sup>

#### Exercise.

VIII 5,1<sup>2)</sup>. Prove that for an  $\tilde{U}_m$  in  $\tilde{U}_n$

$$H_{c \lambda}^{\cdot x} = D_c B_{\lambda}^x$$

if  $\xi^x$  is analytic in  $\eta^\alpha$ . There is no analogous relation for a  $U_m$  in  $U_n$ .

### § 6. Curves in a $\tilde{U}_n$ with a positive definite fundamental tensor.

For  $m=1$  we write  $\eta, \bar{\eta}, d_\eta, d_{\bar{\eta}}$  instead of  $\eta^1, \eta^{\bar{1}}, \partial_1, \partial_{\bar{1}}$  and we suppose that nowhere  $d_\eta \xi^x$  is a nullvector. Then we have

$$(6.1) \quad B_1^x = d_\eta \xi^x; \quad \text{conj.}$$

$$(6.2) \quad 'a_{1\bar{1}} = a_{\lambda\bar{\lambda}} (d_\eta \xi^\lambda) (d_{\bar{\eta}} \xi^{\bar{\lambda}}) = \frac{ds^2}{d\eta d\bar{\eta}} = \left| \frac{ds}{d\eta} \right|^2; \quad 'a^{1\bar{1}} = \left| \frac{ds}{d\eta} \right|^{-2}; \quad \text{conj.}$$

$$(6.3) \quad 'T_{11}^1 = 'a^{\bar{1}1} \partial_\eta 'a_{1\bar{1}} = 2 \partial_\eta \log \left| \frac{ds}{d\eta} \right|; \quad \text{conj.}$$

Here  $ds$  is defined by  $ds^2 = a_{\lambda\bar{\lambda}} d\xi^\lambda d\xi^{\bar{\lambda}}$  and it is not a differential of a function of  $\eta$  or of  $\eta$  and  $\bar{\eta}$ . In fact  $\tilde{U}_1$  is two-dimensional and  $\int ds$  depends on the way of integration. From (5.16) and (6.3) it follows that

$$(6.4) \quad H_{11}^{\cdot x} = \frac{d^2 \xi^x}{d\eta^2} - 2 \frac{d\xi^x}{d\eta} \partial_\eta \log \left| \frac{ds}{d\eta} \right| + T_{\mu\lambda}^x \frac{d\xi^\mu}{d\eta} \frac{d\xi^\lambda}{d\eta}; \quad \text{conj.}$$

<sup>1)</sup> Cf. for imbedding in  $\tilde{U}_n$  SCHOUTEN 1929, 1; SCHOUTEN and v. DANTZIG 1930, 1, p. 335ff.; FUCHS 1935, 1; 1937, 1; ROZENFEL'D 1949, 2; CALABI 1953, 1. Imbedding in  $\tilde{R}_n$  was dealt with by BOCHNER 1947, 1, p. 190ff.

<sup>2)</sup> SCHOUTEN and v. DANTZIG 1930, 1; E II 1938, 2, p. 246.

If instead of  $ds$  we introduce a differential  $dz$  defined by  $ds^2 = dz d\bar{z}$ <sup>1)</sup>,  $dz$  is determined to within a factor  $e^{i\varphi}$  with real  $\varphi$ . In general  $dz$  is not a differential of a function and the expression  $z$  without a letter "d" has no meaning. Using  $dz$  we have instead of (6.2–4) equations of exactly the same form but with  $dz$  instead of  $ds$ .

The tangent unitvector can now be defined by

$$(6.5) \quad j^\kappa \stackrel{\text{def}}{=} \frac{d\xi^\kappa}{dz} = \sigma^{-1} \frac{d\xi^\kappa}{d\eta}; \quad \sigma \stackrel{\text{def}}{=} \frac{dz}{d\eta}; \quad \text{conj.}$$

It is defined to within a factor  $e^{i\varphi}$  with  $\varphi$  real. This is not essentially new because the tangent unitvector of a real curve in  $V_n$  is also fixed to within a factor  $\pm 1$  because its sign depends on the choice of the sense on the curve in which  $s$  increases.

If (6.5) is substituted in (6.4) (for  $d\eta \rightarrow dz$ ) we get

$$(6.6) \quad \begin{cases} H_{ii}^\kappa = \frac{d}{d\eta} \sigma j^\kappa - 2\sigma j^\kappa \partial_\eta \log |\sigma| + \sigma^2 I_{\mu\lambda}^\kappa j^\mu j^\lambda \\ \quad = \sigma^2 j^\mu V_\mu j^\kappa - \sigma^2 j^\mu j^\kappa \partial_\mu \log \bar{\sigma}; \quad \text{conj.} \end{cases}$$

because  $\partial/\partial_\eta = \sigma j^\mu \partial_\mu$ . If the curvature vector of the curve is defined as in V § 9

$$(6.7) \quad u^\kappa \stackrel{\text{def}}{=} j^c j^b H_{cb}^\kappa; \quad \text{conj.}$$

it follows that for a  $\tilde{U}_1$  in a  $\tilde{U}_n$  (cf. V § 9)

$$(6.8) \quad u^\kappa j^\mu V_\mu j^\kappa - j^\mu j^\kappa \partial_\mu \log \bar{\sigma}; \quad \text{conj.}$$

in contradistinction to the case of a  $V_1$  in  $V_n$  where  $u^\kappa$  equals  $j^\mu V_\mu j^\kappa$ . Note that  $u^\kappa$  gets a factor  $e^{2i\varphi}$  if  $j^\kappa$  gets a factor  $e^{i\varphi}$ . The vector  $j^\mu V_\mu j^\kappa$  is not perpendicular to  $j^\kappa$  because

$$(6.9) \quad \begin{cases} a_{\lambda\bar{\lambda}} j^\mu j^{\bar{\lambda}} V_\mu j^\lambda = -j^\mu j_{\bar{\lambda}} V_\mu j^{\bar{\lambda}} = -\sigma^{-1} j_{\bar{\lambda}} \partial_\eta j^{\bar{\lambda}} \\ \quad = -\sigma^{-1} j_{\bar{\lambda}} \frac{d\xi^{\bar{\lambda}}}{d\eta} \partial_\eta \bar{\sigma}^{-1} = j^\mu \partial_\mu \log \bar{\sigma}; \quad \text{conj.} \end{cases}$$

but  $u^\kappa$  is the part of  $j^\mu V_\mu j^\kappa$  perpendicular to  $j^\kappa$  as follows from (6.8, 9)

$$(6.10) \quad j_\lambda u^\lambda = j^\mu \partial_\mu \log \bar{\sigma} - j^\mu \partial_\mu \log \bar{\sigma} = 0; \quad \text{conj.}$$

and also from the fact that the  $\kappa$ -region of  $H_{ii}^\kappa$  is perpendicular to  $j^\kappa$ .

A geodesic  $\tilde{U}_1$  in  $\tilde{U}_n$  is characterized by

$$(6.11) \quad \frac{d^2 \xi^\kappa}{d\eta^2} + I_{\mu\lambda}^\kappa \frac{d\xi^\mu}{d\eta} \frac{d\xi^\lambda}{d\eta} = \alpha \frac{d\xi^\kappa}{d\eta}; \quad \text{conj.}$$

<sup>1)</sup> COBURN 1941, 3.

Now the left hand side of this equation is equal to

$$(6.12) \quad \left\{ \begin{array}{l} \sigma j^\mu \partial_\mu \sigma j^* + \sigma^2 \Gamma_{\mu\lambda}^* j^\mu j^\lambda = \sigma^2 j^\mu V_\mu j^* + \sigma^2 j^* j^\mu \partial_\mu \log \sigma \\ \qquad \qquad \qquad = H_1 j^* + 2\sigma^2 j^* j^\mu \partial_\mu \log |\sigma|; \end{array} \right. \text{conj.}$$

hence the n.a.s. condition for a curve to be a geodesic is (cf. 5.14)

$$(6.13) \quad H_1 j^* = 0; \quad \text{conj.}$$

[equivalent to  $u^* = 0$  according to (6.7)] and every geodesic satisfies the equation

$$(6.14) \quad \frac{d^2 \xi^*}{d\eta^2} + \Gamma_{\mu\lambda}^* \frac{d\xi^\mu}{d\eta} \frac{d\xi^\lambda}{d\eta} = 2\sigma^2 j^* j^\mu \partial_\mu \log |\sigma|; \quad \text{conj.}$$

Note that the right hand side of this equation does not depend on the choice of the free factor in  $j^*$  because  $\sigma j^*$  and  $|\sigma|$  are independent of this choice.

We now consider the special case where the free factor in  $dz$  can be chosen in such a way that  $dz$  is a differential of an analytic function  $z(\eta)$ . Because of (6.5) we then have

$$(6.15) \quad \frac{d\xi^*}{d\eta} = j^* \frac{dz}{d\eta}; \quad \text{conj.}$$

and

$$(6.16) \quad \frac{dz}{d\eta} = j_* \frac{d\xi^*}{d\eta} = j_* B_1^* = j_1; \quad \text{conj.}$$

But (6.16) expresses the fact that the vector  $j_b$  in  $\tilde{U}_1$  is analytic. Conversely, if  $j_b$  is analytic,  $dz/d\eta$  is analytic and  $z$  can be found by integration as an analytic function. It has already been proved in VIII § 3 that  $j_b$  is analytic if and only if the  $\tilde{U}_1$  is an  $\tilde{R}_1$ . Because  $j_1$  is determined to within a factor  $e^{i\varphi}$  with  $\varphi$  real and analytic in  $\eta$ , it follows that  $\varphi$  must be a constant. Hence  $z$  is determined to within an affine transformation ' $z = az + b$ ' with constant coefficients and with  $|a|=1$ . We call  $z$  a *natural parameter* on the  $\tilde{R}_1$  in  $\tilde{U}_n$ . Collecting results we have<sup>1)</sup>:

*If an  $X_1$  in a  $\tilde{U}_n$  is given by the analytic functions  $\xi^*(\eta)$  of a complex parameter  $\eta$ , this  $X_1$  is always a  $\tilde{U}_1$ . If and only if this  $\tilde{U}_1$  is an  $\tilde{R}_1$ , that is, if and only if the induced connexion is analytic and thus integrable, it is possible to find a parameter  $z = z(\eta)$  on this curve analytic in  $\eta$  such that  $ds^2 = dz d\bar{z}$ . This natural parameter is fixed to within an affine transformation ' $z = az + b$ ' with constant coefficients and with  $|a|=1$ .*

Only these  $\tilde{R}_1$ 's in  $\tilde{U}_n$  have an analytic tangent unitvector  $d\xi^*/dz$ . They are in many respects analogous to the  $V_1$ 's in a  $V_n$ .<sup>2)</sup> For instance  $u^* = j^\mu V_\mu j^*$ ; (conj.) comes instead of (6.8).

<sup>1)</sup> SCHOUTEN and v. DANTZIG 1930, 1, p. 343; E II 1938, 2, p. 251. The theorem was proved there without the auxiliary differential  $dz$  of COBURN.

<sup>2)</sup> Cf. COBURN 1941, 3.

We see from (6.14) that the equation of a geodesic with respect to a natural parameter  $z$  (if existing) takes the simple form

$$(6.17) \quad \frac{d^2 \xi^\kappa}{dz^2} + I_{\mu\lambda}^\kappa \frac{d\xi^\mu}{dz} \frac{d\xi^\lambda}{dz} = 0; \quad \text{conj.}$$

There is still another kind of curves in  $\tilde{U}_n$  that arises if we consider equations of the form  $\xi^\kappa = \xi^\kappa(t)$  where  $t$  is a *real* parameter.<sup>1)</sup> These curves are one-dimensional. It is possible to get them from the  $\tilde{U}_1$ 's in  $\tilde{U}_n$  that are  $\tilde{R}_1$ 's, by considering only the points of such an  $\tilde{R}_1$  where some definitely chosen natural parameter takes real values. Of course the natural parameter on such a curve is fixed to within transformations of the form ' $z' = z + b$  with real constant  $b$ .<sup>2)</sup>

## § 7. Conformal transformation of a connexion in $\tilde{U}_n$ .

If the fundamental tensor  $a_{\lambda\bar{\lambda}}$  of a  $\tilde{U}_n$  is transformed into

$$(7.1) \quad 'a_{\lambda\bar{\lambda}} = \sigma a_{\lambda\bar{\lambda}}$$

where  $\sigma$  is an arbitrary *real* semi-analytic scalar, the parameters  $I_{\mu\lambda}^\kappa$ ,  $I_{\bar{\mu}\bar{\lambda}}^\kappa$  transform into

$$(7.2) \quad \begin{cases} \text{a)} & 'I_{\mu\lambda}^\kappa = \bar{a}_{\lambda\bar{\lambda}} \partial_\mu 'a_{\lambda\bar{\lambda}} = I_{\mu\lambda}^\kappa + \sigma_\mu A_\lambda^\kappa; \quad \text{conj.} \\ \text{b)} & 'I_{\bar{\mu}\bar{\lambda}}^\kappa = I_{\bar{\mu}\bar{\lambda}}^\kappa = 0; \quad \text{conj.} \quad \sigma_\mu \stackrel{\text{def}}{=} \partial_\mu \log \sigma; \quad \text{conj.} \end{cases}$$

We call this transformation *conformal* (cf. VI § 5). From (7.2) we get

$$(7.3) \quad 'T_{\mu\lambda}^\mu = I_{\mu\lambda}^\mu + \sigma_\lambda; \quad 'T_{\mu\lambda}^\lambda = I_{\mu\lambda}^\lambda + n \sigma_\mu; \quad \text{conj.}$$

and

$$(7.4) \quad \begin{cases} \text{a)} & 'V_\mu v^\kappa = V_\mu v^\kappa + \sigma_\mu v^\kappa; \quad 'V_{\bar{\mu}} v^\kappa = V_{\bar{\mu}} v^\kappa = \partial_{\bar{\mu}} v^\kappa; \quad \text{conj.} \\ \text{b)} & 'V_\mu w_\lambda = V_\mu w_\lambda - \sigma_\mu w_\lambda; \quad 'V_{\bar{\mu}} w_\lambda = V_{\bar{\mu}} w_\lambda = \partial_{\bar{\mu}} w_\lambda; \quad \text{conj.} \end{cases}$$

The transformation (7.2a) is projective (cf. VI § 1) and leaves parallel fields parallel (cf. VI § 1). If and only if  $\sigma_\lambda = 0$ ,  $I_{\mu\lambda}^\kappa$  and  $'I_{\mu\lambda}^\kappa$  can for  $n > 1$  be symmetric at the same time. Hence

*A conformal transformation of a  $\tilde{U}_n$  is at the same time a restricted projective transformation<sup>3)</sup> and  $\sigma_\lambda$  is the gradient of a **real** scalar. If a*

<sup>1)</sup> SCHOUTEN and v. DANTZIG 1930, 1, p. 343; 1931, 3; COBURN 1941, 1; 3. Cf. COBURN 1942, 2; SUGURI 1951, 1 for FRENET formulae for these curves. VARGA 1939, 1 dealt with curves in the elliptic  $\tilde{R}_2$  and GOLIFMAN 1941, 1 considered one-dimensional curves in the hyperbolic  $\tilde{R}_2$ .

<sup>2)</sup> Congruences of curves of both kinds were studied by COBURN 1943, 1.

<sup>3)</sup> COBURN 1941, 1, p. 32f. Cf. VI § 1; cf. for projective transformations also DE MIRA FERNANDES 1950, 1.

*conformal transformation leaves the connexion of a  $\tilde{V}_n$ ,  $n > 1$ , symmetric,  $a_{\lambda\bar{\lambda}}$  gets a **constant** factor.<sup>1)</sup>*

This latter statement implies that the only conformal transformation that transforms an  $\tilde{S}_n$  into an  $\tilde{S}_n$  is a transformation with  $\sigma = \text{const.}$ . We have seen in VI § 5 that an  $S_n$  can be transformed conformally into every other  $S_n$ . This is quite different in a unitary space. In VI § 5 we also saw that the connexion of a  $V_n$  is determined by its conformal and projective properties together. Now let

$$(7.5) \quad 'T_{\mu\lambda}^{\kappa} = T_{\mu\lambda}^{\kappa} + A_{\lambda}^{\kappa} \partial_{\mu} \log \sigma$$

be a connexion derived from the  $T_{\mu\lambda}^{\kappa}$  of a  $\tilde{U}_n$  by a conformal transformation. Then  $\partial_{\mu} \log \sigma$  can be computed from  $'T_{\mu\lambda}^{\kappa}$  if  $S_{\mu} = \frac{2}{n-1} S_{\mu\lambda}^{\lambda}$  is given. Hence

*The connexion of a  $\tilde{V}_n$  is determined by its conformal properties only, but for the determination of the connexion of a  $\tilde{U}_n$  we also need the field  $S_{\mu}$ .*

$S_{\mu\lambda}^{\lambda}$  and  $S_{\mu}$  transform as follows (cf. 7.2)

$$(7.6) \quad \begin{cases} \text{a)} \quad 'S_{\mu\lambda}^{\lambda} = S_{\mu\lambda}^{\lambda} + \sigma_{[\mu} A_{\lambda]}^{\kappa} \\ \text{b)} \quad 'S_{\mu} = S_{\mu} + \sigma_{\mu} \end{cases}$$

hence

*The connexion of a  $\tilde{U}_n$  can be transformed conformally into a symmetric connexion if and only if it is semi-symmetric and if  $S_{\mu}$  is a gradient of a real scalar.<sup>2)</sup>*

The transformation of the curvature quantities can be derived from (2.17, 21) and (7.2):

$$(7.7) \quad \begin{cases} \text{a)} \quad 'R_{\nu\mu\lambda}^{\kappa} = R_{\nu\mu\lambda}^{\kappa} + A_{\lambda}^{\kappa} \partial_{\nu} \sigma_{\mu}; & \text{conj.} \\ \text{b)} \quad 'R_{\mu\nu\lambda}^{\kappa} = R_{\mu\nu\lambda}^{\kappa} - A_{\lambda}^{\kappa} \partial_{\nu} \sigma_{\mu}; & \text{conj.} \\ \text{c)} \quad 'R_{\nu\lambda} = R_{\nu\lambda} - \partial_{\nu} \sigma_{\lambda}; & \text{conj.} \\ \text{d)} \quad 'V'_{\nu\mu} = V'_{\nu\mu} + n \partial_{\nu} \sigma_{\mu}; & \text{conj.} \\ \text{e)} \quad 'R = \sigma^{-1} R - \sigma^{-1} a^{\nu\lambda} \partial_{\nu} \sigma_{\lambda}; & \text{conj.} \end{cases}$$

A  $\tilde{U}_n$  is called *conformally euclidean* or a  $\tilde{C}_n$  if it can be transformed conformally into an  $\tilde{R}_n$ . From (7.6a, 7a) it follows:

*A  $\tilde{U}_n$  is a  $\tilde{C}_n$  if and only if there exists a real semi-analytic gradient field  $\sigma_{\lambda}$  such that*

$$(7.8) \quad \text{a)} \quad R_{\nu\mu\lambda}^{\kappa} = -A_{\lambda}^{\kappa} \partial_{\nu} \sigma_{\mu}; \quad \text{conj.} \quad \text{b)} \quad S_{\mu\lambda}^{\lambda} = A_{[\mu}^{\kappa} \sigma_{\lambda]}; \quad \text{conj.}$$

<sup>1)</sup> COBURN 1941, 1, p. 31; 1942, 3, p. 140.

<sup>2)</sup> A gradient of a real scalar need not have real components.

For  $n > 1$  it follows from  $A_{[\mu}^{\lambda} \sigma_{\lambda]} = 0$  that  $\sigma_{\lambda} = 0$ , hence

*A  $\tilde{V}_n$  is for  $n > 1$  a  $\tilde{C}_n$  if and only if it is an  $\tilde{R}_n$ .<sup>1)</sup>*

As a corollary we get that no  $\tilde{S}_n$ ,  $n > 1$ , with non vanishing curvature is a  $\tilde{C}_n$ .<sup>1)</sup>

From (7.8) we see that a  $\tilde{C}_n$  is always semi-symmetric and that

$$(7.9) \quad R_{\bar{\nu}\mu\lambda}^{\kappa} = A_{\lambda}^{\kappa} \partial_{\bar{\nu}} S_{\mu}.$$

In VIII § 3 it has been proved that for a semi-symmetric  $\tilde{U}_n$ ,  $n > 2$ ,  $S_{\lambda}$  is always a gradient vector. Now if  $S_{\lambda}$  equals  $\partial_{\lambda} \varphi$  it is also the gradient of the real scalar  $\frac{1}{2}(\varphi + \bar{\varphi})$  if and only if  $\partial_{\bar{\nu}} S_{\lambda} = \partial_{\lambda} S_{\bar{\nu}}$ . Hence<sup>2)</sup>

*A  $\tilde{U}_n$  can be transformed conformally into a  $\tilde{V}_n$  if and only if for  $n > 2$  the connexion is semi-symmetric and  $\partial_{\bar{\mu}} S_{\lambda} = \partial_{\lambda} S_{\bar{\mu}}$ , and for  $n = 2$   $S_{\lambda}$  is a gradient of a real scalar. A  $\tilde{U}_n$  is a  $\tilde{C}_n$  if and only if these conditions are satisfied and if moreover*

$$(7.10) \quad 0 = C_{\bar{\nu}\mu\lambda}^{\kappa} \stackrel{\text{def}}{=} R_{\bar{\nu}\mu\lambda}^{\kappa} - A_{\lambda}^{\kappa} \partial_{\bar{\nu}} S_{\mu}$$

$C_{\bar{\nu}\mu\lambda}^{\kappa}$  is called the *conformal curvature tensor* of the  $\tilde{U}_n$ .

COBURN has dealt with several problems concerning conformal transformations in  $\tilde{U}_n$ . He mostly considers the transformation not as a transformation of the connexion in one and the same space but as a transformation of a space  $\tilde{U}_n$  into another space ' $\tilde{U}_n$ '. But of course this interpretation is not essential. Since such a conformal transformation is always at the same time a restricted projective transformation he requires all those restricted projective transformations of this kind that transform a  $\tilde{U}_n$  into a  $\tilde{V}_n$ .<sup>3)</sup> In the same paper<sup>4)</sup> he considers a *conformally symmetric*  $\tilde{U}_n$ , that is a  $\tilde{U}_n$  with the fundamental tensor

$$(7.11) \quad a_{\lambda\bar{\nu}} = \beta \partial_{\lambda} \partial_{\bar{\nu}} \log \varphi; \quad \text{conj.}; \quad \beta = \text{real}; \quad \varphi = \text{real},$$

subjected to conformal transformations of the form

$$(7.12) \quad 'a_{\lambda\bar{\nu}} = \varphi^{\alpha} a_{\lambda\bar{\nu}}; \quad \text{conj.}; \quad \alpha = \text{real and const.}$$

and deduces invariants for these transformations.

<sup>1)</sup> COBURN 1942, 3, p. 140, only for  $n > 2$ . Remark that every  $S_n$  is a  $C_n$  (cf. VI § 5).

<sup>2)</sup> COBURN 1942, 3, p. 138.

<sup>3)</sup> COBURN 1941, 1, p. 32ff.

<sup>4)</sup> p. 37ff.

### Exercises.

VIII 7,1<sup>1)</sup>. In a  $\tilde{U}_n$  that can be transformed into a  $\tilde{U}_n$  with  $R_{\bar{\nu}\mu}^{\lambda}{}^{\kappa} = 0$  by the conformal transformation  $a_{\lambda\bar{x}} \rightarrow \sigma a_{\lambda\bar{x}}$  the following identities hold

$$\text{VIII 7,1 } \alpha) \quad R_{\bar{\nu}\mu}^{\lambda}{}^{\kappa} = \frac{1}{n} A_{\lambda}^{\kappa} \partial_{\bar{\nu}} \partial_{\mu} \log a; \quad a = |\text{Det}(a_{\lambda\bar{x}})|$$

$$\text{VIII 7,1 } \beta) \quad R_{\nu}^{\lambda}{}^{\kappa} = V_{\nu} \sigma^{\kappa}; \quad \sigma_{\lambda} \stackrel{\text{def}}{=} \partial_{\lambda} \log \sigma.$$

VIII 7,2. A  $\tilde{V}_n$ ,  $n > 1$ , can be transformed conformally into a  $\tilde{U}_n$  with  $R_{\bar{\nu}\mu}^{\lambda}{}^{\kappa} = 0$ , if it is an  $\tilde{R}_n$  (cf. the special case mentioned in the text).

VIII 7,3. If in a semi-symmetric  $\tilde{U}_n$  the vector  $S_{\lambda}$  is a gradient of a real scalar and if there exists a coordinate system ( $x$ ) such that

$$\text{VIII 7,3 } \alpha) \quad I_{(\mu\lambda)}^{\kappa} + A_{(\mu}^{\kappa} S_{\lambda)} \stackrel{*}{=} 0$$

it can be proved that the  $\tilde{U}_n$  is a  $\tilde{C}_n$  and that there exists an  $\tilde{R}_n$  such that the geodesics of  $\tilde{U}_n$  and  $\tilde{R}_n$  are in restricted correspondence.<sup>2)</sup>

VIII 7,4. In a  $\tilde{U}_n$  the tensor  $V'_{\mu\lambda} + n R_{\mu\lambda}$  is a conformal concomitant that vanishes if the  $\tilde{U}_n$  is a  $\tilde{C}_n$ .

### § 8. Conformal unitary connexions<sup>3)</sup>.

The quantity (cf. VI § 7)

$$(8.1) \quad \mathfrak{A}_{\lambda\bar{x}} \stackrel{\text{def}}{=} a^{-\frac{1}{n}} a_{\lambda\bar{x}}; \quad \text{conj.}$$

is a hermitian tensor density of weight  $-\frac{2}{n}$ . It is conformally invariant and its determinant is  $+1$ . Analogous to VI § 7 and (VIII 3.4) we may define from it the conformal parameters

$$(8.2) \quad \overset{\circ}{I}_{\mu\lambda}^{\kappa} = \mathfrak{A}^{\bar{\sigma}\kappa} \partial_{\mu} \mathfrak{A}_{\lambda\bar{\sigma}} = I_{\mu\lambda}^{\kappa} - \frac{1}{n} A_{\lambda}^{\kappa} \partial_{\mu} \log a = I_{\mu\lambda}^{\kappa} - \frac{1}{n} \Gamma_{\mu} A_{\lambda}^{\kappa}; \quad \text{conj.}$$

They transform as follows

$$(8.3) \quad \begin{cases} \text{a) } \overset{\circ}{I}_{\mu'\lambda'}^{\kappa'} = A_{\mu'}^{\mu} A_{\lambda'}^{\lambda} A_{\kappa'}^{\kappa} \overset{\circ}{I}_{\mu\lambda}^{\kappa} + A_{\theta}^{\kappa'} \partial_{\mu'} A_{\lambda}^{\theta} - \frac{1}{n} A_{\lambda'}^{\kappa'} \partial_{\mu'} \log A; & \text{conj.} \\ \text{b) } \overset{\circ}{I}_{\mu'\lambda'}^{\mu} = A_{\lambda'}^{\lambda} \overset{\circ}{I}_{\mu\lambda}^{\mu} + \frac{n-1}{n} \partial_{\lambda'} \log A; & \text{conj.} \end{cases}$$

Now in contradistinction to the conformal parameters belonging to an ordinary metric dealt with in VI § 7 it is possible to derive a connexion

<sup>1)</sup> Cf. Exerc. VI 5,1.

<sup>2)</sup> COBURN 1941, 1 calls the correspondence between geodesics *restricted* if the vector of the projective transformation is a gradient of a real scalar.

<sup>3)</sup> COBURN 1942, 3, p. 129ff. This section is a free interpretation and at some points a simplification of the most important of his results.

from  $\overset{c}{\Gamma}_{\mu\lambda}^{\kappa}$  without using any auxiliary objects or invariant objects of higher order. In fact, if we define for  $n > 1$

$$(8.4) \quad \overset{c}{\Pi}_{\mu\lambda}^{\kappa} \stackrel{\text{def}}{=} \overset{c}{\Gamma}_{\mu\lambda}^{\kappa} + \frac{1}{n-1} A_{\lambda}^{\kappa} \overset{c}{\Gamma}_{\rho\mu}^{\rho} = \overset{c}{\Gamma}_{\mu\lambda}^{\kappa} - S_{\mu} A_{\lambda}^{\kappa}; \quad \text{conj.}$$

the  $\overset{c}{\Pi}_{\mu\lambda}^{\kappa}$  transform just like the  $\overset{c}{\Gamma}_{\mu\lambda}^{\kappa}$  because  $A_{\lambda}^{\kappa} S_{\mu}$  is a tensor, and these new parameters are conformally invariant because of (7.5, 6b). Denoting the covariant derivative belonging to them by  $\overset{c}{V}$  we have

$$(8.5) \quad \begin{cases} \text{a)} & \overset{c}{V}_{\mu} a_{\lambda\bar{\kappa}} = \overset{c}{V}_{\mu} a_{\lambda\bar{\kappa}} + S_{\mu} A_{\lambda}^{\sigma} a_{\sigma\bar{\kappa}} = S_{\mu} a_{\lambda\bar{\kappa}}; \quad \text{conj.} \\ \text{b)} & \overset{c}{V}_{\mu} \alpha = \alpha \overset{c}{V}_{\mu} \log \alpha = \alpha a^{\bar{\kappa}\lambda} \overset{c}{V}_{\mu} a_{\lambda\bar{\kappa}} = n \alpha S_{\mu}; \quad \text{conj.} \end{cases}$$

hence

$$(8.6) \quad \overset{c}{V}_{\mu} \mathfrak{A}_{\lambda\bar{\kappa}} = \overset{c}{V}_{\mu} \alpha^{-\frac{1}{n}} a_{\lambda\bar{\kappa}} = 0; \quad \text{conj.}$$

We call  $\overset{c}{\Pi}_{\mu\lambda}^{\kappa}$  the parameters of the *conformal connexion* belonging to  $[a_{\lambda\bar{\kappa}}]$ . From (8.4) it follows that the conformal connexion is identical with the connexion belonging to  $a_{\lambda\bar{\kappa}}$  if and only if  $S_{\mu} = 0$ , that is (cf. 3.9) if and only if  $a_{\lambda\bar{\kappa}}$  satisfies the differential equation

$$(8.7) \quad \partial_{\mu} \log \alpha = a^{\bar{\kappa}\lambda} \partial_{\lambda} a_{\mu\bar{\kappa}}; \quad \text{conj.}$$

This is always true in a  $\widetilde{V}_n$ , hence the conformal connexion of a  $\widetilde{V}_n$  is the connexion of the  $\widetilde{V}_n$  itself. In fact we have already seen that for a conformal transformation of a  $\widetilde{V}_n$  into a  $\widetilde{V}_n$  always  $\sigma = \text{const}$ . As a generalization we may mention the fact that for a conformal transformation of a  $\widetilde{U}_n$  with  $S_{\mu} = 0$  into a  $\widetilde{U}_n$  with  $S_{\mu} = 0$  also  $\sigma = \text{const}$ . as follows immediately from (7.2a; 6b).

The following conformally invariant tensors can be deduced from  $\overset{c}{\Pi}_{\mu\lambda}^{\kappa}$

$$(8.8) \quad \begin{cases} \text{a)} & \left\{ \begin{array}{l} \overset{c}{R}_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} \stackrel{\text{def}}{=} 2\partial_{[\nu} \overset{c}{\Pi}_{\mu]\lambda}^{\kappa} + 2\overset{c}{\Pi}_{[\nu|\rho}^{\kappa} \overset{c}{\Pi}_{\mu]\lambda}^{\rho} \\ = R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} - 2A_{\lambda}^{\kappa} \partial_{[\nu} S_{\mu]} = -2A_{\lambda}^{\kappa} \partial_{[\nu} S_{\mu]}; \quad \text{conj.} \end{array} \right. \\ \text{b)} & \overset{c}{R}_{\bar{\nu}\mu\lambda}^{\cdot\cdot\cdot\kappa} \stackrel{\text{def}}{=} \partial_{\bar{\nu}} \overset{c}{\Pi}_{\mu\lambda}^{\kappa} = R_{\bar{\nu}\mu\lambda}^{\cdot\cdot\cdot\kappa} - A_{\lambda}^{\kappa} \partial_{\bar{\nu}} S_{\mu} = C_{\bar{\nu}\mu\lambda}^{\cdot\cdot\cdot\kappa}; \quad \text{conj.} \end{cases}$$

The conformal invariance of the first one follows from (7.6b) and the second one equals the conformal curvature tensor defined in (7.10). Both vanish in a  $\widetilde{C}_n$ .

The connexion  $\overset{c}{\Pi}_{\mu\lambda}^{\kappa}$  leaves invariant the hermitian tensor density  $\mathfrak{A}_{\lambda\bar{\kappa}}$  but not in general a hermitian tensor. If in an exceptional case a

tensor  $b_{\lambda\bar{\kappa}}$  with rank  $n$  is invariant

$$(8.9) \quad \overset{c}{V}_\mu b_{\lambda\bar{\kappa}} = \partial_\mu b_{\lambda\bar{\kappa}} - \overset{c}{\Pi}_{\mu\lambda}^\rho b_{\rho\bar{\kappa}} = 0; \quad \text{conj.}$$

the first integrability conditions must be satisfied:

$$(8.10) \quad \begin{cases} \text{a)} & b_{\rho\bar{\kappa}} \overset{c}{R}_{\nu\mu}^{\cdot\cdot\lambda} \overset{e}{=}- b_{\lambda\bar{\kappa}} \partial_{[\nu} S_{\mu]} = 0; \quad \text{conj.} \\ \text{b)} & R_{\bar{\mu}\mu}^{\cdot\cdot\lambda} \overset{e}{b}_{\rho\bar{\kappa}} - R_{\mu\bar{\mu}\bar{\kappa}}^{\cdot\cdot\bar{\lambda}} b_{\lambda\bar{\kappa}} - b_{\lambda\bar{\kappa}} \partial_{\bar{\mu}} S_\mu + b_{\lambda\bar{\kappa}} \partial_\mu S_{\bar{\mu}} = 0; \quad \text{conj.} \end{cases}$$

From (8.10a) and (8.10b) transvected with  $\overset{-1}{b}{}^{\bar{\lambda}}{}^\lambda$  (cf. 3.19d) it follows that  $S_\mu$  must be the gradient of a *real* scalar (cf. VIII § 7). But then the last two terms in (8.10b) vanish and there remains only the algebraic equation

$$(8.11) \quad R_{\bar{\mu}\mu}^{\cdot\cdot\lambda} \overset{e}{b}_{\rho\bar{\kappa}} - R_{\mu\bar{\mu}\bar{\kappa}}^{\cdot\cdot\bar{\lambda}} b_{\lambda\bar{\kappa}} = 0; \quad \text{conj.}$$

Because of (3.22) we have

$$(8.12) \quad R_{\bar{\mu}\mu}^{\cdot\cdot\lambda} \overset{e}{a}_{\rho\bar{\kappa}} = R_{\mu\bar{\mu}\bar{\kappa}}^{\cdot\cdot\bar{\lambda}} a_{\lambda\bar{\kappa}}; \quad \text{conj.}$$

and this proves that (8.11) is satisfied if  $b_{\lambda\bar{\kappa}} = \sigma a_{\lambda\bar{\kappa}}$ , where  $\sigma$  is a *real* coefficient. In fact there exists such a solution if  $S_\mu$  is a gradient of a *real* scalar because if in this case we take  $\sigma_\mu = -S_\mu$  we get

$$(8.13) \quad \overset{c}{V}_\mu \sigma a_{\lambda\bar{\kappa}} = \sigma \sigma_\mu a_{\lambda\bar{\kappa}} + \sigma S_\mu a_{\lambda\bar{\kappa}} = 0; \quad \text{conj.}$$

and  $\sigma a_{\lambda\bar{\kappa}}$  is the fundamental tensor of a  $\tilde{U}_n$  with parameters  $\overset{c}{\Pi}_{\mu\lambda}^\kappa$  in accordance with VIII § 3.

But there may exist other solutions. In order to find them we have to differentiate (8.11) and to eliminate the derivatives of  $b_{\lambda\bar{\kappa}}$  by means of (8.10). Proceeding in this way we get at last a set of algebraic equations for  $b_{\lambda\bar{\kappa}}$  that can be no more extended and that may have solutions with rank  $n$  (cf. the theory of sets of partial differential equations in II § 5). If there is no solution with rank  $n$  the  $\overset{c}{\Pi}_{\mu\lambda}^\kappa$  can not be the parameters of the connexion of a  $\tilde{U}_n$ .

### Exercise.

VIII 8.1. If in a  $\tilde{U}_n$  (cf. Exerc. VIII 7.1)

$$\text{VIII 8.1 } \alpha) \quad R_{\bar{\nu}\mu}^{\cdot\cdot\lambda} \overset{\kappa}{=} \frac{1}{n} A_\lambda^\kappa \partial_{\bar{\nu}} \partial_\mu \log \alpha$$

and if  $\tau = f(\xi^\kappa) g(\xi^{\bar{\kappa}})$  is an ordinary density of weight  $-2$ , the  $\tilde{U}_n$  with the fundamental tensor

$$'a_{\lambda\bar{\kappa}} = \tau^{-\frac{1}{n}} \mathfrak{A}_{\lambda\bar{\kappa}}$$

has ' $R_{\bar{\nu}\mu}^{\cdot\cdot\lambda} \overset{\kappa}{=}$   $0$ '.

### § 9. Spaces of recurrent curvature.<sup>1)</sup>

In III § 2 a recurrent field was defined as a field that at every point and for every direction is proportional to its covariant differential and in VII § 4 we considered, from the point of view of their holonomy group, spaces with a recurrent curvature tensor.<sup>2)</sup> A  $V_n$  is called a  $K_n$  if  $K_{\nu\mu\lambda\nu}$  is recurrent but not covariant constant

$$(9.1) \quad V_\omega K_{\nu\mu\lambda\nu} = k_\omega K_{\nu\mu\lambda\nu}; \quad k_\omega \neq 0$$

and it is said to be a  $K_n^*$  if it is either a  $K_n$  or symmetric

$$(9.2) \quad V_\omega K_{\nu\mu\lambda\nu} = 0$$

and if in this latter case there exists a vector  $k_\omega$  such that

$$(9.3) \quad k_{[\omega} K_{\nu\mu]\lambda\nu} = 0; \quad k_\omega \neq 0.$$

If (9.2, 3) hold,  $k_\omega$  need not be uniquely determined. For instance an  $R_n$  is a  $K_n^*$  with undetermined  $k_\omega$  and an  $S_n$  with  $K \neq 0$  is neither a  $K_n$  nor a  $K_n^*$ .

For  $n=2$  we have (cf. III 5.31):

$$(9.4) \quad K_{\nu\mu\lambda\nu} = -2\nu g_{[\nu} g_{\mu]\lambda\nu}$$

hence

$$(9.5) \quad V_\omega K_{\nu\mu\lambda\nu} = -2(V_\omega \log \nu) K_{\nu\mu\lambda\nu}$$

and this proves that a  $V_2$  is either an  $S_2$  with  $\nu \neq 0$  or a  $K_2$ .

In VII § 4 we proved that  $k_\omega$  is a gradient in a  $K_n$ . In a symmetric  $K_n^*$  the integrability conditions of

$$(9.6) \quad (V_{[\omega} k) K_{\nu\mu]\lambda\nu} = 0$$

are identically satisfied in consequence of (9.2). Hence  $k_\omega$  can always be chosen in such a way that it is a gradient.<sup>3)</sup>

If for  $n > 2$  a  $K_n^*$  is a product  $V_m \times V_{n-m}$  (cf. V § 11) the linear element can be written in the form

$$(9.7) \quad \left\{ \begin{array}{l} ds^2 = g_{\lambda_1\mu_1}(\xi^\kappa) d\xi^{\lambda_1} d\xi^{\mu_1} + g_{\lambda_2\mu_2}(\xi^\kappa) d\xi^{\lambda_2} d\xi^{\mu_2}; \\ \lambda_1, \mu_1 = 1, \dots, m \\ \lambda_2, \mu_2 = m+1, \dots, n \end{array} \right.$$

<sup>1)</sup> This subject was introduced by RUSE 1948, 2; 1949, 2. His first investigations were connected with harmonic spaces and dealt especially with the cases  $n=2, 3, 4$ . WALKER 1950, 1 introduced the  $K_n^*$  and the simple  $K_n^*$  by means of which he and RUSE in 1951, 1; 2 succeeded in giving a classification.

<sup>2)</sup> Spaces with recurrent RICCI tensor were considered by PATTERSON 1952, 2.

<sup>3)</sup> WALKER 1950, 1, p. 46.

and from this we see that  $I_{\mu\lambda}^*$  and  $K_{\nu\mu\lambda\kappa}$  have no components with indices from both ranges. But then it follows from (9.3) that

$$(9.8) \quad K_{\nu_1\mu_1\lambda_1\kappa_1} k_{\omega_2} = 0; \quad K_{\nu_2\mu_2\lambda_2\kappa_2} k_{\omega_1} = 0$$

and this implies that either  $V_m$  or  $V_{n-m}$  or both are euclidean. The only non-trivial case is where only one of them, for instance  $V_{n-m}$  is euclidean. Because the rank of  $g_{\lambda_2\mu_2}$  must be  $n-m$ ,  $V_{n-m}$  must be an  $R_{n-m}$ . Because  $k_{\omega_1} \neq 0$  and independent of the  $\xi^\kappa$ , we have  $k_{[\omega_1} K_{\nu_1\mu_1]\lambda_1\kappa_1} = 0$  and this implies that the  $V_m$  is a  $K_m^*$ . Hence<sup>1)</sup>

*If a  $K_n^*$  is decomposable it is a product of a  $K_m^*$  and an  $R_{n-m}$ .*

$K_n^*$  is called the *flat extension* of  $K_m^*$ . Of course every flat extension of a  $K_m^*$  is some  $K_n^*$  and the same holds for  $K_m$  and  $K_n$ .<sup>2)</sup>

If a  $K_n^*$  contains a parallel field of  $m$ -directions and a field of perpendicular  $(n-m)$ -directions, the first field forms a normal system of geodesic  $V_m$ 's and the second a normal system of geodesic  $V_{n-m}$ 's perpendicular to the first. But then the linear element can be written in the form (9.7). Hence, the  $V_{n-m}$ 's are  $R_{n-m}$ 's and the  $V_m$ 's are  $K_m^*$ 's or vice versa. If the  $V_{n-m}$ 's are  $R_{n-m}$ 's there exist in them  $n-m$  mutually perpendicular non-null vector fields that are not only parallel but also covariant constant.<sup>3)</sup> Hence<sup>4)</sup>

*If a  $K_n^*$  admits exactly  $n-m \geq 1$  mutually perpendicular non-null covariant constant vector fields it is a product of a  $K_m^*$  and an  $R_{n-m}$ .*

This leads to the consideration of a  $K_n^*$  admitting exactly  $r$  mutually perpendicular covariant constant null vector fields. Now EISENHART<sup>5)</sup> proved that in a  $V_n$  of this kind  $n-r$  is always  $\geq r$  and that the linear element can be written in the form

$$(9.9) \quad \left\{ \begin{array}{l} ds^2 = g_{\beta\alpha} d\xi^\beta d\xi^\alpha + 2 \sum d\xi^\chi d\xi^{\chi+n-r}; \\ \alpha, \beta = 1, \dots, n-r \\ \chi = 1, \dots, r \end{array} \right.$$

where the  $g_{\beta\alpha}$  depend on the  $\xi^\alpha$  only. Such a  $V_n$  is called a *null-extension* of the  $V_{n-r}$ , with the linear element  $g_{\beta\alpha} d\xi^\beta d\xi^\alpha$ . This does not imply that the  $V_n$  is decomposable. If this result is applied to a  $K_n^*$  with  $n-m \geq 1$

<sup>1)</sup> WALKER 1950, 1, p. 38.

<sup>2)</sup> RUSE 1949, 2.

<sup>3)</sup> We differ here from RUSE and WALKER. We call a vector field covariant constant if its covariant derivative vanishes at all points, and parallel if at every point and for every direction it is proportional to its covariant differential (cf. III § 2).

<sup>4)</sup> WALKER 1950, 1, p. 39.

<sup>5)</sup> EISENHART 1938, 1; cf. WALKER 1949, 1; 1950, 3; 4.

mutually perpendicular covariant constant vector fields from which  $r$  are null and  $n-m-r$  non-null, we get the linear element<sup>1)</sup>

$$(9.10) \quad \left\{ \begin{array}{l} ds^2 = g_{\alpha\beta} d\xi^\beta d\xi^\alpha + 2\sum d\xi^\chi d\xi^{\chi+m} + \sum_\omega e d\xi^\omega d\xi^\omega \\ \alpha, \beta = 1, \dots, m \\ \chi = 1, \dots, r \\ \omega = r+m+1, \dots, n \\ e = \pm 1. \end{array} \right.$$

A  $V_n$  that admits  $n-2$  independent covariant constant vector fields can be proved to be a  $K_n^*$ . Such a  $K_n^*$  is called *simple*.<sup>2)</sup> The classification of all simple  $K_n^*$ 's makes use of the following propositions due to RUSE

*Every  $V_2$  is a  $K_2^*$ . Every  $V_3$  which is a null-extension of a  $V_2$  is a  $K_3^*$ . Every  $V_4$  which is a null-extension of a  $V_2$  is a  $K_4^*$  and harmonic.*

From this it can be proved that there are only three cases for the simple  $K_n^*$ .<sup>2)</sup> If there are exactly  $n-2$  mutually perpendicular covariant constant vector fields of which exactly  $r$  are null vectors, we have

- a)  $r=0$ :  $K_n^*$  is a flat extension of a  $K_2^*$ ;
- b)  $r=1$ :  $K_n^*$  is a flat extension of a  $K_3^*$  which is a null-extension of a  $K_2^*$ ;
- c)  $r=2$ :  $K_n^*$  is a flat extension of a  $K_4^*$  which is a null-extension of a  $K_2^*$ .

From the linear element (9.10) it can be proved by direct computation that  $K_{\nu\mu\lambda\kappa}$  in a simple  $K_n^*$  is the general product of two equal simple bivectors:

$$(9.11) \quad K_{\nu\mu\lambda\kappa} = u_{[\nu} v_{\mu]} u_{[\lambda} v_{\kappa]}.$$

Conversely, if we know that the curvature tensor of a  $K_n^*$  has the form (9.11) the  $E_{n-2}$  spanned by  $u_\lambda$  and  $v_\lambda$  forms a parallel field. Hence a  $K_n^*$  is simple if and only if its curvature tensor has the form (9.11).<sup>3)</sup>

RUSE<sup>4)</sup> has given the following classification of the simple  $K_n^*$ 's:

I.  $k_\lambda$  non-null;  $K_n^*$  simple

- A.  $K \neq 0$ ;  $K_n^* = K_2^* \times R_{n-2}$ ,
- B.  $K = 0$ ;  $K_{\mu\lambda} \neq 0$ ;  $K_n^* = k_3^* \times R_{n-3}$ ;

1) WALKER 1950, 1.

2) WALKER 1950, 1, p. 40ff.

3) WALKER 1950, 1, p. 44.

4) RUSE 1951, 1; 2.

II.  $k_\lambda$  null

- A.  $K \neq 0$ ;  $K_n^* = K_2^* \times R_{n-2}$ ;
- B.  $K = 0$ ;  $K_{\mu\lambda} \neq 0$ ;  $K_n^* = k_3^* \times R_{n-3}$ ;
- C.  $K_{\mu\lambda} = 0$ ;  $K_n^* = k_4^* \times R_{n-4}$ ;

where  $k_3^*$  and  $k_4^*$  are defined as  $K^*$ -spaces with the linear element

$$(9.12) \quad ds^2 \equiv \psi(x, z) dx^2 + 2dx dy + dz^2$$

and

$$(9.13) \quad ds^2 \equiv \psi(x, z) dx^2 + 2dx dy + 2dz dt$$

respectively. WALKER<sup>1)</sup> proved that the linear element of a non-simple  $K_n^*$  for  $n > 3$  can always be written in the form

$$(9.14) \quad ds^2 \equiv \psi d\xi^1 d\xi^1 + 2d\xi^1 d\xi^2 + k_{\alpha\beta} d\xi^\alpha d\xi^\beta; \quad \alpha, \beta = 3, \dots, n$$

where

$$(9.15) \quad \psi \equiv \theta a_{\alpha\beta} \xi^\alpha \xi^\beta + \chi_\alpha \xi^\alpha; \quad a_{\alpha\beta} \equiv \text{const.}; \quad k_{\alpha\beta} \equiv \text{const.}; \quad |k_{\alpha\beta}| \neq 0$$

and where  $\theta \neq 0$  and the  $\chi_\alpha$  are functions of  $\xi^1$ . The  $K_n^*$  is symmetric if and only if  $\theta = \text{const.}$

In the same paper WALKER also gave particulars about the symmetric  $K_n^*$ , the EINSTEIN  $K_n^*$ <sup>2)</sup> and the  $K_n^*$  which is a  $C_n$ . Moreover he proved some remarkable theorems on harmonic  $K_n^*$ 's, for instance that for  $n > 2$  every harmonic  $K_n^*$  is simply harmonic (cf. VIII § 1) and that for every  $n > 3$  there exists a simply harmonic  $K_n^*$  which is non-decomposable.

### Exercises.

VIII 9,1. Prove that the  $V_n$ 's with the linear elements

$$\text{VIII 9,1 } \alpha) \quad ds^2 \equiv \psi(x, z) dx^2 + 2dx dy + dz^2$$

and

$$\text{VIII 9,1 } \beta) \quad ds^2 \equiv \psi(x, z) dx^2 + 2dx dy + 2dz dt$$

are both  $K_n^*$ 's.

VIII 9,2. Prove that the  $K_4^*$  with the linear element

$$ds^2 \equiv \psi(x, y) dx^2 + 2\varphi(x, y) dx dy + \zeta(x, y) dy^2 + 2dx dz + 2dy dt$$

is simply harmonic (cf. Exerc. VIII 1,2).

VIII 9,3<sup>3)</sup>). Every  $K_n^*$ ,  $n > 2$ , that is an EINSTEIN space, is a special EINSTEIN space.

<sup>1)</sup> WALKER 1950, 1, p. 51.

<sup>2)</sup> Cf. also MOGI 1950, 1.

<sup>3)</sup> WALKER 1950, 1.

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## Index.

- $A_n$  126
- $\tilde{A}_n$  394
- $\mathfrak{A}_n$  61
- $A_n^m$  257, 264
- abbreviations ad hoc 61
- absolute differential invariant 164
- invariance *w. r.* to the streamlines 107
- parallelism 142
- adaptation, theorem of 73
- ADATI, problem of 321 ff.
- addition of quantities 13
- adjoint 34, 38
  - group 198
  - group linear 199
  - system 80
- affine conic 299
- group  $G_a$  1
- length 233
- motions 346f., 370
- parameter 155, 233, 299
- space 1
- alternated derivation 83
- alternating 14
  - differential form 118
  - multiplication 23
  - part 14
  - tensor 22
- analytic connexion in  $\tilde{U}_n$  393, 403
- functions 62
- analyticity of a field 66, 70
- anholonomic coordinate system 99 ff.
  - object  $\Omega_j^k$ , 100
- angles 44
- principal 48
- asymptotic lines 237, 250, 271
  - $V_m$  of order  $\rho$  275
- axes, principal 46, 56
- axial point 272
  
- base 70, 71, 74, — theorems 74, 75, —
  - transformation 75
- basis vectors 7f.
- bending 244
  
- BERTRAND curves 342ff.
- BIANCHI's identity 146ff.
- bivector 23, 35
- blades 35, principal — 46
- box 61
- bra 54
- breakable objects 285
- BRINKMAN, theorem of 314
  
- $C_n$  305
- $\tilde{C}_n$  416
- canonical congruences 245, — forms 33, 93 ff.
- CARTAN's symbolical method 172
- cartesian 42
- cell 65
- centre 2, 224
- centred  $E_n$  2
- characteristic equation 29
- characteristics 90
- CHRISTOFFEL symbol 132
- class, of field 70, of function 62, 66, of pseudo-quantity 11, 69, of vectorfield 89, of  $V_m$  268
- co-, contra-, covariant 6f., 10, 28, 34, 35
- CODAZZI equation 238, 242, 262f., 266f., 311, 410f.
- collecting indices 57
- complementary subspaces 287
- complete systems 81
- components 6, 11, 61, 68
- concircular 322, 330ff., 334
- concurrent 322
- concomitant 15, 164
- conformal connexion 315ff., — unitary connexion 418ff., — curvature tensor 306, 417, — parameters 346
- conformal transformations 134, 304, 312ff., 415ff.
- conformally euclidean 305ff., 416, — geodesic 309f.
- congruences in  $V_n$  244ff.

- connexion 125f., 132f., 139, 187f.  
 connecting objects 13, 68, 74f.  
 "conservation de courbure et de torsion" 173  
 contact of order  $u$  231  
 contain 4  
 contraction 13  
 contragradient 6f.  
 coordinate axes, planes,  $E_p$ 's 3, — systems 42, 54, 62, 64, —  $X_m$ 's 76, — transformations 47, 64  
**COPSON** and **RUSE**, equations of 385  
 covariant constant 125, 187, 322  
 covariant differentiation 122ff., 129f., 336  
 covariant stationary 125f.  
 curvature 228f., absolute, relative, enforced — 271, — figure 271, higher — 275ff., lines of — 237, mean — 250, principal — 250, — regions 271, 275, riemannian — 153, — tensor 139, conformal — tensor 289, projective — tensor 289, recurrent — tensor 359ff., — tensor of valence 3: 256, 276, — vector of curve 228f., — vector, absolute, relative, enforced 250  
 curves 227ff., 412ff.
- D<sub>n</sub>** 290  
 dead indices 7  
 decomposable spaces, objects 285, dito conformally 287  
 definite, indefinite 35, 53  
 deformation 335ff., — of subspaces 352ff.  
 density,  $\Delta$ -density 12  
 derived groups 224, — systems 81  
 differential concomitant 164, — invariant, absolute and relative 164  
 dimension of null manifold 73  
 direction 4  
 "direct" calculus 58  
 displacement, **CARTAN** — of points 127f., linear — 125, parallel — 122ff., — over  $\tau v^x$  104  
 distance 40  
 Div, divergence 84f., theorem of — 85  
 divisor of symmetric tensor 21, — of multivector 23  
 domain with respect to indices 20  
 dragging along of coordinate system 2, 102, — — of field 102, 335, — — over  $v^x dt$  103  
*D*-symbolism 254
- dummy indices 14  
**DUPIN**, indicatrix of 252, theorem of 250  
 duration 44
- E<sub>n</sub>** 1, centred — 2  
 $E_m$ -field 78,  $E_m$ -part 19  
 eigenvalue 29  
 eigenvector 29  
**EINSTEIN** space 148, conformal transformation of — 312ff.  
 elementary divisors 33  
 enveloping 79  
 "équations de structure" 173  
 equiform 11  
 equilibrium, field in 150  
 equipollences in group space 186, in auxiliary  $X_{2n}$  390  
 equivalent systems 73  
 essential parameters of transformation groups 201  
 euclidean  $R_n$  44  
**EULER**, theorem of 253  
 extension 160, flat — 422, null — 422
- FERMI** coordinates 166ff.  
 fixed indices 3  
 flat submanifold 3  
 free indices 14  
**FRENET** formulae for a curve in  $V_n$  229, for a  $V_m$  in  $V_n$  277  
 functional matrix 72  
 fundamental figure in  $R_n$  41, — pseudotensor 135, — tensor 41, 133ff., second — tensor 243, non symmetric — tensor 179ff., hermitian — tensor 54, 396ff., — theorems of transformation groups 203, 206ff.
- G<sub>a</sub>, G<sub>ho</sub>, G<sub>sa</sub>** 1, 2, 12  
**GAUSS**, equations of 237, 240, 242, 261, 266f., 278ff., 311, 410f.  
 general multiplication 13  
 generating of a group 104  
 geodesic 155, — coordinate system 156, — in group space 188, —  $L_n^m$  in  $L_n$  263, —  $V_{n-1}$  in  $V_n$  251, —  $\tilde{L}_m$  in  $\tilde{L}_n$  409, — variation 338  
 geometric manifold 64, — object 61f., — point 64  
 geometry of paths 303

- Grad, gradient 70, — equation 83,  
— product 84, — transformation 91f.  
group, affine 1, homogeneous linear — 2,  
— germ 186,  $r$ -parameter — 185, —  
space 186, 209ff.
- $h$**  100  
harmonic 381 ff., 386  
hermitian 52  
holonomy group 361 ff., 367ff.  
homogeneous equations 80, — linear  
group  $G_h$ , 2  
homologous elements and subgroups 198  
homothetic motions 346  
hybrid quantities 51, connexions for — —  
388ff.  
hyperplane 3
- ideal vectors 58  
identifications 27  
identities for the curvature quantities  
144 ff., 393 ff., 398 ff.  
imbedding 66, 74f., 279 ff., 407  
improper rotations 43, 50  
indices 3, 7, 14, 53, 57  
indicatrix 40  
indefinite 35, 53  
infinitesimal transformations 104 f., 203  
inner orientation 5, — problem 79  
intermediate components 80, 87, 153  
integrable connexion 139, 393, 403,  
— groups 225f.  
integral, — function 80  
invariant 5, 15, — of group 214 ff.,  
220 ff., — subgroup 198, 224  
invertible 53  
irreducible 21  
isomer 13  
isotropic  $E_p$  14, 42  
isotropy groups 373
- join 4
- $K_n$ ,  $K_n^*$**  421  
KÄHLER space 397  
kernel-index method 3  
ket 54  
kind of an object 6, 68  
**KRONECKER symbol** 3
- $L_n$**  125,  $\tilde{L}_n$  393,  $L_m^m$  257, 264  
LAGRANGE, derivative 112 ff., — equation 112  
LAMÉ equations 296  
length 40  
LICHNEROWICZ equation 384, — inequality 388  
LIE derivation 104, 106 ff., 152, 336  
LIE operator, extension of 372  
lie in 4  
linear element of  $U_n$  227, — submanifold 3  
living indices 7  
LIOUVILLE problem, generalization of 312  
LORENTZ transformation 43
- manner of transformation 6  
mapping of spaces 174 ff., 349 f.  
mean curvature 250, 272  
MEUSNIER, theorem of 250  
minimal regular 73, 75, — —  $V_m$  44  
minkowskian  $R_n$  44, — reflexions and  
rotations 43  
minor 1  
mixing 10, 14  
motions, affine 370, conformal 346, homo-  
thetic 346, projective 346, in  $V_n$  and  
 $L_n$  346 ff.  
multiplication, general 13, symmetric 21,  
alternating 23  
multivector 22f.  
multivector-tensor 34
- $n$ -vector** 23  
natural derivative 70, 83 f., — equations  
230, — field value 335, — parameter  
414, — variation 338  
neighbourhood 62, 66  
net 3, 76, 100  
non homogeneous equations 82  
norm 54  
normal coordinates 155 ff., 199, — system  
76, — tensors 159  
normalizing conditions 235, 237  
normals 228 f.  
nullcone 44, 55  
null  $E_p$  44, — manifold 73, — point 73, 80
- object field 70  
osculating  $p$ -direction 232  
opposite 5

- ordinary  $U_n, V_n$  133  
 orientation 5  
 outer orientation 5, — problem 79  
 osculating  $R$ 's 228f.  
 orthogonal coordinate transformation 47,  
   — group 47, 54, — net 246, — system  
   246
- parallel 4, 122ff., 126, 187, 332  
 parameters of displacement 122, pro-  
   jective — 301f, conformal — 346  
 parameter-groups 191ff.  
 parametric form 75  
 part 19, 77, symmetric and alternating —  
   14  
 $p$ -direction 4  
 perpendicular 42, 51  
 PFÄFF's problem 79, 89ff.  
 planar point 272  
 plane 3  
 point 1, 3, — transformation 2  
 pole 317  
 polycylinder 61  
 primitive groups 218  
 principal angles 48, — axes 46, 56, —  
   blades 46, — curvature 237, — direc-  
   tions 146, 153, — multidirections 46,  
   56, — tangents 237, —  $X_n$  in auxiliary  
    $X_{2n}$  390  
 product integral 340, — spaces 285ff.  
 projection 5, 18  
 projective connexion 300ff., — curva-  
   ture tensor 289, — parameters 301f.,  
   — transformations 287ff., 296, 415,  
   restricted — transformations 288  
 projective euclidean 288  
 proper rotations 50  
 pseudo-group 64, -normal vector 235,  
   -quantities 11, 69, -scalar 11, -tensor 11,  
   -vector 11
- quadrivector 23  
 quantities 6, 13, 54, 68  
 quasi-plane curves 298
- $\mathbf{R}_n$  41,  $\tilde{\mathbf{R}}_n$  54  
 raising of indices 44  
 rank 3, 72f.  
 reduction theorem 164
- reciprocal sets 9  
 rectilinear coordinates 1  
 recurrent curvature 359ff., 421ff., —  
   fields 126, 322  
 reduced system 82  
 reflexional orthogonal transformation 50  
 reflexotation 47  
 region 44, 62, 65, 71  
 representative indices 57  
 Ricci equations 261f., 266f., 278ff., 311,  
   410, 412, — space 148, — tensor 141  
 RIEMANN-CHRISTOFFEL tensor 139  
 riemannian curvature 153  
 rigging 17, 77, 79  
 Rot, rotation 83, theorem of — 85  
 rotations 47, proper and improper — 50  
 rotation class 89, coefficients of — 171,  
   minkowskian — 43  
 running indices 3
- $S_n$  148,  $\tilde{S}_n$  404  
 saturated indices 14  
 scalar 6  
 SCHLÄFLI, theorem of 268  
 SCHUR space 252, theorem of — 153  
 screwsense 5  
 section 4, 5, 16, 77f., 89  
 semi-analytic 389, — -definite 35, 53  
 sense 5, — of rotation 5  
 signature 34, 53  
 similarity class 89, — transformation 91f.  
 simple multivector 23  
 simple and semi-simple groups 227  
 skeleton 59, 255  
 space of constant curvature 148  
 spanning 4  
 spatial point 272  
 special affine group  $G_{sa}$  12  
 spur 29  
 stationary 228f.  
 STOKES, theorem of 97  
 strangling 10  
 subgroup 198, 223, invariant — 198, 224,  
   representation of — 221ff.  
 subgeodesic 327  
 submanifold, flat 3  
 subprojective connexion 317ff., — trans-  
   formation 327ff., —  $V_n$  328ff.  
 support of domain 20, — of rotation 90  
 summation convention 1  
 surface 74

|                                                                                                                                   |                                                                     |
|-----------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------|
| structural constants 196, 205, — equations of $L_n$ 173, — formulae of transformation groups after LIE and MAURER-CARTAN 196, 205 | $U_n$ 133, $\tilde{U}_n$ 396                                        |
| symbolical method of CARTAN 117ff.                                                                                                | umbilical point 251, 272, 310                                       |
| symmetric 14, — tensor 21, — multiplication 21, — part 14                                                                         | unified field theory 179ff.                                         |
| tangent $E_m$ 77, principal — 250, — space 69, — vector 235                                                                       | unitary connexion 395ff., — orthogonal group 54, — perpendicular 54 |
| teleparallelism 141                                                                                                               | unitvector 44, 54                                                   |
| tensor 10, 21f., — density 12, unity — 10, — of valence 2: 28ff.                                                                  | unity tensor 10                                                     |
| torse-forming field 322                                                                                                           |                                                                     |
| totally integrable 80                                                                                                             |                                                                     |
| trace 29                                                                                                                          |                                                                     |
| transformation groups 201ff.                                                                                                      |                                                                     |
| transitive group 216f.                                                                                                            |                                                                     |
| translation 4, 349                                                                                                                |                                                                     |
| transposition of variation problem 273, 341                                                                                       |                                                                     |
| transvection 8, 14                                                                                                                |                                                                     |
| trivectors 23, classification of — 35f.                                                                                           |                                                                     |
|                                                                                                                                   | $V_n$ 133, $\tilde{V}_n$ 397, $V_n^m$ 270                           |
|                                                                                                                                   | valence 10                                                          |
|                                                                                                                                   | variation 337f.                                                     |
|                                                                                                                                   | vector 6, — of projective transformation 287                        |
|                                                                                                                                   | volume preserving 144, 155, 288                                     |
|                                                                                                                                   |                                                                     |
|                                                                                                                                   | <b>WEYL</b> geometry 133                                            |
|                                                                                                                                   | $W$ -tensor 12                                                      |
|                                                                                                                                   |                                                                     |
|                                                                                                                                   | $X_n$ 64, $X_n^m$ 253                                               |

## Symbols.

|                 |                                 |                      |                                 |
|-----------------|---------------------------------|----------------------|---------------------------------|
| $\hat{\cdot}$ 3 | $[ ]$ 14                        | $\{ \}$ 340          | $\nabla$ 122, 139, 149, 174ff., |
| $\infty$ 25     | $\{ \}$ 67                      | $\{\{ \}\}$ 132      | 187f., 336, 419.                |
| ( ) 14          | $\{\dot{x}\}_{\mu \lambda}$ 132 | $\mathfrak{f}_v$ 336 |                                 |