Chapter 1

A Brief Introduction

Here we meet string theory for the first time. We see how it fits into the historical development of physics, and how it aims to provide a unified description of all fundamental interactions.

1.1 The road to unification

Over the course of time, the development of physics has been marked by unifications: events when different phenomena were recognized to be related and theories were adjusted to reflect such recognition. One of the most significant of these unifications occurred in the 19th century. After many discoveries in the areas of electricity and magnetism by Coulomb, Ampère, and other scientists, the experiments and studies of Michael Faraday showed that changing magnetic fields generated electric fields. These results led James Clerk Maxwell to a set of equations, now called "Maxwell's equations", that unify electricity and magnetism into a consistent whole. This elegant and aesthetically pleasing unification was not optional. Separate theories of electricity and magnetism would be inconsistent.

Another fundamental unification of two types of phenomena occurred in the 1960's. To appreciate the scope of this unification it is useful to recall what had been understood since the time of Maxwell.

Albert Einstein's Special Theory of Relativity had achieved a striking conceptual unification of the separate notions of space and time. Different from a unification of forces, the merging of space and time into a spacetime continuum represented a new recognition of the nature of the *arena* where physical phenomena take place. In addition, Quantum Theory, as developed by Erwin Schrödinger, Werner Heisenberg, Paul Dirac and others, had been dramatically verified to be the correct framework to describe microscopic phenomena. Finally, four fundamental forces had been recognized to exist in nature.

One of them was the force of gravity, known since antiquity, and first described accurately by Isaac Newton. Gravity underwent a profound reformulation in Albert Einstein's theory of General Relativity. In this theory, the spacetime arena of Special Relativity acquires a life of its own, and gravitational forces arise from the curvature of this dynamical spacetime. The second fundamental force was the electromagnetic force, well described by Maxwell's equations. The third fundamental force was the "weak force". This force is responsible for the process of nuclear beta decay, where a neutron decays into a proton, an electron, and an anti-neutrino. In general, it is responsible for processes involving neutrinos. While nuclear beta decay had been known since the end of the 19th century, the recognition that a new force was at play did not take hold until the middle of the 20th century. Enrico Fermi conjectured the existence of a particle carrying the weak force, and the strength of this force was measured by the Fermi constant. Finally, the fourth and last known force was the strong force, sometimes more poetically called the color force. This force is at play in holding together the constituents of the neutron, the proton, the pions, and many other particles. These constituents, called quarks, are held so tightly by the color force that they cannot be seen in isolation.

Returning to the subject of unification, in the late 1960's the Weinberg-Salam model of *electroweak* interactions put together electromagnetism and the weak force into a unified framework. Again, this unified model was neither dictated nor justified only by considerations of simplicity or elegance. It was necessary for a predictive and consistent theory of the weak interactions. The theory is formulated with four massless particles that carry the forces. A process of symmetry breaking results in three of these particles acquiring large masses. These are the W^+, the W^-, and the Z^0, and they are the carriers of the weak force. The particle that remains massless is the photon, the carrier of the electromagnetic force.

Maxwell's equations are the equations of classical electromagnetism. While classical electromagnetism is consistent with Einstein's theory of Special Relativity, it is not a quantum theory. Physicists have discovered methods, called quantization methods, to turn a classical theory into a quantum theory – a theory that can be calculated using the principles of quantum mechanics. While classical electromagnetism can be used confidently to calculate the transmission of energy in power lines and the radiation patterns of radio antennas, it is neither an accurate nor a correct theory for microscopic phenomena. Quantum electrodynamics (QED), the quantum version of classical electrodynamics, is required for correct computations in this arena. The photon is the quantum of the electromagnetic field. After the recognition of the need for unification with the weak interactions, the correct theory is a quantum theory of the electroweak interactions.

The quantization procedure is also successful in the case of the color force, and the resulting theory has been called quantum chromodynamics (QCD). The carriers of the color force are eight massless particles called gluons, and just as the quarks, they cannot be observed in isolation. While there is just one kind of electric charge, there are three basic types of color charge, or three types of colors.

The electroweak theory, together with QCD, form the Standard Model of particle physics. In the Standard Model there is some interplay between the electroweak sector and the QCD sector arising because some particles feel both types of forces. But there is no real and deep unification of the weak force and the color force.

In the Standard Model there are twelve force carriers: the eight gluons, the W^+, W^-, the Z^0, and the photon. All of them are bosons. There are also many matter particles, all of which are fermions. We have the leptons, which include the electron e^-, the muon μ^-, the tau τ^-, and the associated neutrinos ν_e, ν_μ, ν_τ. If we include their antiparticles, this adds up to twelve leptons. There are also the quarks, which come in three colors and carry electric charge, as well. In addition, quarks also feel the weak interactions, and come in six different types. Poetically called flavors, these types are: up, down, strange, charm, top, and bottom. With three colors and six flavors, and with inclusion of the antiparticles, we get a total of 36 quarks. Adding leptons and quarks together we have 48 matter particles. Despite the large number of particles it describes, the Standard Model is reasonably elegant and very powerful. As a complete theory of physics, however, it has two significant shortcomings. The first one is that it does not include gravity. The second one is that it has nearly twenty parameters that cannot be calculated within its framework. Perhaps the simplest example of such a parameter is the ratio of the mass of the muon to the mass of the electron. The value of this ratio is about 207, and it must be put into the model by hand.

Most physicists believe that the Standard Model is only a step towards the formulation of a complete theory of physics. It is also widely felt that some unification of the weak and strong forces into a Grand Unified Theory (GUT) will prove to be correct. While unification of these two forces may be optional, the inclusion of gravity (with or without unification), is really necessary if one is to have a complete theory. The effects of the gravitational force are presently quite negligible at the microscopic level, but they are crucial in studies of cosmology of the early universe.

There is, however, a major problem in attempting to incorporate gravitational physics into the Standard Model. The Standard Model is a quantum theory while Einstein's General Relativity is a classical theory. It seems very difficult, if not altogether impossible, to have a consistent theory that is partly quantum and partly classical. Given the successes of quantum theory is it widely believed that gravity must be turned into a quantum theory. The procedures of quantization, however, encounter profound difficulties in the case of gravity. The resulting theory of quantum gravity appears to be either incalculable or totally unpredictive, and both options are unacceptable. As a practical matter, in many circumstances one can work confidently with classical gravity coupled to the Standard Model. For example, this is done routinely in present-day descriptions of the universe. A theory of quantum gravity is necessary, however, to study physics at times very near to the Big-Bang, and to study certain properties of black holes. Formulating a quantum theory including both gravity and the other forces seems unavoidable at the foundational level. A *unification* of gravity with the other forces might be required to construct this complete theory.

1.2 String theory as a unified theory of physics

String theory is an excellent candidate for a unified theory of all forces in nature. It is also a rather impressive prototype of a complete theory of physics. In string theory all forces are truly unified in a deep and significant way. In fact, matter is also unified with forces, and they together form a theory that cannot be tinkered with. String theory is a quantum theory, and, because it includes gravitation, it is a quantum theory of gravity. Viewed from this perspective, and recalling the failure of Einstein's gravity to yield a quantum theory, one concludes that all other interactions are necessary for the consistency of the quantum gravitational sector! While it may be difficult to directly measure the effects of quantum gravity, a theory of quantum gravity such as string theory may have testable predictions concerning the other interactions.

Why is string theory truly a unified theory? The reason is simple and goes to the heart of the theory. In string theory, each particle is identified as a particular vibrational mode of an elementary microscopic string. A musical analogy is very apt. Just as a violin string can vibrate in different modes and each mode corresponds to a different sound, the modes of vibration of a fundamental string can be recognized as the different particles we know. One of the vibrational states of strings is the graviton, the quantum of the gravitational field. Since there is just one type of string, and all particles arise from its vibrations, all particles are naturally incorporated together into a single theory. When we think in string theory of a decay process $A \to B + C$, where an elementary particle A decays into particles B and C, we imagine a single string vibrating in such a way that it is identified as particle A that breaks into two strings vibrating in ways that identify them as particles B and C. Since strings are expected to be extremely tiny, perhaps of sizes of about 10^{-33}cm, it will be difficult to observe directly the string-like nature of particles.

Are we sure string theory is a good quantum theory of gravity? There is no complete certainty yet, but the evidence is very good. Indeed, the problems of incalculability or lack of predictability that occur when one tries to quantize Einstein's theory do not seem to appear in string theory.

For a theory as ambitious as string theory, a certain degree of uniqueness is clearly desirable. Having several consistent candidates for a theory of all interactions would be disappointing. The first sign that string theory is rather unique is that it does not have adjustable dimensionless parameters. As we mentioned before, the Standard Model of particle physics has about twenty parameters that must be adjusted to some precise values. A theory with adjustable parameters is not really unique; setting the parameters to different values gives different theories with potentially different predictions.

Another intriguing sign of uniqueness is that the dimension of spacetime is fixed in string theory. Our physical spacetime is four-dimensional, with one time dimension and three space dimensions. In the Standard Model this information has to be built in. In string theory, on the other hand, the number of spacetime dimensions emerges from a calculation. The answer is not four, but rather ten. Some of these dimensions may hide from plain view if they curl up into a space that escapes detection in experiments done with low energies. If string theory is correct, such a mechanism would have to take place to reduce the observable dimension of space-time to the value of four.

The lack of adjustable parameters is a sign of uniqueness: it means a theory cannot be deformed or changed continuously by changing these parameters. But there could be other theories that cannot be reached by continuous deformations. So how many string theories are there?

Let us begin by noting two broad subdivisions. There are open strings, and there are closed strings. Open strings have two endpoints, while closed strings have no endpoints. One can consider theories with only closed strings, and theories with both open and closed strings. There are no theories with only open strings – open strings can always close to form closed strings. The second subdivision is between bosonic string theories, and fermionic string theories. Bosonic strings live in 26 dimensions, and all of their vibrations represent bosons. Lacking fermions, bosonic string theories are not realistic. They are, however, much simpler than the fermionic theories, and most of the important concepts in string theory can be explained in the context of bosonic strings. The fermionic strings, living in ten dimensions, include bosons *and* fermions. In fact, these two kinds of particles are related by a symmetry called supersymmetry.

While in the mid-1980's five supersymmetric ten-dimensional string theories were known to exist, it is now clear that they are only facets of a single theory! This theory has another facet in which it appears to be eleven-dimensional. Dubbed M-theory, for lack of a better name, it seems to encompass all fermionic string theories previously thought to be different. Thus string theory appears to be pretty unique. At present, the less promising bosonic string theories appear to be unrelated to the fermionic theories but this understanding could change in the future.

All in all, we see that string theory is a truly unified and possibly unique theory. It is a candidate for a unified theory of physics, a theory Albert Einstein tried to find ever since his completion of General Relativity. Einstein would have been surprised, or perhaps disturbed, by the prominent role that quantum mechanics plays in string theory. But string theory gives us a glimpse of a worthy successor for General Relativity, and he might have found this pleasing. Paul Dirac's writings on quantization suggest that he felt that deep quantum theories arise from the quantization of classical physics. This is precisely what happens in string theory. This book will explain in detail how string theory, at least in its simplest form, is nothing but the quantum mechanics of classical relativistic strings.

1.3 String theory and its verification

It should be said at the outset that, as of yet, there has been no experimental verification of string theory. Also, no sharp prediction has been derived from string theory that could help us decide if string theory is correct. This situation arises because of two reasons. One is the probable small scale of strings – being many orders of magnitude smaller than the distances that present-day accelerators can probe, most tests of string theory have to be very indirect. The second reason is that string theory is still at an early stage of development. It is not so easy to make predictions with a theory we do not understand very well.

In the absence of sharp predictions of new phenomena, we can ask if string theory does at least reproduce the Standard Model. While string theory certainly has room to include all known particles and interactions, and this is very good news indeed, no one has yet been able to show that they actually emerge in fine detail. Models have been built that resemble the world as we know it, but our inability to do certain calculations in string theory has been an obstacle to a complete and detailed analysis.

In these models, new interactions and particles are typically predicted. Although the predictions are rather model-dependent, if verified, they could provide striking tests of the theory. Most of the models are also supersymmetric. An experimental discovery of supersymmetry in future accelerators would suggest very strongly that we are on the right track.

Despite the still-tenuous connection to experiment, string theory has been a very stimulating and active area of research ever since the mid-1980's when Michael Green and John Schwarz showed that fermionic string theories are not afflicted with fatal inconsistencies that threaten similar particle theories in ten dimensions. Much progress has been made since then.

String theory has turned into a very powerful tool to understand phenomena that conventionally belong to the realm of particle physics. These phenomena concern the behavior of gauge theories, the kind of theories that are used in the Standard Model. Close cousins of these gauge theories arise on string theory D-branes. These D-branes are extended objects, or hyperplanes that can exist in string theory.

String theory has also made good strides towards a statistical mechanics interpretation of black hole entropy. We know from the pioneering work of Jacob Bekenstein and Stephen Hawking that black holes have both entropy and temperature. In statistical mechanics these properties arise if a system can be constructed in many degenerate ways using its basic constituents. Such an interpretation is not available in Einstein's gravitation, where black holes seem to have few, if any constituents. In string theory, however, certain black holes can be built by assembling together various types of D-branes in a controlled manner. For such black holes, the predicted Bekenstein entropy is obtained by counting the ways in which they can be built with their constituent D-branes.

String theory has stimulated a host of ideas that play an important role in attempts to explore physics beyond the Standard Model. String-inspired phenomenology makes use of extra dimensions, sometimes conjectured to be surprisingly large, and of D-branes, sometimes postulating that our four-dimensional universe is a D-brane inside a ten dimensional spacetime. Some of these ideas are testable, and could turn out to be right.

As a quantum gravity theory, string theory should be needed to study cosmology of the Very Early Universe. The deepest mysteries of the universe seem to lie hidden in a regime where classical General Relativity breaks down. String theory should allow us to peer into this unknown realm. Some day, we may be able to understand the nature of the Big-Bang, and know if there is a pre-Big-Bang cosmology.

String theorists sometimes say that string theory *predicts* gravity[1]. There is a bit of jest in saying so – after all, gravity is the oldest-known force in nature. I believe, however, that there is a very substantial point to be made here. String theory is the quantum mechanics of a relativistic string. In no sense whatsoever is gravity put into string theory by hand. It is a complete surprise that gravity emerges in string theory. Indeed, none of the vibrations of the *classical* relativistic string correspond to the particle of gravity. It is a truly remarkable fact that we find the particle of gravity among the *quantum* vibrations of the relativistic string. You will see in detail how this happens as you progress through this book. The striking quantum emergence of gravitation in string theory has the full flavor of a prediction.

[1]This has been emphasized to me by John Schwarz.

Chapter 2

Special Relativity and Extra Dimensions

The word relativistic, as used in relativistic strings, indicates consistency with Einstein's theory of Special Relativity. We review Special Relativity and introduce the light-cone frame, light-cone coordinates, and light-cone energy. We then turn to the idea of additional, compact space-dimensions and show with a quantum mechanical example that, if small, they have little effect at low energies.

2.1 Units and parameters

Units are nothing other than fixed quantities that we use for purposes of reference. A measurement involves finding the unit-free ratio of the observable quantity divided by the appropriate unit. Consider, for example, the definition of a second in the international system of units (SI system). The SI second (sec) is defined to be the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the cesium-133 atom. When we measure the time elapsed between two events, we are really counting a unit-free, or dimensionless number: the number that tells us how many seconds fit between the two events, or alternatively, the number of cesium periods that fit between the two events. The same goes for length. The unit, a meter (m), is nowadays defined as the distance traveled by light in a certain fraction of a second: 1/299792458 of a second, to be precise. Mass introduces a third unit, the prototype kilogram (kg) kept safely in Sèvres, France.

When doing dimensional analysis, we denote the units of length, time and mass by L, T and M respectively. These are called the three basic units. A force, for example has units

$$[F] = M\,L\,T^{-2}\,, \tag{2.1.1}$$

where $[\cdots]$ denotes the units of the quantity under brackets. Equation (2.1.1) follows from Newton's law equating force with mass times acceleration. The Newton (N) is the unit of force, and it equals kg $\cdot$ m /sec^2.

It is interesting that no additional basic units are needed to describe other quantities. Consider, for example, electric charge. Don't we need a new unit to describe charge? Not really. This is simplest to see in gaussian units, which are very convenient in relativity. In these units, Coulomb's law for the force between two charges q_1 and q_2 separated a distance r reads

$$|\vec{F}| = \frac{|q_1 q_2|}{r^2}\,. \tag{2.1.2}$$

The units of charge are fixed in terms of other units because we have a force law where charges appear. The "esu" is the gaussian unit of charge, and it is defined by stating that two charges of one esu each, placed at a distance of one centimeter, repel each other with a force of one dyne. Thus

$$\text{esu}^2 = \text{dyne} \cdot \text{cm}^2 = 10^{-5}\text{N} \cdot (10^{-2}\text{m})^2 = 10^{-9}\,\text{N} \cdot \text{m}^2. \tag{2.1.3}$$

It follows from this equation that

$$[\text{esu}^2] = [\ \text{N} \cdot \text{m}^2]\,, \tag{2.1.4}$$

and using (2.1.1) we finally get

$$[\text{esu}] = M^{1/2} L^{3/2}\, T^{-1}\,. \tag{2.1.5}$$

This expresses the units of the esu in terms of the three basic units.

In SI units charge is measured in coulombs (C). The situation is a little more intricate, but the main point is the same. A coulomb is defined in SI units as the amount of charge carried by a current of one ampere (A) in one second. The ampere itself is defined as the amount of current that produces, when carried by two wires separated a distance of one meter, a force of 2×10^{-7} N/m. The coulomb, as opposed to the esu, is not expressed in terms of meters, kilograms, and seconds. Coulomb's law in SI units is

$$|\vec{F}| = \frac{1}{4\pi\epsilon_0}\frac{|q_1 q_2|}{r^2}\,, \quad \text{with} \quad \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \,\frac{\mathrm{N}\cdot\mathrm{m}^2}{\mathrm{C}^2}\,. \tag{2.1.6}$$

Note the presence of C^{-2} in the definition of the constant prefactor. Since each charge carries one factor of C, the factors of C cancel in the calculation of the force. Two charges of one coulomb each, placed one meter apart, will experience a force of 8.99×10^9 N. This fact allows you to deduce (Problem 2.1) how many esu's there are in a coulomb. Even though we do not write coulombs in terms of other units, this is just a matter of convenience. Coulombs and esu's are related, and esu's are written in terms of the three basic units.

When we speak of parameters in a theory it is convenient to distinguish between dimensionful parameters and dimensionless parameters. Consider, for example, a theory in which there are three types of particles whose masses m_1, m_2, and m_3, take different values. We can think of the theory as having one dimensionful parameter, say the mass m_1 of the first particle, and two dimensionless parameters: the mass ratios m_2/m_1 and m_3/m_1.

String theory is said to have no adjustable parameters. By this it is meant that no dimensionless parameter must be chosen to formulate string theory. String theory, however, has one dimensionful parameter. This is the string length ℓ_s. Adjusting this length sets the scale in which the theory operates. In the early 1970's, when string theory was first formulated, the string length was taken to be comparable to the nuclear scale. String theory was being examined as a theory of hadrons. Nowadays, we think that string theory is a theory of fundamental forces and interactions; accordingly, we set the string length to be much smaller than the nuclear scale.

2.2 Intervals and Lorentz transformations

Special relativity is based on the experimental fact that the speed of light ($c \simeq 3\times 10^8$ m/s) is the same for all inertial observers. This fact leads to some rather surprising conclusions. Newtonian intuition about the absolute nature of time, the concept of simultaneity, and other familiar ideas must be revised.

In comparing the coordinates of events, two inertial observers, henceforth called Lorentz observers, find that the needed coordinate transformations mix space and time.

In special relativity, events are characterized by the values of four coordinates: a time coordinate t, and three spatial coordinates x, y, and z. It is convenient to collect these four numbers in the form (ct, x, y, z), where the time coordinate is scaled by the speed of light so that all coordinates have units of length. To make the notation more uniform, we use indices to relabel the space and time coordinates as follows:

$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z)\,. \tag{2.2.1}$$

Here the superscript μ takes the four values $0, 1, 2$ and 3.

Consider a Lorentz frame S in which two events are represented by coordinates x^μ and $x^\mu + \Delta x^\mu$, respectively. Consider now another Lorentz frame S', where the same events are described by coordinates x'^μ and $x'^\mu + \Delta x'^\mu$, respectively. In general, not only are the coordinates x^μ and x'^μ different, so too are the coordinate differences Δx^μ and $\Delta x'^\mu$. On the other hand, both observers will agree on the value of the invariant interval Δs^2 . This interval is defined as

$$-\,\Delta s^2 \;\;\equiv\;\; -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2\,. \tag{2.2.2}$$

Note the minus sign in front of $(\Delta x^0)^2$, as opposed to the plus sign appearing before spacelike differences $(\Delta x^i)^2$ $(i = 1, 2, 3.)$. This reflects the basic difference between time and space coordinates. The agreement on the value of the intervals is expressed as

$$\begin{aligned}&-(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2\\ &\quad = -(\Delta x'^0)^2 + (\Delta x'^1)^2 + (\Delta x'^2)^2 + (\Delta x'^3)^2\,,\end{aligned} \tag{2.2.3}$$

or, in brief

$$\Delta s^2 = \Delta s'^2\,. \tag{2.2.4}$$

The minus sign in the left hand side of (2.2.2) implies that $\Delta s^2 > 0$ for events that are *timelike separated*. These are events for which

$$(\Delta x^0)^2 > (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2\,. \tag{2.2.5}$$

Any two events on the world-line of a particle are timelike separated because no particle can move faster than light, and therefore the distance light would have traveled in the time that separates the events must be larger than the space separation between the events. This is the content of equation (2.2.5). Events connected by light are said to be *light-like separated*, and we have $\Delta s^2 = 0$. This happens because in this case the two sides of equation (2.2.5) are identical: the spatial separation between the events coincides with the distance that light would have traveled in the time separating the events. A pair of events for which $\Delta s^2 < 0$ are said to be *spacelike separated*. Events simultaneous in a Lorentz frame but occurring at different positions are spacelike separated. It is because of this possibility that Δs^2 is not written generically as $(\Delta s)^2$. On the other hand, for timelike separated events we define

$$\Delta s \equiv \sqrt{\Delta s^2} \quad \text{if} \quad \Delta s^2 > 0 \quad (\text{timelike interval})\,. \tag{2.2.6}$$

Many times it is useful to consider events that are infinitesimally close to each other. Such small coordinate differences are necessary to define velocities, and are also useful in general relativity. Small coordinate differences are written as infinitesimals dx^μ, and the associated invariant interval is written as ds^2. Following (2.2.2) we have

$$-\,ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2\,. \tag{2.2.7}$$

The equality of intervals is the statement

$$ds^2 = ds'^2\,. \tag{2.2.8}$$

A very useful notation can be obtained by trying to write nicely the invariant ds^2. For this we introduce symbols with subscripts instead of superscripts. Let us define

$$dx_0 \equiv -dx^0, \quad dx_1 \equiv dx^1, \quad dx_2 \equiv dx^2, \quad dx_3 \equiv dx^3\,. \tag{2.2.9}$$

The only important change is the inclusion of a minus sign for the zeroth component. All together we write

$$dx_\mu = (dx_0, dx_1, dx_2, dx_3) \equiv (-dx^0, dx^1, dx^2, dx^3)\,. \tag{2.2.10}$$

Now we can rewrite ds^2 in terms of dx^μ and dx_μ:

$$\begin{aligned} -\,ds^2 &= -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2\,, \\ &= dx_0 dx^0 + dx_1 dx^1 + dx_2 dx^2 + dx_3 dx^3\,, \end{aligned} \tag{2.2.11}$$

and we see that the minus sign in (2.2.7) is gone. We can therefore write

$$- ds^2 = \sum_{\mu=0}^{3} dx_\mu dx^\mu \,. \tag{2.2.12}$$

We will use Einstein's summation convention. In this convention, indices that are repeated must be summed over the relevant set of values. We do not consider indices to be repeated when they appear on different terms. For example, there are no sums implied by $a^\mu + b^\mu$, nor by $a^\mu = b^\mu$, but there is a sum in $a^\mu b_\mu$. Repeated indices must appear once as a subscript and once as a superscript, and should not appear more than two times in any term. The letter chosen for the repeated index is not important, thus $a^\mu b_\mu$ is the same as $a^\nu b_\nu$. Because of this, repeated indices are sometimes called dummy indices! Using the summation convention we can write (2.2.12) as

$$-ds^2 = dx_\mu dx^\mu. \tag{2.2.13}$$

Just as we did for arbitrary coordinate differences in (2.2.6), for infinitesimal timelike intervals we define

$$ds \equiv \sqrt{ds^2} \quad \text{if} \quad ds^2 > 0 \quad (\text{timelike interval})\,. \tag{2.2.14}$$

We can also express the interval ds^2 using the Minkowski metric $\eta_{\mu\nu}$. This is done by writing

$$-ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \,. \tag{2.2.15}$$

By definition, the Minkowski metric is symmetric under an exchange in the order of its indices:

$$\eta_{\mu\nu} = \eta_{\nu\mu} \,. \tag{2.2.16}$$

This is reasonable. Any two-index object $M_{\mu\nu}$ can be decomposed into a symmetric part and an antisymmetric part:

$$M_{\mu\nu} = \frac{1}{2}(M_{\mu\nu} + M_{\nu\mu}) + \frac{1}{2}(M_{\mu\nu} - M_{\nu\mu})\,. \tag{2.2.17}$$

The first term in the right hand side, the symmetric part of M, is invariant under the exchange of the indices μ and ν. The second term in the right hand side, the antisymmetric part of M, changes sign under the exchange of the indices μ and ν. If $\eta_{\mu\nu}$ had an antisymmetric part $\xi_{\mu\nu}$ $(= -\xi_{\nu\mu})$, its contribution would drop out of the right hand side in (2.2.15). We can see this as follows:

$$\xi_{\mu\nu}dx^\mu dx^\nu = (-\xi_{\nu\mu})dx^\mu dx^\nu = -\xi_{\mu\nu}\, dx^\nu dx^\mu = -\xi_{\mu\nu}\, dx^\mu dx^\nu\,. \qquad (2.2.18)$$

In the first step we used the antisymmetry of $\xi_{\mu\nu}$. In the second step we relabeled the dummy indices: the μ's were changed into ν's and vice versa. In the third step, we switched the order of the dx^μ and dx^ν factors. The result is that $\xi_{\mu\nu}dx^\mu dx^\nu$ is identical to minus itself, and must therefore vanish.

Since repeated indices are summed over, equation (2.2.15) means

$$-\,ds^2 = \eta_{00}dx^0dx^0 + \eta_{01}dx^0dx^1 + \eta_{10}dx^1dx^0 + \eta_{11}dx^1dx^1 + \cdots\,. \qquad (2.2.19)$$

Comparing with (2.2.11) we see that $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = \eta_{33} = 1$, and all other components vanish. We collect these values in matrix form:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (2.2.20)$$

In this equation, following a common identification of two-index objects with matrices, we think of μ, the first index in η, as the row index, and ν, the second index in η, as the column index. The Minkowski metric can be used to "lower indices". Indeed, equation (2.2.9) can be rewritten as

$$dx_\mu = \eta_{\mu\nu}\, dx^\nu\,. \qquad (2.2.21)$$

If we are handed a set of quantities b^μ, we always define

$$b_\mu \equiv \eta_{\mu\nu} b^\nu\,. \qquad (2.2.22)$$

Given objects a^μ and b^μ, the relativistic *scalar product* $a\cdot b$ is defined as

$$\boxed{a\cdot b \equiv a^\mu b_\mu = \eta_{\mu\nu}\, a^\mu b^\nu = -a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3\,.} \qquad (2.2.23)$$

Applied to (2.2.13), we have $-ds^2 = dx\cdot dx$. Note also that $a^\mu b_\mu = a_\mu b^\mu$.

It is convenient to introduce the matrix inverse for $\eta_{\mu\nu}$. Written conventionally as $\eta^{\mu\nu}$ it is given as

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (2.2.24)$$

You can see by inspection that this matrix is indeed the inverse of the matrix in (2.2.20). When thinking of $\eta^{\mu\nu}$ as a matrix, as before, the first index is a row index and the second index is a column index. In index notation the inverse property is written as

$$\eta^{\nu\rho}\,\eta_{\rho\mu} = \delta^{\nu}_{\mu}\,, \tag{2.2.25}$$

where the Kronecker delta δ^{ν}_{μ} is defined as

$$\delta^{\nu}_{\mu} = \begin{cases} 1\,, & \text{if } \mu = \nu, \\ 0\,, & \text{if } \mu \neq \nu. \end{cases} \tag{2.2.26}$$

Note that the repeated index ρ in (2.2.25) produces the desired matrix multiplication. The Kronecker delta can be thought as the index representation of the identity matrix. The metric with upper indices can be used to "raise indices". Using (2.2.22) and (2.2.25)

$$\eta^{\rho\mu}\,b_{\mu} = \eta^{\rho\mu}\,(\eta_{\mu\nu}\,b^{\nu}) = (\eta^{\rho\mu}\,\eta_{\mu\nu})\,b^{\nu} = \delta^{\rho}_{\nu}\,b^{\nu} = b^{\rho}\,. \tag{2.2.27}$$

The lower μ index of b_{μ} was raised by $\eta^{\rho\mu}$ to become an upper ρ index in the resulting b^{ρ}. The last step in the above calculation needs a little explanation: $\delta^{\rho}_{\nu}\,b^{\nu} = b^{\rho}$ because the sum over ν vanishes unless $\nu = \rho$, in which case the Kronecker delta (2.2.26) gives a unit factor.

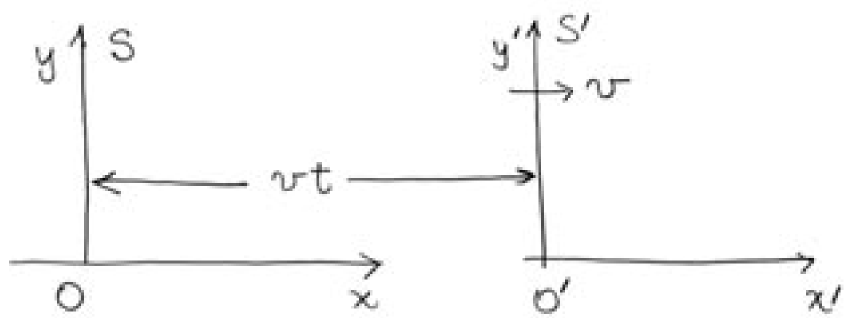

Figure 2.1: Two Lorentz frames connected by a boost. S' is boosted along the $+x$ direction of S with boost parameter β.

Lorentz transformations are the relations between coordinates in two different inertial frames. Consider a frame S and another frame S', which is moving along the x direction of the S frame with a velocity v, as shown in Figure 2.1. Assume the coordinate axes for both systems are parallel, and that their origins coincide at a common time $t = t' = 0$. We say that S' is boosted along the x direction with a boost parameter $\beta \equiv \frac{v}{c}$. The Lorentz transformations in this case read:

$$\begin{aligned} x' &= \gamma(x - \beta ct)\,, \\ ct' &= \gamma(ct - \beta x)\,, \\ y' &= y\,, \\ z' &= z\,, \end{aligned} \tag{2.2.28}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,. \tag{2.2.29}$$

Using indices, and changing the order of the first two equations

$$\begin{aligned} x'^0 &= \gamma(\ \ x^0 \ \ - \ \ \beta x^1)\,, \\ x'^1 &= \gamma(-\beta x^0 + \ \ x^1)\,, \\ x'^2 &= x^2\,, \\ x'^3 &= x^3\,. \end{aligned} \tag{2.2.30}$$

In the above transformations the coordinates x^2 and x^3 remain unchanged. These are the coordinates orthogonal to the boost. The inverse Lorentz transformations give the values of the x coordinates in terms of the x' coordinates. They are readily found by solving for the x's in the above equations. The result is the same set of transformations with x's and x''s exchanged, and with β replaced by $(-\beta)$.

The coordinates in the above equations satisfy the relation

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2\,, \tag{2.2.31}$$

as you can show by direct computation. This is just the statement of invariance of the interval Δs^2 between two events: the first event, represented by $(0,0,0,0)$ in both S and S', and the second event, represented by coordinates x^μ in S and x'^μ in S'. By definition, *Lorentz transformations are linear invertible transformations of coordinates respecting the equality (2.2.31).*

In general, we write a Lorentz transformation as the linear relations

$$x'^{\mu} = L^{\mu}{}_{\nu}\, x^{\nu} \,, \tag{2.2.32}$$

where the entries $L^{\mu}{}_{\nu}$ are constants defining the linear transformations. For the boost in (2.2.30) we have

$$[L] = L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mu \text{ rows}\,, \quad \nu \text{ columns}\,. \tag{2.2.33}$$

In defining the matrix L as $[L] = L^{\mu}{}_{\nu}$, we are following the convention that the first index is a row index and the second index is a column index. This is why the lower index in $L^{\mu}{}_{\nu}$ is written to the right of the upper index.

The coefficients $L^{\mu}{}_{\nu}$ are constrained by equation (2.2.31). In index notation, this equation requires

$$\eta_{\alpha\beta}\, x^{\alpha} x^{\beta} = \eta_{\mu\nu}\, x'^{\mu} x'^{\nu} \,. \tag{2.2.34}$$

Using (2.2.32) twice in the above right hand side,

$$\eta_{\alpha\beta}\, x^{\alpha} x^{\beta} = \eta_{\mu\nu}\, (L^{\mu}{}_{\alpha}\, x^{\alpha})(L^{\nu}{}_{\beta}\, x^{\beta}) = \eta_{\mu\nu}\, L^{\mu}{}_{\alpha}\, L^{\nu}{}_{\beta}\, x^{\alpha}\, x^{\beta} \,. \tag{2.2.35}$$

Since this equation must hold for all values of of the coordinates x, we conclude that L must satisfy the condition

$$\eta_{\mu\nu}\, L^{\mu}{}_{\alpha}\, L^{\nu}{}_{\beta} = \eta_{\alpha\beta} \,. \tag{2.2.36}$$

Rewriting (2.2.36) to make it look more like matrix multiplication, we have

$$L^{\mu}{}_{\alpha}\; \eta_{\mu\nu}\; L^{\nu}{}_{\beta} = \eta_{\alpha\beta} \,. \tag{2.2.37}$$

The sum over the ν index works well: it is a column index in $\eta_{\mu\nu}$, and a row index in $L^{\nu}{}_{\beta}$. The μ index, however, is a row index in $L^{\mu}{}_{\alpha}$, while it should be a column index to match the row index in $\eta_{\mu\nu}$. Moreover, the α index in $L^{\mu}{}_{\alpha}$ is a column index, while it is a row index in $\eta_{\alpha\beta}$. This means that we should exchange the columns and rows of $L^{\mu}{}_{\alpha}$. This is the matrix operation of transposition, and therefore equation (2.2.37) can be rewritten as the matrix equation

$$L^{T}\, \eta\, L = \eta \,. \tag{2.2.38}$$

An important property of Lorentz transformations can be deduced by taking the determinant of each side of this equation. Since the determinant of a product is the product of determinants, we get

$$(\det L^T)(\det \eta)(\det L) = \det \eta \,. \tag{2.2.39}$$

Cancelling the common factor of $\det \eta$ and recalling that the operation of transposition does not change a determinant, we find

$$(\det L)^2 = 1 \quad \rightarrow \quad \det L = \pm 1 \,. \tag{2.2.40}$$

You can check that for the boost in (2.2.33) $\det L = 1$.

Lorentz transformations include boosts along any of the spatial coordinates. They also include rotations of the spatial coordinates. Under a spatial rotation the coordinates (x^0, x^1, x^2, x^3) of a point transform into coordinates (x'^0, x'^1, x'^2, x'^3), where $x^0 = x'^0$, because time is unaffected. Since the spatial distance to the origin is preserved under rotation, we have

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = (x'^1)^2 + (x'^2)^2 + (x'^3)^2 \,. \tag{2.2.41}$$

This together with $x^0 = x'^0$ implies that (2.2.31) holds. Therefore spatial rotations are Lorentz transformations.

Any set of four quantities which transform under Lorentz transformations as the x^μ do is said to be a four-vector. When we use index notation and write b^μ, we mean that b^μ is a four-vector. Taking differentials of the linear equations (2.2.30), we see that the same linear transformations that relate x' to x relate dx' and dx. Therefore the differentials dx^μ define a Lorentz vector. In the spirit of index notation, a quantity with no free indices must be invariant under Lorentz transformations. A quantity h without an index has no free indices, and so does an object having only repeated indices, such as $a^\mu b_\mu$.

A four-vector a^μ is said to be timelike if $a^2 = a \cdot a < 0$. The vector is said to be spacelike if $a^2 > 0$. If $a^2 = 0$, the vector is said to be null. Recalling our discussion below equation (2.2.5), we see that the coordinate differences between timelike-separated events define a timelike vector. Similarly, the coordinate differences between spacelike-separated events define a spacelike vector. Finally, the differences between light-like separated events define a null vector.

Quick Calculation 2.1. Verify that the invariant ds^2 is indeed preserved under the Lorentz transformations (2.2.30).

Quick Calculation 2.2. Consider two Lorentz vectors a^μ and b^μ. Write the Lorentz transformations analogous to (2.2.30). Verify that $a^\mu b_\mu$ is invariant under these transformations.

2.3 Light-cone coordinates

We now discuss a coordinate system that will be extremely useful in our study of string theory. This is the light-cone coordinate system. The quantization of the relativistic string can be worked out most directly using light-cone coordinates. This is the way we will quantize strings in this book, so it is now a good time to introduce some of the features of light-cone coordinates. There is another approach to the quantization of the relativistic string where no special coordinates are used. This approach, called Lorentz covariant quantization, is very elegant, but a proper discussion would require developing a great deal of background material.

We define two light-cone coordinates x^+ and x^- as two independent linear combinations of the time coordinate and one chosen spatial coordinate, conventionally taken to be x^1. This is done by writing:

$$\boxed{\begin{aligned} x^+ &\equiv \frac{1}{\sqrt{2}}\,(x^0 + x^1)\,, \\ x^- &\equiv \frac{1}{\sqrt{2}}\,(x^0 - x^1)\,. \end{aligned}} \tag{2.3.1}$$

The coordinates x^2 and x^3 play no role in this definition. In the light-cone coordinate system, (x^0, x^1) are traded for (x^+, x^-), but we keep the other two coordinates (x^2, x^3). Thus, the complete set of light-cone coordinates is (x^+, x^-, x^2, x^3).

The new coordinates x^+ and x^- are called light-cone coordinates because the associated coordinate axes are the world-lines for beams of light emitted from the origin along the x^1 axis. For a beam of light going in the positive x^1 direction, we have $x^1 = ct = x^0$, and thus $x^- = 0$. The line $x^- = 0$ is, by definition, the x^+ axis (Fig. 2.2). For a beam of light going in the negative x_1 direction, we have $x^1 = -ct = -x^0$, and thus $x^+ = 0$. This corresponds to the x^- axis. The $x^\pm$ axes are lines at 45° with respect to the x^0, x^1 axes.

Can we think of x^+, or perhaps x^-, as a new time coordinate? Yes. In fact, both have equal right to be called a time coordinate, although neither

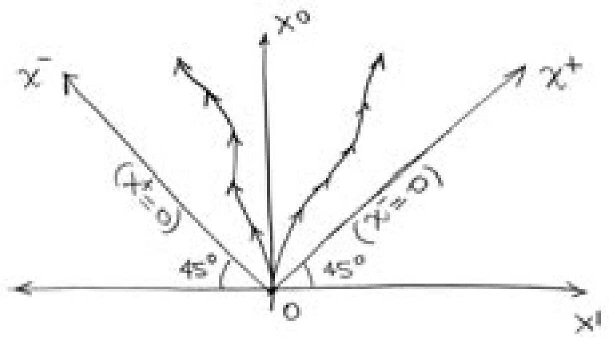

Figure 2.2: A spacetime diagram with x^1 and x^0 represented as orthogonal axes. Shown are the light-cone axes $x^{\pm} = 0$. The curves with arrows are possible world-lines of physical particles.

one is a time coordinate in the standard sense of the word. Light-cone time is not quite the same as ordinary time. Perhaps the most familiar property of time is that it goes forward for any physical motion of a particle. Physical motion starting at the origin is represented in Figure 2.2 as curves that remain within the light cone and whose slopes never go below 45°. For all these curves, as we follow the arrows, both x^+ and x^- increase. The only subtlety is that for special light-rays light-cone time will freeze! As we saw above, for a light ray in the negative x^1 direction, x^+ remains constant, while for a light ray in the positive x^1 direction, x^- remains constant.

For definiteness, we will take x^+ to be the *light-cone time* coordinate. Accordingly, we will think of x^- as a spatial coordinate. Of course, these light-cone time and space coordinates will be somewhat strange.

Taking differentials of (2.3.1) we readily find that

$$2dx^+dx^- = (dx^0 + dx^1)(dx^0 - dx^1) = (dx^0)^2 - (dx^1)^2 \,. \qquad (2.3.2)$$

It follows that the invariant interval (2.2.7), expressed in terms of the light-cone coordinates (2.3.1), takes the form

$$\boxed{-\, ds^2 = -2dx^+dx^- + (dx^2)^2 + (dx^3)^2 \,.} \qquad (2.3.3)$$

The symmetry in the definitions of x^+ and x^- is evident here. Notice that given ds^2, solving for dx^- or for dx^+ does not require taking a square root. This is a very important simplification, as we will see in chapter 9.

How do we represent (2.3.3) with index notation? We still need indices that run over four values, but this time the values will be called

$$\{+, -, 2, 3\}\,. \tag{2.3.4}$$

Just as we did in (2.2.15), we write

$$-\,ds^2 = \widehat{\eta}_{\mu\nu} dx^\mu dx^\nu\,, \tag{2.3.5}$$

where we have introduced a light-cone metric $\widehat{\eta}$, also defined to be symmetric under the exchange of its indices. Expanding this equation and comparing with (2.3.3) we find

$$\widehat{\eta}_{+-} = \widehat{\eta}_{-+} = -1\,, \qquad \widehat{\eta}_{++} = \widehat{\eta}_{--} = 0\,. \tag{2.3.6}$$

In the $(+,-)$ subspace, the diagonal elements of the light-cone metric vanish, but the off-diagonal elements do not. We also find that $\widehat{\eta}$ does not couple the $(+,-)$ subspace to the $(2,3)$ subspace

$$\widehat{\eta}_{+I} = \widehat{\eta}_{-I} = 0\,, \qquad I = 2, 3\,. \tag{2.3.7}$$

The matrix representation of the light-cone metric is

$$\widehat{\eta}_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{2.3.8}$$

For any Lorentz vector a^μ, its light-cone components are defined in analogy with (2.3.1). We set

$$\begin{aligned} a^+ &\equiv \frac{1}{\sqrt{2}}\,(a^0 + a^1)\,, \\ a^- &\equiv \frac{1}{\sqrt{2}}\,(a^0 - a^1)\,. \end{aligned} \tag{2.3.9}$$

The scalar product between vectors shown in (2.2.23) can be written using light-cone coordinates. This time we have

$$\boxed{a \cdot b = -a^- b^+ - a^+ b^- + a^2 b^2 + a^3 b^3 = \widehat{\eta}_{\mu\nu}\, a^\mu b^\nu\,.} \tag{2.3.10}$$

The last equality follows immediately by summing over the repeated indices and using (2.3.8). The first equality needs a small computation. In fact, it suffices to check that

$$-a^- b^+ - a^+ b^- = -a^0 b^0 + a^1 b^1 \,. \tag{2.3.11}$$

This is quickly done using (2.3.9) and the analogous equations for $b^\pm$. We can also introduce lower light-cone indices. Consider $a \cdot b = a_\mu b^\mu$, and expand the sum over the index μ using the light-cone labels:

$$a \cdot b = a_+ b^+ + a_- b^- + a_2 b^2 + a_3 b^3 \,. \tag{2.3.12}$$

Comparing with (2.3.10) we find that

$$a_+ = -a^- \,, \quad a_- = -a^+ \,. \tag{2.3.13}$$

In any Lorentz frame when we lower or raise the zeroth index we get an extra sign. In light-cone coordinates the index changes, and we get an extra sign.

Since physics described using light-cone coordinates looks unusual, we must develop an intuition for it. To do this we will consider an example where the calculations are simple, but the results are surprising.

Consider a particle moving along the x^1-axis with speed parameter $\beta = v/c$. At time $t = 0$, the positions x^1, x^2, and x^3 are all zero. Motion is nicely represented when the positions are expressed in terms of time:

$$x^1(t) = vt = \beta x^0 \,, \quad x^2(t) = x^3(t) = 0 \,. \tag{2.3.14}$$

What happens in light cone coordinates? Since x^+ is time and $x^2 = x^3 = 0$, we must simply express x^- in terms of x^+. Using (2.3.14), we find

$$x^+ = \frac{x^0 + x^1}{\sqrt{2}} = \frac{1+\beta}{\sqrt{2}}\, x^0 \,. \tag{2.3.15}$$

As a result,

$$x^- = \frac{x^0 - x^1}{\sqrt{2}} = \frac{(1-\beta)}{\sqrt{2}}\, x^0 = \frac{1-\beta}{1+\beta}\, x^+ \,. \tag{2.3.16}$$

Since it relates light-cone position to light-cone time, we identify the ratio

$$\frac{dx^-}{dx^+} = \frac{1-\beta}{1+\beta} \,, \tag{2.3.17}$$

as the light-cone velocity. How strange is this light-cone velocity? For light moving to the right, $\beta = 1$, it equals zero. Indeed, light moving to the right has zero light-cone velocity because x^- does not change at all. This is shown as line 1 in Figure 2.3. Suppose you have a particle moving to the right with high conventional velocity, namely β is near one (line 2 in the figure). Its light-cone velocity is then very small. A long light-cone time must pass for this particle to move a little in the x^- direction. Perhaps more interestingly, a static particle in standard coordinates (line 3) is moving quite fast in light-cone coordinates. If $\beta = 0$, we have unit light-cone speed. This light-cone speed increases as β grows negative: the numerator in (2.3.17) is larger than one and increasing, while the denominator is smaller than one and decreasing. As $\beta \to -1$ (line 5), the light-cone velocity becomes infinite! While this seems odd, there is no clash with relativity. Light-cone velocities are just unusual. The light-cone is a frame where kinematics has a non-relativistic flavor, and infinite velocities are possible. Note that light-cone coordinates were introduced as a change of coordinates, not as a Lorentz transformation. There is no Lorentz transformation that takes the coordinates (x^0, x^1) into coordinates $(x'^0, x'^1) = (x^+, x^-)$.

Quick Calculation 2.3. Convince yourself that the last statement above is correct.

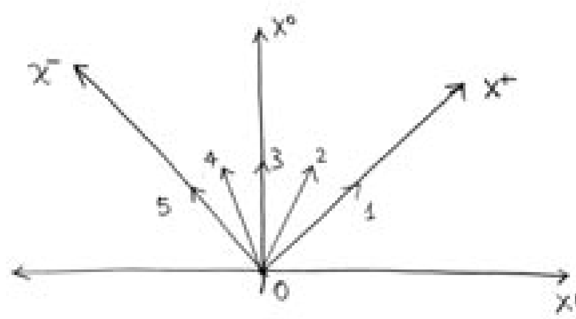

Figure 2.3: World lines of particles with various light-cone velocities. Particle 1 has zero light-cone velocity. The velocities increase until that of particle 5, which is infinite.

2.4 Relativistic energy and momentum

In special relativity there is a basic relation between the rest mass m of a point particle, its relativistic energy E, and its relativistic momentum $\vec{p}$. This relation is

$$\frac{E^2}{c^2} - \vec{p}\cdot\vec{p} = m^2c^2\,. \tag{2.4.1}$$

The relativistic energy and momentum are given in terms of the rest mass and velocity by the following familiar relations:

$$E = \gamma mc^2\,, \quad \vec{p} = \gamma m\vec{v}\,. \tag{2.4.2}$$

Quick Calculation 2.4. Verify that the above E and $\vec{p}$ satisfy (2.4.1).

Energy and momentum can be used to define a momentum *four-vector*, as we will prove shortly. This four-vector is

$$p^\mu = (p^0, p^1, p^2, p^3) \equiv \Big(\frac{E}{c}, p_x, p_y, p_z\Big)\,. \tag{2.4.3}$$

Using the last two equations,

$$p^\mu = \Big(\frac{E}{c}, \vec{p}\Big) = m\gamma(c, \vec{v})\,. \tag{2.4.4}$$

As done in (2.2.22), we can lower the index in p^μ to find

$$p_\mu = (p_0, p_1, p_2, p_3) = \eta_{\mu\nu}p^\nu = \Big(-\frac{E}{c}, p_x, p_y, p_z\Big)\,. \tag{2.4.5}$$

Using the above expressions for p^μ and p_μ,

$$p^\mu p_\mu = -(p^0)^2 + \vec{p}\cdot\vec{p} = -\frac{E^2}{c^2} + \vec{p}\cdot\vec{p}\,, \tag{2.4.6}$$

and making use of (2.4.1) we have

$$p^\mu p_\mu = -m^2c^2\,. \tag{2.4.7}$$

Since $p^\mu p_\mu$ has no free index it must be a Lorentz scalar. Indeed, all Lorentz observers agree on the value of the rest mass of a particle. Using the relativistic scalar product notation, condition (2.4.7) reads

$$\boxed{p^2 \equiv p\cdot p = -m^2c^2\,.} \tag{2.4.8}$$

Another important concept in special relativity is that of *proper time*. Proper time is a Lorentz invariant notion of time. Consider a moving particle and two events along its trajectory. Different Lorentz observers record different values for the time interval between the two events. But now imagine that the moving particle is carrying a clock. The proper time is the time elapsed between the two events *on that clock*. By definition, it is an invariant: all observers of a particular clock must agree on the time elapsed in that clock!

Proper time enters naturally into the calculation of invariant intervals. Consider an invariant interval for the motion of a particle along the x axis:

$$-ds^2 = -c^2dt^2 + dx^2 = -c^2dt^2\,(1-\beta^2)\,. \tag{2.4.9}$$

Now evaluate the interval using a Lorentz frame attached to the particle. This is a frame where the particle does not move, and where time is recorded by a clock moving with the particle. In this frame, $dx = 0$ and $dt = dt_p$ is the proper time elapsed. As a result,

$$-\,ds^2 = -c^2\,dt_p{}^2\,. \tag{2.4.10}$$

We cancel the minus signs and take the square root (using (2.2.14)) to find:

$$ds = c\,dt_p\,. \tag{2.4.11}$$

This shows that for timelike intervals, ds/c is the proper time interval. Similarly, cancelling minus signs and taking the square root of (2.4.9) gives

$$ds = cdt\sqrt{1-\beta^2} \quad \rightarrow \quad \frac{dt}{ds} = \frac{\gamma}{c}\,. \tag{2.4.12}$$

Being a Lorentz invariant, we can use ds to construct new Lorentz vectors starting from old Lorentz vectors, For example, a velocity four-vector u^μ is obtained by taking the ratio of dx^μ and ds. Since dx^μ is a Lorentz vector and ds is a Lorentz scalar, the ratio is also a Lorentz vector:

$$u^\mu = c\frac{dx^\mu}{ds} = c\Big(\frac{d(ct)}{ds}, \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\Big)\,. \tag{2.4.13}$$

The components can be simplified using the chain rule and (2.4.12). For example,

$$\frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{v_x\gamma}{c}\,. \tag{2.4.14}$$

Back in (2.4.13) we find

$$u^\mu = \gamma(c, v_x, v_y, v_z) = \gamma(c, \vec{v})\,. \tag{2.4.15}$$

Comparing with (2.4.4), we see that the momentum four-vector is just mass times the velocity four-vector:

$$p^\mu = m u^\mu\,. \tag{2.4.16}$$

This confirms our earlier assertion that the components of p^μ form a four-vector. Since any four-vector transforms under Lorentz transformations as the x^μ do, we can use (2.2.30) to find that under a boost in the x-direction

$$\begin{aligned} \frac{E'}{c} &= \gamma\Big(\quad\ \frac{E}{c} - \beta\, p_x\Big)\,,\\ p'_x &= \gamma\Big(-\beta\frac{E}{c} + \quad p_x\Big)\,. \end{aligned} \tag{2.4.17}$$

2.5 Light-cone energy and momentum

The light-cone components p^+ and p^- of the momentum Lorentz vector are obtained using the rule (2.3.9):

$$\begin{aligned} p^+ &= \frac{1}{\sqrt{2}}\,(p^0 + p^1) = -p_-\,,\\ p^- &= \frac{1}{\sqrt{2}}\,(p^0 - p^1) = -p_+\,. \end{aligned} \tag{2.5.1}$$

Which component should be identified with light-cone energy? The naive answer would be p^+. Just like time, in any Lorentz frame, energy is the first component of the momentum four vector. Since light-cone time was chosen to be x^+, we might conclude that light-cone energy should be taken to be p^+. This is not appropriate, however. The light-cone frame is not a Lorentz frame, so we should be careful and examine this question in detail. Both $p^\pm$ are energy-like since both are positive for physical particles. Indeed, from (2.4.1), and with $m \neq 0$, we have

$$p^0 = \frac{E}{c} = \sqrt{\vec{p}\cdot\vec{p} + m^2c^2} > |\,\vec{p}\,| \geq |p^1|\,. \tag{2.5.2}$$

As a result $p^0 \pm p^1 > 0$, and thus $p^\pm > 0$. While both are plausible candidates for energy, the physically motivated choice turns out to be $-p_+ = p^-$.

To understand this let us first evaluate $p_\mu x^\mu$, a quantity that will enter into our physical argument. In standard coordinates,

$$p \cdot x = p_0 x^0 + p_1 x^1 + p_2 x^2 + p_3 x^3 \,. \tag{2.5.3}$$

while in light-cone coordinates, using (2.3.10),

$$p \cdot x = p_+ x^+ + p_- x^- + p_2 x^2 + p_3 x^3 \,. \tag{2.5.4}$$

In standard coordinates $p_0 = -E/c$ appears together with the time x^0. In light-cone coordinates p_+ appears together with the light-cone time x^+. We would therefore expect p_+ to be minus the light-cone energy.

Why is this pairing significant? Energy and time are conjugate variables. As you learned in quantum mechanics, the Hamiltonian operator measures energy, and generates time evolution. The wavefunction of a point particle with energy E and momentum $\vec{p}$ is given by

$$\psi(t, \vec{x}) = \exp\Big[-\frac{i}{\hbar}(Et - \vec{p} \cdot \vec{x}) \Big] \,. \tag{2.5.5}$$

Indeed, the eigenvalue of the Hamiltonian $\widehat{H}$ coincides with E

$$\widehat{H}\psi = i\hbar \frac{\partial \psi}{\partial t} = E\psi \,. \tag{2.5.6}$$

For the light-cone Hamiltonian $\widehat{H}_{lc}$ and light-cone energy E_{lc}, the analogous equation would be

$$\widehat{H}_{lc}\psi = i\hbar \frac{\partial \psi}{\partial x^+} = \frac{E_{lc}}{c}\psi \,. \tag{2.5.7}$$

The extra factor of c in the right hand side has been added because x^+, as opposed to t, has units of length. With this factor included, E_{lc} has units of energy. To find the x^+ dependence of the wavefunction we recognize that

$$\psi(t, \vec{x}) = \exp\Big[\frac{i}{\hbar}(p_0 x^0 + \vec{p} \cdot \vec{x}) \Big] = \exp\Big[\ \frac{i}{\hbar} \ p \cdot x \ \Big] \,, \tag{2.5.8}$$

and using (2.5.4) we have

$$\psi(x) = \exp\Big[\frac{i}{\hbar}(p_+ x^+ + p_- x^- + p_2 x^2 + p_3 x^3) \Big] \,. \tag{2.5.9}$$

We can now return to (2.5.7) and evaluate:

$$i\hbar\frac{\partial\psi}{\partial x^+} = -p_+\psi \quad \rightarrow \quad -p_+ = \frac{E_{lc}}{c}\,. \tag{2.5.10}$$

This confirms our identification of $(-p_+)$ with light-cone energy. Since presently $-p_+ = p^-$, it is convenient to use p^- as the light-cone energy in order to eliminate the sign in the above equation:

$$p^- = \frac{E_{lc}}{c}\,. \tag{2.5.11}$$

Some physicists like to raise and lower + and − indices to simplify expressions involving light-cone quantities. While this is consistent, it can easily lead to errors. If you talk with a friend over the phone, and she says "..p-plus times ..," you will have to ask, "plus up, or plus down?" In this book we will not lower the + or − indices. They will always be up, and the energy will be p^-.

We can check that the identification of p^- with energy fits the intuition we have developed for light-cone velocity. For this, we confirm that a particle with small light-cone velocity has small light-cone energy. Suppose we have a particle moving very fast in the $+x^1$ direction. As discussed below (2.3.17), its light-cone velocity is very small. Since p^1 is very large, equation (2.4.1) gives

$$p^0 = \sqrt{(p^1)^2 + m^2c^2} = p^1\sqrt{1 + \frac{m^2c^2}{(p^1)^2}} = p^1 + \frac{m^2c^2}{2p^1} + \mathcal{O}(1/(p^1)^2)\,. \tag{2.5.12}$$

The light-cone energy of the particle is therefore

$$p^- = \frac{1}{\sqrt{2}}\,(p^0 - p^1) \simeq \frac{m^2c^2}{2\sqrt{2}\,p^1}\,. \tag{2.5.13}$$

As anticipated, as p^1 increases, both the light-cone velocity and the light-cone energy decrease.

2.6 Lorentz invariance with extra dimensions

If string theory is correct, we must entertain the possibility that spacetime has more than four dimensions. The number of time dimensions must be kept equal to one – it seems very difficult, if not altogether impossible to construct consistent theories with more than one time dimension. The number of spatial dimensions, however, could be higher than three. Can we have Lorentz invariance in worlds with more than three spatial dimensions? Yes, Lorentz invariance is a concept that admits a very natural generalization to spacetimes with additional dimensions.

We first extend the definition of the invariant interval ds^2 to incorporate the additional space dimensions. In a world with five spatial dimensions, for example, we would write

$$-ds^2 = -c^2dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 + (dx^5)^2\,. \qquad (2.6.1)$$

Lorentz transformations are then defined as the linear changes of coordinates that that leave ds^2 invariant. This ensures that every inertial observer in this six-dimensional spacetime will agree on the value of the speed of light. With more dimensions, there is a larger scope for Lorentz transformations. While in four-dimensional spacetime we have boosts in the x^1, x^2, and x^3 directions, in this new world we have boosts along each of the five spatial dimensions. With three spatial coordinates there are three basic spatial rotations: rotations that mix x^1 and x^2, those that mix x^1 and x^3, and finally those that mix x^2 and x^3. The equality of the number of boosts and the number of rotations is a special feature of four-dimensional spacetime. With five spatial coordinates, we have ten rotations, which is double the number of boosts.

There is a sensible requirement for higher-dimensional Lorentz invariance: if nothing happens along the extra dimensions, then the restrictions of lower dimensional Lorentz invariance must apply. This is the case with (2.6.1). For motion that does not involve the extra dimensions, $dx^4 = dx^5 = 0$, and the expression for ds^2 reduces to that used in four-dimensions.

2.7 Compact extra dimensions

If additional spatial dimensions are undetectable in low-energy experiments, one simple possibility is that they are curled up into a compact space of small volume. In this section we will try to understand what a compact dimension is. We will focus mainly on the case of one dimension. In section 2.9 we will explain why small compact dimensions are hard to detect.

Consider a one-dimensional world, say an infinite line, and let x be a coordinate along this line. For each point P along the line, there is a unique real number $x(P)$ called the x-coordinate of the point P. A good coordinate on this infinite line satisfies two conditions:

- Any two points $P_1 \neq P_2$ have different coordinates: $x(P_1) \neq x(P_2)$.
- The assignment of coordinates to points must be continuous: nearby points must have nearly equal coordinates.

If a choice of origin is made for this infinite line, we can use distance from the origin to define a good coordinate. The coordinate assigned to each point is the distance from the point to the origin, with a sign depending on which side of the origin the point is.

Imagine that you live in one dimension. Suppose you are walking along and notice a strange pattern: the scenery repeats each time you move a distance $2\pi R$, for some value of R that you have measured using a ruler. If you meet your friend Phil, you see that there are Phil clones at distances $2\pi R, 4\pi R, 6\pi R, \cdots$, down the line (see Figure 2.4). In fact, there are clones up the line as well, with the same spacing.

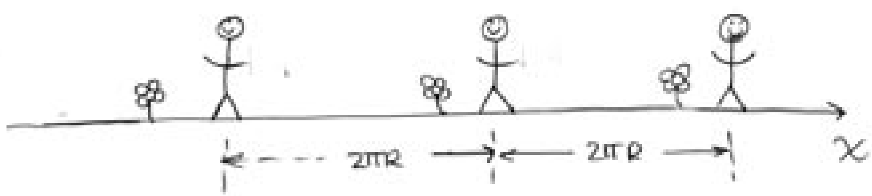

Figure 2.4: A one-dimensional world that repeats each $2\pi R$. Several copies of Phil are shown.

I claim there is no way to distinguish an infinite line with such a strange property from a circle of circumference $2\pi R$. Indeed, saying that this strange line is a circle *explains* the peculiar property – there really are no Phil clones; it is the same Phil you meet again and again as you go around the circle!

How do we express this mathematically? We can think of the circle as the open line with an *identification*. That is, we declare that points whose coordinates differ by $2\pi R$ are the same point. More precisely, two points are declared to be the same point if their coordinates differ by an integer number of $2\pi R$'s:

$$P_1 \sim P_2 \quad \leftrightarrow \quad x(P_1) = x(P_2) + 2\pi R\, n\,, \qquad n \in \mathbb{Z}\,. \tag{2.7.1}$$

This is precise but somewhat cumbersome notation. With no likely confusion, we just write

$$x \sim x + 2\pi R\,, \tag{2.7.2}$$

which should be read as "identify any two points whose coordinates differ by $2\pi R$". With such identification, the open line becomes a circle. The identification has turned a non-compact dimension into a compact one. It may seem to you that a line with identifications is only a complicated way to think of a circle. We will see, however, that many physical problems become clearer when we view a compact dimension as an open one with identifications.

For the line with identification (2.7.2), the interval $0 \leq x < 2\pi R$, is a *fundamental domain* for the identification (see Figure 2.5). A fundamental domain is such that: no two points in it are identified, and, any point in the open space is related by the identification to a point in the fundamental domain. To build the space we take the fundamental domain *with* its boundary: the segment $0 \leq x \leq 2\pi R$, and implement the identifications on the boundary. This glues the point $x = 0$ to the point $x = 2\pi R$ and gives us the circle.

Figure 2.5: The interval $0 \leq x < 2\pi R$ is a fundamental domain of the line with identification (2.7.2). The identified space is a circle of radius R.

On the circle, the coordinate x is no longer a good coordinate. The coordinate x is now either multivalued or discontinuous. This is a problem with any coordinate on a circle, as you may be aware. Consider using angles to assign coordinates on the unit circle (Figure 2.6). Fix a reference point Q on the circle, and let O denote the center of the circle. To any point P on the circle we assign as a coordinate the angle $\theta(P) = \angle POQ$. This angle is naturally multivalued. The reference point, for example, has $\theta(Q) = 0°$ or $\theta(Q) = 360°$. If we force angles not to be multivalued, for example, by restricting $0° \leq \theta < 360°$, then they become discontinuous: two nearby points, Q and Q^- (see Figure 2.6), have very different angles: $\theta(Q) = 0$, while $\theta(Q') \sim 360°$. It is actually easier to work with multivalued functions than to work with discontinuous functions.

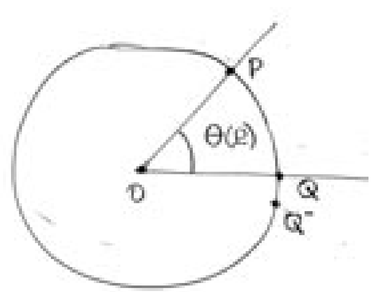

Figure 2.6: Using the angle θ to define a coordinate on a circle. The reference point Q is assigned zero angle: $\theta(Q) = 0$.

In a world with several open dimensions, making the identification in (2.7.2) to one dimension, while doing nothing to the other dimensions, means that the dimension described by x has been turned into a circle while the other dimensions remain open. It is possible, of course, to make more than one dimension compact. Consider, for example, the (x, y) plane, subject to *two* identifications

$$x \sim x + 2\pi R\,, \quad y \sim y + 2\pi R\,. \tag{2.7.3}$$

It is perhaps clearer to write the identifications showing both coordinates

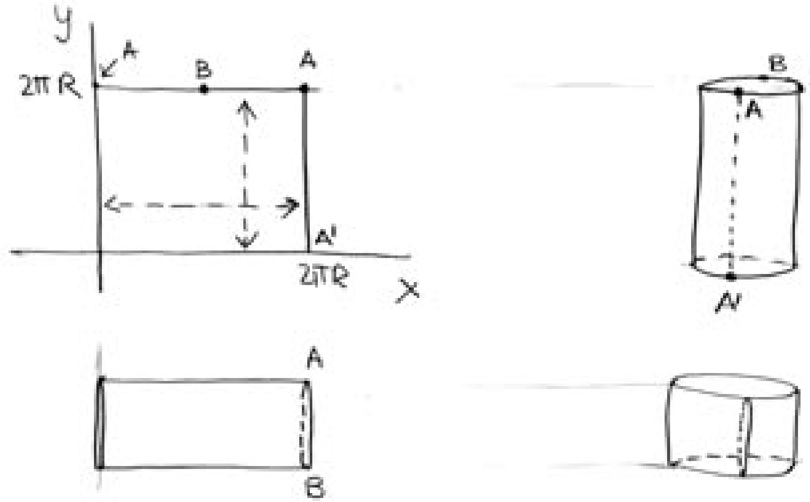

Figure 2.7: A square region in the plane with identifications indicated by the dashed lines. The resulting surface is a torus. The identification of the vertical lines gives a cylinder, shown to the right of the square region. The cylinder, shown horizontally and flattened in the bottom left, must have its edges glued to form the torus.

simultaneously. In that case the two identifications are written as

$$(x, y) \sim (x + 2\pi R\,,\, y)\,, \tag{2.7.4}$$
$$(x, y) \sim (x\,,\, y + 2\pi R)\,. \tag{2.7.5}$$

The first identification implies that we can restrict our attention to $0 \leq x < 2\pi R$, and the second identification implies that we can restrict our attention to $0 \leq y < 2\pi R$. Thus the fundamental domain can be taken to be the square region $0 \leq x, y < 2\pi R$, as shown in Figure 2.7. To build the space we take the fundamental domain *with* its boundary: the square $0 \leq x, y \leq 2\pi R$, and implement the identifications on the boundary. The vertical segments must be identified because they correspond to points $(0, y)$ and $(2\pi R, y)$, and these are the same points on account of (2.7.4). The horizontal segments must be identified because they correspond to points $(x, 0)$ and $(x, 2\pi R)$, which are the same points on account of (2.7.5). The resulting space is called a two dimensional torus. One can visualize the torus by taking the fundamental domain (with its boundary) and gluing the vertical lines as their identification demands. The result is a cylinder, as shown in the top right corner of Figure 2.7 (with the gluing seam dashed). In this cylinder, however, the bottom circle and the top circle must also be glued, since they are nothing other than the horizontal boundaries of the fundamental domain. To do this with paper, you must flatten the cylinder, and then roll it up to glue the circles. The result looks like a flattened doughnut. With a flexible piece of a garden hose, you could simply identify two ends, obtaining the familiar picture of a torus.

We have seen how to compactify coordinates using identifications. Some compact spaces are constructed in other ways. In string theory, however, compact spaces arising from identifications are particularly easy to work with. We shall focus on such spaces throughout this book.

Quick Calculation 2.5. Consider the plane (x, y) with the identification

$$(x\,,\,y) \sim (x + 2\pi R\,,\, y + 2\pi R)\,. \tag{2.7.6}$$

What is the resulting space?

2.8 Quantum mechanics and the square well

Planck's constant $\hbar$ first appeared as the constant of proportionality relating the energy E and the angular frequency ω of a photon:

$$E = \hbar\omega\,. \tag{2.8.1}$$

Since ω has units of T^{-1}, $\hbar$ has units of energy times time. Energy has units of ML^2T^{-2}, and therefore

$$[\hbar] = [\text{Energy}] \times [\text{Time}] = ML^2T^{-1}\,. \tag{2.8.2}$$

The value of Planck's constant is $\hbar = 1.054571596(82) \times 10^{-27}$erg sec.

In the formulation of quantum mechanics, the constant $\hbar$ appears in the basic commutation relations. The Schrödinger position and momentum operators satisfy

$$[x\,, p] = i\hbar\,. \tag{2.8.3}$$

If we have several spatial dimensions, the commutation relations are

$$\boxed{[x^i\,, p_j] = i\hbar\,\delta^i_j\,,} \tag{2.8.4}$$

where the Kronecker delta is defined as in (2.2.26)

$$\delta^i_j = \begin{cases} 1\,, & \text{if } i = j; \\ 0\,, & \text{if } i \neq j. \end{cases} \tag{2.8.5}$$

In three spatial dimensions, the indices i, j run from one to three. The generalization of quantum mechanics to higher dimensions is straightforward. With d spatial dimensions, the indices in (2.8.4) would simply run over d possible values.

To set the stage for the case of a small extra dimension, we review a standard quantum mechanics problem. Consider the time-independent Schrödinger equation

$$\Big[-\frac{\hbar^2}{2m}\nabla^2 + V\,(x)\Big]\psi\,(x) = E\,\psi(x)\,, \tag{2.8.6}$$

for a one-dimensional square well potential of infinite height:

$$V\,(x) = \begin{cases} \;0\,, & \text{if } x \in (0, a) \\ \infty\,, & \text{if } x \notin (0, a)\;. \end{cases} \tag{2.8.7}$$

For $x \notin (0, a)$, the infinite potential implies $\psi(x) = 0$. In particular, $\psi(0) = \psi(a) = 0$. This is just the quantum mechanics of a particle living on a *segment*, as shown in Figure 2.8.

When $x \in (0, a)$, the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi. \tag{2.8.8}$$

The solutions of (2.8.8), consistent with the boundary conditions, are

$$\psi_k\,(x) = \sqrt{\frac{2}{a}}\,\sin\left(\frac{k\pi x}{a}\right)\,, \qquad k = 1, 2, \ldots\,\infty\;. \tag{2.8.9}$$

The value $k = 0$ is not allowed since it would make the wavefunction vanish. By performing the differentiation indicated in (2.8.8) we see that the energy E_k associated with the wavefunction ψ_k is

$$E_k = \frac{\hbar^2}{2m}\left(\frac{k\pi}{a}\right)^2. \tag{2.8.10}$$

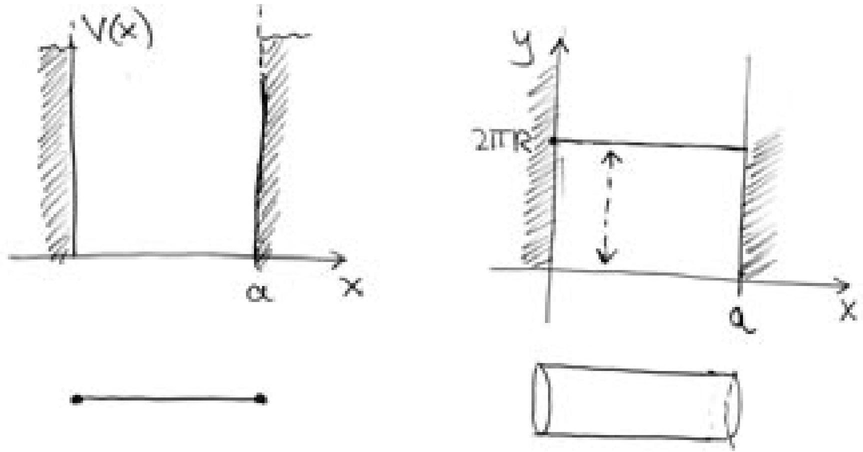

Figure 2.8: Left side: The square well potential in one dimension. Here the particle lives on a segment. Right side: In the x, y plane the particle must remain within $0 < x < a$. The direction y is identified as $y \sim y + 2\pi R$. The particle lives on a cylinder.

2.9 Square well with an extra dimension

We now add an extra dimension to the square well problem. In addition to x, we include a dimension y that is compactified into a small circle of radius R. In other words, we make the identification

$$(x, y) \sim (x\,,\; y + 2\pi R). \tag{2.9.1}$$

The original dimension x has not been changed (see Figure 2.8). Since the y-axis has been turned into a circle of radius R, the space where the particle moves has changed from a segment into a *cylinder*! The cylinder has length a and circumference $2\pi R$. The potential $V(x, y)$ will remain given by (2.8.7) and is y-independent.

We will see that as long as R is small, and as long as we look only at low energies, the quantum mechanics of the particle on the segment is very similar to the quantum mechanics of the particle on the cylinder. The only length scale in this problem is the size a of the segment, so small R means

$R \ll a$.

In two dimensions the Schrödinger equation (2.8.6) becomes

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=E\psi\,. \tag{2.9.2}$$

In order to solve this equation, we use separation of variables. We let $\psi\,(x,y)=\psi\,(x)\,\phi\,(y)$ and find that the equation takes the form

$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2}-\frac{\hbar^2}{2m}\frac{1}{\phi(y)}\frac{d^2\phi(y)}{dy^2}=E. \tag{2.9.3}$$

The x-dependent and y dependent terms of the equation must separately be constant, and the solutions are of the form $\psi_{k,l}(x,y)=\psi_k(x)\phi_l(y)$, where

$$\psi_k\,(x) \;=\; c_k \sin\Big(\frac{k\pi x}{a}\Big)\,, \tag{2.9.4}$$

$$\phi_l\,(y) \;=\; a_l \sin\Big(\frac{ly}{R}\Big)+b_l\cos\Big(\frac{ly}{R}\Big). \tag{2.9.5}$$

The physics along the x dimension is unchanged since the wavefunction must still vanish at the ends of the segment. Therefore (2.9.4) takes the same form as (2.8.9) and $k=1,2,\cdots,\infty$. The boundary condition for $\phi_l(y)$ arises from the identification $y\sim y+2\pi R$. Since y and $y+2\pi R$ are coordinates for the same point, the wavefunction must take the same value at these two arguments:

$$\phi_l(y)=\phi_l(y+2\pi R)\,. \tag{2.9.6}$$

As opposed to $\psi_k(x)$, the function $\phi_l(y)$ need not vanish for any y. As a result, the general periodic solution, recorded in (2.9.5), includes both sines and cosines. The presence of cosines allows a nonvanishing *constant* solution: for $l=0$, we get $\phi_0(y)=b_0$. This solution is key to understanding why a small extra dimension does not change the physics very much.

The energy eigenvalues corresponding to the $\psi_{k,l}$ are

$$E_{k,l}=\frac{\hbar^2}{2m}\Big[\Big(\frac{k\pi}{a}\Big)^2+\Big(\frac{l}{R}\Big)^2\Big]. \tag{2.9.7}$$

These energies correspond to doubly degenerate states when $l\neq 0$ because in this case there are two solutions in (2.9.5). The extra dimension has changed the spectrum dramatically. We will see, however, that, if $R \ll a$, then the *low-lying* part of the spectrum is unchanged. The rest of the spectrum changes, but it is not accessible *at low energies.*

Since $l = 0$ is permitted, the energy levels $E_{k,0}$ coincide with the old energy levels E_k! The new system has all the energy levels of the old system. But it also includes additional energy levels.

What is the lowest *new* energy level? To minimize the energy, each of the terms in (2.9.7) must be as low as possible. This occurs for $k = 1$, since $k = 0$ is not allowed, and for $l = 1$, since $l = 0$ gives us the old levels:

$$E_{1,1} = \frac{\hbar^2}{2m}\Big[\Big(\frac{\pi}{a}\Big)^2 + \Big(\frac{1}{R}\Big)^2\Big]. \tag{2.9.8}$$

When $R \ll a$, the second term is much larger than the first, and

$$E_{1,1} \sim \frac{\hbar^2}{2m}\Big(\frac{1}{R}\Big)^2. \tag{2.9.9}$$

This energy is comparable to that of the level k eigenstate in the original problem (see (2.8.10)), when

$$\frac{k\pi}{a} \sim \frac{1}{R} \quad \to \quad k \sim \frac{1}{\pi}\,\frac{a}{R}. \tag{2.9.10}$$

Since R is much smaller than a, k is a very large number, and the first new energy level appears at an energy far above that of the original, low-lying states. We therefore conclude that an extra dimension can remain hidden from experiments at a particular energy level as long as the dimension is small enough. Once the probing energies become sufficiently high, the effects of an extra dimension can be observed.

Curiously, the quantum mechanics of a string has new features. For an extra dimension *much* smaller than the already small string length ℓ_s (typically thought to be around 10^{-33}cm), new low lying states can appear! These correspond to strings that wrap around the extra dimension. They have no analog in the quantum mechanics of a particle, and we will study them in detail in Chapter 16. The conclusion of the above analysis remains true, with a small qualification. In string theory there are no new low-energy states that arise from an extra dimension if that dimension is small, but still larger than ℓ_s.

Problems

Problem 2.1. *Exercises with units.*

(a) Find the relation between coulombs (C) and esu's.

(b) Explain the meaning of the unit K used for measuring temperatures, and its relation to the basic length, mass and time units.

(c) Construct a dimensionless number out of the charge e of the electron, $\hbar$ and c. Evaluate it.

Problem 2.2. *Simple quantum gravity effects are small.*

(a) What would be the "gravitational" Bohr radius for a hydrogen atom if the attraction binding the electron to the proton was gravitational? The standard Bohr radius is $a_0 = \frac{\hbar^2}{me^2} \simeq 5.29 \times 10^{-9}$cm.

(b) In "units" where $G = \hbar = c = 1$, the temperature of a black hole is given as $kT = \frac{1}{8\pi\, M}$. Insert back the units into this formula, and evaluate the temperature of a black hole of a million solar masses. What is the mass of a black hole whose temperature is room temperature?

Problem 2.3. *Lorentz transformations for light cone coordinates.*

Consider coordinates $x^\mu = (x^0, x^1, x^2, x^3)$ and the associated light cone coordinates (x^+, x^-, x^2, x^3). Write the following Lorentz transformations in terms of light cone coordinates.

(a) A boost with parameter β in the x^1 direction.

(b) A rotation with angle θ in the x^1, x^2 plane.

(c) A boost with parameter β in the x^3 direction.

Problem 2.4. *Lorentz transformations, derivatives, and quantum operators.*

(a) Give the Lorentz transformations for the components a_μ of a vector under a boost along the x^1 axis.

(b) Show that the objects $\frac{\partial}{\partial x^\mu}$ transform under Lorentz transformations in the same way as the a_μ considered in (a) do. Thus, partial derivatives with respect to conventional upper-index coordinates x^μ behave as a four-vector with lower indices – as reflected by writing it as ∂_μ.

(c) Show that in quantum mechanics, the equations giving the energy and momentum as differential operators on functions can be written as

$$p_\mu = \frac{\hbar}{i}\frac{\partial}{\partial x^\mu}\,. \tag{2.9.11}$$

Problem 2.5. *Repeated identifications.*

(a) Consider a circle S^1, presented as the real line with the identification $x \sim x + 2$. Choose $-1 < x \leq 1$ as the fundamental domain. The circle is the space $-1 \leq x \leq 1$ with the points $x = \pm 1$ identified. The space $S^1/\mathbb{Z}_2$ is defined by imposing a (so-called) $\mathbb{Z}_2$ identification: $x \sim -x$. Describe the action of this identification on the circle. Show that there are two points on the circle that are left invariant by the $\mathbb{Z}_2$ action. Find a fundamental domain for the two identifications. Describe the space $S^1/\mathbb{Z}_2$ in simple terms.

(b) Consider a torus T^2 presented as the (x, y) plane with the identifications $x \sim x + 2$, and $y \sim y + 2$. Choose $-1 < x, y \leq 1$ as the fundamental domain of the identifications. The space $T^2/\mathbb{Z}_2$ is defined by imposing a $\mathbb{Z}_2$ identification: $(x, y) \sim (-x, -y)$. Prove that there are four points on the torus that are left invariant by the $\mathbb{Z}_2$ transformation. Show that the space $T^2/\mathbb{Z}_2$ is a two-dimensional sphere, naturally presented as a rectangular pillowcase with seamed edges.

Problem 2.6. *Spacetime diagrams and Lorentz transformations.*

Consider a spacetime diagram where the x^0 and x^1 axes of the Lorentz frame S are represented as vertical and horizontal axes respectively. Show that the $x^{0'}$ and $x^{1'}$ axes of the Lorentz frame S' related to S via (2.2.28) appear in the original string diagram as oblique axes. Find the angle between the axes, and show in detail how they appear when $\beta > 0$ and when $\beta < 0$, indicating in both cases the directions of increasing values of the coordinates.

Problem 2.7. *Light-like compactification.*

The identification

$$x \sim x + 2\pi R\,,$$

is the statement that the coordinate x has been compactified into a circle of radius R. In doing this the time component is left untouched.

Consider now the strange "light-like" compactification where we identify both position and time:

$$\begin{pmatrix} x \\ ct \end{pmatrix} \sim \begin{pmatrix} x \\ ct \end{pmatrix} + 2\pi \begin{pmatrix} R \\ -R \end{pmatrix}\,. \tag{1}$$

(a) Rewrite this identification as identifications in light-cone coordinates.

(b) Consider boosted coordinates (ct', x') related to (ct, x) by a boost with parameter β. Find the corresponding identifications in the primed coordinates.

To interpret (1) physically consider the family of identifications

$$\begin{pmatrix} x \\ ct \end{pmatrix} \sim \begin{pmatrix} x \\ ct \end{pmatrix} + 2\pi \begin{pmatrix} \sqrt{R^2 + R_s^2} \\ -R \end{pmatrix}\,, \tag{2}$$

where R_s is a length that will eventually be taken to zero, in which case (2) reduces to (1).

(c) Show that there is a boosted frame S' where the identification in (2) becomes a standard identification (*i.e.* the space coordinate is identified but time is not). Do this by finding the boost parameter of S' with respect to S and the compactification radius in this Lorentz frame S'.

(d) Represent your answer to part (c) in a space-time diagram, where you should show two points related by the identification (2), and the space and time axes for the Lorentz frame S' where the compactification is standard (see if you get the sign of β right!).

(e) Fill in the blanks in the following statement: Light like compactification with radius R arises by boosting a standard compactification with radius $\cdots$ with boost parameter $\cdots$ in the limit as $\cdots \to 0$.

Problem 2.8. *Extra dimension and statistical mechanics.*

Write a double sum representing the statistical mechanics partition function $Z(a, R)$ for the quantum mechanical system considered in section 2.9. Observe that $Z(a, R)$ factors into

$$Z(a, R) = Z(a)\tilde{Z}(R)\,.$$

(a) Calculate explicitly $Z(a, R)$ in the very high temperature limit ($\beta = \frac{1}{kT} \to 0$). Prove that this partition function coincides with the partition function of a particle in a two-dimensional box with sides a and $2\pi R$. This shows that at high temperatures the effects of the extra dimension are visible.

(b) Assume that $R << a$ in such a way that there are temperatures that are large as far as the box dimension a is concerned, as well as small as far as the compact dimension is concerned. Write an inequality involving kT and other constants to express this possibility. Evaluate $Z(a, R)$ in this regime including only the leading correction due to the small extra dimension.

Chapter 3

Electromagnetism and Gravitation in Various Dimensions

String theory is promising because it includes Maxwell electrodynamics and its nonlinear cousins, as well as gravitation. We review the relativistic formulation of electrodynamics and show how it facilitates the definition of electrodynamics in other dimensions. We give a brief description of Einstein's gravity, and use the Newtonian limit to discuss the relation between Planck's length and the gravitational constant in various dimensions. We study the effect of compactification on the gravitational constant, and explain how large extra dimensions could escape detection.

3.1 Classical Electrodynamics

Unlike Newtonian mechanics, classical electrodynamics is a relativistic theory. In fact, Einstein was led by electrodynamic considerations to formulate the special theory of relativity. Electromagnetism has a particularly elegant formulation where the relativistic character of the theory is manifest. This relativistic formulation allows a natural extension of electromagnetic theory to higher dimensions. Before we discuss the relativistic formulation, however, we will review the equations of Maxwell. These equations describe the dynamics of electric and magnetic fields.

Although most undergraduate and graduate courses in electromagnetism nowadays use the international system of units (SI units), the Gaussian system of units is far more appropriate for discussions involving relativity. In Gaussian units, Maxwell's equations take the following form:

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\,, \tag{3.1.1}$$
$$\nabla \cdot \vec{B} = 0\,, \tag{3.1.2}$$

$$\nabla \cdot \vec{E} = 4\pi\rho\,, \tag{3.1.3}$$
$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\,. \tag{3.1.4}$$

The above equations imply that in Gaussian units $\vec{E}$ and $\vec{B}$ are measured with the *same units*. The first two equations are the source-free Maxwell equations. The second two involve sources: the charge density ρ, with units of charge per unit volume, and the current density $\vec{j}$, with units of current per unit area. The Lorentz force law, which gives the rate of change of relativistic momentum of a charged particle in an electromagnetic field, takes the form

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)\,. \tag{3.1.5}$$

Since the magnetic field $\vec{B}$ is divergenceless, it can be written as the curl of a vector, the well-known vector potential $\vec{A}$:

$$\vec{B} = \nabla \times \vec{A}\,. \tag{3.1.6}$$

In electrostatics, the electric field $\vec{E}$ has zero curl, and is therefore written as (minus) the gradient of a scalar, the well-known scalar potential Φ. In electrodynamics, however, equation (3.1.1) shows that the curl of $\vec{E}$ is not always zero. Substituting (3.1.6) into (3.1.1), we find a combination of $\vec{E}$ and the time derivative of $\vec{A}$ that has zero curl:

$$\nabla \times \left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 0\,. \tag{3.1.7}$$

The object inside parenthesis is set equal to $-\nabla\Phi$, and the electric field $\vec{E}$ can be written in terms of the scalar potential and the vector potential:

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\Phi\,. \tag{3.1.8}$$

Equations (3.1.6) and (3.1.8) express the electric and magnetic fields in terms of potentials. Simply by writing them in this form, equations (3.1.1) and (3.1.2) are automatically satisfied. Thus, the source-free Maxwell equations are solved by using potentials. Equations (3.1.3) and (3.1.4) contain additional information and are used to derive equations for $\vec{A}$ and Φ.

3.2 Electromagnetism in three dimensions

What is electromagnetism in three spacetime dimensions? One way to produce a theory of electromagnetism in three dimensions is to begin with our four-dimensional theory and eliminate one spatial coordinate. This is called doing dimensional reduction.

In four spacetime dimensions, both electric and magnetic fields have three spatial components: (E_x, E_y, E_z) and (B_x, B_y, B_z). It may seem likely that a reduction to a world without a z-coordinate, would require dropping the z-components from the two fields. Surprisingly, this does not work! Maxwell's equations and the Lorentz force law make it impossible.

In order to construct a consistent three-dimensional theory, physically, we must ensure that the dynamics does not depend on the direction that we want to get rid of: the z-direction, in the present case. If there is motion, it must remain restricted to the (x, y) plane. It is thus natural to require that *no quantity should have z-dependence*. This *does not* necessarily mean dropping quantities with a z-index.

The Lorentz force law (3.1.5) is of much help. Suppose there is no magnetic field. Then, in order to keep the z-component of momentum equal to zero we must have $E_z = 0$; the z-component of the electric field must go. The case of the magnetic field is more surprising. Assume the electric field is zero. If the velocity of the particle is a vector in the (x, y) plane, a component of the magnetic field in the plane would generate, via the cross-product, a force in the z-direction. On the other hand, a z-component of the magnetic field would generate a force in the (x, y) plane! We conclude that B_x and B_y must be set to zero, while we can keep B_z. All in all,

$$E_z = B_x = B_y = 0\,. \tag{3.2.1}$$

The left-over fields E_x, E_y, and B_z, can only depend on x and y. In the three-dimensional world with coordinates t, x and y, the z-index of B_z is not a vector index. Therefore, in this reduced world, B_z essentially behaves like a Lorentz scalar. In summary, we have a two-dimensional vector $\vec{E}$ and a scalar field B_z.

We can test the consistency of this truncation by taking a look at the x and y components of (3.1.1). They read

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{1}{c}\frac{\partial B_x}{\partial t}\,, \tag{3.2.2}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{1}{c}\frac{\partial B_y}{\partial t}\,. \tag{3.2.3}$$

Since the right-hand sides are set to zero by our truncation, the left-hand sides must also be zero. Indeed, they are. Each term in the left-hand sides equals zero, either because it contains E_z, or because it contains a z-derivative, and no field depends on z. You may examine all other equations in Problem 3.3.

While setting up three-dimensional electrodynamics was not completely straightforward, it was not too difficult. It is much harder to guess what five-dimensional electrodynamics should be like. As we will see next, the manifestly relativistic formulation of Maxwell's equations immediately gives the appropriate generalization to other dimensions.

3.3 Manifestly relativistic electrodynamics

In the relativistic formulation of Maxwell's equations, neither the electric field nor the magnetic field becomes part of a four-vector. Rather, a four-vector is obtained by combining the scalar potential Φ with the vector potential $\vec{A}$:

$$A^\mu = \left(\Phi, A^1, A^2, A^3\right)\,. \tag{3.3.1}$$

The corresponding object with down indices is therefore

$$A_\mu = \left(-\Phi, A^1, A^2, A^3\right)\,. \tag{3.3.2}$$

From A_μ we create an object known as the electromagnetic *field strength* $F_{\mu\nu}$, defined through the equation

$$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\,.} \tag{3.3.3}$$

Equation (3.3.3) implies that $F_{\mu\nu}$ is antisymmetric:

$$F_{\mu\nu} = -F_{\nu\mu}\,, \tag{3.3.4}$$

and therefore all diagonal components vanish:

$$F_{00} = F_{11} = F_{22} = F_{33} = 0\,. \tag{3.3.5}$$

Let us calculate a few entries in $F_{\mu\nu}$. Letting i denote a space index: $i = 1, 2, 3$, we have

$$F_{0i} = \frac{\partial A_i}{\partial x^0} - \frac{\partial A_0}{\partial x^i} = \frac{1}{c}\frac{\partial A^i}{\partial t} + \frac{\partial \Phi}{\partial x^i} = -E_i\,, \tag{3.3.6}$$

where we made use of (3.1.8). Also, for example,

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 = \partial_x A_y - \partial_y A_x = B_z\,, \tag{3.3.7}$$

comparing with (3.1.6). Continuing in this manner, we can compute all the entries in the matrix $F_{\mu\nu}$:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{3.3.8}$$

The potentials A_μ can be changed by *gauge transformations*. Two sets of potentials A_μ and A'_μ related by gauge transformations are physically equivalent. A necessary condition for physical equivalence is that the potentials A_μ and A'_μ must give identical electric and magnetic fields. On account of (3.3.8), gauge related potentials must give identical field strengths. Gauge transformations take the form

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \epsilon\,. \tag{3.3.9}$$

Here A_μ and A'_μ are the gauge related potentials, and $\epsilon(x)$ is an arbitrary function of the spacetime coordinates. The field strength $F_{\mu\nu}$ is gauge invariant, as we readily verify:

$$\begin{aligned} F_{\mu\nu} \rightarrow F'_{\mu\nu} &\equiv \partial_\mu A'_\nu - \partial_\nu A'_\mu \\ &= \partial_\mu (A_\nu + \partial_\nu \epsilon) - \partial_\nu (A_\mu + \partial_\mu \epsilon) \\ &= F_{\mu\nu} + \partial_\mu \partial_\nu \epsilon - \partial_\nu \partial_\mu \epsilon \\ &= F_{\mu\nu}\,. \end{aligned} \tag{3.3.10}$$

In the last step we used the fact that partial derivatives commute. The equality of $F'_{\mu\nu}$ and $F_{\mu\nu}$ is the statement of gauge invariance of the field strength. We can write the gauge transformations more explicitly by showing the various components. Using (3.3.9) and (3.3.2) we find

$$\begin{aligned} \Phi \to \Phi' &= \Phi - \frac{1}{c}\frac{\partial \epsilon}{\partial t}\,, \\ \vec{A} \to \vec{A}' &= \vec{A} + \nabla\epsilon\,. \end{aligned} \tag{3.3.11}$$

The gauge transformation of $\vec{A}$ may be familiar to you: adding a gradient to a vector does not change its curl, so $\vec{B} = \nabla \times \vec{A}$ is unchanged. The scalar potential Φ also changes under gauge transformations, as you can see above. This is necessary to keep $\vec{E}$ unchanged.

Quick Calculation 3.1. Verify that $\vec{E}$, as given in (3.1.8), is invariant under the gauge transformations (3.3.11).

Recall that the use of potentials to represent $\vec{E}$ and $\vec{B}$ automatically solves the source-free Maxwell equations (3.1.1) and (3.1.2). How are these equations written in terms of the field strength $F_{\mu\nu}$? They must be written in a way that they hold just on account of (3.3.3), the equation that expresses the field strength in terms of potentials. Consider the combination of field strengths of the form

$$T_{\lambda\mu\nu} \equiv \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}\,. \tag{3.3.12}$$

We now show that $T_{\lambda\mu\nu}$ vanishes identically on account of (3.3.3). Indeed,

$$\partial_\lambda\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \partial_\mu\left(\partial_\nu A_\lambda - \partial_\lambda A_\nu\right) + \partial_\nu\left(\partial_\lambda A_\mu - \partial_\mu A_\lambda\right) = 0, \tag{3.3.13}$$

using the commutativity of partial derivatives. The vanishing of $T_{\lambda\mu\nu}$

$$\boxed{\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0\,,} \tag{3.3.14}$$

is a set of differential conditions for the field strength. It turns out that these conditions are precisely those in the source-free Maxwell equations. To make this clear first note that $T_{\lambda\mu\nu}$ satisfies the antisymmetry conditions:

$$T_{\lambda\mu\nu} = -T_{\mu\lambda\nu}\,, \quad T_{\lambda\mu\nu} = -T_{\lambda\nu\mu}\,. \tag{3.3.15}$$

These two equations follow from (3.3.12) and the antisymmetry property $F_{\mu\nu} = -F_{\nu\mu}$ of the field strength. They state that T changes sign under the transposition of any two adjacent indices.

Quick Calculation 3.2. Verify equation (3.3.15).

Any object, with however many indices, that changes sign under the transposition of every pair of adjacent indices will change sign under the transposition of *any* two indices: to exchange any two indices you need an odd number of transpositions of adjacent indices. An object that changes sign under the transposition of any two indices is said to be *totally antisymmetric*. Therefore, T is totally antisymmetric.

Since T is totally antisymmetric, it vanishes when any two of its indices take the same value. T is non-vanishing only when its three indices take different values. In such case, different orderings of any three fixed indices will give T components that differ only by a sign; since we are setting T to zero these various orderings do not give new conditions. Because we have four space-time coordinates, selecting three different indices can only be done in four different ways – leaving out a different index each time. Thus the vanishing of T gives four nontrivial equations. These four equations are equation (3.1.2), and the three components of equation (3.1.1). For example, the vanishing of T_{012} gives us

$$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = \frac{1}{c}\frac{\partial B_z}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0\,. \tag{3.3.16}$$

This is the z-component of equation (3.1.1). You can check that the other three choices of indices lead to the remaining three equations (Problem 3.2).

How can we describe Maxwell equations (3.1.3) and (3.1.4) in our present framework? Since these equations have sources, we must introduce a new four-vector, the current four-vector:

$$j^\mu = \left(c\rho, j^1, j^2, j^3\right)\,, \tag{3.3.17}$$

where ρ is charge density and $\vec{j} = (j^1, j^2, j^3)$ is the current density. In addition, we define a field tensor with upper indices by raising the indices of the field tensor with lower indices

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}\,. \tag{3.3.18}$$

Quick Calculation 3.3. Show that

$$F^{\mu\nu} = -F^{\nu\mu}\,, \quad F^{0i} = -F_{0i}\,, \quad F^{ij} = F_{ij}\,. \tag{3.3.19}$$

Equation (3.3.18), together with the original definition (3.3.3) of $F_{\mu\nu}$, gives

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = \eta^{\mu\alpha}\partial_\alpha(\eta^{\nu\beta}A_\beta) - \eta^{\mu\beta}\partial_\beta(\eta^{\nu\alpha}A_\alpha)\,, \tag{3.3.20}$$

where the constancy of the metric components was used to move them across the derivatives. We can use the rule concerning raising and lowering of indices for the case of partial derivatives. Letting $\partial^\mu \equiv \eta^{\mu\alpha}\partial_\alpha$, we have,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,. \tag{3.3.21}$$

It follows from (3.3.19) and (3.3.8) that

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{3.3.22}$$

With this equation and the definition of the current vector in (3.3.17), we can encapsulate Maxwell equations (3.1.3) and (3.1.4) in the equation

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{4\pi}{c}j^\mu,} \tag{3.3.23}$$

as you will verify in Problem 3.2. In the absence of sources, and using (3.3.21) we find

$$\partial_\nu F^{\mu\nu} = 0 \quad \rightarrow \quad \partial_\nu\partial^\mu A^\nu - \partial^2 A^\mu = 0\,, \tag{3.3.24}$$

where we have written $\partial^2 = \partial^\mu\partial_\mu$.

Equations (3.3.3), which define field strengths in terms of potentials, together with equations (3.3.23) are equivalent to the standard Maxwell equations in four dimensions. We will take these to *define Maxwell theory* in arbitrary dimensions. For example, in three-dimensional spacetime, the matrix $F_{\mu\nu}$ is a 3×3 antisymmetric matrix, obtained from (3.3.8) by dropping the last row and the last column:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & B_z \\ E_y & -B_z & 0 \end{pmatrix}. \tag{3.3.25}$$

This reproduces immediately our analysis in section 3.2, which indicated that we have to set B_x, B_y, and E_z to zero.

In arbitrary dimensions, motivated by (3.3.22), we call F^{0i} the electric field E_i:

$$E_i \equiv F^{0i} = -F_{0i} \,. \tag{3.3.26}$$

The electric field is a spatial vector. Note that (3.3.6) implies that in any number of dimensions

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla \Phi \,, \tag{3.3.27}$$

just as we had in four-dimensional electrodynamics (see (3.1.8)). The magnetic field is identified with the F^{ij} components of the field strength. In four-dimensional spacetime, F^{ij} is a 3×3 antisymmetric matrix, and therefore it has three independent entries – the three components of the magnetic field vector (see (3.3.22)). In dimensions other than four, the magnetic field is no longer a spatial vector. We saw that in three spacetime dimensions it was a single-component object. In five spacetime dimensions it has as many entries as a 4×4 antisymmetric matrix – six entries. Such an object cannot be a spatial vector.

We will now examine the electric field produced by a point charge in arbitrary dimensions. To this end, we must learn how to calculate the volumes of higher-dimensional spheres. We turn to this subject now.

3.4 An aside on spheres in higher dimensions

Since we want to work in various dimensions, we should be careful when speaking about spheres. When we speak loosely, we tend to confuse spheres and *balls*, at least in the precise sense that they are defined in mathematics. When you say that the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$, you really should be saying that this is the volume of the *three-ball* B^3 – the three-dimensional ball enclosed by the two-dimensional *two-sphere* S^2. In three-dimensional space $\mathbb{R}^3$ with coordinates x_1, x_2, and x_3, we write the three-ball as the region defined by

$$B^3(R): \quad x_1^2 + x_2^2 + x_3^2 \leq R^2 \,. \tag{3.4.1}$$

This region is enclosed by the two-sphere:

$$S^2(R): \quad x_1^2 + x_2^2 + x_3^2 = R^2 \,. \tag{3.4.2}$$

The superscripts in B or S denote the dimensionality of the space in question. When we drop the explicit argument R we mean that $R = 1$. Lower-dimensional examples are also familiar. B^2 is a two dimensional disk – the region enclosed in $\mathbb{R}^2$ by the one-dimensional unit radius circle S^1. In, fact, in arbitrary dimensions we define balls and spheres as subspaces of $\mathbb{R}^n$:

$$B^n(R): \quad x_1^2 + x_2^2 + \ldots + x_n^2 \leq R^2\,. \tag{3.4.3}$$

This is the region enclosed by the sphere $S^{n-1}(R)$:

$$S^{n-1}(R): \quad x_1^2 + x_2^2 + \ldots + x_n^2 = R^2\,. \tag{3.4.4}$$

One last piece of advice on terminology: to avoid confusion always speak of volumes! If the space is one-dimensional take volume to mean length. If two-dimensional, take volume to mean area, and so on for higher dimensional spaces. Thus

$$\begin{aligned} \text{vol}(S^1(R)) &= 2\pi R\,, \\ \text{vol}(S^2(R)) &= 4\pi R^2\,. \end{aligned} \tag{3.4.5}$$

Unless you have had the opportunity to work with other spheres before, you probably do not know what the volume of S^3 is. Because volumes must have units of length to the power of the space dimensionality, we have, for example

$$\text{vol}\,(S^{n-1}(R)) = R^{n-1}\ \text{vol}(S^{n-1})\,, \tag{3.4.6}$$

relating the volume of the sphere with radius R to the volume of sphere with unit radius. Thus, it suffices to record the volumes of unit spheres:

$$\begin{aligned} \text{vol}(S^1) &= 2\pi\,, \\ \text{vol}(S^2) &= 4\pi\,. \end{aligned} \tag{3.4.7}$$

We now calculate the volume of S^{n-1}. For this purpose take $\mathbb{R}^n$ with coordinates $x_1, x_2, \ldots, x_n$, and let r be the radial coordinate

$$r^2 = x_1^2 + x_2^2 + \ldots + x_n^2\,. \tag{3.4.8}$$

Consider the integral

$$I_n = \int_{\mathbb{R}^n} dx_1 dx_2 \ldots dx_n\, e^{-r^2}\,, \tag{3.4.9}$$

which we will evaluate in two different ways. First we proceed directly. Using (3.4.8) in e^{-r^2} the integral becomes a product of n gaussian integrals:

$$I_n = \prod_{i=1}^{n} \int_{-\infty}^{\infty} dx_i \, e^{-x_i^2} = (\sqrt{\pi})^n = \pi^{n/2} \,. \tag{3.4.10}$$

Now we proceed indirectly. We do the integral using shells between r and $r + dr$. Since the spaces of constant r are spheres $S^{n-1}(r)$, the volume of a shell equals the volume of $S^{n-1}(r)$ times dr. Therefore,

$$\begin{aligned} I_n &= \int_0^{\infty} dr \, \mathrm{vol}(S^{n-1}(r)) \, e^{-r^2} = \mathrm{vol}\,(S^{n-1}) \int_0^{\infty} dr \, r^{n-1} \, e^{-r^2} \\ &= \frac{1}{2} \mathrm{vol}\,(S^{n-1}) \int_0^{\infty} dt \, t^{\frac{n}{2}-1} \, e^{-t} \,, \end{aligned} \tag{3.4.11}$$

where use was made of (3.4.6), and in the final step we changed the variable of integration to $t = r^2$. The final integral on the right hand side defines a very useful special function: the gamma function. For positive x, the gamma function $\Gamma(x)$ is defined as

$$\Gamma(x) = \int_0^{\infty} dt \, e^{-t} t^{x-1} \,, \qquad x > 0 \,. \tag{3.4.12}$$

Unless $x > 0$, the integral is not convergent near $t = 0$. Using (3.4.12), the value of I_n calculated in (3.4.11) becomes

$$I_n = \frac{1}{2} \mathrm{vol}\,(S^{n-1}) \, \Gamma\big(n/2\big) \,. \tag{3.4.13}$$

Finally, using the earlier evaluation (3.4.10), we get our final result:

$$\boxed{\mathrm{vol}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \,.} \tag{3.4.14}$$

To be really done, however, we must calculate the value of $\Gamma(n/2)$. To find $\Gamma(1/2)$, we use the definition (3.4.12), and let $t = u^2$:

$$\Gamma(1/2) = \int_0^{\infty} dt \, t^{-1/2} \, e^{-t} = 2 \int_0^{\infty} du e^{-u^2} = \sqrt{\pi} \,. \tag{3.4.15}$$

Similarly,

$$\Gamma(1) = \int_0^\infty dt\, e^{-t} = 1\,. \tag{3.4.16}$$

The gamma function satisfies a recursion relation. Consider

$$\Gamma(x+1) = \int_0^\infty dt\, e^{-t} t^x\,, \qquad x > 0\,, \tag{3.4.17}$$

which can be rewritten as

$$\Gamma(x+1) = -\int_0^\infty dt\, \Big(\frac{d}{dt} e^{-t}\Big)\, t^x = -\int_0^\infty dt\, \Big[\frac{d}{dt}(e^{-t}t^x) - x e^{-t} t^{x-1}\Big]\,. \tag{3.4.18}$$

The boundary terms vanish for $x > 0$, and we find

$$\Gamma(x+1) = x\,\Gamma(x)\,, \quad x > 0\,. \tag{3.4.19}$$

Using this property we find, for example, that

$$\Gamma(3/2) = \frac{1}{2}\cdot\Gamma(1/2) = \frac{\sqrt{\pi}}{2}\,, \quad \Gamma(5/2) = \frac{3}{2}\cdot\Gamma(3/2) = \frac{3\sqrt{\pi}}{4}\,.$$

For integer arguments, we simply have the factorial function:

$$\Gamma(5) = 4\cdot\Gamma(4) = 4\cdot 3\cdot\Gamma(3) = 4\cdot 3\cdot 2\cdot\Gamma(2) = 4\cdot 3\cdot 2\cdot 1\cdot\Gamma(1) = 4!\,.$$

Therefore, for $n \in \mathbb{Z}$ and $n \geq 1$, we have

$$\Gamma(n) = (n-1)!\,, \tag{3.4.20}$$

where we recall that $0! = 1$. We can now test our volume formula (3.4.14) in the familiar cases:

$$\begin{aligned} \mathrm{vol}\,(S^1) &= \mathrm{vol}\,(S^{2-1}) = \frac{2\pi}{\Gamma(1)} = 2\pi\,, \\ \mathrm{vol}\,(S^2) &= \mathrm{vol}\,(S^{3-1}) = \frac{2\pi^{3/2}}{\Gamma(3/2)} = 4\pi\,, \end{aligned} \tag{3.4.21}$$

in agreement with the known values. Finally, for the less familiar S^3

$$\mathrm{vol}\,(S^3) = \mathrm{vol}\,(S^{4-1}) = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2\,. \tag{3.4.22}$$

Quick Calculation 3.4. Show that vol $(B^n) = \pi^{n/2}/\Gamma(1+\frac{n}{2})$.

3.5 Electric fields in higher dimensions

Let us now calculate the electric field due to a point charge in a world with an arbitrary but fixed number of spatial dimensions. For this purpose, we consider the zero-th component of equation (3.3.23):

$$\frac{\partial}{\partial x^i} F^{0i} = 4\pi\rho\,. \tag{3.5.1}$$

Since $F^{0i} = E_i$ (see (3.3.26)), this equation is just Gauss's law:

$$\nabla \cdot \vec{E} = 4\pi\rho\,. \tag{3.5.2}$$

Gauss's law is valid in all dimensions! Equation (3.5.2) can be used to determine the electric field of a point charge. Let us review the procedure in the familiar setting of three spatial dimensions.

Consider a point charge Q, a two-sphere $S^2(r)$ of radius r centered on the charge, and the three-ball $B^3(r)$ whose boundary is the two-sphere. We integrate both sides of equation (3.5.2) over the three ball to find

$$\int_{B^3} d(\mathrm{vol}) \nabla \cdot \vec{E} = 4\pi \int_{B^3} d(\mathrm{vol})\, \rho\ . \tag{3.5.3}$$

We use the divergence theorem on the left hand side, and note that the volume integral on the right hand side gives the total charge:

$$\int_{S^2(r)} \vec{E} \cdot \vec{da} = 4\pi Q\,. \tag{3.5.4}$$

Since the magnitude $E(r)$ of $\vec{E}$ is constant over the two-sphere, we get

$$\mathrm{vol}(S^2(r))\, E(r) = 4\pi Q\,. \tag{3.5.5}$$

The volume of the two-sphere is just its area $4\pi r^2$, and therefore

$$E(r) = \frac{Q}{r^2}\,. \tag{3.5.6}$$

This is the familiar result for the electric field of a point charge in our four dimensional spacetime. The electric field magnitude falls off like r^{-2}.

How can we generalize this result to higher dimensions? The starting point (3.5.2) is good in higher dimensions, so we must ask if the divergence theorem holds in higher dimensions. It turns out that it does. We will first state the theorem in general, and then give some justification.

Consider an n-dimensional subspace V^n of $\mathbb{R}^n$ and let ∂V^n denote its boundary. Moreover, let $\vec{E}$ be a vector field in $\mathbb{R}^n$. Then,

$$\int_{V^n} d(\text{vol})\, \nabla \cdot \vec{E} = \text{Flux of } \vec{E} \text{ across } \partial V^n = \int_{\partial V^n} \vec{E} \cdot d\vec{v}\,. \tag{3.5.7}$$

The last right-hand side requires some explanation. At any point on ∂V^n, the space ∂V^n is locally approximated by the $(n-1)$-dimensional tangent hyperplane. For a small piece of ∂V^n around this point, the associated vector $d\vec{v}$ is a vector orthogonal to the hyperplane, pointing out of the volume, and with magnitude equal to the volume of the small piece under consideration. You should be able to see that this explanation is in accord with your experience in $\mathbb{R}^3$, where $d\vec{v}$ goes by the name $d\vec{a}$, for area vector element.

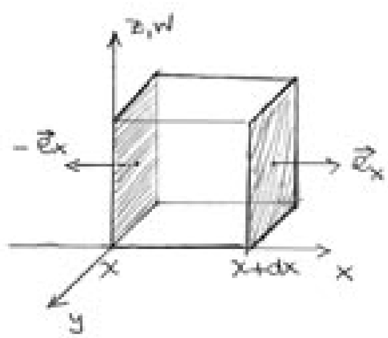

Figure 3.1: An attempt at a representation of a four-dimensional hypercube. The two faces of constant x are shown shaded, with their outgoing normal vectors.

Let us justify the divergence theorem by considering the case of four space dimensions. Following a strategy used in elementary textbooks, it suffices to prove the divergence theorem for a small hypercube – the result for general subspaces follows by breaking such spaces into many small hypercubes. Because it is not easy to imagine a four-dimensional hypercube, we might as well use a three-dimensional picture with four-dimensional labels, as we do in Figure 3.1. We use cartesian coordinates x, y, z, w, and consider a cube whose faces lie on hyperplanes selected by the condition that one of the coordinates is constant. Let one face of the cube, and the face opposite to it, lie on hyperplanes of constant x, and constant $x + dx$, respectively. The outgoing normal vectors are $\vec{e}_x$ for the face at $x + dx$, and $(-\vec{e}_x)$ for the face at x. The volume of each of these two faces equals $dydzdw$, where dy, dz, and dw, together with dx are the lengths of the edges of the cube. For an arbitrary electric field $\vec{E}(x, y, z, w)$, only the x-component contributes to the flux through these two faces. The contribution is

$$[\, E_x(x + dx, y, z, w) - E_x(x, y, z, w)\,]\, dydzdw \simeq \frac{\partial E_x}{\partial x}\, dxdydzdw\,. \tag{3.5.8}$$

Analogous expressions hold for the flux across the three other pairs of faces. The total net flux from the little cube is just

$$\begin{aligned} \text{net flux of } \vec{E} &= \Big(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} + \frac{\partial E_w}{\partial w}\Big)\, dxdydzdw\,, \\ &= \nabla \cdot \vec{E}\, d(\text{vol})\,. \end{aligned} \tag{3.5.9}$$

This result is precisely the divergence theorem stated in (3.5.7) for the case of an infinitesimal hypercube. This is what we wanted to show.

We can now return to the computation of the electric field due to a point charge in a world with n spatial dimensions. Consider a point charge Q, the sphere $S^{n-1}(r)$ of radius r centered on the charge – this is the sphere that surrounds the charge, and the ball $B^n(r)$ whose boundary is the sphere $S^{n-1}(r)$. Again, we integrate both sides of equation (3.5.2) over the ball $B^n(r)$:

$$\int_{B^n} d(\text{vol}) \nabla \cdot \vec{E} = 4\pi \int_{B^n} d(\text{vol})\, \rho\,. \tag{3.5.10}$$

We use the divergence theorem (3.5.7) for the left-hand side, and, as before, the volume integral on the right-hand side gives the total charge:

$$\text{Flux of } \vec{E} \text{ across } S^{n-1}(r) = 4\pi Q\,. \tag{3.5.11}$$

The left hand side is given by the magnitude of the electric field times the volume of the sphere $S^{n-1}(r)$:

$$E(r) \text{vol}\,(S^{n-1}(r)) = 4\pi Q\,. \tag{3.5.12}$$

Making use of (3.4.14) we find

$$E(r) = \frac{2\Gamma(n/2)}{\pi^{\frac{n}{2}-1}} \frac{Q}{r^{n-1}} \,. \tag{3.5.13}$$

For three spatial dimensions $n = 3$, and we recover the inverse squared dependence of the electric field. In higher dimensions the electric field falls off faster at large distances. Near the charge, however, they blow up faster. Finally, we should note that the factor 4π in the right hand side of Gauss' law (3.5.2) was clearly tailored to give a simple expression for the electric field in our physical world (see (3.5.6)). If we lived in four spatial dimensions, we would have certainly used some other constant.

Quick Calculation 3.5. The force $\vec{F}$ on a test charge Q in an electric field $\vec{E}$ is $\vec{F} = Q\vec{E}$. What are the units of charge in various dimensions?

Electrostatic potentials are also of interest. For time independent fields, we have from (3.3.27)

$$\vec{E} = -\nabla\Phi \,, \tag{3.5.14}$$

where Φ is the electrostatic scalar potential. This equation, together with Gauss's law, gives us the Poisson equation:

$$\nabla^2\Phi = -4\pi\rho, \tag{3.5.15}$$

which we can use to calculate the potential due to a charge distribution. The equations above hold in all dimensions using, of course the appropriate definition of the gradient and of the Laplacian.

3.6 Gravitation and Planck's length

Gravitation is described by Einstein's theory of general relativity. This is a very elegant theory where the dynamical variables encode the geometry of spacetime. In here I would like to give you some sense about the concepts involved in this remarkable theory.

When gravitational fields are sufficiently weak and velocities are small, Newtonian gravitation is accurate enough, and one need not work with the more complex machinery of general relativity. We can use Newtonian gravity to understand the definition of Planck's length in various dimensions and its relation to the gravitational constant. These are interesting issues that we will explain here and in the rest of the present chapter. Nevertheless, when gravitation emerges in string theory, it does so in the language of Einstein's theory of general relativity. To be able to recognize the appearance of gravity among the quantum vibrations of the relativistic string you need a little familiarity with the ideas of general relativity.

Most physicists do not expect general relativity to hold at truly small distances nor for extremely large gravitational fields. This is a realm where string theory, the first serious candidate for a quantum theory of gravitation, is necessary. General relativity is the large-distance/weak-gravity limit of string theory. String theory *modifies* general relativity; it must do so to make it consistent with quantum mechanics. The conceptual framework underlying the modifications of general relativity in string theory is not clear yet. It will no doubt emerge as we understand string theory better in the years to come.

The spacetime of special relativity, Minkowski spacetime, is the arena for physics in the *absence* of gravitational fields. The geometrical properties of Minkowski spacetime are encoded by the metric formula (2.2.15), giving the invariant interval separating two nearby events:

$$-ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \,. \tag{3.6.1}$$

Here the Minkowski metric $\eta_{\mu\nu}$ is a constant metric, represented as a matrix with entries $(-1, 1, ..., 1)$ along the diagonal. In the presence of a gravitational field, the metric becomes dynamical! We then write

$$\boxed{-ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \,,} \tag{3.6.2}$$

where the constant $\eta_{\mu\nu}$ is replaced by the metric $g_{\mu\nu}(x)$. If there is a gravitational field, the metric is a function of the space-time coordinates. The metric $g_{\mu\nu}$ is defined to be symmetric

$$g_{\mu\nu}(x) = g_{\nu\mu}(x) \,. \tag{3.6.3}$$

It is also customary to define $g^{\mu\nu}(x)$ as the inverse matrix of $g_{\mu\nu}(x)$:

$$g^{\mu\alpha}(x)\, g_{\alpha\nu}(x) = \delta^\mu_\nu \,. \tag{3.6.4}$$

In many physical phenomena gravity is very weak, and the metric $g_{\mu\nu}(x)$ can be chosen to be very close to the Minkowski metric $\eta_{\mu\nu}$. We then write,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \,, \tag{3.6.5}$$

and we view $h_{\mu\nu}(x)$ as a small fluctuation around the Minkowski metric. This expansion is done, for example, to study gravity waves. Those waves represent small "ripples" on top of the Minkowski metric. Einstein's equations for the gravitational field are written in terms of the spacetime metric $g_{\mu\nu}(x)$. These equations imply that matter or energy sources curve the spacetime manifold. For weak gravitational fields, Einstein's equations can be expanded in terms of $h_{\mu\nu}$ using (3.6.5). To linear order in $h_{\mu\nu}$, and in the absence of sources, the resulting equation is

$$\partial^2 h^{\mu\nu} - \partial_\alpha(\partial^\mu h^{\nu\alpha} + \partial^\nu h^{\mu\alpha}) + \partial^\mu\partial^\nu h = 0\,. \tag{3.6.6}$$

In here $\partial^2 = \partial^\mu\partial_\mu$, and $h \equiv \eta^{\mu\nu}h_{\mu\nu} = -h_{00} + h_{11} + h_{22} + h_{33}$. Equation (3.6.6) is the gravitational analog of equation (3.3.24) for Maxwell fields in the absence of sources. While (3.3.24) is exact, the gravitational equation (3.6.6) is only valid for weak gravitational fields.

Just as in Maxwell theory, in Einstein's gravity there are also gauge transformations. They state that the use of different systems of spacetime coordinates yield equivalent descriptions of gravitational physics. In learning string theory in this book you will get to appreciate the possibilities that arise when you have the freedom to use different sets of coordinates on the surfaces generated by the motion of strings. In general relativity, an infinitesimal change of coordinates

$$x^{\mu\prime} = x^\mu + \epsilon^\mu(x)\,, \tag{3.6.7}$$

can be viewed as an infinitesimal change of the metric $g_{\mu\nu}$, and using (3.6.5), as an infinitesimal change of the fluctuating field $h^{\mu\nu}$. One can show that the change is given as

$$\delta h^{\mu\nu} = \partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu\,. \tag{3.6.8}$$

The linearized equation of motion (3.6.6) is invariant under the gauge transformation (3.6.8). We will check this explicitly in Chapter 10. In Maxwell theory the gauge parameter ϵ has no indices, but in general relativity the gauge parameter has a vector index.

As we mentioned before, Newtonian gravitation emerges from general relativity when describing weak gravitational fields and motion with small velocities. For many purposes Newtonian gravity suffices. Starting now, and for the rest of this chapter, we will use Newtonian gravity to understand the definition of Planck's length in various dimensions, and to investigate how gravitational constants behave when some spatial dimensions are curled up. The results we will obtain hold also in the full theory of general relativity.

Newton's law of gravitation in four-dimensions states that the force of attraction between two masses m_1 and m_2 separated a distance r is given by

$$|\vec{F}^{(4)}| = \frac{G m_1 m_2}{r^2}\,, \tag{3.6.9}$$

where G denotes the four-dimensional Newton constant. As we shall see, this force law is not valid in dimensions other than four. The numerical value for the constant G is experimentally determined:

$$G = 6.67 \times 10^{-11} \frac{\mathrm{N} \cdot \mathrm{m}^2}{(\mathrm{kg})^2}\,. \tag{3.6.10}$$

It follows that the units of the gravitational constant G are

$$[G] = [\mathrm{Force}]\frac{L^2}{M^2} = \frac{ML}{T^2}\frac{L^2}{M^2} = \frac{L^3}{MT^2}\,, \tag{3.6.11}$$

allowing us to write

$$G = 6.67 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2}\,. \tag{3.6.12}$$

In addition, we recall that

$$[c] = \frac{L}{T}\,, \quad [\hbar] = \frac{ML^2}{T}\,. \tag{3.6.13}$$

It is convenient to use a "natural" system of units for the study of gravitation. Since we have three basic units, those of length, mass, and time, we can find new units of length, mass, and time such that the three fundamental constants, $c, \hbar$, and G take the *numerical value of one* in those units. These units are called the Planck length ℓ_P, the Planck time t_P, and the Planck mass m_P, respectively. In those units

$$G = 1 \cdot \frac{\ell_P^3}{m_P t_P^2}\,, \quad c = 1 \cdot \frac{\ell_P}{t_P}\,, \quad \hbar = 1 \cdot \frac{m_P \ell_P^2}{t_P}\,, \tag{3.6.14}$$

without additional numerical constants – as opposed to equation (3.6.12), for example. The above equations allow us to solve for ℓ_P, m_P and t_P in terms of $G, \hbar$ and c. One readily finds

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 1.61 \times 10^{-33}\text{cm}\,, \tag{3.6.15}$$

$$t_P = \frac{\ell_P}{c} = \sqrt{\frac{\hbar G}{c^5}} = 5.4 \times 10^{-44}\text{s}\,, \tag{3.6.16}$$

$$m_P = \sqrt{\frac{\hbar c}{G}} = 2.17 \times 10^{-5}\text{gm}\,. \tag{3.6.17}$$

These numbers represent scales at which relativistic quantum gravity effects can be important. Indeed, the Planck length is an extremely small length, and the Planck time is an incredibly short time – the time it takes light to travel the Planck length! Einstein's gravity can be used down to relatively small distances, and back to relatively early times in the history of the universe, but, to study gravity at the Planck length, or to investigate the universe when it was Planck-time old, a quantum gravity theory such as string theory is needed.

There is an equivalent way to define the Planck length: it is the unique length constructed using only powers of G, c, and $\hbar$. One thus sets

$$\ell_P = (G)^\alpha\, (c)^\beta\, (\hbar)^\gamma\,, \tag{3.6.18}$$

and fixes the constants α, β and γ so that the right-hand side has units of length.

Quick Calculation 3.6. Show that this condition fixes uniquely $\alpha = \gamma = 1/2$, and $\beta = -3/2$, thus reproducing the result in (3.6.15).

It may appear that m_P is not a spectacularly large mass, but it is very large from the viewpoint of elementary particle physics. Indeed, consider the following question: What should be the mass M of the proton so that the gravitational force between two protons cancels the electric repulsion force between them? Equating the magnitudes of the electric and gravitational forces we get

$$\frac{GM^2}{r^2} = \frac{e^2}{r^2} \quad \rightarrow \quad GM^2 = e^2\,. \tag{3.6.19}$$

It is convenient to divide both sides of the equation by $\hbar c$ to find

$$\frac{GM^2}{\hbar c} = \frac{e^2}{\hbar c} \simeq \frac{1}{137} \quad \rightarrow \quad \frac{M^2}{m_P^2} \simeq \frac{1}{137}\,, \tag{3.6.20}$$

where use was made of (3.6.17). We thus find $M \simeq m_P/12$, or about one-tenth of the Planck mass. This is indeed a very large mass, about 10^{18} times the true mass of the proton.

3.7 Gravitational potentials

We want to learn what happens with the gravitational constants when we change the dimensionality of spacetime. To find out, we can use Newtonian gravity, and work with gravitational potentials. Our immediate aim is to find the equation relating the gravitational potential to the mass density in arbitrary number of dimensions. In the following section, this result will be used to define the Planck length in any number of dimensions.

We define a gravity field $\vec{g}$ with units of force per unit mass, just as the electric field has units of force per unit charge. The force on a given test mass m at a point where the gravity field is $\vec{g}$ is given by $m\vec{g}$. We set this $\vec{g}$ equal to minus the gradient of the gravitational potential V_g:

$$\vec{g} = -\nabla V_g \, . \tag{3.7.1}$$

We will take this equation to be true in all dimensions. Equation (3.7.1) has content: if you move a particle along a closed loop in a static gravitational field, the net work you do against the gravitational field is zero.

Quick Calculation 3.7. Prove the above statement.

What are the units for the gravitational potential? Equation (3.7.1) gives

$$\frac{[\text{Force}]}{M} = \frac{[V_g]}{L} \Rightarrow [V_g] = \frac{\text{Energy}}{M} \, . \tag{3.7.2}$$

The gravitational potential has units of energy per unit mass in *any dimension*. The gravitational potential $V_g^{(4)}$ of a point mass in four dimensions is

$$V_g^{(4)} = -\frac{GM}{r} \, . \tag{3.7.3}$$

We can use the electromagnetic analogy to find the equation satisfied by the gravitational potential. In electromagnetism, we found an equation for the electric potential which holds in any dimension. It was given in (3.5.15):

$$\nabla^2 \Phi = -4\pi\rho \, . \tag{3.7.4}$$

The four-dimensional scalar potential for a point charge is

$$\Phi^{(4)} = \frac{Q}{r}\,, \tag{3.7.5}$$

and satisfies (3.7.4). It follows by analogy that the four-dimensional gravitational potential in (3.7.3) satisfies

$$\nabla^2 V_g^{(4)} = 4\pi G\, \rho_m\,, \tag{3.7.6}$$

where ρ_m is the matter density. While this equation is correct in four dimensions, a small modification is needed for other dimensions. Note that the left-hand side has the same units in any number of dimensions: the units of V_g are always the same, and the laplacian always divides by length squared. The right-hand side must also have the same units in any number of dimensions. Since ρ_m is a mass density, it has different units in different dimensions, and as a consequence the units of G must change when the dimensions change. We therefore rewrite the above equation more precisely as

$$\nabla^2 V_g^{(n)} = 4\pi G^{(n)} \rho_m\,, \tag{3.7.7}$$

when working in n-dimensional spacetime. In particular, we identify $G^{(4)}$ as the four-dimensional Newton constant G. Equation (3.7.7) defines Newtonian gravitation in arbitrary number of dimensions.

3.8 The Planck length in various dimensions

We can define the Planck length in any dimension just as we did in four dimensions: the Planck length is the length built using only powers of the gravitational constant $G^{(n)}$, c and $\hbar$. To compute the Planck length we must determine the units of $G^{(n)}$. Let us therefore look at (3.7.7) and recall that the units of the right-hand side must be the same in all dimensions. Comparing the cases of five and four dimensions, for example,

$$[G^{(5)}]\frac{M}{L^4} = [G]\frac{M}{L^3} \quad \rightarrow \quad [G^{(5)}] = L[G]\,. \tag{3.8.1}$$

The units of $G^{(5)}$ have one more factor of length than those of G. Now we can calculate the Planck length in five dimensions. First, we use (3.6.15) to read the units of G in terms of units of length and units of c and $\hbar$:

$$[G] = \frac{[c]^3\, L^2}{[\hbar]}\,. \tag{3.8.2}$$

Using (3.8.1), we get

$$[G^{(5)}] = \frac{[c]^3\, L^3}{[\hbar]}\,. \tag{3.8.3}$$

Since the Planck length is constructed uniquely from the gravitational constant, c, and $\hbar$, we can remove the brackets in the above equation and replace L by the five-dimensional Planck length:

$$(\ell_P^{(5)})^3 = \frac{\hbar G^{(5)}}{c^3}\,. \tag{3.8.4}$$

Reintroducing the four-dimensional Planck length:

$$(\ell_P^{(5)})^3 = \Big(\frac{\hbar G}{c^3}\Big)\frac{G^{(5)}}{G} \quad \to \quad (\ell_P^{(5)})^3 = (\ell_P)^2\frac{G^{(5)}}{G}\,. \tag{3.8.5}$$

Since they do not have the same units, it does not make sense to compare directly the gravitational constants in four and five dimensions. Planck lengths, however, can be compared. If the Planck length in four and five dimensions are the same, $G^{(5)}/G = \ell_P$. In this case the gravitational constants differ by one factor of the common Planck length.

It is not hard to generalize the above equations to n spacetime dimensions:

Quick Calculation 3.8. Show that (3.8.4) and (3.8.5) are replaced by

$$\boxed{\left(\ell_P^{(n)}\right)^{n-2} = \frac{\hbar G^{(n)}}{c^3} = (\ell_P)^2\frac{G^{(n)}}{G}\,.} \tag{3.8.6}$$

3.9 Gravitational constants and compactification

If string theory is correct, our world is really higher dimensional. The fundamental gravity theory is defined in this higher-dimensional world, with some value for the higher-dimensional Planck length. Since we observe only four dimensions, the additional dimensions may be curled up to form a compact space with small volume. We can then ask: what is the effective value of the four-dimensional Planck length? As we shall show here, the effective four-dimensional Planck length depends on the volume of the extra dimensions, as well as on the value of the higher-dimensional Planck length.

These observations raise the possibility that the Planck length in the effectively four-dimensional world – the famous number equal to about 10^{-33}cm – may not coincide with the fundamental Planck length in the original higher-dimensional theory. Is it possible that the fundamental Planck length is much bigger than the familiar, four-dimensional one? We will answer this question in the following section. Here we will now work out the effect of compactification on gravitational constants.

How do we calculate the gravitational constant in four dimensions if we are given the gravitational constant in five? First, we recognize the need to curl up one spatial dimension, otherwise there is no effectively four-dimensional spacetime. As we will see, the size of the extra dimension will enter into the relationship between the gravitational constants. To explore these questions precisely, consider a five-dimensional spacetime where one dimension forms a small circle of radius R. We are given $G^{(5)}$ and would like to calculate $G^{(4)}$.

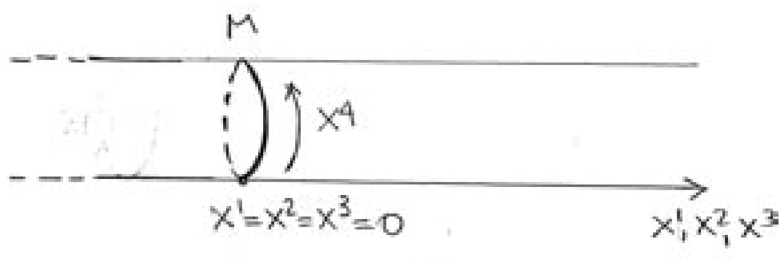

Figure 3.2: A world with four space dimensions, one of which, x^4, is compactified into a circle of radius R. A ring of total mass M wraps around this compact dimension.

Let (x^1, x^2, x^3) denote three spatial dimensions of infinite extent, and x^4 denote a compactified dimension of circumference $2\pi R$ (see Figure 3.2). We place a uniform ring of total mass M all around the circle at $x^1 = x^2 = x^3 = 0$. This is a constant mass distribution along the x^4 dimension. We are interested in the gravitational potential that emerges from such a mass distribution. We could have alternatively placed a point mass at some fixed x^4, but this makes the calculations more involved (see Problem 3.8). In the present case the gravitational potential cannot depend on x^4. The total mass M can be written as

$$\text{Total Mass} = M = 2\pi R m \,, \tag{3.9.1}$$

where m is the mass per unit length.

What is the mass density in the five-dimensional world? It is only nonzero at $x^1 = x^2 = x^3 = 0$. To represent such mass density we must use delta functions. Recall that the delta function $\delta(x)$ can be viewed as a singular function whose value is zero except for $x = 0$ and such that the integral $\int_{-\infty}^{\infty} dx\delta(x) = 1$. This integral implies that if x has units of length, then $\delta(x)$ has units of inverse length. Since the five-dimensional mass density is concentrated at $x^1 = x^2 = x^3 = 0$, it is reasonable to include in its formula the product $\delta(x^1)\delta(x^2)\delta(x^3)$ of three delta functions. We claim that

$$\rho^{(5)} = m\delta(x^1)\delta(x^2)\delta(x^3) \,. \tag{3.9.2}$$

We first check the units. The mass density $\rho^{(5)}$ must have units of M/L^4. This works out since m has units of mass per unit length, and the three delta functions supply an additional factor of L^{-3}. The ansatz in (3.9.2) could still be off by a constant dimensionless factor, a factor of two, for example. As a final check, we integrate $\rho^{(5)}$ all over space. The result should be the total mass:

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx^1 dx^2 dx^3 \int_0^{2\pi R} dx^4 \rho^{(5)} \\
&= m \int_{-\infty}^{\infty} dx^1 \delta(x^1) \int_{-\infty}^{\infty} dx^2 \delta(x^2) \int_{-\infty}^{\infty} dx^3 \delta(x^3) \int_0^{2\pi R} dx^4 \\
&= m\, 2\pi R \,.
\end{aligned} \tag{3.9.3}$$

This is indeed the total mass on account of (3.9.1). To the effectively four-dimensional observer, the correct expression for the mass density would be

$$\rho^{(4)} = M\delta(x^1)\delta(x^2)\delta(x^3) \,. \tag{3.9.4}$$

For this observer, the mass is point-like and it is located at $x^1 = x^2 = x^3 = 0$. Note the relation

$$\rho^{(5)} = \frac{1}{2\pi R}\rho^{(4)} \,. \tag{3.9.5}$$

Let us now use this information in the equations for the gravitational potential. In five-dimensional spacetime equation (3.7.7) gives

$$\nabla^2_{(5)} V_g^{(5)}(x^1, x^2, x^3, x^4) = 4\pi G^{(5)} \rho^{(5)} = \frac{4\pi G^{(5)}}{2\pi R} \rho^{(4)} \,, \tag{3.9.6}$$

where we also used (3.9.5). Since the mass distribution is x^4-independent, the potential $V_g^{(5)}$ is also x^4-independent, and the laplacian becomes the four-dimensional one. We then get

$$\nabla^2_{(4)} V_g^{(5)}(x^1, x^2, x^3) = 4\pi \, \frac{G^{(5)}}{2\pi R} \, \rho^{(4)} \,. \tag{3.9.7}$$

The effective four-dimensional gravitational potential is the five-dimensional one, and the above equation has taken the form of the gravitational equation in four dimensions, where the constant in-between the 4π and the $\rho^{(4)}$ is the four-dimensional gravitational constant. We have therefore shown that

$$G = \frac{G^{(5)}}{2\pi R} \quad \to \quad \boxed{\frac{G^{(5)}}{G} = 2\pi R \equiv \ell_C \,,} \tag{3.9.8}$$

where ℓ_C is the length of the extra compact dimension. This is what we were seeking: a relationship between the strength of the gravitational constants in different dimensions in terms of the size of the extra dimensions.

The generalization of (3.9.8) to the case where there is more than one extra dimension is straightforward. One finds that

$$\frac{G^{(n)}}{G} = (\ell_C)^{n-4} \,, \tag{3.9.9}$$

where ℓ_C is the common length of each of the extra dimensions. When the various dimensions are curled up into circles of different lengths, the above right-hand side must be replaced by the product of the various lengths. This product is, in fact, the volume of the extra dimensions.

3.10 Large extra dimensions

We are now done with all the groundwork. In section 3.8 we found the relation between the Planck length and the gravitational constant in any dimension.

In section 3.9 we determined how gravitational constants are related upon compactification. We are ready to find out how the fundamental Planck length in a higher dimensional theory with compactification is related to the Planck length in the effectively four-dimensional theory.

Consider therefore a five-dimensional world, with Planck length $\ell_P^{(5)}$ and a single spatial coordinate curled up into a circle of circumference ℓ_C. What is then ℓ_P? From (3.8.5) and (3.9.8) we find that

$$(\ell_P^{(5)})^3 = (\ell_P)^2 \, \frac{G^{(5)}}{G} = (\ell_P)^2 \, \ell_C \, . \tag{3.10.1}$$

Solving for ℓ_P, we get

$$\ell_P = \ell_P^{(5)} \sqrt{\frac{\ell_P^{(5)}}{\ell_C}} \, . \tag{3.10.2}$$

This relation enables us to explore the possibility that the world is actually five-dimensional with a fundamental Planck length $\ell_P^{(5)}$ that is much larger than 10^{-33} cm. Of course, we must have $\ell_P \sim 10^{-33}$ cm. This is, after all the four-dimensional Planck length, whose value is given in (3.6.15). What would ℓ_C have to be for us to have an experimentally accessible $\ell_P^{(5)} \sim 10^{-18}$ cm? Substituting $\ell_P^{(5)} \sim 10^{-18}$ cm and $\ell_P \sim 10^{-33}$ cm into equation (3.10.2), we find $\ell_C \sim 10^{12}$ cm $\sim 10^7$ km. This is a more than twenty times the distance from the earth to the moon. Such a large extra dimension would have been detected long ago.

Having failed in five dimensions, let us try *six* space-time dimensions. Generally, equation (3.8.6), together with (3.9.9) gives

$$\left(\ell_P^{(n)}\right)^{n-2} = (\ell_P)^2 \frac{G^{(n)}}{G} = (\ell_P)^2 (\ell_C)^{n-4} \, . \tag{3.10.3}$$

After rearranging terms, we find that

$$\ell_P = \ell_P^{(n)} \left(\frac{\ell_P^{(n)}}{\ell_C}\right)^{\frac{n}{2}-2} . \tag{3.10.4}$$

Let's assume that $\ell_P^{(6)} \sim 10^{-18}$ cm. With $n = 6$, equation (3.10.4) gives

$$10^{-33} \sim 10^{-18} \Big(\frac{\ell_P^{(6)}}{\ell_C}\Big) \, , \tag{3.10.5}$$

which yields

$$\ell_C \sim 10^{-3}\text{cm}\,. \tag{3.10.6}$$

This is a lot more interesting! Could there be extra dimensions 10^{-3} cm long? You might think that this is still too big, since even microscopes probe much smaller distances. In fact, accelerators probe distances of the order of 10^{-18}cm. Surprisingly, it is possible that these “large extra dimensions” exist, and that we have not observed them yet.

How would we confirm the existance of additional dimensions? By testing the force law giving the gravitational attraction between two masses. For distances much larger than the compactification scale ℓ_C the world is effectively four-dimensional, and the force between two masses must follow accurately Newton's law of inverse squared dependence on the separation. On the other hand, for distances smaller than ℓ_C, the world is effectively higher-dimensional, and the force law must change. If we detect that for small distances the force between masses goes like r^{-4}, where r is the separation, then this is consistent with the existance of two compact extra dimensions. Tests of gravity at small distances are very difficult to do. Gravity has not been tested accurately enough below a millimeter. So there is very little evidence that at short distance scales, gravity obeys an inverse squared law.

You might ask: what about forces other than gravity? Electromagnetism has been tested to much smaller distances, and we do know that the electric force obeys an inverse squared law very accurately. This seems to rule out large extra dimensions. The possibility of large extra dimensions, however, exists in string theory, where our spatial world could be a three-dimensional hyperplane transverse to the extra dimension. This hyperplane is called a D3-brane, a D-brane with three spatial dimensions.

Open strings have the remarkable property that their endpoints are stuck to the D-branes. In many phenomenological models of string theory the fluctuations of open strings give rise to the familiar leptons, quarks, and gauge fields, including the Maxwell gauge field. It follows that these fields are bound to the D3-brane and do not feel the extra dimensions. Closed strings are not bound by D3-branes, and therefore gravity, which arises from closed strings, *is* affected by the extra dimensions. The appropriate gravitational experiments, however, do not yet rule out extra dimensions if they are smaller than a tenth of a millimeter.

Even though the Planck length ℓ_P is an important length scale in four dimensions, if there are large extra dimensions, the truly fundamental Planck length would be much bigger than the effective four-dimensional one. The possibility of large extra dimensions is slightly unnatural – why should the extra dimensions be much larger than the fundamental length scale? The most natural possibility is that the extra dimensions have the size of the fundamental length scale, in which case the fundamental length scale remains the fundamental one in lower dimensions (see (3.10.4)). Thus many physicists believe it is unlikely that there are large extra dimensions. What is truly striking is that even if they are there, we would have not detected them yet. Physicists are currently doing experiments to find out if large extra dimensions exist.

Problems

Problem 3.1. *Lorentz covariance for motion in EM fields*

The Lorentz force equation (3.1.5) can be written relativistically as

$$\frac{dp^\mu}{ds} = qF^{\mu\nu}u_\nu\,, \tag{1}$$

where u_ν is the four-velocity, and p^μ is the four-momentum. Check explicitly that this equation reproduces (3.1.5) when μ is a spatial index. What does (1) give when $\mu = 0$? Does it make sense? Is equation (1) gauge invariant?

Problem 3.2. *Maxwell equations in four dimensions*

(a) Show explicitly that the two source-free Maxwell equations emerge from the conditions $T_{\mu\lambda\nu} = 0$.

(b) Show explicitly that the two Maxwell equations with sources emerge from (3.3.23).

Problem 3.3. *E&M in three dimensions*

(a) Begin with Maxwell's equations and the force law in four dimensions, and find the reduced equations in three dimensions by using the ansatz (3.2.1) and assuming that no field can depend on the z-direction.

(b) Repeat the analysis of three-dimensional electromagnetism by starting with the Lorentz covariant formulation. Take $A^\mu = (\Phi, A^1, A^2)$, and examine $F_{\mu\nu}$, the Maxwell equations (3.3.23) and the relativistic form of the force law derived in Problem 3.1.

Problem 3.4. *Electric fields and potentials of point charges.*

(a) Show that for time-independent fields, the Maxwell equation $T_{0ij} = 0$ implies that $\partial_i E_j - \partial_j E_i = 0$. Explain why this condition is satisfied by the ansatz $\vec{E} = -\nabla\Phi$.

(b) Show that with n spatial dimensions, the potential Φ due to a point charge Q is given by

$$\Phi(r) = \frac{\Gamma\left(\frac{n}{2}-1\right)}{\pi^{\frac{n}{2}-1}}\frac{Q}{r^{n-2}}\,.$$

Problem 3.5. *Analytic continuation for gamma functions.*

Consider the definition of the gamma function for complex arguments z whose real part is positive:

$$\Gamma(z) = \int_0^\infty dt\, e^{-t} t^{z-1}\,, \quad \Re(z) > 0\,. \tag{1}$$

Use the above equation to show that for $\Re(z) > 0$

$$\Gamma(z) = \int_0^1 dt\, t^{z-1}\Big(e^{-t} - \sum_{n=0}^{N} \frac{(-t)^n}{n!}\Big) + \sum_{n=0}^{N} \frac{(-1)^n}{n!}\frac{1}{z+n} + \int_1^\infty dt\, e^{-t} t^{z-1}\,.$$

Explain why the above right-hand side is well-defined for $\Re(z) > -n-1$. It follows that this right-hand side provides the analytic continuation of $\Gamma(z)$ for $\Re(z) > -n-1$. Conclude that the gamma function has poles at $0, -1, -2, \ldots$, and give the value of the residue at $z = -n$ (with n a positive integer).

Problem 3.6. *Planetary motion in four and higher dimensions.*

Consider the motion of planets in planar orbits around heavy stars in our four dimensional world and in worlds with additional spatial dimensions. We wish to study the stability of these orbits under perturbations that keep them planar. Such a perturbation would arise, for example, if a meteorite moving on the plane of the orbit hits the planet and changes its angular momentum. Show that while planetary orbits in our four dimensional world are stable under such perturbations, they are not so in five or higher dimensions, where upon meteorite impact the planet would either spiral in towards the star, or spiral out to infinity. [Hint: You may find it useful to use the effective potential for motion in a central force field.].

Problem 3.7. *Gravitational field of a point mass in compactified five dimensional world.*

Consider five dimensional space-time with space coordinates (x, y, z, w) *not yet compactified* and consider a point mass of mass M located at the origin $(x, y, z, w) = (0, 0, 0, 0)$.

(a) Find the gravitational potential $V_g^{(5)}(r)$ due to this point mass. Here $r = (x^2 + y^2 + z^2 + w^2)^{1/2}$, and your answer should be in terms of $G^{(5)}$. You may use the equation $\nabla^2 V_g^{(5)} = 4\pi G^{(5)} \rho_M$, and the divergence theorem.

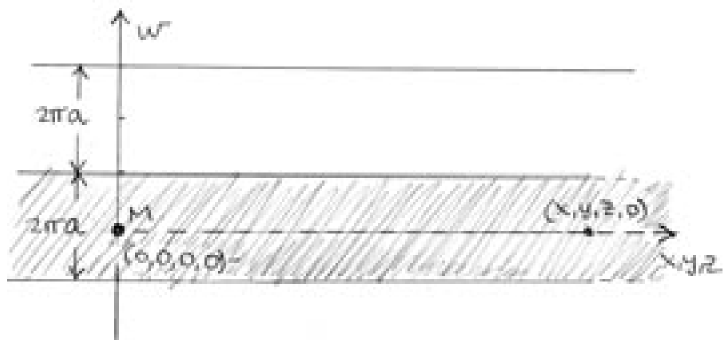

Figure 3.3: Problem 3.8: Point mass M in a five spacetime dimensional world with one extra compact dimension.

Now let w become a circle with radius a keeping the same mass at the same point, as shown in Figure 3.3.

(b) Write an exact expression for the gravitational potential $V_g^{(5)}(x, y, z, 0)$. This potential is in fact a function of $R \equiv (x^2 + y^2 + z^2)^{1/2}$, and can be written as an infinite sum.

(c) Show that for $R >> a$ the above gravitational potential takes the form of a four-dimensional gravitational potential, with Newton's constant $G^{(4)}$ given in terms of $G^{(5)}$ as calculated in class. [Hint: turn the infinite sum into an integral].

These results, confirm both the relation between the four and five dimensional Newton constants for a compactification, and the emergence of a four-dimensional potential at distances large compared to the compactification size.

Problem 3.8. *Exact answer for gravitational potential.*

The infinite sum in Problem 3.8 can be evaluated exactly using the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+(\pi n x)^2} = \frac{1}{x} \coth\Big(\frac{1}{x}\Big) \,.$$

(a) Find an exact expression for the gravitational potential in the compactified theory.

(b) Expand this answer to calculate the leading correction to the gravitational potential in the limit when $a < R$. For what value of R/a is the correction of order 1%?

(c) Use the exact answer in (a) to calculate also the potential when $R < a$, giving the first two terms in the expansion. Do you recognize the leading term?

Chapter 4

Non-Relativistic Strings

A full appreciation for the subtleties of relativistic strings requires an understanding of the basic physics of non-relativistic strings. These strings have mass, tension, and can vibrate both transversely and longitudinally. We study the equations of motion for non-relativistic strings and develop the Lagrangian approach to their dynamics.

4.1 Equations of motion for transverse oscillations

We will begin our study of strings with a look at the transverse fluctuations of a stretched string. The direction along the string is called the longitudinal direction, and the directions orthogonal to the string are called the transverse directions. We consider, for notational simplicity, the case when there is only one transverse direction – the generalization to additional transverse directions is straightforward.

Working in the (x, y) plane, let the classical non-relativistic string have its endpoints fixed at $(0, 0)$, and $(a, 0)$. In the static configuration, the string is stretched along the x-axis between these two points. In a transverse oscillation, the x coordinate of any point on the string does not change in time. The transverse displacement of a point is given by its y-coordinate. The x direction is longitudinal, and the y direction is transverse. To describe the classical mechanics of a homogeneous string, we need two pieces of information: the tension T_0 and the mass per unit length μ_0. The total mass of the string is then $M = \mu_0 a$.

Let us look briefly at the units. Tension has units of force, so

$$[T_0] = [\text{Force}] = \frac{[\text{Energy}]}{L}\,. \tag{4.1.1}$$

If you stretch a string an infinitesimal amount dx, its tension remains approximately constant through the stretching, and the change in energy equals the work done $T_0 dx$. The total mass of the string does not change. If we were considering relativistic strings, however, a static string with more energy would have a larger rest mass. Using (4.1.1), noting that energy has units of mass times velocity squared, and that μ_0 has units of mass per unit length, we have

$$[T_0] = \frac{M}{L}[v]^2 = [\mu_0][v]^2\,. \tag{4.1.2}$$

For a non-relativistic string, both T_0 and μ_0 are adjustable parameters, and the velocity on the right-hand side above will turn out to be the velocity of transverse waves. The above equation suggests that the string tension T_0 and the linear mass density μ_0 in a relativistic string might be related by $T_0 = \mu_0 c^2$, since c is the canonical velocity in relativity. We will see in Chapter 6 that this is indeed the correct relation for a relativistic string.

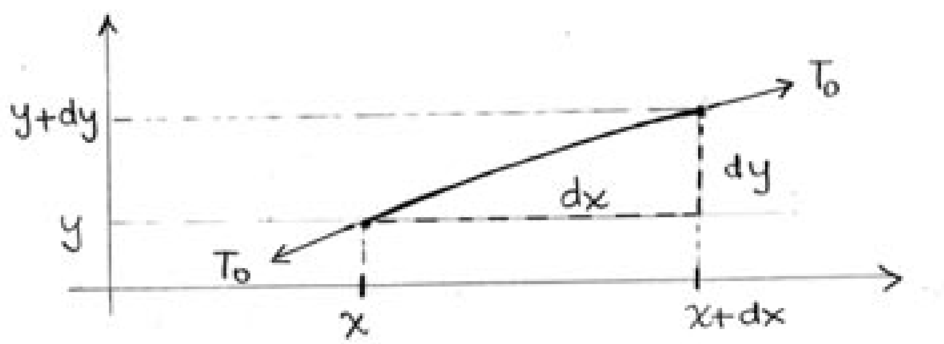

Figure 4.1: A short piece of a classical non-relativistic string vibrating transversally. With different slopes at the two endpoints there is a net vertical force.

Returning to our classical non-relativistic string, let us figure out the equation of motion. Consider a small portion of the string, extending from x to $x + dx$, with $y = 0$, when the string is static. This piece is shown in transverse oscillation in Figure 4.1. At time t, the transverse displacement of the string is $y(t,x)$ at x and $y(t,x+dx)$ at $x+dx$. We will assume that the string oscillations are small, and by this we will mean that at all times

$$\frac{\partial y}{\partial x} << 1\,, \tag{4.1.3}$$

at any point on the string. This guarantees that the transverse displacement of the string is small compared to the length of the string. The length of the string changes little, and we can assume that the tension T_0 is unchanged.

The slope of the string is a bit different at the points x and $x+dx$. This change of slope means that the string tension changes direction and the portion of string under consideration feels a net force. For transverse oscillations we need only calculate the net vertical force – you may show in Problem 4.1 that the horizontal net force is negligible. The vertical force at $(x+dx, y+dy)$ is accurately given by T_0 times $\partial y/\partial x$ evaluated at $x+dx$ and is pointing up; similarly, the vertical force at (x,y) is T_0 times $\partial y/\partial x$ evaluated at x and is pointing down. Therefore the net vertical force dF_v is

$$dF_v = T_0 \frac{\partial y}{\partial x}\Big|_{x+dx} - T_0 \frac{\partial y}{\partial x}\Big|_{x} \simeq T_0 \frac{\partial^2 y}{\partial x^2} dx\,. \tag{4.1.4}$$

The mass dm of this piece of string, originally stretched from x to $x+dx$, is given by the mass density μ_0 times dx. By Newton's law, the net vertical force equals mass times vertical acceleration. So we can simply write

$$T_0 \frac{\partial^2 y}{\partial x^2} dx = (\mu_0 dx)\,\frac{\partial^2 y}{\partial t^2}\,. \tag{4.1.5}$$

We cancel dx on each side and rearrange terms to get

$$\boxed{\frac{\partial^2 y}{\partial x^2} - \frac{\mu_0}{T_0}\frac{\partial^2 y}{\partial t^2} = 0\,.} \tag{4.1.6}$$

This is just a wave equation! Recall that for the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v_0^2}\frac{\partial^2 y}{\partial t^2} = 0\,, \tag{4.1.7}$$

the parameter v_0 is the velocity of the waves. Thus for the transverse waves on our stretched string, the velocity v_0 of the waves is

$$v_0 = \sqrt{T_0/\mu_0}\,. \tag{4.1.8}$$

The higher the tension, or the lighter the string, the faster the waves move.

4.2 Boundary conditions and initial conditions

Since equation (4.1.6) is a partial differential equation involving space and time derivatives, in order to obtain solutions we must in general apply both boundary conditions and initial conditions. Boundary conditions (B.C.) constrain the solution at the boundary of the system, and initial conditions constrain the solution at a fixed starting time. The most common types of boundary conditions are Dirichlet and Neumann boundary conditions.

For our string, Dirichlet boundary conditions specify the positions of the string endpoints. For example, if we attach each end of the string to a wall (see Figure 4.2, left side), we are imposing the Dirichlet boundary conditions

$$y(t, x = 0) = y(t, x = a) = 0\,, \quad \text{Dirichlet B.C.} \tag{4.2.1}$$

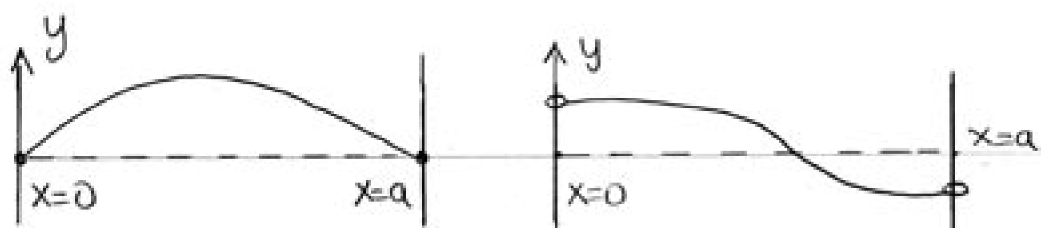

Figure 4.2: Left side: string with Dirichlet boundary conditions at the endpoints. Right side: string with Neumann boundary conditions at the endpoints.

Alternatively, if we attach a massless loop to each end of the string and the loops are allowed to slide along two frictionless poles, we are imposing Neumann boundary conditions. For our string, Neumann boundary conditions specify the values of the derivative $\partial y/\partial x$ at the endpoints. Since the loops are massless and the poles are frictionless, the derivative $\partial y/\partial x$ must vanish at the poles $x = 0, a$ (see Figure 4.2, right side). If this were not the case, then the slope of the string at a pole would be nonzero, and a component of the string tension would accelerate the rings in the y-direction. Since each ring is massless, their acceleration would be infinite. This is not possible, so in effect, we are imposing the Neumann conditions

$$\frac{\partial y}{\partial x}(t, x = 0) = \frac{\partial y}{\partial x}(t, x = a) = 0\,, \quad \text{Neumann B.C.} \tag{4.2.2}$$

Let' s see how we can solve the wave equation for a particular set of initial conditions. The general solution of equation (4.1.6) is of the form

$$y(t,x) = h_+(x - v_0 t) + h_-(x + v_0 t)\,, \tag{4.2.3}$$

where h_+ and h_- are arbitrary functions. This represents a superposition of two waves, h_+ moving to the right and h_- moving to the left. Suppose the initial values of y and $\partial y/\partial t$ are known at time $t = 0$. Using equation (4.2.3) we see that this information yields the equations

$$y(0,x) = h_+(x) + h_-(x)\,, \tag{4.2.4}$$

$$\frac{\partial y}{\partial t}(0,x) = -v_0 h'_+(x) + v_0 h'_-(x)\,, \tag{4.2.5}$$

where the left-hand sides are known functions, and primes denote derivatives with respect to arguments. Using (4.2.4) we can solve for h_- in terms of h_+. Substituting into (4.2.5), we get a first-order ordinary differential equation for h_+. Once we have solved for h_+, using appropriate boundary conditions, we can use (4.2.4) again, this time to find the explicit form of h_-. With h_+ and h_- known, the full solution of the equations of motion is given by (4.2.3).

4.3 Frequencies of transverse oscillation

Suppose we have a string where each point is oscillating in the y-direction sinusoidally and in phase. This means that $y(t,x)$ is of the form

$$y(t,x) = y(x)\sin(\omega t + \phi)\,, \tag{4.3.1}$$

where ω is the angular frequency of oscillation and ϕ is the constant common is phase. Our aim is to find the allowed frequencies of oscillation. Substituting (4.3.1) into (4.1.6) we find, after cancelling the common time dependence

$$\frac{d^2 y(x)}{dx^2} + \omega^2 \frac{\mu_0}{T_0}\, y(x) = 0\,. \tag{4.3.2}$$

This is an ordinary second-order differential equation for the profile $y(x)$ of the oscillations. The allowed frequencies are selected by this equation, together with the boundary conditions. Since ω, μ_0 and T_0 are constants, the differential equation is solved in terms of trigonometric functions. With Dirichlet boundary conditions (4.2.1) we have the nontrivial solutions

$$y_n(x) = A_n \sin\left(\frac{n\pi x}{a}\right) , \qquad n = 1, 2, \ldots , \tag{4.3.3}$$

where A_n is an arbitrary constant. The value $n = 0$ is not included above because it represents a motionless string. Plugging $y_n(x)$ into (4.3.2), we find the allowed frequencies ω_n:

$$\omega_n = \sqrt{\frac{T_0}{\mu_0}}\left(\frac{n\pi}{a}\right) , \quad n = 1, 2, \cdots . \tag{4.3.4}$$

These are the frequencies of oscillation for a Dirichlet string. The strings on a violin are Dirichlet strings. To tune a violin to the right frequency one must adjust the string tension. The higher the tension, the higher is the pitch, as predicted by (4.3.4).

For the case of Neumann boundary conditions (4.2.2), we obtain the spatial solutions

$$y_n(x) = A_n \cos\left(\frac{n\pi x}{a}\right) \qquad n = 1, 2, \ldots . \tag{4.3.5}$$

This time the $n = 0$ case is a little less trivial: the string does not oscillate, but it becomes rigidly translated to $y(t, x) = A_0$. The oscillation frequencies, found by plugging (4.3.5) in (4.3.2), are the same as those in (4.3.4). Therefore, the oscillation frequencies are the same in the Neumann and Dirichlet problems. The Neumann case, however, admits one extra solution not included in our oscillatory ansatz (4.3.1): the string can translate with constant velocity. Indeed, $y(t, x) = at + b$, with a and b arbitrary constants, satisfies both the boundary conditions and the original wave equation (4.1.7).

4.4 More general oscillating strings

Let us discuss briefly problems closely related to the ones considered thus far. For example, we can take the mass density of the string to be a function of position $\mu(x)$. The form (4.1.6) of the wave equation will not change, since it is derived from local considerations: the examination of a little piece of string that can be chosen to be sufficiently small so that the mass density is approximately constant. We therefore get:

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu(x)}{T_0}\frac{\partial^2 y}{\partial t^2} = 0\,. \tag{4.4.1}$$

For normal oscillations, we use the ansatz in (4.3.1) and find

$$\frac{d^2 y}{dx^2} + \frac{\mu(x)}{T_0}\,\omega^2\, y(x) = 0\,. \tag{4.4.2}$$

This equation is no longer simple to solve, and it can only be studied in detail once the function $\mu(x)$ is specified. In Problems 4.4 and 4.5 you will consider some specific mass distributions, and you will explore a variational approach that gives an upper bound for the lowest oscillation frequency.

So far we have only considered strings that are oscillating transversally. Strings also admit longitudinal oscillations, although the relativistic string does not. Imagine a string which at equilibrium lies along the x-axis and consider the infinitesimal segment which at in equilibrium, extends from x to $x+dx$. Suppose now that at time t the ends of this infinitesimal segment are displaced from their equilibrium positions by distances $\eta(t,x)$ and $\eta(t,x+dx)$, respectively. If these two quantities are not the same, the piece of string is being compressed or stretched. An equation of motion can be obtained for this system, much as we did for transverse motion. It is not possible, however, to assume that the tension T is constant throughout the string. For transverse oscillations the net force acting on a little piece of string arose from the different angles at which the tension was applied on opposite ends of the piece. If the string always lies along the x-axis then a net force can act on a segment only if the tension is different on its two ends. Therefore the waves of an oscillating string are accompanied by tension waves! It is an instructive exercise (Problem 4.2) to work out the equations of motion for a longitudinally-oscillating string.

4.5 A brief review of Lagrangian mechanics

The Lagrangian L of a system is defined by

$$L = T - V\,, \tag{4.5.1}$$

where T is the kinetic energy of the system and V is the potential energy of the system. For a point particle of mass m moving along the x axis under the influence of a time-independent potential $V(x)$, the non-relativistic Lagrangian takes the form

$$L(t) = \frac{1}{2} m \, [\, \dot{x}(t)]^2 - V[x(t)] \,, \quad \dot{x}(t) \equiv \frac{dx(t)}{dt} \,. \tag{4.5.2}$$

We must emphasize that the above Lagrangian is implicitly a function of time, but it has no explicit time dependence. All the time dependence arises from the time dependence of the position $x(t)$. The action S is defined as

$$S = \int_{\mathcal{P}} L(t) dt \,, \tag{4.5.3}$$

where $\mathcal{P}$ is a path $x(t)$ between an initial position x_i at an initial time t_i, and a final position x_f at a final time $t_f > t_i$. One such path is shown in Figure 4.3.

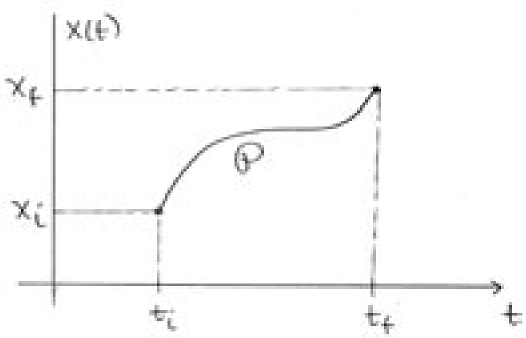

Figure 4.3: A trajectory $\mathcal{P}$ representing possible one-dimensional motion of a particle in the time interval $[t_i, t_f]$.

The action is a *functional*. Whereas a function of a single variable takes one number – the argument – as input and gives another number as output, a functional takes a *function* as the input, and gives a number as output. Since a function is usually defined by an infinitely-many points, we can think of a functional as a function of infinitely-many variables. In our present application, the input for the action functional is the function $x(t)$ which determines the path $\mathcal{P}$. We can emphasize the argument of S by using the notation $S[x]$. Here $[x]$ represents the full function $x(t)$. It is potentially confusing to write $S[x(t)]$, since it suggests that S is ultimately a function of t, which it is not.

More explicitly, for any path $x(t)$, the action is given by

$$S[x] = \int_{t_i}^{t_f} \left\{ \frac{1}{2} m \, [\dot{x}(t)]^2 - V[x(t)] \right\} dt \,. \tag{4.5.4}$$

It is very important to emphasize that the action S can be calculated for any path $x(t)$ and not only for paths that represent physically-realized motion. It is because S can be calculated for all paths that it is a very powerful tool to find the paths that can be physically realized.

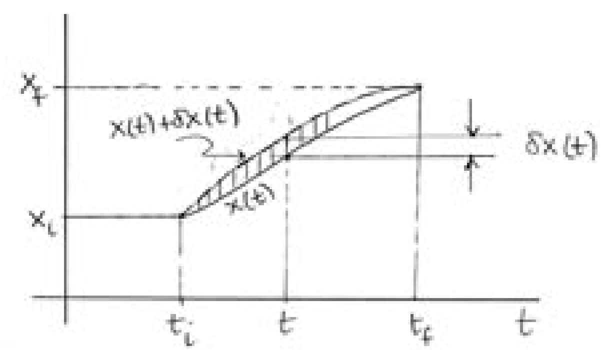

Figure 4.4: A path $x(t)$ and its variation $x(t)+\delta x(t)$. This variation $\delta x(t)$ vanishes at $t = t_i$ and at $t = t_f$.

Hamilton's principle states that the path $\mathcal{P}$ which a system actually takes is one for which the action S is stationary. More precisely, if this path $\mathcal{P}$ is varied infinitesimally, the action does not change to first order in the variation. In terms of the function $x(t)$ specifying the path, the perturbed path takes the form $x(t) + \delta x(t)$, as shown in Figure 4.4. For any time t, the variation $\delta x(t)$ is the vertical distance between the original path and the varied path. As in the figure, we consider variations where the initial and final positions $x_i = x(t_i)$ and $x_f = x(t_f)$ are unchanged:

$$\delta x(t_i) = \delta x(t_f) = 0 \,. \tag{4.5.5}$$

We now calculate the action $S[x+\delta x]$ for the perturbed path $x(t)+\delta x(t)$:

$$\begin{aligned} S[x+\delta x] &= \int_{t_i}^{t_f} \left\{ \frac{m}{2}\left[\frac{d}{dt}(x(t)+\delta x(t))\right]^2 - V[x(t)+\delta x(t)]\right\} dt \qquad (4.5.6) \\ &= S[x] + \int_{t_i}^{t_f} \left\{ m\dot{x}(t)\frac{d}{dt}\delta x(t) - V'[x(t)]\delta x(t)\right\} dt + \mathcal{O}((\delta x)^2)\,, \end{aligned}$$

where in the last equation we have expanded V in a Taylor series about $x(t)$. The terms of order $(\delta x)^2$ and higher are unnecessary to determine whether the action is stationary. We have thus left them undetermined and only indicate them by $\mathcal{O}((\delta x)^2)$. We can write the new action as $S+\delta S$, where δS is linear in δx. From the equation above we see that δS is given by

$$\delta S = \int_{t_i}^{t_f} \left\{ m\dot{x}(t)\frac{d}{dt}\delta x(t) - V'[x(t)]\delta x(t)\right\} dt\,. \qquad (4.5.7)$$

To find the equations of motion, the variation δS must be rewritten in the form $\delta S \sim \int \delta x\{\ldots\}$. In particular, no derivatives must be acting on δx. This can be achieved using integration by parts:

$$\begin{aligned} \delta S &= \int_{t_i}^{t_f} \left\{ \frac{d}{dt}\left[m\dot{x}(t)\delta x(t)\right] - m\ddot{x}(t)\delta x(t) - V'[x(t)]\delta x(t)\right\} dt \\ &= m\dot{x}(t_f)\delta x(t_f) - m\dot{x}(t_i)\delta x(t_i) + \int_{t_i}^{t_f} \delta x(t)\left\{-m\ddot{x}(t) - V'[x(t)]\right\} dt\,. \end{aligned} \qquad (4.5.8)$$

Making use of (4.5.5), the variation reduces to

$$\delta S = \int_{t_i}^{t_f} \delta x(t)\left\{-m\ddot{x}(t) - V'[x(t)]\right\} dt\,. \qquad (4.5.9)$$

The action is stationary if δS vanishes for every variation $\delta x(t)$. For this to happen, the factor multiplying $\delta x(t)$ in the integrand must vanish:

$$m\ddot{x}(t) = -V'[x(t)]\,. \qquad (4.5.10)$$

This is Newton's second law of motion for a particle in a potential $V(x)$. We have recovered the expected equation of motion by requiring that the action be stationary.

Suppose we have determined the path that the particle takes while going from x_i to x_f. As we have seen, the action is then stationary under variations that vanish at the initial and final times. Is the action also stationary under variations that change the initial position at t_i or the final position at t_f? In general, the answer is no. This can be seen from equation (4.5.8). The integral term vanishes by assumption, but if $\delta x(t_f) \neq 0$, the first term in the right-hand side would not vanish unless $m\dot{x}(t_f)$, the final momentum of the particle, happens to vanish. The situation is analogous for $\delta x(t_i) \neq 0$.

4.6 The non-relativistic string Lagrangian

Let's return now to our string with constant mass density μ_0, constant tension T_0, and ends located at $x = 0$ and $x = a$. The kinetic energy is simply the sum of the kinetic energies of all the infinitesimal segments that comprise the string. So it can be written as

$$T = \int_0^a \frac{1}{2}\,(\mu_0 dx)\left(\frac{\partial y}{\partial t}\right)^2 . \tag{4.6.1}$$

The potential energy arises from the work which must be done to stretch the segments. Consider an infinitesimal portion of string which extends from $(x, 0)$ to $(x + dx, 0)$ when the string is in equilibrium. If the string element is momentarily stretched from (x, y) to $(x + dx, y + dy)$, as in Figure 4.1, then the change in length Δl of the infinitesimal segment is given by

$$\begin{aligned} \Delta l &= \sqrt{(dx)^2 + (dy)^2} - dx \\ &= dx\Big(\sqrt{1 + \Big(\frac{\partial y}{\partial x}\Big)^2} - 1\Big) \simeq dx\,\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 , \end{aligned} \tag{4.6.2}$$

where we have used the small oscillation approximation (4.1.3) to discard higher order terms in the expansion of the square root. Since the work done in stretching each infinitesimal unit is $T_0 \Delta l$, we have

$$V = \int_0^a \frac{1}{2} T_0 \left(\frac{\partial y}{\partial x}\right)^2 dx\,. \tag{4.6.3}$$

The Lagrangian for the string is given by $T - V$:

$$L(t) = \int_0^a \Big[\frac{1}{2}\mu_0\Big(\frac{\partial y}{\partial t}\Big)^2 - \frac{1}{2}T_0\Big(\frac{\partial y}{\partial x}\Big)^2\Big] dx \equiv \int_0^a \mathcal{L}\, dx\,, \tag{4.6.4}$$

where $\mathcal{L}$ is referred to as the Lagrangian density:

$$\mathcal{L}\Big(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\Big) = \frac{1}{2}\mu_0 \left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T_0\Big(\frac{\partial y}{\partial x}\Big)^2 . \tag{4.6.5}$$

The action for our string is therefore

$$\boxed{S = \int_{t_i}^{t_f} L(t)dt = \int_{t_i}^{t_f} dt \int_0^a dx \left[\frac{1}{2}\mu_0\Big(\frac{\partial y}{\partial t}\Big)^2 - \frac{1}{2}T_0\Big(\frac{\partial y}{\partial x}\Big)^2\right] .} \tag{4.6.6}$$

In this action the "path" is the function $y(t,x)$. To find the equations of motion, we must examine the variation of the action as we vary: $y(t,x) \to y(t,x) + \delta y(t,x)$. Performing the variation as before, we get

$$\delta S = \int_{t_i}^{t_f} dt \int_0^a dx \left[\mu_0 \frac{\partial y}{\partial t}\frac{\partial(\delta y)}{\partial t} - T_0 \frac{\partial y}{\partial x}\frac{\partial(\delta y)}{\partial x}\right] . \tag{4.6.7}$$

We must have no derivatives acting on the variations, so we rewrite each of the two terms above as a total derivative minus a term in which the derivative does not act on the variation:

$$\begin{aligned}\delta S \;\; = \;\; & \int_{t_i}^{t_f} dt \int_0^a dx \Big[\mu_0 \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\delta y\right) - \mu_0 \frac{\partial^2 y}{\partial t^2}\delta y \\ & \qquad -T_0 \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\delta y\right) + T_0 \frac{\partial^2 y}{\partial x^2}\delta y\Big] .\end{aligned} \tag{4.6.8}$$

The total derivatives can be integrated. The total time derivative on the first line reduces to evaluations at t_f and t_i, while the total space derivative on the second line gives evaluations at the string endpoints:

$$\begin{aligned}\delta S \;\; = \;\; & \int_0^a \mu_0 \left[\frac{\partial y}{\partial t}\delta y\right]_{t=t_i}^{t=t_f} dx - \int_{t_i}^{t_f} T_0 \left[\frac{\partial y}{\partial x}\delta y\right]_{x=0}^{x=a} dt \\ & - \int_{t_i}^{t_f} dt \int_0^a dx \left(\mu_0 \frac{\partial^2 y}{\partial t^2} - T_0 \frac{\partial^2 y}{\partial^2 x}\right)\delta y \, .\end{aligned} \tag{4.6.9}$$

Our final expression for δS contains three terms. Each one must vanish independently. The third term, for example, is determined by the motion of the string for $x \in (0,a)$ and $t \in (t_i, t_f)$. The boundary conditions do not restrict $\delta y(t,x)$ here, so we set to zero the coefficient of δy, and recover our original equation (4.1.6). The first term in (4.6.9) is determined by the configuration of the string at times t_i and t_f. If we specify these configurations, we are in effect setting $\delta y(t_i, x)$ and $\delta y(t_f, x)$ to zero. This causes the first term to vanish. We encountered an analogous situation in our study of the free particle.

The second term in (4.6.9) is new: it concerns the motion of the string endpoints $y(t, 0)$ and $y(t, a)$. We can make this term vanish by specifying either Dirichlet or Neumann boundary conditions. Suppose we impose the Dirichlet boundary conditions (4.2.1). Then the positions of our endpoints are fixed throughout time, so we require that the variation δy vanishes for $x = 0$ and $x = a$. This will cause the second term to vanish. If, on the other hand, we impose the Neumann boundary conditions (4.2.2), then we are setting

$$\boxed{\frac{\partial y}{\partial x}(t, x = 0) = \frac{\partial y}{\partial x}(t, x = a) = 0\,, \quad \text{Neumann B.C.}} \tag{4.6.10}$$

This will also cause the second term to vanish. Dirichlet boundary conditions can be written in a form where the similarity to Neumann boundary conditions is more apparent. If the string endpoints are fixed, the time derivatives of the endpoint coordinates must vanish

$$\boxed{\frac{\partial y}{\partial t}(t, x = 0) = \frac{\partial y}{\partial t}(t, x = a) = 0\,, \quad \text{Dirichlet B.C.}} \tag{4.6.11}$$

The similarity with (4.6.10) is quite striking. The only change is that spatial derivatives were turned into time derivatives. If we write Dirichlet boundary conditions in this form, we must still specify the values of the coordinates of the fixed endpoints.

In order to appreciate further the physical import of boundary conditions, we consider the momentum p_y carried by the string. There is no other component to the momentum, because we have assumed that the motion is restricted to the y-direction. This momentum is simply the sum of the momenta of each infinitesimal segment along the string:

$$p_y = \int_0^a \mu_0 \frac{\partial y}{\partial t} dx\,. \tag{4.6.12}$$

Let us see if this momentum is conserved:

$$\frac{dp_y(t)}{dt} = \int_0^a \mu_0 \frac{\partial^2 y}{\partial t^2} dx = \int_0^a T_0 \frac{\partial^2 y}{\partial x^2} dx = T_0 \left[\frac{\partial y}{\partial x}\right]_{x=0}^{x=a}, \tag{4.6.13}$$

where we used the wave equation (4.1.6). We see that momentum is conserved for Neumann boundary conditions (4.6.10), but for Dirichlet boundary conditions momentum is not conserved! Does this make sense? Certainly. When the endpoints of a string are attached to a wall, the wall is constantly exerting a force on the endpoints of the string in order that they remain fixed. For example, in the lowest normal mode of a Dirichlet string the net momentum constantly oscillates between the $+y$- and $-y$-directions.

Why is this important for string theory? For many years string theorists did not take the possibility of Dirichlet boundary conditions all that seriously. It seemed unphysical that the string momentum could fail to be conserved. Moreover, to what could the endpoints of open strings be attached to? The answer is that they are attached to D-branes – a new kind of dynamical extended object. If a string is attached to a D-brane then momentum can be conserved – the momentum lost by the string is absorbed by the D-brane. A detailed analysis of the spatial boundary term is crucial to recognize the possibility of D-branes in string theory.

We conclude with a more general derivation of the equation of motion for the string. For this we write the action as

$$S = \int_{t_i}^{t_f} dt \int_0^a dx\, \mathcal{L}\Big(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\Big), \tag{4.6.14}$$

where we are using equation (4.6.5). We also define the quantities

$$\mathcal{P}^t \equiv \frac{\partial \mathcal{L}}{\partial \dot{y}}, \qquad \mathcal{P}^x \equiv \frac{\partial \mathcal{L}}{\partial y'}, \tag{4.6.15}$$

with $y' = \partial y/\partial x$. These are simply the derivatives of $\mathcal{L}$ with respect to its first and second arguments, respectively. Explicitly, they are

$$\mathcal{P}^t = \mu_0 \frac{\partial y}{\partial t}, \qquad \mathcal{P}^x = -T_0 \frac{\partial y}{\partial x}. \tag{4.6.16}$$

When we vary the motion by δy, the variation of the action is given by

$$\delta S = \int_{t_i}^{t_f} dt \int_0^a dx \Big[\frac{\partial \mathcal{L}}{\partial \dot{y}} \delta \dot{y} + \frac{\partial \mathcal{L}}{\partial y'} \delta y'\Big] = \int_{t_i}^{t_f} dt \int_0^a dx \Big[\mathcal{P}^t \delta \dot{y} + \mathcal{P}^x \delta y'\Big]. \tag{4.6.17}$$

Using the standard manipulations we find

$$\delta S = \int_{t_i}^{t_f} dt \int_0^a dx \Big[\frac{\partial}{\partial t}(\mathcal{P}^t \delta y) + \frac{\partial}{\partial x}(\mathcal{P}^x\, \delta y) - \delta y\Big(\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x}\Big)\Big]\,. \quad (4.6.18)$$

This gives the equation of motion

$$\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0\,, \quad (4.6.19)$$

Using (4.6.16), one can see that this is the wave equation (4.1.6).

Note that $\mathcal{P}^t$, as given in (4.6.16), coincides with the momentum density used before in equation (4.6.12). This is not a coincidence. In Lagrangian mechanics, the derivative of the Lagrangian with respect to a velocity is the conjugate momentum. For the string, $\dot{y}$ plays the role of a velocity, and therefore $\mathcal{P}^t$, a derivative of the Lagrangian density with respect to a velocity, is a momentum density.

In addition, note that for free string endpoints, the vanishing of the variation δS requires that $\mathcal{P}^x = 0$. As we can see from (4.6.16), this is a Neumann boundary condition. Furthermore, $\mathcal{P}^t$ vanishes at the string endpoints when impose a Dirichlet boundary condition ((4.6.11)). A more detailed analysis of these ideas will be given in Chapter 8, where $\mathcal{P}^t$ and $\mathcal{P}^x$ will be shown to have an interesting two-dimensional interpretation.

Problems

Problem 4.1. *Consistency of small transverse oscillations.*

Reconsider the analysis of transverse oscillations in section 4.1. Calculate the horizontal force dF_h on the little piece of string shown in Figure 4.1. Show that for small oscillations this force is much smaller than the vertical force dF_v responsible for the transverse oscillations.

Problem 4.2. *Longitudinal waves on strings.*

Consider a string with uniform mass density μ_0 stretched between $x = 0$ and $x = a$. Let the equilibrium tension be T_0. Longitudinal waves are only possible if the tension of the string varies as it stretches or compresses. Given a piece of this string with length L and tension T_0, under a small stretching ΔL, the tension changes by ΔT where

$$\frac{1}{\tau_0} = \frac{1}{L}\frac{\Delta L}{\Delta T}$$

Find the equation governing the (small) longitudinal oscillations of this string. Give the velocity of the waves.

Problem 4.3. *Evolving an initial string configuration*

A string with tension T_0, mass density μ_0, and wave velocity $v_0 = \sqrt{T_0/\mu_0}$, is stretched from $(x, y) = (0, 0)$ to $(x, y) = (a, 0)$. The string endpoints are fixed, and the string can vibrate in the y direction.

(a) Show that the above Dirichlet boundary conditions imply that in the notation of equation (4.2.3)

$$h_+(u) = -h_-(-u)\,, \qquad \text{and,} \qquad h_+(u) = h_+(u + 2a)\,. \tag{1}$$

Now consider an initial value problem for this string. At $t = 0$ the transverse displacement is identically zero, and the velocity is

$$\frac{\partial y}{\partial t}(0, x) = v_0\, \frac{x}{a}\Big(1 - \frac{x}{a}\Big)\,, \qquad x \in (0, a)\,. \tag{2}$$

(b) Calculate $h_+(u)$ for $u \in (-a, a)$. Does this define $h_+(u)$ for all x?

(c) Calculate $y(t, x)$ for x and $v_0 t$ in the domain

$$D = \{(x, v_0 t)\big|0 \leq x \pm v_0 t < a\}$$

Show D in a plane with axes x and $v_0 t$.

(d) At $t = 0$ the midpoint $x = a/2$ has the largest velocity of all points in the string. Show that the velocity of the midpoint reaches the value of zero at time $t_0 = a/(2v_0)$ and that $y(t_0, a/2) = a/12$. This is the maximum vertical displacement of the string.

Problem 4.4. *A configuration with two joined strings.*

A string with tension T_0 is stretched from $x = 0$ to $x = 2a$. The part of the string $x \in (0, a)$ has mass density μ_1, and the part of the string $x \in (a, 2a)$ has mass density μ_2. Consider the differential equation (4.4.2) that determines the normal oscillations. What boundary conditions should be imposed on $y(x)$ and $\frac{dy}{dx}(x)$ at $x = a$? Write the conditions that determine the oscillation frequencies. Calculate the lowest frequency of oscillation of this string when $\mu_1 = \mu_0$ and $\mu_2 = 2\mu_0$.

Problem 4.5. *Variational problem for strings.*

Consider a string stretched from $x = 0$ to $x = a$, with a tension T_0 and a position-dependent mass density $\mu(x)$. Equation (4.4.2) determines the transverse oscillation frequencies ω_i and associated profiles $\psi_i(x)$ for this string.

(a) Set up a variational procedure that gives an upper bound on the lowest frequency of oscillation ω_0. (This can be done as in quantum mechanics, where the ground state energy E_0 of a system with Hamiltonian H satisfies $E_0 \leq \frac{\langle\psi, H\psi\rangle}{\langle\psi,\psi\rangle}$). A useful first step may be to define an inner product $\langle\cdot\,,\cdot\rangle$ such that $\langle\psi_i, \psi_j\rangle$ vanishes when $\omega_i \neq \omega_j$. Explain why your variational procedure works.

(b) Consider the case $\mu(x) = \mu_0 \frac{x}{a}$. Use your variational principle to find a simple bound on the lowest oscillation frequency. Compare with the answer $\omega_0^2 = (18.956)\frac{T_0}{\mu_0 a^2}$ obtained by a direct numerical solution of the eigenvalue problem.

Chapter 5

The Relativistic Point Particle

To formulate the dynamics of a system we can write either the equations of motion, or alternatively, an action. In the case of the relativistic point particle, it is rather easy to write the equations of motion. But the action is so physical and geometrical that it is worth pursuing in its own right. More importantly, while it is difficult to guess the equations of motion for the relativistic string, the action is a natural generalization of the relativistic particle action that we will study in this chapter. We conclude with a discussion of the charged relativistic particle.

5.1 Action for a relativistic point particle

How can we find the action S that governs the dynamics of a free relativistic particle? To get started we first think about units. The action is the Lagrangian integrated over time, so the units of action are just the units of the Lagrangian multiplied by the units of time. The Lagrangian has units of energy, so the units of action are

$$[S] = M\,\frac{L^2}{T^2}\,T = \frac{ML^2}{T}\,. \tag{5.1.1}$$

Recall that the action S_{nr} for a free *non-relativistic* particle is given by the time integral of the kinetic energy:

$$S_{nr} = \int \frac{1}{2} m v^2(t)\, dt\,, \quad v^2 \equiv \vec{v}\cdot\vec{v}\,, \quad \vec{v} = \frac{d\vec{x}}{dt}\,. \tag{5.1.2}$$

The equation of motion following by Hamilton's principle is

$$\frac{d\vec{v}}{dt} = 0\,. \tag{5.1.3}$$

The free particle moves with constant velocity and that is the end of the story. Since even a free *relativistic* particle must move with constant velocity, how do we know that the action S_{nr} is not correct in relativity? Perhaps the simplest answer is that this action allows the particle to move with *any* constant velocity, even one that exceeds that of light. The velocity of light does not even appear in this action. S_{nr} cannot be the action for a relativistic point particle.

We now construct a relativistic action S for the free point particle. We will do this by making an educated guess and then showing that it works properly. But first, how should we describe the motion of the particle? Since we are interested in relativistic physics, it is convenient to represent the motion in spacetime. The path traced out in spacetime by the motion of a particle is called its *world-line*. In spacetime even a static particle traces a line, since time always flows.

A key physical requirement is that the action must yield Lorentz invariant equations of motion. Let's elaborate. Suppose a particular Lorentz observer tells you that the particle is moving according to its equations of motion, that is, that the particle is performing physical motion. Then, you should expect that any other Lorentz observer will tell you that the particle is doing physical motion. It would be inconsistent for one observer to state that a certain motion is allowed and for another to state that the same motion is forbidden. If the equations of motion hold in a fixed Lorentz frame, they must hold in all Lorentz frames. This is what is meant by Lorentz invariance of the equations of motion.

We are going to write an action, and we are going to take our time to find the equations of motion. Is there any way to impose a constraint on the action that will result in the Lorentz invariance of the equations of motion? Yes, there is. We require the action to be a Lorentz scalar: for *any* particle world-line all Lorentz observers must compute the same value for the action. Since the action has no spacetime indices, it is indeed reasonable to demand that it be a Lorentz scalar. If the action is a Lorentz scalar, the equations of motion will be Lorentz invariant. The reason is simple and neat. Suppose one Lorentz observer states that, for a given world-line, the action is stationary against all variations. Since all Lorentz observers agree on the values of the action for all world-lines, they will all agree that the action is stationary about the world-line in question. By Hamilton's principle, the world-line that makes the action stationary satisfies the equations of motion, and therefore all Lorentz observers will agree that the equations of motion are satisfied for the world-line in question.

Asking the action to be a Lorentz scalar is a very strong constraint, and there are actually valid grounds for suspecting that it may be too strong. The action in (5.1.2), for example, is *not* invariant under a Galilean boost $\vec{v} \to \vec{v} + \vec{v}_0$ with constant $\vec{v}_0$. Such a boost is a symmetry of the theory, since the equation of motion (5.1.3) *is* invariant. We will find, however, that this complication does not arise in relativity. We can indeed find fully Lorentz-invariant actions.

Quick Calculation 5.1. Show that the variation of the action S_{nr} under a boost is a total time derivative.

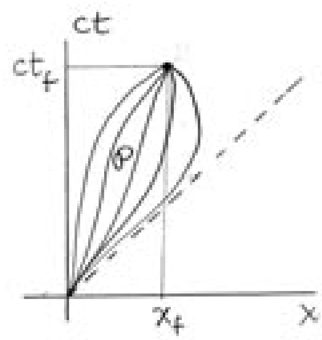

Figure 5.1: A spacetime diagram with a series of world-lines connecting the origin to the spacetime point (ct_f, x_f).

We know that the action is a functional – it takes as input a set of functions describing the world-line and outputs a number S. Now let us imagine a particle that is moving relativistically in spacetime, starting at the initial position $(0, 0)$ and ending at (ct_f, x_f). There are many possible world-lines between the starting and ending points, as shown in Figure 5.1 (our use of just one spatial dimension is only for ease of representation).

We would like our action to assign a number to each of these world-lines, a number which all Lorentz observers agree on. Let $\mathcal{P}$ denote one world-line. What quantity related to $\mathcal{P}$ do all Lorentz observers agree on? The answer is simple: the elapsed proper time! All Lorentz observers agree on the amount of time that elapsed on a clock carried by the moving particle. So let's take the action of the world-line $\mathcal{P}$ to be the proper time associated to it.

To formulate this idea quantitatively, we recall that

$$-ds^2 = -c^2 dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \,, \tag{5.1.4}$$

where the infinitesimal proper time is equal to ds/c. Of course, if we simply take the integral of the proper time, then we will not have the correct units for an action. Suppose instead that we integrate ds, which has the units of length. To get the units of action we need a multiplicative factor with units of mass times velocity. This factor should be Lorentz invariant, to preserve the Lorentz invariance of our partial guess ds. For the mass we can use m, the rest mass of the particle, and for velocity we can use c, the fundamental velocity in relativity. If we had instead used the velocity of the particle, the Lorentz invariance would have been spoiled. Therefore, our guess for the action is the integral of $(mc\, ds)$. Of course, there is still the possibility that a dimensionless coefficient is missing. It turns out that there should be a minus sign, but the unit coefficient is correct. We therefore claim that

$$\boxed{S = -mc \int_{\mathcal{P}} ds \,,} \tag{5.1.5}$$

is the correct action. This action is so simple that it may be baffling! It probably looks nothing like the actions you have seen before. We can make its content more familiar by choosing a particular Lorentz observer and expressing the action as an integral over time. With the help of (5.1.4) we can relate ds to dt as

$$ds = c\, dt \sqrt{1 - \frac{v^2}{c^2}} \,. \tag{5.1.6}$$

This allows us to write the action in (5.1.5) as an integral over time:

$$\boxed{S = -mc^2 \int_{t_i}^{t_f} dt \sqrt{1 - \frac{v^2}{c^2}} \;\;,} \tag{5.1.7}$$

where t_i and t_f are the values of time at the initial and final points of the world-line $\mathcal{P}$, respectively. From this version of the action, we see that the relativistic Lagrangian for the point particle is given by

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}}\,. \tag{5.1.8}$$

The Lagrangian makes no sense when $|\vec{v}| > c$ since it ceases to be real. The constraint of maximal velocity is therefore implemented. This could have been anticipated: proper time is only defined for motion where the velocity does not exceed the velocity of light. The paths shown in Figure 5.1 all represent motion where the velocity of the particle never exceeds the velocity of light. Only for such paths the action is defined. At any point in any of those paths, the tangent vector to the path is a timelike vector.

To show that this Lagrangian gives the familiar physics in the limit of small velocities, we expand the square root assuming $|\vec{v}| \ll c$. Keeping just the first term in the expansion gives

$$L \simeq -mc^2\Big(1-\frac{1}{2}\frac{v^2}{c^2}\Big) = -mc^2 + \frac{1}{2}mv^2\,. \tag{5.1.9}$$

Constant terms in a Lagrangian do not affect the equations of motion, so when velocities are small the relativistic Lagrangian gives the same physics as the the non-relativistic Lagrangian in (5.1.2). This also confirms that we normalized our relativistic Lagrangian correctly.

The canonical momentum is the derivative of the Lagrangian with respect to the velocity. Using (5.1.8) we find

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = -mc^2\Big(-\frac{\vec{v}}{c^2}\Big)\frac{1}{\sqrt{1-v^2/c^2}} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}\,. \tag{5.1.10}$$

This is just the relativistic momentum of the point particle. What about the Hamiltonian? It is given by

$$\begin{aligned} H &= \vec{p}\cdot\vec{v} - L = \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} \\ &= \frac{mc^2}{\sqrt{1-v^2/c^2}}\,. \end{aligned} \tag{5.1.11}$$

This coincides with the relativistic energy (2.4.2) of the point particle.

We have therefore recovered the familiar physics of a relativistic particle from the rather remarkable action (5.1.5). This action is very elegant: it is briefly written in terms of the geometrical quantity ds, it has a clear physical interpretation as total proper time, and it manifestly guarantees the Lorentz invariance of the physics it describes.

5.2 Reparametrization invariance

In this section we will explore an important property of the point particle action (5.1.5). This property is called reparametrization invariance. To evaluate the integral in the action, an observer will find it useful to parametrize the particle world-line. Reparametrization invariance of the action means that the value of the action is independent of parametrization chosen to calculate it. It should be so, since the action is defined independently of any parametrization. We now investigate this point in detail and learn how to rewrite the action in such a way that this property is manifest.

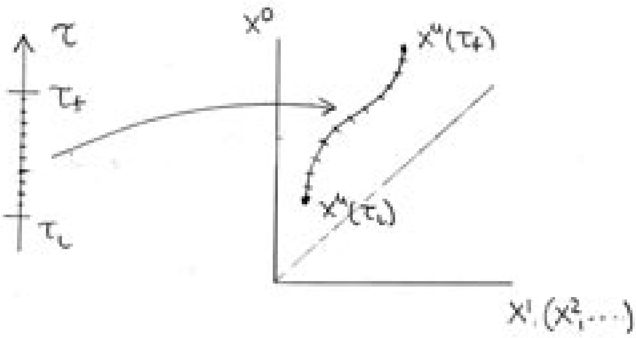

Figure 5.2: A world-line fully parametrized by τ. All spacetime coordinates x^μ are functions of τ.

To integrate ds, we parameterize the world-line $\mathcal{P}$ using a parameter τ, as shown in Figure 5.2. This parameter must be strictly increasing as the world-line goes from the initial point x_i^μ to the final point x_f^μ, but is otherwise arbitrary. As τ ranges in the interval $[\tau_i, \tau_f]$ it describes the motion of the particle. To have a parametrization of the world-line means that we have expressions for the coordinates x^μ as functions of τ:

$$x^\mu = x^\mu(\tau)\,. \tag{5.2.1}$$

We also require

$$x_i^\mu = x^\mu(\tau_i)\,, \quad x_f^\mu = x^\mu(\tau_f)\,. \tag{5.2.2}$$

Note that even the time coordinate x^0 is parameterized. Normally, we use time as a parameter and describe position as a function of time. This is what we did in section 5.1. But if we want to treat space and time coordinates on the same footing, we must parameterize both in terms of an additional parameter τ.

We now reexpress the integrand ds using the parametrized world-line. To this end, we use $ds^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$ to write

$$ds^2 = -\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}(d\tau)^2\,. \tag{5.2.3}$$

For any motion where the velocity does not exceed the velocity of light $ds^2 = (ds)^2$, and therefore the action (5.1.5) takes the form

$$S = -mc\int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\;\,d\tau\,. \tag{5.2.4}$$

This is the explicit form of the action when the path has been specified by a parametrization $x^\mu(\tau)$ with parameter τ.

We have already seen that the value of the action is the same for all Lorentz observers. We have now fixed an observer, who has calculated the action using some parameter τ. Does the value of the action depend on the choice of parameter? It does not. The observer can reparametrize the world-line, and the value of the action will be the same. Thus S is *reparameterization invariant*.

To see this, suppose we change the parameter from τ to τ'. Then, by the chain rule,

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau}\,. \tag{5.2.5}$$

Substituting back into (5.2.4), we get

$$S = -mc\int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau'}\frac{dx^\nu}{d\tau'}}\;\,\frac{d\tau'}{d\tau}d\tau = -mc\int_{\tau_i'}^{\tau_f'} \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau'}\frac{dx^\nu}{d\tau'}}d\tau'\,, \tag{5.2.6}$$

which has the same form as (5.2.4), thus establishing the reparametrization invariance. Because the verification of this property is quite simple, we say that the action is *manifestly* reparameterization invariant.

5.3 Equation of motion

We now move on to the equations of motion. For this we must calculate the variation δS of the action (5.1.5), when the world-line of the particle is varied by a small amount δx^μ. The variation is simply given by

$$\delta S = -mc \int \delta(ds)\,. \tag{5.3.1}$$

The variation of ds can be found from the simpler variation of ds^2. Since $(ds)^2 = ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu$, we find

$$\delta(ds)^2 = 2\, ds\, \delta(ds) = -\eta_{\mu\nu}\delta(dx^\mu dx^\nu) = -2\eta_{\mu\nu}\delta(dx^\mu)dx^\nu\,. \tag{5.3.2}$$

The factor of two on the right-hand side arises because, by symmetry, the variations of dx^ν and dx^ν give the same result. Simplifying a little,

$$\delta(ds) = -\eta_{\mu\nu}\delta(dx^\mu)\frac{dx^\nu}{ds}\,. \tag{5.3.3}$$

We now wish to understand why

$$\delta(dx^\mu) = d(\delta x^\mu)\,. \tag{5.3.4}$$

You may already be familiar with this. A simpler version of this result states that the variation of a velocity is just the derivative with respect to time of the variation of the coordinate. Equation (5.3.4) becomes quite clear when we spell out the meaning of d. For any τ-dependent quantity $A(\tau)$ we have $dA = A(\tau + d\tau) - A(\tau)$. Using this on both sides of equation (5.3.4) we find

$$\delta\left[x^\mu(\tau + d\tau) - x^\mu(\tau)\right] = \delta x^\mu(\tau + d\tau) - \delta x^\mu(\tau)\,. \tag{5.3.5}$$

Since δ is linear, the two sides are indeed equal. We can understand the equality (5.3.4) a little more geometrically by referring to Figure 5.3. The original world-line of the particle is the lower curve, and the world-line after variation is the upper curve. Shown in the lower curve are the points $x^\mu(\tau)$

and $x^\mu(\tau + d\tau)$. By definition, they are separated by the infinitesimal vector dx^μ. The variations $\delta x^\mu(\tau)$ and $\delta x^\mu(\tau + d\tau)$ are also shown as vectors in the figure, with their ends defining the varied points. In the upper curve, the separation between the varied points is the new dx: the old dx plus the variation $\delta(dx)$. The little quadrilateral on the left-side of the figure is shown to the right with labels a, a', b, b' for the vectors that represent the corresponding sides. We see that $\delta(dx^\mu)$ corresponds to $a' - a$. On the other hand $d(\delta x^\mu)$ corresponds to $b' - b$. The equality of these two quantities follows from the vector identity $a + b' = b + a'$.

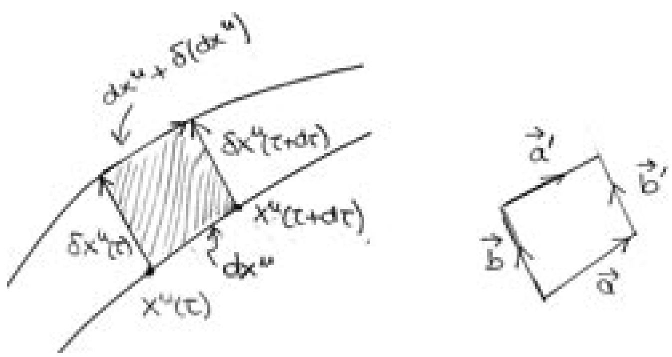

Figure 5.3: Relating $\delta(dx^\mu)$ and $d(\delta x^\mu)$ in the variation of a world-line.

Using (5.3.4) back in (5.3.3), we write the final expression for $\delta(ds)$:

$$\delta(ds) = -\eta_{\mu\nu}\frac{d(\delta x^\mu)}{ds}\frac{dx^\mu}{ds}ds\,. \tag{5.3.6}$$

We can now go ahead and vary the action using (5.3.1):

$$\delta S = mc\int_{s_i}^{s_f}\eta_{\mu\nu}\frac{d(\delta x^\mu)}{ds}\frac{dx^\nu}{ds}ds\,. \tag{5.3.7}$$

We introduced here explicit limits to the integration: s_i and s_f denote the values of the proper time parameter at the initial and final points of the world-line, respectively. To get an equation of motion we need to have δx^μ multiplying an object under the integral – the equation of motion is then simply the vanishing of that object. Since there are still derivatives acting on δx^μ, we must rewrite the integrand as a total derivative plus additional terms where δx^μ appears multiplicatively :

$$\delta S = mc\int_{s_i}^{s_f} ds \frac{d}{ds}\left(\eta_{\mu\nu}\delta x^\mu \frac{dx^\nu}{ds}\right) - \int_{s_i}^{s_f} ds\ \delta x^\mu \left(mc\,\eta_{\mu\nu}\frac{d^2x^\nu}{ds^2}\right). \quad (5.3.8)$$

The first integral just gives some expression evaluated at the boundaries of the world-line. But we fix the coordinates on the boundaries, so the first term vanishes. The variation of the action then reduces to

$$\delta S = -\int_{s_i}^{s_f} ds\ \delta x^\mu(s)\left(mc\,\eta_{\mu\nu}\frac{d^2x^\nu}{ds^2}\right). \quad (5.3.9)$$

Since $\delta x^\mu(s)$ is arbitrary, everything multiplying it must vanish in order for the variation of the action to be zero. We thus find

$$mc\,\eta_{\mu\nu}\frac{d^2x^\nu}{ds^2} = 0\,. \quad (5.3.10)$$

The constants are nonvanishing and can be removed. Even the Minkowski metric can be removed: an equation of the form $0 = \eta_{\mu\nu}b^\nu$, implies that $b^\nu = 0$. Indeed, multiplying of the equation by $\eta^{\rho\mu}$ you find $0 = \eta^{\rho\mu}\eta_{\mu\nu}b^\nu = \delta^\rho_\nu b^\nu = b^\rho$. Therefore, in its simplest form, the equation of motion is

$$\boxed{\frac{d^2x^\mu}{ds^2} = 0\,.} \quad (5.3.11)$$

This is the equation of motion for a free particle. We obtained this equation by varying the relativistic action for the point particle using fully-relativistic notation. Equation (5.3.11) is formulated using the *proper time* parameter s. It tells you that the four-velocity u^μ is constant along the world-line:

$$\frac{d}{ds}\left(\frac{dx^\mu}{ds}\right) = 0 \quad \rightarrow \quad \frac{du^\mu}{ds} = 0\,, \quad u^\mu = \frac{dx^\mu}{ds}\,. \quad (5.3.12)$$

This means that if the path is marked by equal intervals of proper time, the change in x^μ between any successive pair of marks is the same. It also follows from (5.3.12) that the relativistic momentum $p^\mu = mu^\mu$ is constant along the world-line:

$$\frac{dp^\mu}{ds} = 0\,. \quad (5.3.13)$$

What if you used some arbitrary parameter τ instead of s? Equation (5.3.11) would *not hold* with s replaced by τ. This is reasonable: changing arbitrarily the parameter is like changing arbitrarily the marks, and as a consequence, the change in x^μ between any successive pair of new marks will not be the same. It is actually possible to write a slightly more complicated version of the equation of motion (5.3.11). That version uses an arbitrary parameter, and is manifestly reparametrization invariant (see Problem 5.2).

Quick Calculation 5.2. Show that equation (5.3.13) implies that

$$\frac{dp^\mu}{d\tau} = 0\,, \tag{5.3.14}$$

holds for an arbitrary parameter $\tau(s)$. Is there a constraint on $\frac{d\tau}{ds}$?

Our goal in this section has been achieved: we have shown how to derive the physically expected equation of motion (5.3.11) starting from the Lorentz invariant action (5.1.5). As we explained earlier, the resulting equation of motion is guaranteed to be Lorentz invariant. Let us check this explicitly.

Under a Lorentz transformation, the coordinates x^μ transform as indicated in equation (2.2.32): $x'^\mu = L^\mu{}_\nu\, x^\nu$, where the constants $L^\mu{}_\nu$ can be viewed as the entries of an invertible matrix L. Since ds is the same in all Lorentz frames, the equation of motion in primed coordinates is (5.3.11), with x^μ replaced by x'^μ:

$$0 = \frac{d^2 x'^\mu}{ds^2} = \frac{d^2}{ds^2}(L^\mu{}_\nu\, x^\nu) = L^\mu{}_\nu\, \frac{d^2 x^\nu}{ds^2}\,. \tag{5.3.15}$$

Since the matrix L is invertible, the above equation implies equation (5.3.11). Namely, if the equation of motion holds in the primed coordinates, it holds in the unprimed coordinates as well. This is the Lorentz invariance of the equations of motion.

5.4 Relativistic particle with electric charge

The point particle we studied so far in this chapter is free and it moves with constant four-velocity or four-momentum. If a point particle is electrically charged, and there are nontrivial electromagnetic fields, the particle will experience forces and the four-momentum will not be constant. You know, in fact, how the momentum of such particle varies in time. Its time derivative is governed by the Lorentz force equation (3.1.5). The relativistic version of the Lorentz force equation was given in Problem 3.1:

$$\frac{dp^\mu}{ds} = \frac{q}{c^2} F^{\mu\nu} u_\nu \,. \tag{5.4.1}$$

This is an intricate equation. In the spirit of our previous analysis we try to write an action that gives this equation upon variation. The action turns out to be remarkably simple.

Since the Maxwell field couples to the point particle along its world-line, we should add to the action (5.1.5) an integral over $\mathcal{P}$ representing the interaction of the particle with the electromagnetic field. The integral must be Lorentz invariant, and must involve the four-velocity of the particle. Since the four-velocity has one spacetime index, to obtain a Lorentz scalar we must multiply it against another object with one index. The natural candidate is the gauge potential A_μ. I claim that the interaction term in the action is

$$\frac{q}{c} \int_{\mathcal{P}} ds \; A_\mu(x(s)) \frac{dx^\mu}{ds}(s) \,. \tag{5.4.2}$$

Here q is the electric charge, and the integral is over the world-line $\mathcal{P}$, parametrized with proper time. At each s, the four-velocity (dx^μ/ds) is multiplied by the gauge potential A_μ evaluated at the position $x(s)$ of the particle. The integrand can be written more briefly as $A_\mu dx^\mu$, by cancelling the factors of ds. In this form, the interaction term is manifestly independent of parametrization. The world-line of the particle is a one-dimensional space, and the natural field that can couple to a particle in a Lorentz invariant way is a field with one index. This will have an interesting generalization when we consider the motion of strings. Since strings are one-dimensional, they trace out two-dimensional *world-sheets* in spacetime. We will see that they couple naturally to fields with two Lorentz indices!

The full action for the electrically charged point particle is obtained by adding the term in (5.4.2) to (5.1.5):

$$\boxed{S = -mc \int_{\mathcal{P}} ds + \frac{q}{c} \int_{\mathcal{P}} A_\mu(x) dx^\mu \,.} \tag{5.4.3}$$

This Lorentz invariant action is simple and elegant. It is the correct action, and the equation of motion (5.4.1) arises by setting to zero the variation of

S under a change δx^μ of the particle world-line. I do not want to take away from you the satisfaction of deriving this important result. Therefore, I have left to Problem 5.5 the task of varying the action (5.4.3).

Problems

Problem 5.1. *Point particle equation of motion and reparametrizations.*

If we parametrize the path of a point particle using proper time, the equation of motion is (5.3.11). Consider now a new parameter $\tau = f(s)$. What is the most general function f for which (5.3.11) implies

$$\frac{d^2x^\mu}{d\tau^2} = 0\,.$$

Problem 5.2. *Particle equation of motion with arbitrary parametrization.*

Vary the point particle action (5.2.4) to find a *manifestly reparametrization invariant* form of the free particle equation of motion.

Problem 5.3. *Current of a charged point particle.*

Consider a point particle with charge q moving in a $D = d+1$ dimensional spacetime as described by functions $x^\mu(\tau) = \{x^0(\tau), \vec{x}(\tau)\}$ where τ is parameter. Recall that the electromagnetic current j^μ is defined as $j^\mu = (c\rho, \vec{j})$ where ρ is the charge density (charge per unit volume) and $\vec{j}$ is the current density (current per unit area).

(a) Use delta functions to write expressions for $j^0(\vec{x}, t)$ and $j^i(\vec{x}, t)$ describing the electromagnetic current associated to the point particle.

(b) Show that the expressions you wrote in (a) arise from the following integral representation

$$j^\mu(\vec{x}, t) = qc \int d\tau\, \delta^{d+1}(x - x(\tau))\, \frac{dx^\mu(\tau)}{d\tau} \tag{1}$$

Here $\delta^{d+1}(x) \equiv \delta(x^0)\delta(x^1)\cdots\delta(x^d)$.

Problem 5.4. *Hamiltonian for a non-relativistic charged particle.*

The action for a non-relativistic particle of mass m and charge q coupled to electromagnetic fields is obtained by replacing the first term in (5.4.3) by the non-relativistic action for a free point particle:

$$S = \int \frac{1}{2}\, mv^2\, dt + \frac{q}{c} \int A_\mu(x) \frac{dx^\mu}{dt} dt\,. \tag{1}$$

We have also chosen to use time to parametrize the second integral.

(a) Rewrite the above action in term of the potentials $(\Phi, \vec{A})$ and the ordinary velocity $\vec{v}$. What is the Lagrangian?

(b) Calculate the canonical momentum $\vec{p}$ conjugate to the position of the particle and show that it is given by

$$\vec{p} = m\vec{v} + \frac{q}{c}\vec{A}\,. \tag{2}$$

(c) Construct the Hamiltonian for the charged point particle and show it is given by

$$H = \frac{1}{2m}\Big(\vec{p} - \frac{q}{c}\vec{A}\Big)^2 + q\,\Phi\,. \tag{3}$$

Problem 5.5. *Equations of motion for a charged point particle.*

Consider the variation of the action (5.4.3) under a variation $\delta x^\mu(x)$ of the particle trajectory. The variation of the first term in the action was obtained in section 5.3. Vary the second term (written more explicitly in (5.4.2)) and show that the equation of motion for the point particle in the presence of an electromagnetic field is (5.4.1). Begin your calculation by explaining why

$$\delta A_\mu(x(s)) = \frac{\partial A_\mu}{\partial x^\nu}(x(s))\,\delta x^\nu(s)\,.$$

Problem 5.6. *Electromagnetic field dynamics with charged particle.*

The action for the dynamics of *both* a charged point particle and the EM field is given by

$$S' = -mc\int_{\mathcal{P}} ds + \frac{q}{c}\int_{\mathcal{P}} A_\mu(x)dx^\mu - \frac{1}{16\pi c}\int d^{d+1}x\,\Big(F_{\mu\nu}F^{\mu\nu}\Big)\,.$$

Here $d^{d+1}x = dx^0dx^1\cdots dx^d$. Notice that the total action S' is a hybrid. The last term is an integral over spacetime and the first two terms are integrals over the particle world-line. While included for completeness, the first term will play no role here. Vary the action S' under a fluctuation $\delta A_\mu(x)$ and obtain the equation of motion for the electromagnetic field in the presence of the charged point particle. The answer should be equation (3.3.23), where the current is the one calculated in Problem 5.3. [Hint: to vary $A_\mu(x)$ in the world-line action it is useful to first turn this term into a full spacetime integral with the help of delta functions].

Problem 5.7. *Point particle action in curved space*

In section 3.6 we considered the invariant interval $ds^2 = -g_{\mu\nu}(x)dx^\mu dx^\nu$ in curved-space. The motion of a point particle of mass m on curved space is studied using the action

$$S = -mc\int ds\,.$$

Show that the equation of motion takes the form

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0\,, \tag{1}$$

where

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\Big(\frac{\partial g_{\lambda\alpha}}{\partial x^\beta} + \frac{\partial g_{\lambda\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\Big)\,.$$

The Christoffel coefficients Γ are symmetric in the lower indices $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$, and are built from the metric and its first derivatives. When the metric is constant the Christoffel coefficients vanish and we recover the familiar equation of motion of the free point particle in Minkowski space. Equation (1) is called the geodesic equation. It is a rather nontrivial differential equation that follows from the very simple action S.

Chapter 6

Relativistic Strings

We now begin our study of the classical relativistic string – a string that is, in many ways, much more elegant than the non-relativistic one considered before. Inspired by the point particle case, we focus our attention on the surface traced out by the string in spacetime. We use the proper area of this surface as the action, called the Nambu-Goto action. We study the reparameterization property of this action, identify the string tension, and find the equations of motion. For open strings, we focus on the motion of the endpoints, and introduce the concept of D-branes. Finally, we see that the only physical motion is transverse to the string.

6.1 Area functional for spatial surfaces

The action for a relativistic string must be a functional of the string trajectory. Just as a particle traces out a line in spacetime, a string traces out a surface. The line traced out by the particle in spacetime was called the world-line. The two-dimensional surface traced out by a string in spacetime will be called the *world-sheet*. A closed string, for example, will trace out a tube, while an open string will trace out a strip. These two-dimensional world-sheets are shown in the spacetime diagram of Figure 6.1. The lines of constant x^0 in these surfaces are the strings. These are the objects an observer sees at the fixed time x^0. They are open curves for the surface describing the open string evolution (left side), and they are closed curves for the surface describing the closed string evolution (right side).

In Chapter 5 we learned that the point particle action was given by the proper time associated to the point particle world-line. The proper time, multiplied by c, is an invariant "length" associated to the world-line. For strings we will define the Lorentz invariant "proper area" of a world-sheet. The relativistic string action will be proportional to this proper area, and is called the Nambu-Goto action.

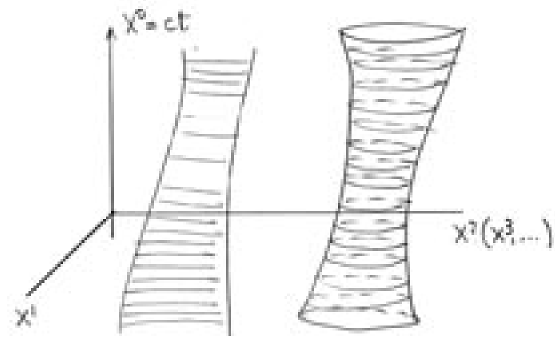

Figure 6.1: The world-sheets traced out by an open string and by a closed string.

Area functionals are useful in other applications: a soap film held between two rings, for example, automatically constructs the surface of minimal area which joins one ring to the other, as in Figure 6.2. The string world-sheet and the soap bubble between two rings are very different types of surfaces. At any given instant of time a Lorentz observer will see the full two-dimensional surface of the soap film, but he or she can only see one string from the two-dimensional world-sheet. Imagine the soap film is static in some Lorentz frame. In this case, time is not relevant to the description of the film, and we think of the film as a *spatial surface*, namely, a surface that extends along two spatial dimensions. The surface exists in its entirety at any instant of time. We will first study these familiar surfaces, and then we will apply our experience to the case of surfaces in spacetime.

A line in space can be parameterized using only one parameter. A surface in space is two-dimensional, so it requires two parameters ξ^1 and ξ^2. Given a parameterized surface, we can draw on that surface the lines of constant ξ^1 and the lines of constant ξ^2. These lines cover the surface with a grid. We call *target space* the world where the two-dimensional surface lives. In the case of a soap bubble in three dimensions, the target space is the three dimensional space x^1, x^2 and x^3. The parameterized surface is described by the collection of functions

$$\vec{x}\,(\xi^1, \xi^2) = \Big(x^1(\xi^1, \xi^2)\,,\; x^2(\xi^1, \xi^2)\,,\; x^3(\xi^1, \xi^2)\Big)\,. \tag{6.1.1}$$

The parameter space is defined by the ranges of the parameters ξ^1 and ξ^2. It may be a square, for example, if we use parameters $\xi^1, \xi^2 \in [0, \pi]$. The real surface is the image of the parameter space under the map $\vec{x}(\xi^1, \xi^2)$. The physical surface is a surface in target space. Alternative, we can view the parameters ξ^1 and ξ^2 as *coordinates* on the physical surface, at least locally. The map inverse to $\vec{x}$ takes the surface to the parameter space. Locally this map is one-to-one and it assigns to each point on the surface two coordinates: the values of the parameters ξ^1 and ξ^2.

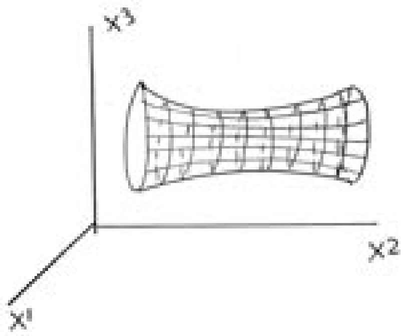

Figure 6.2: A spatial surface stretching between two rings. If the surface is a soap film, it would be a minimal area surface.

We want to calculate the area of a small element of the target space surface. Let's start by looking at an infinitesimal rectangle on the parameter space. Denote the sides of the square by $d\xi^1$ and $d\xi^2$. We want to find dA, the area of the image of this little rectangle in the target space. As shown in Figure 6.3, this is the area of the actual piece of surface that corresponds to the infinitesimal square on parameter space.

Of course, there is no reason why that infinitesimal area element in target space should be a rectangle. In general, it is a parallelogram. Let's call the

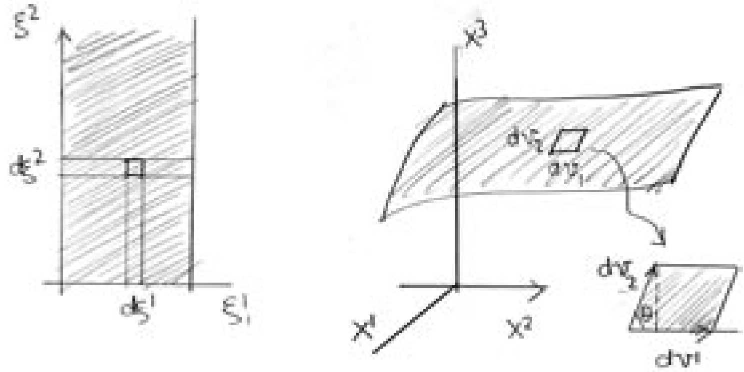

Figure 6.3: Left side: the parameter space, with a little square selected. The target space surface with the image of the little square: a parallelogram whose sides are the vectors $d\vec{v}_1$ and $d\vec{v}_2$ (shown magnified at the end of the wiggly arrow).

sides of this parallelogram $d\vec{v}_1$ and $d\vec{v}_2$. They are the images under the map $\vec{x}$ of the vectors $(d\xi^1, 0)$ and $(0, d\xi^2)$, respectively. We can write them as

$$d\vec{v}_1 = \frac{\partial \vec{x}}{\partial \xi^1}\, d\xi^1\,, \qquad d\vec{v}_2 = \frac{\partial \vec{x}}{\partial \xi^2}\, d\xi^2\,. \tag{6.1.2}$$

This makes sense: $\partial\vec{x}/\partial\xi^1$, for example, represents the rate of variation of the space coordinates with respect to ξ^1. Multiplying this rate by the length $d\xi^1$ of the horizontal side of the tiny parameter-space rectangle, gives us the vector $d\vec{v}_1$ representing this side in the target space. Now let us calculate the area dA. Using the formula for the area of a parallelogram,

$$\begin{aligned} dA &= |d\vec{v}_1||d\vec{v}_2||\sin\theta| = |d\vec{v}_1||d\vec{v}_2|\sqrt{1-\cos^2\theta} \\ &= \sqrt{|d\vec{v}_1|^2|d\vec{v}_2|^2 - |d\vec{v}_1|^2|d\vec{v}_2|^2\cos^2\theta}\,, \end{aligned} \tag{6.1.3}$$

where θ is the angle between the vectors $d\vec{v}_1$ and $d\vec{v}_2$. In terms of spatial dot products, we have

$$dA = \sqrt{(d\vec{v}_1\cdot d\vec{v}_1)(d\vec{v}_2\cdot d\vec{v}_2) - (d\vec{v}_1\cdot d\vec{v}_2)^2}\,. \tag{6.1.4}$$

Finally, using (6.1.2),

$$dA = d\xi^1 d\xi^2\sqrt{\Big(\frac{\partial\vec{x}}{\partial\xi^1}\cdot\frac{\partial\vec{x}}{\partial\xi^1}\Big)\Big(\frac{\partial\vec{x}}{\partial\xi^2}\cdot\frac{\partial\vec{x}}{\partial\xi^2}\Big) - \Big(\frac{\partial\vec{x}}{\partial\xi^1}\cdot\frac{\partial\vec{x}}{\partial\xi^2}\Big)^2}\,. \tag{6.1.5}$$

This is the general expression for the area element of a parameterized spatial surface. The full area functional A is given by

$$A = \int d\xi^1 d\xi^2 \sqrt{\Big(\frac{\partial \vec{x}}{\partial \xi^1}\cdot\frac{\partial \vec{x}}{\partial \xi^1}\Big)\Big(\frac{\partial \vec{x}}{\partial \xi^2}\cdot\frac{\partial \vec{x}}{\partial \xi^2}\Big) - \Big(\frac{\partial \vec{x}}{\partial \xi^1}\cdot\frac{\partial \vec{x}}{\partial \xi^2}\Big)^2}\,. \tag{6.1.6}$$

The integral extends over the relevant ranges of the parameters ξ^1 and ξ^2. The solution of a minimal area problem for a spatial surface is the function $\vec{x}\,(\xi^1, \xi^2)$ that minimizes the functional A.

6.2 Reparameterization invariance of the area

As we have seen, the parameterization of a surface allows us to write the area element in an explicit form. The area of the surface, or even more, the area of any piece of the surface, should be independent of the parameterization chosen to calculate it. This is what we mean when we say that the area must be reparameterization invariant.

Because we will soon equate the relativistic string action to some notion of proper area, it, too, will be reparameterization invariant. This means that we will be free to choose the most useful parameterization without changing the underlying physics. A good choice of parameterization will enable us to solve the equations of motion of the relativistic string in an elegant way.

Reparameterization invariance is thus an important concept so it should be understood thoroughly. To this end we will try to make it manifest in our formulae. The aim of the following analysis is to show how this can be done.

Let's begin by asking: is the area functional A in (6.1.6) reparameterization invariant? We would certainly hope it is. In fact, at first glance it appears to be manifestly reparameterization invariant. After all, if one reparameterizes the surface with $\tilde{\xi}^1(\xi^1)$ and $\tilde{\xi}^2(\xi^2)$, then all of the derivatives introduced by the chain rule cancel appropriately.

Quick Calculation 6.1. Verify the above statement. That is, show that (6.1.6), written fully with tilde parameters $(\tilde{\xi}^1, \tilde{\xi}^2)$ equals (6.1.6) when $\tilde{\xi}^1 = \tilde{\xi}^1(\xi^1)$ and $\tilde{\xi}^2 = \tilde{\xi}^2(\xi^2)$.

The above reparameterization, however, is not completely general for it fails to mix the ξ^1 and ξ^2 coordinates. Suppose, instead, that we make a reparameterization $\tilde{\xi}^1(\xi^1, \xi^2)$ and $\tilde{\xi}^2(\xi^1, \xi^2)$. This time we can verify, using a somewhat laborious computation, that (6.1.6) is invariant under such a reparameterization. But the invariance is no longer intuitively clear. To make the reparameterization invariance of (6.1.6) manifest we will have to rewrite the area functional in a different way.

We begin by observing how the measure of integration transforms. The change-of-variable theorem from calculus tells us that

$$d\xi^1 d\xi^2 = \Big|\det\Big(\frac{\partial \xi^i}{\partial \tilde{\xi}^j}\Big)\Big| d\tilde{\xi}^1 d\tilde{\xi}^2 = |\det M|\, d\tilde{\xi}^1 d\tilde{\xi}^2\,, \tag{6.2.1}$$

where $M = [M_{ij}]$ is the matrix defined by $M_{ij} = \partial\xi^i/\partial\tilde{\xi}^j$. Similarly,

$$d\tilde{\xi}^1 d\tilde{\xi}^2 = \Big|\det\Big(\frac{\partial \tilde{\xi}^i}{\partial \xi^j}\Big)\Big|\; d\xi^1 d\xi^2 = |\det \widetilde{M}| d\xi^1 d\xi^2\,, \tag{6.2.2}$$

where $\widetilde{M} = [\widetilde{M}_{ij}]$ is the matrix defined by $\widetilde{M}_{ij} = \partial\tilde{\xi}^i/\partial\xi^j$. Combining equations (6.2.1) and (6.2.2), we see that

$$|\det M||\det \widetilde{M}| = 1\,. \tag{6.2.3}$$

Let us now consider a target space surface $\mathcal{S}$, described by the mapping functions $\vec{x}(\xi^1,\xi^2)$. Given a vector $d\vec{x}$ tangent to the surface, let ds denote its length. Then we can write

$$ds^2 \equiv (ds)^2 = d\vec{x}\cdot d\vec{x} \tag{6.2.4}$$

For surfaces in space, as we are considering now, it is not customary to add a minus sign in front of ds^2 (compare with (2.2.15)). The vector $d\vec{x}$ can be expressed in terms of partial derivatives and the differentials $d\xi^1$, $d\xi^2$:

$$d\vec{x} = \frac{\partial\vec{x}}{\partial\xi^1}d\xi^1 + \frac{\partial\vec{x}}{\partial\xi^2}d\xi^2 = \frac{\partial\vec{x}}{\partial\xi^i}\,d\xi^i\,. \tag{6.2.5}$$

The repeated index i is summed over its possible values 1 and 2. Back in (6.2.4)

$$ds^2 = \Big(\frac{\partial\vec{x}}{\partial\xi^i}d\xi^i\Big)\cdot\Big(\frac{\partial\vec{x}}{\partial\xi^j}d\xi^j\Big) = \frac{\partial\vec{x}}{\partial\xi^i}\cdot\frac{\partial\vec{x}}{\partial\xi^j}\; d\xi^i d\xi^j\,. \tag{6.2.6}$$

This can be neatly summarized as

$$ds^2 = g_{ij}(\xi)\, d\xi^i d\xi^j\,, \tag{6.2.7}$$

where $g_{ij}(\xi)$ is defined as

$$g_{ij}(\xi) \equiv \frac{\partial \vec{x}}{\partial \xi^i} \cdot \frac{\partial \vec{x}}{\partial \xi^j} \,. \tag{6.2.8}$$

The quantity $g_{ij}(\xi)$ is known as the *induced metric on* $\mathcal{S}$. It is called a metric because (6.2.7) takes, up to a sign, the form of equation (3.6.2), where we introduced the general concept of a metric. It is a metric on $\mathcal{S}$ because, with ξ^i playing the role of coordinates on $\mathcal{S}$, equation (6.2.7) determines distances on $\mathcal{S}$. It is said to be induced because it uses the metric on the ambient space in which $\mathcal{S}$ *lives* to determine distances on $\mathcal{S}$. Indeed, the dot product which appears in definition (6.2.8) is to be performed in the space where $\mathcal{S}$ lives and therefore presupposes that a metric exists on that space. We only have two parameters ξ^1 and ξ^2, so the full matrix g_{ij} takes the form:

$$g_{ij} = \begin{pmatrix} \dfrac{\partial \vec{x}}{\partial \xi^1} \cdot \dfrac{\partial \vec{x}}{\partial \xi^1} & \dfrac{\partial \vec{x}}{\partial \xi^1} \cdot \dfrac{\partial \vec{x}}{\partial \xi^2} \\ \dfrac{\partial \vec{x}}{\partial \xi^2} \cdot \dfrac{\partial \vec{x}}{\partial \xi^1} & \dfrac{\partial \vec{x}}{\partial \xi^2} \cdot \dfrac{\partial \vec{x}}{\partial \xi^2} \end{pmatrix} . \tag{6.2.9}$$

Now we see something truly nice! The determinant of g_{ij} is precisely the quantity which appears under the square root in (6.1.6). Letting

$$g \equiv \det(g_{ij}) \,, \tag{6.2.10}$$

we can write

$$\boxed{A = \int d\xi^1 d\xi^2 \sqrt{g} \,.} \tag{6.2.11}$$

This is an elegant formula for the area in terms of the determinant of the induced metric. Instead of trying to understand the reparameterization invariance of (6.1.6), we now focus on the equivalent but simpler expression (6.2.11).

We are now in position to understand the invariance of the area in terms of the transformation properties of the metric g_{ij}. The key to this lies in equation (6.2.7). The length squared ds^2 is a geometrical property of the vector $d\vec{x}$ that must not depend upon the particular parameterization used to calculate it. For another set of parameters $\tilde{\xi}$ and metric $\tilde{g}(\tilde{\xi})$, the following equality must therefore hold:

$$g_{ij}(\xi) d\xi^i d\xi^j = \tilde{g}_{pq}(\tilde{\xi}) d\tilde{\xi}^p d\tilde{\xi}^q \,. \tag{6.2.12}$$

Making use of the chain rule to express the differentials $d\tilde{\xi}$ in terms of differentials $d\xi$,

$$g_{ij}(\xi)d\xi^i d\xi^j = \tilde{g}_{pq}(\tilde{\xi})\frac{\partial\tilde{\xi}^p}{\partial\xi^i}\frac{\partial\tilde{\xi}^q}{\partial\xi^j}\, d\xi^i d\xi^j\,. \tag{6.2.13}$$

Since this result holds for any choice of differentials $d\xi$, we find a relation between the metric in ξ and $\tilde{\xi}$ coordinates:

$$g_{ij}(\xi) = \tilde{g}_{pq}(\tilde{\xi})\,\frac{\partial\tilde{\xi}^p}{\partial\xi^i}\frac{\partial\tilde{\xi}^q}{\partial\xi^j}\,. \tag{6.2.14}$$

Making use of the definition of $\widetilde{M}$ below (6.2.2), we rewrite the above equation as

$$g_{ij}(\xi) = \tilde{g}_{pq}\,\widetilde{M}_{pi}\,\widetilde{M}_{qj} = (\widetilde{M}^T)_{ip}\,\tilde{g}_{pq}\,\widetilde{M}_{qj}\,. \tag{6.2.15}$$

In matrix notation, the right-hand side is the product of three matrices. Taking the determinant and using the notation in (6.2.10) gives

$$g = (\det\widetilde{M}^T)\,\tilde{g}\,(\det\widetilde{M}) = \tilde{g}(\det\widetilde{M})^2\,. \tag{6.2.16}$$

Taking a square root

$$\sqrt{g} = \sqrt{\tilde{g}}|\det\widetilde{M}|\,, \tag{6.2.17}$$

we obtain the transformation property for the square root of the determinant of the metric.

We are finally ready to appreciate the reparameterization invariance of (6.2.11). Making use of (6.2.1), (6.2.17), and (6.2.3) we have

$$\begin{aligned}\int d\xi^1 d\xi^2\sqrt{g} &= \int d\tilde{\xi}^1 d\tilde{\xi}^2|\det M|\sqrt{\tilde{g}}\,|\det\widetilde{M}| \\ &= \int d\tilde{\xi}^1 d\tilde{\xi}^2\sqrt{\tilde{g}}\,,\end{aligned} \tag{6.2.18}$$

which proves the reparameterization invariance of the area functional. To the trained eye the area formula in (6.2.11) is *manifestly* reparameterization invariant. That is, once you know how metrics transform, the invariance is reasonably simple to establish. No cumbersome calculation is necessary.

Quick Calculation 6.2. Consider the equation $\partial\xi^i/\partial\xi^j = \delta^i_j$ and use the chain rule to show the matrix property

$$M\widetilde{M} = 1\,. \tag{6.2.19}$$

Show that $\widetilde{M}M = 1$ holds as well. Finally, note that as a simple consequence, $\det M\det\widetilde{M} = 1$, a result stronger than the one we proved in (6.2.3).

6.3 Area functional for spacetime surfaces

Let us now move to our case of interest, the case of surfaces in *spacetime*. These surfaces are obtained by representing in spacetime the history of strings, in the same way as a spacetime world-line is obtained by representing the history of a particle. For the case of strings, we obtain a two-dimensional surface called the world-sheet of the string. Spacetime surfaces, such as string world-sheets, are not all that different from the spatial surfaces we considered in the previous section. They are two-dimensional, and require two parameters. Instead of calling the parameters ξ^1 and ξ^2, we give them special names: τ and σ.

Given our usual spacetime coordinates $x^\mu = (x^0, x^1, x^2, x^3, \ldots, x^d)$, the surface is described by the mapping functions

$$x^\mu(\tau, \sigma)\,, \tag{6.3.1}$$

taking some region of the (τ, σ) parameter space into spacetime. Following a standard convention in string theory, we change the notation slightly. We will denote the above mapping functions with the capitalized symbols

$$X^\mu(\tau, \sigma)\,. \tag{6.3.2}$$

We are not changing the meaning of the functions. Given a fixed point (τ, σ) in the parameter space, this point is mapped to a point with spacetime coordinates

$$\big(X^0(\tau,\sigma),\, X^1(\tau,\sigma), \ldots,\, X^d(\tau,\sigma)\big)\,. \tag{6.3.3}$$

Why do we capitalize the X's? Suppose we used the same symbol to denote spacetime coordinates and mapping functions. Then we could still distinguish between them by writing x^μ or $x^\mu(\tau, \sigma)$, but we would not have the luxury of dropping the (τ, σ) arguments. On the other hand, with X^μ we can drop the (τ, σ) arguments and you still know that we are talking about the mapping functions of the string. We will call the X^μ the *string coordinates*.

As before, the parameters τ and σ can be viewed as coordinates on the world-sheet, at least locally. The map inverse to X^μ takes the world-sheet to the parameter space, and locally it assigns to each point on the surface two coordinates: the values of the parameters τ and σ. Introducing some potential for confusion, physicists also use the term world-sheet to denote the two-dimensional parameter space whose image under X^μ gives us the ... world-sheet! [1] Unless explicitly stated, we will reserve the use of the term world-sheet for the spacetime surface. In Figure 6.4 we consider an open string: in the left side you see the parameter space surface, and to the right, the spacetime surface.

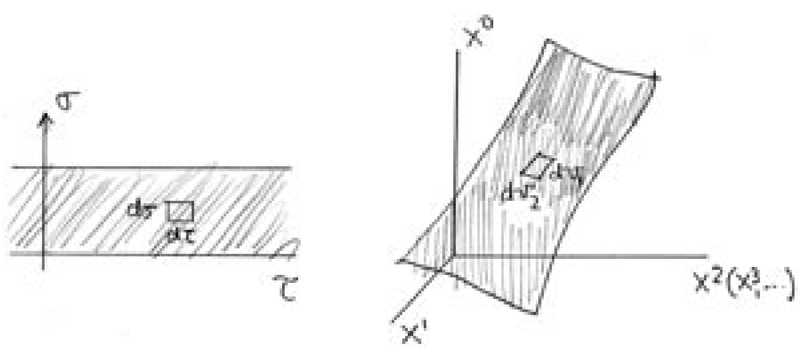

Figure 6.4: Left side: the parameter space (τ, σ), with a little square selected. Right side: The surface in target spacetime with the image of the little square: a parallelogram whose sides are the vectors dv_1^μ and dv_2^μ.

To find the area element, we proceed as in the case of the spatial surface, this time using relativistic notation. The situation is illustrated in Figure 6.4. A little rectangle of sides $d\tau$ and $d\sigma$ in parameter-space, becomes a quadrilateral area element in spacetime. This quadrilateral is spanned by the vectors dv_1^μ and dv_2^μ. Furthermore,

$$dv_1^\mu = \frac{\partial X^\mu}{\partial \tau} d\tau \,, \qquad dv_2^\mu = \frac{\partial X^\mu}{\partial \sigma} d\sigma, \tag{6.3.4}$$

which are analogous to our earlier spatial formulae (6.1.2). We can now use the analog of (6.1.4) as a candidate for the area element dA:

$$dA \stackrel{?}{=} \sqrt{(dv_1 \cdot dv_1)(dv_2 \cdot dv_2) - (dv_1 \cdot dv_2)^2} \,, \tag{6.3.5}$$

[1] Given the spacetime connotation of the term *world*, and the mathematical flavor of the parameter space, I would call the target space surface the world-sheet, and the parameter space surface the *math-sheet*.

where the dot is the relativistic dot product. Using this dot product guarantees that the area element is Lorentz invariant: it is a proper area element. We wrote a question mark on top of the equal sign because there is one problem. Even though this is not obvious to us yet, the sign of the object under the square root is negative. To be able to take the square root we must exchange the two terms under the square root. This change of sign has no effect on the Lorentz invariance. Doing this, and using (6.3.4), we find that the proper area is given as

$$A = \int d\tau d\sigma \sqrt{\Big(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \sigma}\Big)^2 - \Big(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \tau}\Big)\Big(\frac{\partial X^\nu}{\partial \sigma}\frac{\partial X_\nu}{\partial \sigma}\Big)}\,. \tag{6.3.6}$$

Using the relativistic dot product notation,

$$A = \int d\tau d\sigma \sqrt{\Big(\frac{\partial X}{\partial \tau}\cdot\frac{\partial X}{\partial \sigma}\Big)^2 - \Big(\frac{\partial X}{\partial \tau}\Big)^2\Big(\frac{\partial X}{\partial \sigma}\Big)^2}\,. \tag{6.3.7}$$

To understand why the above sign is correct we must convince ourselves that the expression under the square root is positive at any point on the world-sheet of a string.

What characterizes locally the spacetime surface traced by a string? The answer is quite interesting. Consider a point on the world-sheet and the set of all tangent vectors to the surface at that point. These vectors form a two-dimensional vector space. We claim that in this vector space there is a basis made by two vectors, one of which is spacelike, and one of which is timelike. This implies that at each point on the world-sheet there are both timelike and spacelike tangent directions.

The existence of a spacelike direction is easy to visualize: if you took a photograph of the string at some time, every tangent vector along the length of the string would point in a space-like direction. Indeed, in your frame, the events defining the string are simultaneous but spatially-separated.

To appreciate the need for a timelike vector at any point on the world-sheet, consider first the world-line of a point particle. The tangent vector to the world-line is timelike. At each point on the world-line this tangent vector can be used to produce an instantaneous Lorentz observer that sees the particle at rest. Suppose that the tangent vector to the world-line becomes spacelike at some point P. We could imagine at P an infinite collection of Lorentz observers with their (spatial) origin at P, one for each possible velocity. None of them can see the particle at rest at the origin, because the world-line of the origin is timelike for all observers. This is an unphysical situation.

The argument for the string is a little more subtle since there is no way to tell how individual points on the string move. As we shall make abundantly clear later on, the string is not made of constituents whose position we can keep track of (there is just one exception: one can keep track of the motion of the endpoints of an open string). For a closed string world-sheet, for example, consider first the possibility that all along a closed string there is no timelike tangent vector on the world-sheet. This means that we could display all possible Lorentz observers at all points on the string, and no observer could make any point on the string appear to be at rest! A similar unphysical result would occur if any *piece* of the string failed to have timelike tangent vectors on the world-sheet. Since the endpoints of the rest of the string cannot close up the string instantaneously, a piece of the string would have failed to move physically. We must have a timelike vector tangent to the world-sheet at all points on the string.

The existence of both timelike directions and spacelike directions at any point on the world-sheet is our criterion for physical motion. It guarantees that equation(6.3.6) makes sense:

Claim: For a surface where there is at every point P both a timelike direction and a spacelike direction, the quantity under the square root in (6.3.6) is always positive, namely,

$$\Big(\frac{\partial X}{\partial \tau}\cdot\frac{\partial X}{\partial \sigma}\Big)^2 - \Big(\frac{\partial X}{\partial \sigma}\Big)^2\Big(\frac{\partial X}{\partial \tau}\Big)^2 > 0\,. \tag{6.3.8}$$

Proof: We consider every vector tangent to the surface at some point P, and show that in this set there are both spacelike vectors and timelike vectors. First we will parameterize all possible vectors and then search the parameter space. The situation is illustrated in Figure 6.5.

Consider the set of tangent vectors $v^\mu(\lambda)$ at P obtained as:

$$v^\mu(\lambda) = \frac{\partial X^\mu}{\partial \tau} + \lambda\frac{\partial X^\mu}{\partial \sigma}\,, \tag{6.3.9}$$

where λ is a parameter that ranges from minus infinity to plus infinity. Since $\partial X^\mu/\partial\tau$ and $\partial X^\mu/\partial\sigma$ are linearly independent tangent vectors, when we vary λ we get, up to constant scalings, all tangent vectors at P (see Figure 6.5),

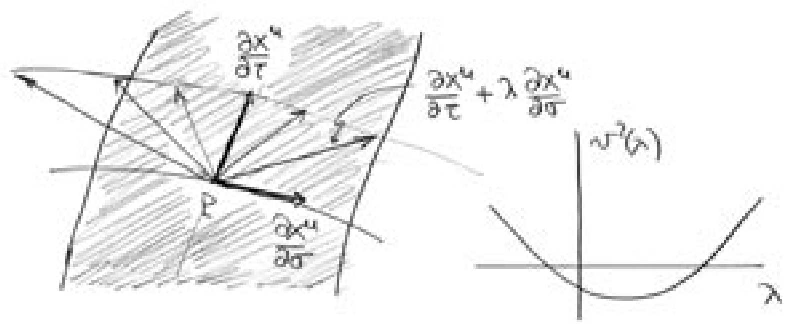

Figure 6.5: Left side: A set of tangent vectors $v(\lambda)$ at a point P on the world-sheet. Right side: a plot of $v^2(\lambda)$ as a function of λ. The vector $v(\lambda)$ may be spacelike or timelike depending on the value of λ.

with the exception of $\partial X^\mu/\partial\sigma$, which is obtained in the limit $\lambda \to \infty$. Constant scalings does not matter in determining whether the vector is timelike or spacelike. To determine whether $v^\mu(\lambda)$ is timelike or spacelike, we consider its square:

$$v^2(\lambda) = v^\mu(\lambda)v_\mu(\lambda) = \lambda^2\Big(\frac{\partial X}{\partial\sigma}\Big)^2 + 2\lambda\Big(\frac{\partial X}{\partial\tau}\cdot\frac{\partial X}{\partial\sigma}\Big) + \Big(\frac{\partial X}{\partial\tau}\Big)^2 . \qquad (6.3.10)$$

The derivatives of X appearing here are just numbers, so we have a quadratic polynomial in λ. To have both timelike and spacelike tangent vectors at P, $v^2(\lambda)$ must take both negative and positive values as we vary λ. In other words, the equation $v^2(\lambda) = 0$ must have two real roots. For this to happen, the discriminant of the quadratic equation $v^2(\lambda) = 0$ must be positive. From (6.3.10) we see that this requires

$$\Big(\frac{\partial X}{\partial\tau}\cdot\frac{\partial X}{\partial\sigma}\Big)^2 - \Big(\frac{\partial X}{\partial\sigma}\Big)^2\Big(\frac{\partial X}{\partial\tau}\Big)^2 > 0\,, \qquad (6.3.11)$$

which is precisely the condition (6.3.8) we set out to prove!

Quick Calculation 6.3. Consider a point on the world-sheet where all tangent vectors are spacelike with the exception of one vector (or any number times the vector) that is null. Why is the quantity under the square root in (6.3.6) zero at this point?

6.4 The Nambu-Goto string action

Now that we are sure that the proper area functional in (6.3.7) is correctly defined, we can introduce the action of the relativistic string. This action is proportional to the proper area of the world-sheet. To have the units of action we must multiply the area functional by some suitable constants.

The area functional in (6.3.7) has units of length squared, as it must be. This is because X^μ has units of length, and each term under the square root has four X's. The units of τ and σ cancel out. Each term in the square root has two σ-derivatives and two τ-derivatives. Their units cancel against the units of the differentials. Nevertheless, we will take σ to have units of length and τ to have units of time. To summarize:

$$[\sigma] = L, \quad [\tau] = T\,, \quad [X^\mu] = L\,, \quad [A] = L^2\,. \tag{6.4.1}$$

Since S must have units of ML^2/T and A has units of L^2, we must multiply the proper area by a quantity with units of M/T. The string tension T_0 has units of force, and force divided by velocity has the desired units of M/T. We can therefore multiply the proper area by T_0/c to get a quantity with the units of action. Making use of (6.3.7) we set the string action equal to

$$\boxed{S = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}\,.} \tag{6.4.2}$$

In writing this action we have introduced some notation for derivatives:

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau}\,, \qquad X^{\mu\prime} \equiv \frac{\partial X^\mu}{\partial \sigma}\,. \tag{6.4.3}$$

Of course, we have not yet confirmed that the symbol T_0 in the string action has the precise interpretation of tension, but we will do so below. We will also confirm that the overall negative sign multiplying the action is correct. The action S is the Nambu-Goto action for the relativistic string.

It is crucial that this action be reparameterization-invariant. We can proceed just as we did with spatial surfaces to write the Nambu-Goto action in a manifestly reparameterization-invariant way. In this case we have

$$-ds^2 = dX^\mu dX_\mu = \eta_{\mu\nu} dX^\mu dX^\nu = \eta_{\mu\nu}\,\frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\nu}{\partial \xi^\beta}\,d\xi^\alpha d\xi^\beta\,. \tag{6.4.4}$$

Here $\eta_{\mu\nu}$ is the target-space metric. Just as in our study of two-dimensional surfaces, we are motivated to define a metric $\gamma = [\gamma_{\alpha\beta}]$ on the world-sheet:

$$\gamma_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\nu}{\partial \xi^\beta} = \frac{\partial X}{\partial \xi^\alpha}\cdot\frac{\partial X}{\partial \xi^\beta}\,. \tag{6.4.5}$$

With $\xi^1 = \tau$ and $\xi^2 = \sigma$, the matrix $\gamma_{\alpha\beta}$ is

$$\gamma_{\alpha\beta} = \begin{bmatrix} (\dot{X})^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & (X')^2 \end{bmatrix}\,. \tag{6.4.6}$$

With the help of this metric we can write the Nambu-Goto action in the manifestly reparameterization-invariant form

$$\boxed{S = -\frac{T_0}{c}\int d\tau d\sigma\sqrt{-\gamma}\,, \quad \gamma = \det(\gamma_{\alpha\beta})\,.} \tag{6.4.7}$$

The analysis in section 6.2 of reparameterization invariance for spatial surfaces holds, without change, in the present case. Not only is the action (6.4.7) manifestly reparameterization-invariant, it is also more compact. In this form, one can readily generalize the Nambu-Goto action to describe the dynamics of objects that have more dimensions than strings. An action of this kind is useful as a first approximation to the dynamics of D-branes.

6.5 Equations of motion, boundary conditions and D-branes

In this section we will obtain the equations of motion that follow by variation of the string action. In doing so we will also have an opportunity to discuss the various boundary conditions that can be imposed on the ends of open strings. Dirichlet boundary conditions will be interpreted to arise due to the existence of D-branes.

Let us begin by writing the Nambu-Goto action (6.4.2) as the double integral of a Lagrangian density $\mathcal{L}$:

$$S = \int_{\tau_i}^{\tau_f} d\tau L = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma\ \mathcal{L}\left(\dot{X}^\mu, X^{\mu\prime}\right), \tag{6.5.1}$$

where $\mathcal{L}$ is given by

$$\mathcal{L}\,(\dot{X}^\mu, X^{\mu\prime}) = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}\,. \tag{6.5.2}$$

We can obtain the equations of motion for the relativistic string by setting the variation of the action (6.5.1) equal to zero. The variation is simply

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}\frac{\partial\,(\delta X^\mu)}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}}\frac{\partial\,(\delta X^\mu)}{\partial \sigma}\right], \tag{6.5.3}$$

where we have used

$$\delta \dot{X}^\mu = \delta\Big(\frac{\partial X^\mu}{\partial \tau}\Big) = \frac{\partial(\delta X^\mu)}{\partial \tau}\,, \tag{6.5.4}$$

and an analogous equation for $\delta X^{\mu\prime}$.

The quantities $\partial\mathcal{L}/\partial\dot{X}^\mu$ and $\partial\mathcal{L}/\partial X'^\mu$ will appear frequently throughout the remainder of our discussion, so it is useful to introduce new symbols for them. This is just what we did when we studied the nonrelativistic string in Section 4.6. This time we find

$$\mathcal{P}^\tau_\mu \;\equiv\; \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}\,, \tag{6.5.5}$$

$$\mathcal{P}^\sigma_\mu \;\equiv\; \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}\,. \tag{6.5.6}$$

Quick Calculation 6.4. Verify equations (6.5.5) and (6.5.6).

Using this notation, the variation δS in (6.5.3) becomes

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_f} d\sigma \left[\frac{\partial}{\partial \tau}(\delta X^\mu \mathcal{P}^\tau_\mu) + \frac{\partial}{\partial \sigma}\left(\delta X^\mu \mathcal{P}^\sigma_\mu\right) - \delta X^\mu\Big(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma}\Big)\right]. \tag{6.5.7}$$

The first term on the right-hand side, being a full derivative in τ, will contribute terms proportional to $\delta X^\mu(\tau_f, \sigma)$ and $\delta X^\mu(\tau_i, \sigma)$. If the initial and final states of the string are specified, we can restrict ourselves to variations for which $\delta X^\mu(\tau_f, \sigma) = \delta X^\mu(\tau_i, \sigma) = 0$. We will always assume such variations, so we can forget about these terms. The variation then becomes

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \left[\delta X^\mu \mathcal{P}^\sigma_\mu\right]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma\, \delta X^\mu \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma}\right). \tag{6.5.8}$$

The first term on the right-hand side has to do with the string endpoints. As before, there are two natural types of boundary conditions which one can impose on the endpoints. The first is Dirichlet boundary conditions, which require that the endpoints of the string remain fixed throughout the motion:

$$\boxed{\text{Dirichlet Boundary Condition:} \quad \frac{\partial X^\mu}{\partial \tau}(0,\tau) = \frac{\partial X^\mu}{\partial \tau}(\sigma_1,\tau) = 0\,.} \tag{6.5.9}$$

Alternatively, rather than requiring that the τ-derivatives vanish, we could simply specify constant values for $X^\mu(0,\tau)$ and $X^\mu(\sigma_1,\tau)$. If the string endpoints are fixed, the variations are set to vanish at the endpoints: $\delta X^\mu(0,\tau) = 0$, and $\delta X^\mu(\sigma_1,\tau) = 0$. This will guarantee that the first term in δS vanishes. Alternatively, setting

$$\boxed{\text{Free Boundary Condition :} \quad \mathcal{P}^\sigma_\mu(0,\tau) = \mathcal{P}^\sigma_\mu(\sigma_1,\tau) = 0\,,} \tag{6.5.10}$$

would also result in the vanishing of the boundary term. This is the "free-endpoints" boundary condition for the relativistic string. For the non-relativistic string, the free-endpoints boundary condition implies the vanishing of $\mathcal{P}^x$, which imposes a Neumann boundary condition on the string coordinate (see (4.6.16)). While it will take us some work to get there, we will eventually understand (6.5.10) in terms of a Neumann boundary condition. Similarly, the Dirichlet boundary (6.5.9) will be shown to imply the vanishing of $\mathcal{P}^\tau_\mu$ at the string endpoints.

The above boundary conditions can be imposed in many possible ways. We need not use the same boundary condition for all values of the index μ. Some string coordinates may have a Dirichlet-type condition, and some others may have a free-type condition. Even more, for any given μ, the two endpoints of the open string need not satisfy the same boundary conditions: one end could be fixed and the other free. For closed strings there are no boundary conditions.

Let us digress for a while on the case of Dirichlet boundary conditions. It is clear from the study of non-relativistic strings that Dirichlet boundary conditions arise if string endpoints are attached to some physical objects. Consider, for example, Figure 4.2. On the left side of the figure, the string is attached to two points. On the right side of the figure the string is free to slide up and down at the endpoints, but the string endpoints are forced to stay on one-dimensional lines – horizontal motion of the endpoints is forbidden.

The objects where open string endpoint must lie on, are characterized by their dimensionality, more precisely, by the number of spatial dimensions they have. They are called D-branes, where the D stands for Dirichlet. The objects fixing the string on the left-side of Figure 4.2, are zero-dimensional. They are called D0-branes. The lines fixing the string on the right-side of the figure, are one-dimensional. They are called D1-branes.

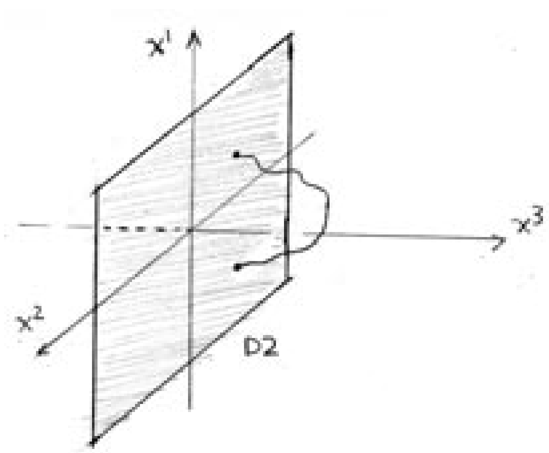

Figure 6.6: A D2-brane stretched over the (x^1, x^2) plane. The endpoints of the open string can move freely on the plane, but must remain attached to it. The coordinate x^3 of the endpoints must vanish at all times. This is a Dirichlet boundary condition for the string coordinate X^3.

A Dp-brane is an object with p spatial dimensions. Since the string endpoints must lie on the Dp-brane, a set of Dirichlet boundary conditions are specified. A flat D2-brane in a three-dimensional space, for example, is specified by one condition, say $x^3 = 0$ (see Figure 6.6). This means that the D2-brane extends over the (x^1, x^2) plane. The Dirichlet boundary condition applies to the string coordinate X^3, which must vanish for the string endpoints. Since the motion of the string endpoint is free along the directions of the brane, the string coordinates X^1 and X^2 satisfy free boundary conditions. When the open string endpoints have free boundary conditions along all spatial directions, we still have a D-brane, but this time it is a space-filling D-brane. The D-brane extends all over space, and since open string endpoints can be anywhere on the D-brane, open string endpoints are completely free.

For (quantum) relativistic strings the consistency of Dirichlet boundary conditions allows one to discover the properties of D-branes. D-branes are physical objects that exist in a theory of strings, and they are not introduced by hand. They have calculable energy densities, and a host of remarkable properties. We will study them in more detail beginning in Chapter 12.

Returning after this long aside to the variation of the action, since the second term in (6.5.8) must vanish for all variations of the motion, we set

$$\boxed{\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0\,.} \tag{6.5.11}$$

This is the equation of motion for the relativistic string, open or closed. A quick glance at definitions (6.5.5) and (6.5.6) shows that this equation is incredibly complicated. The key to its solution will lie in the reparameterization invariance of the Nambu-Goto action. Choosing a clever parameterization will simplify our work enormously.

6.6 The static gauge

To make progress in understanding the action for the relativistic string, we must parameterize the string surface in a useful way. We are allowed to freely choose the parameterization because of the reparameterization invariance of the string action. Reparameterization invariance in string theory is analogous to gauge invariance in electrodynamics. Maxwell's equations possess a symmetry under gauge transformations that allows us to use different potentials A_μ to represent the same electromagnetic fields $\vec{E}$ and $\vec{B}$. A suitable choice of gauge helps to uncover the physics. Similarly, we may use many different grids on the world-sheet to describe the same physical motion of the string. A suitable choice of grid can make this task much easier. A good choice of parameterization was useful even for the relativistic point particle – its equation of motion is simplest when the trajectory is parameterized by proper time.

In this section, we will discuss only a partial parameterization on the world-sheet. We will fix the lines of constant τ by relating τ to the time coordinate $X^0 = ct$, the time in some chosen Lorentz frame.

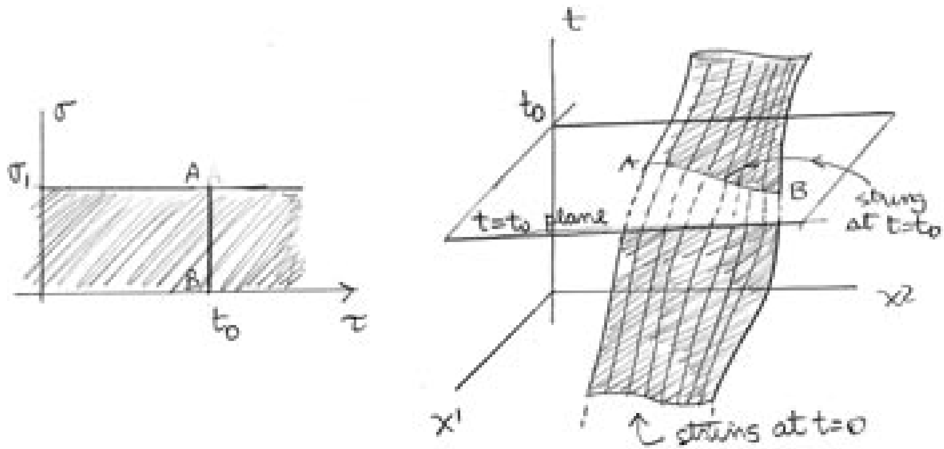

Figure 6.7: Left side: the parameter space strip for an open string. The vertical segment AB is the line $\tau = t_0$. Right side: the open string world-sheet in target space. The string at time $t = t_0$ is the intersection of the world-sheet with the hyperplane $t = t_0$. In the static gauge, the string at time $t = t_0$ is the image of the $\tau = t_0$ segment AB.

We proceed as in Figure 6.7. Suppose we draw a hyperplane of constant t in the target space, say the $t = t_0$ plane. This plane will intersect the world-sheet along a curve – *the string at time* t_0 according to observers in our chosen Lorentz frame. We declare this curve to be a curve of constant τ; in fact, we declare it to be the curve $\tau = t_0$. Extending this definition to all times t, we declare that for any point Q on the world-sheet

$$\tau(Q) = t(Q). \tag{6.6.1}$$

This choice of τ parameterization is called the *static gauge* because lines of constant τ are "static strings" in the chosen Lorentz frame.

We will not try to make a sophisticated choice of σ at this time. For an open string, we will choose one edge of the worldsheet to be the curve $\sigma = 0$, and the other edge to be the curve $\sigma = \sigma_1$:

$$\sigma \in [0, \sigma_1] , \quad \text{for an open string.} \tag{6.6.2}$$

We draw lines of constant σ on the surface quite arbitrarily, provided, of course, that constant σ lines vary smoothly, do not intersect, and are consistent with the two curves which are the boundary of the world-sheet. Drawing constant σ lines is equivalent to giving an explicit σ-parameterization to all the strings.

For closed strings the same ideas apply, but there is a significant proviso: there must be an identification in the (τ, σ) parameter space. The σ direction must be made into a circle, making the (τ, σ) parameter space into a cylinder. This is needed because the closed string world-sheet is topologically a cylinder. Letting σ_c denote the circumference of the σ circle, the identification is

$$(\sigma, \tau) \sim (\sigma + \sigma_c, \tau) \,. \tag{6.6.3}$$

Points that are identified by this relation on the parameter space map to the same point on the closed string world-sheet. The closed strings can be parameterized using any σ interval of length σ_c, for example

$$\sigma \in [0, \sigma_c] \,, \quad \text{for a closed string.} \tag{6.6.4}$$

Let us now explore the implications of our choice of τ. We can write (6.6.1) as

$$X^0(\sigma, \tau) \equiv c\, t(\sigma, \tau) = c\, \tau \,, \tag{6.6.5}$$

or simply

$$\tau = t \,. \tag{6.6.6}$$

We can thus describe the collection of string coordinates X^μ as

$$X^\mu(\tau, \sigma) = X^\mu(t, \sigma) = \{c\, t, \vec{X}(t, \sigma)\} \,, \tag{6.6.7}$$

letting the vector $\vec{X}$ represent the spatial string coordinates. We then find

$$\begin{aligned} \frac{\partial X^\mu}{\partial \sigma} &= \Big(\frac{\partial X^0}{\partial \sigma}, \frac{\partial \vec{X}}{\partial \sigma}\Big) = \Big(0, \frac{\partial \vec{X}}{\partial \sigma}\Big), \\ \frac{\partial X^\mu}{\partial \tau} &= \Big(\frac{\partial X^0}{\partial t}, \frac{\partial \vec{X}}{\partial t}\Big) = \Big(c\,, \frac{\partial \vec{X}}{\partial t}\Big) \,. \end{aligned} \tag{6.6.8}$$

As you can see, this parameterization separates the time and space components quite neatly.

Now that we have made a choice of τ coordinates, we can do a simple test to confirm that we got the right sign under the radical in the Nambu-Goto action (6.4.2). Imagine a little piece of string with no velocity. Because it is not moving, $\partial\vec{X}/\partial t = 0$, and using (6.6.8), the square root in (6.4.2) becomes

$$\sqrt{0 - \Big(\frac{\partial\vec{X}}{\partial\sigma}\Big)^2(-c^2)}. \tag{6.6.9}$$

The quantity under the square root is positive, just as we expected. If some day you forget the sign under the radical in the string action, this is a good way to check it quickly.

6.7 Tension and energy of a stretched string

Let us now do our first calculation with the Nambu-Goto action – our first calculation in string theory! We are going to analyze a stretched relativistic string. The endpoints of the string are fixed at $x^1 = 0$, and at $x^1 = a > 0$, with vanishing values for the coordinates of the additional spatial dimensions. We therefore write the string endpoints as the (space) points $(0, \vec{0})$ and $(a, \vec{0})$. The inclusion of the common $(d-1)$-dimensional vector $\vec{0}$ tells us that the string is only stretched along the first spatial coordinate.

We now evaluate the string action for this stretched string using the static gauge $X^0 = c\tau$. Because this is a static string stretched from $x^1 = 0$ to $x^1 = a$, we can write

$$X^1(t,\sigma) = f(\sigma)\,, \quad X^2 = X^3 = \cdots = X^d = 0\,, \tag{6.7.1}$$

where

$$f(0) = 0\,, \quad f(\sigma_1) = a\,, \tag{6.7.2}$$

and the function $f(\sigma)$ is strictly increasing and continuous on the interval $\sigma \in [0, \sigma_1]$. The setup is illustrated in Figure 6.8. The function f must be strictly increasing to ensure that each point along the string is assigned a unique σ coordinate.

It now follows that

$$\dot{X}^\mu = (c, 0, \vec{0}), \qquad X'^\mu = (0, f', \vec{0}), \tag{6.7.3}$$

with $f' = df/d\sigma$. Therefore

$$(\dot{X})^2 = -c^2\,, \quad (X')^2 = (f')^2\,, \quad \dot{X}\cdot X' = 0. \tag{6.7.4}$$

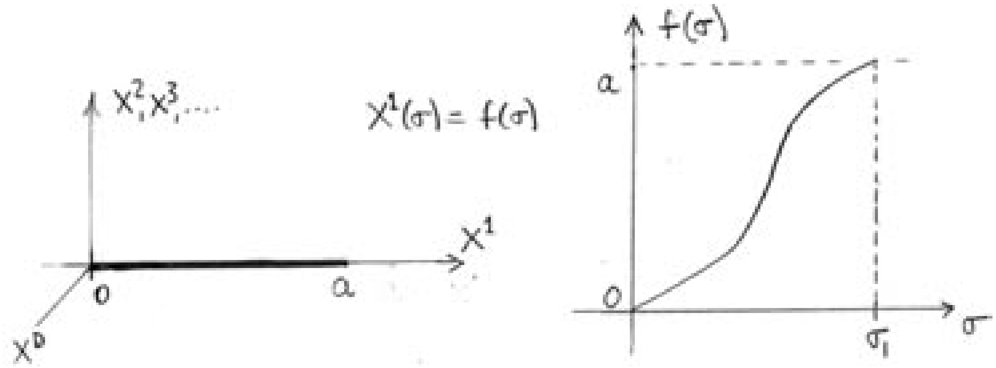

Figure 6.8: A string of length a stretched along the x^1-axis. The string is parameterized as $X^1(t,\sigma) = f(\sigma)$.

We can now evaluate the action (6.4.2):

$$S = -\frac{T_0}{c}\int_{t_i}^{t_f} dt \int_0^{\sigma_1} d\sigma \sqrt{0 - (-c^2)(f')^2} = -T_0 \int_{t_i}^{t_f} dt \int_0^{\sigma_1} d\sigma \frac{df}{d\sigma}\,. \quad (6.7.5)$$

The σ integrand is a total derivative, so

$$S = -T_0 \int_{t_i}^{t_f} dt\,(f(\sigma_1) - f(0)) = \int_{t_i}^{t_f} dt(-T_0 a)\,, \quad (6.7.6)$$

where we used (6.7.2). It is interesting to see that the value of the action does not depend on the function f used to parameterize the string. This is an explicit confirmation of the reparameterization invariance of the string action.

We would like to interpret our result. For this, recall that the action is the time integral of the Lagrangian L. When the kinetic energy vanishes, $L = -V$, where V is the potential energy. Since our string is static, there is no kinetic energy, so

$$S = \int_{t_i}^{t_f} dt(-V). \quad (6.7.7)$$

Comparing this with (6.7.6) we conclude that

$$V = T_0\, a\,. \quad (6.7.8)$$

The potential energy of our stretched string is just $T_0 a$. What does this mean? If the tension of a static string is T_0, regardless of its length, then $T_0 a$ is the amount of energy you must spend to create a string of length a. Imagine that you start with an infinitesimal string and you start pulling it. As you do work you are giving energy to the string, in fact, you are creating rest energy, or rest mass. The rest mass μ_0 per unit length is

$$\mu_0 c^2 = \frac{V}{a} = T_0 \quad \rightarrow \quad \mu_0 = \frac{T_0}{c^2} . \tag{6.7.9}$$

The mass (or rest energy) arises only because the string has a tension. Because of this, the relativistic string is sometimes referred to as a massless string. The above calculation also confirms that the minus sign in front of the action (6.4.2) is necessary – otherwise the potential energy of the stretched string would have come out negative. Finally, the constant T_0 was identified as the string tension.

There is one point we have glossed over. We assumed in our analysis that the configuration in (6.7.1) satisfies the string equations of motion. If it does not, then the configuration cannot be physically realized. Let us check that the equations of motion are satisfied.

First note that on account of (6.7.3) neither $\dot{X}^\mu$ nor $X^{\mu\prime}$ has τ dependence. Therefore neither $\mathcal{P}^\tau$ nor $\mathcal{P}^\sigma$ has τ dependence (see (6.5.5) and (6.5.6)). This being the case, the equation of motion (6.5.11) reduces to

$$\frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0 . \tag{6.7.10}$$

This requires that $\mathcal{P}^\sigma_\mu$ be σ-independent. We look again at (6.5.6) and use (6.7.4) to find

$$\mathcal{P}^\sigma_\mu = -\frac{T_0}{c} \frac{c^2 X'_\mu}{\sqrt{c^2 (f')^2}} = -T_0 \frac{X'_\mu}{f'} . \tag{6.7.11}$$

This is non-vanishing only for $\mu = 1$, in which case $X'_1 = f'$, so $\mathcal{P}^\sigma$ is indeed σ-independent. Thus the equation of motion is satisfied. Even the boundary conditions are satisfied. As we discussed in section 6.5, there is no condition to be checked for string coordinates when the endpoints satisfy Dirichlet boundary conditions. In our problem this means that there are no extra conditions to be checked for any of the spatial coordinates. On the other hand, for the zeroth coordinate our choice of gauge required $X^0 = c\tau$. This is not a Dirichlet boundary condition since X^0 is not a constant anywhere on the string. It follows that X^0 must be treated as a coordinate with free endpoints, and the condition in (6.5.10) must be checked. This just requires $\mathcal{P}_0^\sigma = 0$, a fact that holds on account of (6.7.11).

6.8 Action in terms of transverse velocity

We have begun to choose a specific parameterization of the string surface by fixing τ via the condition $X^0 = ct = c\tau$. With this choice, a line of constant τ on the string spacetime surface corresponds to the string as seen by our chosen Lorentz observer at the particular time $t = \tau$.

Can we define some sort of string velocity? Since $\vec{X}(t, \sigma)$ are the string spatial coordinates, the derivative $\partial\vec{X}/\partial t$ seems to be the closest thing we have to a velocity. This velocity, however, depends upon the choice of σ. Its direction, for example, goes along the lines of constant σ. Since σ can be chosen quite arbitrarily, keeping σ constant in taking the derivative is clearly not very physically significant!

Fixing physically the σ parameterization of a string is subtle because the string is an object with no substructure. When comparing a string at two nearby times, it is not possible to say that a point moved from one location to the next. To speak of points on the string we need a σ parameterization, and reparameterization invariance makes it clear to us that this parameterization is not unique. This suggests that longitudinal motion on the string is not physically meaningful.

There is an invariant velocity that can be defined on the string. This is, however, a *transverse* velocity. We consider the string motion in *space*, and imagine that each point on the string moves transversely to the string. Consider a string at some fixed time and pick a point p on it. Draw the hyperplane orthogonal to the string at p. An infinitesimal instant later the string has moved, but it will still intersect the plane, this time at a point p'. The transverse velocity is what we get if we presume that the point p moved to p'. No string parameterization is needed to define this velocity.

When speaking of evolving strings there are two surfaces we can discuss. One is the world-sheet, the surface in spacetime which represents the history of the string. The other is a surface in *space*. This *spatial surface* is put together by combining the strings that we observe at all times. This is the surface that would be generated if the string were to leave a wake as it moved.

The transverse velocity at any point on the string is a vector orthogonal to the string and tangent to the string spatial surface.

We are therefore motivated to define, for each point on the string, a velocity $\vec{v}_\perp$ perpendicular to the string itself. Our discussion above indicates that this is a reparameterization-invariant notion of velocity, and therefore we expect it to enter naturally into the evaluation of the string action for a string moving arbitrarily.

In order to define the perpendicular velocity, it is useful to introduce a unit vector tangent to the string. To this end, we now introduce a parameter s which is more physical than our nearly-arbitrary σ. Let us work with a fixed string, and define $s(\sigma)$ to be the length of the string in the interval $[0, \sigma]$. Thus, for example, $s(0) = 0$, and $s(\sigma_1)$ is the length of an entire open string. Since ds is the length of the infinitesimal vector $d\vec{X}$ arising from a world-sheet segment $d\sigma$ along the string, we have:

$$ds = |d\vec{X}| = \left|\frac{\partial \vec{X}}{\partial \sigma}\right| |d\sigma| \,. \tag{6.8.1}$$

Now consider the quantity $\partial \vec{X}/\partial s$, which is the variation of $\vec{X}$ with the length of the string. First note that it is a unit vector:

$$\frac{\partial \vec{X}}{\partial s} \cdot \frac{\partial \vec{X}}{\partial s} = \frac{\partial \vec{X}}{\partial \sigma} \cdot \frac{\partial \vec{X}}{\partial \sigma}\Big(\frac{d\sigma}{ds}\Big)^2 = \left|\frac{\partial \vec{X}}{\partial \sigma}\right|^2 \Big(\frac{d\sigma}{ds}\Big)^2 = 1\,. \tag{6.8.2}$$

The derivative $\partial \vec{X}/\partial \sigma$, as the notation indicates, is taken with t held fixed, and therefore it lies along a line of constant t. Since the lines of constant t are precisely the strings, it is tangent to the string. In addition

$$\frac{\partial \vec{X}}{\partial s} = \frac{\partial \vec{X}}{\partial \sigma}\frac{d\sigma}{ds}\,, \tag{6.8.3}$$

and thus $\partial \vec{X}/\partial s$ is also tangent to the string. Because it has unit length,

$$\frac{\partial \vec{X}}{\partial s} \quad \text{is a unit vector tangent to the string}\,. \tag{6.8.4}$$

We define $\vec{v}_\perp$ to be the component of the velocity $\partial \vec{X}/\partial t$, in the direction perpendicular to the string (see Figure 6.9). For any vector $\vec{u}$, its component perpendicular to a unit vector $\vec{n}$ is $\vec{u} - (\vec{u}\cdot\vec{n})\vec{n}$. Therefore, using our unit vector $\partial\vec{X}/\partial s$ along the string, we have

$$\vec{v}_\perp = \frac{\partial\vec{X}}{\partial t} - \Big(\frac{\partial\vec{X}}{\partial t}\cdot\frac{\partial\vec{X}}{\partial s}\Big)\frac{\partial\vec{X}}{\partial s}\,. \tag{6.8.5}$$

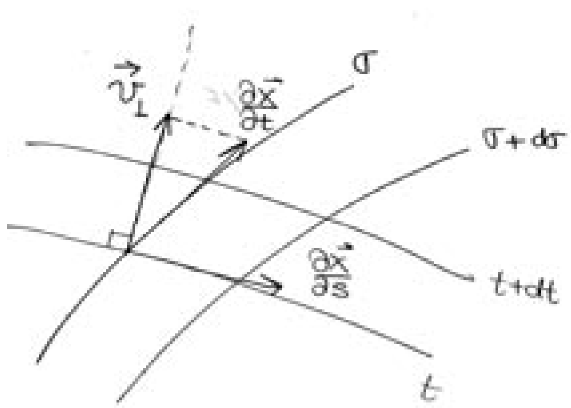

Figure 6.9: A small piece of the world-sheet showing the vector $\partial\vec{X}/\partial t$, the transverse velocity $\vec{v}_\perp$ and the unit vector $\partial\vec{X}/\partial s$.

For future use, let's calculate $v_\perp^2$:

$$\begin{aligned} v_\perp^2 &= \Big(\frac{\partial\vec{X}}{\partial t}\Big)^2 - 2\Big(\frac{\partial\vec{X}}{\partial t}\cdot\frac{\partial\vec{X}}{\partial s}\Big)\Big(\frac{\partial\vec{X}}{\partial t}\cdot\frac{\partial\vec{X}}{\partial s}\Big) + \Big(\frac{\partial\vec{X}}{\partial t}\cdot\frac{\partial\vec{X}}{\partial s}\Big)^2, \\ &= \Big(\frac{\partial\vec{X}}{\partial t}\Big)^2 - \Big(\frac{\partial\vec{X}}{\partial t}\cdot\frac{\partial\vec{X}}{\partial s}\Big)^2. \end{aligned} \tag{6.8.6}$$

The string action depends only upon products of the vectors $\dot{X}^\mu$ and $X^{\mu\prime}$. Our goal now is to write it in terms of $\vec{v}_\perp$ and other quantities, if necessary. Using the static gauge $\tau = t$, and equations (6.6.8), we find

$$(\dot{X})^2 = -c^2 + \Big(\frac{\partial\vec{X}}{\partial t}\Big)^2, \; (X')^2 = \Big(\frac{\partial\vec{X}}{\partial\sigma}\Big)^2, \quad \dot{X}\cdot X' = \frac{\partial\vec{X}}{\partial t}\cdot\frac{\partial\vec{X}}{\partial\sigma}\,. \tag{6.8.7}$$

With these relations we simplify the square root in the string action:

$$
\begin{aligned}
(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2 &= \Big(\frac{\partial\vec{X}}{\partial t}\cdot\frac{\partial\vec{X}}{\partial\sigma}\Big)^2 + \Big[c^2 - \Big(\frac{\partial\vec{X}}{\partial t}\Big)^2\Big]\Big(\frac{\partial\vec{X}}{\partial\sigma}\Big)^2 \\
&= \Big(\frac{ds}{d\sigma}\Big)^2\Big[\Big(\frac{\partial\vec{X}}{\partial t}\cdot\frac{\partial\vec{X}}{\partial s}\Big)^2 + c^2 - \Big(\frac{\partial\vec{X}}{\partial t}\Big)^2\Big]. \qquad (6.8.8)
\end{aligned}
$$

The terms on the right-hand side above can be neatly expressed in terms of $v_\perp^2$. Making use of (6.8.6),

$$
(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2 \;=\; \Big(\frac{ds}{d\sigma}\Big)^2\big(c^2 - v_\perp^2\big)\,, \qquad (6.8.9)
$$

or, alternatively,

$$
\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2} = c\,\frac{ds}{d\sigma}\sqrt{1-\frac{v_\perp^2}{c^2}}\,. \qquad (6.8.10)
$$

This simple expression for the string Lagrangian density shows that $\vec{v}_\perp$ is a natural dynamical variable. Moreover, the longitudinal component of the velocity is completely irrelevant. Now we can write the string action as

$$
\boxed{S = -T_0\int dt\int_0^{\sigma_1} d\sigma\Big(\frac{ds}{d\sigma}\Big)\sqrt{1-\frac{v_\perp^2}{c^2}}\,.} \qquad (6.8.11)
$$

Here $ds/d\sigma = |\partial\vec{X}/\partial\sigma|$. Moreover, we did not cancel the $d\sigma$'s because it is typically useful to have an integral over a fixed parameter range. While the range of σ is constant, the length s of a string is time-dependent.

The associated Lagrangian is given by

$$
L = -T_0\int ds\sqrt{1-\frac{v_\perp^2}{c^2}}\,. \qquad (6.8.12)
$$

This formula was written as an integral over the length parameter in order to give an interpretation. For each piece of string, $T_0 ds$ is its rest energy. As a result, the Lagrangian is an integral over the string of (minus) the rest energy times a local relativistic factor. In this form, we recognize (6.8.12) as the natural generalization of the relativistic particle Lagrangian (5.1.8).

The action (6.8.11) is valid both for open strings and for closed strings. Although relatively simple, it still leads to rather complicated equations of motion in all but the most symmetrical situations. In order to obtain simple equations of motion, we will have to be clever in our choice of σ. For open strings, in addition, we must understand how the endpoints move. We turn now to this question.

6.9 Motion of open string endpoints

We will now analyze the motion of the endpoints of an open relativistic string. We consider endpoints that are free to move in all directions. Given our discussion in section 6.5, this means that we have a space-filling D-brane. Free endpoints are specified by the boundary conditions (6.5.10), which require the vanishing of $\mathcal{P}^\sigma_\mu$ at the endpoints. We will discover two important properties of the free motion of open string endpoints:

- The endpoints move with the speed of light.
- The endpoints move transversely to the string.

On the interior of the string the notion of a velocity was ambiguous. For the string endpoints, however, the velocity is well-defined – there is no ambiguity defining the velocity of points! Therefore, our statements about endpoint motion have content. In the second statement, motion transverse to the string means that the velocity of an endpoint is orthogonal to the tangent to the string at the endpoint.

To prove the above properties, we investigate our expression (6.5.6) for $\mathcal{P}^{\sigma\mu}$, which we know must vanish at the endpoints. The denominator of $\mathcal{P}^{\sigma\mu}$ is given in (6.8.10) and the numerator is simplified using relations (6.8.7). We find

$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c}\,\frac{\left(\frac{\partial\vec{X}}{\partial\sigma}\cdot\frac{\partial\vec{X}}{\partial t}\right)\dot{X}^\mu - \left(-c^2 + (\frac{\partial\vec{X}}{\partial t})^2\right)X^{\mu\prime}}{c\frac{ds}{d\sigma}\sqrt{1-\frac{v_\perp^2}{c^2}}}\,. \tag{6.9.1}$$

Bringing the $ds/d\sigma$ from the denominator up to the numerator, we can turn derivatives with respect to σ into derivatives with respect to s:

$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c^2}\,\frac{\left(\frac{\partial\vec{X}}{\partial s}\cdot\frac{\partial\vec{X}}{\partial t}\right)\dot{X}^\mu + \left(c^2 - (\frac{\partial\vec{X}}{\partial t})^2\right)\frac{\partial X^\mu}{\partial s}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\,. \tag{6.9.2}$$

Now consider the $\mu = 0$ component of this quantity. In this case we can make some simplifications: $\dot{X}^0 = c$ and $\partial X^0/\partial s = c\,\partial t/\partial s = 0$. We find that

$$\mathcal{P}^{\sigma,0} = -\frac{T_0}{c}\,\frac{(\frac{\partial\vec{X}}{\partial s}\cdot\frac{\partial\vec{X}}{\partial t})}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\,. \tag{6.9.3}$$

Since $\mathcal{P}^{\sigma,0}$ vanishes at the endpoints, and the square root in the denominator is manifestly finite, we deduce that

$$\frac{\partial \vec{X}}{\partial s} \cdot \frac{\partial \vec{X}}{\partial t} = 0 \text{ at the endpoints}. \tag{6.9.4}$$

Since $\partial \vec{X}/\partial s$ is a unit vector tangent to the string, and $\partial \vec{X}/\partial t$ is the endpoint velocity, this equation proves that the endpoints move transversely to the string – one of our two claims. In agreement with this interpretation, using (6.9.4) in (6.8.5) we see that at the endpoints $\vec{v} = \vec{v}_\perp$. Equation (6.9.4) actually allows for vanishing endpoint velocity, in which case the transversality property would be trivially satisfied. But this cannot happen; the endpoints move with the speed of light, as we now show.

Using (6.9.4), we simplify the expression (6.9.2) for $\mathcal{P}^{\sigma\mu}$ at the endpoints:

$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c^2}\frac{c^2(1-\frac{v^2}{c^2})(\frac{\partial X^\mu}{\partial s})}{\sqrt{1-\frac{v^2}{c^2}}} = -T_0\sqrt{1-\frac{v^2}{c^2}}\,\frac{\partial X^\mu}{\partial s}\,, \quad \text{at the endpoints.} \tag{6.9.5}$$

For the space coordinates, $\mu = 1, 2, 3, \ldots, d$, equation (6.9.5) gives

$$\vec{\mathcal{P}}^{\sigma} = -T_0\sqrt{1-\frac{v^2}{c^2}}\,\frac{\partial \vec{X}}{\partial s} = 0\,, \quad \text{at the endpoints.} \tag{6.9.6}$$

Since $\partial \vec{X}/\partial s$ is a unit vector, we conclude that

$$v^2 = c^2\,. \tag{6.9.7}$$

This proves that free endpoints move with the speed of light.

Problems

Problem 6.1. *Stretched string and the non-relativistic limit.*

A Nambu-Goto string with endpoints attached at $(0, \vec{0})$ and $(a, \vec{0})$ (as in section 6.7) is vibrating non-relativistically. Show that the action (6.8.11) reduces to that of a non-relativistic string with transverse oscillations. What is the tension and linear mass density of that non-relativistic string?

Problem 6.2. *Time evolution of a closed circular string.*

At $t = 0$, a closed string forms a circle of radius R on the (x, y) plane and has zero velocity. The time development of this string can be studied using the action (6.8.11). The string will remain circular, but its radius will be time-dependent. Calculate the radius and velocity as functions of time. Sketch the spacetime surface traced by the string in a 3-dimensional plot with x, y, and ct axes.

Problem 6.3. *Covariant analysis of open string endpoint motion.*

Use the explicit form of $\mathcal{P}^\sigma_\mu$ to calculate $\mathcal{P}^\sigma_\mu \mathcal{P}^{\sigma\mu}$ explicitly. Show that the result of this calculation can be used to prove that free open string endpoints move with the speed of light.

Problem 6.4. *Hamiltonian density for relativistic strings.*

Consider the string Lagrangian density $\mathcal{L}$ in the static gauge, and written in terms of $\partial_\sigma \vec{X}$ and $\partial_t \vec{X}$. Calculate the canonical momentum $\vec{\mathcal{P}}(\sigma, t)$:

$$\vec{\mathcal{P}}(\sigma, t) = \frac{\partial \mathcal{L}}{\partial(\partial_t \vec{X})}\,.$$

Recall that $\mathcal{L}$ can be written in terms of $\vec{v}_\perp$ and $\frac{ds}{d\sigma}$. Show that this is also possible for $\vec{\mathcal{P}}$. Calculate the Hamiltonian density $\mathcal{H}$. Write the total Hamiltonian as $H = \int d\sigma \mathcal{H} = \int ds(\cdots)$ and show that your answer is consistent with the interpretation that the energy of the string arises as energy of transverse motion of a string whose rest mass arises solely from the tension.

Problem 6.5. *Open strings ending on D-branes of various dimensions.*

Consider a world with d spatial dimensions. A Dp-brane is an extended object with p spatial dimensions: a p-dimensional hyperplane inside the d-dimensional space. We will examine properties of strings ending on a Dp-brane, where $0 \leq p < d$. The case $p = d$, where the D-brane is space-filling was discussed in section 6.9.

For a Dp-brane, let x^i, with $i = 1, 2, \cdots, p$, correspond to directions on the Dp-brane, and x^a with $a = p+1, p+2, \cdots, d$, correspond to directions orthogonal to the Dp-brane. The Dp-brane position would be specified, for example by $x^a = 0$, with $a = p+1, \cdots, d$. Open string endpoints must lie on the Dp-brane, and, focusing on the $\sigma = 0$ endpoint we have

$$X^a(\sigma = 0, t) = 0\,, \qquad a = p+1, p+2, \cdots, d\,.$$

There are no constraints on $X^i(\sigma = 0, t)$.

(a) State the boundary conditions that the various components of $\mathcal{P}^\sigma_\mu$ must satisfy. Distinguish three cases: the case of $\mathcal{P}^\sigma_0$, the case of the components $\mathcal{P}^\sigma_i$, and the case of the components $\mathcal{P}^\sigma_a$.

(b) Show that all constraints are automatically satisfied when the string ends on a D0-brane.

(c) Show that for a string ending on a D1-brane, the tangent to the string at the endpoint must be orthogonal to the D1-brane, and the endpoint velocity is unconstrained.

(d) For $p \geq 2$ show that there are two possibilities:

  (i) the string is orthogonal to the Dp-brane at the endpoint and the endpoint velocity is unconstrained, or,

  (ii) the tangent to the string at the endpoint is not orthogonal to the Dp-brane and the endpoint moves with the speed of light.

Chapter 7

String parameterization and classical motion

We require that the lines of constant σ be perpendicular to the lines of constant τ and use the energy carried by the string to fix the σ parameterization. The resulting system of equations includes wave equations and two nonlinear constraints. We solve the general equations of motion for open strings.

7.1 Choosing a σ parameterization

We have already learned a few facts about the motion of relativistic strings. In particular, we have investigated the motion of free open string endpoints. This was all done using the static gauge $X^0 \equiv ct = c\tau$, which partially fixed the parameterization on the string surface. Once we have chosen this gauge, the string motion is defined by the functions $\vec{X}(t, \sigma)$. As we vary t and σ, we get the string spatial surface – the surface in space consisting of the strings at all times.

It is now time to find a useful σ parameterization of the string spatial surface. If we know the σ parameterization of the string spatial surface, we also know the σ parameterization of the world-sheet – the spacetime surface generated by the string motion. In this chapter, when speaking about the string surface, we will generally be speaking about the string spatial surface. For example, the statement that the open string endpoints move transversely to the string, implies that the vectors tangent to the boundary of the string spatial surface are orthogonal to the strings.

We will now show how to use a particular σ parameterization of a single string to construct a useful σ parameterization of all strings, and thus of the entire string surface. Suppose the $t = 0$ string is given some σ parameterization with $\sigma \in [0, \sigma_1]$. Now, consider the string at $t = \Delta$, with Δ infinitesimal. On the string spatial surface we can draw short segments perpendicular to the $t = 0$ string. Let these segments intersect the $t = \Delta$ string. Consider a point σ_0 on the $t = 0$ string, and the short perpendicular above it. We declare the intersection of this perpendicular with the $t = \Delta$ string to also have $\sigma = \sigma_0$. We do this all over the $t = 0$ string, obtaining a parameterization of the $t = \Delta$ string. We then repeat this procedure, using the $t = \Delta$ string to parameterize the $t = 2\Delta$ string. We continue in this way, working in the limit of very small Δ. The result is a set of lines of constant σ that are everywhere orthogonal to the strings (that is, orthogonal to the lines of constant t). This construction can be done both for open and for closed strings. For closed strings the σ-range $[0, \sigma_c]$ of the $t = 0$ string becomes the range for all other strings. For open strings the σ-range $[0, \sigma_1]$ of the $t = 0$ string also becomes the range of all other strings. This happens because the boundaries of the string surface are orthogonal to the strings, and as a result they are lines of constant σ.

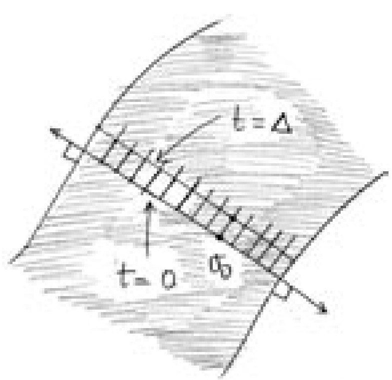

Figure 7.1: Using the parameterization of the $t = 0$ string to construct the parameterization of the $t = \Delta$ string. On the string spatial surface, the lines of constant σ are chosen to be orthogonal to the lines of constant t.

In summary, the σ parameterization of a given string can be used to construct on the surface lines of constant σ that are always perpendicular to the lines of constant t. In this parameterization, the tangent $\partial\vec{X}/\partial\sigma$ to the strings and the tangent $\partial\vec{X}/\partial t$ to the lines of constant σ are perpendicular to each other:

$$\boxed{\frac{\partial\vec{X}}{\partial\sigma}\cdot\frac{\partial\vec{X}}{\partial t}=0\,.} \tag{7.1.1}$$

Since the velocity $\partial\vec{X}/\partial t$ is perpendicular to the string, it coincides with $\vec{v}_\perp$ (see (6.8.5))

$$\boxed{v_\perp=\frac{\partial\vec{X}}{\partial t}\,,\quad \text{at all points,}} \tag{7.1.2}$$

and not only at the endpoints, where it happens independently of parameterization. Condition (7.1.1) allows us to simplify various expressions that enter into the action and into the equations of motion:

$$\dot{X}\cdot X'=\Big(c,\frac{\partial\vec{X}}{\partial t}\Big)\cdot\Big(0,\frac{\partial\vec{X}}{\partial\sigma}\Big)=\frac{\partial\vec{X}}{\partial t}\cdot\frac{\partial\vec{X}}{\partial\sigma}=0\,, \tag{7.1.3}$$

$$\dot{X}^2=-c^2+\Big(\frac{\partial\vec{X}}{\partial t}\Big)^2=-c^2+v_\perp^2\,, \tag{7.1.4}$$

and, as before,

$$X'^2=\Big(\frac{\partial\vec{X}}{\partial\sigma}\Big)^2=\Big(\frac{ds}{d\sigma}\Big)^2, \tag{7.1.5}$$

where s is the length parameter along the string. Using the above equations, we find that (6.5.5) simplifies to

$$\mathcal{P}^{\tau\mu}=-\frac{T_0}{c}\frac{-(\frac{ds}{d\sigma})^2\dot{X}^\mu}{c(\frac{ds}{d\sigma})\sqrt{1-\frac{v_\perp^2}{c^2}}}=\frac{T_0}{c^2}\frac{(\frac{ds}{d\sigma})}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\frac{\partial X^\mu}{\partial t}\,. \tag{7.1.6}$$

Similarly, (6.5.6) becomes

$$\mathcal{P}^{\sigma\mu}=-\frac{T_0}{c^2}\frac{c^2(\sqrt{1-\frac{v_\perp^2}{c^2}})\frac{\partial X^\mu}{\partial s}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}=-T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\,\frac{\partial X^\mu}{\partial s}. \tag{7.1.7}$$

Equation (7.1.7) held at the open string endpoints regardless of parameterization (see (6.9.5)); now it holds all over the string.

7.2 Physical interpretation of the string equation of motion

Having used our σ parameterization to simplify some of our previous expressions, we now look into the string equations of motion (6.5.11). With $t = \tau$ we have

$$\frac{\partial \mathcal{P}^{\tau\mu}}{\partial t} = -\frac{\partial \mathcal{P}^{\sigma\mu}}{\partial \sigma}\,. \tag{7.2.1}$$

Let's first consider the $\mu = 0$ component of this equation. It follows from (7.1.7) that $\mathcal{P}^{\sigma,0} = 0$. Furthermore, equation (7.1.6) gives

$$\mathcal{P}^{\tau,0} = \frac{T_0}{c}\frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\,. \tag{7.2.2}$$

Back into the equation of motion (7.2.1), we obtain

$$\frac{\partial \mathcal{P}^{\tau,0}}{\partial t} = \frac{\partial}{\partial t}\Big(\frac{\frac{ds}{d\sigma}T_0}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\Big) = 0\,. \tag{7.2.3}$$

To understand this result physically consider a small piece $d\sigma$ of the string. The motion of this particular piece of string is well-defined now that we have fixed the lines of constant σ. If we multiply the expression inside the derivative in (7.2.3) by the constant $d\sigma$, we conclude that

$$\frac{T_0\,ds}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\,, \tag{7.2.4}$$

is constant in time. Here ds is the (possibly time-dependent) length of the piece of string $d\sigma$. The quantity in (7.2.4) has units of energy, suggesting that it may be the relativistic energy associated with this piece of the string. Indeed, this is in agreement with our findings in section 6.7 where we saw that the rest energy of a static stretched string is given by its length times the tension T_0. The rest energy in the above expression is $T_0 ds$, and the relativistic factor in the denominator makes (7.2.4) the total energy. Equation (7.2.3) therefore states that the energy in each piece $d\sigma$ of the string is conserved. This is a very interesting fact.

The above interpretation is also confirmed by a calculation of the energy E of a relativistic string (Problem 6.4). You found that

$$E = \int_{string} \frac{T_0 ds}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} \,, \tag{7.2.5}$$

showing that (7.2.4) is the energy in a piece of string.

Now we turn to the space components of the string equation of motion. The space components $\vec{\mathcal{P}}^\tau$ of $\mathcal{P}^{\tau\mu}$ can be read from (7.1.6):

$$\vec{\mathcal{P}}^\tau = \frac{T_0}{c^2} \frac{\frac{ds}{d\sigma}}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} \vec{v}_\perp \,, \tag{7.2.6}$$

and similarly, from (7.1.7),

$$\vec{\mathcal{P}}^\sigma = -T_0 \sqrt{1 - \frac{v_\perp^2}{c^2}} \, \frac{\partial \vec{X}}{\partial s} \,. \tag{7.2.7}$$

Now we can substitute these expressions back into (7.2.1) to find that

$$\begin{aligned} \frac{\partial}{\partial \sigma}\Big[T_0 \sqrt{1 - \frac{v_\perp^2}{c^2}} \frac{\partial \vec{X}}{\partial s}\Big] &= \frac{\partial}{\partial t}\Big[\frac{T_0}{c^2} \frac{\frac{ds}{d\sigma}}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} \vec{v}_\perp\Big] \\ &= \frac{T_0}{c^2} \frac{\frac{ds}{d\sigma}}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} \frac{\partial \vec{v}_\perp}{\partial t} \,, \end{aligned} \tag{7.2.8}$$

where the final step used equation (7.2.3). It is possible to loosely interpret this equation in terms of an "effective" non-relativistic string. Recall that the equations of motion for a classical nonrelativistic string are

$$\mu_0 \frac{\partial^2 \vec{y}}{\partial t^2} = T_0 \frac{\partial^2 \vec{y}}{\partial x^2} = \frac{\partial}{\partial x}\Big[T_0 \frac{\partial \vec{y}}{\partial x}\Big] \,. \tag{7.2.9}$$

How do we recast (7.2.8) to resemble (7.2.9)? We use the $ds/d\sigma$ factor in the right-hand side to transform the σ-derivative on the left-hand side into an s-derivative:

$$\frac{T_0}{c^2} \frac{1}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} \frac{\partial \vec{v}_\perp}{\partial t} = \frac{\partial}{\partial s}\Big[T_0 \sqrt{1 - \frac{v_\perp^2}{c^2}} \Big(\frac{\partial \vec{X}}{\partial s}\Big)\Big] \,. \tag{7.2.10}$$

Comparing this equation with (7.2.9), we find that the relativistic string has a velocity-dependent effective tension T_{eff}, and a velocity-dependent effective mass density μ_{eff} given by

$$T_{\text{eff}} = T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\,, \qquad \mu_{\text{eff}} = \frac{T_0}{c^2}\frac{1}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\,. \tag{7.2.11}$$

Since free open string endpoints move with $v_\perp = c$, the effective tension of the string goes to zero at the endpoints. One could say that the endpoints have to move at the speed of light in order to make the tension vanish at the endpoints. This is the only way the relativistic string can have a tension and still make sense with free open ends. The effective mass density diverges at the endpoints. This is not a problem, the same divergence is present for the energy density appearing as the integrand in (7.2.5). Despite the singular behavior at the endpoints, the integral turns out to be finite, as required by consistency since, after all, we are trying to describe strings with finite energy. An explicit verification of this is given in Problem 7.2.

7.3 Wave equation and constraints

Equation (7.2.10) is still fairly complicated. It may seem that by requiring the lines of constant σ to be orthogonal to the lines of constant t, we have ran out of tricks to simplify the equations using reparameterization invariance. This is not the case, however. We showed how to construct the lines of constant σ if we have one parameterized string. We must now try to parameterize this first string in the best possible way!

Here is the physical way to do so: we will parameterize the string so that each string segment of equal length in σ carries the same amount of energy. We will parameterize the string using the energy! Now let us see how this idea comes about mathematically.

First we rewrite (7.2.10) in a more suggestive way:

$$\frac{1}{c^2}\Bigg[\frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\Bigg]\frac{\partial^2\vec{X}}{\partial t^2} = \frac{\partial}{\partial\sigma}\Bigg[\frac{\sqrt{1-\frac{v_\perp^2}{c^2}}}{\left(\frac{ds}{d\sigma}\right)}\ \frac{\partial\vec{X}}{\partial\sigma}\Bigg]\,. \tag{7.3.1}$$

Let us now define the quantity $A(\sigma)$ by

$$A(\sigma) \equiv \frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}. \tag{7.3.2}$$

We already showed in (7.2.3) that A is independent of time. With this definition the equation of motion becomes

$$\frac{1}{c^2}\frac{\partial^2 X}{\partial t^2} = \frac{1}{A(\sigma)}\frac{\partial}{\partial\sigma}\Big[\frac{1}{A(\sigma)}\frac{\partial\vec{X}}{\partial\sigma}\Big]. \tag{7.3.3}$$

This equation is begging for the introduction of a new σ variable. If we introduce a σ' variable as

$$d\sigma' = A(\sigma)d\sigma\,, \tag{7.3.4}$$

the equation of motion becomes

$$\frac{1}{c^2}\frac{\partial^2\vec{X}}{\partial t^2} = \frac{\partial^2\vec{X}}{\partial\sigma'^2}\,, \tag{7.3.5}$$

which is our familiar wave equation! This is the simplest form of the equations for the relativistic string. To elucidate the properties of the σ' coordinate we replace (7.3.2) into (7.3.4) and find:

$$d\sigma' = \frac{ds}{\sqrt{1-\frac{v_\perp^2}{c^2}}} = \frac{1}{T_0}dE\,. \tag{7.3.6}$$

The last equality follows from the identification of (7.2.4) with the energy dE carried by the little piece of string. We see that the parameter length $d\sigma'$ assigned to this piece of string is proportional to its energy dE. In the σ' parameterization, the energy density $dE/d\sigma'$ is a constant equal to the tension. Assign $\sigma'=0$ to one endpoint of the open string, and consider a point Q on the string. Equation (7.3.6) can be integrated to give

$$\sigma'(Q) = \frac{E(Q)}{T_0}\,. \tag{7.3.7}$$

The coordinate $\sigma'(Q)$ assigned Q equals the energy $E(Q)$ carried by the portion of the string stretching from the selected endpoint up to Q, divided by the tension. It also follows from the above equation that

$$\sigma' \in [0,\sigma_1\,]\,, \qquad \sigma_1 = \frac{E}{T_0}\,, \tag{7.3.8}$$

where E is the total energy of the string.

We have changed σ into the new parameter σ'. Does the orthogonality (7.1.1) hold for the new parameter? This is easily confirmed:

$$\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma} = \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma'} \Big(\frac{d\sigma'}{d\sigma} \Big) = 0\,, \tag{7.3.9}$$

leading to an equivalent condition involving σ':

$$\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma'} = 0\,. \tag{7.3.10}$$

The preservation of the orthogonality is intuitively clear – the line of constant σ' is the same line as the line of constant $\sigma(\sigma')$.

Our parameterization condition (7.3.6) for σ' is actually equivalent to a differential constraint on the embedding coordinates $\vec{X}$. We first rewrite the first equality in (7.3.6) as

$$\Big(\frac{ds}{d\sigma'} \Big)^2 + \frac{1}{c^2}\, v_\perp^2 = 1\,. \tag{7.3.11}$$

With the help of (7.1.2) and (7.1.5), this equation is recognized as

$$\Big(\frac{\partial \vec{X}}{\partial \sigma'} \Big)^2 + \frac{1}{c^2} \Big(\frac{\partial \vec{X}}{\partial t} \Big)^2 = 1\,. \tag{7.3.12}$$

Finally, let us examine the boundary conditions. From (7.2.7) we have

$$\vec{\mathcal{P}}^\sigma = -T_0 \sqrt{1 - \frac{v_\perp^2}{c^2}}\, \frac{d\sigma'}{ds}\, \frac{\partial \vec{X}}{\partial \sigma'} = -T_0\, \frac{\partial \vec{X}}{\partial \sigma'}\,. \tag{7.3.13}$$

Therefore, the free endpoints boundary condition is very simple:

$$\frac{\partial \vec{X}}{\partial \sigma'} = 0\,, \quad \text{at the endpoints.} \tag{7.3.14}$$

This is just a Neumann boundary condition. We have finally shown that the condition of free endpoints, first introduced in (6.5.10), takes the form of a Neumann boundary condition.

All in all we have four equations to solve in order to find the motion of a relativistic string: (7.3.5), (7.3.10), (7.3.12) and (7.3.14). Calling henceforth σ' by the name σ these equations become:

$$\begin{aligned}
\text{Wave equation :}\quad & \frac{\partial^2 \vec{X}}{\partial \sigma^2} - \frac{1}{c^2}\frac{\partial^2 \vec{X}}{\partial t^2} = 0\,, & (7.3.15)\\
\text{Parameterization condition :}\quad & \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma} = 0\,, & (7.3.16)\\
\text{Parameterization condition :}\quad & \Big(\frac{\partial \vec{X}}{\partial \sigma}\Big)^2 + \frac{1}{c^2}\Big(\frac{\partial \vec{X}}{\partial t}\Big)^2 = 1\,, & (7.3.17)\\
\text{Boundary condition :}\quad & \frac{\partial \vec{X}}{\partial \sigma}\Big|_{\sigma=0} = \frac{\partial \vec{X}}{\partial \sigma}\Big|_{\sigma=\sigma_1} = 0\,. & (7.3.18)
\end{aligned}$$

For a string with energy E, the above equations will require $\sigma_1 = E/T_0$. For the record we also note that (7.3.6) now reads

$$\frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v_\perp^2}{c^2}}} = 1\,, \tag{7.3.19}$$

and from equations (7.1.6) and (7.1.7) we get

$$\begin{aligned}
\mathcal{P}^{\tau\mu} &= \frac{T_0}{c^2}\,\frac{\partial X^\mu}{\partial t}\,, & (7.3.20)\\
\mathcal{P}^{\sigma\mu} &= -T_0\,\frac{\partial X^\mu}{\partial \sigma}\,. & (7.3.21)
\end{aligned}$$

7.4 General motion of an open string

In this section, our goal is to describe the general motion of open strings with free boundary conditions. We will therefore examine in detail how to solve the relevant equations (7.3.15) – (7.3.18).

Let us focus first on the wave equation for $\vec{X}$. This is readily solved in terms of arbitrary vector functions of $(ct \pm \sigma)$. We thus write

$$\vec{X}(t,\sigma) = \frac{1}{2}\big(\vec{F}(ct+\sigma) + \vec{G}(ct-\sigma)\big)\,. \tag{7.4.1}$$

Let us now use the boundary condition at the $\sigma = 0$ endpoint:

$$\frac{\partial \vec{X}}{\partial \sigma}\Big|_{\sigma=0} = 0 \quad \to \quad \vec{F}'(ct) - \vec{G}'(ct) = 0\,, \tag{7.4.2}$$

where prime denotes derivative with respect to the argument. Since ct takes all possible values, the above equation holds for all values of the argument. Calling u the argument, we write

$$\frac{d\vec{F}(u)}{du} = \frac{d\vec{G}(u)}{du} \quad \rightarrow \quad \vec{G}(u) = \vec{F}(u) + \vec{a}_0 \,, \tag{7.4.3}$$

where $\vec{a}_0$ is a constant vector. Back in (7.4.1) we now have

$$\vec{X}(t,\sigma) = \frac{1}{2}\Big(\vec{F}(ct+\sigma) + \vec{F}(ct-\sigma) + \vec{a}_0\Big) \,. \tag{7.4.4}$$

We can absorb the constant vector $\vec{a}_0$ into the definition of $\vec{F}$ (call $\vec{F}(u)+\vec{a}_0/2$ the new $\vec{F}$), and therefore our solution so far takes the form

$$\vec{X}(t,\sigma) = \frac{1}{2}\Big(\vec{F}(ct+\sigma) + \vec{F}(ct-\sigma)\Big) \,. \tag{7.4.5}$$

Consider now the boundary condition at $\sigma = \sigma_1$. Using (7.4.5) we find

$$\frac{\partial \vec{X}}{\partial \sigma}\bigg|_{\sigma=\sigma_1} = 0 \quad \longrightarrow \quad \vec{F}'(ct+\sigma_1) - \vec{F}'(ct-\sigma_1) = 0 \,. \tag{7.4.6}$$

Letting $u = ct - \sigma_1$, the above condition becomes

$$\frac{d\vec{F}}{du}(u + 2\sigma_1) = \frac{d\vec{F}}{du}(u) \,. \tag{7.4.7}$$

This equation tells us that the derivative of $\vec{F}$ is periodic with period $2\sigma_1$. Integrating, the function $\vec{F}$ must be quasi-periodic: after a period $2\sigma_1$ it changes by a fixed constant. We write

$$\vec{F}(u + 2\sigma_1) = \vec{F}(u) + 2\sigma_1 \frac{\vec{v}_0}{c} \,, \tag{7.4.8}$$

where $\vec{v}_0$ is a vector constant of integration with units of velocity, and the constants have been added for convenience. This concludes our analysis of the boundary conditions.

We now examine what restrictions the parameterization conditions (7.3.16) and (7.3.17) impose on the function $\vec{F}$. A standard trick is to add and subtract the first equation to the second, as in

$$\Big(\frac{\partial \vec{X}}{\partial \sigma}\Big)^2 \pm 2\, \frac{\partial \vec{X}}{\partial \sigma} \cdot \frac{1}{c}\frac{\partial \vec{X}}{\partial t} + \frac{1}{c^2}\Big(\frac{\partial \vec{X}}{\partial t}\Big)^2 = 1 \,, \tag{7.4.9}$$

which can then be written more briefly as

$$\Big(\frac{\partial\vec{X}}{\partial\sigma}\pm\frac{1}{c}\frac{\partial\vec{X}}{\partial t}\Big)^2=1\,. \tag{7.4.10}$$

Note that this is equivalent to the two constraints (7.3.16) and (7.3.17). Using (7.4.5) we can evaluate the derivatives that enter into the constraints:

$$\begin{aligned}\frac{\partial\vec{X}}{\partial\sigma}&=\frac{1}{2}\big(\vec{F}'(ct+\sigma)-\vec{F}'(ct-\sigma)\big)\,,\\ \frac{1}{c}\frac{\partial\vec{X}}{\partial t}&=\frac{1}{2}\big(\vec{F}'(ct+\sigma)+\vec{F}'(ct-\sigma)\big)\,.\end{aligned} \tag{7.4.11}$$

As a result, the two linear combinations of derivatives entering the constraints become:

$$\frac{\partial\vec{X}}{\partial\sigma}\pm\frac{1}{c}\frac{\partial\vec{X}}{\partial t}=\pm\vec{F}'(ct\pm\sigma)\,. \tag{7.4.12}$$

It follows that the constraints (7.4.10) require

$$\vec{F}'\cdot\vec{F}'=1\,, \tag{7.4.13}$$

for all arguments of $\vec{F}$. This is more explicitly written as

$$\Big|\frac{d\vec{F}(u)}{du}\Big|^2=1\,. \tag{7.4.14}$$

This is good progress: the constraints give a single tractable condition for $\vec{F}(u)$. Since $\vec{F}$ is a vector function of the parameter u, we can visualize $\vec{F}(u)$ as a parameterized curve in space. Equation (7.4.14) has a simple interpretation:

$$\boxed{u\ \text{ is a length parameter along the curve } \vec{F}(u)\,.} \tag{7.4.15}$$

This is explained as follows. Consider two nearby points $\vec{F}(u+du)$ and $\vec{F}(u)$ on the curve. Their vector separation $d\vec{F}=\vec{F}(u+du)-\vec{F}(u)$ has length $|d\vec{F}|$. Equation (7.4.14) implies that $|d\vec{F}|=|du|$, showing that the parameter change $|du|$ is the distance between the two nearby points.

We can now summarize our analysis of the equations of motion. We have that the general solution describing the motion of an open string with free endpoints is given as

$$\boxed{\vec{X}(t,\sigma)=\frac{1}{2}\Big(\vec{F}(ct+\sigma)+\vec{F}(ct-\sigma)\Big)\,,} \tag{7.4.16}$$

where the vector function $\vec{F}$ satisfies the conditions

$$\boxed{\left|\frac{d\vec{F}(u)}{du}\right|^2 = 1\,, \quad \text{and} \quad \vec{F}(u+2\sigma_1) = \vec{F}(u) + 2\sigma_1\frac{\vec{v}_0}{c}\,.} \tag{7.4.17}$$

The problem has been reduced to finding a vector function $\vec{F}$ satisfying equations (7.4.17). The second of these equations tells us that it suffices to find $\vec{F}(u)$ for $u \in [0, 2\sigma_1]$, where $\sigma_1 = E/T_0$ and E is the energy of the string. This determines $\vec{F}(u)$ for all u, and thus it determines $\vec{X}(t,\sigma)$ completely. The interpretation of $\vec{v}_0$ will be given below. An illustration of $\vec{F}$ is shown in Figure 7.2.

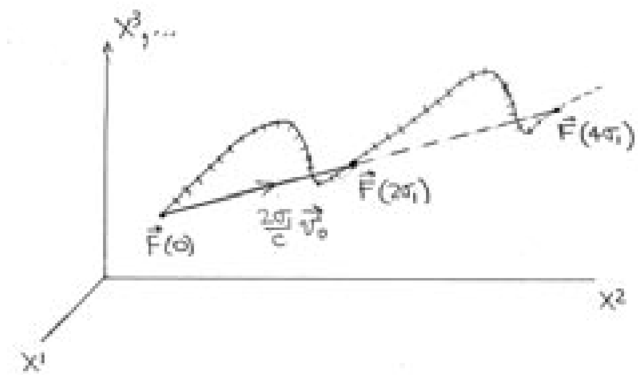

Figure 7.2: The vector function $\vec{F}(u)$ changes by a constant vector $(2\sigma_1\vec{v}_0/c)$ when $u \to u + 2\sigma_1$. The parameterized curve $\vec{F}(u)$ encodes the full motion of an open string with free endpoints.

We can give a physical interpretation to $\vec{F}(u)$. It follows from (7.4.16) that the motion of the $\sigma = 0$ endpoint of the open string is described by

$$\vec{X}(t,0) = \vec{F}(ct) \tag{7.4.18}$$

Therefore, we see that:

$$\boxed{\vec{F}(u) \;\text{ is the position of the } \sigma = 0 \text{ endpoint at time } u/c\,.} \tag{7.4.19}$$

Additionally, we can give a physical interpretation to the constant velocity $\vec{v}_0$. From the second equation in (7.4.17) we have

$$\vec{F}(2\sigma_1) = \vec{F}(0) + 2\sigma_1 \frac{\vec{v}_0}{c}\,, \tag{7.4.20}$$

and expressing the $\vec{F}$'s in terms of the position of the $\sigma = 0$ endpoint we find

$$\vec{X}\Big(t = \frac{2\sigma_1}{c}, 0\Big) = \vec{X}(t = 0, 0) + \Big(\frac{2\sigma_1}{c}\Big)\vec{v}_0\,. \tag{7.4.21}$$

This shows that $\vec{v}_0$ is the average velocity of the $\sigma = 0$ endpoint.

Quick Calculation 7.1. Show that $\vec{v}_0$ is, in fact, the average velocity of any point σ on the string calculated over any time interval of duration $2\sigma_1/c$.

Since $\vec{F}$ can be reconstructed by looking only at the $\sigma = 0$ endpoint, we may ask: How long do we need to observe this endpoint in order to determine the full motion, past and future, of an open string with energy E? Since the motion is determined if we know $\vec{F}(u)$ from $u = 0$ to $u = 2\sigma_1$, we must observe $\vec{X}(t, 0)$ from $t = 0$ to $t = 2\sigma_1/c$. Since $\sigma_1 = E/T_0$, we need to observe the endpoint for a time interval $\Delta t = 2E/cT_0$. This is twice the time light would take to travel a length E/T_0, the length of a static string of energy E.

We now use the above construction to describe the motion of a straight open string with energy E rotating rigidly about its midpoint in the (x, y) plane, as shown in Figure 7.3.

Our first goal is to produce the function $\vec{F}(u)$. This function can be easily reconstructed because we know a lot about the motion of the endpoints. Assuming the string is of total length ℓ and rotates with angular frequency ω we write:

$$\vec{X}(t, 0) = \frac{\ell}{2}\big(\cos\omega t\,, \sin\omega t\big)\,, \tag{7.4.22}$$

where we use vector notation with two components, since the motion is restricted to the (x, y) plane. Given that $\vec{F}(ct) = \vec{X}(t, 0)$ we have

$$\vec{F}(u) = \frac{\ell}{2}\Big(\cos\frac{\omega u}{c}\,, \sin\frac{\omega u}{c}\Big)\,. \tag{7.4.23}$$

Note that the function $\vec{F}$ is strictly periodic. This is reasonable since the endpoints of the string trace a stationary circle. As a result their average

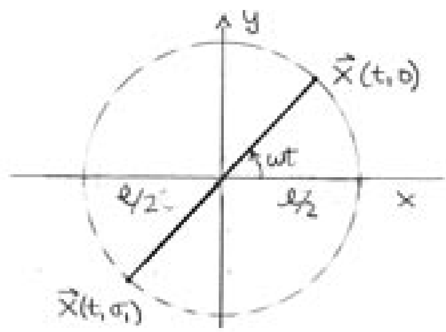

Figure 7.3: An open string of length ℓ rotating rigidly in the (x, y) plane with angular velocity ω.

velocity is zero. This means that $\vec{v}_0$ equals zero, and from equation (7.4.17) we find strict periodicity with period $2\sigma_1$. Equating this period with the period $2\pi c/\omega$ of (7.4.23) we find

$$\frac{2\pi c}{\omega} = 2\sigma_1 \quad \longrightarrow \quad \frac{\omega}{c} = \frac{\pi}{\sigma_1} = \frac{\pi T_0}{E}\,. \tag{7.4.24}$$

This gives the angular frequency of the motion. The first condition in (7.4.17) determines the length ℓ. Indeed,

$$\frac{d\vec{F}}{du} = \frac{\omega\ell}{2c}\Big(-\sin\frac{\pi u}{\sigma_1}, \cos\frac{\pi u}{\sigma_1}\Big)\,, \tag{7.4.25}$$

and, as a result,

$$\Big|\frac{d\vec{F}}{du}\Big|^2 = \Big(\frac{\omega\ell}{2c}\Big)^2 = 1 \quad \to \quad \ell = \frac{2c}{\omega} = \frac{2\sigma_1}{\pi} = \frac{2}{\pi}\frac{E}{T_0}\,. \tag{7.4.26}$$

This length is smaller, by a factor $2/\pi$, than the length of a static string with energy E. This is sensible since this string has kinetic energy. Note also that $\omega(\ell/2) = c$, telling us that the endpoints move with the speed of light.

Having determined ω and ℓ, we really know the motion of the string. It is of interest, however, to write the complete expression for the motion of the *parameterized* string $\vec{X}(t,\sigma)$. In terms of σ_1, the vector $\vec{F}$ in (7.4.23) is now given as

$$\vec{F}(u) = \frac{\sigma_1}{\pi}\Big(\cos\frac{\pi u}{\sigma_1}\,,\,\sin\frac{\pi u}{\sigma_1}\Big)\,. \tag{7.4.27}$$

Finally, using (7.4.5) we obtain

$$\vec{X}(t,\sigma) = \frac{\sigma_1}{2\pi}\bigg(\cos\frac{\pi(ct+\sigma)}{\sigma_1} + \cos\frac{\pi(ct-\sigma)}{\sigma_1}\,,\;\sin\frac{\pi(ct+\sigma)}{\sigma_1} + \sin\frac{\pi(ct-\sigma)}{\sigma_1}\bigg) \tag{7.4.28}$$

and after simplification

$$\vec{X}(t,\sigma) = \frac{\sigma_1}{\pi}\cos\frac{\pi\sigma}{\sigma_1}\bigg(\cos\frac{\pi ct}{\sigma_1}\,,\,\sin\frac{\pi ct}{\sigma_1}\bigg)\,. \tag{7.4.29}$$

The parameterized string is of interest because we know that the energy density is a constant function of the parameter σ. In Problem 7.2 you will examine the energy $\mathcal{E}(s)$ per unit length on the string as a function of the distance s to the center. You will show that

$$\mathcal{E}(s) = \frac{T_0}{\sqrt{1-\frac{4s^2}{\ell^2}}}\,. \tag{7.4.30}$$

At the center of the string, $s = 0$, and the energy density $\mathcal{E}(0)$ coincides with T_0. This had to be so, because at the center, the string does not move. The energy density diverges at the string endpoints $s = \pm\ell/2$. The total energy, however, is finite.

Problems

Problem 7.1. *Four short proofs concerning open strings with free endpoints.*

(a) Prove that if the motion of the endpoint of an open string is restricted to a hyperplane, the motion of the full open string is also restricted to the same hyperplane.

(b) Prove that if the motion of the endpoint of an open string is restricted to lie within a distance R from a point P_0, then the full string lies at all times within a distance R from P_0.

(c) Prove that if the motion of the endpoint of an open string is restricted to lie within a convex subspace, then the full string lies at all times within that convex subspace. [This is the general version of the results proven in (a) and (b)].

(d) Show that the length ℓ of an open string parameterized with energy, as discussed in section 7.3, is given by

$$\ell = \int_0^{\sigma_1} \sqrt{1 - \frac{v_\perp^2}{c^2}}\, d\sigma \,.$$

Problem 7.2. *Exploring further the rigidly rotating string.*

Let $s \in (-\ell/2, \ell/2)$ be a length parameter on the rigidly rotating string studied in section 7.4, with $s = 0$ chosen to be the fixed center of the string. Let $\mathcal{E}(s)$ denote the energy per unit length as a function of s.

(a) Show that

$$\mathcal{E}(s) = \frac{T_0}{\sqrt{1 - \frac{4s^2}{\ell^2}}} \,.$$

Verify that the total energy of the string is $\frac{\pi}{2}\ell T_0$. Plot $\mathcal{E}(s)$. Note that $\mathcal{E}(s)$ has integrable singularities at the string endpoints.

(b) For what points on the string is the local energy density equal to the average energy density?

(c) Calculate the energy $\widehat{E}(s)$ carried by the string in the interval $[-s, s]$. What is the value of s for this energy to be half of the total energy of the string? How about 90% of the energy?

Problem 7.3. *Time evolution of an initially static closed relativistic string.*

For the time development of closed strings in the static gauge, equations (7.3.15), (7.3.16), and (7.3.17) apply.

(a) Given $\frac{\partial \vec{X}}{\partial t}(t=0,\sigma)=0$, write the general solution for $\vec{X}(t,\sigma)$ in terms of a vector function $\vec{F}(u)$ of a single variable. What do the parameterization conditions require on $\vec{F}$?

(b) Since we have a closed string the parameter σ lives on a circle $\sigma \sim \sigma+\sigma_1$. What condition would you impose on $\vec{X}(t,\sigma)$ to implement this feature ? What are the implications for $\vec{F}$?

(c) Consider an initially static closed string at $t=t_0$ tracing a particular closed curve γ of length ℓ. Calculate the minimum time Δt that must elapse for the closed string to trace the curve γ again. Is $\vec{X}(t_0+\Delta t,\sigma)$ equal to $\vec{X}(t_0,\sigma)$?

(d) Sketch the steps you would take with a computer (that can do integrals and invert functions) to produce the time evolution of an initially static closed string of arbitrary shape lying on the (x,y) plane. Assume the initial string is given to you as the parameterized closed curve $(x(\lambda),y(\lambda))$, with some parameter $\lambda\in[0,\lambda_0]$.

Problem 7.4. *Relativistic jumping rope.*

Consider a relativistic open string with fixed endpoints:

$$\vec{X}(t,\sigma=0)=\vec{x}_1\,,\qquad \vec{X}(t,\sigma=\sigma_1)=\vec{x}_2\,. \tag{1}$$

The boundary condition at $\sigma=0$ is satisfied by the solution

$$\vec{X}(t,\sigma)=\vec{x}_1+\frac{1}{2}\Big(\vec{F}(ct+\sigma)-\vec{F}(ct-\sigma)\Big)\,, \tag{2}$$

where $\vec{F}$ is a vector function of a single variable.

(a) Use (2) and the boundary condition at $\sigma=\sigma_1$ to derive a condition on $\vec{F}(u)$.

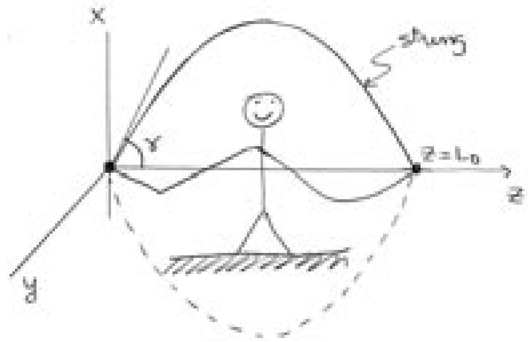

Figure 7.4: Problem 7.4: Kasey's relativistic jumping rope.

(b) Write down the constraint on $\vec{F}(u)$ that arises from the parameterization conditions

$$\left(\frac{\partial \vec{X}}{\partial \sigma} \pm \frac{1}{c}\frac{\partial \vec{X}}{\partial t}\right)^2 = 1\,. \tag{3}$$

As an application, consider Kasey's attempts to use a relativistic open string as a jumping rope! For this purpose she holds the open string (in three spatial dimensions) stretched with her right hand at the origin $\vec{x}_1 = (0,0,0)$ and her left hand at the point $z = L_0$ on the z-axis, or $\vec{x}_2 = (0,0,L_0)$ (see Figure 7.4). As she starts jumping we observe that the string is moving so that the tangent vector $\vec{X}'$ to the string at the origin rotates around the z axis forming an angle γ with it, as shown in Figure 7.4.

(c) Use the above information to write an expression for $\vec{F}'(u)$.

(d) Find σ_1 in terms of the length L_0 and the angle γ.

(d) Give the solution $\vec{X}(t,\sigma)$ for the motion of Kasey's relativistic jumping rope.

Chapter 8

World-sheet Currents

For physical insight, physicists often turn to ideas of symmetry and invariance. Symmetry properties of dynamical systems and conserved quantities are closely related. We will learn that in string theory there are currents that flow on the two-dimensional world-sheet traced out by the string in spacetime. The conserved charges associated with these currents are key quantities that characterize the free motion of strings. We will discover a nice physical interpretation of the objects $\mathcal{P}^\tau$ and $\mathcal{P}^\sigma$ which we encountered earlier.

8.1 Electric charge conservation

We begin our study by reviewing the physics and the mathematics of charge conservation in the context of Maxwell theory. This classic example will help us to develop a more general understanding of the concept of conserved currents.

In electromagnetism, the conserved current is the four vector $j^\alpha = (c\rho, \vec{j}\,)$, where ρ is the charge density and $\vec{j}$ is the current density. Why do we say that j^α a conserved current? By definition, j^α is a conserved current because it satisfies the equation

$$\partial_\alpha j^\alpha = 0\,. \tag{8.1.1}$$

Any four-vector which satisfies this equation is called a conserved current. The term "conserved current" is a little misleading, but it is a convention. More precisely, we should say that we have a conserved *charge*, because it is really the charge associated with the current that is conserved. Let us see how this conservation arises.

When we separate the space and time indices in (8.1.1) we get

$$\partial_0 j^0 + \partial_i j^i = \frac{\partial j^0}{\partial x^0} + \nabla \cdot \vec{j} = 0\,. \tag{8.1.2}$$

Why is this equation a statement of charge conservation? In electromagnetism, the total charge $Q(t)$ in a volume V is just the integral of the charge density ρ over the volume:

$$Q(t) = \int_V \rho(t,\vec{x})\, d^3x = \int_V \frac{j^0(t,\vec{x})}{c}\; d^3x\,. \tag{8.1.3}$$

Up to a constant, the charge is the integral over space of the first component of the current. Its time derivative is given by

$$\frac{dQ}{dt} = \int_V \frac{\partial j^0}{\partial x^0}\; d^3x\,. \tag{8.1.4}$$

Using equation (8.1.2), we can write

$$\frac{dQ}{dt} = -\int_V \nabla \cdot \vec{j}\; d^3x\,. \tag{8.1.5}$$

Letting S denote the boundary of V, the divergence theorem gives:

$$\frac{dQ}{dt} = \int_S \vec{j} \cdot d\vec{a}\,. \tag{8.1.6}$$

This equation encodes the physical statement of charge conservation: the charge inside a volume V can only change if there is an appropriate flux of current across the surface S bounding the volume. In many cases, we take V to be so large that the current $\vec{j}$ vanishes on the surface S. In these cases,

$$\frac{dQ}{dt} = 0\,. \tag{8.1.7}$$

The charge Q is then time-independent and is said to be "conserved".

8.2 Conserved charges from Lagrangian symmetries

One of the most useful properties of Lagrangians is that they can be used to deduce the existence of conserved quantities. Conserved quantities can be quite helpful in understanding the dynamics of a system. In this section we begin our work in the context of Lagrangian mechanics, learning how to construct the conserved quantity associated to a symmetry. We then turn to Lagrangian densities, and show how to construct the conserved current associated to a symmetry.

Let $L(q, \dot{q}; t)$ be a Lagrangian that depends on a coordinate q, a velocity $\dot{q}$, and possibly, time. Consider, moreover, a variation of the coordinate of the form

$$q(t) \to q(t) + \delta q(t)\,, \tag{8.2.1}$$

where $\delta q(t)$ is some specific infinitesimal variation. Infinitesimal variations can be written as

$$\delta q(t) = \epsilon\, h(q(t), \dot{q}(t); t)\,, \tag{8.2.2}$$

where ϵ is an infinitesimal constant and $h(q, \dot{q}; t)$ is a function. As a result of the variation in (8.2.1), there is a corresponding variation in the velocity $\dot{q}$:

$$\dot{q}(t) \to \dot{q}(t) + \frac{d(\delta q(t))}{dt}\,. \tag{8.2.3}$$

Suppose you wish to determine how $L(q, \dot{q}; t)$ changes as a result of the variations (8.2.1) and (8.2.3). Because δq is infinitesimal, the variation of the Lagrangian consists only of terms which are linear in δq. If those terms vanish, the Lagrangian is said to be invariant.

We claim that if the Lagrangian is indeed invariant under the variations (8.2.1) and (8.2.3), then the quantity Q, defined by

$$\boxed{\epsilon\, Q \equiv \frac{\partial L}{\partial \dot{q}}\, \delta q\,,} \tag{8.2.4}$$

is conserved in time for *physical motion*. That is, for any motion $q(t)$ satisfying the equations of motion, the 'charge' Q is a constant:

$$\boxed{\frac{dQ}{dt} = 0\,.} \tag{8.2.5}$$

Note that the ϵ on the left-hand side of (8.2.4) cancels with the ϵ in the definition (8.2.2) of δq.

To prove the conservation of Q, consider the Euler-Lagrange equations:

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) - \frac{\partial L}{\partial q} = 0\,. \tag{8.2.6}$$

Since the Lagrangian does not change under the coordinate and velocity variations (8.2.1) and (8.2.3), we must have

$$\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q) = 0\,. \tag{8.2.7}$$

We now prove our claim by direct calculation of the time derivative of Q:

$$\epsilon\,\frac{dQ}{dt} = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big)\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q) = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q) = 0, \tag{8.2.8}$$

where we used (8.2.6) and (8.2.7). We see that our formulation of charge conservation is indeed correct.

Let's apply this new perspective on charge conservation to a Lagrangian that depends only on the velocity: $L = L(\dot{q})$. How can we vary q but leave $L(\dot{q})$ unchanged? One way is to apply $q(t) \to q(t)+\epsilon$, where ϵ is any constant. Correspondingly, we see that $\dot{q}(t) \to \dot{q}(t) + d\epsilon/dt = \dot{q}(t)$. Because $\dot{q}$ does not change, $L(\dot{q})$ does not change. Making use of (8.2.4), we find

$$\epsilon\, Q = \frac{\partial L}{\partial \dot{q}}\delta q = \frac{\partial L}{\partial \dot{q}}\epsilon \quad\longrightarrow\quad Q = \frac{\partial L}{\partial \dot{q}}\,. \tag{8.2.9}$$

We recognize Q as the momentum associated to the coordinate q. This quantity is conserved because the position q does not appear in the Lagrangian. Note that the conservation equation $dQ/dt = 0$ coincides with the Euler-Lagrange equation (8.2.6). This example illustrates a familiar result in Lagrangian mechanics: if the Lagrangian of a system does not contain a given coordinate, the momentum conjugate to that coordinate is conserved. For a free nonrelativistic particle, for example, $L = \frac{1}{2}m(\dot{q})^2$. In this case $Q = m\dot{q}$.

Now let's consider Lagrangian *densities* with symmetries. While a symmetry of a Lagrangian guarantees the existence of conserved *charges*, a symmetry of a Lagrangian density guarantees the existence of conserved *currents*. We write the action as the integral of the Lagrangian density over the full set of coordinates ξ^α of some relevant "world":

$$S = \int d\xi^0 d\xi^1 \cdots d\xi^k\ \mathcal{L}(\phi^a, \partial_\alpha\phi^a)\,. \tag{8.2.10}$$

Here k denotes the number of space dimensions of the world. The world could be the full Minkowski spacetime, some subspace thereof, or, for example, the two-dimensional parameter space of the string world-sheet. The fields $\phi^a(\xi)$ are functions of the coordinates, and

$$\partial_\alpha \phi^a = \frac{\partial \phi^a}{\partial \xi^\alpha}\,, \tag{8.2.11}$$

are derivatives of the fields with respect to the coordinates. Each value of the index a corresponds to a field. Consider now the infinitesimal variation

$$\phi^a(\xi) \;\to\; \phi^a(\xi) + \delta\phi^a(\xi)\,, \tag{8.2.12}$$

and the associated variation for the field derivatives $\partial_\alpha \phi^a$. The infinitesimal variations are conveniently written as

$$\delta\phi^a = \epsilon^i \; h_i^a(\phi, \partial\phi)\,, \tag{8.2.13}$$

where ϵ^i are a set of infinitesimal constants, and, for brevity, we ommitted all indices in the arguments of h_i^a. We have included an index in ϵ^i because the variation may involve several parameters. A spacetime translation, for example, would involve as many parameters as there are spacetime dimensions. Since the index i is repeated in (8.2.13), it is summed over. You must distinguish clearly the various types of indices we are working with:

$$\begin{array}{cl} \alpha & \text{index to label world coordinates } \xi^\alpha, \text{ or vector components,} \\ i & \text{index to label parameters in the symmetry transformation,} \\ a & \text{index to label various fields in the Lagrangian.} \end{array} \tag{8.2.14}$$

If $\mathcal{L}$ is invariant under (8.2.12) and the associated variations of the field derivatives, then the quantities j_i^α defined by

$$\boxed{\epsilon^i \, j_i^\alpha \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \delta\phi^a\,,} \tag{8.2.15}$$

are conserved currents:

$$\boxed{\partial_\alpha \, j_i^\alpha = 0\,.} \tag{8.2.16}$$

We will prove this shortly. In (8.2.15) the repeated field index a is summed over. If the index i is present in (8.2.13), then we have *several* currents indexed by i, one for each parameter of the variation. The components of the currents, as many as the number of dimensions in the world, are indexed by α. It is important not to confuse the very different roles that these two kinds of indices play:

$$\begin{aligned} j_i^\alpha \ : \ & i \text{ labels the various currents,} \\ & \alpha \text{ labels the components of the currents.} \end{aligned} \tag{8.2.17}$$

We showed in section 8.1 that a conserved current gives rise to a conserved charge. The charge was the integral over space of the zeroth component of the current. It follows that the currents j_i^α give rise to the conserved charges

$$Q_i = \int d\xi^1 d\xi^2 \cdots d\xi^k \ j_i^0 \,. \tag{8.2.18}$$

We have as many conserved charges as there are parameters in the symmetry transformation.

Quick Calculation 8.1. Verify that (8.2.16) implies that

$$\frac{dQ_i}{d\xi^0} = 0 \,, \tag{8.2.19}$$

when the currents j_i^α vanish sufficiently rapidly at spatial infinity.

To prove (8.2.16), we write both the Euler-Lagrange equations associated to the action (8.2.10) and the statement of invariance:

$$\partial_\alpha\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\Big) - \frac{\partial\mathcal{L}}{\partial\phi^a} \ = \ 0 \,, \tag{8.2.20}$$

$$\frac{\partial\mathcal{L}}{\partial\phi^a}\,\delta\phi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\partial_\alpha(\delta\phi^a) \ = \ 0 \,. \tag{8.2.21}$$

The proof is simple. Using (8.2.15) and the two equations above, we find

$$\begin{aligned} \epsilon^i\,\partial_\alpha j_i^\alpha \ &= \ \partial_\alpha\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\Big)\delta\phi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\,\partial_\alpha(\delta\phi^a) \\ &= \ \frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\,\partial_\alpha(\delta\phi^a) = 0 \,. \end{aligned} \tag{8.2.22}$$

We will use (8.2.15) to construct conserved currents which live on the string world-sheet. Conserved charges and conserved currents exist under conditions less stringent than those considered here. These are explored in Problem 8.5 and Problem 8.6.

8.3 Conserved currents on the world-sheet

To each string we would like to assign a relativistic momentum p_μ which is conserved if the string is moving freely. Even though the momentum p_μ carries an index, it is not a current, but rather a charge. In fact, since each component of p_μ should be separately conserved, this is a case where we have a set of conserved charges.

In the notation of the previous section, Q_i denote the various charges, so we see that the μ index in p_μ is playing the role of the i index in Q_i; it labels the various charges. What then is the α index in j_i^α? We will see that this index labels the coordinates on the world-sheet. The currents live on the world-sheet!

In the string action (6.4.2), the Lagrangian density is integrated over the world-sheet coordinates τ and σ, and not over the spacetime coordinates x^μ. In this example, the world of (8.2.10) is two-dimensional, and the index α in (8.2.14) takes two values. As a result, the conserved currents will live on the world-sheet: they have two components and they are functions of the world-sheet coordinates. More explicitly,

$$S = \int d\xi^0 d\xi^1 \mathcal{L}\left(\partial_0 X^\mu, \partial_1 X^\mu\right), \quad \text{with} \quad (\xi^0\,, \xi^1) = (\tau\,, \sigma)\,, \tag{8.3.1}$$

and $\partial_\alpha = \partial/\partial\xi^\alpha$. Comparing with (8.2.10), we see that the field variables ϕ^a are simply the string coordinates X^μ. Note that the string action only depends on derivatives of the string coordinates.

To find conserved currents, we need a field variation δX^μ that does not change the Lagrangian density. One such variation is given by

$$\delta X^\mu(\xi) = \epsilon^\mu\,, \tag{8.3.2}$$

where ϵ^μ is a constant – that is, it does not depend on τ or σ. The Lagrangian density is invariant because it only depends on $\partial_\alpha X^\mu$, and the variation of this quantity vanishes: $\delta(\partial_\alpha X^\mu) = \partial_\alpha(\delta X^\mu) = \partial_\alpha \epsilon^\mu = 0$. The role of the various indices in (8.2.14) is now clear: α is a world-sheet index, i takes the values of the Minkowski spacetime index μ, and so does the field index a.

Now let's construct the conserved current. Using (8.2.15), and letting i and a indices run over the values of μ, we have

$$\epsilon^\mu\, j_\mu^\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha X^\mu)}\delta X^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha X^\mu)}\,\epsilon^\mu\,. \tag{8.3.3}$$

Cancelling the common ϵ^μ factor on the two sides of this equation, we find an expression for the currents:

$$j^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \quad \longrightarrow \quad (j^0_\mu\,, j^1_\mu) = \Big(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}\,, \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}}\Big)\,. \tag{8.3.4}$$

We have seen such derivatives of $\mathcal{L}$ before: they appeared in equations (6.5.5) and (6.5.6). We can in fact identify

$$\boxed{j^\alpha_\mu = \mathcal{P}^\alpha_\mu \quad \longrightarrow \quad (j^0_\mu\,, j^1_\mu) = (\mathcal{P}^\tau_\mu\,, \mathcal{P}^\sigma_\mu)\,.} \tag{8.3.5}$$

This is really interesting: the τ and σ superscripts in $\mathcal{P}_\mu$ label the components of a current that lives on the world-sheet! The equation for current conservation is

$$\partial_\alpha \mathcal{P}^\alpha_\mu = \frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0. \tag{8.3.6}$$

This is just the equation of motion (6.5.13) for the relativistic string!

Since our currents $\mathcal{P}^\alpha_\mu$ are indexed by μ, the conserved charges are also indexed by μ. Following (8.2.18), to get the charges we must integrate over space the zeroth components $\mathcal{P}^\tau_\mu$ of the currents. In the present case, this means integrating over σ. Therefore,

$$p_\mu(\tau) = \int_0^{\sigma_1} \mathcal{P}^\tau_\mu(\tau, \sigma) d\sigma\,. \tag{8.3.7}$$

This integral is done with τ held constant. We have called the conserved charges p_μ, because we expect them to correspond to the momentum carried by the string. Indeed, $\mathcal{P}^\tau_\mu$ is the derivative of the Lagrangian density with respect to the velocity $\dot{X}^\mu$, so it has the interpretation of momentum density. A similar identification was obtained for the non-relativistic strings of section 4.6: $\mathcal{P}^t$, defined in (4.6.16), has the interpretation of momentum density, on account of equation (4.6.12).

To check conservation, we differentiate with respect to τ:

$$\frac{dp_\mu}{d\tau} = \int_0^{\sigma_1} \frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} d\sigma = -\int_0^{\sigma_1} d\sigma \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = -\mathcal{P}^\sigma_\mu\Big|_0^{\sigma_1}\,, \tag{8.3.8}$$

where we made use of the equation of motion (8.3.6). For a closed string the coordinates $\sigma = 0$ and $\sigma = \sigma_1$ represent the same point on the worldsheet, and the right hand side vanishes. For an open string with free endpoints, the boundary condition (6.5.11) states that $\mathcal{P}^\sigma_\mu$ vanishes at the endpoints, so, again, the right hand side vanishes. In both of these cases, p_μ is conserved:

$$\frac{dp_\mu}{d\tau} = 0\,. \tag{8.3.9}$$

There is more to this equation than our previous statement of charge conservation (8.2.5). The derivative in (8.3.9) is with respect to τ and not t. We will discuss this at length in the next section.

For open strings with Dirichlet boundary conditions along some space directions, the momentum carried by the string along those directions may fail to be conserved. We noted this fact earlier, in the case of the non-relativistic string (section 4.6). In string theory, Dirichlet boundary conditions represent open strings attached to D-branes. The momentum of the string can fail to be conserved, but the total momentum of the string *and* the D-brane is conserved.

8.4 The complete momentum current

Equation (8.3.9) is very intriguing. It is a conservation law on the world-sheet rather than in spacetime! But we could not have expected otherwise – the currents, after all, live on the world-sheet: their indices are world-sheet indices, and their arguments are world-sheet coordinates. As seen in spacetime, the currents vanish everywhere except on the surface traced out by the string.

If we trust the reparameterization invariance of the physics, however, we can easily obtain a standard *spacetime* conservation law by choosing the static gauge. The integral in (8.3.7) is then an integral over the strings, the lines of constant time as seen by the chosen Lorentz observer. The conservation law (8.3.9) becomes

$$\boxed{\frac{dp_\mu}{dt} = 0\,.} \tag{8.4.1}$$

The Lorentz observer confirms that momentum is conserved in *time*.

Equation (8.3.9), together with (8.3.7), tells us that for any fixed world-sheet parameterization, we can compute uniquely a quantity p_μ using any line of constant τ. This quantity must coincide with the time-independent momentum p_μ obtained using the static gauge, as we now explain.

Consider a string propagating in spacetime, a fixed Lorentz observer, and a particular choice of parameterization on the world-sheet. In this parameterization, over some region of the surface the lines of constant τ are lines of constant t. As a result, over this region we have the static gauge. Over the rest of the surface the parameterization changes smoothly: the lines of constant τ are not anymore lines of constant time. On the static-gauge region the integral (8.3.7) gives us the time-independent momentum carried by the strings. Because of (8.3.9), on the rest of the surface the integral (8.3.7) must still give the same value for p_μ, even though the lines of constant τ are not strings.

This argument suggests that any curve on the world-sheet can be used to calculate the conserved momentum p_μ. Formula (8.3.7), however, only tells us how to calculate the momentum if the curve is a curve of constant τ. We now show how to generalize equation (8.3.7) to be able to compute the momentum p_μ using an (almost) arbitrary curve on the world-sheet *together* with an arbitrary parameterization of the world-sheet. When we deal with open strings, the curve must stretch from one boundary to the other. When we deal with closed strings, the curve must be a closed non-contractible curve.

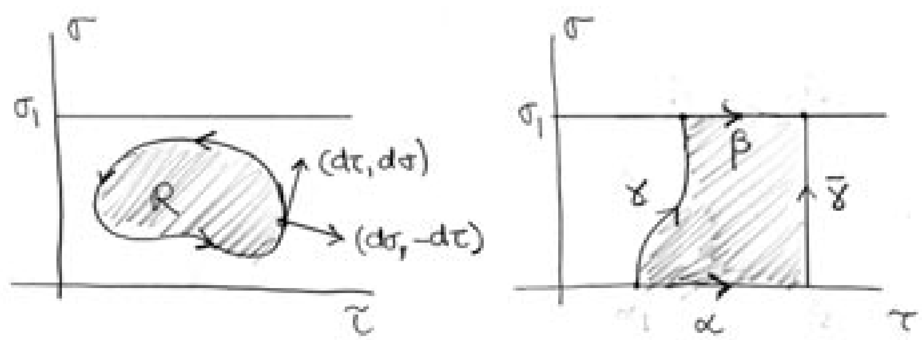

Figure 8.1: Left side: The total momentum flow out of a simply connected region $\mathcal{R}$ of the world-sheet is zero. Right side: A simply connected region including an arbitrary curve γ and a string $\bar{\gamma}$.

Let us reconsider equation (8.3.7), where we integrate the τ-component of the two-dimensional current $(\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu)$ over a curve of constant τ. The quantity we are calculating is actually the *flux* of the current across the curve. Since the σ-component $\mathcal{P}^\sigma_\mu$ is tangent to the curve of constant τ, it does not contribute to the flux. More generally, consider an infinitesimal segment $(d\tau, d\sigma)$ along an oriented closed curve Γ that encloses a simply connected region $\mathcal{R}$ of the world-sheet (see Figure 8.1). Since $(d\tau, d\sigma)$ is parallel to the oriented tangent, the outgoing normal to the segment is $(d\sigma, -d\tau)$. The outgoing flux of the current across the segment is given by the dot product of the current vector with the outgoing normal vector:

$$\text{infinitesimal flux} = (\mathcal{P}^\tau_\mu\,,\,\mathcal{P}^\sigma_\mu)\cdot(d\sigma, -d\tau) = \mathcal{P}^\tau_\mu\,d\sigma - \mathcal{P}^\sigma_\mu\,d\tau\,. \tag{8.4.2}$$

We now show that the current flux across a contractible closed curve Γ on the world-sheet is zero. This is a reasonable result because a contractible curve surrounds a domain $\mathcal{R}$, and we do not expect a domain to be a momentum source or sink. The outgoing flux across Γ is written as

$$p_\mu(\Gamma) = \oint_\Gamma \left(\mathcal{P}^\tau_\mu\,d\sigma - \mathcal{P}^\sigma_\mu\,d\tau\right)\,. \tag{8.4.3}$$

By the two-dimensional version of the divergence theorem, the flux of the current $\mathcal{P}^\alpha_\mu$ out of $\mathcal{R}$ is equal to the integral of the divergence of $\mathcal{P}^\alpha_\mu$ over $\mathcal{R}$:

$$p_\mu(\Gamma) = \int_{\mathcal{R}} \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma}\right) d\tau d\sigma = 0\,, \tag{8.4.4}$$

since $\mathcal{P}^\alpha_\mu$ is a conserved current. This is what we wanted to show.

We now generalize (8.3.7) as follows. For an arbitrary curve γ that starts on the $\sigma = 0$ boundary of the world-sheet and ends on the $\sigma = \sigma_1$ boundary, we define

$$p_\mu(\gamma) = \int_\gamma \left(\mathcal{P}^\tau_\mu\,d\sigma - \mathcal{P}^\sigma_\mu\,d\tau\right)\,. \tag{8.4.5}$$

If γ is a curve of constant τ, then $d\tau = 0$ all along γ, and $p_\mu(\gamma)$ reduces to (8.3.7). We now prove that $p_\mu(\gamma)$, as defined in all generality by (8.4.5), actually coincides with p_μ, as defined in (8.3.7). Consider a curve γ that stretches from one side of the world-sheet to the other, and a curve $\bar{\gamma}$ of constant τ, as shown on the right side of Figure 8.1. Let α and β denote oriented paths along the world-sheet boundary such that the closed curve Γ indicated in the figure is given by

$$\Gamma = \bar{\gamma} - \beta - \gamma + \alpha\,. \tag{8.4.6}$$

The curve Γ is oriented counterclockwise. Applying (8.4.5) to Γ:

$$\begin{aligned} p_\mu(\Gamma) &= \int_\Gamma \left(\mathcal{P}^\tau_\mu\, d\sigma - \mathcal{P}^\sigma_\mu\, d\tau\right) \\ &= \left(\int_{\bar\gamma} - \int_\gamma + \int_\alpha - \int_\beta\right) \left(\mathcal{P}^\tau_\mu\, d\sigma - \mathcal{P}^\sigma_\mu\, d\tau\right) = 0\,, \end{aligned} \tag{8.4.7}$$

because Γ is closed and contractible. Since α and β are curves where $d\sigma$ vanishes, only $\mathcal{P}^\sigma_\mu\, d\tau$ contributes to the integrals. But $\mathcal{P}^\sigma_\mu$ vanishes at the string endpoints (for free boundary conditions), so these integrals vanish identically. Only the integrals over γ and $\bar\gamma$ survive, so we have

$$\int_\gamma \left(\mathcal{P}^\tau_\mu\, d\sigma - \mathcal{P}^\sigma_\mu\, d\tau\right) = \int_{\bar\gamma} \left(\mathcal{P}^\tau_\mu\, d\sigma - \mathcal{P}^\sigma_\mu\, d\tau\right) = \int_{\bar\gamma} \mathcal{P}^\tau_\mu\, d\sigma = p_\mu\,, \tag{8.4.8}$$

where we noted that $d\tau = 0$ on $\bar\gamma$ and used (8.3.7). This proves that $p_\mu(\gamma) = p_\mu$ for any contour connecting the $\sigma = 0$ and $\sigma = \sigma_1$ boundaries of the world-sheet. We can thus rewrite (8.4.5) as

$$\boxed{p_\mu = \int_\gamma \left(\mathcal{P}^\tau_\mu\, d\sigma - \mathcal{P}^\sigma_\mu\, d\tau\right)\,.} \tag{8.4.9}$$

The statement of conservation has become the fact that the integral above is independent of the chosen curve γ (as long as the endpoints of the curve lie on the boundary components of the world-sheet).

The arguments are similar in the case of closed strings. We consider an arbitrary nontrivial closed curve γ winding once around the world-sheet, and another similarly nontrivial closed curve $\bar\gamma$ of constant τ, for some arbitrary, but fixed, parameterization. The two curves form the boundary of an annular region $\mathcal{R}$ (Figure 8.2). An argument completely analogous, based on the divergence theorem, shows that both contours give the same result for p_μ. Therefore, we can calculate the momentum of the closed string using any closed curve that winds once around the world-sheet.

How does an arbitrary Lorentz observer use equation (8.4.9)? The observer looks at a string at some time t, and asks for its momentum. This requires the use of (8.4.9), with a curve γ corresponding to the string in question. With an arbitrary parameterization γ need not be a constant τ curve. At some later time t', the observer asks again for the momentum. At this time, a string corresponds to a curve γ', generically different from γ. By the curve independence of (8.4.9), the observer concludes that the momentum did not change in time. The momentum was conserved.

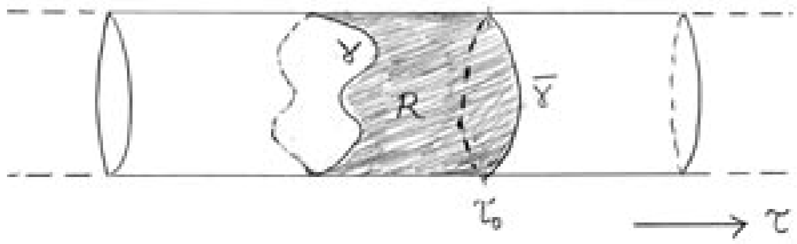

Figure 8.2: A closed string world-sheet with an arbitrary nontrivial closed curve γ and a closed curve $\bar{\gamma}$ at constant τ. The region in between the contours is $\mathcal{R}$.

8.5 Lorentz symmetry and associated currents

By construction, the action of the relativistic string is Lorentz invariant. It is written in terms of Lorentz vectors that are contracted to build Lorentz scalars. Concretely, this means that Lorentz transformations of the coordinates X^μ leave the action invariant. In this section we will construct the conserved charges associated with Lorentz symmetry.

These charges will be particularly useful when we study quantum string theory in Chapter 12. Whenever we quantize a classical system, there is the possibility that crucial symmetries of the classical theory will be lost. If Lorentz invariance were lost upon quantization, quantum string theory would be very problematic, to say the least. We will have to make sure that the quantum theory is Lorentz invariant.

To calculate the conserved charges, we first need the infinitesimal form of the general Lorentz transformations. We recall (section 2.2) that Lorentz transformations are linear transformations of the coordinates X^μ that leave the quadratic form $\eta_{\mu\nu}X^\mu X^\nu$ invariant. Every infinitesimal linear transformation is of the form $X^\mu \to X^\mu + \delta X^\mu$, where

$$\delta X^\mu = \epsilon^{\mu\nu} X_\nu \,. \tag{8.5.1}$$

Here $\epsilon^{\mu\nu}$ is a matrix of infinitesimal constants. Lorentz invariance imposes conditions on the $\epsilon^{\mu\nu}$. We require $\delta(\eta_{\mu\nu}X^\mu X^\nu) = 0$, and therefore

$$2\eta_{\mu\nu}(\delta X^\mu)X^\nu = 2\eta_{\mu\nu}(\epsilon^{\mu\rho}X_\rho)\,X^\nu = 2\,\epsilon^{\mu\rho}X_\rho X_\mu = 0\,. \tag{8.5.2}$$

Imagine decomposing the matrix ϵ into its antisymmetric part and its symmetric part. The antisymmetric part would not contribute to $\epsilon^{\mu\rho}X_\rho X_\mu$. The vanishing of $\epsilon^{\mu\rho}X_\rho X_\mu$, for all X, implies that the symmetric part of ϵ is zero. It follows that the general solution is represented by an antisymmetric $\epsilon^{\mu\nu}$:

$$\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}\,. \tag{8.5.3}$$

Infinitesimal Lorentz transformations are very simple: they are $\delta X^\mu = \epsilon^{\mu\nu}X_\nu$, with $\epsilon^{\mu\nu}$ antisymmetric.

Quick Calculation 8.2. Consider a fixed two-by-two matrix A^{ab} $(a, b = 1, 2)$, that satisfies $A^{ab}v_a v_b = 0$, for all values of v_1 and v_2. Write out the four terms involved in the left hand side of the vanishing condition and confirm explicitly that the matrix A^{ab} must be antisymmetric.

Quick Calculation 8.3. Examine the boost in (2.2.30) for β very small. Write $x'^\mu = x^\mu + \epsilon^{\mu\nu}x_\nu$ and calculate the entries of the matrix $\epsilon^{\mu\nu}$. Show that $\epsilon^{10} = -\epsilon^{01} = \beta$, and that all other entries are zero, thus confirming that the matrix $\epsilon^{\mu\nu}$ is antisymmetric for an infinitesimal boost.

Let us now show explicitly that the string Lagrangian density is invariant under Lorentz transformations. All terms that appear there are of the form

$$\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial X^\nu}{\partial\xi^\beta}\,, \tag{8.5.4}$$

where ξ^α and ξ^β are either τ or σ. We claim that any such term is Lorentz invariant. Indeed

$$\begin{aligned}
\delta\Big(\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial X^\nu}{\partial\xi^\beta}\Big) &= \eta_{\mu\nu}\Big(\frac{\partial\delta X^\mu}{\partial\xi^\alpha}\frac{\partial X^\nu}{\partial\xi^\beta} + \frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial\delta X^\nu}{\partial\xi^\beta}\Big)\\
&= \eta_{\mu\nu}\Big(\epsilon^{\mu\rho}\frac{\partial X_\rho}{\partial\xi^\alpha}\frac{\partial X^\nu}{\partial\xi^\beta} + \epsilon^{\nu\rho}\frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial X_\rho}{\partial\xi^\beta}\Big)\\
&= \epsilon_{\nu\rho}\frac{\partial X^\rho}{\partial\xi^\alpha}\frac{\partial X^\nu}{\partial\xi^\beta} + \epsilon_{\mu\rho}\frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial X^\rho}{\partial\xi^\beta}\,,
\end{aligned} \tag{8.5.5}$$

where we used η to lower the indices on the ϵ constants. Letting $\mu \to \rho$ and $\rho \to \nu$ in the second term we get

$$\delta\Big(\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial X^\nu}{\partial\xi^\beta}\Big) = (\epsilon_{\nu\rho} + \epsilon_{\rho\nu})\frac{\partial X^\rho}{\partial\xi^\alpha}\frac{\partial X^\nu}{\partial\xi^\beta} = 0\,, \tag{8.5.6}$$

because of the antisymmetry of ϵ. This explicitly proves the Lorentz invariance of the string action.

We can now use equation (8.2.15) to write the currents. It follows from (8.5.1) that the role of the small parameter ϵ^i is played here by $\epsilon^{\mu\nu}$. We therefore have

$$\epsilon^{\mu\nu}\, j^\alpha_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)}\delta X^\mu = \mathcal{P}^\alpha_\mu\, \epsilon^{\mu\nu} X_\nu\,. \tag{8.5.7}$$

The current $j^\alpha_{\mu\nu}$, since it is multiplied by the antisymmetric matrix $\epsilon^{\mu\nu}$, can be defined to be antisymmetric – any symmetric part would drop out of the left hand side. Using the antisymmetry of $\epsilon^{\mu\nu}$, the right hand side is written as

$$\epsilon^{\mu\nu}\, j^\alpha_{\mu\nu} = (-\tfrac{1}{2}\epsilon^{\mu\nu})\left(X_\mu \mathcal{P}^\alpha_\nu - X_\nu \mathcal{P}^\alpha_\mu\right)\,. \tag{8.5.8}$$

The currents can be read directly from these equations because the factor multiplying $\epsilon^{\mu\nu}$ in the right hand side is explicitly antisymmetric. Since the overall normalization of the currents is for us to choose, we define the currents $\mathcal{M}^\alpha_{\mu\nu}$ by

$$\boxed{\mathcal{M}^\alpha_{\mu\nu} = X_\mu \mathcal{P}^\alpha_\nu - X_\nu \mathcal{P}^\alpha_\mu\,.} \tag{8.5.9}$$

By construction,

$$\mathcal{M}^\alpha_{\mu\nu} = -\mathcal{M}^\alpha_{\nu\mu}\,. \tag{8.5.10}$$

The equation of current conservation is

$$\frac{\partial \mathcal{M}^\tau_{\mu\nu}}{\partial \tau} + \frac{\partial \mathcal{M}^\sigma_{\mu\nu}}{\partial \sigma} = 0\,, \tag{8.5.11}$$

and the charges, in analogy with (8.4.9), are given by

$$\boxed{M_{\mu\nu} = \int_\gamma \left(\mathcal{M}^\tau_{\mu\nu} d\sigma - \mathcal{M}^\sigma_{\mu\nu} d\tau\right)\,.} \tag{8.5.12}$$

As we can see, the charges, just as the currents, are antisymmetric:

$$M_{\mu\nu} = -M_{\nu\mu}\,. \tag{8.5.13}$$

The conservation of $M_{\mu\nu}$ is a result of the contour-independence of the definition (8.5.12). For closed strings, contour independence is guaranteed by the argument given earlier in the case of momentum charges. For free open strings, one point must be addressed to ensure contour independence. The $M_{\mu\nu}$ integrals must receive no contributions from contours on the boundary of the world-sheet. This requires the vanishing of $\mathcal{M}^\sigma_{\mu\nu}$ on the boundary. We can easily verify that this holds: $\mathcal{M}^\sigma_{\mu\nu}$ involves $\mathcal{P}^\sigma$ multiplicatively, and $\mathcal{P}^\sigma$ vanishes on the world-sheet boundary. As explained for the case of the momentum charges, a Lorentz observer measuring $M_{\mu\nu}$ using strings at different times would conclude that $dM_{\mu\nu}/dt = 0$.

We can also compute the Lorentz charges $M_{\mu\nu}$ using constant τ lines. In that case,

$$\boxed{M_{\mu\nu} = \int \mathcal{M}^\tau_{\mu\nu}(\tau, \sigma)\, d\sigma = \int (X_\mu \mathcal{P}^\tau_\nu - X_\nu \mathcal{P}^\tau_\mu)\, d\sigma\,.} \tag{8.5.14}$$

Since the $M_{\mu\nu}$ are antisymmetric, in four dimensions we have six conserved charges. Letting i and j denote space indices that take the three possible values, M_{0i} are the three charges associated to the three boosts and M_{ij} are the three charges associated to the three rotations. Since $\vec{\mathcal{P}}^\tau$ is the momentum density, the normalization chosen in (8.5.9) ensures that $\mathcal{M}^\tau_{ij}$ is precisely the angular momentum density. As a consequence, the components M_{ij} measure the string angular momentum $\vec{L}$ via the usual relations $L_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$. Here ϵ_{ijk} is a totally antisymmetric object satisfying $\epsilon_{123} = 1$. More explicitly, we have $L_1 = M_{23}, L_2 = M_{31}$ and $L_3 = M_{12}$.

8.6 The slope parameter α'

The string tension T_0 is the only dimensionful parameter in the string action. In this section we will motivate the definition of an alternative parameter: the slope parameter α'. These two parameters are related, we can use one or the other. The parameter α' has an interesting physical interpretation, used since the early days of string theory. If we consider a rigidly rotating open string, α' is the proportionality constant relating the angular momentum J of the string to the square of its energy E. More explicitly,

$$\frac{J}{\hbar} = \alpha' E^2\,. \tag{8.6.15}$$

Since angular momentum has the same units as $\hbar$, the units of α' are those of inverse energy-squared:

$$[\alpha'] = \frac{1}{[E]^2}. \tag{8.6.16}$$

To verify equation (8.6.15) we consider a straight open string of energy E rotating rigidly on the (x, y) plane. This is precisely the problem examined in section 7.4. The only non-vanishing component of angular momentum is M_{12}, and its magnitude is denoted by $J = |M_{12}|$. Equation (8.5.14) tells us that

$$M_{12} = \int_0^{\sigma_1} (X_1 \mathcal{P}_2^\tau - X_2 \mathcal{P}_1^\tau)\, d\sigma\,. \tag{8.6.17}$$

To evaluate this integral we need formulae for the position and momenta of the rotating string. We recall equation (7.4.29):

$$\vec{X}(t,\sigma) = \frac{\sigma_1}{\pi} \cos\frac{\pi\sigma}{\sigma_1} \left(\cos\frac{\pi ct}{\sigma_1}, \sin\frac{\pi ct}{\sigma_1} \right), \tag{8.6.18}$$

which records the components (X_1, X_2) of the rotating string. Using equation (7.3.20), we find

$$\vec{\mathcal{P}}^\tau = \frac{T_0}{c^2}\frac{\partial \vec{X}}{\partial t} = \frac{T_0}{c} \cos\frac{\pi\sigma}{\sigma_1} \left(-\sin\frac{\pi ct}{\sigma_1}, \cos\frac{\pi ct}{\sigma_1} \right), \tag{8.6.19}$$

where the right-hand side gives the components $(\mathcal{P}_1^\tau, \mathcal{P}_2^\tau)$. The integral in (8.6.17) is thus given by

$$M_{12} = \frac{\sigma_1}{\pi}\frac{T_0}{c} \int_0^{\sigma_1} \cos^2\frac{\pi\sigma}{\sigma_1}\, d\sigma = \frac{\sigma_1^2 T_0}{2\pi c}\,, \tag{8.6.20}$$

where the time dependence dissappeared, as expected for a conserved charge. Since $J = |M_{12}|$ and $\sigma_1 = E/T_0$, we have found that

$$J = \frac{E^2}{2\pi T_0 c} \quad \rightarrow \quad \frac{J}{\hbar} = \frac{1}{2\pi T_0 \hbar c} E^2\,. \tag{8.6.21}$$

As anticipated, the angular momentum is proportional to the square of the energy of the string. Comparing with equation (8.6.15) we deduce that

$$\boxed{\alpha' = \frac{1}{2\pi T_0 \hbar c} \quad \text{and} \quad T_0 = \frac{1}{2\pi\alpha'\hbar c}\,.} \tag{8.6.22}$$

These equations relate the slope parameter α' to the string tension T_0.

The name slope parameter arises because α' is the slope of the lines of $J/\hbar$, when plotted as a function of energy-squared. In fact, Regge trajectories are approximate lines that arise when plotting angular momentum as a function of energy-squared for hadronic excitations. In the early 1970's, when string theory was investigated as a theory of strong interactions, the slope parameter α' was the experimentally-measured quantity that could be used to define the string action. The action, previously given in (6.4.2) in terms of T_0, takes the form

$$S = -\frac{1}{2\pi\alpha'\hbar c^2}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\pi} d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}\,. \tag{8.6.23}$$

It turns out that most of the modern work on string theory uses the slope parameter α' as opposed to the string tension T_0. Recall that in section 3.6 we used $\hbar, c$, and Newton's constant G to calculate a characteristic length called Planck's length ℓ_P. In string theory, we can use $\hbar, c$, and the dimensionful parameter α' to calculate a characteristic length called the string length ℓ_s.

Quick Calculation 8.4. Show that

$$\boxed{\ell_s = \hbar c\sqrt{\alpha'}\,.} \tag{8.6.24}$$

As the above equation shows, up to factors of $\hbar$ and c, the string length ℓ_s is the square root of α'. This connection to a fundamental length scale in string theory provides a physical interpretation to the slope parameter α'.

Problems

Problem 8.1. *Angular momentum as conserved charge.*

Consider a Lagrangian L that depends only on the *magnitude* of the velocity $\dot{\vec{q}}(t)$ of a particle moving in ordinary three-dimensional space.

(a) Write an infinitesimal variation $\delta\vec{q}(t)$ that represents a small rotation of the vector $\vec{q}(t)$. Explain why it leaves the Lagrangian invariant.

(b) Construct the conserved quantity associated with this symmetry transformation. Verify that this conserved quantity is the (vector) angular momentum.

Problem 8.2. *Lorentz charges for the relativistic point particle*

Consider the point particle action $S = -mc\int d\tau\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$ introduced in (5.2.4). Note that $\dot{x}^\mu(\tau) = dx^\mu(\tau)/d\tau$.

(a) Show that $x^\mu(\tau) \to x^\mu(\tau) + \epsilon^\mu$, with ϵ^μ constant is a symmetry. Find the conserved charges associated to this symmetry and verify explicitly that they are conserved. Compare with the momenta $p_\mu(\tau)$ defined canonically from the Lagrangian.

(b) Write the infinitesimal Lorentz transformations of $x^\mu(\tau)$, and explain why the action is invariant under such transformations.

(c) Find an expression for the Lorentz charges in terms of $x^\mu(\tau)$ and $p^\mu(\tau)$. Make sure that your Lorentz charges coincide with angular momentum charges for the appropriate values of the indices. Verify explicitly their conservation.

Problem 8.3. *Simple estimates.*

(a) In hadronic physics the slope parameter α' is approximately given as $\alpha' \simeq 0.95\text{GeV}^{-2}$. Calculate the hadronic string tension in tons, and the string length in centimeters.

(b) Assume the string length is $\ell_s = 10^{-30}$cm. Calculate α' in GeV^{-2}, and the string tension in tons.

Problem 8.4. *Angular momentum of the jumping rope*

Consider the relativistic jumping rope solution of Problem 7.4. Calculate the magnitude J of the angular momentum of this string as a function of the length parameters L_0 and the angle γ. What is the relation between the angular momentum J of this string and its energy? Compare your result with equation (8.6.15), and show that in this problem $J/\hbar < \alpha' E^2$.

Problem 8.5. *Generalizing the construction of conserved charges.*

Examine the setup that led to the conserved charge Q defined in (8.2.4). Assume now that the symmetry transformation (8.2.1) does not leave the Lagrangian invariant, but rather the change in L is a total time derivative

$$\delta L = \frac{d}{dt}(\epsilon\,\Lambda)\,, \tag{1}$$

where Λ is some calculable function of coordinates, velocities, and possibly, of time. Show that there is a modified conserved charge taking the form

$$\epsilon\,Q = \frac{\partial L}{\partial \dot{q}}\,\delta q - \epsilon\,\Lambda\,. \tag{2}$$

As an application consider a Lagrangian $L(q(t), \dot{q}(t))$ that has no explicit time dependence. Show that the transformations $q(t) \to q(t) + \epsilon \dot{q}(t)$, with ϵ constant, is a symmetry in the sense of (1). Calculate Λ and construct the conserved charge Q. Is the result familiar?

Problem 8.6. *Generalizing the construction of conserved currents.*

Assume now that the symmetry transformation (8.2.12) does not leave the Lagrangian density invariant, but rather the change in $\mathcal{L}$ is a total derivative

$$\delta\mathcal{L} = \frac{\partial}{\partial \xi^\alpha}(\epsilon^i\,\Lambda_i^\alpha)\,, \tag{1}$$

where the Λ_i^α are a set of calculable functions of fields, field derivatives, and possibly, coordinates. Show that there is a conserved current taking the form

$$\epsilon^i\,j_i^\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha \phi^a)}\delta\phi^a - \epsilon^i\,\Lambda_i^\alpha\,. \tag{2}$$

As an application consider a Lagrangian density $\mathcal{L}(\phi^a, \partial_\alpha \phi^a)$ that has no explicit dependence on the world coordinates ξ^α. Show that the transformations $\phi^a \to \phi^a + \epsilon^\beta \partial_\beta \phi^a$, with ϵ^β constant, are a symmetry in the sense of (1), with $\Lambda_\beta^\alpha = \delta_\beta^\alpha \mathcal{L}$. Construct the conserved currents j_β^α. Is j_0^0 familiar? The set of quantities j_β^α define the energy-momentum tensor T_β^α.

Chapter 9

Light-cone Relativistic Strings

A class of gauges is introduced that fix the parameterization of the world-sheet, lead to a pair of constraints, and give equations of motion that are wave equations. One of these gauges, the light-cone gauge, sets X^+ proportional to τ and allows for a complete and explicit solution to the equations of motion. In this gauge, the dynamics of the string is determined by the motion of the transverse directions and two zero modes. We encounter the classical Virasoro modes as the oscillation modes of the X^- coordinate and learn how to calculate the mass of an arbitrary string configuration.

9.1 A class of choices for τ

Our first encounter with classical string dynamics was simplified by our use of the static gauge. Recall that in this gauge the world-sheet time τ is identified with the spacetime time-coordinate X^0 by

$$X^0(\tau, \sigma) = c\tau\,. \tag{9.1.1}$$

We are now going to consider more general gauges. Among the class of gauges we will consider, one of them will turn out to be particularly useful: the light-cone gauge. Using this gauge we will solve the equations of motion of the string in a complete and explicit fashion. We will go further than we could with the static gauge. Using the static gauge we did not solve the equations of motion completely; instead, we obtained a differential equation for a function $\vec{F}$ which defines the motion (see (7.4.17)).

The gauges we will examine are those for which τ is set equal to a linear combination of the string coordinates. This can be written as

$$n_\mu X^\mu(\tau,\sigma) = \lambda\tau\,. \tag{9.1.2}$$

If we choose $n_\mu = (1,0,..,0)$, and $\lambda = c$, this equation becomes (9.1.1). To understand the meaning of (9.1.2), consider first the related equation

$$n_\mu x^\mu = \lambda\tau\,, \tag{9.1.3}$$

where we write x^μ, as opposed to X^μ, to emphasize that we are dealing with the general spacetime coordinates. If x_1^μ and x_2^μ are two points satisfying (9.1.3) for the same value of τ, we see that $n_\mu(x_1^\mu - x_2^\mu) = 0$. This shows that any vector joining points in the space (9.1.3) is orthogonal to n_μ. The set of all points satisfying (9.1.3) form a hyperplane normal to n_μ.

We can now make clear the meaning of equation (9.1.2). The points X^μ satisfying $n_\mu X^\mu = \tau$ are points that lie both on the world-sheet and on the hyperplane (9.1.3). Equation (9.1.2) states that all these points must be assigned the same value τ of the world-sheet coordinate. If we define a string to be the set of points $X^\mu(\tau,\sigma)$ with constant τ, in the gauge (9.1.2), strings lie on hyperplanes of the form (9.1.3). The string with world-sheet time τ is the intersection of the world-sheet with the hyperplane $n\cdot x = \lambda\tau$, as illustrated in Figure 9.1.

We want strings to be spacelike objects. More precisely, the interval ΔX^μ between any two points on a string should be spacelike, perhaps null in some limit, but never timelike. We will now see that in the gauge (9.1.2), a timelike n^μ guarantees that the string is spacelike. In this gauge any interval ΔX^μ along the string satisfies $n\cdot\Delta X = 0$. Since this condition is Lorentz invariant, it can be analyzed in a Lorentz frame where the only non-zero component of n^μ is the time component. In this frame ΔX cannot have a time component and is therefore spacelike. If n^μ is a null vector ($(1,1,0...0)$, for example), one can show (Problem 9.1) that $n\cdot\Delta X = 0$ implies that ΔX^μ is generally spacelike and ocassionally null. We will allow n^μ to be null in (9.1.2).

At this point, it is convenient to streamline the way we deal with units. While we have been using the convention that τ and σ have units of time and length, respectively, starting now *τ and σ will be dimensionless.* We proved in Chapter 8 that for open strings with free endpoints, there is a well-defined conserved momentum p^μ. We will use this Lorentz-vector to rewrite

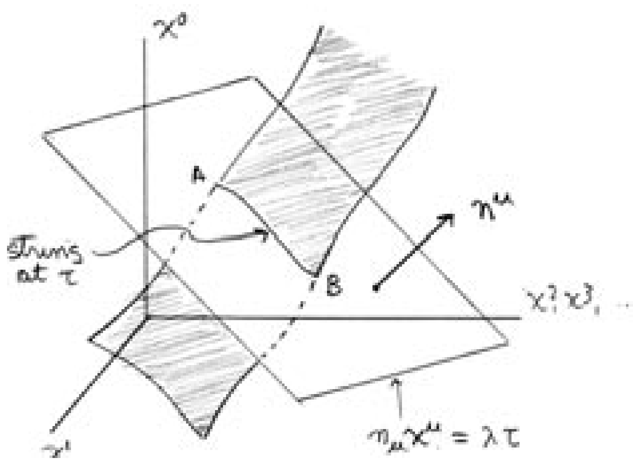

Figure 9.1: The gauge condition $n \cdot X = \lambda\tau$ fixes the strings to be the curves at the intersection of the world-sheet with the hyperplanes orthogonal to the vector n^μ.

our gauge condition (9.1.2):

$$n \cdot X(\tau, \sigma) = \tilde{\lambda}\,(n \cdot p)\,\tau\,. \tag{9.1.4}$$

When open strings are attached to D-branes not all components of the string momentum are conserved. Since we want our analysis to hold even in this case, we will assume that the vector n^μ is chosen in such a way that $n \cdot p$ *is conserved*. This condition is weaker than the condition of momentum conservation.

By involving the vector n^μ explicitly on both sides of equation (9.1.4), the scale of n^μ has been made irrelevant. Only the direction of n^μ matters. Bearing in mind that $n \cdot X$ has units of length, $n \cdot p$ has units of momentum, and τ is dimensionless, imagine dividing both sides of this equation by the unit of time. We see that $\tilde{\lambda}$ has units of velocity divided by force. The canonical choice for velocity is c, and the canonical choice for force is T_0, the string tension. It is therefore natural to set

$$\tilde{\lambda} \sim \frac{c}{T_0} = 2\pi\alpha'\hbar c^2\,, \tag{9.1.5}$$

using equation (8.6.22), which relates the string tension to α'. Before fixing $\tilde{\lambda}$ precisely, let's further simplify our treatment of units.

At stake is our ability to track the units of different physical quantities. We can simplify this matter by deciding to track just *one* unit, instead of the three units of length, time, and mass. By convention, we do this by setting two of the fundamental constants equal to one:

$$\hbar = c = 1\,, \tag{9.1.6}$$

as if these constants had no units! This has two consequences. First, any $\hbar$ or c in our formulae disappear without a trace. This is not a serious problem, if we know the full units of an expression where $\hbar$ and c have been set equal to one, we can reconstruct the dependence on $\hbar$ and c unambiguously. Second, the units become dependent, and we are left with just one independent unit. Since $[c] = L/T$, the condition $c = 1$ implies that

$$L = T\,. \tag{9.1.7}$$

At this stage $[\hbar] = ML^2/T$ becomes $[\hbar] = ML$. With $\hbar = 1$ we get

$$M = \frac{1}{L}\,. \tag{9.1.8}$$

Thus we can write all units in terms of mass, or length (for some reason nobody uses time!). We will say that we are working with *natural units* when we set $\hbar = c = 1$ and track just one unit.

Back to (9.1.5), the units of α' have now become

$$[\alpha'] = \frac{1}{[T_0]} = \frac{L}{M} = L^2. \tag{9.1.9}$$

While the complete units of α' are those of inverse energy squared, we see that in natural units, α' has units of length-squared. This is in agreement with our result in (8.6.24). In natural units the string length is

$$\boxed{\ell_s = \sqrt{\alpha'}\,.} \tag{9.1.10}$$

For reference, in natural units the Nambu-Goto action (8.6.23) takes the form

$$\boxed{S = -\frac{1}{2\pi\alpha'}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\pi} d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}\,.} \tag{9.1.11}$$

In natural units equation (9.1.5) sets $\tilde{\lambda}$ proportional to α'. For open strings we choose $\tilde{\lambda} = 2\alpha'$, and equation (9.1.4) becomes

$$n \cdot X(\sigma, \tau) = 2\alpha'(n \cdot p)\,\tau \quad \text{(open strings)}\,. \tag{9.1.12}$$

This is the final form of our gauge condition which fixes the τ parameterization of the world-sheet.

9.2 The associated σ parameterization

Having fixed the τ parameterization, let's determine the appropriate parameterization of σ. In the static gauge we required that the energy density be constant over the strings. The energy density is the $\mu = 0$ component of the momentum density $\mathcal{P}^{\tau\mu}$. Since the static gauge uses $n^\mu = (1, 0, ..., 0)$, we were actually demanding the constancy of $n_\mu \mathcal{P}^{\tau\mu}$.

The proper generalization to situations where n^μ is arbitrary is to demand the constancy of $n_\mu \mathcal{P}^{\tau\mu} = n \cdot \mathcal{P}^\tau$ over the strings (the curves of constant τ). Assume, additionally, that we require a parameterization range $\sigma \in [0, \pi]$ for all open strings. We will see that these conditions are both satisfied if σ satisfies the following equation

$$(n \cdot p)\,\sigma = \pi \int_0^\sigma d\tilde{\sigma}\; n \cdot \mathcal{P}^\tau(\tau, \tilde{\sigma}) \quad \text{(open strings)}\,. \tag{9.2.1}$$

The constant $(n \cdot p)$ has been added to the left hand side of (9.2.1) because σ is dimensionless and the right-hand side of has units of momentum.

Let us first understand the meaning of this equation. Consider a point Q on the string of constant τ, as shown in Figure 9.2. We apply equation (9.2.1) to this point by letting σ be $\sigma(Q)$, both on the left-hand side, and on the upper limit of the integral. We then see that $\sigma(Q)$ is proportional to the integral of the momentum density $n \cdot \mathcal{P}^\tau$ over the portion of the string $\sigma \in [\,0, \sigma(Q)]$. This is the statement made by equation (9.2.1).

For any two points on a string separated by an infinitesimal $d\sigma$, equation (9.2.1) also tells us that

$$(n \cdot p)\, d\sigma = \pi\, d\sigma\; n \cdot \mathcal{P}^\tau(\tau, \sigma) \tag{9.2.2}$$

Cancelling the common factor of $d\sigma$, we see that $n \cdot \mathcal{P}^\tau$ is σ independent and thus, as claimed, constant over the strings. In fact, even more is true. Since

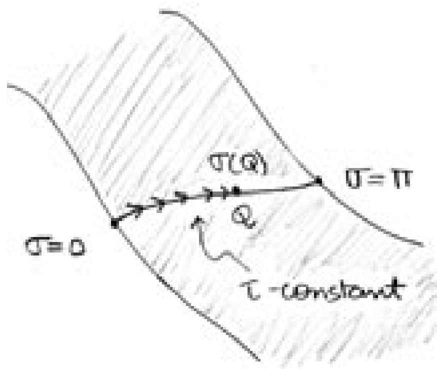

Figure 9.2: The σ parameterization of the string is fixed by requiring that $\sigma(Q)$ be proportional to the integral of the momentum density $n \cdot \mathcal{P}^\tau$ over the portion of the string extending from $\sigma = 0$ up to Q.

$n \cdot p$ is τ independent, so is $n \cdot \mathcal{P}^\tau$. We thus conclude that

$$\boxed{\quad n \cdot \mathcal{P}^\tau \quad \text{is a world-sheet constant.}} \tag{9.2.3}$$

Equation (9.2.1) also implies that $\sigma \in [0, \pi]$. Indeed, if σ_1 denotes the maximum value of σ, then

$$(n \cdot p)\sigma_1 = \pi\, n_\mu \int_0^{\sigma_1} d\tilde{\sigma}\; \mathcal{P}^{\tau\mu}(\tau, \tilde{\sigma}) = \pi\, n_\mu\, p^\mu = \pi\,(n \cdot p)\,, \tag{9.2.4}$$

since the integral over the string of $\mathcal{P}^{\tau\mu}$ is the momentum p^μ. It follows from (9.2.4) that $\sigma_1 = \pi$. This completes our confirmation that equation (9.2.1) has the expected consequences.

Let's now examine the equation of motion $\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu = 0$. Dotting this equation with n^μ we arrive at the condition

$$\frac{\partial}{\partial\tau}(n \cdot \mathcal{P}^\tau) + \frac{\partial}{\partial\sigma}(n \cdot \mathcal{P}^\sigma) = 0\,. \tag{9.2.5}$$

The first term vanishes on account of (9.2.3), and we are left with

$$\frac{\partial}{\partial\sigma}(n \cdot \mathcal{P}^\sigma) = 0\,. \tag{9.2.6}$$

This means that $n \cdot \mathcal{P}^\sigma$ is independent of σ.

We now explain that for open strings, $n \cdot \mathcal{P}^\sigma = 0$. Because of equation (9.2.6), it suffices to show that $n \cdot \mathcal{P}^\sigma$ vanishes at some point on each string. Indeed, $n \cdot \mathcal{P}^\sigma$ is required to vanish at the string endpoints to guarantee the conservation of $n \cdot p = \int d\sigma \, n \cdot \mathcal{P}^\tau$ (recall Chapter 8). Thus, at least for open strings,

$$n \cdot \mathcal{P}^\sigma = 0 \,. \tag{9.2.7}$$

Closed strings are dealt with in a similar way. We also parameterize closed strings making the momentum density $n \cdot \mathcal{P}^\tau$ constant. On the other hand, for closed strings we want a range $\sigma \in [0, 2\pi]$. These conditions require

$$(n \cdot p)\, \sigma = 2\pi \int_0^\sigma d\tilde{\sigma} \;\, n \cdot \mathcal{P}^\tau(\tau, \tilde{\sigma}) \quad \text{(closed strings)}\,. \tag{9.2.8}$$

The steps leading to (9.2.3) also hold for the above equation. As a result, $n \cdot \mathcal{P}^\tau$ is also constant on closed string world-sheets. Since closed strings are parameterized from zero to 2π, it will be convenient to set the τ parameterization of closed strings as

$$n \cdot X(\tau, \sigma) = \alpha'(n \cdot p)\, \tau \quad \text{(closed strings)}\,. \tag{9.2.9}$$

While equation (9.2.6) also holds for closed strings, we cannot prove that $n \cdot \mathcal{P}^\sigma = 0$. There is no special point on a closed string that can be of help. We note, in addition, a related difficulty with equation (9.2.8): it is not clear how to select the point $\sigma = 0$ on each string. The two problems can be solved at once. We will select $\sigma = 0$ on *one* string arbitrarily. We will then select $\sigma = 0$ on all other strings by requiring $n \cdot \mathcal{P}^\sigma = 0$.

To show how this is done, we begin by considering $\mathcal{P}^{\sigma\mu}$ from (6.5.6):

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} \frac{(\dot{X} \cdot X')\dot{X}^\mu - (\dot{X})^2 \, X^{\mu\prime}}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}\,. \tag{9.2.10}$$

We find that $n \cdot \mathcal{P}^\sigma$ is given by

$$n \cdot \mathcal{P}^\sigma = -\frac{1}{2\pi\alpha'} \frac{(\dot{X} \cdot X')\partial_\tau(n \cdot X) - (\dot{X})^2 \partial_\sigma(n \cdot X)}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}\,. \tag{9.2.11}$$

From (9.2.9) we see that $\partial_\sigma(n\cdot X)=0$, and therefore $n\cdot\mathcal{P}^\sigma$ is

$$n\cdot\mathcal{P}^\sigma = -\frac{1}{2\pi\alpha'}\frac{(\dot X\cdot X')\,\partial_\tau(n\cdot X)}{\sqrt{(\dot X\cdot X')^2-(\dot X)^2(X')^2}}\,. \qquad (9.2.12)$$

It suffices to prove that we can make $n\cdot\mathcal{P}^\sigma$ vanish at one point on each string. Since $\partial_\tau(n\cdot X)$ is a constant, we must show that $\dot X\cdot X'=0$ at some point on each string.

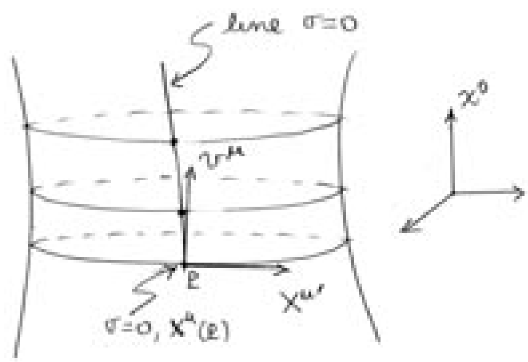

Figure 9.3: Fixing the $\sigma=0$ line on the closed string spacetime surface, after declaring P to be a point with $\sigma=0$.

Pick an arbitrary point P on a given string, and declare it to be a point for which $\sigma=0$ (see Figure 9.3). In addition, let X^μ denote the coordinates of P. We claim that, at P, there is a unique direction on the world-sheet for which the tangent v^μ is orthogonal to the spacelike vector $X^{\mu\prime}$ that is tangent to the string. This is not difficult to see. The world-sheet has a timelike tangent vector t^μ at each point. Since $X^{\mu\prime}$ and t^μ are not parallel, they generate the tangent space to the world-sheet at P. If $t_\mu X^{\mu\prime}=0$, then t^μ is the desired vector v^μ. If $t_\mu X^{\mu\prime}\neq 0$, we can define

$$v^\mu = t^\mu + \beta X^{\mu\prime}\,, \qquad (9.2.13)$$

and solve for β such that $v_\mu X^{\mu\prime}=0$:

$$t\cdot X' + \beta X'\cdot X' = 0 \quad\longrightarrow\quad v^\mu = t^\mu - \frac{t\cdot X'}{X'\cdot X'}X^{\mu\prime}\,. \qquad (9.2.14)$$

The point $\sigma = 0$ on neighboring strings is declared to be given by $X^\mu + \epsilon\, v^\mu$, for infinitesimal ϵ. The vector v^μ is therefore tangent to the desired $\sigma = 0$ line at P. The full $\sigma = 0$ line is constructed by requiring that at each point its tangent be orthogonal to $X^{\mu\prime}$. Since the tangent to the $\sigma = 0$ line is proportional to $\dot{X}^\mu$, we guarantee that $\dot{X} \cdot X' = 0$ along the $\sigma = 0$ line. Thus we can ensure that $\dot{X} \cdot X'$, and consequently $n \cdot \mathcal{P}^\sigma$, vanishes at one point on each string. As a result, we have $n \cdot \mathcal{P}^\sigma = 0$ everywhere. Equation (9.2.7) can therefore be used both for open and for closed strings:

$$n \cdot \mathcal{P}^\sigma = 0 \qquad \text{(open and closed strings)}\,. \tag{9.2.15}$$

Quick Calculation 9.1. Show that the vector v^μ in (9.2.14) is timelike.

We have now concluded the description of the parameterizations of open and closed strings. The defining equations, in both cases, are summarized by

$$\boxed{\begin{aligned} n \cdot X(\tau, \sigma) =&\ \beta\alpha'(n \cdot p)\, \tau\,, \\ (n \cdot p)\, \sigma \ =&\ \frac{2\pi}{\beta} \int_0^\sigma d\tilde{\sigma}\ n \cdot\ \mathcal{P}^\tau(\tau, \tilde{\sigma})\,, \end{aligned}} \tag{9.2.16}$$

where

$$\beta = \begin{cases} 2, & \text{for open strings,} \\ 1, & \text{for closed strings.} \end{cases} \tag{9.2.17}$$

Taking the σ derivative of the second equation in (9.2.16) we find

$$n \cdot p = \frac{2\pi}{\beta}\, n \cdot \mathcal{P}^\tau\,. \tag{9.2.18}$$

Even though we succeeded in defining the $\sigma = 0$ line on the closed string worldsheet consistently, our construction has an obvious ambiguity. We had to choose one arbitrary point on one string. Any other point on that one string could have been used as $\sigma = 0$. This means that the parameterization of the closed string world-sheet can be shifted rigidly along the σ direction. There is no way to avoid this ambiguity. The gauge conditions do not fix uniquely the parameterization of the closed string world-sheet. This will have implications for the theory of closed strings.

9.3 Constraints and wave equations

Let us now explore the constraints on X' and $\dot{X}$ that are implied by our chosen parameterization. The vanishing of $n\cdot\mathcal{P}^\sigma$, together with (9.2.12) and the recognition that $\partial_\tau(n\cdot X)$ is a non-vanishing constant, leads to

$$\dot{X}\cdot X' = 0\,. \tag{9.3.1}$$

In the static gauge, $X^{0\prime} = 0$, and (9.3.1) reduces to $\dot{\vec{X}}\cdot\vec{X}' = 0$, which we obtained before in (7.1.1). Equation (9.3.1) is a constraint that follows from our parameterization.

We now use (9.3.1) to simplify the expression (6.5.5) for $\mathcal{P}^\tau$:

$$\mathcal{P}^\tau_\mu = \frac{1}{2\pi\alpha'}\frac{X'^2\dot{X}^\mu}{\sqrt{-\dot{X}^2X'^2}}\,. \tag{9.3.2}$$

With the help of this result, equation (9.2.18) gives

$$n\cdot p = \frac{1}{\beta\alpha'}\frac{X'^2(n\cdot\dot{X})}{\sqrt{-\dot{X}^2X'^2}}\,. \tag{9.3.3}$$

Since $n\cdot\dot{X} = \beta\alpha'(n\cdot p)$ (see (9.2.16)), the factors of β cancel, and we find

$$1 = \frac{X'^2}{\sqrt{-\dot{X}^2X'^2}} \quad\longrightarrow\quad \dot{X}^2 + X'^2 = 0\,. \tag{9.3.4}$$

Aside from the units that are now different, this is consistent with the earlier result (7.3.12), which we obtained using the static gauge. Equations (9.3.1) and (9.3.4) are the constraint equations that follow from our choice of parameterization. Together they read

$$\boxed{\dot{X}\cdot X' = 0\,,\qquad \dot{X}^2 + X'^2 = 0\,.} \tag{9.3.5}$$

These two conditions are conveniently packaged together as

$$\boxed{(\dot{X}\pm X')^2 = 0\,.} \tag{9.3.6}$$

Given these constraints, the momentum densities $\mathcal{P}^\tau$ and $\mathcal{P}^\sigma$ simplify considerably. To use the constraints to simplify (9.3.2), we must take the *positive* square root in the denominator. Since $X'^2 > 0$, using (9.3.4) gives

$$\sqrt{-\dot{X}^2 X'^2} = \sqrt{X'^2 X'^2} = X'^2 \,. \tag{9.3.7}$$

Back in (9.3.2) we therefore have

$$\boxed{\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\,\dot{X}^\mu \,.} \tag{9.3.8}$$

The momentum density $\mathcal{P}^{\sigma\mu}$ was recorded in (9.2.10). It simplifies down to

$$\mathcal{P}^{\sigma\mu} = \frac{1}{2\pi\alpha'}\frac{\dot{X}^2 X^{\mu\prime}}{\sqrt{-\dot{X}^2 X'^2}} = \frac{1}{2\pi\alpha'}\frac{\dot{X}^2 X^{\mu\prime}}{X'^2}\,, \tag{9.3.9}$$

and, using (9.3.4):

$$\boxed{\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}\,X^{\mu\prime} \,.} \tag{9.3.10}$$

The momentum densities are simple derivatives of the coordinates. Using these expressions in the field equation $\partial_\tau \mathcal{P}^{\tau\mu} + \partial_\sigma \mathcal{P}^{\sigma\mu} = 0$, we find

$$\boxed{\ddot{X}^\mu - X^{\mu\prime\prime} = 0 \,.} \tag{9.3.11}$$

With our parameterization conditions the equations of motion are just wave equations! Notice that the minus sign on the right hand side of (9.3.10) was necessary for us to get a wave equation. For open strings with free endpoints, the wave equations are supplemented by the requirement that the $\mathcal{P}^{\sigma\mu}$, and therefore the X'^μ, vanish at the endpoints.

9.4 Wave equation and mode expansions

We will now explicitly solve the wave equation (9.3.11) in full generality for the case of open strings. In doing so we will introduce some of the basic notation that is used in string theory. We will assume we have a space-filling D-brane. As a result, all string coordinates X^μ will satisfy free-boundary conditions.

We know that the most general $X^\mu(\tau,\sigma)$ that solves the wave equation (9.3.11) is

$$X^\mu(\tau,\sigma) = \frac{1}{2}\Big(f^\mu(\tau+\sigma) + g^\mu(\tau-\sigma)\Big)\,, \tag{9.4.1}$$

where f^μ and g^μ are arbitrary functions of a single argument. Bearing in mind (9.3.10), the vanishing of $\mathcal{P}^{\sigma\mu}$ required for free endpoints, implies the Neumann boundary conditions

$$\frac{\partial X^\mu}{\partial\sigma} = 0\,, \quad \text{at } \sigma = 0, \pi\,. \tag{9.4.2}$$

The boundary condition at $\sigma = 0$ gives us

$$\frac{\partial X^\mu}{\partial\sigma}(\tau,0) = \frac{1}{2}\Big(f'^\mu(\tau) - g'^\mu(\tau)\Big) = 0\,. \tag{9.4.3}$$

Since the derivatives of f^μ and g^μ coincide, f^μ and g^μ can differ only by a constant c^μ. After replacing $g^\mu = f^\mu + c^\mu$ in (9.4.1), the constant c^μ can be reabsorbed into the definition of f^μ. The result is

$$X^\mu(\tau,\sigma) = \frac{1}{2}\Big(f^\mu(\tau+\sigma) + f^\mu(\tau-\sigma)\Big)\,. \tag{9.4.4}$$

Now let us consider the boundary condition at $\sigma = \pi$:

$$\frac{\partial X^\mu}{\partial\sigma}(\tau,\pi) = \frac{1}{2}\Big(f'^\mu(\tau+\pi) - f'^\mu(\tau-\pi)\Big) = 0\,. \tag{9.4.5}$$

Since this equation must hold for all τ, we learn that f'^μ *is periodic with period* 2π. Since 2π is a natural period, our decision to parameterize the open string with $\sigma \in [0,\pi]$ has paid off.

We now write the general Fourier series for the periodic function $f'(u)$:

$$f'^\mu(u) = f_1^\mu + \sum_{n=1}^{\infty}\Big(a_n^\mu \cos nu + b_n^\mu \sin nu\Big)\,. \tag{9.4.6}$$

Integrating this equation we get the expansion of f^μ:

$$f^\mu(u) = f_0^\mu + f_1^\mu u + \sum_{n=1}^{\infty}\Big(A_n^\mu \cos nu + B_n^\mu \sin nu\Big)\,, \tag{9.4.7}$$

where we have absorbed the constants arising from integration into new coefficients. We substitute this expression for $f(u)$ back into (9.4.4) and simplify to get

$$X^\mu(\tau,\sigma) = f_0^\mu + f_1^\mu \tau + 2\sum_{n=1}^{\infty}\Big(A_n^\mu \cos n\tau + B_n^\mu \sin n\tau\Big)\cos(n\sigma)\,. \tag{9.4.8}$$

We want to use in equation (9.4.8) coefficients that have a simple physical interpretation. Our first step is introducing constants a_n^μ as

$$\begin{aligned} 2A_n^\mu \cos(n\tau) + 2B_n^\mu \sin(n\tau) &= (-i)\Big((B_n^\mu + iA_n^\mu)e^{in\tau} - (B_n^\mu - iA_n^\mu)e^{-in\tau}\Big) \\ &\equiv (-i)\left(a_n^{\mu *}e^{in\tau} - a_n^\mu e^{-in\tau}\right)\frac{\sqrt{2\alpha'}}{\sqrt{n}}\,. \end{aligned} \tag{9.4.9}$$

Here $*$ denotes complex conjugation. The purpose of the $\sqrt{2\alpha'}$ factor is to make the constants a_n^μ dimensionless. These constants, and their complex conjugates, will turn into annihilation and creation operators when we consider the quantum theory. Equation (9.4.9) introduces the notation that string theorists conventionally use.

In equation (9.4.8) the constant f_1^μ has a simple physical interpretation. Using (9.3.8), the momentum density is given by

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu = \frac{1}{2\pi\alpha'}\,f_1^\mu + \cdots\,. \tag{9.4.10}$$

where the dots denote terms having a $\cos(n\sigma)$ dependence. To find the total momentum p^μ, we need to integrate over $\sigma \in [0,\pi]$. Happily, the terms represented by the dots do not contribute as the integral of $\cos(n\sigma)$ vanishes over the interval. We get the simple result that

$$p^\mu = \int_0^\pi \mathcal{P}^{\tau\mu} d\sigma = \frac{1}{2\pi\alpha'}\pi f_1^\mu \quad\longrightarrow\quad f_1^\mu = 2\alpha' p^\mu\,. \tag{9.4.11}$$

This identifies f_1^μ as a quantity proportional to the spacetime momentum carried by the string. Declaring $f_0^\mu = x_0^\mu$, and collecting all the above information, equation (9.4.8) now takes the conventional form

$$\boxed{X^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p^\mu \tau - i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\Big(a_n^{\mu *}\, e^{in\tau} - a_n^\mu e^{-in\tau}\Big)\frac{\cos n\sigma}{\sqrt{n}}\,.} \tag{9.4.12}$$

The terms on the right-hand side clearly correspond to the zero mode, to the momentum, and to the oscillations of the string. If all the coefficients a_n^μ of the oscillations vanish, the equation represents the motion of a point particle.

Quick Calculation 9.2. Verify explicitly that $X^\mu(\tau,\sigma)$ is real.

Let's now introduce some notation that will allow us to write simple expressions for the τ and σ derivatives of $X^\mu(\tau,\sigma)$. We start by defining

$$\boxed{\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu\,.} \tag{9.4.13}$$

Furthermore, we define

$$\boxed{\alpha_n^\mu = a_n^\mu \sqrt{n}\,, \qquad \alpha_{-n}^\mu = a_n^{\mu *}\sqrt{n}\,, \qquad n \geq 1\,.} \tag{9.4.14}$$

It is important to note that

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^*\,. \tag{9.4.15}$$

Moreover, while the a_n^μ are only defined when n is a positive integer, the α_n^μ are defined for any integer n, including zero. Using these new names, we can rewrite X^μ as

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\tau - i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\Big(\alpha_{-n}^\mu e^{in\tau} - \alpha_n^\mu e^{-in\tau}\Big)\frac{\cos n\sigma}{n}\,. \tag{9.4.16}$$

It is convenient to sum over all integers except zero:

$$\boxed{X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\, e^{-in\tau}\cos n\sigma\,.} \tag{9.4.17}$$

This completes the solution of the wave equations with Neumann boundary conditions. In the above equation, a solution is defined once we specify the constants x_0^μ and α_n^μ for $n \geq 0$.

It will be of use later to record here the τ and σ derivatives of X^μ. From (9.4.17) we see that

$$\dot{X}^\mu = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu \cos n\sigma e^{-in\tau}\,, \tag{9.4.18}$$

$$X^{\mu\prime} = -i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu \sin n\sigma e^{-in\tau}\,. \tag{9.4.19}$$

where $\mathbb{Z}$ denotes the set of all integers (positive, negative and zero). Finally, two linear combinations of the above derivatives are particularly nice:

$$\boxed{\dot{X}^\mu \pm X^{\mu\prime} = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^\mu \, e^{-in(\tau\pm\sigma)} \, .} \tag{9.4.20}$$

We have solved the wave equations satisfying the relevant boundary conditions, but we must also ensure that the constraints (9.3.5) are satisfied. If we specify arbitrarily all constants α_n^μ, the constraints will not be satisfied. We will use the light-cone gauge to find a solution that satisfies the wave equations as well as the constraints.

9.5 Light-cone solution of equations of motion

The light-cone solution of the equations of motion involves using light-cone coordinates to represent the motion of strings, and, imposing a set of conditions that define the light-cone gauge. We have seen in Chapter 2 that using light-cone coordinates means using x^+ and x^- instead of x^0 and x^1 – this is just a change of coordinates. Imposing a light-cone gauge condition is a more substantial step. The gauges we have examined in this chapter represent very specific choices of world-sheet coordinates. One of these choices is called the light-cone gauge.

Selecting the light-cone gauge means imposing the conditions (9.2.16) with a choice of vector n^μ such that $n \cdot X = X^+$. Taking

$$n_\mu = \Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \ldots 0\Big) \, , \tag{9.5.1}$$

we indeed find

$$n \cdot X = \frac{X^0 + X^1}{\sqrt{2}} = X^+ \, , \quad n \cdot p = \frac{p^0 + p^1}{\sqrt{2}} = p^+ \, . \tag{9.5.2}$$

Using these relations in (9.2.16) we have

$$X^+(\tau,\sigma) = \beta\alpha' p^+ \tau \, , \qquad p^+ \sigma = \frac{2\pi}{\beta} \int_0^\sigma d\tilde{\sigma} \; \mathcal{P}^{\tau +}(\tau, \tilde{\sigma}) \, , \tag{9.5.3}$$

where $\beta = 2, 1$, in the cases of open and closed strings, respectively. The second equation tells us that the density of p^+ is constant along the string.

The strategy behind the light-cone gauge is to use the especially simple form of X^+ to show that there is no dynamics in X^- (up to a zero mode) and that all the dynamics is in the *transverse coordinates* $X^2,\ X^3, ..., X^D$. These transverse coordinates will be denoted by X^I. In order to proceed we must look at the constraint equations (9.3.6). Using the definition (2.3.10) of the relativistic dot-product in light-cone coordinates, we can write these constraints as

$$-2(\dot{X}^+ \pm X^{+\prime})(\dot{X}^- \pm X^{-\prime}) + (\dot{X}^I \pm X^{I\prime})^2 = 0\,, \tag{9.5.4}$$

where $(a^I)^2 = a^I a^I$, and, as usual, repeated indices imply summation. Since $X^{+\prime} = 0$ and $\dot{X}^+ = \beta\alpha' p^+$, we in fact have

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{\beta\alpha'}\frac{1}{2p^+}(\dot{X}^I \pm X^{I\prime})^2\,. \tag{9.5.5}$$

In writing the above, we have assumed $p^+ \neq 0$. While p^+ certainly satisfies $p^+ \geq 0$, it can happen that p^+ is equal to zero. For this, the momentum p^1 must cancel the energy, and this can only occur if we have a massless particle travelling exactly in the negative x^1-direction. Since the vanishing of p^+ is thus not a common occurrence, we will take p^+ to be always positive. If we come across a situation where p^+ is zero, the light-cone formalism will not apply. Equations (9.5.5) determine both $\dot{X}^-$ and $X^{-\prime}$ in terms of the X^I, and therefore they determine X^- up to a single integration constant. All that is needed is the value of X^- at some point on the world-sheet – namely, a constant.

Note the crucial role played by both the choice of light-cone coordinates and the choice of light-cone gauge in allowing us to solve for the derivatives of X^-. Light-cone coordinates were useful because the off-diagonal metric in the $+,-$ sector allowed us to solve for the derivatives of X^- without having to take a square root! We just had to divide by $\dot{X}^+$. Here the light-cone *gauge* was useful, since it makes this factor equal to a constant.

Our analysis indicates that the full evolution of the string is determined by the following set of objects:

$$X^I(\tau, \sigma), \quad p^+, \quad x_0^-\,, \tag{9.5.6}$$

where x_0^- is the constant of integration needed for X^-.

Let's focus on the case of open strings ($\beta = 2$). We consider the explicit solution for the transverse coordinates X^I and calculate the associated X^-. Making use of the general solution in (9.4.17) we have

$$X^I(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\,\alpha_0^I\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I e^{-in\tau}\cos n\sigma\,. \tag{9.5.7}$$

Moreover, for the X^+ coordinate the gauge condition gives

$$X^+(\tau,\sigma) = 2\alpha' p^+\,\tau = \sqrt{2\alpha'}\,\alpha_0^+\tau\,. \tag{9.5.8}$$

As we can see, in the light-cone gauge the oscillations of the X^+ coordinate have been set to equal to zero:

$$\alpha_n^+ = \alpha_{-n}^+ = 0\,, \qquad n = 1, 2, \cdots\infty\,. \tag{9.5.9}$$

What about X^-? Being a linear combination of X^0 and X^1, the coordinate X^- satisfies the same wave equation and the same boundary conditions as all the other coordinates. We can therefore use the same expansion as in (9.4.17) to write

$$X^-(\tau,\sigma) = x_0^- + \sqrt{2\alpha'}\,\alpha_0^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^- e^{-in\tau}\cos n\sigma\,. \tag{9.5.10}$$

Using equation (9.4.20) with $\mu = -$ and $\mu = I$, we find

$$\dot{X}^- \pm X^{-\prime} \;=\; \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^-\,e^{-in(\tau\pm\sigma)}\,, \tag{9.5.11}$$

$$\dot{X}^I \pm X^{I\prime} \;=\; \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^I\,e^{-in(\tau\pm\sigma)}\,. \tag{9.5.12}$$

We use these equations and (9.5.5) to solve for the minus oscillators:

$$\begin{aligned}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^-\,e^{-in(\tau\pm\sigma)} &= \frac{1}{2p^+}\sum_{p,q\in\mathbb{Z}}\alpha_p^I\alpha_q^I e^{-i(p+q)(\tau\pm\sigma)}\,,\\ &= \frac{1}{2p^+}\sum_{n,p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I e^{-in(\tau\pm\sigma)}\,,\\ &= \frac{1}{2p^+}\sum_{n\in\mathbb{Z}}\Big(\sum_{p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I\Big)\;e^{-in(\tau\pm\sigma)}\,.\end{aligned} \tag{9.5.13}$$

It follows that we can identify α_n^- consistently as

$$\sqrt{2\alpha'}\,\alpha_n^- = \frac{1}{2p^+}\sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p^I\,. \tag{9.5.14}$$

This represents a complete solution! We now have explicit expressions for the minus oscillators α_n^- in terms of the transverse oscillators. On the right-hand side the spacetime indices are only to be summed over the labels of the transverse coordinates.

The general solution representing an allowed motion is fixed by specifying arbitrary values for p^+, x_0^-, x_0^I, and for all the constants α_n^I. This clearly determines $X^I(\tau,\sigma)$ in (9.5.7), and $X^+(\tau,\sigma)$ in (9.5.8). Using (9.5.14) we can calculate the constants α_n^-, which together with x_0^- determine $X^-(\tau,\sigma)$ in (9.5.10). The full solution is thus constructed.

The quadratic combinations of oscillators appearing in (9.5.14) are remarkably useful, so they have been given a name. They are the *transverse Virasoro* modes $L_n^\perp$:

$$\boxed{\sqrt{2\alpha'}\,\alpha_n^- = \frac{1}{p^+}L_n^\perp\,, \qquad L_n^\perp \equiv \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p^I\,.} \tag{9.5.15}$$

In particular, for $n = 0$ we have

$$\sqrt{2\alpha'}\,\alpha_0^- = 2\alpha' p^- = \frac{1}{p^+}L_0^\perp \quad \longrightarrow \quad \boxed{2p^+p^- = \frac{1}{\alpha'}L_0^\perp\,.} \tag{9.5.16}$$

With the definition in (9.5.15), equations (9.5.11) and (9.5.5) give

$$\boxed{\dot{X}^- \pm X^{-\prime} = \frac{1}{p^+}\sum_{n\in\mathbb{Z}} L_n^\perp\, e^{-in(\tau\pm\sigma)} = \frac{1}{4\alpha' p^+}(\dot{X}^I \pm X^{I\prime})^2\,.} \tag{9.5.17}$$

Quick Calculation 9.3. Show that

$$X^-(\tau,\sigma) = x_0^- + \frac{1}{p^+}L_0^\perp \tau + \frac{i}{p^+}\sum_{n\neq 0}\frac{1}{n}L_n^\perp e^{-in\tau}\cos n\sigma\,. \tag{9.5.18}$$

This equation demonstrates explicitly the claim that the Virasoro modes are the expansion modes of the coordinate $X^-(\tau,\sigma)$.

It is instructive to compute the mass of the string for an arbitrary motion. The mass can be calculated using the relativistic equation

$$M^2 = -p^2 = 2p^+p^- - p^I p^I \,. \tag{9.5.19}$$

Since the mass must be a constant of the motion, we anticipate that it will depend on the constant coefficients a_n^I introduced to define a classical solution. To evaluate the mass we begin by calculating $2p^+p^-$ as given in (9.5.16):

$$\begin{aligned} 2p^+p^- = \frac{1}{\alpha'}L_0^\perp &= \frac{1}{\alpha'}\Big(\frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{n=1}^{\infty}\alpha_n^{I*}\,\alpha_n^I\Big) \\ &= p^I p^I + \frac{1}{\alpha'}\sum_{n=1}^{\infty} n\,a_n^{I*}\,a_n^I \,. \end{aligned} \tag{9.5.20}$$

Here we made use of (9.5.15) to read $L_0^\perp$ and of the definitions in (9.4.13) and (9.4.14). Finally, back in (9.5.19) we now have

$$\boxed{M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} n\,a_n^{I*}\,a_n^I \,.} \tag{9.5.21}$$

This is a very interesting result. Since M^2 is written as a sum of terms each of which is positive (being of the form $a^*a = |a|^2$), we find $M^2 \geq 0$. This shows that the classical string mass $M = \sqrt{M^2}$ takes real values. Such a result is hard to obtain without using the light-cone gauge. We also see that we can adjust the a_n^I to obtain classical string solutions with arbitrary values of the mass.

If all the coefficients a_n^I vanish, the result is a massless object $M^2 = 0$. Indeed, when all a_n^I vanish the string collapses to a moving point: equation (9.5.7) gives $X^I(\sigma,\tau) = x_0^I + \sqrt{2\alpha'}\,\alpha_0^I\tau$, and the σ dependence disappears.

The classical result (9.5.21) for M^2 does not survive quantization. First, M^2 will become quantized, and string states will not exhibit a continuous spectrum of masses. This is good because we do not observe in nature particle states taking continuous values of the mass. Even more, equation (9.5.21) does not give enough massless states. As we saw above, massless states are particle-like, and while this is not clear to us yet, they do not behave at all like the massless states of Maxwell theory, or like those of gravitation. Quantum mechanics will add an extra constant to M^2 that will enable us to find states that correspond to those of physical theories. It is because quantization changes (9.5.21) in a subtle way that string theory has a chance to describe gauge fields and gravity.

Problems

Problem 9.1. *Vectors orthogonal to null vectors are null or spacelike.*

Let n^μ be a non-zero null vector ($n_\mu n^\mu = 0$) in D-dimensional Minkowski space. In addition, let b^μ be a vector that satisfies $n_\mu b^\mu = 0$. Prove that

(a) The vector b^μ is either spacelike or null.

(b) If b^μ is null, then $b^\mu = \lambda n^\mu$ for some constant λ.

(c) The set of vectors b^μ satisfying $n_\mu b^\mu = 0$ form a vector space V of dimension $(D-1)$. The subset of null vectors b^μ form a vector subspace of V of dimension one.

This result shows that for gauges (9.1.2) with n^μ null, and $D > 2$, strings are almost always spacelike objects. Moreover, the hyperplane orthogonal to n^μ contains n^μ. It is reassuring to confirm this last statement for $D = 2$:

(d) Consider a spacetime diagram such as the one in Figure 2.2. What is the null vector n^μ such that $n \cdot X = X^+$. Confirm that n^μ points along the lines of constant X^+.

Problem 9.2. *Consistency checks on the solution for X^-.*

(a) Show that X^-, as calculated in (9.5.5), satisfies the wave equation if the transverse coordinates X^I satisfy the wave equation.

(b) Assume that at the open string endpoints some of the transverse light-cone coordinates X^I satisfy Neumann boundary conditions, and some satisfy Dirichlet boundary conditions. Prove that X^-, as calculated in (9.5.5), will always satisfy Neumann boundary conditions.

Problem 9.3. *Rotating open string in the light-cone gauge.*

Consider string motion defined by $x_0^- = x_0^I = 0$, and the vanishing of all coefficients α_n^I with the exception of

$$\alpha_1^{(2)} = \alpha_{-1}^{(2)*} = a\,, \qquad \alpha_1^{(3)} = \alpha_{-1}^{(3)*} = ia\,. \tag{1}$$

Here a is a dimensionless real constant. We want to construct a solution that represents an open string that is rotating on the (x^2, x^3) plane.

(a) What is the mass (or energy) of this string?

(b) Construct the explicit functions $X^{(2)}(\tau,\sigma)$ and $X^{(3)}(\tau,\sigma)$. What is the length of the string in terms of a and α'?

(c) Calculate the $L_n^\perp$ modes for all n. Use your result to construct $X^-(\tau,\sigma)$. Your answer should be σ-independent!

(d) Determine the value of p^+ using the condition that the string has $X^1(\tau,\sigma)=0$. Find the relation between t and τ.

(e) Confirm that in your solution the energy of the string and its angular frequency of rotation are related to its length as in (7.4.26).

Problem 9.4. *Generating consistent open string motion.*

Consider string motion defined by $x_0^- = x_0^I = 0$, and the vanishing of all coefficients α_n^I with the exception of

$$\alpha_1^{(2)} = \alpha_{-1}^{(2)*} = a\,.$$

Here a is a dimensionless real constant. We want to construct of a solution that represents an open string oscillating on the (x^1,x^2) plane and having *zero momentum* in this plane.

(a) Show that the string motion is described by

$$\begin{aligned}
\frac{1}{\sqrt{2\alpha'}}\frac{1}{a}X^0(\tau,\sigma) &= \sqrt{2}\left(\tau+\frac{1}{4}\sin 2\tau\cos 2\sigma\right),\\
\frac{1}{\sqrt{2\alpha'}}\frac{1}{a}X^1(\tau,\sigma) &= -\frac{1}{2\sqrt{2}}\,\sin 2\tau\cos 2\sigma\,,\\
\frac{1}{\sqrt{2\alpha'}}\frac{1}{a}X^2(\tau,\sigma) &= 2\,\sin\tau\cos\sigma\,.
\end{aligned}$$

(b) Confirm that τ flows t flows. In the chosen Lorentz frame strings are lines on the world-sheet that lie at constant time t (or X^0). Find the values of τ for which constant τ lines are strings. Describe those strings.

(c) At $\tau=0$ the string has zero length. Study in detail the motion for $\tau \ll 1$. Calculate $\tau=\tau(t,\sigma)$ and use this result to find $X^1(t,\sigma)$ and $X^2(t,\sigma)$. Prove that as the string expands from zero size, it lies on the portion $\cos\theta \geq -1/3$ of a circle whose radius grows at the speed of light. Note that the endpoints move transversely to the string.

(d) Use your favorite mathematical software package to do a parametric plot of the string world-sheet as a surface developing in three dimensions. Use X^1, X^2 and X^0 as x, y and z axes, respectively. The parameters are τ and σ.

Chapter 10

Light-Cone Fields and Particles

We study the classical equations of motion for scalar fields, Maxwell fields, and gravitational fields. We use the light-cone gauge to find plane-wave solutions to their equations of motion and the number of degrees of freedom that characterize them. We explain how the quantization of such classical field configurations gives rise to particle states – scalar particles, photons, and gravitons. In doing so we prepare the ground for the later identification of such states among the quantum states of relativistic strings.

10.1 Introduction

In our investigation of classical string motion we had a great deal of freedom in choosing the coordinates on the world-sheet. This freedom was a direct consequence of the reparameterization invariance of the action, and we exploited it to simplify the equations of motion tremendously. Reparameterization invariance is an example of a *gauge invariance*, and a choice of parameterization is an example of a choice of *gauge*. We saw that the light-cone gauge – a particular parameterization in which τ is related to the light-cone time X^+, and σ is chosen so that the p^+-density is constant – was useful to obtain a complete and explicit solution of the equations of motion.

Classical field theories sometimes have gauge invariances. Classical electrodynamics, for example, is described in terms of gauge potentials A_μ. The gauge invariance of this description is often used to great advantage. The classical theory of a *scalar* field is simpler than classical electromagnetism. This theory is not studied at the undergraduate level, however, because elementary scalar particles – the kind of particles associated to the quantum theory of scalar fields – have not been detected yet. On the other hand, photons – the particles associated to the quantum theory of the electromagnetic field – are found everywhere! Scalar particles may play an important role in the Standard Model of particle physics, where they can help trigger symmetry breaking. Thus physicists may detect scalar particles sometime in the future. The field theory of a single scalar field has no gauge invariance. We will study it because it is the simplest field theory and because scalar particles arise in string theory. The most famous scalar particle in string theory is the tachyon. Also important is the dilaton, a massless scalar particle.

Einstein's classical field theory of gravitation is more complicated than classical electromagnetism. In gravity, as we explained in section 3.6, the dynamical variable is the two-index metric field $g_{\mu\nu}(x)$. Gravitation has a very large gauge invariance. The gauge transformations involve reparameterizations of spacetime.

We will consider scalar fields, electromagnetic fields, and gravitational fields. The light-cone gauge will allow us to dramatically simplify the (linearized) equations of motion, find their plane-wave solutions, and count the number of degrees of freedom that characterize the solutions. We will also briefly consider how the quantization of plane-wave solutions gives rise to particle states. These are the quantum states associated to the field theories. We quantize the relativistic string in Chapter 12. There we relate the quantum states of the string to the particle states of the field theories that we study in the present chapter. We use the light-cone gauge here because we will quantize the relativistic string in the light-cone gauge.

10.2 An action for scalar fields

A scalar field is simply a single real function of spacetime. It is written as $\phi(t, \vec{x})$, or more briefly, as $\phi(x)$. The term scalar means scalar under Lorentz transformations. Physically, two Lorentz observers will agree on the value of the scalar field at any fixed point in spacetime. Scalar fields have no Lorentz indices.

Let's now motivate the simplest kind of action principle that can be used to define the dynamics of a scalar field. Consider first the kinetic energy. In mechanics, the kinetic energy of a particle is proportional to its velocity squared. For a scalar field, the kinetic energy density T is declared to be proportional to the square of the rate of change of the field with time:

$$T = \frac{1}{2}\,(\partial_0\phi)^2\,. \tag{10.2.1}$$

We speak of densities because, at any fixed time, T is a function of position. The total kinetic energy will be the integral over space of the density T.

Now consider the potential energy density. There is one class of term that is natural. Suppose the equilibrium state of the field is $\phi = 0$. For a simple harmonic oscillator with equilibrium position $x = 0$, the potential energy goes like $V \sim x^2$. If we want the field to prefer its equilibrium state, then this must be encoded in the potential. The simplest potential which does this is harmonic:

$$V = \frac{1}{2}\,m^2\phi^2\,. \tag{10.2.2}$$

It is interesting to note that the constant m introduced here has the units of mass! Indeed, since the expressions on the right-hand side of the two equations above must have the same units ($[T] = [V]$), and both have two factors of ϕ, we require $[m] = [\partial_0] = L^{-1} = M$.

We could now attempt to form a Lagrangian density by combining the two energies above and setting

$$\mathcal{L} \stackrel{?}{=} T - V = \frac{1}{2}\,(\partial_0\phi)^2 - \frac{1}{2}\,m^2\phi^2\,. \tag{10.2.3}$$

This Lagrangian density, however, is not Lorentz invariant. The second term on the right-hand side is a scalar, but the first term is not, for it treats time as special. We are missing a contribution representing the energy cost when the scalar field varies in space. This is eminently reasonable in special relativity: if it costs energy to have the field vary in time, it must also cost energy to have the field vary in space. The extra contribution is therefore associated to spatial derivatives of the scalar field, and can be written as

$$V' = \frac{1}{2}\,\sum_i(\partial_i\phi)^2 = \frac{1}{2}(\nabla\phi)^2\,, \tag{10.2.4}$$

where ∂_i are derivatives with respect to spatial coordinates. We have written this term as a new contribution V' to the potential energy, rather than as a contribution to the kinetic energy. There are several reasons for this. First, it is needed for Lorentz invariance. The two options lead to contributions of opposite signs in the Lagrangian, and only one sign is consistent with Lorentz invariance. Second, kinetic energy is always associated with time derivatives. Third, the calculation of the total energy vindicates the correctness of the choice. Indeed, with this additional term the Lagrangian density becomes

$$\mathcal{L} = T - V' - V = \frac{1}{2}\,\partial_0\phi\,\partial_0\phi - \frac{1}{2}\,\partial_i\phi\,\partial_i\phi - \frac{1}{2}\,m^2\phi^2\,, \tag{10.2.5}$$

where the repeated spatial index i denotes summation. The relative sign between the first two terms on the right-hand side allows us to rewrite them as a single term involving the Minkowski metric $\eta^{\mu\nu}$:

$$\mathcal{L} = -\frac{1}{2}\,\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\,m^2\phi^2\,. \tag{10.2.6}$$

This makes it clear that the Lagrangian density is Lorentz invariant. The associated action is therefore

$$S = \int d^Dx\,\Big(-\frac{1}{2}\,\eta^{\mu\nu}\,\partial_\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2\Big), \tag{10.2.7}$$

where $d^Dx = dx^0dx^1\cdots dx^d$, and $D = d+1$, is the number of spacetime dimensions. This is called the action for a *free* scalar field with mass m. A field is said to be free when its equations of motion are linear. If each term in the action is quadratic in the field, as is the case in (10.2.7), the equations of motion will be linear in the field. A field that is not free is said to be interacting, in which case the action contains terms of order three or higher in the field.

To calculate the energy density in this field we find the Hamiltonian density $\mathcal{H}$. The momentum Π conjugate to the field is defined as

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi\,, \tag{10.2.8}$$

and the Hamiltonian density is constructed as

$$\mathcal{H} = \Pi\partial_0\phi - \mathcal{L}\,. \tag{10.2.9}$$

Quick Calculation 10.1. Show that the Hamiltonian density takes the form

$$\mathcal{H} = \frac{1}{2}\,\Pi^2 + \frac{1}{2}\,(\nabla\phi)^2 + \frac{1}{2}\,m^2\phi^2\,. \tag{10.2.10}$$

The three terms in $\mathcal{H}$ are identified as T, V' and V respectively. This is what we expected physically for the energy density. The total energy E is given by the Hamiltonian H, which in turn, is the spatial integral of the Hamiltonian density $\mathcal{H}$:

$$E = H = \int d^dx \Big(\frac{1}{2}\,\partial_0\phi\,\partial_0\phi + \frac{1}{2}\,(\nabla\phi)^2 + \frac{1}{2}\,m^2\phi^2\Big)\,. \qquad (10.2.11)$$

To find the equations of motion from the action (10.2.7), we consider a variation $\delta\phi$ of the field, and set the variation of the action equal to zero:

$$\begin{aligned}\delta S &= \int d^Dx \Big(-\eta^{\mu\nu}\partial_\mu(\delta\phi)\partial_\nu\phi - m^2\phi\delta\phi\Big) \\ &= \int d^Dx\,\delta\phi\Big(\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi\Big) = 0\,, \end{aligned} \qquad (10.2.12)$$

where we discarded a total derivative. The equation of motion for ϕ is thus

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi = 0\,. \qquad (10.2.13)$$

If we define $\partial^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$, then we have

$$(\partial^2 - m^2)\phi = 0\,. \qquad (10.2.14)$$

Separating out time and space derivatives, this equation is recognized as the Klein-Gordon equation:

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi - m^2\phi = 0\,. \qquad (10.2.15)$$

We will now study some classical solutions of this equation.

10.3 Classical plane-wave solutions

We can find plane-wave solutions to the classical scalar field equation (10.2.15). Consider, for example, the expression

$$\phi(t,\vec{x}) = a\,e^{-iEt+i\vec{p}\cdot\vec{x}}\,, \qquad (10.3.1)$$

where a and E are constants and $\vec{p}$ is an arbitrary vector. The field equation (10.2.15) fixes the possible values of E:

$$E^2 - \vec{p}^{\,2} - m^2 = 0 \quad\longrightarrow\quad E = \pm E_p \equiv \pm\sqrt{\vec{p}^{\,2} + m^2}\,. \qquad (10.3.2)$$

The square root is chosen to be positive, so $E_p > 0$.There is a small problem with the solution in (10.3.1). While ϕ is a real field, the solution (10.3.1) is not real. To make it real, we just add to it its complex conjugate:

$$\phi(t, \vec{x}) = a\, e^{-iE_pt + i\vec{p}\cdot\vec{x}} + a^*\, e^{iE_pt - i\vec{p}\cdot\vec{x}}\,. \tag{10.3.3}$$

This solution depends on the complex number a. A general solution to the equation of motion (10.2.15) is obtained by superimposing solutions, such as those above, for all values of $\vec{p}$. Since $\vec{p}$ can be varied continuously, the general superposition is actually an integral. The classical field does not have a simple quantum mechanical interpretation. If the two terms above were to be thought of as wavefunctions, the first would represent the wavefunction of a particle with momentum $\vec{p}$ and positive energy E_p. The second would represent the wavefunction of a particle with momentum $-\vec{p}$ and *negative* energy $-E_p$. This is not acceptable. To do quantum mechanics with a classical field one must quantize the field. The result is particle states with positive energy, as we will discuss briefly in the following section.

An analysis of the classical field equation (in a practical way that applies elsewhere) uses the Fourier transformation of the scalar field $\phi(x)$:

$$\phi(x) = \int \frac{d^Dp}{(2\pi)^D}\; e^{ip\cdot x}\, \phi(p)\,. \tag{10.3.4}$$

Here $\phi(p)$ is the Fourier transform of $\phi(x)$. We will always show the argument of ϕ so no confusion should arise between the spacetime field and the momentum-space field. Note that we are performing the Fourier transform over all space-time coordinates, time included: $p\cdot x = -p^0x^0 + \vec{p}\cdot\vec{x}$. The reality of $\phi(x)$ means that $\phi(x) = (\phi(x))^*$. Using equation (10.3.4), this condition yields

$$\int \frac{d^Dp}{(2\pi)^D}\; e^{ip\cdot x}\, \phi(p) = \int \frac{d^Dp}{(2\pi)^D}\; e^{-ip\cdot x}\, (\phi(p))^*\,. \tag{10.3.5}$$

We let $p \to -p$ on the left-hand side of this equation. This change of integration variable does not affect the integration $\int d^Dp$, and results in

$$\int \frac{d^Dp}{(2\pi)^D}\; e^{-ip\cdot x} \Big(\phi(-p) - (\phi(p))^*\Big) = 0\,, \tag{10.3.6}$$

where we collected all terms on the left-hand side. This left-hand side is a function of x that must vanish identically. It is also, the Fourier transform of the momentum-space function in parenthesis. This function must therefore vanish:

$$(\phi(p))^* = \phi(-p)\,. \qquad (10.3.7)$$

This is the reality condition in momentum space.

Substituting (10.3.4) into (10.2.14) and letting ∂^2 act on $e^{ip\cdot x}$ we find

$$\int \frac{d^D p}{(2\pi)^D}\; (-p^2 - m^2)\phi(p)e^{ipx} = 0\,. \qquad (10.3.8)$$

Since this must hold for all values of x, this equation requires

$$(p^2 + m^2)\phi(p) = 0 \qquad \text{for all } \; p\,. \qquad (10.3.9)$$

This is a simple equation: $(p^2 + m^2)$ is just a number multiplying $\phi(p)$ and the product must vanish. Solving this equation means specifying the values of $\phi(p)$ for all values of p. Since either factor may vanish we must consider two cases:

(i) $p^2 + m^2 \neq 0$. In this case the scalar field vanishes: $\phi(p) = 0$.

(ii) $p^2 + m^2 = 0$. In this case the scalar field $\phi(p)$ is arbitrary.

In momentum space the hypersurface $p^2 + m^2 = 0$ is called the *mass-shell*. With $p^\mu = (E, \vec{p}\,)$, the mass-shell is the locus of points in momentum space where $E^2 = \vec{p}^{\,2} + m^2$. The mass-shell is therefore the set of points $(\pm E_p, \vec{p}\,)$, for all values of $\vec{p}$. We have learned that $\phi(p)$ vanishes off the mass-shell and is arbitrary (up to the reality condition) on the mass-shell.

We now introduce the idea of *classical degrees of freedom*. For a point p^μ on the mass-shell, the solution is determined by specifying the complex number $\phi(p)$. This number determines as well the solution at the point $(-p^\mu)$, also belonging to the mass shell: $\phi(-p) = (\phi(p))^*$. So, a complex number fixes the values of the field for *two* points on the mass-shell. We need, on average, one real number for each point on the mass-shell. We will say that a field satisfying equation (10.3.9) represents *one degree of freedom per point on the mass-shell*.

We conclude this section by writing the scalar field equation of motion in light-cone coordinates. Let $\vec{x}_T$ denote a vector whose components are the transverse coordinates x^I:

$$\vec{x}_T = (x^2, x^3, \ldots, x^d)\,. \tag{10.3.10}$$

In this notation, the collection of spacetime coordinates becomes $(x^+, x^-, \vec{x}_T)$. Equation (10.2.14) expanded in light-cone coordinates is

$$\Big(-2\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-} + \frac{\partial}{\partial x^I}\frac{\partial}{\partial x^I} - m^2\Big)\,\phi(x^+, x^-, \vec{x}_T) = 0\,. \tag{10.3.11}$$

To simplify this equation, we Fourier transform the *spatial* dependence of the field, changing x^- into p^+, and x^I into p^I. Letting $\vec{p}_T$ denote the vector whose components are the transverse momenta p^I

$$\vec{p}_T = (p^2, p^3, \ldots, p^d)\,, \tag{10.3.12}$$

the Fourier transform is written as

$$\phi(x^+, x^-, \vec{x}_T) = \int \frac{dp^+}{2\pi} \int \frac{d^{D-2}\vec{p}_T}{(2\pi)^{D-2}}\; e^{-ix^-p^+ + i\,\vec{x}_T\cdot\vec{p}_T}\;\phi(x^+, p^+, \vec{p}_T)\,. \tag{10.3.13}$$

We now substitute this form of the scalar field into (10.3.11) to get

$$\Big(-2\frac{\partial}{\partial x^+}(-ip^+) - p^Ip^I - m^2\Big)\,\phi(x^+, p^+, \vec{p}_T) = 0\,, \tag{10.3.14}$$

and dividing by $2p^+$ we find

$$\Big(\,i\,\frac{\partial}{\partial x^+} - \frac{1}{2p^+}(p^Ip^I + m^2)\Big)\,\phi(x^+, p^+, \vec{p}_T) = 0\,. \tag{10.3.15}$$

This is the equation we were after. As opposed to the original covariant equation of motion, which had second derivatives with respect to time, the light-cone equation is a first-order differential equation in light-cone time. Equation (10.3.15) has the formal structure of a Schrödinger equation, which is also first-order in time. This fact will prove useful when we study how the quantum point particle is related to the scalar field.

Another version of equation (10.3.15) will be needed in our later work. Using a new time parameter τ related to x^+ as $x^+ = p^+\tau/m^2$ we obtain

$$\Big(\,i\,\frac{\partial}{\partial \tau} - \frac{1}{2m^2}(p^Ip^I + m^2)\Big)\,\phi(\tau, p^+, \vec{p}_T) = 0\,. \tag{10.3.16}$$

10.4 Quantum scalar fields and particle states

Quantum field theory is a natural language for discussing the quantum behavior of elementary particles and their interactions. Quantum field theory is quantum mechanics applied to classical fields. In quantum mechanics, classical dynamical variables turn into operators. The position and momentum of a classical particle, for example, turn into position and momentum operators. If our dynamical variables are classical fields, the quantum operators will be *field operators*. Thus, in quantum field theory, the fields are operators. The state space in a quantum field theory is typically described using a set of *particle states*. Quantum field theory also has energy and momentum operators. When they act on a particle state, these operators give the energy and momentum of the particles described by the state.

In this section, we examine briefly how the features just described arise concretely. Inspired by the plane-wave solution (10.3.3), which describes a superposition of complex waves with momenta $\vec{p}$ and $-\vec{p}$, we consider a classical field configuration $\phi_p(t,\vec{x})$ of the form

$$\phi_p(t,\vec{x}) = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2E_p}}\Big(a(t)\,e^{i\vec{p}\cdot\vec{x}} + a^*(t)\,e^{-i\vec{p}\cdot\vec{x}}\Big)\,. \tag{10.4.1}$$

There have been two changes. First, the time dependence has been made more general by introducing the function $a(t)$ and its complex conjugate $a^*(t)$. The function $a(t)$ is the dynamical variable that determines the field configuration. Second, we have placed a normalization factor $\sqrt{V}$, where V is assumed to be the volume of space. The normalization factor also includes a square root of the energy E_p defined in equation (10.3.2).

We can imagine space as a box with sides $L_1, L_2, \ldots L_d$, in which case $V = L_1L_2\ldots L_d$. When we put a field on a box we usually require it to be periodic. The field ϕ_p is periodic if each component p_i of $\vec{p}$ satisfies

$$p_iL_i = 2\pi n_i\,, \quad i = 1,2,\ldots,d\,. \tag{10.4.2}$$

Here the n_i's are integers. Each component of the momentum is quantized.

We now try to do quantum mechanics with the field configuration (10.4.1). To this end, we evaluate the scalar field action (10.2.7) for $\phi = \phi_p(t,\vec{x})$:

$$S = \int dt \int d^dx\Big(\frac{1}{2}(\partial_0\phi_p)^2 - \frac{1}{2}(\nabla\phi_p)^2 - \frac{1}{2}m^2\phi_p^2\Big)\,. \tag{10.4.3}$$

The evaluation involves squaring the field, squaring its time derivative, and squaring its gradient. In squaring any of these, we obtain from the cross multiplication two types of terms: those with spatial dependence $\exp(\pm 2i\vec{p}\cdot\vec{x})$ and those without spatial dependence.

Assuming $\vec{p} \neq 0$, we claim that the spatial integral $\int d^dx$ of the terms with spatial dependence is zero, so these terms cannot contribute. Indeed, the quantization conditions in (10.4.2) imply that

$$\int_0^{L_1} dx^1 \cdots \int_0^{L_d} dx^d \, \exp(\pm 2i\,\vec{p}\cdot\vec{x}) = 0\,. \tag{10.4.4}$$

For the terms without spatial dependence, the spatial integral gives a factor of the volume V, which cancels with the product of $\sqrt{V}$ factors we introduced in (10.4.1). The result is

$$S = \int dt \left(\frac{1}{2E_p}\, \dot{a}^*(t)\dot{a}(t) - \frac{1}{2}\, E_p\, a^*(t)a(t) \right). \tag{10.4.5}$$

Quick Calculation 10.2. Verify that equation (10.4.5) is correct.

Similarly, if we evaluate the energy of the field, as given by H in equation (10.2.11), we get

$$H = \frac{1}{2E_p}\, \dot{a}^*(t)\dot{a}(t) + \frac{1}{2}\, E_p\, a^*(t)a(t)\,. \tag{10.4.6}$$

Quick Calculation 10.3. Verify that equation (10.4.6) is correct.

To find the equation of motion satisfied by $a(t)$, we can vary $a^*(t)$ in the action (10.4.5) while keeping $a(t)$ fixed. In other words, we calculate using a variation δa^*, while keeping $\delta a = 0$. In this way the variation of S is

$$\delta S = \int dt \left(-\frac{1}{2E_p}\, \delta a^*(t)\ddot{a}(t) - \frac{1}{2}\, E_p\, \delta a^*(t)\, a(t) \right), \tag{10.4.7}$$

where we discarded a total time derivative. The equation of motion is

$$\ddot{a}(t) = -E_p^2\, a(t)\,. \tag{10.4.8}$$

The equation of motion for $a^*(t)$ is just the complex conjugate of the above equation, and arises by varying $a(t)$, while keeping $a^*(t)$ fixed. You may be puzzled by the above procedure. If a and a^* are complex conjugates of each other, we cannot really vary one and keep the other unchanged. Nevertheless, the above procedure gives the correct answer. To verify this do the following quick computation:

Quick Calculation 10.4. Define $a(t) = u(t) + iv(t)$ and $a^*(t) = u(t) - iv(t)$, where u and v are real functions of t. Write the action (10.4.5) in terms of $u, \dot{u}, v$, and $\dot{v}$. Find the equations of motion for u and v. Confirm that (10.4.8) holds.

Equation (10.4.8) is readily solved in terms of exponentials. Since it is a second order equation, there are two solutions:

$$a(t) = a_p\, e^{-iE_pt} + a^*_{-p}\, e^{iE_pt}\,. \tag{10.4.9}$$

There is no reality condition here since $a(t)$ is complex. In writing the above solution we introduced two independent complex constants a_p and a^*_{-p}. Substituting this solution into (10.4.6), we find

$$H = E_p \left(a^*_p a_p + a^*_{-p} a_{-p}\right)\,. \tag{10.4.10}$$

The quantization is done by turning the classical variables into quantum operators, and setting up commutation relations. We declare a_p and a_{-p} to be annihilation operators as in the simple harmonic oscillator. We then turn a^*_p and a^*_{-p} into the creation operators $a^\dagger_p$ and $a^\dagger_{-p}$, respectively. These oscillators are defined to satisfy the commutation relation

$$[\, a_p\,,\, a^\dagger_p\,] = 1\,, \qquad [\, a_{-p}\,,\, a^\dagger_{-p}\,] = 1\,. \tag{10.4.11}$$

All commutators involving an operator with subscript p and an operator with subscript $(-p)$ are declared to vanish. The quantum Hamiltonian then becomes

$$H = E_p \left(a^\dagger_p a_p + a^\dagger_{-p} a_{-p}\right)\,, \tag{10.4.12}$$

which is the Hamiltonian for a pair of simple harmonic oscillators, each of which has frequency E_p. When we built states, each creation operator contributes an energy E_p to the state, as we will see below.

In the classical scalar field theory, there is an integral expression that gives the momentum $\vec{P}$ carried by the field. By a generalization of the methods introduced in Chapter 8 one can show that

$$\vec{P} = -\int d^dx (\partial_0\phi)\nabla\phi\,. \tag{10.4.13}$$

In Problem 10.1 you are asked to evaluate $\vec{P}$ for the field configuration (10.4.1), when $a(t)$ is given by (10.4.9). The result that you will find is

$$\vec{P} = \vec{p}\left(a^*_p a_p - a^*_{-p} a_{-p}\right)\,. \tag{10.4.14}$$

In the quantum theory, the momentum becomes the operator

$$\vec{P} = \vec{p}\left(a_p^\dagger a_p - a_{-p}^\dagger a_{-p}\right) . \tag{10.4.15}$$

Note that the oscillators with subscripts $(-p)$ contribute with a negative sign to the momentum. This equation will help our interpretation below.

As a check of our quantum setup, let's show that the chosen Hamiltonian leads to Heisenberg operators with the expected time dependence. We use (10.4.9) to note that at $t = 0$, the quantum operator $a(0)$ must be:

$$a(0) = a_p + a_{-p}^\dagger \, . \tag{10.4.16}$$

We would like to show that the associated Heisenberg operator

$$a(t) \equiv e^{iHt}\, a(0)\, e^{-iHt} = e^{iHt}\, a_p\, e^{-iHt} + e^{iHt}\, a_{-p}^\dagger\, e^{-iHt} \, , \tag{10.4.17}$$

takes the form suggested by the classical solution (10.4.9). The first term in the above right-hand side is

$$a_p(t) \equiv e^{iHt}\, a_p\, e^{-iHt} \, . \tag{10.4.18}$$

Taking a derivative with respect to time, we obtain

$$\frac{da_p(t)}{dt} = i\, e^{iHt}\, [\, H \, , \, a_p]\, e^{-iHt} \, . \tag{10.4.19}$$

Since $[H, a_p] = -E_p a_p$, the above equation becomes

$$\frac{da_p(t)}{dt} = -iE_p\, e^{iHt}\, a_p\, e^{-iHt} = -iE_p\, a_p(t) \, . \tag{10.4.20}$$

The solution, subject to the initial condition $a_p(0) = a_p$ is

$$a_p(t) = e^{iHt}\, a_p\, e^{-iHt} = a_p\, e^{-iE_p t} \, . \tag{10.4.21}$$

A completely analogous calculation shows that

$$a_{-p}^\dagger(t) = e^{iHt}\, a_{-p}^\dagger\, e^{-iHt} = a_{-p}^\dagger\, e^{iE_p t} \, . \tag{10.4.22}$$

Replacing the above results back into (10.4.17):

$$a(t) = a_p\, e^{-iE_p t} + a_{-p}^\dagger\, e^{iE_p t} \, . \tag{10.4.23}$$

This result is the expected one on account of (10.4.9).

If we substitute (10.4.23) back into the field configuration (10.4.1) we find

$$\begin{aligned}\phi_p(t,\vec{x}) = &\frac{1}{\sqrt{V}}\frac{1}{\sqrt{2E_p}}\Big(a_p\, e^{-iE_pt+i\vec{p}\cdot\vec{x}} + a_p^\dagger\, e^{iE_pt-i\vec{p}\cdot\vec{x}}\Big)\\ &+\frac{1}{\sqrt{V}}\frac{1}{\sqrt{2E_p}}\Big(a_{-p}\, e^{-iE_pt-i\vec{p}\cdot\vec{x}} + a_{-p}^\dagger\, e^{iE_pt+i\vec{p}\cdot\vec{x}}\Big)\,.\end{aligned}\tag{10.4.24}$$

We see that the classical field configuration ϕ_p has become an operator. It is, in fact, a spacetime dependent operator, or a field operator. The second line in (10.4.24) is obtained from the first line by the replacement $\vec{p} \to -\vec{p}$, which does not affect E_p. In full generality, the quantum field $\phi(x)$ includes contributions from all values of the spatial momentum $\vec{p}$. As a result, one can write

$$\boxed{\phi(t,\vec{x}) = \frac{1}{\sqrt{V}}\sum_{\vec{p}}\frac{1}{\sqrt{2E_p}}\Big(a_p\, e^{-iE_pt+i\vec{p}\cdot\vec{x}} + a_p^\dagger\, e^{iE_pt-i\vec{p}\cdot\vec{x}}\Big)\,.}\tag{10.4.25}$$

The commutation relations for oscillators are then

$$\boxed{[a_p\,, a_k^\dagger\,] = \delta_{p,k}\,,\quad [a_p\,, a_k\,] = [a_p^\dagger\,, a_k^\dagger\,] = 0\,.}\tag{10.4.26}$$

All subscripts here are spatial vectors, written without the arrows to avoid cluttering the equations. The Kronecker delta $\delta_{p,k}$ is zero unless $\vec{p} = \vec{k}$, in which case it equals one. Once we consider contributions from all values of the momenta, the previous expression for the Hamiltonian in (10.4.12) and that for the momentum operator in (10.4.14) must be changed. One can show that

$$H = \sum_{\vec{p}} E_p\, a_p^\dagger a_p\,,\tag{10.4.27}$$

$$\vec{P} = \sum_{\vec{p}} \vec{p}\, a_p^\dagger a_p\,.\tag{10.4.28}$$

We will not derive these expressions, but they should seem quite plausible.

The state space of this quantum system is built in the same way as the state space of the simple harmonic oscillator. We assume the existence of a vacuum state $|\Omega\rangle$, which acts just as the simple harmonic oscillator ground state $|0\rangle$ in that it is annihilated by the annihilation operators a_p: $a_p|\Omega\rangle = 0$ for all $\vec{p}$. It follows that $H|\Omega\rangle = 0$, which makes the vacuum a zero-energy state. This vacuum state is interpreted as a state in which there are no particles. On the other hand, the state

$$a_p^\dagger\,|\Omega\rangle\,, \tag{10.4.29}$$

is interpreted as a state with precisely one particle. We claim that it is a particle with momentum $\vec{p}$. To verify this, we act on the state with the momentum operator (10.4.28) and use (10.4.26) to find

$$\vec{P}a_p^\dagger\,|\Omega\rangle = \sum_{\vec{k}} \vec{k}a_k^\dagger\,[a_k\,, a_p^\dagger]\,|\Omega\rangle = \vec{p}\,a_p^\dagger\,|\Omega\rangle\,. \tag{10.4.30}$$

The energy of the state is similarly computed by acting on the state with the Hamiltonian H:

$$Ha_p^\dagger\,|\Omega\rangle = \sum_{\vec{k}} E_k\,a_k^\dagger\,[a_k\,, a_p^\dagger]\,|\Omega\rangle = E_p\,a_p^\dagger\,|\Omega\rangle\,. \tag{10.4.31}$$

The state $a_p^\dagger|\Omega\rangle$ has positive energy. While the quantum field has both positive and negative energy components, the states representing the particles have positive energy. The states $a_p^\dagger\,|\Omega\rangle$ are the *one-particle states.*

The state space contains multiparticle states as well. These are states built by acting on the vacuum with a collection of creation operators:

$$a_{p_1}^\dagger\,a_{p_2}^\dagger \ldots a_{p_k}^\dagger|\Omega\rangle\,. \tag{10.4.32}$$

This state, with k creation operators acting on the vacuum, represents a state with k particles. The particles have momenta $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_k$ and energies $E_{p_1}, E_{p_2}, \ldots, E_{p_k}$.

Quick Calculation 10.5. Show that the eigenvalues of $\vec{P}$ and H acting on (10.4.32) are $\sum_{n=1}^k \vec{p}_n$ and $\sum_{n=1}^k E_{p_n}$, respectively.

Quick Calculation 10.6. Convince yourself that $N = \sum_{\vec{p}} a_p^\dagger a_p$ is a number operator: acting on a state it gives us the number of particles it contains.

Our analysis of classical solutions in the previous section led to the conclusion that there is one degree of freedom per point on the mass-shell. At the quantum level, we focus on the one-particle states. Consequently, we restrict ourselves to the physical part of the mass-shell, the part where the energy is positive ($p^0 = E > 0$). We have a single one-particle state for each point on the physical mass-shell. This state is labelled by its spatial momentum $\vec{p}$.

To describe the particle states in light-cone coordinates, the changes are minimal. The physical part of the mass-shell is parameterized by the transverse momenta $\vec{p}_T$ and the light-cone momenta p^+ for which $p^+ > 0$. The value of the light-cone energy p^- is then fixed. Thus, instead of labeling the oscillators with $\vec{p}$, we simply label them with p^+ and $\vec{p}_T$. The one-particle states are written as

$$\boxed{\text{One-particle states of a scalar field:} \quad a^{\dagger}_{p^+,\,p_T}\,|\Omega\rangle\,.} \tag{10.4.33}$$

The momentum operator given in (10.4.28) has a natural light-cone version. The various components of the operator take the form

$$\begin{aligned} \widehat{p}^{\,+} &= \sum_{p^+, p_T} p^+ \; a^{\dagger}_{p^+,\,p_T} a_{p^+,\,p_T}\,, \\ \widehat{p}^{\,I} &= \sum_{p^+, p_T} p^I \; a^{\dagger}_{p^+,\,p_T} a_{p^+,\,p_T}\,, \\ \widehat{p}^{\,-} &= \sum_{p^+, p_T} \frac{1}{2p^+}\Big(p^I p^I + m^2\Big)\, a^{\dagger}_{p^+,\,p_T} a_{p^+,\,p_T}\,. \end{aligned} \tag{10.4.34}$$

In the last equation $(p^I p^I + m^2)/(2p^+)$ is the value of p^- in terms of p^+ and $\vec{p}_T$, determined from the mass-shell condition. This last equation is analogous to (10.4.27) where E_p is the energy determined from the mass-shell condition.

10.5 Maxwell fields and photon states

We now turn to an analysis of Maxwell fields and their corresponding quantum states. As opposed to the case of the scalar field, where there is no gauge invariance, electromagnetic fields have a gauge invariance that will make our analysis more subtle and interesting. In order to study the field equations in a convenient way, we will impose the gauge condition that defines the light-cone gauge. We will then be able to describe the quantum states of the Maxwell field.

The field equations for electromagnetism are written in terms of the electromagnetic vector potential $A_\mu(x)$. As we reviewed in section 3.3, the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant under the gauge transformation

$$\delta A_\mu = \partial_\mu \epsilon \,, \tag{10.5.1}$$

where ϵ is the gauge parameter. The field equations take the form

$$\partial_\nu F^{\mu\nu} = 0 \quad \longrightarrow \quad \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0 \,, \tag{10.5.2}$$

and can be written as

$$\partial^2 A^\mu - \partial^\mu(\partial \cdot A) = 0 \,. \tag{10.5.3}$$

Compare this equation with equation (10.2.14) for the scalar field. There is no indication of a mass term for the Maxwell field – such a term would be recognized as one without spacetime derivatives. We will confirm below that the Maxwell field is, indeed, massless.

We Fourier transform all the components of the vector potential in order to determine the equation of motion in momentum space:

$$A^\mu(x) = \int \frac{d^D p}{(2\pi)^D}\, e^{ipx} A^\mu(p) \,, \tag{10.5.4}$$

where reality of $A^\mu(x)$ implies $A^\mu(-p) = (A^\mu(p))^*$. Substituting this into (10.5.3) we obtain the equation:

$$p^2 A^\mu - p^\mu(p \cdot A) = 0 \,. \tag{10.5.5}$$

The gauge transformation (10.5.1) can also be Fourier transformed. In momentum space, the gauge transformation relates $\delta A_\mu(p)$ to the Fourier transform $\epsilon(p)$ of the gauge parameter:

$$\delta A_\mu(p) = i p_\mu \, \epsilon(p) \,. \tag{10.5.6}$$

Since the gauge parameter $\epsilon(x)$ is real, we have $\epsilon(-p) = \epsilon^*(p)$. The gauge transformation (10.5.6) is consistent with the reality of $\delta A_\mu(x)$. Indeed,

$$(\delta A_\mu(p))^* = -ip_\mu(\epsilon(p))^* = i(-p_\mu)\epsilon(-p) = \delta A_\mu(-p) \,. \tag{10.5.7}$$

Note the role of the factor of i in getting the signs to work out.

Being done with preliminaries, we can analyze (10.5.5) subject to the gauge transformations (10.5.6). At this point, it is more convenient to work with the light-cone components of the gauge field:

$$A^+(p)\,, \quad A^-(p)\,, \quad A^I(p)\,. \tag{10.5.8}$$

The gauge transformations (10.5.6) then read

$$\delta A^+ = ip^+\epsilon\,, \quad \delta A^- = ip^-\epsilon\,, \quad \delta A^I = ip^I\epsilon\,. \tag{10.5.9}$$

We now impose a gauge condition. As we have emphasized before, when working with the light-cone formalism we always assume $p^+ \neq 0$. The above gauge transformations now make it clear that we can set A^+ to zero by choosing ϵ correctly. Indeed, if we apply a gauge transformation

$$A^+ \to A'^+ = A^+ + ip^+\epsilon\,, \tag{10.5.10}$$

then the $+$ component of the new gauge field A' vanishes if we choose $\epsilon = iA^+/p^+$. In other words, we can always make the $+$ component of the Maxwell field zero by applying a gauge transformation. This will be our defining condition for the light-cone gauge in Maxwell theory:

$$\boxed{\text{light-cone gauge condition :} \quad A^+(p) = 0\,.} \tag{10.5.11}$$

Setting A^+ to zero determines the gauge parameter ϵ, and no additional gauge transformations are possible: if $A^+ = 0$, any further gauge transformation will make it non-zero. The similarities with light-cone gauge string theory are noteworthy. In light-cone gauge open string theory all world-sheet reparameterization invariances are fixed. Moreover, while not equal to zero, X^+ is very simple: the corresponding zero mode and oscillators vanish.

The gauge condition (10.5.11) simplifies the equation of motion (10.5.5) considerably. Taking $\mu = +$ we find

$$p^+(p\cdot A) = 0 \quad \longrightarrow \quad p\cdot A = 0\,. \tag{10.5.12}$$

This equation can be expanded out using light-cone indices:

$$-p^+A^- - p^-A^+ + p^IA^I = 0\,. \tag{10.5.13}$$

Since $A^+ = 0$, this equation determines A^- in terms of the transverse A^I:

$$A^- = \frac{1}{p^+}\,(p^I A^I)\,. \tag{10.5.14}$$

This is reminiscent of our light-cone analysis of the string, where X^- was solved for in terms of the transverse coordinates (and a zero mode). Using (10.5.12) back in (10.5.5), all that remains from the field equation is

$$p^2 A^\mu(p) = 0\,. \tag{10.5.15}$$

For $\mu = +$ this equation is trivially satisfied, since $A^+ = 0$. For $\mu = I$ we get a set of nontrivial conditions:

$$p^2 A^I(p) = 0\,. \tag{10.5.16}$$

For $\mu = -$, we get $p^2 A^-(p) = 0$. This is automatically satisfied on account of (10.5.14) and (10.5.16).

For each value of I, equation (10.5.16) takes the form of the equation of motion for a massless scalar. Thus, $A^I(p) = 0$ when $p^2 \neq 0$. This makes $A^- = 0$, and since A^+ is zero, the full gauge field vanishes. For $p^2 = 0$, the $A^I(p)$ are unconstrained, and the $A^-(p)$ are determined as a function of the A^I (see (10.5.14)). The degrees of freedom of the Maxwell field are thus carried by the $(D-2)$ transverse fields $A^I(p)$, for $p^2 = 0$. We say that we have $(D-2)$ degrees of freedom per point on the mass-shell.

It is actually possible to show that there are no degrees of freedom for $p^2 \neq 0$, without having to make a choice of gauge. Although not every field is zero, every field is *gauge equivalent* to the zero field when $p^2 \neq 0$. If a field differs from the zero field by only a gauge transformation, we say that the field is *pure gauge*. Recall that fields A_μ and A'_μ are gauge equivalent if $A_\mu = A'_\mu + \partial_\mu \chi$, for some scalar function χ. Taking $A'_\mu = 0$, we learn that $A_\mu = \partial_\mu \chi$ is gauge-equivalent to the zero field, and is therefore pure gauge. The term pure gauge is suitable: A_μ takes the form of a gauge transformation. In momentum space, a pure gauge is a field that can be written as

$$\text{pure gauge:} \quad A_\mu(p) = ip_\mu \chi(p)\,, \tag{10.5.17}$$

for some choice of χ. Rewrite now the equation of motion (10.5.5) as

$$p^2 A_\mu = p_\mu (p \cdot A)\,. \tag{10.5.18}$$

Since $p^2 \neq 0$, we can write

$$A_\mu = ip_\mu\Big(\frac{-ip\cdot A}{p^2}\Big)\,. \tag{10.5.19}$$

Comparing with (10.5.17), we see that A_μ is pure gauge. It means that there are no degrees of freedom in the Maxwell field when $p^2 \neq 0$. For all intents and purposes, there is no field.

Let us now discuss briefly photon states. Each of the independent classical fields A^I can be expanded as we did for the scalar field in (10.4.25). To do so, we would introduce – as you can infer by analogy – oscillators a_p^I and $a_p^{I\dagger}$. Here the subscripts p represent the values of p^+ and $\vec{p}_T$. We thus get $(D-2)$ species of oscillators. Introducing a vacuum $|\Omega\rangle$, the one-photon states would be written as

$$a^{I\dagger}_{p^+,\,p_T}|\Omega\rangle\,. \tag{10.5.20}$$

Here the label I is a polarization label. The photon state (10.5.20) is said to be polarized in the I'th direction. Since we have $(D-2)$ possible polarizations, *we have $(D-2)$ linearly independent one-photon states for each point on the physical sector of the mass-shell.* A general one-photon state with momentum $(p^+, \vec{p}_T)$ would be a linear superposition of the above states:

$$\boxed{\text{one-photon states:}\quad \sum_{I=2}^{D-1}\xi_I\, a^{I\dagger}_{p^+,\,p_T}|\Omega\rangle\,.} \tag{10.5.21}$$

Here the transverse vector ξ_I is called the polarization vector.

In four dimensional spacetime, Maxwell theory gives rise to $D-2=2$ single photon states for any fixed spatial momentum. This is indeed familiar to you, at least classically. An electromagnetic plane wave propagating in a fixed direction and having some fixed wavelength (*i.e.* fixed momentum), can be written as a superposition of two plane waves representing independent polarization states.

10.6 Gravitational fields and graviton states

Gravitation emerges in string theory in the language of Einstein's theory of general relativity. We discussed briefly this language in section 3.6. The dynamical field variable is the spacetime metric $g_{\mu\nu}(x)$, which in the approximation of weak gravitational fields can be taken to be of the form $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$. Both $g_{\mu\nu}$ and $h_{\mu\nu}$ are symmetric under the exchange of their indices. The field equations for $g_{\mu\nu}$ – Einstein's equations – can be used to derive a linearized equation of motion for the fluctuations $h_{\mu\nu}$. This equation was given in (3.6.6). Defining $h_{\mu\nu}(p)$ to be the Fourier transform of $h_{\mu\nu}(x)$, the momentum-space version of this equation is

$$S^{\mu\nu}(p) \equiv p^2 h^{\mu\nu} - p_\alpha(p^\mu h^{\nu\alpha} + p^\nu h^{\mu\alpha}) + p^\mu p^\nu h = 0\,. \tag{10.6.1}$$

If we were considering Einstein's equations in the presence of sources, the right-hand side of the equation would include terms related to the energy-momentum tensor that is associated to the sources. In the above equation $h = \eta^{\mu\nu} h_{\mu\nu} = h^\mu_\mu$, and indices on $h_{\mu\nu}$ can be raised or lowered using the Minkowski metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$. Since every term in (10.6.1) contains two derivatives, this suggests that the fluctuations $h_{\mu\nu}$ are associated to massless excitations.

As we will see shortly, the equation of motion (10.6.1) is invariant under the gauge transformations

$$\delta h^{\mu\nu}(p) = ip^\mu \epsilon^\nu(p) + ip^\nu \epsilon^\mu(p)\,. \tag{10.6.2}$$

The infinitesimal gauge parameter $\epsilon^\mu(p)$ is a vector. In gravitation, the gauge invariance is reparameterization invariance: the choice of coordinate system used to parameterize spacetime does not affect the physics.

Let's verify that (10.6.1) is invariant under the gauge transformation (10.6.2). First we compute δh and find that

$$\delta h = i\eta^{\mu\nu}(p_\mu \epsilon_\nu + p_\nu \epsilon_\mu) = 2ip\cdot\epsilon\,. \tag{10.6.3}$$

The resulting variation in $S^{\mu\nu}$ is therefore given by

$$\begin{aligned}\delta S^{\mu\nu} = {} & ip^2(p^\mu\epsilon^\nu + p^\nu\epsilon^\mu) - ip_\alpha p^\mu(p^\nu\epsilon^\alpha + p^\alpha\epsilon^\nu) \\ & - ip_\alpha p^\nu(p^\mu\epsilon^\alpha + p^\alpha\epsilon^\mu) + 2ip^\mu p^\nu p\cdot\epsilon\,.\end{aligned} \tag{10.6.4}$$

But we can rewrite

$$\begin{aligned}\delta S^{\mu\nu} = {} & ip^2(p^\mu\epsilon^\nu + p^\nu\epsilon^\mu) - ip^\mu p^\nu(p\cdot\epsilon) - ip^2 p^\mu\epsilon^\nu \\ & - ip^\mu p^\nu(p\cdot\epsilon) - ip^2 p^\nu\epsilon^\mu + 2ip^\mu p^\nu p\cdot\epsilon\,.\end{aligned} \tag{10.6.5}$$

It is readily seen that all the terms in (10.6.5) cancel, so $\delta S^{\mu\nu} = 0$. The equation of motion exhibits the proper gauge invariance.

Since the metric $h^{\mu\nu}$ is symmetric and has two indices, each running over $(+, -, I)$, we have the following objects to consider:

$$(h^{IJ}, h^{+I}, h^{-I}, h^{+-}, h^{++}, h^{--})\,. \tag{10.6.6}$$

We shall try to set to zero all the fields in (10.6.6) that contain a + index. For this, we use (10.6.2) to examine their gauge transformations:

$$\begin{aligned} \delta h^{++} &= 2ip^+\epsilon^+\,, & (10.6.7)\\ \delta h^{+-} &= ip^+\epsilon^- + ip^-\epsilon^+\,, & (10.6.8)\\ \delta h^{+I} &= ip^+\epsilon^I + ip^I\epsilon^+\,. & (10.6.9)\end{aligned}$$

As before, we assume $p^+ \neq 0$. From (10.6.7), we see that a judicious choice of ϵ^+ will permit us to gauge away h^{++}. This fixes our choice of ϵ^+. Looking at equation (10.6.8), we see that although we have fixed ϵ^+, we can still find an ϵ^- that will set the h^{+-} term to zero. This fixes ϵ^-. Similarly, we can use (10.6.9) and a suitable choice of ϵ^I to set h^{+I} to zero. We have used the full gauge freedom to set to zero all of the entries in $h^{\mu\nu}$ with + indices. This defines the light-cone gauge for the gravity field:

$$\boxed{\text{light-cone gauge conditions}: \quad h^{++} = h^{+-} = h^{+I} = 0\,.} \tag{10.6.10}$$

The remaining degrees of freedom are carried by

$$(h^{IJ}, h^{-I}, h^{--})\,. \tag{10.6.11}$$

We must now see what is implied by the equations of motion (10.6.1). Bearing in mind the gauge conditions (10.6.10), when $\mu = \nu = +$ we find

$$(p^+)^2 h = 0 \quad \longrightarrow \quad h = 0\,. \tag{10.6.12}$$

This means that $h^{\mu\nu}$ is traceless. More explicitly,

$$h = \eta_{\mu\nu}h^{\mu\nu} = -2h^{+-} + h^{II} = 0 \quad \longrightarrow \quad h^{II} = 0\,, \tag{10.6.13}$$

since $h^{+-} = 0$ in our gauge. With $h = 0$, the equation of motion (10.6.1) reduces to

$$p^2 h^{\mu\nu} - p^\mu(p_\alpha h^{\nu\alpha}) - p^\nu(p_\alpha h^{\mu\alpha}) = 0\,. \tag{10.6.14}$$

Now set $\mu = +$. We obtain $p^+(p_\alpha h^{\nu\alpha}) = 0$, and as a result

$$p_\alpha h^{\nu\alpha} = 0\,. \qquad (10.6.15)$$

If (10.6.15) holds, equation (10.6.14) reduces to

$$p^2 h^{\mu\nu} = 0\,. \qquad (10.6.16)$$

This is all that remains of the equation of motion! Before delving into this familiar equation, let's investigate the implications of (10.6.15). The only free index here is ν. For $\nu = +$ the equation is trivial, since the $h^{+\alpha}$ are zero in our gauge. Consider now $\nu = I$. This gives $p_\alpha h^{I\alpha} = 0$, which we can expand as

$$-p^+ h^{I-} - p^- h^{I+} + p_J h^{IJ} = 0 \quad \longrightarrow \quad h^{I-} = \frac{1}{p^+} p_J h^{IJ}\,. \qquad (10.6.17)$$

Similarly, with $\nu = -$ we get $p_\alpha h^{-\alpha} = 0$, which expands to

$$-p^+ h^{--} - p^- h^{-+} + p_I h^{-I} = 0 \quad \longrightarrow \quad h^{--} = \frac{1}{p^+} p_I h^{-I}\,. \qquad (10.6.18)$$

Equations (10.6.17) and (10.6.18) give us the h's with $-$ indices in terms of the transverse h^{IJ}. There is no more content to (10.6.15).

We now return to equation (10.6.16). This equation holds trivially for any field with a $+$ index. The equation is nontrivial for transverse indices:

$$p^2 h^{IJ}(p) = 0\,. \qquad (10.6.19)$$

Equations $p^2 h^{I-} = 0$ and $p^2 h^{--} = 0$ are automatically satisfied on account of our solutions (10.6.17) and (10.6.18), together with (10.6.19). Equation (10.6.19) implies that $h^{IJ}(p) = 0$ for $p^2 \neq 0$, in which case all other components of $h^{\mu\nu}$ also vanish. For $p^2 = 0$, the $h^{IJ}(p)$ are unconstrained, except for the tracelessness condition $h_{II}(p) = 0$. All other components are determined in terms of the transverse components.

We conclude that the degrees of freedom of the classical D-dimensional gravitational field $h^{\mu\nu}$ are carried by a *symmetric, traceless, transverse tensor field*, the components of which satisfy the equation of motion of a massless scalar. This tensor has as many components as a symmetric traceless square matrix of size $(D-2)$. The number of components $n(D)$ in this matrix is

$$n(D) = \frac{1}{2}\,(D-2)(D-1) - 1 = \frac{1}{2}\,D(D-3)\,. \qquad (10.6.20)$$

Moreover, as before, we count a massless scalar as one degree of freedom per point on the mass-shell. Therefore we say that a classical gravity wave has $n(D)$ degrees of freedom per point on the mass-shell. In four-dimensional spacetime there are two transverse directions, and a symmetric traceless 2×2 matrix has two independent components. In four dimensions we thus have $n(4) = 2$ degrees of freedom. In five dimensions we have $n(5) = 5$ degrees of freedom, in ten dimensions $n(10) = 35$ degrees of freedom, and in twenty-six dimensions $n(26) = 299$ degrees of freedom.

To obtain graviton states, each of the independent classical fields h^{IJ} fields are expanded in terms of creation and annihilation operators, just as we did for the scalar field in (10.4.25). To do so we need oscillators $a^{IJ}_{p^+,p_T}$ and $a^{IJ\dagger}_{p^+,p_T}$. We introduce a vacuum $|\Omega\rangle$, and a basis of states

$$a^{IJ\,\dagger}_{p^+,p_T}|\Omega\rangle\,. \tag{10.6.21}$$

A one-graviton state with momentum $(p^+,\vec{p}_T)$ is a linear superposition of the above states:

$$\boxed{\text{one-graviton states}: \ \sum_{I,J=2}^{D-1}\xi_{IJ}\,a^{IJ\,\dagger}_{p^+,p_T}|\Omega\rangle \quad \xi_{II}=0\,.} \tag{10.6.22}$$

Here ξ_{IJ} is the graviton polarization tensor. The classical tracelessness condition we found earlier becomes in the quantum theory the tracelessness $\xi_{II} = 0$ of the polarization tensor. Since ξ_{IJ} is a traceless symmetric matrix of size $(D-2)$, we have $n(D)$ linearly independent graviton states for each point on the physical mass-shell.

Problems

Problem 10.1. *Momentum for the classical scalar field.*

Show that the integral (10.4.13) for the momentum carried by the scalar field, when evaluated for the field configuration (10.4.1) gives

$$\vec{P} = -\frac{i\,\vec{p}}{2E_p}\,(\dot{a}^* a - a^* \dot{a}) \,.$$

Use (10.4.9) to show that $\vec{P} = \vec{p}\left(a_p^* a_p - a_{-p}^* a_{-p}\right)$, as quoted in (10.4.14).

Problem 10.2. *Commutator for the quantum scalar field.*

(a) Consider a periodic function $f(\vec{x})$ on the box described above equation (10.4.2). Such a function can be expanded as the Fourier series

$$f(\vec{x}) = \sum_{\vec{p}} f(\vec{p})\, e^{i\vec{p}\cdot\vec{x}} \,. \tag{1}$$

Show that

$$f(\vec{p}) = \frac{1}{V}\int d\vec{x}'\, f(\vec{x}')e^{-i\vec{p}\cdot\vec{x}'} \,. \tag{2}$$

Plug (2) back into (1) to discover a representation for the d-dimensional delta function $\delta^d(\vec{x}-\vec{x}')$.

(b) Consider the complete scalar field expansion in (10.4.25). Calculate the corresponding expansion of $\Pi(t,\vec{x}) = \partial_0\phi(t,\vec{x})$. Show that

$$[\phi(t,\vec{x})\,,\,\Pi(t,\vec{x}')\,] = i\,\delta^d(\vec{x}-\vec{x}')\,. \tag{3}$$

This is the equal-time commutator between the field operator and its corresponding canonical momentum. Most discussions of quantum field theory begin by postulating this commutator.

Problem 10.3. *Constant electric field in light-cone gauge.*

Find the potentials that describe a uniform constant electric field $\vec{E} = E_0\vec{e}_x$ in the light-cone gauge ($A^+ = (A^0 + A^1)/\sqrt{2} = 0$). Write A^- and A^I in terms of the light-cone coordinates x^+, x^-, and x^I.

Problem 10.4. *Gravitational fields that are pure gauge.*

Following the discussion of Maxwell fields that are pure gauge, define gravitational fields that are pure gauge. Prove that any gravitational field $h_{\mu\nu}(p)$ satisfying the equations of motion is pure gauge when $p^2 \neq 0$.

Problem 10.5. *Kalb-Ramond field $B_{\mu\nu}$.*

Here we examine the field theory of a massless antisymmetric tensor gauge field $B_{\mu\nu} = -B_{\nu\mu}$. This gauge field is a tensor analog of the Maxwell gauge field A_μ. In Maxwell theory we defined the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. For $B_{\mu\nu}$ we define a field strength $H_{\mu\nu\rho}$:

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \,. \tag{1}$$

(a) Show that $H_{\mu\nu\rho}$ is totally antisymmetric. Prove that $H_{\mu\nu\rho}$ is invariant under the *gauge transformations*

$$\delta B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu \,. \tag{2}$$

(b) The above gauge transformations are peculiar: the gauge parameters themselves have a gauge invariance! Show that ϵ'_μ given as

$$\epsilon'_\mu = \epsilon_\mu + \partial_\mu \lambda \,, \tag{3}$$

generates the *same* gauge transformations as ϵ_μ.

(c) Use light-cone coordinates and momentum space to argue that $\epsilon^+(p)$ can be set to zero for a suitable choice of $\lambda(p)$. Thus, the effective gauge symmetry of the Kalb-Ramond field is generated by the gauge parameters $\epsilon^I(p)$ and $\epsilon^-(p)$.

(d) Consider the spacetime action principle

$$S \sim \int d^D x \Big(-\frac{1}{6} H_{\mu\nu\rho} H^{\mu\nu\rho} \Big) \,. \tag{4}$$

Find the field equation for $B_{\mu\nu}$, and write it in momentum space.

(e) What are the suitable light-cone gauge conditions for $B^{\mu\nu}$? Bearing in mind the results of part (c), show that these gauge conditions can be implemented using the gauge invariance. Analyze the equations of motion and find the components of $B^{\mu\nu}$ representing truly independent degrees of freedom.

(f) Argue that the one-particle states of the Kalb-Ramond field are

$$\sum_{I,J=2}^{D-1} \zeta_{IJ} \, a^{IJ\dagger}_{p^+,\, p_T} |\Omega\rangle \,. \tag{5}$$

What kind of matrix is ζ_{IJ}?

Problem 10.6. *Massive vector field.*

The purpose of this problem is to understand the massive version of Maxwell fields. We will see that in D-dimensional spacetime a massive vector field has $(D-1)$ degrees of freedom.

Consider the action $S = \int d^Dx\, \mathcal{L}$ with

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu\, \partial^\mu A^\nu + \frac{1}{2}\partial_\nu A_\mu\, \partial^\mu A^\nu - \frac{1}{2}m^2 A_\mu A^\mu - \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - (\partial\cdot A)m\phi \quad (1)$$

The first two terms in $\mathcal{L}$ are the familiar ones for the Maxwell field. The third looks like a mass term for the Maxwell field, but alone would not suffice. The additional terms show a scalar, the one that is *eaten* to give the gauge field a mass, as we will see.

(a) Show that the action S is invariant under the gauge transformations

$$\delta A_\mu = \partial_\mu \epsilon\,, \qquad \delta\phi = \beta\, m\, \epsilon\,, \quad (2)$$

where β is a constant you will determine. While the gauge field has the familiar Maxwell gauge transformation, the fact that the scalar field transforms under the gauge transformation is quite unusual.

(b) Vary the action and write down the field equations for A^μ and for ϕ.

(c) Argue that the gauge transformations allow us to set $\phi = 0$. Since the field ϕ disappears from sight, we say it was *eaten*. What do the field equations in part (b) simplify into?

(d) Write the simplified equations in momentum space and show that for $p^2 \neq -m^2$ there are no nontrivial solutions, while for $p^2 = -m^2$ the solution implies that there are $D-1$ degrees of freedom (It may be useful to use a Lorentz transformation to represent the vector p^μ satisfying $p^2 = -m^2$ as a vector having a component only in one direction).

Chapter 11

The Relativistic Quantum Point Particle

To prepare ourselves for quantizing the string, we study the light-cone gauge quantization of the relativistic point particle. We set up the quantum theory by requiring that the Heisenberg operators satisfy the classical equations of motion. We show that the quantum states of the relativistic point particle coincide with the one-particle states of the quantum scalar field. Moreover, the Schrödinger equation for the particle wavefunctions coincides with the classical scalar field equations. Finally, we set up light-cone gauge Lorentz generators.

11.1 Light-cone point particle

In this section we study the classical relativistic point particle using the light-cone gauge. This is, in fact, a much easier task than the one we faced in Chapter 9, where we examined the classical relativistic string in the light-cone gauge. Our present discussion will allow us to face the complications of quantization in the simpler context of the particle. Many of the ideas needed to quantize the string are also needed to quantize the point particle.

The action for the relativistic point particle was studied in Chapter 5. Let's begin our analysis with the expression given in equation (5.2.4), where an arbitrary parameter τ is used to parameterize the motion of the particle:

$$S = -m \int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\; d\tau \,. \tag{11.1.1}$$

In writing the above action, we have set $c = 1$. We will also set $\hbar = 1$ when appropriate. Finally, the time parameter τ will be dimensionless, just as it was for the relativistic string. We can simplify our notation by writing

$$\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \eta_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\mu = \dot{x}^2 \,. \tag{11.1.2}$$

Thinking of τ as a time variable and of the $x^\mu(\tau)$ as coordinates, the action S defines a Lagrangian L as

$$S = \int_{\tau_i}^{\tau_f} L \; d\tau \,, \qquad L = -m\sqrt{-\dot{x}^2} \,. \tag{11.1.3}$$

As usual, the momentum is obtained by differentiating the Lagrangian with respect to the velocity:

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \,. \tag{11.1.4}$$

The Euler-Lagrange equations arising from L are

$$\frac{dp_\mu}{d\tau} = 0 \,. \tag{11.1.5}$$

To define the light-cone gauge for the particle, we set the coordinate x^+ of the particle proportional to τ:

$$\boxed{\text{Light-cone gauge condition:} \quad x^+ = \frac{1}{m^2}\, p^+ \tau \,.} \tag{11.1.6}$$

The factor of m^2 on the right hand side is needed to get the units to work. Now consider the + component of equation (11.1.4):

$$p^+ = \frac{m}{\sqrt{-\dot{x}^2}}\; \dot{x}^+ = \frac{1}{\sqrt{-\dot{x}^2}}\; \frac{p^+}{m} \,. \tag{11.1.7}$$

Cancelling the common factor of p^+, and squaring, we find the constraint

$$\dot{x}^2 = -\,\frac{1}{m^2} \,. \tag{11.1.8}$$

This result helps us simplify the expression (11.1.4) for the momentum:

$$p_\mu = m^2\, \dot{x}_\mu \,. \tag{11.1.9}$$

The appearance of m^2, as opposed to m, is due to our choice of unit-less τ. The equation of motion (11.1.5) then gives

$$\ddot{x}_\mu = 0\,. \tag{11.1.10}$$

Using (11.1.9) we can rewrite the constraint equation (11.1.8) as

$$p^2 + m^2 = 0\,. \tag{11.1.11}$$

Expanding in light-cone components,

$$-2p^+p^- + p^Ip^I + m^2 = 0 \quad \longrightarrow \quad p^- = \frac{1}{2p^+}(p^Ip^I + m^2)\,. \tag{11.1.12}$$

Having solved for p^-, equation (11.1.9) gives

$$\frac{dx^-}{d\tau} = \frac{1}{m^2}\,p^-\,, \tag{11.1.13}$$

which is integrated to find

$$x^-(\tau) = x_0^- + \frac{p^-}{m^2}\,\tau\,, \tag{11.1.14}$$

where x_0^- is a constant of integration. Equation (11.1.9) also gives $dx^I/d\tau = p^I/m^2$, which is integrated to give

$$x^I(\tau) = x_0^I + \frac{p^I}{m^2}\,\tau\,, \tag{11.1.15}$$

where x_0^I is a constant of integration. Note that the light-cone gauge condition (11.1.6) implies that $x^+(\tau)$ has no constant piece x_0^+.

The specification of the motion of the point particle is now complete. Equation (11.1.12) tells us that the momentum is completely determined once we fix p^+ and the components p^I of the transverse momentum $\vec{p}_T$. The motion in the x^- direction is determined by (11.1.14), once we fix the value of x_0^-. The transverse motion is determined by the $x^I(\tau)$, or the x_0^I, since we presume to know the p^I. For a symmetric treatment of coordinates versus momenta in the quantum theory, we choose the x^I as dynamical variables. Our independent dynamical variables for the point particle are therefore

$$\text{Dynamical variables:} \quad \Big(\, x^I, \quad x_0^-\,, \quad p^I\,, \quad p^+ \Big)\,. \tag{11.1.16}$$

11.2 Heisenberg and Schrödinger pictures

Traditionally, there are two main approaches to the understanding of time evolution in quantum mechanics. In the Schrödinger picture, the state of a system evolves in time, while operators remain unchanged. In the Heisenberg picture, it is the operators which evolve in time, while the state remains unchanged. Of the two, the Heisenberg picture is more closely related to classical mechanics, where the dynamical variables (which become operators in quantum mechanics) evolve in time. Both the Schrödinger and the Heisenberg pictures will be useful in developing the quantum theories of the relativistic point particle and the relativistic string. Because we would like to exploit our understanding of classical dynamics in developing the quantum theories, we will begin by focusing on the Heisenberg picture.

Both the Heisenberg and the Schrödinger picture make use of the same state space. Whereas in the Heisenberg picture the state representing a particular physical system is fixed in time, in the Schrödinger picture the state of a system is constantly changing direction in the state space in a manner which is determined by the Schrödinger equation. Although we generally think of the operators in the Schrödinger picture as being time-independent, there are those which depend *explicitly* on time and therefore have time dependence. These operators are formed from time-independent operators and the variable t. For example, the position and momentum operators q and p are time-independent. But the operator $\mathcal{O} = q + p\,t$ has an explicit time dependence. If it has explicit time dependence, even the Hamiltonian $H(p, q; t)$ can be a time-dependent operator.

Now, as we move from the Schrödinger to the Heisenberg picture, we will encounter operators with two types of time dependence. As we noted earlier, Heisenberg operators have time dependence, but this time dependence can be both *implicit* and *explicit*. The Heisenberg-equivalent of a time-independent Schrödinger operator is said to have implicit time dependence. This implicit time dependence is due to our folding into the operators the time dependence which, in the Schrödinger picture, is present in the state. If a Heisenberg operator is explicitly time dependent it is because the explicit time dependence of the corresponding Schrödinger operator has been carried over.

For example, when we pass from the Schrödinger to the Heisenberg picture, the time-independent Schrödinger operators q and p become $q(t)$ and $p(t)$, respectively. The Schrödinger commutator $[q, p] = i$ turns into the commutator

$$[\, q(t)\, ,\, p(t)\,] = i\, . \tag{11.2.1}$$

Although $q(t)$ and $p(t)$ depend on time, their time dependence is implicit. If $\xi(t)$ is a Heisenberg operator arising from a time-independent Schrödinger operator, the time evolution of $\xi(t)$ is governed by

$$i\frac{d\xi(t)}{dt} = \Big[\, \xi(t)\, ,\, H(p(t), q(t); t)\Big]\, . \tag{11.2.2}$$

Here $H(p(t), q(t); t)$ is the Heisenberg Hamiltonian corresponding to the possibly time-dependent Schrödinger Hamiltonian $H(p, q; t)$.

If $\mathcal{O}(t)$ is the Heisenberg operator which corresponds to an explicitly time-dependent Schrödinger operator, then the time evolution of $\mathcal{O}(t)$ is given by

$$i\,\frac{d\mathcal{O}(t)}{dt} = i\,\frac{\partial\mathcal{O}}{\partial t} + \Big[\mathcal{O}(t)\, ,\, H(p(t), q(t); t)\Big]\, . \tag{11.2.3}$$

This equation reduces to (11.2.2) when the operator has no explicit time dependence. If the Hamiltonian $H(p(t), q(t))$ has no explicit time dependence, then we can use (11.2.2) with $\xi = H$, to find

$$\frac{d}{dt}H(p(t), q(t)) = 0\, . \tag{11.2.4}$$

In this case the Hamiltonian is a constant of the motion.

The discussion above is easily made very explicit when the Schrödinger Hamiltonian $H(p, q)$ is time independent. In this case a state $|\Psi\rangle$ at time $t = 0$, evolves in time becoming, at time t,

$$|\Psi, t\rangle = e^{-iHt}|\Psi\rangle\, . \tag{11.2.5}$$

Quick Calculation 11.1. Confirm that $|\Psi, t\rangle$ satisfies the Schrödinger equation

$$i\frac{d}{dt}|\Psi, t\rangle = H|\Psi, t\rangle\, . \tag{11.2.6}$$

It is clear from (11.2.5) that the operator e^{iHt} brings time dependent states to rest:

$$e^{iHt}|\Psi, t\rangle = |\Psi\rangle\, . \tag{11.2.7}$$

If we act with this operator on the product $\alpha|\Psi, t\rangle$, where α is a Schrödinger operator, we find

$$e^{iHt}\alpha|\Psi, t\rangle = e^{iHt}\,\alpha\, e^{-iHt}|\Psi\rangle \equiv \alpha(t)|\Psi\rangle\,, \tag{11.2.8}$$

where $\alpha(t) = e^{iHt}\alpha\, e^{-iHt}$ is the Heisenberg operator corresponding to the Schrödinger operator α. This definition applies both if α has or does not have explicit time dependence. This simple relation ensures that if a set of Schrödinger operators satisfy certain commutation relations, the corresponding Heisenberg operators satisfy the same commutation relations:

Quick Calculation 11.2. If $[\alpha_1, \alpha_2] = \alpha_3$ holds for Schrödinger operators α_1, α_2, and α_3, show that $[\alpha_1(t), \alpha_2(t)] = \alpha_3(t)$ holds for the corresponding Heisenberg operators.

This result holds even if the Hamiltonian is time dependent (see Problem 11.2). It justifies the commutator in (11.2.1), noting that the constant right-hand side is not affected by the rule turning a Schrödinger operator into a Heisenberg operator.

11.3 Quantization of the point particle

We now develop a quantum theory from the classical theory of the relativistic point particle. We will define the relevant Schrödinger and Heisenberg operators, including the Hamiltonian, and describe the state space. All of this will be done in the light-cone gauge.

Our first step is to choose a set of time-independent Schrödinger operators. A reasonable choice is provided by the dynamical variables in (11.1.16):

$$\boxed{\text{Time-independent Schrödinger ops.}: \Big(x^I, \quad x_0^-\,, \quad p^I\,, \quad p^+\Big)\,.} \tag{11.3.1}$$

We could include hats to distinguish the operators from their eigenvalues, but this will not be necessary in most cases. We parameterize the trajectory of a point particle using τ, so the associated Heisenberg operators are:

$$\boxed{\text{Heisenberg ops.}: \Big(x^I(\tau), \quad x_0^-(\tau)\,, \quad p^I(\tau)\,, \quad p^+(\tau)\Big)\,.} \tag{11.3.2}$$

We postulate the following commutation relations for the Schrödinger operators:

$$\boxed{[\, x^I, p^J] = i\,\eta^{IJ}\,, \quad [\, x_0^-, p^+] = i\,\eta^{-+} = -i\,,} \tag{11.3.3}$$

with all other commutators set equal to zero. The first commutator is the familiar commutator of spatial coordinates with the corresponding spatial momenta (recall that $\eta^{IJ} = \delta^{IJ}$). The second commutator is well motivated, after all, x_0^- is treated as a spatial coordinate in the light-cone, and p^+ is the corresponding conjugate momentum. The second commutator, just as the first one, has an η carrying the indices of the coordinate and the momentum.

The Heisenberg operators, as explained earlier, satisfy the same commutation relations as the Schrödinger operators:

$$[\, x^I(\tau), p^J(\tau)] = i\,\eta^{IJ}\,, \quad [\, x_0^-(\tau), p^+(\tau)] = -i\,, \tag{11.3.4}$$

with all other commutators set equal to zero.

We have discussed the operators that correspond to the independent observables of the classical theory. But just as there are classical observables which depend on those independent ones, there are also quantum operators which are constructed from the set of independent Schrödinger operators, and time. These additional operators are $x^+(\tau), x^-(\tau)$ and p^-. The definitions of these operators are postulated to be the quantum analogues of equations (11.1.6), (11.1.14), and (11.1.12). These give us the operator equations

$$x^+(\tau) \;\equiv\; \frac{p^+}{m^2}\,\tau\,, \tag{11.3.5}$$

$$x^-(\tau) \;\equiv\; x_0^- + \frac{p^-}{m^2}\,\tau\,, \tag{11.3.6}$$

$$p^- \;\equiv\; \frac{1}{2p^+}\Big(p^I p^I + m^2\Big)\,. \tag{11.3.7}$$

Note that p^- is time-independent. Both $x^+(\tau)$ and $x^-(\tau)$ are time-dependent Schrödinger operators.

The commutation relations involving the operators $x^+(\tau), x^-(\tau)$ and p^- are determined by the postulated commutation relations in (11.3.3), along with the defining equations (11.3.5) – (11.3.7). The decision to choose the operators in (11.3.1) as the independent operators of our quantum theory was very significant. For example, if we had chosen x^+ and p^- to be independent operators, we might have been led to write a commutation relation

$[x^+, p^-] = -i$. In our present framework, however, this quantity vanishes, since $[\,p^+, p^I] = 0$.

We have not yet determined the Hamiltonian H. Since p^- is the light-cone energy (see (2.5.11)), we expect it to generate x^+ evolution:

$$\frac{\partial}{\partial x^+} \quad \longleftrightarrow \quad p^- \,. \tag{11.3.8}$$

Although x^+ is light-cone time, we are parameterizing our operators with τ, so we expect H to generate τ evolution, which is related, but is not the same as x^+ evolution. Since $x^+ = p^+\tau/m^2$, we can anticipate that τ evolution will be generated by

$$\frac{\partial}{\partial \tau} = \frac{p^+}{m^2}\frac{\partial}{\partial x^+} \quad \longleftrightarrow \quad \frac{p^+}{m^2}\, p^- \,, \tag{11.3.9}$$

We therefore postulate the Heisenberg Hamiltonian

$$\boxed{H(\tau) = \frac{p^+(\tau)}{m^2}\; p^-(\tau) = \frac{1}{2m^2}\Big(p^I(\tau)p^I(\tau) + m^2\Big)\,.} \tag{11.3.10}$$

Note that $H(\tau)$ has no explicit time dependence. Equation (11.2.4) applies, and as a result, the Hamiltonian is actually time independent.

Let's now make sure that this Hamiltonian generates the expected equations of motion. First we check that H gives the correct time evolution of the Heisenberg operators (11.3.2) which arise from the time-independent Schrödinger operators. The equation governing the time evolution of those operators is (11.2.2). Let us begin with p^+ and p^I:

$$\begin{aligned} i\,\frac{dp^+(\tau)}{d\tau} &= \big[\,p^+(\tau)\,, H(\tau)\,\big] = 0\,, \\ i\,\frac{dp^I(\tau)}{d\tau} &= \big[\,p^I(\tau)\,, H(\tau)\big] = 0\,. \end{aligned} \tag{11.3.11}$$

Both of these commutators vanish because H is a function of $p^I(\tau)$ alone, and all the momenta commute. Equations (11.3.11) are good news, because the classical momenta p^+ and p^I are constants of the motion. This allows us to write $p^I(\tau) = p^I$ and $p^+(\tau) = p^+$. We now test the τ-development of the Heisenberg operator $x^I(\tau)$:

$$i\,\frac{dx^I(\tau)}{d\tau} = \Big[\,x^I(\tau)\,,\, \frac{1}{2m^2}\Big(p^Jp^J + m^2\Big)\Big] = i\,\frac{p^I}{m^2}\,. \tag{11.3.12}$$

Here, we have used $[x^I, p^J p^J] = [x^I, p^J]p^J + p^J[x^I, p^J] = 2i\, p^I$. Cancelling the common factor in (11.3.12) we find

$$\frac{dx^I(\tau)}{d\tau} = \frac{p^I}{m^2}\,. \tag{11.3.13}$$

This result is in accord with our classical expectations and allows us to write

$$x^I(\tau) = x_0^I + \frac{p^I}{m^2}\,\tau\,, \tag{11.3.14}$$

where x_0^I is an operator without any time dependence. Finally, we must examine $x_0^-(\tau)$. Since $x_0^-(\tau)$ commutes with $p^I(\tau)$,

$$i\,\frac{dx_0^-(\tau)}{d\tau} = \Big[\, x_0^-(\tau)\,,\, \frac{1}{2m^2}\Big(p^I p^I + m^2\Big)\Big] = 0\,. \tag{11.3.15}$$

As expected, this operator is a constant of the motion, and we can write $x_0^-(\tau) = x_0^-$. So as far as the operators in (11.3.1) are concerned, our ansatz for H functions properly as a Hamiltonian.

We now turn to the remaining operators $x^+(\tau)$, $x^-(\tau)$, and $p^-(\tau)$. Of these, $p^-(\tau)$ is a function of the p^I only and is therefore time independent. It is easy to to see that the commutator with H vanishes, so we have nothing left to check for this operator. The Heisenberg operators $x^+(\tau)$ and $x^-(\tau)$ both arise from Schrödinger operators with explicit time dependence, so we use (11.2.3) to calculate their time evolution. For example:

$$i\frac{dx^-(\tau)}{d\tau} = i\frac{\partial x^-}{\partial\tau} + \big[\, x^-(\tau)\,,\, H(\tau)\,\big]\,. \tag{11.3.16}$$

Since $x^-(\tau) \equiv x_0^- + p^-\tau/m^2$ and both x_0^- and p^- commute with the p^I, we see that $[x^-(\tau), H(\tau)] = 0$. Consequently,

$$\frac{dx^-(\tau)}{d\tau} = \frac{p^-}{m^2}\,, \tag{11.3.17}$$

which is the expected result. Similarly, since $x^+(\tau) = p^+\tau/m^2$, we find that $[x^+(\tau), H(\tau)] = 0$, and therefore

$$\frac{dx^+(\tau)}{d\tau} = \frac{\partial x^+}{\partial\tau} = \frac{p^+}{m^2}\,. \tag{11.3.18}$$

These computations show that our ansatz (11.3.10) for the Hamiltonian generates the expected equations of operator evolution.

Quick Calculation 11.3. We introduced x_0^I in (11.3.14) as a constant operator. Show that $dx_0^I/d\tau$ must be calculated by viewing x_0^I as an explicitly time-dependent Heisenberg operator defined by (11.3.14).

Our final step in constructing the quantum theory of the point particle is to develop the state space. The states are labeled by the eigenvalues of a maximal set of commuting operators. For the set of operators we have introduced in (11.3.1), a maximal commuting subset can include only one element from the pair (x^-, p^+), and one element from each of the pairs (x^I, p^I). Because it is convenient to work in momentum space, we will work with the operators p^+ and p^I. So we write the states as

$$\boxed{\text{States of the quantum point particle:} \quad |p^+, \vec{p}_T\rangle\,,} \tag{11.3.19}$$

where p^+ is the eigenvalue of the p^+ operator, and $\vec{p}_T$ is the transverse momentum, the components of which are the eigenvalues of the p^I operators:

$$\widehat{p}^{\,+}|p^+, \vec{p}_T\rangle = p^+|p^+, \vec{p}_T\rangle\,, \quad \widehat{p}^{\,I}|p^+, \vec{p}_T\rangle = p^I|p^+, \vec{p}_T\rangle\,. \tag{11.3.20}$$

In light of (11.3.7), these equations imply

$$\widehat{p}^{\,-}|p^+, \vec{p}_T\rangle = \frac{1}{2p^+}\Big(p^Ip^I + m^2\Big)|p^+, \vec{p}_T\rangle\,. \tag{11.3.21}$$

To write a Schrödinger equation for the point particle, we consider time-dependent states. These are formed as time-dependent superpositions of the basis states in (11.3.19):

$$|\Psi, \tau\rangle = \int dp^+ d\vec{p}_T\, \psi(\tau, p^+, \vec{p}_T)|p^+, \vec{p}_T\rangle\,. \tag{11.3.22}$$

Since p^+ and $\vec{p}_T$ are continuous variables, an integral is necessary. To produce a general τ-dependent superposition, we introduced the arbitrary function $\psi(\tau, p^+, \vec{p}_T)$. In fact, this function is the momentum-space wavefunction associated to the state $|\Psi, \tau\rangle$. Indeed, with dual bras $\langle p^+, \vec{p}_T|$ defined to satisfy

$$\langle p^{+\prime}, \vec{p}_T{}'|\, p^+, \vec{p}_T\rangle = \delta(p^{+\prime} - p^+)\, \vec{\delta}\,(\vec{p}_T{}' - \vec{p}_T)\,, \tag{11.3.23}$$

we see that

$$\langle p^+, \vec{p}_T|\, \Psi, \tau\rangle = \psi(\tau, p^+, \vec{p}_T)\,. \tag{11.3.24}$$

The Schrödinger equation for the state $|\Psi,\tau\rangle$ is

$$i\frac{\partial}{\partial\tau}|\Psi,\tau\rangle = H|\Psi,\tau\rangle\,. \tag{11.3.25}$$

Using the state in (11.3.22) and the Hamiltonian in (11.3.10), we find

$$\int dp^+ d\vec{p}_T\,\Big[i\frac{\partial}{\partial\tau}\psi(\tau,p^+,\vec{p}_T) - \frac{1}{2m^2}\Big(p^Ip^I + m^2\Big)\psi(\tau,p^+,\vec{p}_T)\Big]|p^+,\vec{p}_T\rangle = 0\,. \tag{11.3.26}$$

Since the basis vectors $|p^+,\vec{p}_T\rangle$ are all linearly independent, the expression within brackets must vanish for all values of the momenta:

$$i\frac{\partial}{\partial\tau}\,\psi\,(\tau,p^+,\vec{p}_T) = \frac{1}{2m^2}\Big(p^Ip^I + m^2\Big)\psi(\tau,p^+,\vec{p}_T)\,. \tag{11.3.27}$$

We recognize this equation as a Schrödinger equation for the momentum-space wavefunction $\psi(\tau,p^+,\vec{p}_T)$. We have thus developed a theory of the quantum point particle.

11.4 Quantum particle and scalar particles

The states of the quantum point particle given in (11.3.19) may remind you of the one-particle states (10.4.34) in the quantum theory of the scalar field. This is actually a fundamental correspondence:

There is a natural identification of the quantum states of a relativistic point particle of mass m with the one-particle states of the quantum theory of a scalar field of mass m:

$$\boxed{\;|p^+,\,\vec{p}_T\rangle \quad\longleftrightarrow\quad a^\dagger_{p^+,\,p_T}|\Omega\rangle\,.\;} \tag{11.4.1}$$

The identification is possible because the labels of the point particle states correspond to the labels of the creation operators which generate the one-particle states of the scalar quantum field theory. The correspondence between the quantum point particle and the quantum scalar field theory can be extended from the state space to the operators that act on the state spaces. The quantum point particle theory has operators p^+, p^I, and p^-, and so does the quantum field theory, as shown in (10.4.35). If we identify the state spaces using (11.4.1), then the two sets of operators give the same eigenvalues. This makes the identification natural.

The above observations lead us to conclude that the states of the quantum point particle and the one-particle states of the scalar field theory are indistinguishable. Because it contains creation operators that can act multiple times on the vacuum state, the scalar field theory has multiparticle states that did not arise in our quantization of the point particle. Indeed, there are no creation operators in the theory of the quantum point particle. Because it provides a natural description of multi-particle states, the scalar field theory can be said to be a more complete theory.

How could we have anticipated that the one-particle states of a quantum *scalar field theory* would match those of the quantum point particle? The answer is quite interesting: the Schrödinger equation for the quantum point particle wavefunctions has the form of the classical field equation for the scalar field. More precisely:

There is a canonical correspondence between the quantum point particle wavefunctions and the classical scalar field, such that the Schrödinger equation for the quantum point particle wavefunctions, becomes the classical field equation for the scalar field.

One element of this correspondence is the classical field equation for the scalar field. In light-cone gauge, this equation takes the form (10.3.16):

$$\Big(i\,\frac{\partial}{\partial\tau} - \frac{1}{2m^2}(p^Ip^I + m^2)\Big)\,\phi(\tau, p^+, \vec{p}_T) = 0\,. \tag{11.4.2}$$

This differential equation is first order in τ. The other element in the correspondence is this Schrödinger equation (11.3.27). The two equations are identical once we identify the wavefunction $\psi(\tau, p^+, \vec{p}_T)$ and the scalar field $\phi(\tau, p^+, \vec{p}_T)$:

$$\boxed{\psi(\tau, p^+, \vec{p}_T) \quad\longleftrightarrow\quad \phi(\tau, p^+, \vec{p}_T)\,.} \tag{11.4.3}$$

This is the claimed correspondence.

The quantization of the point particle is an example of *first quantization*. In first quantization, the coordinates and momenta of classical mechanics are turned into quantum operators and a state space is constructed. Generically, the result is a set of one-particle states. *Second quantization* refers to the quantization of a classical field theory, the result of which is a quantum field theory with field operators and multi-particle states. Our analysis allows us to see how second quantization follows after first quantization. A first-quantization of the classical point-particle mechanics gives one-particle states. We then re-interpret the Schrödinger equation for the associated wavefunctions as the classical field equation for a scalar field. A second-quantization, this time of the classical field theory, gives us the set of multi-particle states.

So far we have only quantized the *free* relativistic point particle. All quantum states, including the multi-particle ones obtained by second quantization, represent free particles. How do we get interactions between the particles? Such processes are included in the scalar field theory by adding interaction terms to the action. So far, all the terms that we have included are quadratic in the fields. The interaction terms include three or more fields. Since the quantum point particle state space does not include multi-particle states, the description of interactions in the language of first quantization is not straightforward. On the other hand, in the framework of quantum field theory interactions are dealt with very naturally.

11.5 Light-cone momentum generators

Since the point particle Lagrangian L in (11.1.3) depends only on τ-derivatives of the coordinates, it is invariant under the translations

$$\delta x^\mu(\tau) = \epsilon^\mu \,, \tag{11.5.1}$$

with ϵ^μ constant. The conserved charge associated to this symmetry transformation is the momentum p_μ of the particle. This follows from (8.2.9) and (11.1.4).

What happens to conserved charges in the quantum theory? They become quantum operators with a remarkable property: they generate, via commutation, a quantum version of the symmetry transformation that gave rise to them classically!

This property is most apparent if we use a framework where the manifest Lorentz invariance of the classical theory is preserved in the quantization. This is *not* the framework we have used to quantize the point particle. In light-cone gauge quantization, the x^0 and x^1 coordinates of the particle are afforded special treatment. This hides the Lorentz invariance of the theory from plain view. We will not discuss fully the Lorentz covariant quantization of the point particle. A few remarks will suffice for our present purposes. The covariant quantization of the string is discussed in some detail in Chapter 21.

In the Lorentz covariant quantization of the point particle, we have Heisenberg operators $x^\mu(\tau)$ and $p^\mu(\tau)$. Note that even the time coordinate $x^0(\tau)$ becomes an operator! The commutation relations are

$$[\, x^\mu(\tau), p^\nu(\tau)] = i\,\eta^{\mu\nu}\,, \tag{11.5.2}$$

as well as

$$[\, x^\mu(\tau), x^\nu(\tau)] = 0 \quad \text{and} \quad [\, p^\mu(\tau), p^\nu(\tau)] = 0\,. \tag{11.5.3}$$

Equation (11.5.2) is reasonable. The indices match, which ensures consistency with Lorentz covariance. Moreover, when μ and ν take spatial values, the commutation relations are the familiar ones. We already know that (11.5.2) is not consistent with the light-cone gauge commutators of section 11.3. We saw there that $[x^+(\tau), p^-(\tau)] = 0$, while (11.5.2) would predict a nonzero result. An equality of two objects carrying Lorentz indices can be used letting the indices run over the light-cone values $+, -$, and I. The equation $R^{\mu\nu} = S^{\mu\nu}$ gives, for example, $R^{+-} = S^{+-}$ (see Problem 10.3). As a result, equation (11.5.2), indeed gives $[x^+(\tau), p^-(\tau)] = i\eta^{+-} = -i$.

Let us now check that the operator $p^\mu(\tau)$ generates translations. More precisely, we check that $i\epsilon_\rho\, p^\rho(\tau)$ generates the translation (11.5.1):

$$\delta x^\mu(\tau) = [\, i\epsilon_\rho\, p^\rho(\tau)\,, x^\mu(\tau)\,] = i\epsilon_\rho\,(-i\eta^{\rho\mu}) = \epsilon^\mu\,. \tag{11.5.4}$$

This is an elegant result, but it is by no means clear that it carries over to our light-cone gauge quantization. We must find out if the light-cone gauge momentum operators generate translations.

For this purpose, we expand the generator $i\epsilon_\rho\, p^\rho(\tau)$ in light-cone components:

$$i\epsilon_\rho\, p^\rho(\tau) = -i\epsilon^-\, p^+ - i\epsilon^+\, p^- + i\epsilon^I\, p^I\,. \tag{11.5.5}$$

We have dropped the τ arguments from the momenta because they are τ-independent. Note that in here, p^- is given by (11.3.7). Let's test (11.5.4) with $\epsilon^I \neq 0$, and $\epsilon^+ = \epsilon^- = 0$:

$$\delta x^\mu(\tau) = \, i\epsilon^I [\, p^I\,, x^\mu(\tau)\,]\,. \tag{11.5.6}$$

We would expect that $\delta x^J(\tau) = \epsilon^J$ and that $\delta x^+(\tau) = \delta x^-(\tau) = 0$. All these expectations are realized. Choosing $\mu = J$, and using the commutator

(11.3.4) we find $\delta x^J(\tau) = \epsilon^J$. To compute the action on $x^+(\tau)$ and $x^-(\tau)$, we must use their definitions:

$$x^+(\tau) = \frac{p^+}{m^2}\tau\,, \qquad x^-(\tau) = x_0^- + \frac{p^-}{m^2}\tau\,. \tag{11.5.7}$$

Recalling that p^I commutes with all momenta and with x_0^-, we confirm that $\delta x^+(\tau) = \delta x^-(\tau) = 0$.

Quick Calculation 11.4. Test (11.5.4) with $\epsilon^- \neq 0$ and $\epsilon^+ = \epsilon^I = 0$. To do this compute $\delta x^\mu(\tau) = -i\epsilon^-[\,p^+\,, x^\mu(\tau)\,]$. Confirm that $\delta x^-(\tau) = \epsilon^-$ and that all other coordinates are not changed.

It remains to see if p^- generates the expected translations. Since p^- is a nontrivial function of other momenta, there is some scope for complications! This time we consider the transformations that are generated using (11.5.4) with $\epsilon^+ \neq 0$ and $\epsilon^- = \epsilon^I = 0$:

$$\delta x^\mu(\tau) = \,-i\epsilon^+[\,p^-\,, x^\mu(\tau)\,]\,. \tag{11.5.8}$$

The naive expectation $\delta x^+(\tau) = \epsilon^+$ is not realized: choosing $\mu = +$ and using (11.5.7) we see that

$$\delta x^+(\tau) = -i\epsilon^+\Big[\,p^-\,, p^+\frac{\tau}{m^2}\Big] = 0\,. \tag{11.5.9}$$

Not only is $x^+(\tau)$ left unchanged, but the other components, which naively should be left unchanged, are not:

$$\delta x^I(\tau) = -i\epsilon^+\Big[\,p^-, x^I(\tau)\Big] = -i\epsilon^+\frac{1}{2p^+}\,(-2ip^I)\, = -\epsilon^+\frac{p^I}{p^+}\,, \tag{11.5.10}$$

$$\delta x^-(\tau) = -i\epsilon^+\Big[\,p^-, x_0^- + \frac{p^-}{m^2}\tau\Big] = -i\epsilon^+[\,p^-, x_0^-] = -\epsilon^+\frac{p^-}{p^+}\,. \tag{11.5.11}$$

In these calculations only one step requires some explanation. How do we find $[\,p^-, x_0^-]$? The only reason p^- does not commute with x_0^- is that p^- depends on p^+. In fact, what we need to know is the commutator $[x_0^-, 1/p^+]$. This can be done as follows:

$$\begin{aligned}\Big[\,x_0^-\,,\,\frac{1}{p^+}\Big] &= x_0^-\,\frac{1}{p^+} - \frac{1}{p^+}x_0^- = \frac{1}{p^+}\,p^+x_0^-\frac{1}{p^+} - \frac{1}{p^+}x_0^-p^+\frac{1}{p^+}\\ &= \frac{1}{p^+}\,[\,p^+\,, x_0^-\,]\,\frac{1}{p^+} = \frac{i}{{p^+}^2}\,.\end{aligned} \tag{11.5.12}$$

Quick Calculation 11.5. Use (11.5.12) to show that

$$[\, x_0^- \, , \, p^- \,] = i \, \frac{p^-}{p^+} \, . \tag{11.5.13}$$

Equations (11.5.9), (11.5.10), and (11.5.11) show that p^- does not generate the expected transformations. What happened? It turns out that p^- actually generates both a translation and a reparameterization of the world-line of the particle. We know that the particle action is invariant under changes of parameterization $\tau \to \tau'(\tau)$. When we described symmetries in Chapter 8, however, we exhibited them as changes in the dynamical variables of the system. A change in parameterization can also be described in that way. Writing $\tau \to \tau' = \tau + \lambda(\tau)$, with λ infinitesimal, we note that the plausible change

$$x^\mu(\tau) \;\to\; x^\mu(\tau + \lambda(\tau)) = x^\mu(\tau) + \lambda(\tau)\partial_\tau x^\mu(\tau) \, , \tag{11.5.14}$$

leads us to write

$$\delta x^\mu(\tau) = \lambda(\tau)\partial_\tau x^\mu(\tau) \, . \tag{11.5.15}$$

We claim that these are *symmetries* of the point particle theory. Actually, the variation (11.5.15) does not leave the point particle Lagrangian invariant. The Lagrangian changes into a total τ-derivative (Problem 11.4), and this, in fact, suffices to have a symmetry (Problem 8.5).

Let's now show that p^- generates a translation plus a reparameterization. The expected translation was $\delta x^+ = \epsilon^+$. On the other hand, from (11.5.15), a reparameterization of x^+ gives $\delta x^+ = \lambda \partial_\tau x^+$. Bearing in mind (11.5.9), the expected translation plus the reparameterization give zero variation, so,

$$0 = \epsilon^+ + \lambda \partial_\tau x^+(\tau) = \epsilon^+ + \lambda \, \frac{p^+}{m^2} \quad \longrightarrow \quad \lambda = -\frac{m^2}{p^+}\epsilon^+ \, . \tag{11.5.16}$$

The reparameterization parameter λ turns out to be a constant. We can now use this result to "explain" the transformations (11.5.10) and (11.5.11) that p^- generates on x^I and on x^-. For these coordinates there is no translation, but the reparameterization still applies. Therefore,

$$\delta x^I(\tau) = \lambda \, \partial_\tau x^I(\tau) = -\frac{m^2}{p^+}\epsilon^+ \, \frac{p^I}{m^2} = -\epsilon^+ \frac{p^I}{p^+} \, , \tag{11.5.17}$$

$$\delta x^-(\tau) = \lambda \, \partial_\tau x^-(\tau) = -\frac{m^2}{p^+}\epsilon^+ \, \frac{p^-}{m^2} = -\epsilon^+ \frac{p^-}{p^+} \, , \tag{11.5.18}$$

in perfect agreement with the transformations generated by p^-. We can also understand why p^- does not change x^+. If x^+ had been changed by a constant ϵ^+, the new x^+ coordinate would not satisfy the light-cone gauge condition whereby x^+ is just proportional to τ. In fact, p^- generates a translation plus the compensating transformation needed to preserve the light-cone gauge condition! That transformation turned out to be a reparameterization of the world-line.

One final remark about momentum operators. The Lorentz covariant momentum operators that we used to motivate our analysis generate simple translations and commute among each other. It follows directly that, using light-cone *coordinates*, the operators $p^\pm = (p^0 \pm p^1)/\sqrt{2}$ and the transverse p^I all commute. The light-cone *gauge* momentum operators we discussed above are completely different objects. They had an intricate action on coordinates, and p^- was defined in terms of the transverse momenta and p^+. Nevertheless, all the light-cone gauge momentum operators still commute. They obey the same commutation relations that the covariant operators do when expressed using light-cone coordinates.

11.6 Light-cone Lorentz generators

In section 8.5 we determined the conserved charges that are associated with the Lorentz invariance of the relativistic string Lagrangian. Similar charges exist for the relativistic point particle. As we found in (8.5.1), the infinitesimal Lorentz transformations of the point particle coordinates $x^\mu(\tau)$ take the form

$$\delta x^\mu(\tau) = \epsilon^{\mu\nu} x_\nu(\tau)\,, \tag{11.6.1}$$

where $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ are a set of infinitesimal constants. The associated Lorentz charges are given by

$$M^{\mu\nu} = x^\mu(\tau)p^\nu(\tau) - x^\nu(\tau)p^\mu(\tau)\,, \tag{11.6.2}$$

as you may have derived in Problem 8.2. These charges are conserved classically. The quantum charges are expected to generate Lorentz transformations of the coordinates. Again, it is straightforward to see this using the operators of Lorentz-covariant quantization. In this case, the quantum charges are given by (11.6.2) with $x^\mu(\tau)$ and $p^\mu(\tau)$ taken to be the Heisenberg operators introduced earlier and satisfying the commutation relations (11.5.2) and (11.5.3). Both $x^\mu(\tau)$ and $p^\mu(\tau)$ are Hermitian operators. The Lorentz charges $M^{\mu\nu}$ are Hermitian as well:

$$(M^{\mu\nu})^\dagger = p^\nu(\tau)x^\mu(\tau) - p^\mu(\tau)x^\nu(\tau) = M^{\mu\nu}\,, \tag{11.6.3}$$

since the two constants induced by rearranging the coordinates and momenta back to the original form cancel out.

Quick Calculation 11.6. Show that

$$[\,M^{\mu\nu}\,,\,x^\rho(\tau)\,] = i\,\eta^{\mu\rho}\,x^\nu(\tau) - i\,\eta^{\nu\rho}\,x^\mu(\tau)\,. \tag{11.6.4}$$

This commutator helps us check that the quantum Lorentz charges generate Lorentz transformations:

$$\begin{aligned} \delta x^\rho(\tau) &= \Big[-\frac{i}{2}\,\epsilon_{\mu\nu}M^{\mu\nu}\,,\,x^\rho(\tau)\Big], \\ &= \frac{1}{2}\,\epsilon_{\mu\nu}\Big(\eta^{\mu\rho}\,x^\nu(\tau) - \,\eta^{\nu\rho}\,x^\mu(\tau)\Big)\,, \\ &= \frac{1}{2}\,\epsilon^{\rho\nu}x_\nu(\tau) + \frac{1}{2}\,\epsilon^{\rho\mu}\,x_\mu(\tau) = \epsilon^{\rho\nu}x_\nu(\tau)\,. \end{aligned} \tag{11.6.5}$$

Equation (11.6.4) can be used in light-cone *coordinates* by simply using light-cone indices. For example,

$$[M^{-I}, x^+(\tau)] = i\eta^{-+}\,x^I(\tau) - i\eta^{I+}x^-(\tau) = -i\,x^I(\tau)\,, \tag{11.6.6}$$

since $\eta^{I+} = 0$. The operator M^{-I} here is a Lorentz-covariant generator expressed in light-cone coordinates. It is *not* a light-cone gauge Lorentz generator. Those we have not yet constructed.

Given a set of quantum operators, it is interesting to calculate their commutators. In quantum mechanics, for example, you learned that the components L_x, L_y, and L_z of the angular momentum satisfy a set of commutation relations ($[L_x, L_y] = iL_z$, and others) that define the Lie algebra of angular momentum. The momentum operators p^μ considered earlier define a very simple Lie algebra; they all commute. We would like to know what is the commutator of two Lorentz generators. The computation takes a few steps (see Problem 11.5). Using equation (11.6.4), and a similar equation for $[M^{\mu\nu}\,,\,p^\rho]$, one finds that the commutator can be written as a linear combination of four Lorentz generators:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\rho}M^{\mu\sigma} + i\eta^{\mu\sigma}M^{\rho\nu} - i\eta^{\nu\sigma}M^{\rho\mu}\,. \tag{11.6.7}$$

This result defines the Lorentz Lie algebra. Equation (11.6.7) must be satisfied by the analogous operators $M^{\mu\nu}$ of any Lorentz-invariant quantum theory. If it is not possible to construct such operators, the theory is not Lorentz-invariant. This will be crucial to our quantization of the string, for requiring that (11.6.7) holds imposes additional restrictions, which have significant physical consequences.

Quick Calculation 11.7. Since $M^{\mu\nu} = -M^{\nu\mu}$, the left hand side of (11.6.7) changes sign under the exchange of μ and ν. Verify that the right-hand side also changes sign under this exchange.

We can now use (11.6.7) to determine the commutators of Lorentz charges in light-cone *coordinates*. The Lorentz generators are given by

$$M^{IJ}\,, \quad M^{+I}\,, \quad M^{-I}\,, \text{ and } \quad M^{+-}\,. \tag{11.6.8}$$

Consider, for example the commutator $[M^{+-}, M^{+I}]$. To use (11.6.7) notice the structure of its right-hand side: each η contains one index from each of the generators in the left-hand side. For $[M^{+-}, M^{+I}]$, the only way to get a nonvanishing η is to use the $-$ from the first generator and the $+$ from the second generator. The nonvanishing term is the second one on the right hand side of (11.6.7), and we find

$$[M^{+-}, M^{+I}] = -i\,\eta^{-+}M^{+I} = iM^{+I}\,. \tag{11.6.9}$$

Similarly,

$$[M^{-I}, M^{-J}] = 0\,. \tag{11.6.10}$$

Here η must use the I and J indices, but then the other two indices must go into M giving us M^{--}, which vanishes by antisymmetry.

So far, we have considered the covariant Lorentz charges in light-cone *coordinates*. We must now find Lorentz charges for our light-cone *gauge* quantization of the particle. Our earlier discussion of the momenta suggests that we really face three questions:

(1) How are these charges going to be defined?

(2) What kind of transformations will they generate?

(3) Which commutation relations will they satisfy?

In the remaining of this section we will explore question (1) in detail. Before doing so, let's give brief answers to questions (2) and (3), leaving further analysis of these questions to Problems 11.6 and 11.7. The light-cone gauge Lorentz generators are expected to generate Lorentz transformations of coordinates and momentum, but in some cases, these transformations will be accompanied by reparameterizations of the world-line. Regarding (3), the light-cone gauge Lorentz generators will satisfy the same commutation relations that the covariant operators in light-cone coordinates do. This establishes that Lorentz symmetry holds in the light-cone theory of the quantum point particle. The success of the construction is not obvious *a priori*. It is not clear that the reduced set of light-cone gauge operators suffices to construct quantum Lorentz charges that generate Lorentz transformations (plus other transformations) and satisfy the Lorentz algebra.

The simplest guess for the light-cone gauge generators is to use light-cone coordinates in the covariant formula (11.6.2) and then replace $x^+(\tau)$ and p^- using their light-cone gauge definitions in (11.3.5) and (11.3.7). Let's try this prescription with M^{+-}:

$$\begin{aligned} M^{+-} &\overset{?}{=} x^+(\tau)\, p^-(\tau) - x^-(\tau)\, p^+(\tau)\,, \\ &\overset{?}{=} \frac{p^+\tau}{m^2}\, p^- - \Big(x_0^- + \frac{p^-}{m^2}\tau\Big)p^+\,, \\ &\overset{?}{=} -\, x_0^-\, p^+\,. \end{aligned} \tag{11.6.11}$$

Since x_0^- and p^+ are τ-independent, so too is M^{+-}. We have a minor complication, however. The operator M^{+-} is not Hermitian: $(M^{+-})^\dagger - M^{+-} = [x_0^-, p^+] \neq 0$. This failure of Hermiticity illustrates how the use of the light-cone gauge can affect basic properties of operators. The covariant Lorentz generators were automatically Hermitian, the light-cone gauge generators are not. We are therefore motivated to define a Hermitian M^{+-} as

$$M^{+-} = -\frac{1}{2}(x_0^- p^+ + p^+ x_0^-)\,. \tag{11.6.12}$$

We take this to be the light-cone gauge Lorentz generator M^{+-}.

The most complicated of all generators is M^{-I}. It is also the most interesting one as well. The prescription used for M^{+-} this time gives

$$\begin{aligned} M^{-I} &\stackrel{?}{=} x^-(\tau)\, p^I - x^I(\tau)\, p^- \,, \\ &\stackrel{?}{=} \Big(x_0^- + \frac{p^-}{m^2}\tau\Big) p^I - \Big(x_0^I + \frac{p^I \tau}{m^2}\Big) p^- \,, \\ &\stackrel{?}{=} x_0^-\, p^I - x_0^I\, p^- \,. \end{aligned} \tag{11.6.13}$$

As before, the τ dependence vanishes, but we are left with a complicated result, since p^- is a nontrivial function of the other momenta. We define M^{-I} as the Hermitian version of the operator obtained above:

$$M^{-I} \equiv x_0^- p^I - \frac{1}{2}\left(x_0^I\, p^- + p^-\, x_0^I\right). \tag{11.6.14}$$

If the light-cone gauge Lorentz charges are to satisfy the Lorentz algebra we must have

$$[\, M^{-I} \,,\, M^{-J}\,] = 0 \,, \tag{11.6.15}$$

as we noted in (11.6.10). Does M^{-I}, as defined by (11.6.14), satisfy this equation? The answer is yes, as you will see for yourself in Problem 11.6. This result is necessary to ensure Lorentz invariance of the quantum theory.

When we quantize the string, the calculation of $[\, M^{-I} \,,\, M^{-J}\,]$ will be fairly complicated. But the answer is very interesting. It turns out that the commutator is zero only if the string lives in a particular spacetime dimension and, furthermore, only if the definition of mass is changed in such a way that we can find massless gauge fields in the spectrum of the open string! String theory is such a constrained theory that it is only Lorentz invariant for a fixed spacetime dimensionality.

Problems

Problem 11.1. *Equation of motion for Heisenberg operators.*

Assume the Schrödinger Hamiltonian $H(p,q)$ is time independent. In this case the time-independent Schrödinger operator ξ yields a Heisenberg operator $\xi(t) = e^{iHt}\xi e^{-iHt}$. Show that this operator satisfies the equation

$$i\frac{d\xi(t)}{dt} = \Big[\,\xi(t)\,,\ H(p(t), q(t))\Big]\,.$$

This computation proves that equation (11.2.2) holds for time-independent Hamiltonians.

Problem 11.2. *Heisenberg operators and time-dependent Hamiltonians.*

When the Schrödinger Hamiltonian $H = H(p,q;t)$ is time dependent, time evolution of states is generated by a unitary operator $U(t)$:

$$|\Psi, t\rangle = U(t)|\Psi\rangle\,, \tag{1}$$

where $U(t)$ bears some nontrivial relation to H. Here $|\Psi\rangle$ denotes the state at zero time.

(a) Use the Schrödinger equation to show that

$$i\frac{dU(t)}{dt} = HU(t)\,. \tag{2}$$

Let $U \equiv U(t)$, for brevity. Since U^{-1} acting on $|\Psi, t\rangle$ gives a time-independent state, considerations similar to those given for (11.2.8) lead us to define the Heisenberg operator corresponding to the Schrödinger operator α as

$$\alpha(t) = U^{-1}\alpha\, U\,. \tag{3}$$

(b) Let ξ be a time-independent Schrödinger operator, and $\xi(t)$ the corresponding Heisenberg operator, defined using (3). Show that

$$i\frac{d\xi(t)}{dt} = \Big[\,\xi(t)\,,\ H(p(t), q(t);t)\Big]\,.$$

This computation proves that equation (11.2.2) holds for time-dependent Hamiltonians.

(c) If $[\alpha_1, \alpha_2] = \alpha_3$ holds for arbitrary Schrödinger operators α_1, α_2, and α_3, show that $[\alpha_1(t), \alpha_2(t)] = \alpha_3(t)$ holds for the corresponding Heisenberg operators.

Problem 11.3. *Classical dynamics in Hamiltonian language.*

Consider a classical phase space (q, p), a trajectory $(q(t), p(t))$, and an observable $v\,(q(t), p(t);\, t)$. From the standard rules of differentiation,

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial p}\frac{dp}{dt} + \frac{\partial v}{\partial q}\frac{dq}{dt}\,. \tag{1}$$

With the Poisson bracket defined as

$$\{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}\,, \tag{2}$$

show that

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \{v, H\}\,. \tag{3}$$

Comparing this result to (11.2.3) we see the parallel between the time evolution of a general operator $\mathcal{O}$ and the classical Hamiltonian evolution of an observable v in phase space.
To derive (3) you need the classical equations of motion in Hamiltonian language. These can be obtained by demanding that

$$\int dt\Big(p(t)\dot{q}(t) - H(p(t), q(t); t)\Big)\,,$$

be stationary for independent variations $\delta q(t)$ and $\delta p(t)$.

Problem 11.4. *Reparameterization symmetries of the point particle.*

Show that the variation $\delta x^\mu(\tau) = \lambda(\tau)\partial_\tau x^\mu(\tau)$ induces a variation δL of the point particle Lagrangian that can be written as

$$\delta L(\tau) = \partial_\tau\Big(\lambda(\tau)L(\tau)\Big)\,. \tag{11.6.16}$$

This proves that the reparameterizations δx^μ are symmetries of the point particle theory, in the sense defined in Problem 8.5. Show, however, that the charges associated to these reparameterization symmetries vanish. When λ is τ-independent, the reparameterization is an infinitesimal constant τ translation. The conserved charge is then the Hamiltonian. Show directly that the Hamiltonian defined canonically from the point particle Lagrangian vanishes.

Problem 11.5. *Lorentz generators and Lorentz algebra.*

In this problem we consider the Lorentz-covariant charges (11.6.2).

(a) Calculate the commutator $[\, M^{\mu\nu}\, ,\, p^\rho]$.

(b) Calculate the commutator $[\, M^{\mu\nu}\, ,\, M^{\rho\sigma}]$ and verify that (11.6.7) holds.

(c) Consider the Lorentz algebra in light cone *coordinates*. Give

$$[\, M^{\pm I}\, ,\, M^{JK}]\,, \quad [\, M^{\pm I}\, ,\, M^{\mp J}]\,, \quad [M^{+-}\, ,\, M^{\pm I}]\,, \quad \text{and} \quad [\, M^{\pm I}\, ,\, M^{\pm J}]\,.$$

Problem 11.6. *Light-cone gauge commutator $[M^{-I}, M^{-J}]$ for the particle.*

The purpose of the present calculation is to show that

$$[\, M^{-I}\, ,\, M^{-J}\,] = 0\,. \tag{1}$$

(a) Verify that the light-cone gauge operator M^{-I} takes the form

$$M^{-I} = (x_0^- p^I - x_0^I p^-) + \frac{i}{2}\frac{p^I}{p^+}\,. \tag{2}$$

Set up now the computation of (1) distinguishing the two kinds of terms in (2). Calculate the contributions to the commutator from mixed terms and from the last term.

(b) Complete the computation of (1) by finding the contribution from the first term in the right hand side of (2).

Problem 11.7. *Transformations generated by the light-cone gauge Lorentz generators M^{+-} and M^{-I}.*

(a) Calculate the commutators of M^{+-}, defined in (11.6.12), with the light-cone coordinates $x^+(\tau), x^-(\tau)$, and $x^I(\tau)$. Show that M^{+-} generates the expected Lorentz transformations of these coordinates.

(b) Calculate the commutators of M^{-I} with the light-cone coordinates $x^+(\tau), x^-(\tau)$, and $x^J(\tau)$. Show that M^{-I} generates the expected Lorentz transformations together with a compensating reparameterization of the world-line. Calculate the parameter λ for this reparameterization.

(c) Calculate $[M^{-I}\, ,\, p^+]$. Combine this result with your result for the commutator of M^{-I} with $x^+(\tau)$ to argue that the generator M^{-I} preserves the light-cone gauge condition.

Chapter 12

The Relativistic Quantum Open String

We finally quantize the relativistic open string. We use the light-cone gauge to set up commutation relations and define a Hamiltonian using the Heisenberg picture. We discover an infinite set of creation and annihilation operators, labelled by an integer and a transverse vector index. The oscillators corresponding to the X^- direction are transverse Virasoro operators. The ambiguities we encounter in defining the quantum theory are fixed by requiring that the theory be Lorentz invariant. Among these ambiguities, the dimensionality of spacetime is fixed to the value 26, and the mass formula is shifted slightly from its classical counterpart such that the spectrum admits massless photon states. The spectrum also contains a tachyon state, which indicates the instability of the D25-brane.

12.1 Light-cone Hamiltonian and commutators

We are at long last in a position to quantize the relativistic string. We have acquired considerable intuition for the dynamics of classical relativistic strings, and we have examined in detail how to quantize the simpler, but still nontrivial, relativistic point particle. Moreover, having taken a brief look into the basics of scalar, electromagnetic, and gravitational quantum fields in the light-cone gauge, we will be able to appreciate the implications of quantum open string theory. In this chapter we will deal with open strings.

We will assume throughout the presence of a space-filling D-brane. In the next chapter we will quantize the closed string.

Just as before, we will interpret the classical equations of motion in the light-cone gauge as equations for the appropriate Heisenberg operators. It is therefore necessary for us to review the results of our light-cone analysis of the classical relativistic string.

We found a class of world-sheet parameterizations (9.2.16) for which the equations of motion are wave equations $\ddot{X}^\mu - X^{\mu''} = 0$. This remarkable simplification came at the expense of two constraints: $(\dot{X} \pm X')^2 = 0$. With these constraints, the momentum densities became simple derivatives of the coordinates:

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X^{\mu\prime}, \quad \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu. \tag{12.1.1}$$

These equations hold in any gauge which is of the class that we considered. In particular, they are true in the light-cone gauge. For open strings in the light-cone gauge, we set $X^+ = 2\alpha' p^+ \tau$ and solved for X^- in terms of the transverse coordinates X^I. Indeed, using (9.5.5) with $\beta = 2$, we have

$$\dot{X}^- = \frac{1}{2\alpha'} \frac{1}{2p^+} (\dot{X}^I \dot{X}^I + X^{I\prime} X^{I\prime}). \tag{12.1.2}$$

This gives us an explicit expression for $\mathcal{P}^{\tau -}$:

$$\begin{aligned} \mathcal{P}^{\tau -} &= \frac{1}{2\pi\alpha'} \dot{X}^- = \frac{1}{2\pi\alpha'} \frac{1}{2\alpha'} \frac{1}{2p^+} (2\pi\alpha')^2 \Big(\mathcal{P}^{\tau I} \mathcal{P}^{\tau I} + \frac{X^{I\prime} X^{I\prime}}{(2\pi\alpha')^2} \Big) \\ &= \frac{\pi}{2p^+} \Big(\mathcal{P}^{\tau I} \mathcal{P}^{\tau I} + \frac{X^{I\prime} X^{I\prime}}{(2\pi\alpha')^2} \Big). \end{aligned} \tag{12.1.3}$$

These equations will soon become useful.

As a first step in defining a quantum theory of the light-cone relativistic string, we must give the list of Schrödinger operators. Motivated by the list (11.3.1) of Schrödinger operators for the quantum point particle, we choose our τ-independent Schrödinger operators to be

$$\boxed{\text{Schrödinger ops.} : \Big(\; X^I(\sigma), \quad x_0^- \, , \quad \mathcal{P}^{\tau I}(\sigma) \, , \quad p^+ \Big) .} \tag{12.1.4}$$

The associated Heisenberg operators are then

$$\boxed{\text{Heisenberg ops.}: \Big(X^I(\tau,\sigma), \quad x_0^-(\tau)\,, \quad \mathcal{P}^{\tau I}(\tau,\sigma)\,, \quad p^+(\tau)\,\Big).} \quad (12.1.5)$$

Because the operators (12.1.4) have no explicit τ-dependence, neither do the Heisenberg operators (12.1.5). As in the case of the point particle, we expect x_0^- and p^+ to be fully τ-independent Heisenberg operators.

Now we set up the commutation relations. For the Schrödinger operators $X^I(\sigma)$ and $\mathcal{P}^{\tau I}(\sigma)$ we must face the fact that these operators have σ dependence. It is reasonable to demand that such operators fail to commute only if they are at the same point along the string. We do not expect (simultaneous) measurements at different points on the string to interfere with each other. Therefore we set

$$\big[\,X^I(\sigma)\,, \mathcal{P}^{\tau J}(\sigma')\,\big] = i\eta^{IJ}\,\delta(\sigma-\sigma')\,. \quad (12.1.6)$$

Here the delta function is being used to implement the constraint that the commutator must vanish when $\sigma \neq \sigma'$. We had to use a Dirac delta, as opposed to a Kronecker delta, since σ is a continuous variable. Equation (12.1.6) is naturally supplemented with the commutation relations

$$\big[\,X^I(\sigma)\,, X^J(\sigma')\,\big] = \big[\,\mathcal{P}^{\tau I}(\sigma)\,, \mathcal{P}^{\tau J}(\sigma')\,\big] = 0\,, \quad (12.1.7)$$

and

$$\big[\,x_0^-\,,\,p^+\big] = -i\,. \quad (12.1.8)$$

The operators x_0^- and p^+ commute with all of the other Schrödinger operators:

$$\big[\,x_0^-\,, X^I(\sigma)\big] = \big[\,x_0^-\,, \mathcal{P}^{\tau I}(\sigma)\big] = \big[\,p^+\,, X^I(\sigma)\,\big] = \big[\,p^+\,, \mathcal{P}^{\tau I}(\sigma)\,\big] = 0\,. \quad (12.1.9)$$

For the associated Heisenberg operators, the only non-vanishing equal-time commutation relations are therefore

$$\boxed{\big[X^I(\tau,\sigma)\,, \mathcal{P}^{\tau J}(\tau,\sigma')\big] = i\,\eta^{IJ}\,\delta(\sigma-\sigma')\,,} \quad (12.1.10)$$

as well as

$$\big[\,x_0^-(\tau)\,,\,p^+(\tau)\big] = -i\,. \quad (12.1.11)$$

All other commutators vanish:

$$\begin{aligned}
\big[X^I(\tau,\sigma), X^J(\tau,\sigma')\big] &= \big[\mathcal{P}^{\tau I}(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')\big] = 0\,,\\
\big[x_0^-(\tau)\,, X^I(\tau,\sigma)\big] &= \big[x_0^-(\tau)\,, \mathcal{P}^{\tau I}(\tau,\sigma)\,\big] = 0\,,\\
\big[p^+(\tau)\,, X^I(\tau,\sigma)\big] &= \big[p^+(\tau)\,, \mathcal{P}^{\tau I}(\tau,\sigma)\,\big] = 0\,.
\end{aligned} \tag{12.1.12}$$

We must now invent the Hamiltonian. Our Hamiltonian should generate τ translation. From our experience with the point particle, we know that p^- generates X^+ translation. But in the light-cone gauge $X^+ = 2\alpha' p^+\tau$, so

$$\frac{\partial}{\partial\tau} = \frac{\partial X^+}{\partial\tau}\frac{\partial}{\partial X^+} = 2\alpha' p^+ \frac{\partial}{\partial X^+}\,. \tag{12.1.13}$$

It follows that the Hamiltonian that generates change in τ is

$$H = 2\alpha' p^+ p^- \ = 2\alpha' p^+ \int_0^\pi d\sigma\, \mathcal{P}^{\tau -}\,. \tag{12.1.14}$$

This will indeed turn out to be the correct string Hamiltonian. Using (12.1.3), the Hamiltonian can be written more explicitly as the Heisenberg operator

$$\boxed{H(\tau) = \pi\alpha' \int_0^\pi d\sigma \left(\mathcal{P}^{\tau I}(\tau,\sigma)\mathcal{P}^{\tau I}(\tau,\sigma) + \frac{X^{I\prime}(\tau,\sigma)X^{I\prime}(\tau,\sigma)}{(2\pi\alpha')^2}\right).} \tag{12.1.15}$$

H must generate quantum equations of motion that are operator versions of the classical equations of motion. It is worth recognizing that H is very simple when expressed in terms of the transverse Virasoro modes that we introduced in Chapter 9. There we saw that $L_0^\perp = 2\alpha' p^+ p^-$ ((9.5.16)), so (12.1.14) immediately gives

$$H = L_0^\perp\,. \tag{12.1.16}$$

This expression for the Hamiltonian is perhaps the most memorable one, though, as we will see later on, the true Hamiltonian, is changed slightly. The operator products $\mathcal{P}\mathcal{P}$ and $X'X'$ in (12.1.15) are actually ambiguous operators and need careful definition. Additionally, Lorentz invariance will require the subtraction of a calculable constant from H.

Now that we have a plausible candidate for the Hamiltonian, we have to derive the equations of motion. Any Heisenberg operator $\xi(\tau)$ which arises from a Schrödinger operator ξ that does not have explicit τ dependence must satisfy

$$i\,\dot{\xi}(\tau,\sigma) = [\,\xi(\tau,\sigma)\,,\, H(\tau)\,]\,, \tag{12.1.17}$$

where $H(\tau)$ is given in (12.1.15). Since $H(\tau)$ is built from Heisenberg operators that have no explicit time dependence, we can substitute $H(\tau)$ for $\xi(\tau,\sigma)$ in (12.1.17). We conclude that it is completely time independent: $H(\tau) = H$. Furthermore, we can see that $x_0^-(\tau)$ and $p^+(\tau)$ commute with H. They are therefore time independent operators, so we will henceforth denote them by x_0^- and p^+. The commutator (12.1.11) then becomes

$$\boxed{\;\big[\,x_0^-\,,\,p^+\big] = -i\,.} \tag{12.1.18}$$

The Heisenberg equation of motion for $X^I(\tau,\sigma)$ is

$$i\dot{X}^I(\tau,\sigma) = [X^I(\tau,\sigma)\,,\,H(\tau)] = \Big[\,X^I(\tau,\sigma), \pi\alpha'\int_0^\pi d\sigma'\,\mathcal{P}^{\tau J}(\tau,\sigma')\mathcal{P}^{\tau J}(\tau,\sigma')\Big],$$

where we dropped the second term in H since it commutes with $X^I(\tau,\sigma)$:

$$\Big[X^I(\tau,\sigma)\,,\,X^{J\prime}(\tau,\sigma')\Big] = \frac{\partial}{\partial\sigma'}\big[X^I(\tau,\sigma)\,,\,X^J(\tau,\sigma')\big] = 0\,. \tag{12.1.19}$$

We also reinserted the time parameter in $H(\tau)$, choosing a time that gives easily evaluated equal-time commutators. Making use of (12.1.10) we find

$$\begin{aligned} i\dot{X}^I(\tau,\sigma) &= \pi\alpha'\cdot 2\cdot\int_0^\pi d\sigma'\,\mathcal{P}^{\tau J}(\tau,\sigma')\,i\,\eta^{IJ}\delta(\sigma-\sigma') \\ \longrightarrow\quad \dot{X}^I(\tau,\sigma) &= 2\pi\alpha'\,\mathcal{P}^{\tau I}(\tau,\sigma)\,. \end{aligned} \tag{12.1.20}$$

Happily, this coincides with the classical equation of motion (12.1.1). The other equations of motion can be checked in a similar fashion. For example, you can calculate $\dot{\mathcal{P}}^{\tau I}$ and use the result to verify that

$$\ddot{X}^I - X^{I\prime\prime} = 0\,, \tag{12.1.21}$$

is the quantum equation of motion (Problem 12.1). As we turn classical string theory into a quantum theory, the classical boundary conditions become operator equations. For example, the Neumann boundary conditions

$$\partial_\sigma X^I(\tau,\sigma) = 0\,,\quad \sigma = 0,\pi\,, \tag{12.1.22}$$

are taken literally as the condition that the operator $\partial_\sigma X^I(\tau,\sigma)$ vanishes at the open string endpoints.

12.2 Commutation relations for oscillators

The commutation relations (12.1.10) are delicate to handle because they involve field operators and use delta functions. They are an infinite set of relations holding for continuous values of σ and σ'. It is therefore useful to recast them in discrete form; namely, as a denumerable set of commutation relations. For this purpose we will consider the mode expansions of section 9.4. These followed from the classical wave equations and the boundary conditions for a space-filling D-brane. Since the wave equations and the boundary conditions continue to hold in the quantum theory, we can use the mode expansions in the quantum theory. The classical modes α_n^I, however, become quantum operators with nontrivial commutation relations.

Recall our solution (9.5.7) to the wave equation with Neumann boundary conditions:

$$X^I(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\,\alpha_0^I\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I\cos n\sigma\, e^{-in\tau}\,. \tag{12.2.1}$$

In addition, from (9.5.12) we have

$$\dot{X}^I \pm X^{I\prime} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^I\, e^{-in(\tau\pm\sigma)}\,. \tag{12.2.2}$$

While $\sigma\in[0,\pi]$ for open strings, the above right-hand side naturally defines a periodic function of σ with period 2π. Select the top sign and consider this function over the interval $[-\pi,\pi]$. The left-hand side tells us what the function is when $\sigma\in[0,\pi]$. For the full interval we have

$$\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^I\, e^{-in(\tau+\sigma)} = \begin{cases}(\dot{X}^I + X^{I\prime})(\tau,\,\sigma), & \sigma\in[\,0\,,\,\pi\,]\,,\\ (\dot{X}^I - X^{I\prime})(\tau,-\sigma), & \sigma\in[-\pi,0]\,.\end{cases} \tag{12.2.3}$$

To write this for $\sigma\in[-\pi,0]$, we used the lower-sign version of (12.2.2). Note that the σ arguments in the right-hand side of (12.2.3) are always positive. This has to be so because the string coordinates are only defined for $\sigma\in[0,\pi]$.

Our goal is to find the commutation relations for the operators α_n^I that follow from the basic commutator (12.1.10). We will do this by examining the commutators among the linear combinations of derivatives $(\dot{X}^I \pm X^{I\prime})$. We begin by using (12.1.20) to rewrite the commutator (12.1.10) as

$$\Big[\,X^I(\tau,\sigma),\dot{X}^J(\tau,\sigma')\,\Big] = 2\pi\alpha'\,i\,\eta^{IJ}\,\delta(\sigma-\sigma')\,. \tag{12.2.4}$$

Taking the σ-derivative of this equation yields

$$\Big[X^{I'}(\tau,\sigma), \dot{X}^J(\tau,\sigma') \Big] = 2\pi\alpha'\, i\, \eta^{IJ}\, \frac{d}{d\sigma}\delta(\sigma-\sigma')\,. \tag{12.2.5}$$

Differentiating (12.1.12), we find that τ- and σ-derivatives of the coordinates separately commute among themselves:

$$\Big[X^{I'}(\tau,\sigma), X^{J'}(\tau,\sigma') \Big] = \Big[\dot{X}^I(\tau,\sigma)\,, \dot{X}^J(\tau,\sigma') \Big] = 0\,. \tag{12.2.6}$$

Now we examine the commutator

$$\Big[(\dot{X}^I + X^{I'})(\tau,\sigma)\,,\, (\dot{X}^J + X^{J'})(\tau,\sigma') \Big]\,, \tag{12.2.7}$$

which as a consequence of (12.2.6) equals

$$\Big[\dot{X}^I(\tau,\sigma)\,,\, X^{J'}(\tau,\sigma') \Big] + \Big[X^{I'}(\tau,\sigma)\,,\, \dot{X}^J(\tau,\sigma') \Big]\,. \tag{12.2.8}$$

The second term is given by (12.2.5). The first term equals

$$\begin{aligned} -\Big[X^{J'}(\tau,\sigma')\,, \dot{X}^I(\tau,\sigma) \Big] &= -(2\pi\alpha')i\eta^{JI}\frac{d}{d\sigma'}\delta(\sigma'-\sigma)\,, \\ &= 2\pi\alpha'\, i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma')\,. \end{aligned} \tag{12.2.9}$$

In obtaining this result we noted that a σ'-derivative can be traded for minus a σ-derivative when acting on a function of $(\sigma-\sigma')$. Moreover, we used $\delta(x)=\delta(-x)$. We now see that both terms in (12.2.8) are equal, so

$$\Big[(\dot{X}^I + X^{I'})(\tau,\sigma)\,,\, (\dot{X}^J + X^{J'})(\tau,\sigma') \Big] = 4\pi\alpha'\, i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma')\,. \tag{12.2.10}$$

In fact, more generally, we have found that

$$\boxed{\Big[(\dot{X}^I \pm X^{I'})(\tau,\sigma)\,,\, (\dot{X}^J \pm X^{J'})(\tau,\sigma') \Big] = \pm 4\pi\alpha'\, i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma')\,,} \tag{12.2.11}$$

since only the cross terms contribute. Finally,

$$\Big[(\dot{X}^I \pm X^{I'})(\tau,\sigma)\,,\, (\dot{X}^J \mp X^{J'})(\tau,\sigma') \Big] = 0\,. \tag{12.2.12}$$

The three equations above hold for $\sigma, \sigma' \in [0, \pi]$.

We consider now the commutator of the function defined in (12.2.3) at σ, with the same function at σ'. If $\sigma, \sigma' \in [0, \pi]$ we can use the top right-hand side of (12.2.3) together with (12.2.2) to find

$$
\begin{aligned}
2\alpha' \sum_{m', n' \in \mathbb{Z}} e^{-im'(\tau+\sigma)} e^{-in'(\tau+\sigma')} [\alpha^I_{m'}\,, \alpha^J_{n'}\,] = \\
&= \Big[\, (\dot{X}^I + X^{I'})(\tau, \sigma)\,,\, (\dot{X}^J + X^{J'})(\tau, \sigma') \Big]\,, \qquad (12.2.13)\\
&= 4\pi\alpha' \, i\eta^{IJ} \frac{d}{d\sigma} \delta(\sigma - \sigma')\,.
\end{aligned}
$$

Cancelling the common factor of $2\alpha'$,

$$
\sum_{m', n' \in \mathbb{Z}} e^{-im'(\tau+\sigma)} e^{-in'(\tau+\sigma')} [\alpha^I_{m'}\,, \alpha^J_{n'}\,] = 2\pi \, i\eta^{IJ} \frac{d}{d\sigma} \delta(\sigma - \sigma')\,. \qquad (12.2.14)
$$

We now claim that this equation actually holds for $\sigma, \sigma' \in [-\pi, \pi]$. If $\sigma \in [-\pi, 0]$ and $\sigma' \in [0, \pi]$, or viceversa, the commutator we get in (12.2.13) is that in (12.2.12), which vanishes. Accordingly, the right hand of (12.2.14) also vanishes since in this case σ and σ' cannot be equal. When both $\sigma, \sigma' \in [-\pi, 0]$, the second line in (12.2.13) is replaced by

$$
\Big[\, (\dot{X}^I - X^{I'})(\tau, -\sigma)\,,\, (\dot{X}^J - X^{J'})(\tau, -\sigma') \Big]\,, \qquad (12.2.15)
$$

and making use of (12.2.10), this equals

$$
= -4\pi\alpha' \, i\eta^{IJ} \frac{d}{d(-\sigma)} \delta(-\sigma + \sigma') = 4\pi\alpha' \, i\eta^{IJ} \frac{d}{d\sigma} \delta(\sigma - \sigma')\,, \qquad (12.2.16)
$$

which coincides with the final right-hand side in (12.2.13). We have therefore shown that (12.2.14) holds for $\sigma, \sigma' \in [-\pi, \pi]$.

In order to extract detailed information from (12.2.14) we will apply two integral operations both on the left-hand side and on the right-hand side. The operations are

$$
\frac{1}{2\pi} \int_0^{2\pi} d\sigma \, e^{im\sigma} \;\cdot\; \frac{1}{2\pi} \int_0^{2\pi} d\sigma' \, e^{in\sigma'}\,. \qquad (12.2.17)
$$

On the left-hand side of (12.2.14) the integrals pick the term with $m' = m$ and $n' = n$:

$$e^{-i(m+n)\tau}[\alpha_m^I \,, \alpha_n^J\,]\,. \tag{12.2.18}$$

On the right-hand side of (12.2.14) the integrals give

$$\begin{aligned}
&i\eta^{IJ}\frac{1}{2\pi}\int_0^{2\pi} d\sigma\, e^{im\sigma}\,\frac{d}{d\sigma}\int_0^{2\pi} d\sigma'\, e^{in\sigma'}\,\delta(\sigma-\sigma')\,,\\
&= i\eta^{IJ}\frac{1}{2\pi}\int_0^{2\pi} d\sigma\, e^{im\sigma}\,\frac{d}{d\sigma}e^{in\sigma} = -n\,\eta^{IJ}\frac{1}{2\pi}\int_0^{2\pi} d\sigma\, e^{i(m+n)\sigma}\\
&= -n\,\eta^{IJ}\delta_{m+n,0} = m\,\eta^{IJ}\delta_{m+n,0}\,.
\end{aligned} \tag{12.2.19}$$

Equating our results (12.2.18) and (12.2.19), we find

$$[\alpha_m^I \,, \alpha_n^J\,] = m\,\eta^{IJ}\delta_{m+n,0}\, e^{+i(m+n)\tau} = m\,\eta^{IJ}\delta_{m+n,0}\,, \tag{12.2.20}$$

since the Kronecker delta can be used to set $m = -n$. Therefore, the commutation relation is

$$\boxed{\;\big[\,\alpha_m^I \,, \alpha_n^J\,\big] = m\,\eta^{IJ}\delta_{m+n,0}\,.} \tag{12.2.21}$$

This is the fundamental commutation relation between α-modes. Note that α_0^I commutes with all other oscillators. This is quite reasonable: as shown in (9.4.13), α_0^I is proportional to the momentum of the string

$$\boxed{\;\alpha_0^I = \sqrt{2\alpha'}\,p^I\,,\;} \tag{12.2.22}$$

and it is expected to have a nontrivial commutator with x_0^J only. Indeed, to complete the list of all possible commutators we must find the commutators between x_0^I and the oscillators α_n^J. For this, we consider equation (12.2.4), and integrate both sides of the equation over $\sigma \in [0, \pi]$. On the left-hand side, the terms with oscillators in $X^I(\tau, \sigma)$ give no contribution, and on the right-hand side the delta function dissappears giving a factor of one:

$$\Big[\,x_0^I + \sqrt{2\alpha'}\,\alpha_0^I\,\tau\,,\,\dot{X}^J(\tau,\sigma')\Big] = 2\alpha'\,i\,\eta^{IJ}\ . \tag{12.2.23}$$

Since $\dot{X}^J$ is a sum of α_n^J's, $[\alpha_0^I \,, \dot{X}^I] = 0$. Additionally, using the mode expansion of $\dot{X}^J$, equation (12.2.23) becomes

$$\sum_{n'\in\mathbb{Z}}\,\big[\,x_0^I\,, \alpha_{n'}^J\big]\cos n'\sigma\, e^{-in'\tau} = \sqrt{2\alpha'}\,i\,\eta^{IJ}\,. \tag{12.2.24}$$

Reorganizing the left-hand side of this equation we find

$$[x_0^I\,,\alpha_0^J] + \sum_{n'=1}^{\infty} \Big[x_0^I\,, \alpha_{n'}^J e^{-in'\tau} + \alpha_{-n'}^J e^{in'\tau} \Big] \cos n'\sigma = \sqrt{2\alpha'}\, i\, \eta^{IJ}\,. \quad (12.2.25)$$

We apply to both sides of this equation the integral operation $\frac{1}{\pi}\int_0^\pi d\sigma \cos n\sigma$, with $n \geq 1$. We then find that

$$\big[x_0^I\,, \alpha_n^J e^{-in\tau} + \alpha_{-n}^J e^{in\tau} \big] = 0\,, \quad (12.2.26)$$

or equivalently

$$\big[x_0^I\,, \alpha_n^J \,\big]\, e^{-in\tau} + \big[x_0^I\,, \alpha_{-n}^J \,\big]\, e^{in\tau} = 0\,. \quad (12.2.27)$$

Since the left-hand side must vanish for all values of τ, each term must vanish separately (prove this!). It follows that

$$\big[x_0^I\,, \alpha_n^J \,\big] = 0 \;\text{ for } \quad n \neq 0\,. \quad (12.2.28)$$

Additionally, equation (12.2.25) gives

$$\big[x_0^I, \alpha_0^J \,\big] = \sqrt{2\alpha'}\, i\eta^{IJ}\,. \quad (12.2.29)$$

This, together with (12.2.22), gives the expected commutator

$$\boxed{\big[x_0^I\,, p^J \,\big] = i\eta^{IJ}\,.} \quad (12.2.30)$$

As in familiar quantum mechanics, the operators x_0^I and p^I are Hermitian:

$$(x_0^I)^\dagger = x_0^I\,, \qquad (p^I)^\dagger = p^I\,. \quad (12.2.31)$$

The calculations we performed to obtain the commutation relations took quite a few steps, which we explained in detail. When we will discuss closed strings, or open strings on general D-brane configurations, similar questions will arise. We will then be able to answer them using, with minimal modifications, the calculations we just did.

It is useful at this point to examine in detail the commutation relations (12.2.21) for the α_n^I modes. As we will show below, they are equivalent to those of an infinite set of creation and annihilation operators. To see this, we begin by defining *oscillators*, taking our inspiration from the classical variables introduced in (9.4.14):

$$\alpha_n^\mu = a_n^\mu \sqrt{n}\,, \qquad \alpha_{-n}^\mu = a_n^{*\mu}\sqrt{n}\,, \qquad n \geq 1\,. \tag{12.2.32}$$

In these equations, both the α's and the a's are classical variables. Now they must become operators. Classical variables that are complex conjugates of each other must become operators that are Hermitian conjugates of each other in the quantum theory. We can therefore preserve the first of the above definitions, but the second must be changed. For our light-cone modes $\mu = I$ we take

$$\alpha_n^I = a_n^I \sqrt{n} \qquad \text{and} \qquad \alpha_{-n}^I = a_n^{I\dagger}\sqrt{n}\,, \qquad n \geq 1\,. \tag{12.2.33}$$

Note that, with this definition,

$$\boxed{(\alpha_n^I)^\dagger = \alpha_{-n}^I\,, \qquad n \in \mathbb{Z}\,.} \tag{12.2.34}$$

This equation holds for $n = 0$ because α_0^I, being proportional to p^I, is also Hermitian. It is useful to emphasize that, while the α_n^I modes are defined for all integers n, the a_n^I and $a_n^{I\dagger}$ operators are only defined for positive n.

An important consequence of the above Hermiticity properties is that $X^I(\tau,\sigma)$, which used to be real in the classical theory, is now a Hermitian operator:

Quick Calculation 12.1. Use the expansion (12.2.1) and the Hermiticity conditions (12.2.31) and (12.2.34) to show that

$$(X^I(\tau,\sigma))^\dagger = X^I(\tau,\sigma)\,. \tag{12.2.35}$$

Note that the factor of i in front of the sum in (12.2.1), is needed for this calculation to work out.

We can now rephrase the commutation relation for the α-modes in terms of the oscillators $(a_n^I, a_n^{I\dagger})$. For this purpose, rewrite (12.2.21) as

$$\big[\,\alpha_m^I\,,\alpha_{-n}^J\,\big] = m\delta_{m,n}\eta^{IJ}\,. \tag{12.2.36}$$

When m and n are integers of opposite signs the right-hand side vanishes, and the two operators in the commutator have mode numbers of the same sign. Therefore, we learn that

$$\big[\,a_m^I\,,a_n^J\,\big] = \big[\,a_m^{I\dagger}\,,\,a_n^{J\dagger}\big] = 0\,. \tag{12.2.37}$$

If both m and n are positive in (12.2.36) we find

$$\left[\,\sqrt{m}\,a_m^I\,,\sqrt{n}\,a_n^{J\dagger}\,\right] = m\delta_{m,n}\eta^{IJ}\,. \tag{12.2.38}$$

Moving the square roots to the right-hand side

$$\left[\,a_m^I\,,a_n^{J\dagger}\,\right] = \frac{m}{\sqrt{mn}}\,\delta_{m,n}\eta^{IJ}\,. \tag{12.2.39}$$

Since the right-hand side vanishes unless $m = n$, it simplifies to

$$\boxed{\left[\,a_m^I\,,\,a_n^{J\dagger}\,\right] = \delta_{m,n}\,\eta^{IJ}\,.} \tag{12.2.40}$$

This, together with (12.2.37), shows that $(a_m^I, a_m^{I\dagger})$ satisfy the commutation relations of the canonical annihilation and creation operators of a quantum simple harmonic oscillator. There is a pair of creation and annihilation operators for each value $m \geq 1$ of the mode number and for each transverse light-cone direction I. The commutation relations are diagonal: oscillators corresponding to different mode numbers, or to different light-cone coordinates commute. If the mode numbers and the coordinate labels agree, the commutator is equal to one. In terms of the α-operators, with $n \geq 1$:

$$\begin{array}{ll} \alpha_n^I & \text{are destruction operators, and,} \\ \alpha_{-n}^I & \text{are creation operators.} \end{array} \tag{12.2.41}$$

For future reference let's rewrite the expansion of $X^I(\tau,\sigma)$ in (12.2.1) in terms of creation and annihilation operators. Separating out the sum over all integers into sums over positive and negative integers, and using (12.2.22), we find

$$X^I(\tau,\sigma) = x_0^I + 2\alpha'\,p^I\tau + i\sqrt{2\alpha'}\,\sum_{n=1}^{\infty}\Big(\frac{1}{n}\alpha_n^I\,e^{-in\tau} - \frac{1}{n}\alpha_{-n}^I\,e^{in\tau}\Big)\cos n\sigma\,.$$

Replacing α-modes by the corresponding oscillators we obtain

$$X^I(\tau,\sigma) = x_0^I + 2\alpha'\,p^I\tau + i\sqrt{2\alpha'}\,\sum_{n=1}^{\infty}\Big(a_n^I\,e^{-in\tau} - a_n^{I\dagger}\,e^{in\tau}\Big)\frac{\cos n\sigma}{\sqrt{n}}\,. \tag{12.2.42}$$

This is the expansion of the coordinate operator in terms of creation and annhilation operators.

Let us take stock of what we have learned. The list of operators we started with was given in (12.1.5). We have seen that the operators $X^I(\tau,\sigma)$ and $\mathcal{P}^{\tau I}(\tau,\sigma)$ can be traded for an infinite collection of oscillators, plus pairs of zero modes (x_0^I, p^I). Since the other two operators in the list, x_0^- and p^+, are also zero-modes, the full set of basic operators of string theory are a collection of zero modes plus an infinite set of creation and annihilation operators. This result is so important that we will now derive it in a different way, showing explicitly how the quantum simple harmonic oscillators arise.

12.3 Strings as harmonic oscillators

Our aim here is to give a more physical derivation of the results obtained in the previous section, in particular, of the mode expansion (12.2.42) and the commutation relations between the operators in that expansion. These results followed from the fundamental commutation relation (12.1.10) together with the operator equations of motion (12.1.21) and the operator boundary conditions (12.1.22). Of these, the commutation relations (12.1.10) are perhaps the least intuitive, as they involve a delta function. In the derivation below there will be no delta function.

Here is our strategy. We will invent a simple Lagrangian that describes the dynamics of the light-cone coordinates X^I. This is not such a difficult task since we know the equations of motion of the X^I, their boundary conditions, and the definition of the canonical momenta $\mathcal{P}^{\tau I}$. Then we will expand the coordinate $X^I(\tau,\sigma)$ as a function of σ but with τ-dependent expansion coefficients. Using the Lagrangian we will show that those expansion coefficients are actually the coordinates of harmonic oscillators that have ever-increasing energy! We will conclude by relating these oscillators to the creation and annihilation operators obtained in the previous analysis.

To set up the notation, we begin by reviewing the basic properties of the quantum harmonic oscillator. Let $q_n(t)$ be the coordinate of a classical simple harmonic oscillator, and let the action be given by

$$S_n = \int L_n(t)\, dt = \int dt \Big(\frac{1}{2n}\, \dot{q}_n^2(t) - \frac{n}{2}\, q_n^2(t) \Big)\,. \tag{12.3.1}$$

We recognize this as a harmonic oscillator because the kinetic energy is proportional to the velocity squared, and the potential energy is proportional to the coordinate squared. For this Lagrangian, the momentum p_n conjugate to the coordinate q_n is

$$p_n = \frac{\partial L}{\partial \dot{q}_n} = \frac{1}{n}\,\dot{q}_n\,. \tag{12.3.2}$$

A little calculation now gives the Hamiltonian as

$$H_n(p_n, q_n) = p_n\,\dot{q}_n - L_n = \frac{n}{2}(p_n^2 + q_n^2)\,. \tag{12.3.3}$$

In this equation, n plays the role of the frequency ω of the harmonic oscillator. To define the quantum oscillator, we introduce Schrödinger operators q_n and p_n, with the canonical commutation relation

$$[\,q_n\,, p_n] = i\,. \tag{12.3.4}$$

Creation and annihilation operators can be introduced as

$$a_n = \frac{1}{\sqrt{2}}(p_n - iq_n)\,, \qquad a_n^\dagger = \frac{1}{\sqrt{2}}(p_n + iq_n)\,. \tag{12.3.5}$$

You should check that as a consequence of (12.3.4) the creation and annihilation operators satisfy the commutation relation

$$[a_n\,, a_n^\dagger] = 1\,. \tag{12.3.6}$$

Inverting the relations in (12.3.5), we find

$$q_n = \frac{i}{\sqrt{2}}(a_n - a_n^\dagger)\,, \qquad p_n = \frac{1}{\sqrt{2}}(a_n + a_n^\dagger)\,. \tag{12.3.7}$$

These can be used to rewrite the Hamiltonian H_n in terms of the creation and annihilation operators. We find the familiar result

$$H_n = n\Big(\,a_n^\dagger a_n + \frac{1}{2}\Big)\,. \tag{12.3.8}$$

We can now consider the Heisenberg operators $(a_n(t), a_n^\dagger(t))$ that are associated with the Schrödinger operators $(a_n, a_n^\dagger)$. As emphasized in section 11.2, the Heisenberg operators satisfy the same commutation relations as the Schrödinger operators:

$$\big[\,a_n(t)\,, a_n^\dagger(t)\big] = 1\,. \tag{12.3.9}$$

The Heisenberg equation of motion for $a_n(t)$ is

$$\dot{a}_n(t) = i\,[\,H_n(t)\,, a_n(t)\,] = in\,\left[\,a_n^\dagger(t)a_n(t)\,, a(t)\right] = -in\,a_n(t)\,. \qquad (12.3.10)$$

This differential equation is solved by

$$a_n(t) = e^{-int}a_n(0) = e^{-int}a_n\,, \qquad (12.3.11)$$

where a_n is the constant Heisenberg operator that equals $a_n(t)$ at $t = 0$. A similar calculation gives

$$a_n^\dagger(t) = e^{int}a_n^\dagger(0) = e^{int}a_n^\dagger\,. \qquad (12.3.12)$$

As you can see, the angular frequency of oscillation is indeed equal to n. Finally, with these results and (12.3.7), we can find the explicit time dependence of the operator $q_n(t)$:

$$q_n(t) = \frac{i}{\sqrt{2}}\,(a_n(t) - a_n^\dagger(t)\,) = \frac{i}{\sqrt{2}}\left(a_n\,e^{-int} - a_n^\dagger\,e^{int}\right). \qquad (12.3.13)$$

This concludes our review of the quantum simple harmonic oscillator.

We now turn to the discussion of an action that encodes the dynamics of the transverse light-cone coordinates $X^I(\tau,\sigma)$. We claim that the action is simply given by

$$S = \int d\tau d\sigma\,\mathcal{L} = \frac{1}{4\pi\alpha'}\int d\tau\int_0^\pi d\sigma\Big(\dot{X}^I\,\dot{X}^I - X^{I\prime}\,X^{I\prime}\Big)\,. \qquad (12.3.14)$$

This action is much simpler than the Nambu-Goto action: it has no square root, for example. The first term, having time derivatives, represents kinetic energy. The second term, having spatial derivatives, represents potential energy. The canonical momentum associated to X^I coincides with the momentum density $\mathcal{P}^{\tau I}$:

$$\frac{\partial\mathcal{L}}{\partial\dot{X}^I} = \frac{1}{2\pi\alpha'}\dot{X}^I = \mathcal{P}^{\tau I}\,, \qquad (12.3.15)$$

as we see comparing with (12.1.1). This confirms the correctness of the chosen normalization for $\mathcal{L}$. The equations of motion for X^I follow by variation:

$$\delta S = \frac{1}{2\pi\alpha'}\int d\tau\int_0^\pi d\sigma\Big(\partial_\tau(\delta X^I)\,\dot{X}^I - \partial_\sigma(\delta X^I)\,X^{I\prime}\Big)\,. \qquad (12.3.16)$$

Dropping the total τ-derivatives, by restricting ourselves to variations where the initial and final positions are fixed, we find

$$\delta S = -\frac{1}{2\pi\alpha'}\int d\tau \left[(X^{I\prime}\delta X^I)\Big|_0^\pi + \int_0^\pi d\sigma\, \delta X^I\Big(\ddot{X}^I - X^{I\prime\prime}\Big)\right]. \qquad (12.3.17)$$

It is clear that the requirement that the action be stationary gives us both the wave equation (12.1.21) for the coordinates and boundary conditions at the string endpoints. As a final check of the consistency of the action we calculate the Hamiltonian:

$$H = \int_0^\pi d\sigma \mathcal{H} = \int_0^\pi d\sigma\Big(\mathcal{P}^{\tau I}\dot{X}^I - \mathcal{L}\Big). \qquad (12.3.18)$$

Writing the τ-derivative of X^I in terms of $\mathcal{P}^{\tau I}$ we find

$$H = \int_0^\pi d\sigma\Big(\pi\alpha'\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + \frac{1}{4\pi\alpha'}X^{I\prime}X^{I\prime}\Big). \qquad (12.3.19)$$

This Hamiltonian coincides with the one we postulated and tested in section 12.1.

Let's now use the action (12.3.14) to quantize the theory. For this purpose we replace the dynamical variable $X^I(\tau,\sigma)$ by a collection of dynamical variables that have no σ dependence. This is done by writing the expansion

$$X^I(\tau,\sigma) = q^I(\tau) + 2\sqrt{\alpha'}\sum_{n=1}^{\infty} q_n^I(\tau)\frac{\cos n\sigma}{\sqrt{n}}\,. \qquad (12.3.20)$$

This is the most general expression that satisfies Neumann boundary conditions at the endpoints. The particular normalization used to introduce the expansion coefficients was chosen for convenience.

Our next step is to evaluate the action (12.3.14) using the above expansion for $X^I(\tau,\sigma)$. For this we use

$$\begin{aligned}\dot{X}^I &= \dot{q}^I(\tau) + 2\sqrt{\alpha'}\sum_{n=1}^{\infty}\dot{q}_n^I(\tau)\frac{\cos n\sigma}{\sqrt{n}}\,,\\ X^{I\prime} &= -2\sqrt{\alpha'}\sum_{n=1}^{\infty} q_n^I(\tau)\sqrt{n}\,\sin n\sigma\,.\end{aligned} \qquad (12.3.21)$$

The evaluation of the action S using the above expansions is quite straighforward because the σ-integrals of $(\cos n\sigma \cos m\sigma)$ and $(\sin n\sigma \sin m\sigma)$ vanish unless $n = m$. We find

$$S = \int d\tau \Big[\frac{1}{4\alpha'}\,\dot{q}^I(\tau)\,\dot{q}^I(\tau) + \sum_{n=1}^{\infty}\Big(\frac{1}{2n}\,\dot{q}_n^I(\tau)\,\dot{q}_n^I(\tau) - \frac{n}{2}\,q_n^I(\tau)\,q_n^I(\tau)\Big)\Big]\,. \tag{12.3.22}$$

Quick Calculation 12.2. Prove equation (12.3.22).

Comparing this action with the one recorded in (12.3.1) we see that the $q_n^I(\tau)$, with $n \geq 1$, are the coordinates of simple harmonic oscillators. The frequency of oscillation of $q_n^I(\tau)$ is n. This is the physical interpretation of the expansion coefficients in (12.3.20). Since the action for $q_n^I(\tau)$ coincides exactly with the action S_n, no new work is necessary to compute the Hamiltonian, except for the zero mode q^I:

$$p^I = \frac{\partial L}{\partial \dot{q}^I} = \frac{1}{2\alpha'}\dot{q}^I \qquad \text{and} \qquad [\,q^I\,, p^J] = i\eta^{IJ}\,. \tag{12.3.23}$$

The Hamiltonian is then given by

$$H = \alpha'\,p^I\,p^I + \sum_{n=1}^{\infty}\frac{n}{2}\Big(p_n^I\,p_n^I + q_n^I\,q_n^I\Big)\,, \tag{12.3.24}$$

where we used (12.3.3) to write the part of the Hamiltonian that arises from the oscillators. The earlier analysis of the Heisenberg operator $q_n(t)$ led to the solution (12.3.13). This means that for the $q_n^I(\tau)$ oscillators we must find

$$q_n^I(\tau) = \frac{i}{\sqrt{2}}\left(a_n^I\,e^{-in\tau} - a_n^{I\dagger}e^{in\tau}\right)\,, \tag{12.3.25}$$

where $(a_n^I, a_n^{I\dagger})$ are canonically normalized annihilation and creation operators. For the Heisenberg operator $q^I(\tau)$ we have

$$\dot{q}^I(\tau) = i\,[\,H, q^I(\tau)] = i\alpha'\,[\,p^J\,p^J(\tau)\,, q^I(\tau)] = 2\alpha'\,p^I(\tau)\,. \tag{12.3.26}$$

Note that p^I is a τ-independent Heisenberg operator. We solve this differential equation for $q^I(\tau)$ by writing

$$q^I(\tau) = x_0^I + 2\alpha'\,p^I\,\tau\,. \tag{12.3.27}$$

Here x_0^I is a constant operator that on account of (12.3.23) satisfies $[x_0^I, p^J] = i\eta^{IJ}$. Finally, we can substitute our solutions (12.3.25) and (12.3.27) into the expansion (12.3.20) for X^I to find

$$X^I(\tau,\sigma) = x_0^I + 2\alpha' \, p^I \tau + i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \Big(a_n^I \, e^{-in\tau} - a_n^{I\dagger} \, e^{in\tau} \Big) \frac{\cos n\sigma}{\sqrt{n}} \, , \quad (12.3.28)$$

in exact agreement with the previously derived (12.2.42). We have therefore given a physical derivation of the mode expansion and commutation relations. We identified the classical variables that become oscillators, and we did not have to use delta functions. Having done so in this case, when we quantize other string configurations we will simply use the abstract approach of the previous section. It gives a direct and quick route to the desired answers.

12.4 Transverse Virasoro operators

We have written mode expansions for the transverse coordinates $X^I(\tau,\sigma)$ and we have seen quite explicitly the connection to harmonic oscillators. How about the other light-cone coordinates, $X^+(\tau,\sigma)$ and $X^-(\tau,\sigma)$? The expansion of X^+ is truly simple:

$$X^+(\tau,\sigma) = 2\alpha' p^+ \, \tau = \sqrt{2\alpha'} \, \alpha_0^+ \, \tau \, . \quad (12.4.1)$$

As discussed for the classical case in section 9.5, this means that we are setting

$$x_0^+ = 0 \, , \quad \alpha_n^+ = 0 \, , \quad n \neq 0 \, . \quad (12.4.2)$$

For the X^- coordinate, the situation is quite different. We gave a mode expansion for X^- in (9.5.10):

$$X^-(\sigma,\tau) = x_0^- + \sqrt{2\alpha'} \, \alpha_0^- \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^- e^{-in\tau} \cos n\sigma \, . \quad (12.4.3)$$

Moreover, we used the constraints to solve for X^- in terms of the X^I, p^+ and a constant of integration x_0^-. This meant that the α_n^- modes could be written in terms of the α_n^I modes, as shown in equation (9.5.15):

$$\sqrt{2\alpha'} \, \alpha_n^- = \frac{1}{p^+} L_n^\perp \, , \quad (12.4.4)$$

where

$$L_n^\perp \equiv \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha^I_{n-p}\alpha^I_p\,. \tag{12.4.5}$$

The repeated index I is summed over the transverse light-cone directions. In Chapter 9 we called the $L_n^\perp$ transverse Virasoro modes. Having seen that the α-modes become operators, we will call the $L_n^\perp$ transverse Virasoro *operators*. The steps leading to (12.4.5) remain valid in the quantum theory, except for one issue. In deriving this equation the α-modes were treated as commuting classical variables. We now know that the α-operators do not commute. We must therefore question whether the ordering of the two α-operators appearing in (12.4.5) is the correct one. A better question is whether the ordering matters. Since two α-operators fail to commute only when their mode numbers add up to zero, the two operators in $L_n^\perp$ fail to commute only when $n = 0$. So $L_0^\perp$ is the only ambiguous operator.

There is plenty at stake in ordering $L_0^\perp$ correctly. The operator $L_0^\perp$ is in fact the light-cone Hamiltonian, as we showed in equation (12.1.16). Moreover, we saw at the end of Chapter 9 that $L_0^\perp$ enters directly into the calculation of the mass of string states. We also mentioned at that time that the quantum theory would bring a subtlety into the calculation of the mass. Well, the subtlety has arrived: we must define the quantum operator $L_0^\perp$! Let us therefore look at $L_0^\perp$ in more detail:

$$L_0^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha^I_{-p}\alpha^I_p = \frac{1}{2}\alpha^I_0\alpha^I_0 + \frac{1}{2}\sum_{p=1}^{\infty}\alpha^I_{-p}\alpha^I_p + \frac{1}{2}\sum_{p=1}^{\infty}\alpha^I_p\alpha^I_{-p}\,. \tag{12.4.6}$$

The first sum on the right-hand side is normal-ordered: annihilation operators are to the right of the creation operators. It is useful to work with normal-ordered operators since they act in a simple manner on the vacuum state. We cannot use operators that do not have a well-defined action on the vacuum state. Since the last sum on the right-hand side of (12.4.6) is not a normal-ordered operator, we rewrite it as

$$
\begin{aligned}
\frac{1}{2}\sum_{p=1}^{\infty}\alpha_p^I\alpha_{-p}^I &= \frac{1}{2}\sum_{p=1}^{\infty}\Big(\alpha_{-p}^I\alpha_p^I + [\alpha_p^I\,,\alpha_{-p}^I]\Big)\,,\\
&= \frac{1}{2}\sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I + \frac{1}{2}\sum_{p=1}^{\infty}p\,\eta^{II}\,, \qquad (12.4.7)\\
&= \frac{1}{2}\sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I + \frac{1}{2}(D-2)\sum_{p=1}^{\infty}p\,.
\end{aligned}
$$

If you look at the last term of the above equation you will note that it is divergent; it involves the sum of all positive integers! This is clearly problematic. How do we deal with this? One option is to simply ignore this difficulty, claiming that it is really up to us how we define $L_0^\perp$. There is a kernel of truth to this option, but it is not completely correct. Adding a constant to $L_0^\perp$ changes the values of the masses of the string states, and if anything, the above computation has alerted us to the fact that this additive constant could be non-zero, or even infinite. Taken at face value, the above computation gives

$$L_0^\perp = \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I + \frac{1}{2}(D-2)\sum_{p=1}^{\infty}p\,. \qquad (12.4.8)$$

The operator $L_0^\perp$ enters into our computation of the mass via the definition of p^-. From (12.4.4), with $n=0$:

$$\sqrt{2\alpha'}\,\alpha_0^- = 2\alpha' p^- = \frac{1}{p^+}L_0^\perp\,. \qquad (12.4.9)$$

This suggests a strategy. First, we *define*, once and for all, $L_0^\perp$ to be the normal-ordered operator in (12.4.8) *without* including the ordering constant:

$$L_0^\perp \equiv \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I = \alpha'\,p^I\,p^I + \sum_{p=1}^{\infty}p\,a_p^{I\dagger}\,a_p^I\,. \qquad (12.4.10)$$

Note that $L_0^\perp$ is Hermitian: $(L_0^\perp)^\dagger = L_0^\perp$. Second, we introduce an ordering constant a into (12.4.9):

$$2\alpha'\,p^- \equiv \frac{1}{p^+}\left(L_0^\perp - a\right)\,. \qquad (12.4.11)$$

Of course, if we took seriously our attempt to order $L_0^\perp$, we would have to conclude that

$$a \stackrel{?}{=} -\frac{1}{2}(D-2)\sum_{p=1}^{\infty} p\,. \qquad (12.4.12)$$

We will see below one remarkable interpretation of this equation which does, in fact, give the correct result. More pragmatically, we will take a to be an undetermined constant. As we will show in section 12.5, the quantum consistency of string theory will fix the constant a to an interesting finite value. Before proceeding further, let us investigate how the inclusion of a modifies the computation of the mass-squared operator. Working from the definition $M^2 = -p^2$, and using (12.4.10) and (12.4.11), we find

$$\begin{aligned} M^2 = -p^2 = 2p^+p^- - p^Ip^I &= \frac{1}{\alpha'}(L_0^\perp - a\,) - p^Ip^I \\ &= \frac{1}{\alpha'}\Big(-a + \sum_{n=1}^{\infty} n a_n^{I\,\dagger} a_n^I\Big)\,. \end{aligned} \qquad (12.4.13)$$

As expected, a introduces a constant shift into the mass-squared operator.

It is impossible to resist the temptation of trying to interpret (12.4.12). An important result in mathematics suggests a finite value for the right-hand side. For this we consider the zeta-function $\zeta(s)$, which is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\,, \qquad \Re(s) > 1\,. \qquad (12.4.14)$$

The argument s of the zeta-function is assumed to be a complex number, but as indicated, the above sum only converges if the real part of the argument is greater than one. We can use analytic continuation to define the zeta-function for all possible values of the argument. $\zeta(s)$ turns out to be finite for all values of s except $s = 1$. In particular, as you will see in Problem 12.4, $\zeta(-1) = -1/12$. On account of (12.4.14) this suggests that

$$\zeta(-1) = -\frac{1}{12} \stackrel{?}{=} 1 + 2 + 3 + 4 + \cdots\,. \qquad (12.4.15)$$

This is a surprising interpretation for the infinite sum $\sum_{p=1}^{\infty} p$. Not only is the result finite, but it is also negative! Substituting back in (12.4.12), it gives us

$$a = \frac{1}{24}(D-2)\,. \qquad (12.4.16)$$

This is actually the correct value of a, as we will explain in section 12.5. The inspired guess gave the right answer. We will also see that consistency requires $D = 26$, so that in fact, $a = 1$. This value for the shift in the mass-squared operator is precisely what is needed for open strings to include massless photon states!

Having discussed $L_0^\perp$ in detail, let's consider the other transverse Virasoro operators. Since $(\alpha_n^J)^\dagger = \alpha_{-n}^J$, we may expect that $(\alpha_n^-)^\dagger = \alpha_{-n}^-$, or equivalently, on account of (12.4.4), that

$$(L_n^\perp)^\dagger = L_{-n}^\perp \,. \tag{12.4.17}$$

We have already verified this equation for $n = 0$. For $n \neq 0$, we can easily prove this Hermiticity property using (12.4.5):

$$(L_n^\perp)^\dagger = \frac{1}{2}\sum_{p\in\mathbb{Z}}(\alpha_{n-p}^I\alpha_p^I)^\dagger = \frac{1}{2}\sum_{p\in\mathbb{Z}}(\alpha_p^I)^\dagger(\alpha_{n-p}^I)^\dagger = \frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_{-p}^I\alpha_{-n+p}^I \,. \tag{12.4.18}$$

Since the oscillators in each term of the sum commute, we can exchange them. By also letting $p \to -p$, we get the expected result:

$$(L_n^\perp)^\dagger = \frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_{-n-p}^I\alpha_p^I = L_{-n}^\perp \,. \tag{12.4.19}$$

Perhaps the most interesting property of the Virasoro operators is that they do not commute. We have seen that α_m^I and α_n^I commute except when $m + n$ equals zero. This is not the case with the α_n^- modes. Two Virasoro operators $L_m^\perp$ and $L_n^\perp$ never commute when $m \neq n$. The commutation properties of Virasoro operators are a bit intricate, so we will consider them in steps of increasing generality.

As a warmup, let us consider the commutator between a Virasoro operator and an oscillator α_n^J. We have

$$\begin{aligned}[L_m^\perp\,, \alpha_n^J] &= \frac{1}{2}\sum_{p\in\mathbb{Z}}[\alpha_{m-p}^I\alpha_p^I\,, \alpha_n^J] \\ &= \frac{1}{2}\sum_{p\in\mathbb{Z}}\Big(\alpha_{m-p}^I[\alpha_p^I\,, \alpha_n^J] + [\alpha_{m-p}^I\,, \alpha_n^J]\,\alpha_p^I\Big)\,. \end{aligned} \tag{12.4.20}$$

Evaluating the commutators and recalling that $\eta^{IJ} = \delta^{IJ}$, we find

$$[L^\perp_m\,, \alpha^J_n] = \frac{1}{2}\sum_{p\in\mathbb{Z}}\Big(p\,\delta_{p+n,0}\,\alpha^J_{m-p} + (m-p)\delta_{m-p+n,0}\,\alpha^J_p\Big)\,. \tag{12.4.21}$$

Because of the Kronecker deltas only one term contributes in each sum: $p = -n$ in the first and $p = m+n$ in the second. We thus find

$$\big[\,L^\perp_m\,, \alpha^J_n\,\big] = \frac{1}{2}\Big(-n\alpha^J_{m+n} - n\,\alpha^J_{m+n}\Big)\,. \tag{12.4.22}$$

Our final result is therefore

$$\boxed{\big[\,L^\perp_m\,, \alpha^J_n\,\big] = -n\,\alpha^J_{m+n}\,.} \tag{12.4.23}$$

The mode number on the right-hand side is the sum of the mode numbers on the left-hand side. This could not be otherwise, since the basic commutator of α-operators trades two operators with opposite mode number for a constant, and in doing so, the total mode number is conserved. Moreover, the spatial index on the oscillator is preserved.

Quick Calculation 12.3. Show that

$$\boxed{\big[\,L^\perp_m\,, x^I_0\big] = -i\sqrt{2\alpha'}\,\alpha^I_m\,.} \tag{12.4.24}$$

Let us now consider the commutator of two Virasoro operators $L^\perp_m$ and $L^\perp_n$ with $m+n \neq 0$. This condition on the mode numbers avoids subtle complications that can arise during the calculation of the commutator. We do this computation by writing one of the Virasoro operators in terms of oscillators:

$$\big[\,L^\perp_m, L^\perp_n\big] = \frac{1}{2}\sum_{p\in\mathbb{Z}}\,\big[\,L^\perp_m\,, \alpha^I_{n-p}\alpha^I_p\big]\;. \tag{12.4.25}$$

Even if $n = 0$, writing $L^\perp_n$ in this form is safe: the ordering constant, whatever its value, will commute with $L^\perp_m$. Expanding the commutator and using (12.4.23), we find

$$\begin{aligned}\big[\,L^\perp_m, L^\perp_n\,\big] &= \frac{1}{2}\sum_{p\in\mathbb{Z}}\Big(\,[L^\perp_m\,, \alpha^I_{n-p}]\alpha^I_p + \alpha^I_{n-p}\,[L^\perp_m\,, \alpha^I_p]\Big)\,,\\ &= \frac{1}{2}\sum_{p\in\mathbb{Z}}\Big(\,(p-n)\alpha^I_{m+n-p}\,\alpha^I_p - p\,\alpha^I_{n-p}\,\alpha^I_{m+p}\Big)\,.\end{aligned} \tag{12.4.26}$$

Letting $p \to p - m$ in the second sum gives

$$\begin{aligned} \big[\, L_m^\perp, L_n^\perp \,\big] &= \frac{1}{2}\sum_{p\in\mathbb{Z}}\Big(\, (p-n)\alpha^I_{m+n-p}\,\alpha^I_p - (p-m)\alpha^I_{m+n-p}\,\alpha^I_p\Big) \\ &= (m-n)\,\frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha^I_{m+n-p}\,\alpha^I_p\,. \end{aligned} \tag{12.4.27}$$

The right-hand side contains as a factor the Virasoro operator $L^\perp_{m+n}$, with $m+n \neq 0$. We have therefore shown that

$$\big[\, L_m^\perp, L_n^\perp \,\big] = (m-n)L^\perp_{m+n}\,, \qquad m+n\neq 0\,. \tag{12.4.28}$$

The commutator of two Virasoro operators is a Virasoro operator with mode number equal to the sum of the mode numbers of the operators entering the commutator. The above computation is not correct when $m+n=0$, in which case the answer is different. Nevertheless, as a mathematical construct, a set of operators $L_n^\perp$ with $n \in \mathbb{Z}$, satisfying (12.4.28) for *all* m and n, defines an interesting Lie algebra (see Problem 12.5). This algebra is called the *Virasoro algebra without central extension* or the Witt algebra.

Let us look into the commutator of two Virasoro operators $L_m^\perp$ and $L_n^\perp$ with $m+n=0$. If we try to repeat the above computation, each of the two sums in (12.4.26) will require ordering, each giving infinite constant contributions. While it is *not* straightforward to find the correct answer proceeding in this way, we see that, when $m+n=0$, the result in (12.4.28) must be corrected by the addition of a constant term:

$$\big[\, L_m^\perp, L_n^\perp \,\big] = (m-n)L^\perp_{m+n} + A^\perp(m)\delta_{m+n,0}\,. \tag{12.4.29}$$

A set of operators $L_n^\perp$, with $n\in\mathbb{Z}$, satisfying (12.4.29) defines the *centrally extended Virasoro algebra*. The term $A^\perp(m)\delta_{m+n,0}$ is said to be central because it commutes with all the operators. Our goal now is to compute $A^\perp(m)$. We will do this by examining the commutator of $L_m^\perp$ with $L_{-m}^\perp$, for $m\geq 1$:

$$[L_m^\perp, L_{-m}^\perp] = 2mL_0^\perp + A^\perp(m)\,. \tag{12.4.30}$$

As a warmup exercise we will first take $m=2$. Then we will discuss the general case when m is an arbitrary positive integer. To make the computation less cumbersome, let us consider the case when the index I on the oscillators just takes one value. You can quickly see that (12.4.28) holds regardless of the number of transverse light-cone directions. The result for $A^{\perp}(m)$ *will* depend on the number of such directions, but in a simple manner. Let us denote by L_m Virasoro operators built with just one type of oscillator α_n carrying no superscript:

$$L_n \equiv \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{n-p}\alpha_p\,. \tag{12.4.31}$$

Of course, we take $[\alpha_m\,,\alpha_n] = m\,\delta_{m+n,0}$. The Virasoro commutation relations then take the form

$$[\,L_m, L_n\,] = (m-n)L_{m+n} + A(m)\delta_{m+n,0}\,. \tag{12.4.32}$$

Quick Calculation 12.4. Show that L_2 and L_{-2} can be written as

$$L_2 = \frac{1}{2}\,\alpha_1\,\alpha_1 \quad + \Big(\alpha_0\alpha_2 \;+\; \alpha_{-1}\alpha_3 + \alpha_{-2}\alpha_4 + \cdots\Big)\,, \tag{12.4.33}$$

$$L_{-2} = \;\frac{1}{2}\alpha_{-1}\alpha_{-1} + \Big(\alpha_{-2}\alpha_0 + \;\alpha_{-3}\alpha_1 + \alpha_{-4}\alpha_2 + \cdots\Big)\,. \tag{12.4.34}$$

How do we compute $[L_2, L_{-2}]$? Note that each Virasoro operator splits into a term with oscillators whose modes have the same sign and the rest, enclosed by parenthesis. We know from (12.4.30) that

$$[\,L_2, L_{-2}\,] = 4L_0 + A(2)\,, \tag{12.4.35}$$

and we are just interested in finding the value of $A(2)$.

First note that the cross terms between (12.4.33) and (12.4.34) commute: the term outside of the parenthesis in L_2, for example, commutes with all the terms inside of the parenthesis in L_{-2}. This is because the former has two oscillators with positive mode numbers each of which is smaller than two. On the other hand, each term in the latter has one positively moded oscillator and one negatively moded oscillator with mode smaller or equal to minus two.

We find nontrivial commutators only between the terms inside of the parenthesis in L_2 and L_{-2}, and between the terms outside of the parenthesis. Let us consider the former first. Here we see that the nonvanishing commutators occur only when we take terms that are directly on top of each other. They are terms of the form $\alpha_{-p}\alpha_q$ and $\alpha_{-q}\alpha_p$, where both $p, q \geq 0$ and

$p \neq q$. Can the commutator of these two terms be normal-ordered without contributing an additional constant? Let' s see:

$$\begin{aligned}
[\alpha_{-p}\alpha_q\,,\alpha_{-q}\alpha_p] &= \alpha_{-p}\,\underline{\alpha_q\,\alpha_{-q}}\,\alpha_p - \alpha_{-q}\,\underline{\alpha_p\alpha_{-p}}\,\alpha_q \\
&= q\alpha_{-p}\alpha_p + \alpha_{-p}\alpha_{-q}\,\alpha_q\alpha_p - p\,\alpha_{-q}\alpha_q - \alpha_{-q}\alpha_{-p}\,\alpha_p\alpha_q\,,
\end{aligned}$$

where each of the underlined terms was replaced by the commutator plus the operators with reverse ordering. As you can see, the second and fourth terms cancel to give

$$[\alpha_{-p}\alpha_q\,,\alpha_{-q}\alpha_p] = q\,\alpha_{-p}\alpha_p - p\,\alpha_{-q}\alpha_q\,. \tag{12.4.36}$$

The result is two normal-ordered terms, with *no* additional constant. These commutators build the $4L_0$ on the right-hand side of (12.4.35). It is therefore clear that a constant can only arise from the commutator of $\frac{1}{2}\alpha_1\alpha_1$ with $\frac{1}{2}\alpha_{-1}\alpha_{-1}$:

$$\begin{aligned}
\tfrac{1}{4}\,[\,\alpha_1\alpha_1\,,\alpha_{-1}\alpha_{-1}\,] &= \tfrac{1}{4}\left([\alpha_1\alpha_1\,,\alpha_{-1}]\alpha_{-1} + \alpha_{-1}[\alpha_1\alpha_1\,,\alpha_{-1}]\right), \\
&= \tfrac{1}{2}\alpha_1\alpha_{-1} + \tfrac{1}{2}\alpha_{-1}\alpha_1\,, \\
&= \tfrac{1}{2} + \alpha_{-1}\alpha_1\,,
\end{aligned} \tag{12.4.37}$$

where the constant term arises because the first term in the second line of the right-hand side is not normal-ordered. As a result, we now know that

$$[L_2\,,L_{-2}] = 4L_0 + \tfrac{1}{2}\,. \tag{12.4.38}$$

What happens when we consider the same commutator but with the transverse Virasoro operators? Only the central term is affected. As you can see from equation (12.4.37), to get a contant term from the commutator of two objects that are quadratic in oscillators, the commutation relation between oscillators must be used *two* times. Since we are commuting $\alpha^I_{\cdots}\alpha^I_{\cdots}$ with $\alpha^J_{\cdots}\alpha^J_{\cdots}$, each commutator gives an η^{IJ}. The product $\eta^{IJ}\eta^{IJ}$ equals η^{II}, which is simply the number of transverse light-cone directions. Therefore, for transverse Virasoro operators, the constant calculated in (12.4.37) must be multiplied by $(D-2)$:

$$[L^\perp_2\,,L^\perp_{-2}] = 4L^\perp_0 + \tfrac{1}{2}(D-2)\,. \tag{12.4.39}$$

Quick Calculation 12.5. Reconsider the computation in (12.4.37), this time including Lorentz indices for the oscillators. Verify that the constant term is multiplied by $(D-2)$.

We have now developed enough confidence to compute the commutator $[L_m\,, L_{-m}]$. As before, the first step is to separate out the two kinds of terms that are involved.

Quick Calculation 12.6. Show that L_m and L_{-m} can be written as

$$L_m = \frac{1}{2}\sum_{n=1}^{m-1} \alpha_n\alpha_{m-n} \quad + \Big(\alpha_0\alpha_m \; + \; \alpha_{-1}\alpha_{m+1} \; + \cdots \Big)\,, \tag{12.4.40}$$

$$L_{-m} = \frac{1}{2}\sum_{p=1}^{m-1} \alpha_{-p}\alpha_{-(m-p)} + \Big(\alpha_{-m}\alpha_0 + \alpha_{-(m+1)}\alpha_1 + \cdots \Big)\,. \tag{12.4.41}$$

The remarks we made below (12.4.35) regarding L_2 and L_{-2} apply here as well (you should review them explicitly!). The central term in the commutator, denoted as $[L_m, L_{-m}]_{\text{cen}}$, arises only from the terms outside of the parenthesis. We therefore have

$$A(m) = [\, L_m\,, L_{-m}\,]_{\text{cen}} = \frac{1}{4}\sum_{n,p=1}^{m-1} [\alpha_n\alpha_{m-n}\,, \alpha_{-p}\alpha_{-(m-p)}]_{\text{cen}}\,. \tag{12.4.42}$$

Note that over the range of summation, the numbers $n, m-n, p$, and $m-p$ appearing on the oscillators are all positive. This commutator can be studied using the identity

$$[AB\,, CD] = CA[B, D] + C[A, D]B + A[B, C]D + [A, C]BD\,, \tag{12.4.43}$$

where we identify

$$A = \alpha_n\,, \quad B = \alpha_{m-n}\,, \quad C = \alpha_{-p}\,, \quad D = \alpha_{-(m-p)}\,. \tag{12.4.44}$$

Since the commutator of two oscillators is a number, and CA, as well as CB are normal-ordered, the first two terms in (12.4.43) do not contribute to the central term. The contribution to the central term from the last two terms is obtained by replacing AD with $[A, D]$ and BD with $[B, D]$. We thus find that

$$\begin{aligned} [AB\,, CD]_{\text{cen}} &= [A, D][B, C] + [A, C][B, D]\,, \\ &= n\delta_{n,m-p}\;(m-n)\delta_{m-n,p} + n\delta_{n,p}\;(m-n)\delta_{n,p}\,, \\ &= n(m-n)\delta_{n,m-p} + n(m-n)\delta_{n,p}\,, \\ &= n(m-n)\left(\delta_{n,m-p} + \delta_{n,p}\right). \end{aligned} \tag{12.4.45}$$

In the second line each term has two equivalent Kronecker deltas, so we replaced them by one Kronecker delta. We can now substitute this result into (12.4.42):

$$A(m) = \frac{1}{4}\sum_{n,p=1}^{m-1} n(m-n)\left(\delta_{n,m-p} + \delta_{n,p}\right). \tag{12.4.46}$$

Each of the Kronecker deltas simply deletes the sum over p; the first sets $p = m - n$, which is in the sum range, and the second sets $p = n$, which is also in the sum range. We therefore get twice the sum over n alone:

$$A(m) = \frac{1}{2}\sum_{n=1}^{m} n(m-n) = \frac{1}{2} m \sum_{n=1}^{m} n - \frac{1}{2}\sum_{n=1}^{m} n^2 , \tag{12.4.47}$$

where, for simplicity, the limit of summation was extended to include $n = m$; this is allowed since the corresponding summand is zero. To complete this calculation we need to know the sum of squares of integers:

Quick Calculation 12.7. Use mathematical induction to prove that

$$\sum_{n=1}^{m} n^2 = \frac{1}{6}(2m^3 + 3m^2 + m) . \tag{12.4.48}$$

Making use of this sum, we finally find

$$\begin{aligned} A(m) &= \frac{1}{4} m^2(m+1) - \frac{1}{12}(2m^3 + 3m^2 + m) \\ &= \frac{1}{12}(m^3 - m) . \end{aligned} \tag{12.4.49}$$

The central term vanishes for $m = 0$ and for $m = \pm 1$. There is therefore no central term in the commutator $[L_1, L_{-1}]$. For $m = 2$, the central term equals $(8-2)/12 = 1/2$, in agreement with (12.4.38). Having computed the central term $A(m)$ of the Virasoro algebra, we can now write the general commutation relations (12.4.32) explicitly:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{1}{12}\left(m^3 - m\right) \delta_{m,-n} . \tag{12.4.50}$$

For the transverse Virasoro operators, the central term is multiplied by the number of transverse light-cone directions. We therefore find

$$\boxed{\left[L_m^{\perp}, L_n^{\perp}\right] = (m-n)\, L_{m+n}^{\perp} + \frac{D-2}{12}\left(m^3 - m\right) \delta_{m,-n} .} \tag{12.4.51}$$

The Virasoro algebra is perhaps the most important algebra in string theory. In the light-cone quantization of string theory – our subject in this chapter – the Virasoro operators enter into the definition of the Lorentz generators, as we will see in section 12.5.

We conclude this section by studying how the Virasoro operators act on the string coordinates. Since quantum operators act via commutators, we must find the commutator of a Virasoro operator with the coordinate operator $X^I(\tau,\sigma)$. We will see that the Virasoro operators actually generate reparameterizations of the world-sheet.

Making use of the coordinate expansion (12.2.1) we find

$$\begin{aligned}[\, L_m^\perp \,, X^I(\tau,\sigma)\,] &= [L_m^\perp, x_0^I] + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\cos n\sigma\, e^{-in\tau}[L_m^\perp,\alpha_n^I]\,,\\ &= -i\sqrt{2\alpha'}\alpha_m^I - i\sqrt{2\alpha'}\sum_{n\neq 0}\cos n\sigma\, e^{-in\tau}\,\alpha_{m+n}^I\,, \end{aligned} \tag{12.4.52}$$

where we used (12.4.23) and (12.4.24) to evaluate the commutators. The right-hand side above can be written as a single sum:

$$\begin{aligned}[\, L_m^\perp \,, X^I(\tau,\sigma)\,] &= -i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\cos n\sigma\, e^{-in\tau}\,\alpha_{m+n}^I\,,\\ &= -i\sqrt{2\alpha'}\,\frac{1}{2}\sum_{n\in\mathbb{Z}}(e^{-in(\tau-\sigma)}+e^{-in(\tau+\sigma)})\,\alpha_{m+n}^I\,,\\ &= -i\sqrt{2\alpha'}\,\frac{1}{2}\sum_{n\in\mathbb{Z}}(e^{-i(n-m)(\tau-\sigma)}+e^{-i(n-m)(\tau+\sigma)})\,\alpha_n^I\,, \end{aligned}$$

where in the last step we let $n\to n-m$. Finally,

$$\begin{aligned}[\, L_m^\perp \,, X^I(\tau,\sigma)\,] = &-\frac{i}{2}e^{im(\tau-\sigma)}\,\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}e^{-in(\tau-\sigma)}\alpha_n^I\\ &-\frac{i}{2}e^{im(\tau+\sigma)}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}e^{-in(\tau+\sigma)}\,\alpha_n^I\,. \end{aligned}$$

To interpret this result it is necessary to express the right-hand side in terms of derivatives of the string coordinates. This is done with the help of (12.2.2):

$$\begin{aligned}[\, L_m^\perp \,, X^I(\tau,\sigma)\,] &= -\frac{i}{2}e^{im(\tau-\sigma)}\,(\dot X^I - X^{I\prime}) - \frac{i}{2}e^{im(\tau+\sigma)}(\dot X^I + X^{I\prime})\,,\\ &= -ie^{im\tau}\cos m\sigma\,\dot X^I + e^{im\tau}\sin m\sigma\, X^{I\prime}\,. \end{aligned} \tag{12.4.53}$$

This equation has taken the form

$$[\, L_m^\perp \,, X^I(\tau,\sigma)\,] = \xi_m^\tau \,\dot{X}^I + \xi_m^\sigma \,X^{I'} \,, \tag{12.4.54}$$

where

$$\begin{aligned} \xi_m^\tau(\tau,\sigma) &= -ie^{im\tau}\, \cos m\sigma \,, \\ \xi_m^\sigma(\tau,\sigma) &= \quad\; e^{im\tau}\, \sin m\sigma \,. \end{aligned} \tag{12.4.55}$$

We now claim that the interpretation of (12.4.54) is that the Virasoro operators generate reparameterizations of the world-sheet. In particular, they change the τ and σ coordinates as

$$\begin{aligned} \tau &\to \tau + \epsilon\, \xi_m^\tau(\tau,\sigma) \,, \\ \sigma &\to \sigma + \epsilon\, \xi_m^\sigma(\tau,\sigma) \,, \end{aligned} \tag{12.4.56}$$

where ϵ is an infinitesimal parameter. In order to see this, note that Taylor expansion gives

$$\begin{aligned} X^I\big(\tau + \epsilon\, \xi_m^\tau \,, \sigma + \epsilon\, \xi_m^\sigma\big) &= X^I(\tau,\sigma) + \epsilon\, (\xi_m^\tau \,\dot{X}^I + \,\xi_m^\sigma \,X^{I'}) \,, \\ &= X^I(\tau,\sigma) + \epsilon\, \big[\, L_m^\perp \,, X^I(\tau,\sigma)\,\big] \;. \end{aligned} \tag{12.4.57}$$

This equation states that the action of the Virasoro operators on the string coordinates generates the same change that would occur as a result of a reparameterization of the world-sheet. This is what we wanted to show.

What is the reparameterization generated by $L_0^\perp$? Setting $m = 0$ in (12.4.55), we find $\xi_0^\tau = -i$ and $\xi_0^\sigma = 0$. As a result, (12.4.54) gives

$$[L_0^\perp, X^I] = -i\partial_\tau X^I \,, \tag{12.4.58}$$

which we recognize as the Heisenberg equation of motion for X^I. Indeed, $L_0^\perp$ is the string Hamiltonian, and as such, it must generate time translations. It is also interesting to note that for all m, ξ_m^σ vanishes at $\sigma = 0$ and at $\sigma = \pi$. This means that the reparameterizations generated by the Virasoro operators do not change the σ coordinates at the string endpoints. The range of σ remains $[0, \pi]$.

The functions ξ_m^τ and ξ_m^σ in (12.4.55) are not real, and used in (12.4.56) they spoil the reality of the coordinates τ and σ. This complication is familiar to you from quantum mechanics. Real transformations are generated by anti-Hermitian operators. The momentum operator $\vec{p} = -i\nabla$ is Hermitian, and therefore it is the anti-Hermitian combination $i\vec{p} = \nabla$ that generates real translations. Out of the operators $L_m^\perp$ and $L_{-m}^\perp$, we can generate two anti-Hermitian combinations:

$$L_m^\perp - L_{-m}^\perp \qquad \text{and} \qquad i\,(L_m^\perp + L_{-m}^\perp)\,. \tag{12.4.59}$$

Consider the first combination. It follows from (12.4.54) that the parameters for the transformation generated by $(L_m^\perp - L_{-m}^\perp)$ are

$$\begin{aligned} \xi^\tau &= \xi_m^\tau - \xi_{-m}^\tau = 2\sin m\tau\,\cos m\sigma\,, \\ \xi^\sigma &= \xi_m^\sigma - \xi_{-m}^\sigma = 2\cos m\tau\,\sin m\sigma\,. \end{aligned} \tag{12.4.60}$$

Quick Calculation 12.8. Show that the parameters for the transformation generated by $i\,(L_m^\perp + L_{-m}^\perp)$ are

$$\begin{aligned} \xi^\tau &= \quad 2\cos m\tau\,\cos m\sigma\,, \\ \xi^\sigma &= -2\sin m\tau\,\sin m\sigma\,. \end{aligned} \tag{12.4.61}$$

Our discussion of the Virasoro operators has been quite detailed. We have examined their precise definition and seen how they affect the computation of the mass. We have determined their commutator algebra, and shown how they act on string coordinates. In the following section we will see that the Virasoro operators also enter into the definition of the operators that generate Lorentz transformations.

12.5 Lorentz generators

In Chapter 8, the Lorentz invariance of the string action allowed us to find a set of conserved world-sheet currents $\mathcal{M}_{\mu\nu}^\alpha$ labeled by the indices μ and ν, with $\mu \neq \nu$. The resulting conserved charges $M_{\mu\nu}$ were given in (8.5.14), and for open strings with $\sigma \in [0, \pi]$ they read

$$M_{\mu\nu} = \int_0^\pi \mathcal{M}_{\mu\nu}^\tau(\tau,\sigma)\,d\sigma = \int_0^\pi (X_\mu \mathcal{P}_\nu^\tau - X_\nu \mathcal{P}_\mu^\tau)\,d\sigma\,. \tag{12.5.1}$$

Making use of (12.1.1), and raising the spacetime indices,

$$M^{\mu\nu} = \frac{1}{2\pi\alpha'}\int_0^\pi \Big(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\Big)\,d\sigma\,. \tag{12.5.2}$$

Constructing suitable quantum operators can be delicate, so let us gain some intuition by thinking classically. Explicit mode expressions for X^μ and $\dot{X}^\nu$ are given in equations (9.4.17) and (9.4.18). Since $M^{\mu\nu}$ is guaranteed to be τ-independent, to evaluate (12.5.2) it suffices to pick up the τ-independent terms that arise in the products. For example,

$$X^\mu \dot{X}^\nu = x_0^\mu(\sqrt{2\alpha'}\,\alpha_0^\nu) + i\,2\alpha' \sum_{n\neq 0} \alpha_n^\mu \alpha_{-n}^\nu \frac{\cos^2 n\sigma}{n} + \cdots \tag{12.5.3}$$

where the dots represent τ-dependent terms that must fail to contribute in the calculation of $M^{\mu\nu}$. With this equation, and a similar one with μ and ν exchanged we find that upon integration (12.5.2) gives us

$$M^{\mu\nu} = x_0^\mu\, p^\nu - x_0^\nu\, p^\mu - i\sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu\right) . \tag{12.5.4}$$

Quick Calculation 12.9. Prove equation (12.5.4).

Equation (12.5.4) gives the classical Lorentz charges in terms of oscillation modes. We should ask ourselves if we can use such an expression for the quantum Lorentz charges, taking the α's to be operators. There would not be an ordering ambiguity since $\mu \neq \nu$, which implies that the oscillators commute. We will use (12.5.4) to suggest the form of the quantum Lorentz generators in light-cone gauge string theory. As was the case for the quantum particle, in addition to using light-cone coordinates we are in fact working in the light-cone *gauge*. Accordingly, the canonical structure of the theory is unusual, and there is no guarantee that we can build consistent quantum Lorentz charges. But even though we are working in the light-cone gauge, the physics of string theory must be Lorentz invariant. An inability to construct quantum Lorentz charges would mean that quantum string theory fails to be physically Lorentz invariant.

In light-cone gauge, the most delicate quantum Lorentz generator is M^{-I}. This is because the X^- coordinate is a nontrivial function of the transverse coordinates. A consistent M^{-I} must generate Lorentz transformations on the string coordinates, possibly accompanied with world-sheet reparameterizations. Indeed, in the simpler context of the point particle, the action of M^{-I} includes world-line reparameterizations. The generator M^{-I} must also satisfy the commutation relation

$$\left[\, M^{-I}, M^{-J} \,\right] = 0\,. \tag{12.5.5}$$

To find a candidate for M^{-I}, we consider equation (12.5.4) and write

$$M^{-I} \stackrel{?}{=} x_0^- p^I - x_0^I p^- - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^- \alpha_n^I - \alpha_{-n}^I \alpha_n^- \right) . \tag{12.5.6}$$

This is just a first guess, though it is a pretty good one. A satisfactory M^{-I} should be both Hermitian and normal-ordered. Let us consider Hermiticity first. The first term in the right-hand side of (12.5.6) is Hermitian since x_0^- and p^I commute. The second term, however, is not Hermitian, since x_0^I and p^- do not commute. A simple solution is to symmetrize the term by writing

$$M^{-I} \stackrel{?}{=} x_0^- \, p^I - \frac{1}{2} \left(x_0^I \, p^- + p^- x_0^I \right) - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^- \alpha_n^I - \alpha_{-n}^I \alpha_n^- \right) . \tag{12.5.7}$$

The last term above is fully Hermitian since $(\alpha_n^I)^\dagger = \alpha_{-n}^I$ and $(\alpha_n^-)^\dagger = \alpha_{-n}^-$. Consider now normal ordering. Do all of the annihilation operators appear to the right of the creation operators? They do, because the α^- oscillators are normal-ordered Virasoro operators. Finally, to be complete, we must give the definition of p^-. As stated in (12.4.11), p^- includes an undetermined constant a that reflects our difficulties in ordering the Virasoro operator $L_0^\perp$. With this definition, and writing the other minus oscillators in terms of Virasoro operators, we find

$$\begin{aligned} M^{-I} = x_0^- \, p^I &- \frac{1}{4\alpha' p^+} \left(x_0^I \left(L_0^\perp - a \right) + \left(L_0^\perp - a \right) x_0^I \right) \\ &- \frac{i}{\sqrt{2\alpha'}\, p^+} \sum_{n=1}^{\infty} \frac{1}{n} \left(L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp \right) . \end{aligned} \tag{12.5.8}$$

Now that we have a candidate for the quantum Lorentz charge M^{-I}, we can proceed to the computation of $[M^{-I}\,, M^{-J}]$.

There is much at stake in this calculation. It is in fact, one of the most important calculations in string theory. Our Lorentz charge has two undetermined parameters: the dimension D of spacetime, implicit in the sums over transverse directions, and the constant a affecting the mass of the particles. The calculation is long and uses many of our previously derived results, including the Virasoro commutation relations. We will not attempt to do it here, but the result is

$$\begin{aligned}\left[M^{-I}, M^{-J}\right] =& \frac{1}{\alpha' p^{+2}} \sum_{m=1}^{\infty} \left(\alpha_{-m}^I \alpha_m^J - \alpha_{-m}^J \alpha_m^I\right) \\ &\times \left\{ m\left[1 - \frac{1}{24}(D-2)\right] + \frac{1}{m}\left[\frac{1}{24}(D-2) - a\right]\right\}.\end{aligned} \tag{12.5.9}$$

The right hand side is a sum of terms, each of which contains the operator $\left(\alpha_{-m}^I \alpha_m^J - \alpha_{-m}^J \alpha_m^I\right)$ for a different value of m. Such terms cannot cancel each other, so the commutator above can vanish if and only if the coefficient in large braces vanishes for all positive integers m:

$$m\left[1 - \frac{1}{24}(D-2)\right] + \frac{1}{m}\left[\frac{1}{24}(D-2) - a\right] = 0\,, \quad \forall\, m \in \mathbb{Z}^+\,. \tag{12.5.10}$$

It suffices to examine this condition for $m = 1$ and $m = 2$ to conclude that each of the terms in brackets must simply vanish. We therefore have

$$1 - \frac{1}{24}(D-2) = \frac{1}{24}(D-2) - a = 0\,. \tag{12.5.11}$$

The vanishing of the first term fixes the dimension of spacetime:

$$\boxed{D = 26\,.} \tag{12.5.12}$$

The vanishing of the second term then fixes

$$\boxed{a = \frac{1}{24}(D-2) = \frac{24}{24} = 1\,.} \tag{12.5.13}$$

This value of a coincides with the one obtained in (12.4.16) by ordering $L_0^\perp$ and using the zeta function to interpret the resulting infinity. For future reference, with $a = 1$ the expression for p^- in (12.4.11) becomes

$$2\alpha' p^- \equiv \frac{1}{p^+}\left(L_0^\perp - 1\right)\,. \tag{12.5.14}$$

In addition, because of (12.1.14) the string Hamiltonian is now just

$$\boxed{H = L_0^\perp - 1\,.} \tag{12.5.15}$$

Here, of course, $L_0^\perp$ is the normal-ordered operator without additional constants. The above equation is the precise version of equation (12.1.16).

In summary, we have seen that the condition of Lorentz invariance of quantum string theory simultaneously fixes the dimension of spacetime and the constant shift in the masses of the particles. In superstring theory a similar calculation fixes the dimensionality of spacetime to the value $D = 10$. The fact that string theory cannot be a good Lorentz invariant quantum theory in any arbitrary dimension shows that string theory is very constrained.

12.6 Constructing the state space

The classical open string does not provide a reasonable theory of physics because the mass of string states assumes a continuous range of values. Only the ground state is massless in the classical theory, and this ground strate does not include any polarization labels. As a result, classical open strings have no states that can be identified as photons.

The miracle of quantum string theory is that both of these problems are solved. The continuous spectrum disappears after quantization. Candidate photon states emerge because the downward shift of the squared masses gives us massless states with polarization labels.

Let's begin by introducing the ground states of the quantum string. The quantum string shares with the quantum point particle the same set of zero modes. We have the canonical pairs (x_0^I, p^I) and (x_0^-, p^+). Therefore, just as for the point particle (see (11.3.19)), we introduce states

$$|p^+, \vec{p}_T\rangle \,, \tag{12.6.1}$$

The above states are called ground states for all values of the momenta indicated by the labels. They are also declared to be vacuum states for all the oscillators in string theory. Thus, by definition, they are annihilated by all the a_n^I:

$$a_n^I|p^+, \vec{p}_T\rangle = 0\,, \quad n \geq 1, \quad I = 2, \ldots, 25. \tag{12.6.2}$$

How do we create states from the $|p^+, \vec{p}_T\rangle$? We simply act on them with the creation operators. There are infinitely many of them, and we can operate on each state arbitrarily many times with any particular creation operator. The list of creation operators at our disposal is infinite, but it can be organized as follows:

$$\begin{array}{cccc} a_1^{(2)\dagger} & a_1^{(3)\dagger} & \cdots & a_1^{(25)\dagger} \\ a_2^{(2)\dagger} & a_2^{(3)\dagger} & \cdots & a_2^{(25)\dagger} \\ \vdots & \vdots & \vdots & \vdots \\ a_n^{(2)\dagger} & a_n^{(3)\dagger} & \cdots & a_n^{(25)\dagger} \\ \vdots & \vdots & \vdots & \vdots \end{array} \tag{12.6.3}$$

Above, the polarization index I has been enclosed by a parenthesis. The general basis state $|\lambda\rangle$ of the state space can be written as

$$|\lambda\rangle = \prod_{n=1}^{\infty}\prod_{I=2}^{25}\left(a_n^{I\,\dagger}\right)^{\lambda_{n,I}} |p^+,\, \vec{p}_T\rangle\,. \tag{12.6.4}$$

Here the non-negative integer $\lambda_{n,I}$ denotes the number of times that we act with the creation operator $a_n^{I\,\dagger}$. As you see, the state $|\lambda\rangle$ is specified by stating how many times each of the oscillators in the list (12.6.3) acts on the vacuum. This information is given by the list of non-negative integers $\lambda_{n,I}$ for all $n \geq 1$ and all $I = 2, \ldots, 25$. Since all the creation operators commute among each other, the order in which they appear is irrelevant. We restrict ourselves to the case where states only have a finite number of creation operators acting on the ground states. This means that for each state $|\lambda\rangle$ only a finite number of $\lambda_{n,I}$'s are different from zero. The string Hilbert space is an infinite-dimensional vector space: it is spanned by an infinite set of linearly independent basis states $|\lambda\rangle$. This is why string theory describes an infinite number of different particles! A general state in the Hilbert space is a linear superposition of the basis states $|\lambda\rangle$.

To understand the physical significance of the above states, consider the mass-squared operator (12.4.13), with our new found knowledge that $a = 1$:

$$M^2 = \frac{1}{\alpha'}\Big(-1 + \sum_{n=1}^{\infty} n a_n^{I\,\dagger} a_n^I\Big)\,. \tag{12.6.5}$$

The sum appearing in (12.6.5) is important enough to have its own name; it is called the number operator $N^\perp$:

$$N^\perp \equiv \sum_{n=1}^{\infty} n a_n^{I\,\dagger} a_n^I\,, \qquad M^2 = \frac{1}{\alpha'}(-1 + N^\perp)\,. \tag{12.6.6}$$

$N^\perp$ is the sum of standard number operators, one for each harmonic oscillator in the string. The most important property of $N^\perp$ is that its commutator with a creation operator gives the mode number of that operator:

$$\left[N^\perp, a_n^{I\,\dagger}\right] = n\, a_n^{I\,\dagger}\,, \tag{12.6.7}$$

as you can readily verify. In addition,

$$\left[N^\perp, a_n^{I}\right] = -n\, a_n^{I}\,. \tag{12.6.8}$$

Since the number operator is normal-ordered it annihilates the ground states:

$$N^\perp|p^+,\, \vec{p}_T\rangle = 0\,. \tag{12.6.9}$$

Note, incidentally, that the number operator $N^\perp$ enters into the definition of $L_0^\perp$ in (12.4.10). We can write

$$L_0^\perp = \alpha'\, p^I p^I + N^\perp\,. \tag{12.6.10}$$

Let's get some practice using $N^\perp$ by computing its action on some basis states. Consider, for example, its action on $a_2^{I\dagger}|p^+,\, \vec{p}_T\rangle$:

$$N^\perp a_2^{I\dagger}|p^+,\, \vec{p}_T\rangle = \left[N^\perp, a_2^{I\dagger}\right]|p^+,\, \vec{p}_T\rangle + a_2^{I\dagger} N^\perp|p^+,\, \vec{p}_T\rangle = 2a_2^{I\dagger}|p^+,\, \vec{p}_T\rangle\,.$$

The state is an eigenstate of $N^\perp$ with eigenvalue 2. Now let's try a more complicated state:

$$\begin{aligned} N^\perp a_3^{J\dagger} a_2^{I\dagger}|p^+, \vec{p}_T\rangle &= \left[N^\perp, a_3^{J\dagger}\right] a_2^{I\dagger}|p^+, \vec{p}_T\rangle + a_3^{J\dagger} N^\perp a_2^{I\dagger}|p^+, \vec{p}_T\rangle \\ &= 5\, a_3^{J\dagger} a_2^{I\dagger}|p^+, \vec{p}_T\rangle\,. \end{aligned} \tag{12.6.11}$$

It is clear that when the number operator acts on a basis state, the eigenvalue is the sum of the mode numbers of the creation operators appearing in the state. In general, for the basis state $|\lambda\rangle$ in (12.6.4) we have

$$N^\perp|\lambda\rangle = N_\lambda^\perp|\lambda\rangle\,, \tag{12.6.12}$$

where $N_\lambda^\perp$ is given by

$$N_\lambda^\perp = \sum_{n=1}^{\infty}\sum_{I=2}^{25} n\lambda_{n,I}\,. \tag{12.6.13}$$

Since $N^\perp$ enters additively into the mass-squared operator (12.6.6), we see that the oscillator with mode number n contributes n units of $1/\alpha'$ to M^2. The eigenvalues of $N^\perp$ are non-negative integers, so for all string states $M^2 \geq -1/\alpha'$.

We are now ready to discuss particular states in some detail. We will begin with the simplest ones, the gound states. These are the unique states with $N^\perp = 0$. As in the case of the point particle, the states $|p^+, \vec{p}_T\rangle$ are the one-particle states of a scalar field. They are states of a scalar particle. What is the mass of this particle? To find out, we act with M^2 on the states:

$$M^2|p^+,\, \vec{p}_T\rangle = \frac{1}{\alpha'}(-1 + N^\perp)|p^+,\, \vec{p}_T\rangle = -\frac{1}{\alpha'}\,|p^+,\, \vec{p}_T\rangle\,. \tag{12.6.14}$$

The value of M^2 is all due to the ordering constant. If this constant had vanished the mass would have been zero. In fact, massless scalars are problematic – they have not been observed in nature. Because of the ordering constant, the result is strange: $M^2 = -1/\alpha' < 0$. The wavefunction of the state tells the same story: $\psi(\tau, p^+, \vec{p}_T)$ can be set in correspondance with a classical scalar field. This scalar field, which has a negative mass-squared, is called a *tachyon*. A negative mass-squared is a sign of instability: the potential for a scalar field goes like $V = \frac{1}{2}M^2\phi^2$ ((10.2.2)), so a negative M^2 simply means that the stationary point $\phi = 0$ is unstable. The energy can be reduced by having $\phi \neq 0$. We will study this further in section 12.8.

Let us consider now the excited states with lowest M^2. Those will arise when $N^\perp$ takes the smallest possible, non-zero value, one. Remarkably, due to the ordering constant, the $N^\perp = 1$ states have $M^2 = 0$. They are massless states. Had the ordering constant taken a non-integer value, quantum string theory would have no massless states. We get states with $N^\perp = 1$ when we act with any of the transverse oscillators $a_1^{I\dagger}$ on the ground states $|p^+, \vec{p}_T\rangle$. That means that we have $D - 2 = 24$ massless states:

$$a_1^{I\dagger}|p^+,\, \vec{p}_T\rangle\,, \qquad M^2 a_1^{I\dagger}|p^+,\, \vec{p}_T\rangle = 0\,. \tag{12.6.15}$$

The general massless state is a linear combination of the above basis states:

$$\sum_{I=2}^{25} \xi_I\, a_1^{I\dagger}|p^+,\, \vec{p}_T\rangle\,. \tag{12.6.16}$$

The above expression may remind you of the photon states (10.5.21) that we found in our light-cone analysis of Maxwell theory:

$$\sum_{I=2}^{D-1} \xi_I \, a^{I\dagger}_{p^+,\,p_T} |\Omega\rangle \,. \tag{12.6.17}$$

We have a matching of states: in both cases ξ_I is an arbitrary transverse vector, and the states correspond to one another:

$$a_1^{I\,\dagger} |p^+, \vec{p}_T\rangle \quad \longleftrightarrow \quad a^{I\dagger}_{p^+,\,p_T} |\Omega\rangle \,. \tag{12.6.18}$$

Both states have exactly the same Lorentz labels, they carry the same momenta, and have the same mass. This proves a remarkable result. The open string theory quantum states include photon states! Open string theory, which started from Nambu-Goto action having no hint whatsoever of electromagnetic gauge invariance, has shown to contain Maxwell field excitations. This astonishing result is a consequence of the mass shift encountered in passing from the classical to the quantum theory of the open string.

It is worth belaboring the point. In Chapter 10 we showed that the quantum states of free Maxwell theory, the photon states, were $(D-2)$ massless states labelled by a transverse Lorentz index. The index is important; it indicates that these states transform into each other under Lorentz transformations. Exactly this kind of states have appeared in our quantization of the string. Additionally, the collection of wavefunctions $\psi^I(\tau, p^+, \vec{p}_T)$ associated to the states in (12.6.15) matches with the components A^I of the Maxwell gauge field. Finally, the Schrödinger equation for these wavefunctions matches the light-cone-gauge field equation for the Maxwell field. We will show this in the following section.

Let's conclude with an examination of the states with $N^\perp = 2$. They are built by acting on the ground states with $a_1^{I\,\dagger} a_1^{J\,\dagger}$ or with $a_2^{I\,\dagger}$. There are

$$\frac{1}{2}(D-2)(D-1) + (D-2) = \frac{1}{2}(D-2)(D+1) \tag{12.6.19}$$

such states, and their mass-squared is given by $M^2 = 1/\alpha'$. These particles are known as massive tensors, and in $D = 26$ there are 324 such states. Our results for all states with $N^\perp \le 2$ are summarized in Table 12.1.

Each state $|\lambda\rangle$ of the quantum string represents a one-particle state. Thus, the $a_1^{I\,\dagger} |p^+, \vec{p}_T\rangle$ are one-photon states, and the $a_1^{I\,\dagger} a_1^{J\,\dagger} |p^+, \vec{p}_T\rangle$ are one-particle

$N^\perp$	$\|\lambda\rangle$	$\alpha' M^2$	number of states	Wavefunction
0	$\|p^+, \vec{p}_T\rangle$	-1	1	$\psi(\tau, p^+, \vec{p}_T)$
1	$a_1^{I\dagger}\|p^+, \vec{p}_T\rangle$	0	$D-2$	$\psi^I(\tau, p^+, \vec{p}_T)$
2	$a_1^{I\dagger}a_1^{J\dagger}\|p^+, \vec{p}_T\rangle$ $a_2^{I\dagger}\|p^+, \vec{p}_T\rangle$	1	$\frac{1}{2}(D-2)(D+1)$	$\psi^{IJ}(\tau, p^+, \vec{p}_T)$ $\psi^I(\tau, p^+, \vec{p}_T)$

Table 12.1: List of open string states with $N^\perp \leq 2$.

tensor states (*not* two-photon states). There is one wavefunction for each set of discrete labels defining the states, as you can see in the table. Accordingly, there is one quantum field for each set of discrete labels. The multiparticle states are described using these quantum fields. The total quantum field theory which describes the whole set of quantum fields associated to the one-particle states of the string is called *string field theory.*

12.7 Equations of motion

Let's now elaborate on the correspondence between string states and quantum fields by considering the Schrödinger equations satisfied by the string wavefunctions. We saw in Chapter 11 that the Schrödinger equation for the point particle wavefunction is isomorphic to the classical field equation of a scalar field. We want to repeat such an analysis for the string.

To construct general time-dependent states from the string basis states we need wavefunctions. Consider, for example, a basis state

$$a_{n_1}^{I_1\,\dagger} \cdots a_{n_p}^{I_p\,\dagger} |p^+, \vec{p}_T\rangle\,. \tag{12.7.1}$$

The general time-dependent state built by superposition is

$$|\Psi, \tau\rangle = \int dp^+\, d\vec{p}_T\, \psi_{I_1,\ldots,I_p}(\tau, p^+, \vec{p}_T)\; a_{n_1}^{I_1\,\dagger} \cdots a_{n_p}^{I_p\,\dagger} |p^+, \vec{p}_T\rangle\,. \tag{12.7.2}$$

Note that the polarization indices carried by the oscillators match with the index labels of the wavefunction $\psi_{I_1,\ldots,I_p}(\tau, p^+, \vec{p}_T)$. This equation is the string analog of (11.3.22), which gave the general time-dependent state of the point particle. For general tachyon states (12.7.2) becomes

$$|\text{tachyon}, \tau\rangle = \int dp^+ \, d\vec{p}_T \, \psi(\tau, p^+, \vec{p}_T) \, |p^+, \vec{p}_T\rangle \,. \tag{12.7.3}$$

For photon states we find

$$|\text{photon}, \tau\rangle = \int dp^+ \, d\vec{p}_T \, \psi_I(\tau, p^+, \vec{p}_T) \; a_1^{I\dagger} |p^+, \vec{p}_T\rangle \,. \tag{12.7.4}$$

The Schrödinger equation satisfied by the general states (12.7.2) is

$$i\frac{\partial}{\partial \tau}|\Psi, \tau\rangle = H|\Psi, \tau\rangle \,. \tag{12.7.5}$$

In here, the Hamiltonian is given by

$$H = (L_0^\perp - 1) = \alpha' p^I p^I + N^\perp - 1 = \alpha'(p^I p^I + M^2) \,, \tag{12.7.6}$$

where we used (12.5.15) and (12.6.10). Using the explicit expression (12.7.2) for the states, equation (12.7.5) gives:

$$i\frac{\partial}{\partial \tau}\psi_{I_1,\ldots,I_p} = \left(\alpha' p^I p^I + N^\perp - 1\right) \psi_{I_1,\ldots,I_p} \,, \tag{12.7.7}$$

where $N^\perp$ denotes the eigenvalue of the operator $N^\perp$ on the state (12.7.2).

Quick Calculation 12.10. Show that equation (12.7.7) emerges. The calculation paralells that which gave (11.3.27).

For the tachyon states (12.7.3) $N^\perp = 0$ and we get

$$i\frac{\partial \psi}{\partial \tau} = \left(\alpha' p^I p^I - 1\right) \psi \,. \tag{12.7.8}$$

For the photon states (12.7.4) $N^\perp = 1$ and we get

$$i\frac{\partial \psi_I}{\partial \tau} = \alpha' p^J p^J \, \psi_I \,. \tag{12.7.9}$$

Let us now compare these Schrödinger equations with the relevant classical field equations. We showed in Chapter 10 that the scalar field equation

$$\left(\partial^2 - m^2\right) \phi = 0 \,, \tag{12.7.10}$$

could be written as (10.3.15):

$$\Big(i\,\frac{\partial}{\partial x^+} - \frac{1}{2p^+}(p^I p^I + m^2)\Big)\, \phi(x^+, p^+, \vec{p}_T) = 0\,. \tag{12.7.11}$$

Letting $x^+ = 2\alpha' p^+ \tau$ we now have

$$\Big(i\frac{\partial}{\partial \tau} - \alpha'(p^I p^I + m^2)\Big)\, \phi(\tau, p^+, \vec{p}_T) = 0\,. \tag{12.7.12}$$

This equation is precisely the same as (12.7.8) when $m^2 = -1/\alpha'$, confirming the identification of the tachyon with a scalar field. Perhaps more surprisingly, equation (12.7.12) is structurally equivalent to the Schrödinger equation (12.7.7) satisfied by *any* string wavefunction. The only difference is that the wavefunctions carry indices. As a result, the classical field equation for the field corresponding to any string state, must take the form (12.7.10), with the field carrying some indices.

This may seem strange: isn't the Maxwell classical field equation, for example, more complicated than the field equation for a scalar? Not in the light-cone gauge. We noticed this before: equation (10.5.16) showed that the transverse components of the gauge field satisfy $p^2 A^I(p) = 0$. This is of the form (12.7.10) with $m^2 = 0$. The steps leading from (12.7.10) to (12.7.12), when applied to $\partial^2 A^I = 0$ give

$$\Big(i\frac{\partial}{\partial \tau} - \alpha' p^J p^J\Big)\, A^I(\tau, p^+, \vec{p}_T) = 0\,. \tag{12.7.13}$$

This classical field equation for the Maxwell field is in complete correspondence with the Schrödinger equation (12.7.9) for the $N^\perp = 1$ wavefunctions.

12.8 Tachyons and D-brane decay

We conclude this chapter by discussing the physics of the tachyon. We explained earlier that the tachyon state has the lowest value of M^2:

$$M^2 |p^+, \vec{p}_T\rangle = -\frac{1}{\alpha'} |p^+, \vec{p}_T\rangle\,. \tag{12.8.1}$$

The field associated with this state is a scalar field. What does it mean for this scalar to have a negative M^2? The physics of the open string tachyon was a mystery ever since the discovery of string theory. A series of developments starting in 1999 have essentially elucidated the role of the open string tachyon. Let's discuss what has been learned.

Our first goal is to understand the instability of a theory with a tachyon. For this purpose we consider the Lagrangian density for a classical scalar field, along the lines of section 10.2. In some generality,

$$\mathcal{L} = -\frac{1}{2}\,\eta^{\mu\nu}\,(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi) = \frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}|\nabla\phi|^2 - V(\phi)\,, \qquad (12.8.2)$$

where $V(\phi)$ is the *potential* for the scalar field. For spatially-homogeneous field configurations, $\nabla\phi = 0$, and the potential energy density is given by the potential $V(\phi)$. The equation of motion following from variation is

$$\partial^2\phi - V'(\phi) = 0\,, \qquad (12.8.3)$$

where prime denotes derivative with respect to the argument. More explicitly,

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi - V'(\phi) = 0\,. \qquad (12.8.4)$$

Quick Calculation 12.11. Prove that equation (12.8.3) arises from variation of the action $S = \int d^D x \mathcal{L}$.

To understand the instability of the tachyon scalar field theory it suffices to consider the free part of the tachyon Lagrangian; interactions will feature later. For a free scalar field theory, the potential $V(\phi)$ takes the form

$$V(\phi) = \frac{1}{2}M^2\phi^2\,. \qquad (12.8.5)$$

Here M^2 is the mass-squared of the scalar field. The potential will change with the inclusion of interactions. When $M^2 > 0$, the potential $V(\phi)$ has a stable minimum at $\phi = 0$. When $M^2 < 0$, $V(\phi)$ has an unstable maximum at $\phi = 0$ (see Figure 12.1). We can understand the implications of such potentials by studying the equation of motion for the field. Using the specified form of V, equation (12.8.4) gives

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi - M^2\phi = 0\,. \qquad (12.8.6)$$

To make our analysis simpler, let's assume that the field ϕ depends only on time. The equation of motion then becomes

$$\frac{d^2\phi(t)}{dt^2} + M^2\phi(t) = 0\,. \tag{12.8.7}$$

When $M^2 = M \cdot M > 0$, the solutions of this equation represent oscillations:

$$\phi = A\cos(Mt) + B\sin(Mt) = C\sin(Mt + \alpha_0)\,. \tag{12.8.8}$$

This is the interpretation of a scalar field with a "good" mass-squared. The scalar field could sit at $\phi = 0$ forever because it is a stable point; if it is displaced, it simply oscillates around $\phi = 0$.

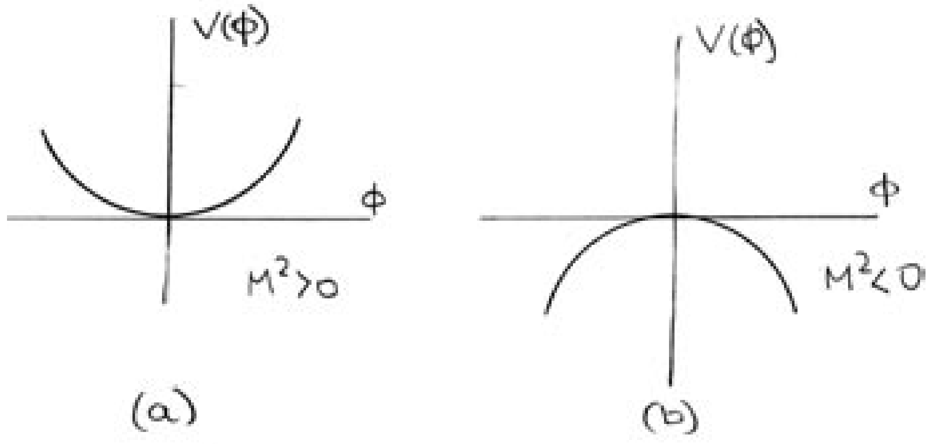

Figure 12.1: (a) A potential $V(\phi) = \frac{1}{2}M^2\phi^2$ with positive mass-squared M^2. The value $\phi = 0$ is a stable critical point. (b) A potential $V(\phi) = \frac{1}{2}M^2\phi^2$ with negative mass-squared M^2. The value $\phi = 0$ is an unstable critical point.

Consider, on the other hand, the tachyon, which is an example of a scalar with negative mass-squared. In this case it is convenient to write $M^2 = -\beta^2 = -\beta \cdot \beta$, and equation (12.8.7) becomes

$$\frac{d^2\phi(t)}{dt^2} - \beta^2\phi(t) = 0\,, \tag{12.8.9}$$

with $\beta^2 > 0$. This time the solutions are

$$\phi(t) = A\cosh(\beta t) + B\sinh(\beta t)\,. \tag{12.8.10}$$

Consider the solution $\phi(t) = \sinh(\beta t)$. At time zero ϕ is zero, but as time goes to infinity ϕ also goes to infinity. We can imagine this as the field rolling to the right of the potential in Figure 12.1(b). In fact, any nontrivial solution must necessarily have ϕ reach infinity, either in the far past or in the far future. The tachyon could stay at $\phi = 0$ forever, using the trivial solution $\phi(t) = 0$, but any infinitesimal perturbation would set it on a course to a dramatic rolloff. The value $\phi = 0$ is an allowed critical point, but it is unstable. We cannot realistically expect the tachyon to stay near $\phi = 0$ for an indefinite length of time. This is the instability of a theory that contains a tachyon. Since the mass-squared of the open string tachyon is equal to $(-1/\alpha')$, the free part of the tachyon potential is

$$V_{\text{tach}}^{\text{free}}(\phi) = -\frac{1}{2\alpha'}\phi^2\,. \tag{12.8.11}$$

A mechanical analogy works for arbitrary potentials $V(\phi)$. You can visualize the spatially-homogeneous rolling of a scalar field on a potential $V(\phi)$ by considering the motion of a particle on the potential $V(x)$, where x, replacing ϕ, is the coordinate along the motion. Indeed, the relevant equations match. For an arbitrary potential $V(\phi)$, homogeneous rolling is governed by

$$\frac{d^2\phi}{dt^2} = -V'(\phi)\,, \tag{12.8.12}$$

while the rolling of a unit-mass particle on a potential $V(x)$ is governed by Newton's second law:

$$\frac{d^2x}{dt^2} = -V'(x)\,. \tag{12.8.13}$$

The presence of a tachyon signals an instability of open string theory. More precisely, there is some instability in the theory of open strings on the background of a space-filling D25-brane. It is clear that we should try to understand the fate of this instability: once the tachyon begins to roll, where does it end? For a while, not everyone agreed that this was an urgent question. Some argued that along with the lack of fermions, the tachyon was another good reason to consider this open string theory unrealistic and not worth of much study. Some even saw the tachyon as a sign that open bosonic string theory is simply inconsistent. For quite a few years, superstring theories, the kind of string theories that also include fermions, seemed blessedly devoid of tachyons. Later studies, however, showed that open superstring tachyons appear when we try to construct realistic models based on superstrings. It then became clear that we must try to understand tachyons.

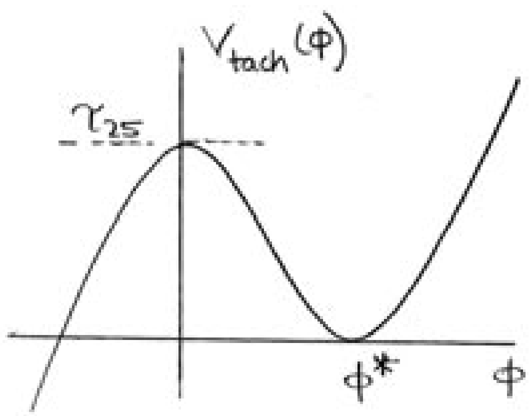

Figure 12.2: The tachyon potential for the open string theory based on a D25 brane. The configuration $\phi = 0$ represents the unstable D-brane. The stable critical point ϕ^* has zero energy.

The open string theory we have in our hands is the theory of strings on a D25-brane, a D-brane that fills all of the space dimensions. The D25-brane is a physical object, not just a mathematical construct, so it has a constant energy density $\mathcal{T}_{25}$ which, in fact, can be calculated exactly. The key insight can now be stated: the theory of open strings is, in some sense, the theory of the D25-brane itself! We have viewed tachyons as states of strings attached to a D-brane. A D-brane with open strings attached, it turns out, is an excited state of the D-brane. If this is so, a tachyon state represents an excitation that can lower the energy of the D-brane. The existence of the tachyon is telling us that the D25-brane is unstable!

Now that we see that the tachyon describes the physics of the D25-brane, the energy density of this brane is a contribution to the potential energy of the system, and must be incorporated into the tachyon potential. As a result, the potential in (12.8.11) is changed into

$$V_{\rm tach}(\phi) = \mathcal{T}_{25} - \frac{1}{2\alpha'}\,\phi^2 + \beta\phi^3 + \cdots\,. \tag{12.8.14}$$

We have also included in the tachyon potential a cubic term and represented other possible terms by dots. All the terms that are cubic or higher order in the field, represent the effect of interactions. The above potential describes correctly our statements about the D25-brane. The unstable point $\phi = 0$ represents the world with a D25-brane, and therefore has an energy density $\mathcal{T}_{25}$. To find out what happens when the tachyon starts rolling down, we need to calculate the full tachyon potential $V_{\rm tach}(\phi)$.

The physics can be anticipated before computing this potential. If the D25-brane is unstable, it will decay. The stable endpoint of this process would be a world without the D25-brane. If this is so, the tachyon potential must have a stable critical point at some $\phi = \phi^*$ with $V_{\rm tach}(\phi^*) = 0$. That stable critical point would represent a background with zero energy, a background where the D25-brane, rendered unstable by the tachyon, has disappeared completely! The conjectured form of the tachyon potential is shown in Figure 12.2.

To test directly this proposal, we must calculate the full tachyon potential in open string theory. Calculating the tachyon potential is something physicists only succeeded in doing recently. The tachyon potential was calculated using the field theory of open strings, and a critical point was found. The calculation is complicated enough that at some stage computers must be used. The proposal requires zero energy at the stable critical point, and calculations done in 2002 do in fact give an energy very close to zero. In units of the D25-brane energy, the energy at the stable vacuum is smaller than one part in ten-thousand. It is expected that an exact calculation will give exactly zero energy. With this evidence, and additional evidence obtained by other means, physicists are quite confident that the tachyon instability is the instability of the D25-brane.

What happens when the tachyon rolls down to the stable minimum and the D25-brane disappears? All the open strings must also disappear, because open string endpoints are confined to D-branes. Only closed strings can exist in the absence of D-branes. Presumably all the energy initially stored in the

D25-brane goes into the closed strings, although this has not been proven yet. While the potential at the stable critical point ϕ^* may appear from the figure to be that of a scalar with some finite and positive mass-squared, this cannot quite be correct. All particles arising as open string excitations, including the tachyon, must disappear. This shows that the theory near ϕ^* is quite subtle and novel. Physicists are trying to understand *vacuum string field theory*, which is the formulation of string theory at the vacuum ϕ^* where both the D-branes and the open strings disappear.

Further interesting facts about tachyons and D-branes have emerged. It has been shown that Dp-branes with $p < 25$ are themselves large and coherent excitations of the tachyon field. In some sense, D-branes are made of tachyons! This is also true, with minor modifications, in superstring theory. In superstring theory certain D-branes carry charge and therefore charge conservation ensures that they are stable against decay. In fact, the open string theory in the background of any such D-brane has no tachyons. However, a configuration consisting of a D-brane and a coincident, oppositely-charged anti-D-brane, is unstable: the two objects can annihilate without violating charge conservation. The open strings stretching from the D-brane to the anti-D-brane contain a tachyon – a superstring tachyon! This tachyon describes the instability of the D-brane/anti-D-brane pair. The study of D-brane/anti-D-brane annihilation plays an important role in attempts to use string theory to describe the early universe. It is thus quite possible that the tachyon will end up playing a prominent role in string cosmology.

Problems

Problem 12.1. *Heisenberg equation for momentum density.*

We verified in (12.1.20) that $\dot{X}^I = 2\pi\alpha'\mathcal{P}^{\tau I}$ follows from the Heisenberg equation $i\dot{\xi} = [\xi, H]$ applied to X^I. Calculate $\dot{\mathcal{P}}^{\tau I}$ similarly. Use this to verify that the classical equation of motion $\ddot{X}^I - X^{I\prime\prime} = 0$ holds as an operator equation.

Problem 12.2. *Testing explicitly some vanishing commutators.*

Use the mode expansion (12.2.1) and the commutation relations of the α-operators to check explicitly that equation (12.2.6) holds, namely,

$$[X^{I\prime}(\tau,\sigma), X^{J\prime}(\tau,\sigma')] = [\dot{X}^I(\tau,\sigma)\,, \dot{X}^J(\tau,\sigma')] = 0\,.$$

Problem 12.3. *Testing explicitly the main commutator.*

(a) Use the explicit mode expansions of X^I and $\mathcal{P}^{\tau J}$, together with the commutation relations (12.2.21) and (12.2.30) to show that

$$\big[\, X^I(\tau,\sigma)\,, \mathcal{P}^{\tau J}(\tau,\sigma')\,\big] = i\,\eta^{IJ}\;\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \cos n\sigma\;\cos n\sigma'\,.$$

(b) If the above result agrees with (12.1.10), we must have

$$\delta(\sigma-\sigma') = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} \cos n\sigma\;\cos n\sigma'\,. \tag{1}$$

This equation follows from the completeness of the functions $\cos n\sigma$ with $n \geq 0$ in the interval $\sigma \in [0,\pi]$. The completeness is readily explained: any function $f(\sigma)$ defined over $\sigma \in [0,\pi]$ can be extended to a function over $\sigma \in [-\pi,\pi]$ by letting $f(-\sigma) \equiv f(\sigma)$ for $\sigma \in [0,\pi]$. The resulting function is an even function of σ and by the basic result of Fourier series it can be expanded in terms of cosines. We we can therefore expand any function $f(\sigma)$ with $\sigma \in [0,\pi]$ as

$$f(\sigma) = \sum_{n=0}^{\infty} A_n \cos n\sigma\,. \tag{2}$$

Prove (1) by calculating the coefficients A_n, and substituting the result back into the right hand side of (2).

Problem 12.4. *Analytic continuation of the ζ function.*

Consider the definition of the gamma function $\Gamma(s) = \int_0^\infty dt\, e^{-t} t^{s-1}$. Let $t \to nt$ in this integral, and use the resulting equation to prove that

$$\Gamma(s)\,\zeta(s) = \int_0^\infty dt \frac{t^{s-1}}{e^t - 1}\,, \quad \Re(s) > 1\,. \tag{1}$$

Verify also that

$$\frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + \mathcal{O}(t^2)\,. \tag{2}$$

Use the above equations to show that for $\Re(s) > 1$

$$\begin{aligned}\Gamma(s)\,\zeta(s) = &\int_0^1 dt\, t^{s-1}\Big(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\Big)\\ &+ \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12\,(s+1)} + \int_1^\infty dt \frac{t^{s-1}}{e^t-1}\,.\end{aligned}$$

Explain why the above right-hand side is well-defined for $\Re(s) > -2$. It follows that this right-hand side defines the analytic continuation of the left-hand side to $\Re(s) > -2$. Recall the pole structure of $\Gamma(s)$ (see Problem 3.6) and use it to show that $\zeta(0) = -1/2$ and that $\zeta(-1) = -1/12$.

Problem 12.5. *The Virasoro algebra is a Lie algebra.*

Given a vector space L with elements $x, y, z, \cdots$ and a bilinear bracket $[\cdot\,,\,\cdot]$ that takes two elements of L and yields another element of L, we have a Lie algebra if

(i) Antisymmetry: $[x, y] = -[y, x]$ for all elements x, y of L, and,

(ii) Jacobi Identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, for all elements x, y, z of L.

Consider the vector space L spanned by the Virasoro operators with modes $n \in \mathbb{Z}$. Show that the commutators in (12.4.28), assumed to hold for all values of m and n, define a Lie algebra. Then, consider the commutators in (12.4.50), and show that they also define a Lie algebra.

Problem 12.6. *Consistency conditions on the Virasoro anomaly.*

The Virasoro commutation relations take the form

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0}\,, \tag{1}$$

where $A(m)$ is a function of m that was calculated directly in this chapter. The purpose of this problem is to find the constraints on $A(m)$ that follow from the condition that (1) defines a Lie algebra.

(a) What does the antisymmetry requirement on a Lie algebra tell you about $A(m)$? What is $A(0)$?

(b) Consider now the Jacobi identity for generators L_m, L_n, L_k, with $m+n+k=0$. Show that

$$(m-n)A(k) + (n-k)A(m) + (k-m)A(n) = 0\,. \tag{2}$$

(c) Use equation (2) to show that $A(m) = \alpha\, m$ and $A(m) = \beta m^3$, for constants α and β, yield consistent central extensions.

(d) Consider equation (2) with $k=1$. Show that $A(1)$ and $A(2)$ determine all $A(n)$'s.

Problem 12.7. *Exercises with Virasoro operators.*

(a) Use the Virasoro algebra (12.4.50) to show that if a state is annihilated by L_1 and L_2 it is annihilated by all L_n's with $n \geq 1$.

(b) Consider the Virasoro operators L_0, L_1 and L_{-1}. Write out the three relevant commutators. Do these operators form a subalgebra of the Virasoro algebra ? Is there a central term in here ? Calculate the result of acting with each of these three operators on the zero-momentum vacuum state $|0\rangle$.

Problem 12.8. *Reparameterizations generated by Virasoro operators.*

(a) Consider the string at $\tau = 0$. Which of the combinations in (12.4.59) reparameterize the σ coordinate of the string while keeping $\tau = 0$? When $\tau = 0$ is preserved, the world-sheet reparameterization is actually a *string* reparameterization. Show that the generators of these reparameterizations form a subalgebra of the Virasoro algebra.

(b) Find the complete set of *world-sheet* reparameterizations that leave the midpoint $\sigma = \pi/2$ of the $\tau = 0$ open string fixed.

Problem 12.9. *Reparameterizations and constraints.*

(a) Verify that the reparameterization parameters in (12.4.55) satisfy the relations (ommitting the subscript m for convenience)

$$\dot{\xi}^{\tau} = \xi^{\sigma\prime}\,, \qquad \dot{\xi}^{\sigma} = \xi^{\tau\prime}\,.$$

(b) Think of the reparameterizations (12.4.56) generated by the Virasoro operators as a change of coordinates

$$\tau' = \tau + \epsilon\,\xi^{\tau}(\tau,\sigma)\,, \quad \sigma' = \sigma + \epsilon\,\xi^{\sigma}(\tau,\sigma)\,.$$

Note that for infinitesimal ϵ the above equations also imply that

$$\tau = \tau' - \epsilon\,\xi^{\tau}(\tau',\sigma')\,, \quad \sigma = \sigma' - \epsilon\,\xi^{\sigma}(\tau',\sigma')\,.$$

Show that the classical constraints

$$\partial_\tau X \cdot \partial_\sigma X = 0\,, \qquad (\partial_\tau X)^2 + (\partial_\sigma X)^2 = 0\,.$$

assumed to hold in (τ,σ) coordinates, also hold in (τ',σ') coordinates (to order ϵ).

Problem 12.10. *Unoriented open strings.*

The open string $X^\mu(\tau,\sigma)$ with $\sigma \in [0,\pi]$ and fixed τ, is a parameterized curve in spacetime. The orientation of a string is the direction of increasing σ.

(a) Consider now the open string $X^\mu(\tau,\pi-\sigma)$ at the same τ. How is this second string related to the first string above? How are their endpoints and orientations related? Make a rough sketch showing the original string as a continuous curve in spacetime, and the second string as a dashed curve in spacetime.

Assume there is an orientation reversing *twist* operator Ω such that

$$\Omega\, X^I(\tau,\sigma)\,\Omega^{-1} = X^I(\tau,\pi-\sigma)\,. \tag{1}$$

Moreover, assume that

$$\Omega\, x_0^-\,\Omega^{-1} = x_0^-\,, \quad \Omega\, p^+\,\Omega^{-1} = p^+\,. \tag{2}$$

(b) Use the open string oscillator expansion (12.2.1) to calculate

$$\begin{aligned} &\Omega\, x_0^I\, \Omega^{-1} = \cdots \\ &\Omega\, \alpha_0^I\, \Omega^{-1} = \cdots \\ &\Omega\, \alpha_n^I\, \Omega^{-1} = \cdots \qquad n \neq 0\,. \end{aligned}$$

(c) Show that $\Omega\, X^-(\tau,\sigma)\, \Omega^{-1} = X^-(\tau, \pi-\sigma)$. Since $\Omega\, X^+(\tau,\sigma)\, \Omega^{-1} = X^+(\tau,\pi-\sigma)$, equation (1) actually holds for all string coordinates.

We say that twist, or orientation reversal, is a symmetry of open string theory because the transformations $X^\mu(\tau,\sigma) \to X^\mu(\tau,\pi-\sigma)$ leave the string action invariant (do you see this?).

(d) Assume that the ground states are twist invariant:

$$\Omega|p^+,\vec{p}_T\rangle = \Omega^{-1}|p^+,\vec{p}_T\rangle = |p^+,\vec{p}_T\rangle$$

List the open string states for $N^\perp \leq 3$, and give their twist eigenvalues. Prove that, in general,

$$\Omega = (-1)^{N^\perp}\,.$$

(e) A state is said to be *unoriented* if it is twist invariant. If you are commissioned to build a theory of unoriented open strings, which of the states in part (d) would you have to discard? In general, which levels of the original string state space must be discarded?

Problem 12.11. *Tachyon potentials.*

Consider scalar field theories of the form

$$S = \int d^D x \Big(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\Big)\,. \tag{1}$$

We will examine three examples of scalar potentials

$$V_1(\phi) = \frac{1}{\alpha'}\,\frac{1}{3\phi_0}\,(\phi-\phi_0)^2\Big(\phi + \frac{1}{2}\phi_0\Big) \tag{2a}$$

$$V_2(\phi) = -\frac{1}{4\alpha'}\,\phi^2\,\ln\Big(\frac{\phi^2}{\phi_0^2}\Big) \tag{2b}$$

$$V_3(\phi) = \frac{1}{4\alpha'}\;\phi_0^2\,\Big(\frac{\phi^2}{\phi_0^2} - 1\Big)^2 \tag{2c}$$

where ϕ_0 is a (positive) constant. For each of the three potentials V_i:

(a) Plot $V_i(\phi)$ as a function of ϕ.

(b) Find the critical points of the potential and the values of the potentials at those points.

(c) Each critical point represents a possible background for the scalar field theory. At each critical point $\bar{\phi}$ expand the action for fluctuations of ϕ around this point, that is, let $\phi = \bar{\phi} + \psi$ where the fluctuation ψ is small. The quadratic term in ψ (with no derivatives) can be used to read the mass of the scalar particle.

The potential V_1 is a rough model for the tachyon potential on an unstable bosonic D-brane. V_2 is the true tachyon potential on an unstable bosonic D-brane! The potential V_3 is a rough model for the tachyon potential on a pair of coincident D-brane and anti-D-brane in superstring theory.

Chapter 13

Quantum Relativistic Closed Strings

Except for shared position and momentum zero modes, the operator content of quantum closed strings contains two commuting copies of the open string operators. Even in the light-cone gauge the reparameterization invariance cannot be fully fixed: there is no natural way to choose a starting point for a closed string. As a result, the closed string spectrum is subject to the constraint $L_0^\perp - \overline{L}_0^\perp = 0$*, selecting the states that are invariant under rigid rotations of the string. We find that the massless closed string quantum states include one-particle graviton states, making string theory a quantum gravity theory. Additionally, we find massless Kalb-Ramond and dilaton states. The dilaton state controls the strength of string interactions. We conclude by giving a brief introduction to superstring theory.*

13.1 Mode expansions and commutation relations

When it was first discovered, string theory was thought to be a theory of strongly-interacting particles. A theory of open strings is only consistent if closed strings are also included. But there was a problem with closed strings: among the excitations of closed strings there were massless states with spin two. No known particle had these properties. Despite much effort, all attempts to eliminate these closed string states from the spectrum failed.

It turns out that these massless states can be identified as graviton states, and physicists soon realized that closed strings could be a theory of quantum gravity. In this chapter we quantize the relativistic closed string and see how graviton states emerge. Much of the quantization procedure will parallel our quantization of the open string in Chapter 12, but there are a number of new features.

Let us begin by recalling some of the important facts about closed strings that you learned in Chapter 9. We considered at that time a family of gauges (see (9.2.16)) defined by the conditions

$$n \cdot X = \alpha'(n \cdot p)\,\tau\,, \qquad (n \cdot p)\,\sigma = 2\pi \int_0^\sigma n \cdot \mathcal{P}(\tau, \tilde{\sigma})\, d\tilde{\sigma}\,. \tag{13.1.1}$$

The second condition indicates that the closed string parameter σ spans an interval of length 2π. We could take, for example,

$$\sigma \in [0, 2\pi]\,, \tag{13.1.2}$$

with $\sigma = 0$ and $\sigma = 2\pi$ representing the same point on the closed string. The range $\sigma \in [0, 2\pi]$ for the closed string is twice the range $\sigma \in [0, \pi]$ used for open strings. We found that the conditions (13.1.1) did not fully fix the parameterization of the closed strings. Unlike open strings, closed strings do not have a special point that can be selected as $\sigma = 0$. We used this arbitrariness to our advantage: once we selected some $\sigma = 0$ on one closed string, we could impose the constraint $X' \cdot \dot{X} = 0$ by suitably choosing the $\sigma = 0$ point on all of the other closed strings on the world-sheet. After this, we still had the ability to let $\sigma \to \sigma + \sigma_0$, with some constant σ_0 that is the same for all strings. This rigid rotation of the lines of constant σ is a reparameterization invariance of the closed string action that cannot be fixed. When we build the quantum states of the closed string, this will result in a constraint on states.

The condition $X' \cdot \dot{X} = 0$, together with the parameterization conditions (13.1.1) implied $X'^2 + \dot{X}^2 = 0$. We thus obtained the familiar conditions

$$(\dot{X} \pm X')^2 = 0\,, \tag{13.1.3}$$

and the momentum densities became simple derivatives of the coordinates:

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X'^\mu, \quad \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu. \tag{13.1.4}$$

Finally, all the string coordinates satisfy the wave equation

$$\Big(\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial\sigma^2}\Big)X^\mu=0\,. \tag{13.1.5}$$

Let us now consider the classical solution to the equation of motion for the closed string. The general solution to the wave equation is

$$X^\mu(\tau,\sigma)=X_L^\mu(\tau+\sigma)+X_R^\mu(\tau-\sigma)\,, \tag{13.1.6}$$

where X_L^μ (the L stands for left-moving) is a wave moving towards more-negative σ and X_R^μ (the R stands for right-moving) is a wave moving towards more-positive σ. For open strings, the left-moving and right-moving waves were related to each other by the boundary conditions at the endpoints. The closed string has no endpoints, but we do have a periodicity condition to work with. To describe closed strings we must compactify the world-sheet coordinate σ:

$$\sigma\sim\sigma+2\pi\,. \tag{13.1.7}$$

Two points on the world-sheet whose difference of σ-coordinates is a multiple 2π are the same point. We can use any interval of the form $[\sigma_0,\sigma_0+2\pi]$ to describe the closed strings. When we include the τ-coordinate, the identification of points on the world-sheet is given by

$$(\tau,\sigma)\sim(\tau,\sigma+2\pi)\,. \tag{13.1.8}$$

We demand that X^μ assumes the same value at any two coordinates that represent the same point on the worldsheet:

$$X^\mu(\tau,\sigma)=X^\mu(\tau,\sigma+2\pi)\,,\qquad \text{for all}\;\;\sigma,\tau\,. \tag{13.1.9}$$

This condition is both easier to deal with and easier to interpret than the naive condition $X^\mu(\tau,0)=X^\mu(\tau,2\pi)$. The periodicity condition (13.1.9) is appropriate for strings that are propagating in a simply-connected space, a space where every closed string can be continuously shrunk to a point. Minkowski space, for example, is simply-connected. But if there is a spatial direction that has been curled up into a circle, then closed strings wrapped around the circle cannot be shrunk away. In such a case, the space is not simply-connected, and the periodicity condition (13.1.9) must be modified.

We will consider this important possibility in detail in Chapter 16.

We will now show that the periodicity condition (13.1.9) induces a small but significant constraint relating the left-moving and the right-moving waves. Let's define two new variables,

$$\begin{aligned} u &= \tau + \sigma\,, \\ v &= \tau - \sigma\,. \end{aligned} \tag{13.1.10}$$

In terms of these variables, equation (13.1.6) becomes

$$X^\mu(u,v) = X_L^\mu(u) + X_R^\mu(v)\,. \tag{13.1.11}$$

When $\sigma \to \sigma + 2\pi$, the variables u and v increase and decrease by 2π, respectively. As a result, the periodicity condition (13.1.9) gives

$$X_L^\mu(u) + X_R^\mu(v) = X_L^\mu(u+2\pi) + X_R^\mu(v-2\pi)\,, \tag{13.1.12}$$

or, equivalently,

$$X_L^\mu(u+2\pi) - X_L^\mu(u) = X_R^\mu(v) - X_R^\mu(v-2\pi)\,. \tag{13.1.13}$$

This equation establishes that the left-moving and right-moving waves are in fact dependent on each other: if one fails to be periodic, the other has to fail by the same amount. Since u and v are independent variables, both the u-derivative of the right-hand side and the v-derivative of the left-hand side must vanish. As a consequence, we find that both $X_L^{\mu\prime}(u)$ and $X_R^{\mu\prime}(v)$ are strictly periodic functions with period 2π (for functions of a single variable primes denote derivatives with respect to the argument). We can therefore write the mode expansions

$$\begin{aligned} X_L^{\mu\prime}(u) &= \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \bar{\alpha}_n^\mu e^{-inu}\,, \\ X_R^{\mu\prime}(v) &= \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-inv}\,. \end{aligned} \tag{13.1.14}$$

A set of barred α-modes was introduced for the expansion of $X_L^{\mu\prime}(u)$. Even though they are written identically, the un-barred α-modes used in the expansion of $X_R^{\mu\prime}(v)$ have no relation to the open string modes of Chapter 12.

In closed string theory we need two sets of α-modes, barred and un-barred. We integrate equations (13.1.14) to find

$$\begin{aligned} X_L^\mu(u) &= \frac{1}{2}\,x_0^{L\mu} + \sqrt{\frac{\alpha'}{2}}\,\bar\alpha_0^\mu\,u + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\bar\alpha_n^\mu}{n}e^{-inu}\,,\\ X_R^\mu(v) &= \frac{1}{2}\,x_0^{R\mu} + \sqrt{\frac{\alpha'}{2}}\,\alpha_0^\mu\,v + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-inv}\,, \end{aligned} \tag{13.1.15}$$

where the coordinate zero modes $x_0^{L\mu}$ and $x_0^{R\mu}$ have appeared as constants of integration. These are somewhat puzzling, after all, in open string theory there was a single coordinate zero mode whose canonical conjugate was the momentum of the string. We will see that only the sum of the two zero modes plays a role here. If the space is not simply connected, however, each of the coordinate zero modes plays a role, as we will see in Chapter 16.

The aperiodicity of X_R^μ and of X_L^μ is a consequence of the linear terms appearing in (13.1.15). Condition (13.1.13) constrains these terms giving

$$2\pi\sqrt{\frac{\alpha'}{2}}\;\bar\alpha_0^\mu = 2\pi\sqrt{\frac{\alpha'}{2}}\;\alpha_0^\mu\,, \tag{13.1.16}$$

and therefore

$$\boxed{\bar\alpha_0^\mu = \alpha_0^\mu\,.} \tag{13.1.17}$$

Due to this equality, quantum closed string theory has only *one* momentum operator. As we will soon see, this means that canonical quantization works consistently with only *one* coordinate zero-mode operator.

We can now assemble the mode expansion for $X^\mu(\tau,\sigma)$ by substituting (13.1.15) into (13.1.6):

$$\begin{aligned} X(\tau,\sigma) = \frac{1}{2}\,x_0^{L\mu} &+ \sqrt{\frac{\alpha'}{2}}\,\bar\alpha_0^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\bar\alpha_n^\mu}{n}e^{-in(\tau+\sigma)}\\ + \frac{1}{2}\,x_0^{R\mu} &+ \sqrt{\frac{\alpha'}{2}}\,\alpha_0^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in(\tau-\sigma)}\,. \end{aligned} \tag{13.1.18}$$

With $\bar\alpha_0^\mu = \alpha_0^\mu$, we find

$$X^\mu(\tau,\sigma) = \frac{1}{2}(x_0^{L\mu} + x_0^{R\mu}) + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\alpha_n^\mu e^{in\sigma} + \bar\alpha_n^\mu e^{-in\sigma})\,. \tag{13.1.19}$$

As expected, X^μ is a periodic function of σ with period 2π. The canonically conjugate momentum density is

$$\mathcal{P}^{\tau\mu}(\tau,\sigma) = \frac{1}{2\pi\alpha'}\dot{X}^\mu(\tau,\sigma) = \frac{1}{2\pi\alpha'}(\sqrt{2\alpha'}\,\alpha_0^\mu + \cdots)\,, \tag{13.1.20}$$

where the dots represent the terms in $\dot{X}^\mu$ that integrate to zero over the interval $\sigma \in [0, 2\pi]$ and therefore do not contribute to the evaluation of the total momentum:

$$p^\mu = \int_0^{2\pi} \mathcal{P}^{\tau\mu}(\tau,\sigma)d\sigma = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\,\sqrt{2\alpha'}\,\alpha_0^\mu = \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu\,. \tag{13.1.21}$$

Thus we have the relation

$$\boxed{\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}\;p^\mu\,.} \tag{13.1.22}$$

This differs from the open string analog (12.2.22) by a factor of two, but the idea is the same: α_0^μ is proportional to the spacetime momentum carried by the string.

There is only one momentum variable, and thus in the quantum theory there is only one momentum operator. We should also have only one conjugate coordinate zero mode. Thus, despite our left-right decomposition of the solution to the wave equation, x_0^L and x_0^R cannot both be independent variables. Their sum is the only object that appears in (13.1.19), so it must be the relevant coordinate zero mode. Therefore, without loss of generality, we may set

$$x_0^{L\mu} = x_0^{R\mu} \equiv x_0^\mu\,. \tag{13.1.23}$$

On this account, equation (13.1.19) can be put into its final form:

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma})\,. \tag{13.1.24}$$

It is convenient at this stage to record the τ- and σ-derivatives of the coordinates. With the help of (13.1.6) we note that

$$\begin{aligned}\dot{X}^\mu &= X_L^{\mu\prime}(\tau+\sigma) + X_R^{\mu\prime}(\tau-\sigma)\,,\\ X^{\mu\prime} &= X_L^{\mu\prime}(\tau+\sigma) - X_R^{\mu\prime}(\tau-\sigma)\,.\end{aligned} \tag{13.1.25}$$

Adding and subtracting these equations, and using (13.1.14), we find

$$\begin{aligned} \dot{X}^\mu + X^{\mu\prime} &= 2X_L^{\mu\prime}(\tau+\sigma) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \bar{\alpha}_n^\mu \, e^{-in(\tau+\sigma)}\,, \\ \dot{X}^\mu - X^{\mu\prime} &= 2X_R^{\mu\prime}(\tau-\sigma) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-in(\tau-\sigma)}\,. \end{aligned} \tag{13.1.26}$$

Note that the barred oscillators do not mix with the un-barred oscillators in these combinations of derivatives. We have tailored the normalization constants to arrive at the above relations. They are completely analogous to the open string expansions (12.2.2). This will allow us to obtain some closed string commutators without doing any new computations. .

Let us now turn to the quantization of the closed string theory. The canonical commutation relations take the same form as in open string theory. For the transverse light-cone coordinates and momenta we set

$$[X^I(\tau,\sigma)\,, \mathcal{P}^{\tau J}(\tau,\sigma')] = i\delta(\sigma-\sigma')\eta^{IJ}\,, \tag{13.1.27}$$

and, as usual, we set to zero the commutator of coordinates with coordinates and the commutator of momenta with momenta. For zero-modes, we also have $[x_0^-\,, p^+] = -i$. Since the commutation relations did not change, equations (12.2.11) and (12.2.12) are valid here. The first of these is

$$[\,(\dot{X}^I \pm X^{I'})(\tau,\sigma)\,,\,(\dot{X}^J \pm X^{J'})(\tau,\sigma')\,] = \pm 4\pi\alpha'\, i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma')\,. \tag{13.1.28}$$

In fact, this time the situation is simpler. Equations (13.1.28) hold for $\sigma,\sigma' \in [0,2\pi]$ since the string coordinates are defined for this full interval. Moreover, the mode expansions (13.1.26) also hold for the full interval. Since the combinations of derivatives take the same exact form they did for open strings, the above equation leads to identically-looking commutation relations. The oscillators, however, are barred when we use the top sign and unbarred when we use the lower sign. The result is therefore

$$\boxed{\begin{aligned} \left[\,\bar{\alpha}_m^I, \bar{\alpha}_n^J\,\right] &= m\,\delta_{m+n,0}\,\eta^{IJ}\,, \\ \left[\,\alpha_m^I, \alpha_n^J\,\right] &= m\,\delta_{m+n,0}\,\eta^{IJ}\,. \end{aligned}} \tag{13.1.29}$$

On account of the expansions (13.1.26), we call the $\bar{\alpha}$ operators left-moving operators, and we call the α operators right-moving operators. Each of these sets matches the operator content of an open string theory. The commutation relations also take the form we would expect from open string theory. Closed string theory thus has the operator content of two copies of open string theory, except for zero modes. The momentum zero modes are equal ($\alpha_0^I = \bar{\alpha}_0^I$), and there is only one set of coordinate zero modes x_0^I, x_0^-.

Equation (12.2.12) states that combinations of derivatives with opposite signs commute. In the present case this leads to the result that left-moving and right-moving oscillators commute:

$$\boxed{\left[\, \alpha_m^I \,, \bar{\alpha}_n^J \,\right] = 0 \,.} \tag{13.1.30}$$

We can define canonical creation and annihilation operators just as we did for open strings:

$$\begin{aligned} \alpha_n^I &= a_n^I \sqrt{n}\,, \qquad \text{and} \qquad \alpha_{-n}^I = a_n^{I\dagger} \sqrt{n}\,, \qquad n \geq 1\,. \\ \bar{\alpha}_n^I &= \bar{a}_n^I \sqrt{n}\,, \qquad \text{and} \qquad \bar{\alpha}_{-n}^I = \bar{a}_n^{I\dagger} \sqrt{n}\,, \qquad n \geq 1\,. \end{aligned} \tag{13.1.31}$$

The commutation relations are then the expected

$$\begin{aligned} \left[\, \bar{a}_m^I \,, \bar{a}_n^J \,\right] &= \delta_{m,n}\, \eta^{IJ} \,, \\ \left[\, a_m^I \,, a_n^J \,\right] &= \delta_{m,n}\, \eta^{IJ} \,, \\ \left[\, a_m^I \,, \bar{a}_n^J \,\right] &= 0 \,. \end{aligned} \tag{13.1.32}$$

The commutators involving x_0^I can be found following steps analogous to those used for open strings. This time (see Problem 13.1) we find the vanishing of $[x_0^I, \alpha_n^J]$ and $[x_0^I, \bar{\alpha}_n^J]$ when $n \neq 0$, and

$$\left[\, x_0^I \,, \alpha_0^J \,\right] = \left[\, x_0^I \,, \bar{\alpha}_0^J \,\right] = i\sqrt{\frac{\alpha'}{2}}\, \eta^{IJ} \quad \longrightarrow \quad \left[\, x_0^I \,, p^J \,\right] = i\eta^{IJ} \,, \tag{13.1.33}$$

where the expression to the right arises because of (13.1.22).

What is the light-cone closed string Hamiltonian? We know that p^- generates X^+ translations, and that $X^+ = \alpha' p^+ \tau$. As a result $\partial_\tau = \alpha' p^+ \partial_{X^+}$, and the Hamiltonian must be given by

$$H = \alpha' p^+ p^- \,. \tag{13.1.34}$$

In order to find the normal-ordered version of this Hamiltonian, we now turn to the transverse Virasoro operators of closed string theory.

13.2 Closed string Virasoro operators

We learned in Chapter 12 that the open string transverse Virasoro operators were just the modes α_n^- of the light-cone coordinate X^-. For closed strings coordinates we have two types of modes, barred and unbarred. This is also true for the closed string X^- coordinates: we have α_n^- and $\bar{\alpha}_n^-$ modes, and therefore we expect to have two sets of Virasoro operators. On the other hand, since $\alpha_0^- = \bar{\alpha}_0^-$, a surprise awaits us regarding the Virasoro operators with mode number zero.

To begin our analysis we need an expression that relates X^- to the transverse coordinates. The requisite formula was obtained in (9.5.5). With $\beta = 1$, as appropriate for closed strings, it reads

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{\alpha'}\frac{1}{2p^+}(\dot{X}^I \pm X^{I\prime})^2 \,. \tag{13.2.1}$$

We define Virasoro operators following the pattern in equation (9.5.17):

$$\begin{aligned}(\dot{X}^I + X^{I\prime})^2 &= 4\alpha' \sum_{n\in\mathbb{Z}}\Big(\frac{1}{2}\sum_{p\in\mathbb{Z}} \bar{\alpha}_p^I \bar{\alpha}_{n-p}^I\Big) e^{-in(\tau+\sigma)} \equiv 4\alpha' \sum_{n\in\mathbb{Z}} \overline{L}_n^\perp e^{-in(\tau+\sigma)} \,,\\ (\dot{X}^I - X^{I\prime})^2 &= 4\alpha' \sum_{n\in\mathbb{Z}}\Big(\frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_p^I \alpha_{n-p}^I\Big) e^{-in(\tau-\sigma)} \equiv 4\alpha' \sum_{n\in\mathbb{Z}} L_n^\perp e^{-in(\tau-\sigma)} \,.\end{aligned} \tag{13.2.2}$$

In each of the above equations, the equality requires a small calculation using (13.1.26), and the second relation is a definition. More explicitly,

$$\overline{L}_n^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}} \bar{\alpha}_p^I \bar{\alpha}_{n-p}^I \,, \qquad L_n^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_p^I \alpha_{n-p}^I \,. \tag{13.2.3}$$

These are the two sets of Virasoro operators of closed string theory. Plugging the definitions in (13.2.2) back into (13.2.1) we obtain

$$\dot{X}^- + X^{-\prime} = \frac{2}{p^+}\sum_{n\in\mathbb{Z}} \overline{L}_n^\perp \, e^{-in(\tau+\sigma)} \,, \quad \dot{X}^- - X^{-\prime} = \frac{2}{p^+}\sum_{n\in\mathbb{Z}} L_n^\perp \, e^{-in(\tau-\sigma)} \,. \tag{13.2.4}$$

On the other hand, the derivatives of X^-, as those of any other closed string coordinate, can be expanded along the lines of (13.1.26) to give

$$\dot{X}^- + X^{-\prime} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \bar{\alpha}_n^-\, e^{-in(\tau+\sigma)}\,, \quad \dot{X}^- - X^{-\prime} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^-\, e^{-in(\tau-\sigma)}\,. \tag{13.2.5}$$

We compare equations (13.2.4) and (13.2.5) to read the expressions for the minus oscillators:

$$\sqrt{2\alpha'}\,\bar{\alpha}_n^- = \frac{2}{p^+}\,\overline{L}_n^\perp\,, \qquad \sqrt{2\alpha'}\,\alpha_n^- = \frac{2}{p^+}\,L_n^\perp\,, \qquad \forall n\,. \tag{13.2.6}$$

For $n=0$, however, there is a constraint. Since $\alpha_0^- = \bar{\alpha}_0^-$, we must have

$$\boxed{\; L_0^\perp = \overline{L}_0^\perp\,. \;} \tag{13.2.7}$$

If you look at the definitions of $L_0^\perp$ and $\overline{L}_0^\perp$ in (13.2.3), you will realize that these two operators are clearly very different from each other. What does it mean that they must be equal? Since operators are ultimately defined by how they act on states, the meaning of the equality (13.2.7) is that any state $|\lambda,\bar{\lambda}\rangle$ of the closed string must satisfy $L_0^\perp|\lambda,\bar{\lambda}\rangle = \overline{L}_0^\perp|\lambda,\bar{\lambda}\rangle$. This is therefore a constraint on the state space of the theory: "states" that do not satisfy this constraint do not in fact belong to the state space.

To fix the ordering ambiguities in the operators $\overline{L}_0^\perp$ and $L_0^\perp$ we define them to be ordered operators without any additional constants:

$$L_0^\perp = \frac{\alpha'}{4}p^Ip^I + N^\perp\,, \qquad \overline{L}_0^\perp = \frac{\alpha'}{4}p^Ip^I + \overline{N}^\perp\,. \tag{13.2.8}$$

Here $\overline{N}^\perp$ and $N^\perp$ are the number operators that are associated with the barred and un-barred operators, respectively:

$$N^\perp \equiv \sum_{n=1}^{\infty} n a_n^{I\dagger}\, a_n^I\,, \qquad \overline{N}^\perp \equiv \sum_{n=1}^{\infty} n\bar{a}_n^{I\dagger}\,\bar{a}_n^I\,. \tag{13.2.9}$$

While we will not go through the trouble of proving it, the critical dimension for closed strings turns out to be $D=26$. This follows from the requirement that the quantum theory be Lorentz invariant. It is no coincidence that the critical dimension for closed strings coincides with the critical dimension for open strings. It means the both types of strings can co-exist. In fact, since open strings can, in general, close to form closed strings, it would have been quite strange if the critical dimensions did not agree.

The constant ambiguities due to the ordering of $\overline{L}_0^\perp$ and $L_0^\perp$ are also fixed by the condition of Lorentz invariance, just as it happened for open strings. The answer could be anticipated, since the left and right sectors of closed string theory behave like open strings. In addition, the naïve argument based on zeta functions suggests that the ordering constants for $L_0^\perp$ and $\overline{L}_0^\perp$ are the same and equal to that for the $L_0^\perp$ operator of the open string. These constants are included in the relation between α_0^- and $L_0^\perp$ and in the corresponding barred relation. Therefore equations (13.2.6), for $n = 0$, become

$$\sqrt{2\alpha'}\,\bar{\alpha}_0^- = \frac{2}{p^+}(\overline{L}_0^\perp - 1)\,, \qquad \sqrt{2\alpha'}\,\alpha_0^- = \frac{2}{p^+}(L_0^\perp - 1)\,. \tag{13.2.10}$$

The constraint $L_0^\perp = \overline{L}_0^\perp$, which emerged from $\alpha_0^- = \bar{\alpha}_0^-$, remains unchanged by the constant shifts. On account of (13.2.8), this constraint can be written more simply as

$$N^\perp = \overline{N}^\perp\,. \tag{13.2.11}$$

Averaging the two expressions for α_0^- in (13.2.10), we can find a symmetric expression:

$$\sqrt{2\alpha'}\alpha_0^- \equiv \frac{1}{p^+}(L_0^\perp + \overline{L}_0^\perp - 2) = \alpha' p^-\,, \tag{13.2.12}$$

where the relation to p^- follows from (13.1.22). With p^- known, we can calculate the mass squared:

$$M^2 = -p^2 = 2p^+p^- - p^Ip^I = \frac{2}{\alpha'}(L_0^\perp + \overline{L}_0^\perp - 2) - p^Ip^I\,, \tag{13.2.13}$$

and substituting the values of $L_0^\perp$ and $\overline{L}_0^\perp$ given in (13.2.8) yields

$$\boxed{M^2 = \frac{2}{\alpha'}\left(N^\perp + \overline{N}^\perp - 2\right).} \tag{13.2.14}$$

This is the mass formula for closed string states. The closed string Hamiltonian (13.1.34) can be written in terms of Virasoro operators using (13.2.12). The answer is very simple:

$$\boxed{H = \alpha' p^+ p^- = L_0^\perp + \overline{L}_0^\perp - 2\,.} \tag{13.2.15}$$

This Hamiltonian is the sum of an "open string" Hamiltonian $L_0^\perp - 1$ for the right-moving operators, and an "open string" Hamiltonian $\overline{L}_0^\perp - 1$ for the left-moving operators. Using (13.2.8) the Hamiltonian can be written as

$$H = \frac{\alpha'}{2} p^I p^I + N^\perp + \overline{N}^\perp - 2 \,. \tag{13.2.16}$$

We conclude this section with a study of the Virasoro action on closed string coordinates. The commutation of the closed string Virasoro operators with the oscillators follows the pattern of equation (12.4.23). We have

$$\begin{aligned} \big[\, L_m^\perp \,, \alpha_n^J \,\big] &= -n\, \alpha_{m+n}^J \,, \\ \Big[\, \overline{L}_m^\perp \,, \bar{\alpha}_n^J \,\Big] &= -n\, \bar{\alpha}_{m+n}^J \,, \end{aligned} \tag{13.2.17}$$

and, in addition,

$$\big[\, L_m^\perp \,, \bar{\alpha}_n^J \big] = \Big[\, \overline{L}_m^\perp \,, \alpha_n^J \,\Big] = 0 \,. \tag{13.2.18}$$

On the other hand, both $L_m^\perp$ and $\overline{L}_m^\perp$ have a nontrivial commutator with x_0^I:

Quick Calculation 13.1. Verify that

$$\big[\, L_m^\perp \,, x_0^I \,\big] = -i\sqrt{\frac{\alpha'}{2}}\, \alpha_m^I \,, \qquad \Big[\, \overline{L}_m^\perp \,, x_0^I \,\Big] = -i\sqrt{\frac{\alpha'}{2}}\, \bar{\alpha}_m^I \,. \tag{13.2.19}$$

The closed string Virasoro operators $L_m^\perp$ and $\overline{L}_m^\perp$ both satisfy the Virasoro algebra (12.4.51) and commute. Let us focus here only on the action of $L_0^\perp$ and $\overline{L}_0^\perp$ on the string coordinates. It is a straightforward exercise to confirm that:

Quick Calculation 13.2. Verify that

$$\begin{aligned} \big[\, L_0^\perp \,, X^I(\tau,\sigma) \,\big] &= -\frac{i}{2}\,(\dot{X}^I - X^{I\prime}) \,, \\ \Big[\, \overline{L}_0^\perp \,, X^I(\tau,\sigma) \,\Big] &= -\frac{i}{2}\,(\dot{X}^I + X^{I\prime}) \,. \end{aligned} \tag{13.2.20}$$

Adding the two equations in (13.2.20), we find

$$\Big[\, L_0^\perp + \overline{L}_0^\perp \,,\; X^I(\tau,\sigma) \,\Big] = -i \frac{\partial X^I}{\partial \tau} \,. \tag{13.2.21}$$

This equation is consistent with the Heisenberg equation of motion for X^I, since the closed string Hamiltonian is $(L_0^\perp + \overline{L}_0^\perp - 2)$. Subtracting the two equations in (13.2.20), we find a more surprising result:

$$\Big[L_0^\perp - \overline{L}_0^\perp \,, X^I(\tau,\sigma) \Big] = i\,\frac{\partial X^I}{\partial \sigma}\,. \tag{13.2.22}$$

This equation indicates that $L_0^\perp - \overline{L}_0^\perp$ generates constant translations along the string. Indeed, for infinitesimal ϵ,

$$X^I(\tau,\sigma) + \Big[-i\epsilon(L_0^\perp - \overline{L}_0^\perp)\,, X^I(\tau,\sigma) \Big] = X^I(\tau,\sigma+\epsilon)\,. \tag{13.2.23}$$

More generally, a finite translation along the string can be obtained by acting on the string coordinate with exponentials of $L_0^\perp - \overline{L}_0^\perp$. Writing

$$P \equiv L_0^\perp - \overline{L}_0^\perp\,, \tag{13.2.24}$$

we find that (Problem 13.3)

$$e^{-iP\sigma_0}\, X^I(\tau,\sigma)\, e^{iP\,\sigma_0} = X^I(\tau,\sigma+\sigma_0)\,, \tag{13.2.25}$$

for any finite σ_0. For $\sigma_0 = \epsilon$, infinitesimal, this general result reduces to (13.2.23). The operator P generates the one reparameterization symmetry that cannot be fixed even in the light-cone gauge. Since P annihilates all closed string states (13.2.7), we conclude that closed string states are invariant under rigid σ translations. By this we mean that $\exp(-iP\sigma_0)|\Psi\rangle = |\Psi\rangle$, for any closed string state $|\Psi\rangle$.

In the two-dimensional (τ,σ) parameter space of the closed string world-sheet, the operator $L_0^\perp + \overline{L}_0^\perp$ is a generator of τ translations. It is therefore a world-sheet energy. Since the gauge condition relates τ to the light-cone time, this world-sheet energy turns out to give us the spacetime Hamiltonian, the generator of light-cone time evolution. The other combination $L_0^\perp - \overline{L}_0^\perp = P$ generates translations along the world-sheet coordinate σ. It can therefore be viewed as a *world-sheet momentum*. This momentum should not be confused with the spacetime momentum of the string. For closed string states the world-sheet momentum must in fact vanish, and this is a nontrivial constraint. In open string theory the operator analogous to P vanishes identically. All the open string states that we have built automatically satisfy the condition of zero world-sheet momentum: the σ coordinate describes a finite interval and naturally states have zero net momentum in the σ-direction. This does not happen with the closed string. States with non-zero momentum along σ can be build, but they do not belong to the closed string state space.

13.3 Closed string state space

We are now ready to build the state space of the quantum closed string. The ground states are $|p^+, \vec{p}_T\rangle$ and they are annihilated by both the left-moving and the right-moving annihilation operators. To generate all of the basis states we must act on the ground states with the creation operators $a_n^{I\dagger}$ and $\bar{a}_n^{I\dagger}$. The general *candidate* basis vector is

$$|\lambda, \bar{\lambda}\rangle = \Big[\prod_{n=1}^{\infty}\prod_{I=2}^{25}(a_n^{I\dagger})^{\lambda_{n,I}}\Big] \times \Big[\prod_{m=1}^{\infty}\prod_{J=2}^{25}(\bar{a}_m^{J\dagger})^{\bar{\lambda}_{m,J}}\Big]|p^+, \vec{p}_T\rangle. \tag{13.3.1}$$

Just as with open strings, the occupation numbers $\lambda_{n,I}$ and $\bar{\lambda}_{n,I}$ are non-negative integers. The number operators act on $|\lambda, \bar{\lambda}\rangle$ with eigenvalues

$$N^\perp = \sum_{n=1}^{\infty}\sum_{I=2}^{25} n\lambda_{n,I}\,, \qquad \overline{N}^\perp = \sum_{m=1}^{\infty}\sum_{J=2}^{25} m\bar{\lambda}_{m,J}\,. \tag{13.3.2}$$

Except for the momentum labels, the above states are those that one would obtain by combining *multiplicatively* arbitrary states built from the left-moving and from the right-moving operators (compare with (12.6.4)). Not all of the states in (13.3.1) belong to the closed string state space. The constraint (13.2.11) must be satisfied by the true states of the theory. A basis vector $|\lambda, \bar{\lambda}\rangle$ belongs to the state space *if and only if* it satisfies

$$N^\perp = \overline{N}^\perp\,. \tag{13.3.3}$$

This constraint cannot be “solved”: it cannot be implemented by eliminating some oscillators from the list of operators that can act on the ground states. It must be implemented in a case-by-case fashion. The masses of the states are obtained from (13.2.14):

$$\frac{1}{2}\alpha' M^2 = N^\perp + \overline{N}^\perp - 2\,. \tag{13.3.4}$$

$N^\perp, \overline{N}^\perp$	$\lvert\lambda, \bar{\lambda}\rangle$	$\frac{1}{2}\alpha' M^2$	number of states	wavefunction
$0, 0$	$\lvert p^+, \vec{p}_T\rangle$	-2	1	$\psi(\tau, p^+, \vec{p}_T)$
$1, 1$	$a_1^{I\dagger}\bar{a}_1^{J\dagger}\lvert p^+, \vec{p}_T\rangle$	0	$(D-2)^2$	$\psi^{IJ}(\tau, p^+, \vec{p}_T)$

Table 13.1: The states with $N^\perp + \overline{N}^\perp \leq 2$ in the closed string spectrum.

Let's identify the first few basis states, give their masses, and explain what fields they represent. The results are tabulated in Table 13.1.

The ground states in the first row of Table 13.1 are the one-particle states of a quantum scalar field. For such states $N^\perp = \overline{N}^\perp = 0$ and $M^2 = -4/\alpha' < 0$, so these are closed string tachyons; they are in fact completely analogous to the tachyons of open string theory. The mass-squared of the closed string tachyon is four times larger than that of the open string tachyon. The closed string tachyon is far less understood than the open string tachyon. In particular, the closed string tachyon potential has not been calculated yet. The instabilities associated to closed string tachyons remain largely mysterious.

The next excited states must be built with *two* oscillators acting on the ground states. This is because we must satisfy the constraint $N^\perp = \overline{N}^\perp$. One oscillator must be from the left-sector and one from the right-sector, both with the lowest possible mode number – mode number one. This gives the states described in the second line of the table. All these states have $M^2 = 0$, so they are of great interest. Since I and J are completely arbitrary labels attached to *different* oscillators, the number of states is $(D-2)^2$.

Let us consider the general state of fixed momentum at the massless level. We write it as

$$\sum_{I,J} R_{IJ}\; a_1^{I\dagger}\bar{a}_1^{J\dagger}\lvert p^+, \vec{p}_T\rangle\,. \tag{13.3.5}$$

Here R_{IJ} are the elements of an arbitrary square matrix of size $(D-2)$. Any square matrix can be decomposed into its symmetric part and its antisymmetric part:

$$R_{IJ} = \frac{1}{2}\,(R_{IJ} + R_{JI}) + \frac{1}{2}\,(R_{IJ} - R_{JI}) \equiv S_{IJ} + A_{IJ}\,, \tag{13.3.6}$$

where S_{IJ} and A_{IJ} are the symmetric and antisymmetric parts of R_{IJ} respectively. The symmetric part S_{IJ} can in fact be decomposed further:

$$S_{IJ} = \Big(S_{IJ} - \frac{1}{D-2}\delta_{IJ}S\Big) + \frac{1}{D-2}\,\delta_{IJ}\,S\,, \quad S \equiv S^{II} = \delta^{IJ}S_{IJ}\,. \tag{13.3.7}$$

The first term on the right-hand side is traceless:

$$\delta^{IJ}\Big(S_{IJ} - \frac{1}{D-2}\delta_{IJ}\,S\Big) = S - \frac{1}{D-2}\,\delta_{IJ}\,\delta^{IJ}S = 0\,, \tag{13.3.8}$$

since $\delta_{IJ}\,\delta^{IJ} = D-2$. Therefore (13.3.7) is a natural decomposition of S into a traceless matrix plus a multiple of the unit matrix. Let $\widehat{S}_{IJ}$ denote the traceless part of S_{IJ} and let $S' = S/(D-2)$. All in all, we have decomposed R_{IJ} as

$$R_{IJ} = \widehat{S}_{IJ} + A_{IJ} + S'\delta_{IJ}\,. \tag{13.3.9}$$

This is the standard decomposition of a matrix into a symmetric-traceless part, an antisymmetric part, and a trace part. Each of the three pieces can be specified independently when writing a general matrix R. Therefore the states in (13.3.5) can be split into three groups of linearly-independent states:

$$\sum_{I,J} \widehat{S}_{IJ}\, a_1^{I\dagger}\bar{a}_1^{J\dagger}|p^+,\vec{p}_T\rangle\,, \tag{13.3.10}$$

$$\sum_{I,J} A_{IJ}\, a_1^{I\dagger}\bar{a}_1^{J\dagger}|p^+,\vec{p}_T\rangle\,, \tag{13.3.11}$$

$$S'\; a_1^{I\dagger}\bar{a}_1^{I\dagger}\;|p^+,\vec{p}_T\rangle\,. \tag{13.3.12}$$

We now make a remarkable claim: the states (13.3.10) represent one-particle graviton states! We examined one-particle graviton states in section 10.6. In the quantum theory of the free gravitational field these states were given as (10.6.22):

$$\sum_{I,J=2}^{D-1} \xi_{IJ}\, a^{IJ\dagger}_{p^+,\,p_T}|\Omega\rangle\,, \tag{13.3.13}$$

where ξ_{IJ} was an arbitrary symmetric, traceless matrix. Since $\widehat{S}_{IJ}$ is also a symmetric, traceless matrix, the identification of states is possible if we identify the basis states via

$$a_1^{I\dagger}\bar{a}_1^{J\dagger}|p^+,\vec{p}_T\rangle \quad \longleftrightarrow \quad a^{IJ\dagger}_{p^+,\,p_T}|\Omega\rangle \tag{13.3.14}$$

This identification is possible because the two sets of states have the same Lorentz labels, they carry the same momentum, and they have the same mass (both zero). This shows that the closed string has graviton states. Gravity has appeared in string theory! We never put in a dynamical metric and we never spoke about general covariance, yet somehow the quantum states of the gravitational field have emerged!

The set of states in (13.3.11) corresponds to the one-particle states of the Kalb-Ramond field, an antisymmetric tensor with two indices. The light-cone analysis of this field was discussed in Problem 10.6 (see, in particular, parts (e) and (f)). The Kalb-Ramond field is in many ways the tensor generalization of the Maxwell gauge field A^I. The Kalb-Ramond field couples to strings in a way that is analogous to the way the Maxwell field couples to particles. Thus, as we will explore in detail in Chapter 15, strings carry Kalb-Ramond charge.

There is one state remaining to discuss. Expression (13.3.12) has no free indices (the index I on the oscillators is summed over) and hence represents only one state. It corresponds to a one-particle state of a massless scalar field. This field is called the *dilaton*.

The above discussion of particle states is supplemented by an analysis of wavefunctions and field equations. Such analysis follows closely the treatment in section 12.7. Wavefunctions $\psi_{IJ}(\tau, p^+, p_T)$ describe the general time-dependent states at the massless level of the closed string:

$$|\Psi, \tau\rangle = \int dp^+ d\vec{p}_T \;\psi_{IJ}(\tau, p^+, \vec{p}_T) a_1^{I\dagger} \bar{a}_1^{J\dagger} |p^+, \vec{p}_T\rangle \,. \tag{13.3.15}$$

The Schrödinger equation satisfied by the states is $i\partial_\tau |\Psi, \tau\rangle = H|\Psi, \tau\rangle$. Using (13.2.16) and noting that for the states in question $N^\perp = \overline{N}^\perp = 1$, we find

$$i\frac{\partial \psi_{IJ}}{\partial \tau} = \frac{\alpha'}{2}\, p^K p^K \,\psi_{IJ} \,. \tag{13.3.16}$$

The wavefunctions $\psi_{IJ}(\tau, p^+, p_T)$ become the fields of the classical field theories, with the Schrödinger equations interpreted as classical field equations. The symmetric-traceless part of ψ_{IJ} becomes the graviton field, the antisymmetric part becomes the Kalb-Ramond field, and the trace part becomes the dilaton field. The Schrödinger equations for the wavefunctions of the graviton states, the Kalb-Ramond states, and the dilaton state, are all included in (13.3.16), and can be separated by selecting the symmetric-traceless components, the antisymmetric components, and the trace component of ψ_{IJ}. On the other hand, the massless scalar field equation $\partial^2\phi = 0$ in light-cone coordinates takes the form (10.3.15):

$$\left(i\,\frac{\partial}{\partial x^+} - \frac{1}{2p^+}\,p^K p^K \right) \phi(x^+, p^+, \vec{p}_T) = 0\,. \tag{13.3.17}$$

Setting $x^+ = \alpha' p^+ \tau$, this equation becomes

$$\left(i\,\frac{\partial}{\partial \tau} - \frac{\alpha'}{2}\,p^K p^K \right) \phi(\tau, p^+, \vec{p}_T) = 0\,. \tag{13.3.18}$$

This equation is of the same form as (13.3.16). In fact, in the light-cone gauge, graviton fields, Kalb-Ramond fields, and the dilaton field, all satisfy the simple equation $\partial^2\phi^{\cdots} = 0$, where the dots refer to the relevant indices. This is manifestly true for the massless dilaton, which is a scalar. For graviton fields it was shown in equation (10.6.19) and for Kalb-Ramond fields it was part of the analysis in Problem 10.6.

In summary, at the massless level of the closed string we found the graviton, the Kalb-Ramond antisymmetric tensor, and a massless scalar called the dilaton. Each of these fields deserves intense study. The graviton field is studied in General Relativity. We will study the Kalb-Ramond field to understand the concept of string charge. The dilaton is a massless scalar field with subtle properties. A proper study of the dilaton belongs to an advanced course, but in the following section we give you some idea of the role it plays in string theory.

13.4 String coupling and the dilaton

The massless scalar field called the dilaton has a fascinating property: its expectation value controls the string coupling! This coupling is a dimensionless number that sets the strength of string interactions.

You may be familiar with the dimensionless fine-structure constant $g_e^2 \equiv e^2/\hbar c$ of electromagnetism. This coupling constant controls the strength of electromagnetic interactions. It enters, for example, in the quantum Hamiltonian of the hydrogen atom via the term that represents the electostatic interaction energy between the proton and the electron. The mass-scale in the hydrogen atom is set by the mass m of the electron. Physical, dimensionful quantities, such as the binding energy E of the ground state, depend on the dimensionful parameter m of the theory, the fundamental constants $\hbar, c$, *and* the dimensionless coupling g_e^2:

$$E = \frac{e^2}{2a_0} = \frac{1}{2}\Big(\frac{e^2}{\hbar c}\Big)^2 mc^2 = \frac{1}{2}\, g_e^2\,(mc^2)\,, \tag{13.4.1}$$

where $a_0 = \hbar^2/me^2$ is the Bohr radius. If the fine structure constant g_e^2 were taken to be zero, the binding energy would also vanish, and the Bohr radius would be infinite. This is what becomes of the hydrogen atom as we turn off electromagnetic interactions.

In string theory the story is not so different at first. The dimensionful parameter can be taken to be α', which defines the string length $\ell_s = \sqrt{\alpha'}$, working with $\hbar = c = 1$. Let g denote the dimensionless coupling for closed string interactions. If g was set to zero, then strings would not interact. Interactions in gravity are determined by the value of Newton's gravitational constant. If closed strings do not interact, then gravitation would emerge without interactions, and the value of Newton's constant in string theory would be zero. Since it must vanish when $g \to 0$, Newton's constant may be proportional to some positive power of g. It turns out that it is proportional to g^2. Dimensional analysis fixes the α' dependence of Newton's constant. Equation (3.8.6) shows that the 26-dimensional Newton constant $G^{(26)}$ has (natural) units of length to the power 24. Since α' has units of length squared, we find

$$G^{(26)} \sim g^2\,(\alpha')^{12}\,. \tag{13.4.2}$$

Most phenomenological studies of string theory begin with ten-dimensional superstring theories. These theories contain both bosonic and fermionic excitations, so they include all the particles we observe in nature. The ten-dimensional Newton constant $G^{(10)}$ in superstring theory is given by

$$G^{(10)} \sim g^2\,(\alpha')^4\,. \tag{13.4.3}$$

Using equation (3.8.6), $G^{(10)}$ can be expressed in terms of the ten-dimensional Planck length $\ell_P^{(10)}$:

$$\big(\ell_P^{(10)}\big)^8 \sim g^2\,(\alpha')^4 \quad \to \quad \ell_P^{(10)} \sim g^{1/4}\sqrt{\alpha'} = g^{1/4}\,\ell_s\,. \tag{13.4.4}$$

If the string coupling g is a small number, the string length can be larger than the Planck length. If g is of order unity, the string length and the Planck length are comparable. To find four-dimensional equations, we can use the relation between Newton constants that results from compactification. Assume that six of the dimensions of the ten-dimensional world are compactified into a space with volume $V^{(6)}$. Because of (3.9.9), the four-dimensional Newton constant G would then be related to the ten-dimensional one by

$$G = \frac{G^{(10)}}{V^{(6)}} \sim g^2\,\alpha'\,\frac{1}{V^{(6)}/(\alpha')^3}\,. \tag{13.4.5}$$

The ratio $V^{(6)}/(\alpha')^3$ is a dimensionless number that is typically assumed to be large. As a result, for a compactification of fixed volume in units of the string length, the four-dimensional Newton constant behaves like

$$G \sim g^2\,\alpha'\,. \tag{13.4.6}$$

In a theory with both open and closed strings, the open string coupling g_o is actually determined in terms of the closed string coupling g. One can prove that

$$g_o^2 \sim g\,. \tag{13.4.7}$$

This relation arises because of the topological properties of two-dimensional world-sheets. We will discuss it in Chapter 23.

The coupling controlling the strength of an interaction may sometimes fail to be constant. Consider adding to a free Hamiltonian H_0 an interaction term gH_{int} that is proportional to a dimensionless coupling g. If g is constant, you must specify its value to define the complete Hamiltonian $H_0 + gH_{\text{int}}$. But suppose that g is not a constant but rather a dynamical variable $g(t)$, and that the full Hamiltonian includes an additional term H_g that gives dynamics to g. In this case you would not need to specify g to define the theory. The coupling would be determined, perhaps uniquely, or perhaps not, by the Hamiltonian equations derived from $H_0 + gH_{\text{int}} + H_g$.

This is precisely what happens in string theory. The closed string coupling g depends on the value of the dilaton field $\phi(x)$ via an equation of the form

$$g \sim e^{\phi}\,. \tag{13.4.8}$$

As a result, the string coupling g is not an adjustable parameter in string theory; rather, it is a *calculable* parameter. This is a theoretically ideal situation, but it has been difficult in practice to calculate the values ϕ might take. One possibility is that other fields generate a potential $V(\phi)$ for the dilaton. If this potential has a stable critical point ϕ_* the value of the dilaton may be set equal to ϕ_*. Even more, around this critical point, the dilaton field would acquire a mass. This is necessary for a realistic model of physics because there are no known massless scalars in nature.

The study of string interactions is fascinating. If the string coupling is small, the quantum mechanical amplitudes for interactions can be calculated accurately using known results about Riemann surfaces. Moreover, you can understand why the infinities that plague quantum amplitudes in general relativity do not appear in string theory. This point will be discussed in Chapter 23.

13.5 A brief look at superstring theories

We have so far studied bosonic string theories, both open and closed. These string theories live in 26-dimensional spacetime, and all of their quantum states represent bosonic particle states. Among them, we found important bosonic particles, such as the graviton, and the photon. Non-abelian gauge bosons are needed to transmit the strong and weak forces. They too arise in bosonic string theory, as we will see in Chapter 14.

Realistic string theories, however, must also have quantum states that represent fermionic particle states. You may recall that a quantum state of identical bosonic particles is symmetric under the exchange of any two of the particles. A quantum state of identical fermionic particles, however, is antisymmetric under the exchange of any two of the particles. Quarks and leptons are fermionic particles. To obtain them, we need fermionic string theories, or *superstring* theories. We will not study superstrings in detail in this book. An proper explanation of the needed background material would take us too long. In here, we would like to give you some idea about superstrings. At later points in this book, some applications involve superstrings, and the physical ideas will be clear once you have a little familiarity with superstring terminology.

How do we get fermions from string theory? For classical bosonic strings, we described the position of the string using the classical variable $X^\mu(\tau, \sigma)$. For superstrings, we need new dynamical variables $\psi_1^\mu(\tau, \sigma)$ and $\psi_2^\mu(\tau, \sigma)$. The novel thing about the classical ψ_α^μ variables ($\alpha = 1, 2$) is that they are not ordinary commuting variables, but rather *anticommuting ones* (two variables b_1 and b_2 are said to be anticommuting if $b_1 b_2 = -b_2 b_1$). In fact, for each μ, the two components of ψ_α^μ comprise a fermion on the (τ, σ) world, that is, a *world-sheet* fermion. Remarkably, the quantization of such object results in particle states that behave as *spacetime* fermions, which is what we need.

In light-cone quantization X^+ was set proportional to τ and X^- could be solved for. For superstrings, this remains true, but in addition, the light-cone gauge condition sets $\psi_\alpha^+ = 0$ and allows one to solve for ψ_α^-. Both X^- and ψ_α^- receive contributions from the transverse X^I and ψ_α^I. Since both ψ_α^μ and X^μ are spacetime Lorentz vectors, both enter into the definition of the light-cone Lorentz generator M^{-I}. It follows that the commutator $[M^{-I}, M^{-J}]$ gets contributions from both ψ_α^μ and X^μ. The requirement that this commutator vanishes gives answers different from those obtained for bosonic strings. The number of spacetime dimensions is no longer twenty six, but rather ten. The downward shift of the mass-squared equals $-1/2$, rather than minus one. The spectrum of fermionic strings can be truncated consistently in such a way that the tachyon disappears and supersymmetry emerges. Supersymmetry is a symmetry relating the bosonic and fermionic quantum states of the theory. In a superstring theory, at any level, we find equal number of fermionic and bosonic states.

Let us discuss briefly the quantization of the variables $\psi_\alpha^\mu(\tau, \sigma)$. For open string theory in the light-cone gauge, the equations of motion imply that ψ_1^I is a right-mover and ψ_2^I is a left-mover:

$$\psi_1^I(\tau, \sigma) = \Psi_1^I(\tau - \sigma), \qquad \psi_2^I(\tau, \sigma) = \Psi_2^I(\tau + \sigma). \tag{13.5.1}$$

Additionally, the boundary conditions require that

$$\psi_1^I(\tau, \sigma_*)\, \delta\psi_1^I(\tau, \sigma_*) - \psi_2^I(\tau, \sigma_*)\, \delta\psi_2^I(\tau, \sigma_*) = 0\,, \tag{13.5.2}$$

at both endpoints $\sigma_* = 0$ and $\sigma_* = \pi$. Focus on one endpoint σ_*. It suffices to require $\psi_1^I(\tau, \sigma_*) = \pm\psi_2^I(\tau, \sigma_*)$, for then the variations will also respect this condition: $\delta\psi_1^I(\tau, \sigma_*) = \pm\delta\psi_2^I(\tau, \sigma_*)$, and as a result (13.5.2) will hold for either choice of sign. The overall relative sign between ψ_1^I and ψ_2^I happens to be conventional and thus we can set the components to be equal at one endpoint, $\sigma_* = 0$, for example:

$$\psi_1^I(\tau, 0) = \psi_2^I(\tau, 0)\,. \tag{13.5.3}$$

The choice of sign at the other endpoint is then relevant:

$$\psi_1^I(\tau,\pi) = \pm\psi_2^I(\tau,\pi)\,. \tag{13.5.4}$$

The full superstring theory state space breaks into two subspaces or, as they are typically called, two sectors: a Ramond (R) sector containing the states that arise via quantization using the top choice of sign, and a Neveu-Schwarz (NS) sector containing the states that arise via quantization using the lower choice of sign. These boundary conditions can be understood better by assembling a fermion field over the full interval $\sigma \in [-\pi, \pi]$:

$$\Psi^I(\tau,\sigma) \equiv \begin{cases} \psi_1^I(\tau,\,\sigma), & \sigma \in [\,0\,,\,\pi\,]\,, \\ \psi_2^I(\tau,-\sigma), & \sigma \in [-\pi, 0]\,. \end{cases} \tag{13.5.5}$$

This construction is reminiscent of (12.2.3), which gave a field defined over $\sigma \in [-\pi, \pi]$ for bosonic open strings. The boundary condition (13.5.3) guarantees that Ψ^I is continuous as $\sigma = 0$. Moreover, on account of (13.5.1), Ψ^I is only a function of $\tau - \sigma$. Finally, the boundary condition (13.5.4) gives

$$\Psi^I(\tau,\pi) = \psi_1^I(\tau,\pi) = \pm\psi_2^I(\tau,\pi) = \pm\Psi^I(\tau,-\pi)\,. \tag{13.5.6}$$

We thus learn that a periodic fermion Ψ^I corresponds to Ramond boundary conditions and an antiperiodic fermion Ψ^I corresponds to Neveu-Schwarz boundary conditions:

$$\begin{aligned} &\Psi^I(\tau,\pi) = +\Psi^I(\tau,-\pi) \qquad \text{Ramond boundary condition,} \\ &\Psi^I(\tau,\pi) = -\Psi^I(\tau,-\pi) \qquad \text{Neveu- Schwarz boundary condition.} \end{aligned} \tag{13.5.7}$$

Let us consider first the case of NS boundary conditions. Since Ψ^I is antiperiodic, it can be expanded with fractionally-moded exponentials:

$$\Psi^I(\tau,\sigma) = \sum_{r\in\mathbb{Z}+1/2} b_r^I\, e^{-ir(\tau-\sigma)}\,, \tag{13.5.8}$$

which guarantee that Ψ^I changes sign when $\sigma \to \sigma + 2\pi$. The negatively-moded coefficients $b^I_{-1/2}$, $b^I_{-3/2}$, $b^I_{-5/2}, \ldots$, are creation operators and they act on the Neveu-Schwarz vacuum $|\text{NS}\rangle$. These creation operators are all anticommuting, implying that the product of two identical ones vanishes. As a result, each b^I_{-r} can appear at most once in any state. Since the $X^I(\tau,\sigma)$

are quantized as usual, we still have the α^I_{-n} creation operators. As a result, the NS sector states are of the form:

$$\text{NS sector:} \quad \prod_{I=2}^{9}\prod_{n=1}^{\infty}(\alpha^I_{-n})^{\lambda_{n,I}} \prod_{J=2}^{9} \prod_{r=\frac{1}{2},\frac{3}{2}\cdots} (b^J_{-r})^{\rho_{r,J}}|\text{NS}\rangle \otimes |p^+, \vec{p}_T\rangle\,, \quad (13.5.9)$$

Here the $\rho_{r,J}$ are either zero or one. We have written the full ground state as a "product" $\otimes$ of the ground state $|\text{NS}\rangle$ that the b^I_{-r} act upon and the ground state$|p^+, \vec{p}_T\rangle$ that the α^I_{-n} act upon. The order in which the b operators appear in the state does not matter when we consider a single state. Since all the b's anticommute, different orderings can only differ only by an overall sign, and no new states are obtained.

With Ramond boundary conditions, Ψ^I is periodic and is expanded as

$$\Psi^I(\tau,\sigma) = \sum_{n\in\mathbb{Z}} d^I_n\, e^{-in(\tau-\sigma)}\,, \quad (13.5.10)$$

The modes d^I_n are integrally moded. The creation operators $d^I_{-1}, d^I_{-2}, d^I_{-3}, \ldots$ are all anticommuting and as a result can only act at most once in any given state. Ramond fermions are more complicated than NS fermions because the eight zero-mode fermionic oscillators d^I_0 must be treated with care. It turns out that these eight operators can be organized into four creation operators and four annihilation operators. Being zero modes, these four creation operators do not contribute to the mass-squared of the states. Postulating a unique vacuum, we get in fact $16 = 2^4$ degenerate Ramond ground states states: each of the four creation operators may or may not be act on the vacuum state. The degenerate Ramond vacua are denoted as $|\text{R}^A\rangle$ with $A = 1, \ldots, 16$.

Quick Calculation 13.3. Label the four creation operators as $\xi_1, \xi_2, \xi_3, \xi_4$ and the vacuum as $|0\rangle$. Exhibit explicitly the 16 ground states. Show that eight of them have an even number of ξ's acting on the vacuum, and eight of them have an odd number of ξ's acting on the vacuum.

Thus the sixteen ground states $|\text{R}^A\rangle$ can be split into eight ground states $|\text{R}^a_1\rangle$ $(a = 1, \ldots, 8)$ having an even number of fermionic operators and eight ground states $|\text{R}^a_2\rangle$ having an odd number of fermionic operators.

The Ramond sector of the state space contains the states:

$$\text{R sector:} \quad \prod_{I=2}^{9}\prod_{n=1}^{\infty}(\alpha^I_{-n})^{\lambda_{n,I}} \prod_{J=2}^{9} \prod_{m=1}^{\infty} (d^J_{-m})^{\rho_{m,J}}|\text{R}^A\rangle \otimes |p^+, \vec{p}_T\rangle\,. \quad (13.5.11)$$

Here the $\rho_{m,J}$ are either zero or one.

The open superstring theory state space is obtained by combining additively a subset of states from the Ramond sector and a subset of states from the Neveu-Schwarz sector. From the NS sector we keep only the states that have an odd-number of fermions b^I_{-r} acting on the ground states. From the Ramond sector the rule is a little more intricate. We first select either the $|\mathrm{R}^a_1\rangle$ or the $|\mathrm{R}^a_2\rangle$ to play the role of ground states. In open superstrings this choice does not matter, so let's say we select $|\mathrm{R}^a_1\rangle$. The Ramond states that we keep are then those built on the selected ground-states $|\mathrm{R}^a_1\rangle$ with an *even* number of fermions d^I_{-m}, *plus* the states build on the other ground states $|\mathrm{R}^a_2\rangle$ with an *odd* number of number of fermions d^I_{-m}.

With this truncation, all the spacetime bosonic states arise from the NS sector, and the fermionic states arise from R sector (even though fermionic operators are used in both sectors!). With this truncation, the spectrum has supersymmetry, in particular, at each mass level one has the same number of bosonic and fermionic states. In the NS sector the states

$$b^I_{-1/2}|\mathrm{NS}\rangle \otimes |p^+, \vec{p}_T\rangle \,, \tag{13.5.12}$$

represent the eight massless photon states that arise from a Maxwell gauge field. These states are massless because in the NS sector the mass-squared formula is shifted downwards by $-1/2$:

$$\text{NS sector:} \quad M^2 = \frac{1}{\alpha'}\Big(-\frac{1}{2} + N^\perp\Big)\,, \tag{13.5.13}$$

where $N^\perp$ is a number operator defined to kill $|\mathrm{NS}\rangle$. As the notation suggests, b^I_{-r} has $N^\perp$ eigenvalue equal to $+r$. Then, as stated, the states in (13.5.12) are seen to have $M^2 = 0$. The would-be tachyon state $|\mathrm{NS}\rangle \otimes |p^+, \vec{p}_T\rangle$ of $\alpha' M^2 = -1/2$ is removed from the spectrum by the truncation.

Quick Calculation 13.4. Explain why all states in the truncated NS sector have half-integer $N^\perp$ eigenvalues and integrally valued $\alpha' M^2$.

In the Ramond sector there is no shift in the mass-squared:

$$\text{R sector:} \quad M^2 = \frac{1}{\alpha'} N^\perp \,, \tag{13.5.14}$$

where $N^\perp$ is a number operator defined to kill $|\mathrm{R}^a_i\rangle$. As the notation suggests, d^I_{-n} has $N^\perp$ eigenvalue equal to $+n$. The states

$$|\mathrm{R}^a_i\rangle \otimes |p^+, \vec{p}_T\rangle \,, \tag{13.5.15}$$

therefore represent eight massless fermionic states. These fermionic states are the one-particle states of a fermion field in ten dimensions. Thus the massless fields of open superstring theory on a D9-brane (a space-filling brane) are a Maxwell field and a fermion field.

What about closed string theory? As we saw in this chapter, closed strings are roughly obtained by combining multiplicatively left-moving and right-moving copies of an open string theory. The same is true in superstring theory. Since an open superstring has two sectors (NS and R), closed strings sectors can be formed in four ways. Suppose that in the left-moving factor we select ground states $|\mathrm{NS}\rangle_L$ and $|\mathrm{R}_i^a\rangle_L$ and that in the right-moving factor we select ground states $|\mathrm{NS}\rangle_R$ and $|\mathrm{R}_j^a\rangle_R$. We speak of selecting ground states since we have a choice of i, j being equal to one or two. The ground states in the resulting four closed-string sectors are

$$|\mathrm{NS}\rangle_L \otimes |\mathrm{NS}\rangle_R \otimes |p^+, \vec{p}_T\rangle \,, \tag{13.5.16}$$

$$|\mathrm{NS}\rangle_L \otimes |\mathrm{R}_j^a\rangle_R \otimes |p^+, \vec{p}_T\rangle \,, \tag{13.5.17}$$

$$|\mathrm{R}_i^a\rangle_L \otimes |\mathrm{NS}\rangle_R \otimes |p^+, \vec{p}_T\rangle \,, \tag{13.5.18}$$

$$|\mathrm{R}_i^a\rangle_L \otimes |\mathrm{R}_j^a\rangle_R \otimes |p^+, \vec{p}_T\rangle \,. \tag{13.5.19}$$

Here i and j are indices that can take the values one or two, and reflect choices of R ground states. A closed superstring theory is obtained by combining additively the states that are built on top of these four sets of ground states. The earlier truncations are still needed: states must have an odd number of fermions acting on each NS ground state, and an even number of fermions acting on the selected R ground states or, an odd number of fermions on the alternate R ground states. If the truncations are done in other ways the resulting theory is not supersymmetric. As befits closed strings, there is also the $L_0^\perp - \overline{L}_0^\perp = 0$ constraint.

The states built on (13.5.16) comprise the NS-NS sector of the closed superstring. These states are bosons and include at the massless level exactly the same states we found in the bosonic closed string: the graviton, the Kalb-Ramond field, and the dilaton:

$$\text{NS-NS massless fields}: \quad g_{\mu\nu}\,,\ B_{\mu\nu}\,,\ \phi\,. \tag{13.5.20}$$

The set of massless states, as you may guess from (13.5.12) are

$$b_{-1/2}^I\, \bar{b}_{-1/2}^J |\mathrm{NS}\rangle_L \otimes |\mathrm{NS}\rangle_R \otimes |p^+, \vec{p}_T\rangle \,, \tag{13.5.21}$$

and since I, J run over eight values, there are 64 bosonic states.

Quick Calculation 13.5. Count the number of graviton, Kalb-Ramond, and dilaton states in ten dimensions. Add these numbers up and confirm that you get 64.

The states built on (13.5.17) and (13.5.18) comprise the NS-R and R-NS sectors of the closed superstring, respectively. Since the ground states include only one Ramond vacuum, the states built on them are fermions. The massless states are of the form

$$\bar{b}^I_{-1/2}\,|\mathrm{NS}\rangle_L \otimes |\mathrm{R}^a_j\rangle_R \otimes |p^+, \vec{p}_T\rangle\,, \quad b^I_{-1/2}\,|\mathrm{R}^a_i\rangle_L \otimes |\mathrm{NS}\rangle_R \otimes |p^+, \vec{p}_T\rangle\,. \quad (13.5.22)$$

Since $a = 1, \ldots, 8$, these are a total of $2 \times 8 \times 8 = 128$ fermionic states.

Finally, the states built on (13.5.19) comprise the RR sector of the closed superstring. Since the ground states include the product of two R ground states, they are 'doubly' fermionic, and the states in this sector are *bosons*. The massless RR bosonic states are in fact those in (13.5.19), and there are 64 of them. Together with the states arising from the NS-NS sector, the closed superstring has a total of 128 massless bosonic states. As required by supersymmetry, these match with the 128 massless fermionic states of the R-NS and NS-R sectors.

The closed string theory we have just described is called a type II theory. There are two inequivalent type II theories: type IIA theory and type IIB. In type IIA theory the left and right Ramond ground states are different: $i \neq j$. The same IIA theory emerges for $i = 1, j = 2$ and for $i = 2, j = 1$. In type IIB theory the left and right Ramond ground states are the same: $i = j$. The same IIB theory emerges for $i = j = 1$ and for $i = j = 2$.

While the NS-NS bosons of type IIA and type IIB theories are the same ((13.5.20)), the RR bosons are rather different. In the type IIA theory, one finds a Maxwell field A_μ, and a three-index antisymmetric field $A_{\mu\nu\rho}$. On the other hand, in the type IIB theory, we get a scalar field A, a Kalb-Ramond field $A_{\mu\nu}$, and an antisymmetric field $A_{\mu\nu\rho\sigma}$ with four indices. Summarizing

$$\text{RR massless fields in type IIA :} \qquad A_\mu\,, \quad A_{\mu\nu\rho}\,. \qquad (13.5.23)$$

$$\text{RR massless fields in type IIB :} \qquad A \;\;,\quad A_{\mu\nu}\,, \quad A_{\mu\nu\rho\sigma}\,. \quad (13.5.24)$$

These RR fields are deeply related to the existence of *stable* D-branes in the type II superstrings. We will discuss this matter further in Chapter 15.

Recall that in bosonic string theory the D25-brane is unstable. In fact, in bosonic string theory all Dp-branes are unstable.

In addition to the type II theories, there are also two heterotic superstrings. These are rather peculiar closed string theories. While the type II closed superstrings arise by combining together left-moving and right-moving copies of open superstrings, in the heterotic string we combine a left moving open *bosonic* string with a right-moving *superstring*! From the 26 left-moving bosonic coordinates only ten of them are matched by the right-moving bosonic coordinates in the superstring factor. As a result, this theory effectively lives in ten-dimensional spacetime. Heterotic strings come in two versions: $E_8 \times E_8$ type, and SO(32) type. Finally, there is the type I theory. This is a supersymmetric theory of open and closed unoriented strings. A string theory is said to be unoriented (see Problems 12.10 and 13.5) if the states of the theory are invariant under an operation that reverses the orientation of the strings. Both the type II theories and the heterotic theories are theories of oriented closed strings. The complete list of ten-dimensional supersymmetric string theories is therefore

- Type IIA
- Type IIB
- Type I
- SO(32) heterotic,
- $E_8 \times E_8$ heterotic.

These five theories were all known since the middle 1980's. Some relations between them were soon found, but a clearer picture only emerged in the late 1990's. The limit of type IIA theory as the string coupling is taken to infinity was shown to give a theory in eleven dimensions. This theory is called M-theory, with the meaning of M to be decided when the nature of the theory becomes clear. It is known, however, that M-theory is *not* a string theory. M-theory may end up playing a prominent role in understanding string theory. The discovery of many other relations between the five string theories listed above and M-theory has made it clear that we really have just *one theory*. This is a fundamental result: there is a unique theory, and the five superstrings and M-theory are different limits of this unique theory. The bosonic strings do not appear to be a part of this interrelated set of theories. This seems a little puzzling, but perhaps we have not heard the last word.

Problems

Problem 13.1. *Commutation relations for oscillators*

(a) Use the lower-sign version of equation (13.1.28) and the appropriate mode expansion to verify explicitly that the un-barred commutation relations of (13.1.29) emerge.

(b) The set of functions $e^{in\sigma}$, $n \in \mathbb{Z}$, is complete in the interval $\sigma \in [0, 2\pi]$. Use this fact to prove that

$$\delta(\sigma - \sigma') = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{in(\sigma-\sigma')} \,. \tag{1}$$

(c) Compute explicitly the commutator $[X^I(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')]$ using the mode expansions of X and $\mathcal{P}$, and the commutation relations (13.1.29) and (13.1.30). Use equation (1) to confirm that the expected answer (13.1.27) emerges.

(d) Prove the zero-mode commutation relations (13.1.33) starting with a derivation of

$$\Big[\, x_0^I + \sqrt{2\alpha'}\alpha_0^I \tau \,,\, \dot{X}^J(\tau,\sigma') \Big] = i\alpha' \eta^{IJ} \,,$$

which is the closed string analog of equation (12.2.23).

Problem 13.2. *A projector into physical states.*

Consider the vector space $\mathcal{H}$ spanned by the set of states $|\lambda, \bar\lambda\rangle$ in equation (13.3.1). Explain why for any state $|\lambda, \bar\lambda\rangle \in \mathcal{H}$ the eigenvalue of $L_0^\perp - \overline{L}_0^\perp$ is an integer. Show that

$$\mathcal{P}_0 = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{-i(L_0^\perp - \overline{L}_0^\perp)\theta} \,,$$

is a projector from $\mathcal{H}$ into the vector subspace where $P = 0$. Thus $\mathcal{P}_0$ projects into the state space of closed strings.

Problem 13.3. *Action of* $L_0^\perp - \overline{L}_0^\perp$.

(a) Prove that equation (13.2.25) holds for finite σ. You may find it useful to define $f(\sigma_0) = e^{-iP\sigma_0} X^I(\tau,\sigma)\, e^{iP\,\sigma_0}$ and to calculate multiple derivatives of f evaluated at $\sigma_0 = 0$.

(b) Explain why

$$e^{-iP\sigma_0}\,(\dot{X}^I \pm X^{I'})(\tau,\sigma)\, e^{iP\,\sigma_0} = (\dot{X}^I \pm X^{I'})(\tau, \sigma+\sigma_0)\,. \qquad (1)$$

(c) Use equation (1) to calculate $e^{-iP\sigma_0}\,\alpha_n^I\, e^{iP\,\sigma_0}$ and $e^{-iP\sigma_0}\,\bar{\alpha}_n^I\, e^{iP\,\sigma_0}$. In doing this you are finding the action of a σ-translation on the oscillators.

(d) Consider the state

$$|U\rangle = \alpha_{-m}^I \bar{\alpha}_{-n}^J |p^+, \vec{p}_T\rangle\,, \qquad m,n>0\,.$$

Use the results of (c) to calculate $e^{-iP\sigma_0}|U\rangle$. What is the condition that makes the state $|U\rangle$ invariant under σ-translations?

Problem 13.4. $L_0^\perp - \overline{L}_0^\perp$ *as world-sheet momentum.*

(a) Use equations (13.2.2) to show that

$$L_0^\perp - \overline{L}_0^\perp = -\frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\, \dot{X}^I X^{I'}\,. \qquad (1)$$

(b) The dynamics of the transverse light-cone string coordinates are governed by the Lagrangian density (12.3.14)

$$\mathcal{L} = \frac{1}{4\pi\alpha'}\Big(\dot{X}^I\,\dot{X}^I - X^{I'}\,X^{I'}\Big)\,.$$

The discussion in Problem 8.6 applies to $\mathcal{L}$, which has a symmetry under the σ-translation $\delta X^I = \epsilon\,\partial_\sigma X^I$, where ϵ is an infinitesimal constant. Calculate the charge associated to this symmetry transformation and show that it is proportional to $L_0^\perp - \overline{L}_0^\perp$, as given in (1).

Problem 13.5. *Unoriented closed strings.*

This problem is the closed string version of Problem 12.10. The closed string $X^\mu(\tau, \sigma)$ with $\sigma \in [0, 2\pi]$ and fixed τ, is a parameterized closed curve in spacetime. The orientation of a string is the direction of increasing σ.

(a) Consider now the closed string $X^\mu(\tau, 2\pi - \sigma)$ at the same τ. How is this second string related to the first string above? How are their orientations related? Make a rough sketch showing the original string as a continuous line, and the second string as a dashed line.

Assume there is an orientation reversing *twist* operator Ω such that

$$\Omega\, X^I(\tau, \sigma)\, \Omega^{-1} = X^I(\tau, 2\pi - \sigma)\,. \tag{1}$$

Moreover, assume that

$$\Omega\, x_0^-\, \Omega^{-1} = x_0^-\,, \quad \Omega\, p^+\, \Omega^{-1} = p^+\,. \tag{2}$$

(b) Use the closed string oscillator expansion (13.1.24) to calculate

$$\Omega\, x_0^I\, \Omega^{-1}\,, \quad \Omega\, \alpha_0^I\, \Omega^{-1}\,, \quad \Omega\, \alpha_n^I\, \Omega^{-1}\,, \text{ and } \Omega\, \bar{\alpha}_n^I\, \Omega^{-1}\,.$$

(c) Show that $\Omega\, X^-(\tau, \sigma)\, \Omega^{-1} = X^-(\tau, \pi - \sigma)$. Since $\Omega\, X^+(\tau, \sigma)\, \Omega^{-1} = X^+(\tau, \pi - \sigma)$, equation (1) actually holds for all string coordinates.

We say that twist, or orientation reversal, is a symmetry of closed string theory because the transformations $X^\mu(\tau, \sigma) \to X^\mu(\tau, 2\pi - \sigma)$ leave the string action invariant (do you see this?).

(d) Assume that the ground states are twist invariant. List the closed string states for $N^\perp \leq 2$, and give their twist eigenvalues. If you are commissioned to build a theory of *unoriented* closed strings, which of the states would you have to discard? What are the massless fields of unoriented closed string theory?

Problem 13.6. *Orientifold Op-planes.*

An orientifold Op-plane is a a hyperplane with p spatial dimensions, just as a Dp-brane has p spatial dimensions. The Op-plane arises when we do a truncation that keeps closed string states that are invariant under a symmetry transformation that combines string orientation-reversal and reflection of the coordinates normal to the Op-plane.

For an Op-plane, let $x^1, x^2, ..., x^p$, be directions on the Op-plane, and let $x^{p+1}, \ldots, x^d$ with $d = 25$ be directions orthogonal to the Op-plane. The Op-plane position is defined by $x^a = 0$ for $a = p+1, \cdots, d$. We will organize the string coordinates as $X^+, X^-, \{X^i\}, \{X^a\}$ with $i = 2, \ldots, p$, and $a = p+1, \ldots, d$. Let Ω_p denote the operator generating the transformation

$$\Omega_p\, X^a(\tau,\sigma)\, \Omega_p^{-1} = -X^a(\tau, 2\pi - \sigma)\,. \tag{1}$$

$$\Omega_p\, X^i(\tau,\sigma)\, \Omega_p^{-1} = \quad X^i(\tau, 2\pi - \sigma)\,. \tag{2}$$

Moreover, assume that

$$\Omega_p\, x_0^-\, \Omega_p^{-1} = x_0^-\,, \quad \Omega_p\, p^+\, \Omega_p^{-1} = p^+\,. \tag{3}$$

(a) For an O23-plane, the two normal directions x^{24}, x^{25} can be represented in a plane. A closed string at a fixed τ appears as a parametrized closed curve $X^a(\tau,\sigma)$ in this plane. Draw such an oriented closed string that lies fully in the first quadrant of the (x^{24}, x^{25}) plane. Draw also the string $\widetilde{X}^a(\tau,\sigma) = -X^a(\tau, 2\pi - \sigma)$.

(b) For any operator $\mathcal{O}$ the action of Ω_p is defined as $\mathcal{O} \to \Omega_p \mathcal{O} \Omega_p^{-1}$. Use the expansion (13.1.24) to calculate the action of Ω_p on the operators:

$$x_0^a\,,\ p^a\,,\ \alpha_n^a\,,\ \bar{\alpha}_n^a\,, \quad x_0^i\,,\ p^i\,,\ \alpha_n^i\,,\ \bar{\alpha}_n^i$$

Show that $\Omega_p\, X^\pm(\tau,\sigma)\, \Omega_p^{-1} = X^\pm(\tau, \pi - \sigma)$. Explain why the orientifold transformations are symmetries of closed string theory.

(c) Denote the ground states as $|p^+, p^i, p^a\rangle$. Assume that the states $|p^+, p^i, \vec{0}\,\rangle$ are invariant under Ω_p: $\Omega_p|p^+, p^i, \vec{0}\,\rangle = |p^+, p^i, \vec{0}\,\rangle$. Prove that

$$\Omega_p|p^+, p^i, p^a\,\rangle = |p^+, p^i, -p^a\,\rangle\,.$$

The massless states of the oriented closed string theory are given by

$$|\Phi\rangle = \int dp^+ d\vec{p}^{\,i}\, d\vec{p}^{\,a}\, \Phi^{\pm}_{IJ}(\tau, p^+, p^i, p^a)\left(\alpha^I_{-1}\bar{\alpha}^J_{-1} \pm \alpha^J_{-1}\bar{\alpha}^I_{-1}\right)|p^+, p^i, p^a\rangle,$$

where $\Phi^{\pm}_{IJ}$ are wavefunctions and the transverse light-cone indices I, J run over all the values that the indices i and a take.

Now consider the truncation to Ω_p invariant states: $\Omega_p|\Phi\rangle = |\Phi\rangle$. The resulting string theory is the theory in the presence of the orientifold Op-plane. Intuitively, for an invariant state, the amplitudes for the string to lie along each of the two related curves of part (a) are equal.

(d) Find the conditions that must be satisfied by $\Phi^{\pm}_{ab}, \Phi^{\pm}_{ia}, \Phi^{\pm}_{ij}$ to guarantee Ω_p invariance. All of the conditions are of the form

$$\Phi^{\pm}_{IJ}(\tau, p^+, p^i, p^a) = \ldots \Phi^{\pm}_{IJ}(\tau, p^+, p^i, -p^a)\,,$$

where the dots represent a sign factor that you must determine for each case.

Remarks: In coordinate space, letting $x^m = \{x^0, \ldots, x^p\}$, the above invariance conditions require that

$$\Phi^{\pm}_{IJ}(x^m, x^a) = \ldots \Phi^{\pm}_{IJ}(x^m, -x^a)\,.$$

In the presence of an orientifold plane, the values of the fields at (x^m, x^a) determine the values of the fields at the reflected point $(x^m, -x^a)$. The fields are either even or odd under $x^a \to -x^a$. The orientifold plane is some kind of mirror that relates the physics at reflected points, thus effectively cutting the space by half. In one half of the space and away from the orientifold, there are no constraints, so one has the full set of fields of oriented closed strings. An O25-plane is space-filling. Since it has no normal directions, the orientifold symmetry includes only string orientation reversal. This case was studied in Problem 13.5.

Problem 13.7. *Massive level in the open superstring.*

(a) Consider eight anticommuting variables b^i, with $i = 1, \ldots 8$. Ignoring signs, how many inequivalent products of the form $b^{i_1}b^{i_2}$ can be built? How many $b^{i_1}b^{i_2}b^{i_3}$? How many $b^{i_1}b^{i_2}b^{i_3}b^{i_4}$?

(b) Consider the first excited level of the open superstring ($\alpha' M^2 = 1$). List the states in the NS sector and in the R sector. Confirm that you get the same number of states.

Chapter 14

D-branes and Gauge Fields

The open strings we have studied so far were described by coordinates all of which satisfy Neumann boundary conditions. They represent open strings moving on the world-volume of a space-filling D25-brane. Here we quantize open strings attached to more general D-branes. We begin with the case of a single Dp-brane, with $1 \leq p \leq 25$. We then turn to the the case of multiple, parallel Dp branes, where we see the appearance of interacting gauge fields, and the possibility of massive gauge fields. We conclude with the case of parallel D-branes of different dimensionalities.

14.1 Dp-branes and boundary conditions

A Dp-brane is an extended object with p *spatial* dimensions. In bosonic string theory, where the number of spatial dimensions is 25, a D25-brane is a space-filling brane. The label D in Dp-brane stands for Dirichlet. In the presence of a D-brane, the enpoints of open strings must lie on the brane. As we will see in more detail below, this requirement imposes a number of Dirichlet boundary conditions on the motion of the open string endpoints.

Not all extended objects in string theory are D-branes. Strings, for example, are 1-branes because they are extended objects with one spatial dimension, but they are not D1-branes. Branes with p spatial dimensions are generically called p-branes. A 0-brane is some kind of particle. Just as the world-line of a particle is one dimensional, the world-volume of a p-brane is $(p+1)$-dimensional. Of these $p+1$ dimensions, one is the time dimension and the other p are spatial dimensions. We first discussed the concept of

D-branes in section 6.5. In addition, Problem 6.6 examined the motion of open strings ending on D-branes of various dimensions. Our main subject in the present chapter is the quantization of open strings in the presence of various kinds of D-branes. This is a rich subject with important applications for the problem of constructing realistic models of strings. Even more, configurations of D-branes in interplay with gravitation have led to surprising new insights in the study of strongly interacting gauge theories.

Our immediate goal is to set up the notation to describe D-branes and to state the appropriate boundary conditions. We let d denote the total number of spatial dimensions in the theory; in the present case, $d = 25$. A Dp-brane with $p < 25$ extends over a p-dimensional subspace of the 25-dimensional space. We will focus on simple Dp-branes: those that are p-dimensional *hyperplanes* inside the d-dimensional space. How can we specify such hyperplanes? We need $(d - p)$ linear conditions. In three spatial dimensions $(d = 3)$, a 2-brane $(p = 2)$ is a plane, and it is specified by one linear condition $(d - p = 3 - 2 = 1)$. For example, $z = 0$, specifies the (x, y) plane. Similarly, a string along the z axis $(p = 1)$ is specified by 2 linear conditions $(d - p = 3 - 1 = 2)$: $x = 0, y = 0$. We need as many conditions as there are spatial coordinates normal to the brane.

Consider a Dp-brane in $D = d + 1 = 26$ dimensional spacetime. We will define coordinates x^μ, with $\mu = 0, 1, \ldots, 25$, that are split into two groups. The first group includes the coordinates tangential to the brane *world-volume*. These are the time coordinate and p spatial coordinates. The second group includes the $(d-p)$ coordinates normal to the brane world-volume. We write

$$\underbrace{x^0,\, x^1,\, \ldots\, ,x^p}_{\text{D}p \text{ tangential coordinates}} \qquad \underbrace{x^{p+1},\, x^{p+2},\, \ldots,\, x^d}_{\text{D}p \text{ normal coordinates}} . \tag{14.1.1}$$

The location of the Dp-brane is specified by fixing the values of the coordinates normal to the brane. With this split in mind we write

$$x^a = \bar{x}^a\,, \qquad a = p + 1, p + 2, \cdots, d\,. \tag{14.1.2}$$

Here the $\bar{x}^a$ are a set of $(d-p)$ constants. In a completely analogous fashion, the string coordinates $X^\mu(\tau, \sigma)$ are split as

$$\underbrace{X^0,\, X^1,\, \ldots\, ,X^p}_{\text{D}p \text{ tangential coordinates}} \qquad \underbrace{X^{p+1},\, X^{p+2},\, \ldots,\, X^d}_{\text{D}p \text{ normal coordinates}} . \tag{14.1.3}$$

Since the endpoints of the open string must lie on the D-brane, the string coordinates normal to the brane must satisfy Dirichlet boundary conditions

$$X^a(\tau,\sigma)\Big|_{\sigma=0} = X^a(\tau,\sigma)\Big|_{\sigma=\pi} = \bar{x}^a\,, \qquad a = p+1, p+2, \cdots d\,. \tag{14.1.4}$$

The string coordinates X^a are called DD coordinates, because both endpoints satisfy a Dirichlet boundary condition. The open string enpoints can move freely along the directions tangential to the D-brane. As a result, the string coordinates tangential to the D-brane satisfy Neumann boundary conditions:

$$X^{m\prime}(\tau,\sigma)\Big|_{\sigma=0} = X^{m\prime}(\tau,\sigma)\Big|_{\sigma=\pi} = 0\,, \qquad m = 0, 1, \ldots, p\,. \tag{14.1.5}$$

These string coordinates are called NN coordinates because both endpoints satisfy a Neumann boundary condition. We see that the split (14.1.3) into tangential and normal coordinates is also a split into coordinates which satisfy Neumann and Dirichlet boundary conditions, respectively:

$$\underbrace{X^0,\, X^1,\, \ldots\,, X^p}_{\text{NN coordinates}} \;\; \underbrace{X^{p+1},\, X^{p+2},\, \ldots,\, X^d}_{\text{DD cooordinates}}\;. \tag{14.1.6}$$

In order to use the light-cone gauge we need at least one spatial NN coordinate that can be used together with X^0 to define the coordinates $X^\pm$. We therefore need to take $p \geq 1$, and our analysis does not apply to the case of a D0-brane. We will label the light-cone coordinates as

$$\underbrace{X^+, X^-, \{X^i\}}_{\text{NN}}\underbrace{\{X^a\}}_{\text{DD}} \quad i = 2, \ldots, p, \;\; \text{and} \;\; a = p+1, \ldots, d\,. \tag{14.1.7}$$

14.2 Quantizing open strings on Dp-branes

Having specified the boundary conditions on the various string coordinates we can proceed to the quantization of open strings in the presence of a Dp-brane. The purpose of the analysis that follows is to determine the spectrum of open string states, and to use this result to understand more deeply what goes on in the world-volume of a D-brane.

Our earlier work in Chapter 13 is quite useful here. The NN coordinates $X^i(\tau,\sigma)$ satisfy exactly the same conditions that the light-cone coordinates

X^I satisfy in the quantization of open strings attached to a D25-brane. All expansions and commutation relations for the X^i coordinates can be obtained from those of X^I by replacing $I \to i$ in the relevant equations.

Additionally, we recall that the X^- coordinate was determined in terms of the transverse light-cone coordinates as in equation (9.5.5):

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{2\alpha'}\frac{1}{2p^+}(\dot{X}^I \pm X^{I\prime})^2 \,. \tag{14.2.1}$$

Moreover, the mode expansion of $\dot{X}^I \pm X'^I$ was given in (9.5.12):

$$\dot{X}^I \pm X^{I\prime} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^I \, e^{-in(\tau\pm\sigma)} \,. \tag{14.2.2}$$

A completely analogous expression holds for the mode expansion of the coordinate X^-. These equations led eventually to equations (12.4.10) and (12.4.11), summarized here as

$$2p^+ p^- \equiv \frac{1}{\alpha'}\Big(\frac{1}{2}\,\alpha_0^I\,\alpha_0^I + \sum_{n=1}^{\infty}\,\alpha_{-n}^I\,\alpha_n^I - a\Big) \,. \tag{14.2.3}$$

The subtraction constant a was determined to be equal to one for the quantization of strings on a D25-brane. The light-cone index $I = 2, \ldots, 25$, takes values that presentely run over NN coordinates labelled by i and DD coordinates labeled by a. As a result, (14.2.1) becomes

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{2\alpha'}\frac{1}{2p^+}\Big\{(\dot{X}^i \pm X^{i\prime})^2 + (\dot{X}^a \pm X^{a\prime})^2\Big\} \,. \tag{14.2.4}$$

As explained before, the X^i coordinates are expanded as

$$\dot{X}^i \pm X^{i\prime} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^i \, e^{-in(\tau\pm\sigma)} \,. \tag{14.2.5}$$

The X^a coordinates are the ones we must investigate. If an expansion analogous to (14.2.5) holds for X^a, we will be able to find p^- by letting $I \to (i, a)$ in (14.2.3), just as we did to obtain equation (14.2.4).

We are finally in a position to address the novel part of the quantization of open strings attached to a Dp-brane. The coordinates X^a transverse to the brane satisfy the wave equation. The general solution is a superposition of two waves:

$$X^a(\tau,\sigma) = \frac{1}{2}\Big(f^a(\tau+\sigma) + g^a(\tau-\sigma)\Big)\,, \tag{14.2.6}$$

Let's examine the boundary conditions (14.1.4). At $\sigma = 0$ we obtain

$$X^a(\tau,0) = \frac{1}{2}\left(f^a(\tau) + g^a(\tau)\right) = \bar{x}^a\,, \tag{14.2.7}$$

so that $g^a(\tau) = -f^a(\tau) + 2\bar{x}^a$, and as a result

$$X^a(\tau,\sigma) = \bar{x}^a + \frac{1}{2}\Big(f^a(\tau+\sigma) - f^a(\tau-\sigma)\Big)\,. \tag{14.2.8}$$

The boundary condition at $\sigma = \pi$ then gives us

$$f^a(\tau+\pi) = f^a(\tau-\pi)\,. \tag{14.2.9}$$

This simply means that $f^a(u)$ is a periodic function with period 2π. This information is incorporated into the following expansion for $f(u)$:

$$f^a(u) = \widetilde{f}^a_0 + \sum_{n=1}^{\infty}(\widetilde{f}^a_n \cos nu + \widetilde{g}^a_n \sin nu)\,. \tag{14.2.10}$$

It is interesting to note that there is no term linear in u. Such a term was present when the coordinate satisfied a Neumann boundary condition because in that case the derivative $f'(u)$ was periodic. Returning to (14.2.8), with (14.2.10) we have, after some trigonometric simplification,

$$X^a(\tau,\sigma) = \bar{x}^a + \sum_{n=1}^{\infty}\Big(-\widetilde{f}^a_n \sin n\tau \sin n\sigma + \widetilde{g}^a_n \cos n\tau \sin n\sigma\Big)\,. \tag{14.2.11}$$

Redefining the expansion coefficients that are arbitrary anyway, we can write

$$X^a(\tau,\sigma) = \bar{x}^a + \sum_{n=1}^{\infty}\Big(f^a_n \cos n\tau + \widetilde{f}^a_n \sin n\tau\Big)\sin n\sigma\,. \tag{14.2.12}$$

There is no term linear in τ here. This means that the string has no momentum in the direction x^a. This is reasonable: strings must remain attached to the brane. With a $p^a\tau$ term, the endpoint $\sigma = 0$ would not remain at

$x^a = \bar{x}^a$ for $\tau \neq 0$.

In order to define the quantum theory associated to X^a, we focus on the classical parameters describing the motion of the open string in equation (14.2.12). Since we are trying to quantize strings attached to a *fixed* Dp-brane, the values $\bar{x}^a$ are not parameters that can be adjusted to describe various open string motions. The $(f^a, \widetilde{f}^a)$ are on a different footing; they are parameters of the classical motion of the open string. Therefore, in quantizing the open string, the $\bar{x}^a$ *remain numbers and do not become operators*, while the f^a and $\widetilde{f}^a$ turn into operators.

We now rewrite (14.2.12) in terms of oscillators, defined conveniently to obtain familiar commutation relations:

$$X^a(\tau, \sigma) = \bar{x}^a + \sqrt{2\alpha'} \sum_{n\neq 0}^{\infty} \frac{1}{n} \alpha_n^a e^{-in\tau} \sin n\sigma \,. \tag{14.2.13}$$

The string coordinate X^a is Hermitian if $(\alpha_n^a)^\dagger = \alpha_{-n}^a$, the usual Hermiticity property of oscillators. Note that the zero mode α_0^a does not exist. Additionally,

$$\dot{X}^a = -i\sqrt{2\alpha'} \sum_{n\neq 0} \alpha_n^a e^{-in\tau} \sin n\sigma \,, \quad X^{a\prime} = \sqrt{2\alpha'} \sum_{n\neq 0} \alpha_n^a e^{-in\tau} \cos n\sigma \,, \tag{14.2.14}$$

and therefore

$$X^{a\prime} \pm \dot{X}^a = \sqrt{2\alpha'} \sum_{n\neq 0} \alpha_n^a e^{-in(\tau\pm\sigma)} \,. \tag{14.2.15}$$

The analogy with (14.2.5) is quite close, but there are two differences. First, when the lower sign applies, the combinations of derivatives differ by an overall minus sign. Second, the zero mode is absent for (14.2.15).

The quantization is now straightforward. With $\mathcal{P}^a(\tau, \sigma) = \dot{X}^a / 2\pi\alpha'$, the non-vanishing commutators are postulated to be

$$\Big[X^a(\tau, \sigma) \,, \dot{X}^b(\tau, \sigma') \Big] = 2\pi\alpha' \, i \, \delta^{ab} \, \delta(\sigma - \sigma') \,. \tag{14.2.16}$$

Following the analysis of section 12.2, this commutator can be rewritten in the form (12.2.11), with (I, J) replaced by (a, b). Since the mode expansions (14.2.15) take the standard form, the earlier analysis applies. The overall sign difference alluded above is of no import since such terms appear twice is the relevant formulae. We thus find

$$[\alpha_m^a, \alpha_n^b] = m\,\delta^{ab}\,\delta_{m+n,0}\,, \qquad m, n \neq 0\,. \tag{14.2.17}$$

There is no mismatch for zero modes: $\bar{x}^a$ is a constant, and there is no conjugate momentum since $\alpha_0^a \equiv 0$. The sign difference is also immaterial for the evaluation of (14.2.4). As a result, equation (14.2.3) can be split as

$$2p^+\,p^- \equiv \frac{1}{\alpha'}\Big(\alpha'\,p^i\,p^i + \sum_{n=1}^{\infty}\Big[\alpha_{-n}^i\,\alpha_n^i + \alpha_{-n}^a\,\alpha_n^a\Big] - 1\Big)\,. \tag{14.2.18}$$

A few comments are needed here. Since the momentum $p^a \sim \alpha_0^a \equiv 0$, the term $\frac{1}{2}\alpha_0^I\alpha_0^I$ simply became $\alpha' p^i p^i$ (recall that $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$). The ordering constant has been set to minus one, as for the D25-brane. Nor is the critical dimension changed. This is reasonable since only the zero mode structure differs between the X^a and the X^i coordinates. In particular, note that the naive contributions needed to normal order $L_0^\perp$ are the same for X^a and for X^i. If follows from (14.2.18) that

$$M^2 = -p^2 = 2p^+p^- - p^ip^i = \frac{1}{\alpha'}\Big(\sum_{n=1}^{\infty}\Big[\alpha_{-n}^i\alpha_n^i + \alpha_{-n}^a\alpha_n^a\Big] - 1\Big)\,. \tag{14.2.19}$$

Using creation and annihilation operators, we get

$$M^2 = \frac{1}{\alpha'}\Big(-1 + \sum_{n=1}^{\infty}\sum_{i=2}^{p} n a_n^{i\,\dagger} a_n^i + \sum_{m=1}^{\infty}\sum_{a=p+1}^{d} m a_m^{a\,\dagger} a_m^a\Big)\,. \tag{14.2.20}$$

Let us now consider the state space for the quantum string. The ground states of the quantum string in the D25-brane background were $|p^+, \vec{p}_T\rangle$, where $\vec{p}_T = (p^2, \ldots, p^{25})$ is the vector with components p^I. The I index now runs over i and a values, but there are no p^a momenta since no p^a operators exist. Therefore the ground states of the theory are labelled by p^+ and p^i

$$|p^+, \vec{p}\rangle \quad \text{with} \quad \vec{p} = (p^2, \ldots, p^p)\,. \tag{14.2.21}$$

We build states by acting with oscillators on the ground states. We have oscillators along the brane:

$$a_n^{i\,\dagger}\,, \quad n \geq 1, \quad i = 2, \ldots, p\,, \tag{14.2.22}$$

and oscillators transverse to the brane:

$$a_n^{a\,\dagger}\,, \quad n \geq 1, \quad a = p+1, \ldots, d\,. \tag{14.2.23}$$

So the states take the form

$$\Big[\prod_{n=1}^{\infty}\prod_{i=2}^{p}\Big(a_n^{i\,\dagger}\Big)^{\lambda_{n,i}}\Big]\Big[\prod_{m=1}^{\infty}\prod_{a=p+1}^{d}\big(a_m^{a\,\dagger}\big)^{\lambda_{m,a}}\Big]|p^+,\vec{p}\rangle\,. \tag{14.2.24}$$

Schrödinger wavefunctions take the schematic form

$$\psi_{i_1\ldots i_p\,a_1\ldots a_q}\,(\tau, p^+, \vec{p}\,)\,. \tag{14.2.25}$$

Just as the indices on the oscillators, the indices on the wavefunctions are of two types: indices along the directions tangent to the brane (i-type), and indices along the directions normal to the brane (a-type).

In considering the field theories that arise from string quantization, we have seen that the Schrödinger wavefunctions become the fields. We can therefore ask: where do these fields live? Do these fields live over all of spacetime, or only in some subspace thereof? Since a field is a function, we are really asking about the arguments of this function. Is it a function over all of spacetime, or only over some subspace thereof?

Since the wavefunctions depend on all the p^i, we have dependence on all the x^i coordinates. Because of the τ and p^+ arguments, we have x^+ and x^- dependence. All together, we have fields that depend on x^+, x^-, and x^i, with $i = 2, \ldots, p$. These are precisely the $(p+1)$ coordinates that span the world-volume of the Dp-brane. It is reasonable to conclude that the fields actually live on the Dp-brane. Indeed, this world-volume is the only natural candidate for a $(p+1)$-dimensional subspace of spacetime.

Our analysis suggests, but does not prove that the fields live on the Dp-brane; the positions $\bar{x}^a$ of the Dp-brane did not appear in the states nor in the wavefunctions. How could we *prove* that the fields in question live on the Dp-brane? We would have to study interactions. Since closed strings have no endpoints, they are not fixed by D-branes and can exist over all of spacetime. By scattering closed strings off the Dp-brane we can investigate if the interactions between fields from the closed string sector and fields from the open string sector take place on the D-brane world-volume. The answer appears to be yes, at least in some computational schemes. It is likely, however, that statements about where open string fields live are ambiguous or even gauge dependent. Different answers could be completely consistent.

We conclude our analysis of the Dp-brane by giving a list and detailed description of the fields that have $M^2 \leq 0$. Since all these fields live on the Dp brane, we must decide whether they are scalars or vectors with respect to the Lorentz symmetry of the Dp-brane. Let us begin with the simplest states, the ground states

$$|p^+, \vec{p}\rangle\,, \qquad M^2 = -\frac{1}{\alpha}\,. \tag{14.2.26}$$

These states are tachyon states on the brane, and have exactly the same mass as the tachyon states we found on the D25-brane. The corresponding tachyon field, of course, is just a Lorentz scalar on the brane.

The next states have one oscillator acting on them. Consider first the case when the oscillator arises from the coordinates tangent to the brane:

$$a_1^{i\,\dagger}|p^+, \vec{p}\rangle\,, \quad i = 2, \ldots p, \qquad M^2 = 0\,, \tag{14.2.27}$$

For any momenta, these give $(p+1) - 2$ massless states. Moreover, they carry one index, which lives on the brane. They are therefore states that transform as a Lorentz vector. Since the number of states equals the spacetime dimensionality of the brane minus two, these are clearly photon states. The associated field is a Maxwell gauge field living on the brane. This is a fundamental result:

$$\boxed{\text{A D}p\text{-brane has a Maxwell field living on its world-volume.}} \tag{14.2.28}$$

Finally, let us consider the case when the oscillator acting on the vacuum states arises from coordinates normal to the brane:

$$a_1^{a\dagger}\,|p^+, \vec{p}\rangle\,, \quad a = p+1, \ldots, d, \qquad M^2 = 0\,. \tag{14.2.29}$$

For any momenta, these are $(d-p)$ states living on the brane. Since the index a is not a Lorentz index for the brane, as far as the brane is concerned this index is just a counting label. The brane considers these states to be states that transform as Lorentz *scalars*. Therefore, we get a massless scalar field for each direction normal to the Dp-brane:

$$\boxed{\text{A D}p\text{-brane has a massless scalar for each normal direction.}} \tag{14.2.30}$$

These massless scalars have an interpretation. In section 12.8 we indicated that open string states represent D-brane excitations. Our Dp-brane and a slightly displaced parallel Dp-brane are actually states of the same energy. Such a displacement, constant over all of the Dp-brane, corresponds to a zero-momentum excitation with zero energy. These excitations arise from the massless scalars: massless excitations obey $E = p$, and in the limit of zero-momentum they have zero energy. Supporting this interpretation, we have as many massless fields as there are directions normal to the Dp-brane. Those are the independent directions in which the Dp-brane can be moved. Finally, note that the space-filling D25-brane has no massless scalars on its world-volume, consistent with the fact that this D-brane cannot be displaced.

All in all, the massless states on the Dp-brane are $(p-1)$ photon states and $(d-p)$ scalar field states. Apart from the momentum labels which are different, we have the same number of massless states as on the D25-brane. The $(d-1)$ states on the D25-brane are accounted on the Dp-brane by $(p-1)$ photon states and $(d-p)$ scalar states.

14.3 Open strings between parallel Dp-branes

We will now consider the quantization of open strings that can exist between two parallel Dp-branes. In describing such branes we will continue to use the notation of the previous sections. Two parallel branes of the same dimensionality have the same set of longitudinal coordinates and the same set of normal coordinates. Recall that the values $\bar{x}^a$ of the normal coordinates specify the position of a Dp-brane. This time the first Dp-brane is located at $x^a = \bar{x}_1^a$ and the second at $x^a = \bar{x}_2^a$. If we happen to have that $\bar{x}_1^a = \bar{x}_2^a$ for all a, the two Dp-branes coincide – they are on top of each other. Otherwise, they are separated. In Figure 14.1 we illustrate the situation with an example of two parallel, separated D2-branes.

What kinds of open strings does this configuration of parallel Dp-branes support? There are actually four different classes of strings, each of which must be analyzed separately. The first two classes are made up of open strings that begin and end on the same D-brane, either brane one or brane two. These strings we already studied and quantized in the previous section. The other two classes are made up of strings that start on one of the branes and end on the other brane. These are *stretched strings*. The strings that begin on brane one and end on brane two are different from the strings that

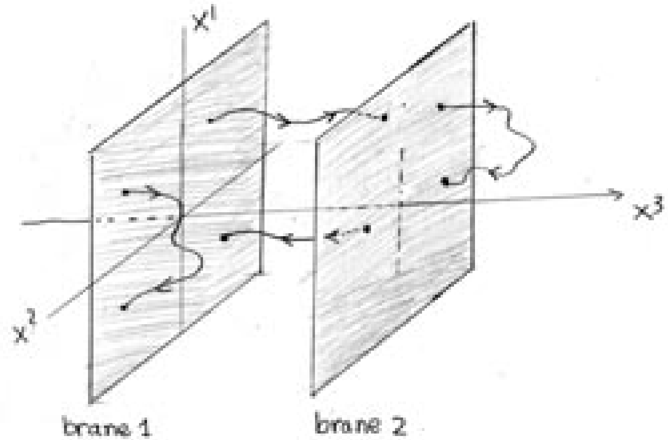

Figure 14.1: Two parallel D2-branes. Here x^1 and x^2 are longitudinal coordinates, and x^3 is a normal coordinate. The positions of the branes are specified by the coordinates $\bar{x}_1^3$ and $\bar{x}_2^3$ of brane one and brane two, respectively. We show the four types of strings that this configuration supports.

begin on brane two and end on brane one. This is clear from the coordinate expansions that we will write. These strings are oppositely oriented, and the orientation of the string (the direction of increasing σ) matters. As we will show in Chapter 15, the string charge of a string changes sign when we reverse its orientation. The classes of open strings that exist in a configuration of D-branes are called *sectors*. The quantum theory of open strings in a configuration of two parallel Dp-branes has four sectors. In Figure 14.1 we show a string for each of the four sectors.

Let us consider the sector where open strings begin on brane one and end on brane two. The NN string coordinates X^+, X^-, and X^i are quantized just as before, since the boundary conditions are not changed from (14.1.5). On the other hand, for the DD string coordinates equation (14.1.4) is changed into

$$X^a(\tau,\sigma)\Big|_{\sigma=0} = \bar{x}_1^a\,, \quad X^a(\tau,\sigma)\Big|_{\sigma=\pi} = \bar{x}_2^a\,. \tag{14.3.1}$$

The solution of the wave equation subject to these boundary conditions can be studied starting from (14.2.8), which already incorporates the boundary condition at $\sigma = 0$. In the present case we just change $\bar{x}^a$ to $\bar{x}_1^a$:

$$X^a(\tau, \sigma) = \bar{x}_1^a + \frac{1}{2}\Big(f^a(\tau + \sigma) - f^a(\tau - \sigma)\Big) . \qquad (14.3.2)$$

The boundary condition at $\sigma = \pi$ then gives us

$$f^a(\tau + \pi) - f^a(\tau - \pi) = 2\,(\bar{x}_2^a - \bar{x}_1^a)\,, \qquad (14.3.3)$$

or, equivalently,

$$f^a(u + 2\pi) - f^a(u) = 2\,(\bar{x}_2^a - \bar{x}_1^a)\,. \qquad (14.3.4)$$

This means that the derivative $f^{a\prime}(u)$ is a periodic function with period 2π, and has an expansion of the type indicated in (14.2.10). Integrating, the function $f^a(u)$ must have an expansion of the form

$$f^a(u) = f_0^a\, u + \sum_{n=1}^{\infty}(h_n^a \cos nu + g_n^a \sin nu)\,. \qquad (14.3.5)$$

We have not included a constant term because it would drop out of X^a, as can be seen in (14.3.2). The constant f_0^a is fixed by the boundary condition (14.3.4):

$$f_0^a = \frac{1}{\pi}(\bar{x}_2^a - \bar{x}_1^a)\,. \qquad (14.3.6)$$

It is now possible to substitute $f^a(u)$ into (14.3.2). Except for the zero modes, the computations are identical to those that led to (14.2.12). This time,

$$X^a(\tau, \sigma) = \bar{x}_1^a + (\bar{x}_2^a - \bar{x}_1^a)\frac{\sigma}{\pi} + \sum_{n=1}^{\infty}\Big(f_n^a \cos n\tau + \widetilde{f}_n^a \sin n\tau\Big)\sin n\sigma\,. \quad (14.3.7)$$

Note that the boundary conditions are manifestly satisfied. For strings that go from brane two to brane one we would have to exchange $\bar{x}_1^a$ and $\bar{x}_2^a$ in the equation above. Using (14.2.13) as a model, we can rewrite (14.3.7) in terms of oscillators:

$$X^a(\tau, \sigma) = \bar{x}_1^a + (\bar{x}_2^a - \bar{x}_1^a)\frac{\sigma}{\pi} + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_m^a\, e^{-im\tau}\sin m\sigma\,. \qquad (14.3.8)$$

As before, the constants x_1^a and x_2^a are not parameters of the open string fluctuations for fixed D-branes, and therefore they do not become quantum operators. Note the absence of terms linear in τ: the open strings do not have momentum in the x^a directions. Even though we are not giving the oscillators above different names, they are not the same operators we had for the quantization of the strings beginning and ending on the same Dp-brane. The oscillators in different sectors must not be confused. This time the derivatives give

$$\dot{X}^a = -i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^a e^{-in\tau}\sin n\sigma\,, \quad X^{a\prime} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^a e^{-in\tau}\cos n\sigma\,, \tag{14.3.9}$$

where

$$\sqrt{2\alpha'}\alpha_0^a = \frac{1}{\pi}\left(\bar{x}_2^a - \bar{x}_1^a\right). \tag{14.3.10}$$

Although the strings do not carry momentum in the x^a direction, there is an α_0^a. The interpretation of α_0 as momentum requires that α_0 appear in $\dot{X}$. As you can see, α_0^a appears in $X^{a\prime}$, but does not appear in $\dot{X}^a$. A non-vanishing α_0^a implies stretched strings: α_0^a vanishes if the two D-branes coincide. Similar facts emerge for closed strings that wrap around compact dimensions (see Chapter 17).

The two derivatives in (14.3.9) can be combined into

$$X^{a\prime} \pm \dot{X}^a = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^a e^{-in(\tau\pm\sigma)}\,. \tag{14.3.11}$$

It follows from this result, and our comments in the previous section, that the oscillators satisfy the expected commutation relations. To calculate the mass squared operator, we reconsider equation (14.2.3). As before, we let $I \to (i,a)$, and set the subtraction constant a equal to one, finding

$$2p^+ p^- = \frac{1}{\alpha'}\Big(\alpha' p^i p^i + \frac{1}{2}\alpha_0^a\,\alpha_0^a + \sum_{n=1}^{\infty}\Big[\alpha_{-n}^i\,\alpha_n^i + \alpha_{-n}^a\,\alpha_n^a\Big] - 1\Big)\,. \tag{14.3.12}$$

We therefore have

$$M^2 = 2p^+p^- - p^ip^i = \frac{1}{2\alpha'}\alpha_0^a\alpha_0^a + \frac{1}{\alpha'}\Big(\sum_{n=1}^{\infty}\Big[\alpha_{-n}^i\alpha_n^i + \alpha_{-n}^a\alpha_n^a\Big] - 1\Big)\,. \tag{14.3.13}$$

Using the explicit value of α_0^a we finally get:

$$M^2 = \Big(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\Big)^2 + \frac{1}{\alpha'}(N^\perp - 1)\,, \tag{14.3.14}$$

where

$$N^\perp = \sum_{n=1}^{\infty}\sum_{i=2}^{p} n a_n^{i\,\dagger} a_n^i + \sum_{m=1}^{\infty}\sum_{a=p+1}^{d} m a_m^{a\dagger}\, a_m^a\,. \tag{14.3.15}$$

The first term in the right hand side of (14.3.14) is a new contribution to the mass-squared of the states. Since the string tension is $T_0 = 1/(2\pi\alpha')$, the term is simply the square of the energy of a classical static string stretched between the two D-branes. It is eminently reasonable to find that the mass-squared operator is changed by the addition of this constant. If the branes coincide the constant vanishes.

Let's now consider the ground states. In fact, let's consider the ground states for the four sectors describing open strings on this D-brane configuration. The momenta labels of these states are the same: p^+ and $\vec{p}$. To distinguish the various sectors, we include as additional ground-state labels two integers $[ij]$ with values one or two. For any open string sector, the first integer denotes the brane where the $\sigma = 0$ endpoint lies, and the second integer denotes the brane where the $\sigma = \pi$ endpoint lies. Alternatively, in the $[ij]$ sector, the open strings go from brane i to brane j. The ground states are thus written as $|p^+, \vec{p};\, [ij]\rangle$ and the four types of vacuum states are

$$|p^+, \vec{p};\, [11]\rangle\,, \quad |p^+, \vec{p};\, [22]\rangle\,, \quad |p^+ \vec{p};\, [12]\rangle\,, \quad |p^+, \vec{p};\, [21]\rangle\,. \tag{14.3.16}$$

The states of open strings that are fully attached to the first brane are built with oscillators acting on the first of the above states. The states of open strings that are fully attached to the second brane are built on the second of the above states. The states of open strings stretched from brane one to brane two are built on the third of the above ground states. Finally, the states of open strings stretched from brane two to brane one are built on the last of the above ground states. These four sets of states all take the form indicated in (14.2.24), with the exception that the vacuum state is replaced by the appropriate $|p^+, \vec{p};\, [ij]\rangle$. The oscillators in the four sectors are the same in number and in type, but are really different operators. We could label them with the $[ij]$ labels, but this is seldom necessary.

Where do the fields corresponding to the states built on $|p^+, \vec{p};\, [12]\rangle$ live? This question is difficult to answer. They are clearly $(p+1)$-dimensional fields, since the momentum structure of the states is the same as the one we had for string states fully-attached to a single brane. Since the two D-branes are on a similar footing as far as the stretched strings is concerned, we cannot say that the fields live in any single one of the two D-branes. In some sense, the fields must live on both of them. Operationally, the fields are declared to live on some fixed $(p+1)$-dimensional space (not necessarily identified with any of the two D-branes), and are seen to have non-local interactions reflecting the fact that the D-branes are separated. More conceptually, the issue appears to require a new way of thinking, the basis of which may be provided by a branch of mathematics called non-commutative geometry.

We continue our discussion of the state space by giving a list and a detailed description of the fields making up the two lowest levels of the stretched strings. Just as in the case of the single brane, we will determine whether they are scalars or vectors with respect to $(p+1)$-dimensional Lorentz symmetry.

The simplest states are the ground states

$$|p^+, \vec{p}; [12]\rangle\,, \qquad M^2 = -\frac{1}{\alpha} + \Big(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\Big)^2. \tag{14.3.17}$$

These states are tachyon states of the usual mass-squared if the separation between the branes is zero. If the branes are separated the mass-squared gets a positive contribution. In fact, for the critical separation

$$|\bar{x}_2^a - \bar{x}_1^a| = 2\pi\sqrt{\alpha'}\,, \tag{14.3.18}$$

the ground states represent a massless scalar field. For larger separations, the ground states represent a massive scalar.

The next states have one oscillator acting on them. Assume, until stated otherwise, that the separation between the branes is non-zero. If the oscillator acting on the ground states arises from the coordinates normal to the brane we have

$$a_1^{a\dagger}\,|p^+, \vec{p}; [12]\rangle\,, \quad a = p+1, \ldots, d, \qquad M^2 = \Big(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\Big)^2. \tag{14.3.19}$$

For any momenta, these are $(d-p)$ massive states. Since the index a is not a Lorentz index for the $(p+1)$-dimensional spacetime, these states are Lorentz scalars. Therefore, we get $(d-p)$ massive scalar fields. If the oscillator arises from the coordinates tangent to the brane we have

$$a_1^{i\,\dagger}|p^+, \vec{p}; [12]\rangle\,, \quad i = 2, \ldots p, \qquad M^2 = \Big(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\Big)^2, \tag{14.3.20}$$

For any momenta, these are $(p+1)-2 = p-1$ massive states. Moreover, they carry an index corresponding to the $(p+1)$-dimensional spacetime. We might think that these states make up a massive Maxwell gauge field, but this is not correct.

A massive gauge field has more degrees of freedom than a massless gauge field. The results of Problem 10.7 indicated that a massive gauge field has, for each value of the momentum, *one more state* than a massless gauge field. In a D-dimensional spacetime, a massless gauge field has, for each momentum, $D-2$ states, while a massive gauge field has $D-1$ states. Therefore, in the case at hand, one of the states in (14.3.19) must join the $(p-1)$ states in (14.3.20) to form the massive vector. At the end we have one massive vector and $(d-p-1)$ massive scalars.

Can we make an educated guess as to which scalar state in (14.3.19) becomes part of the massive gauge field? If $p = d-1$ the answer is simple. The D-branes are only separated along one coordinate, and there is just one scalar in (14.3.19). The scalar uses the oscillator labelled with the direction along which the branes are separated. For $p < d-1$ there are several states in (14.3.19). The scalar state that becomes part of the vector must be that which arises as the linear combination

$$\sum_a (\bar{x}_2^a - \bar{x}_1^a)\, a_1^{a\dagger}\, |p^+, \vec{p}; [12]\rangle\,. \tag{14.3.21}$$

From all directions normal to the D-branes, the direction defined by the spatial vector with non-vanishing components $\bar{x}_2^a - \bar{x}_1^a$ is unique: it takes us from one brane to the other one. To visualize this concretely you can think of two parallel D1-branes in three-dimensional space. There are clearly many normal directions that do not take us from one brane to the other, and just one direction that does. More generally, since the vectors with nonvanishing components $\bar{x}_1^a$ and $\bar{x}_2^a$ are points on the first and second branes, respectively, the vector difference manifestly joins one brane to the other. This direction can be said to be the direction along which the branes are separated. The guess (14.3.21) can be proven to be the correct one.

In the limit as the separation between the branes goes to zero, we obtain a very interesting situation. Even though the D-branes are coincident, they are still distinguishable and we have the four open-string sectors. The massless open-string states representing strings from brane one to brane two include a massless gauge field and $(d-p)$ massless scalars. This is the same field content as that for the sector where strings begin and end on the same D-brane. When the two D-branes coincide we therefore get a total of four massless gauge fields. These gauge fields actually interact with one another – in the string picture they do so by the process of joining endpoints. Theories of interacting gauge fields are called Yang-Mills theories. They were discovered in the 1950's and later used successfully to build the theories of electroweak and of strong interactions. In the world-volume of two coincident D-branes we indeed get a particular Yang-Mills theory: it is a $U(2)$ Yang-Mills theory. The two in $U(2)$ is there precisely because we have two coincident D-branes.

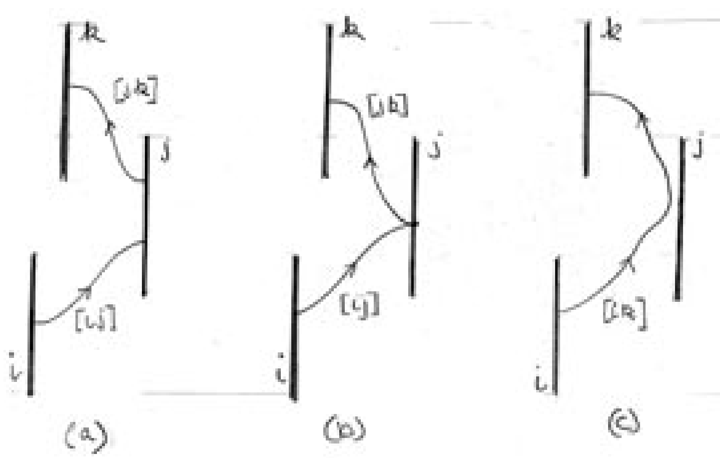

Figure 14.2: The interaction where a string in the sector $[ij]$ combines with a string in the sector $[jk]$ to form a string in the sector $[ik]$. The interaction occurs in (b), when the end of one string coincides with the beginning of the next.

Suppose we have N Dp-branes. This time the sectors will be labeled by pairs $[ij]$ where i and j are integers that run from one to N. The $[ij]$ sector represents open strings starting on the i-th brane and ending on the j-th brane. It is clear that there are as many sectors as there are entries in an $N \times N$ matrix, thus N^2 sectors. String interactions can be visualized neatly. In a typical process, a first open string joins a second open string to form a third open string. To do so, the *end* of the first string ($\sigma = \pi$) joins with the *beginning* of the second string ($\sigma = 0$) to form a third open string. If the open strings are stretched between D-branes, a first string from the $[ij]$ sector can be joined by a second open string from the $[jk]$ sector to give a product open string in the $[ik]$ sector. We write this possible interaction as

$$[ij] * [jk] = [ik] \,, \qquad j \text{ not summed} \,. \tag{14.3.22}$$

This interaction is possible since both the end of the first string and the beginning of the second string lie on the same D-brane, the j D-brane. The physical process can be imagined to take place as in Figure 14.2. In part (a) we see the three D-branes, labelled i, j, and k, and two strings in the $[ij]$ and $[jk]$ sectors, respectively. The end of the $[ij]$ string meets the beginning of the $[jk]$ string in (b). At this stage the interaction takes place and the strings join to form a single string. The resulting string does not remain attached to the j D-brane since the joining point is not anymore the endpoint of any string. As the string moves away from the j D-brane in (c), the string is clearly recognized to belong to the $[ik]$ sector.

If the N Dp-branes are coincident the N^2 sectors result in N^2 interacting massless gauge fields. This defines a $U(N)$ Yang-Mills theory on the world-volume of the N coincident D-branes:

$$\boxed{N \text{ coincident D-branes carry } U(N) \text{ massless gauge fields.}} \tag{14.3.23}$$

The full spectrum of the open string theory consists of N^2 copies of the spectrum of a single Dp-brane. Each copy is a sector of the theory that carries the appropriate $[ij]$ labels.

If we have a single brane, N equals one, and we get a $U(1)$ Yang-Mills theory. The $U(1)$ Yang-Mills theory, having just one massless gauge field, coincides with the Maxwell theory. This is consistent with (14.2.28). Here $U(1)$ denotes a group; the group whose elements are complex numbers of unit length, and where group multiplication is just multiplication. The relevance of the $U(1)$ group arises because the gauge parameters in Maxwell theory are actually elements of the group. So far, our study of Maxwell gauge transformations has made no use of the $U(1)$ group structure. We will see in Chapter 18 that this group structure is needed to understand the gauge symmetry of Maxwell theory in the presence of compact spatial dimensions. For the case of $U(N)$ Yang-Mills theory, $U(N)$ is also a group of symmetries: the group whose elements are $N \times N$ unitary matrices (matrices whose Hermitian conjugates and inverses coincide) and where group multiplication is matrix multiplication. The gauge symmetries of $U(N)$ Yang-Mills theory are described by the group $U(N)$, each group element giving rise to a gauge transformation.

Quick Calculation 14.1. Consult a math book for the definition of a group, and verify that $U(1)$ and $U(N)$, as described above, are groups.

The discrete labels i, j used to label the branes and the various open string sectors are sometimes called *Chan-Paton* indices. They were introduced much before we knew about D-branes precisely in order to obtain Yang-Mills theories with open strings. With the discovery of D-branes it became clear that the Chan-Paton indices were simply labels for the various D-branes in a multi-D-brane configuration.

The appearance of Yang-Mills theories on the world-volume of D-brane configurations is of great relevance because Yang-Mills theories are used to describe the Standard Model of particle physics. Electromagnetism is a $U(1)$ gauge theory, and the photon γ is the quantum of the electromagnetic field. The electroweak theory is described by a $U(2)$ Yang-Mills theory. The four gauge bosons of this theory include the photon γ, the W^+, the W^-, and the Z^0. The latter three are massive gauge fields. The mechanism by which massless gauge fields become massive, known as the Higgs mechanism of field theory, has a D-brane counterpart. It corresponds to separating the D-branes that, when coincident, gives the corresponding massless gauge particles. If we have two coincident D3-branes, we obtain a $U(2)$ Yang-Mills theory, with four massless gauge fields living on the four-dimensional world-volume of the brane. Is this a good model for the electroweak gauge theory? Not quite. If we separate the D3-branes to give mass to some gauge bosons, two of them acquire a mass – the two arising from the stretched strings – and two remain massless. In the electroweak gauge theory only one gauge field remains massless. A more sophisticated D-brane configuration is needed to produce a model of the electroweak theory.

14.4 Strings between Dp-, and Dq-branes

In this section we consider a fairly general case: the case of two parallel D-branes of different dimensionality. Let p and q be two integers both of which satisfy $1 \leq p, q \leq 25$. We are interested in a configuration where we have two D-branes: a Dp-brane and a Dq-brane. We take $p > q$, since the case of branes of equal dimensionality was considered before. The branes can be coincident, if the Dq-brane world-volume is fully contained in the Dp-brane world-volume, or they can be separate. The branes are taken to be parallel.

Coordinate	t	x	y	z
D2	$\times$	$\times$	$\times$	$\bullet$
D1	$\times$	$\times$	$\bullet$	$\bullet$
BC. [D2 , D1]	NN	NN	ND	DD

Table 14.1: In the second and third row we note by $\times$ coordinates along which the D-brane stretches, and by $\bullet$ coordinates normal to the brane. In the last row we indicate the boundary conditions (BC) for open strings belonging to the sector [D2 , D1] of strings that stretch from the D2-brane to the D1-brane.

This means the same as what we mean when we say that a line is parallel to a plane: a second plane parallel to the given plane contains the line. Thus, if separate, there is a p-dimensional hyperplane, parallel to the p-dimensional Dp-brane, that contains the Dq-brane.

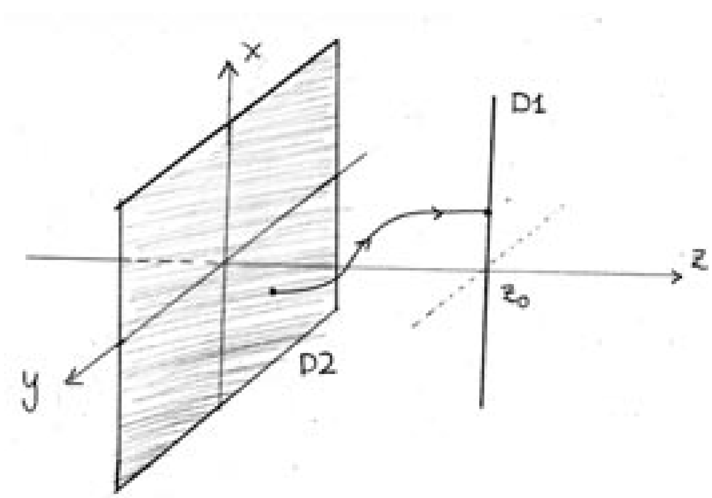

Figure 14.3: A D2-brane stretched on the (x, y) plane and a parallel D1-brane stretched along the x-axis, and located at $y = 0$, $z = z_0$. Also shown is an open string going from the D2-brane to the D1-brane. For such a string, the string coordinate Y is of ND type.

A simple case that can be easily visualized is that of a D2-brane and a parallel D1-brane, as illustrated in Figure 14.3. The D2-brane stretches along the x- and y-directions and is located at $z = 0$. The D1 stretches along the x-direction and is located at $y = 0$ and at $z = z_0$. This D1-brane is parallel to the D2-brane, and they are not coincident when $z_0 \neq 0$. In table 14.1 the relevant information about the D-branes is summarized. With a $(\times)$ we indicate the directions along the world-volume of the D-brane. With a $(\bullet)$ we indicate the directions normal to the D-brane. As a result, the coordinates t and x are common tangential directions. The coordinate y is a mixed direction: one brane extends on this direction, the other brane does not. Finally the z-coordinate is a common normal direction. More generally, for the Dp-, Dq-brane configuration, we have (with $p > q$)

$$\underbrace{x^0,\, x^1,\, \ldots\,, x^q}_{\text{common tangential coordinates}} \qquad \underbrace{x^{q+1},\, x^{q+2},\, \ldots,\, x^p}_{\text{mixed cooordinates}} \qquad \underbrace{x^{p+1},\, x^{p+2},\, \ldots,\, x^d}_{\text{common normal cooordinates}} \; . \tag{14.4.1}$$

We have $(q + 1)$ common tangential coordinates – all the world-volume coordinates of the Dq-brane. We have $(p - q)$ directions that are tangential to the Dp-brane and normal to the Dq-brane. Finally, we have $(d - p)$ common normal directions.

We have already studied in some detail strings that begin and end on the same D-brane, so we focus here on the strings that go from one D-brane to the other. For definiteness, consider the strings that stretch from the Dp-brane to the Dq-brane. We already have partial knowledge about these strings. For example the common tangential coordinates are of NN type since they have Neumann boundary conditions on both endpoints. The common normal coordinates are of DD type, since they have Dirichlet boundary conditions on both endpoints. We had the opportunity to study those coordinates explicitly in the previous sections. There is, however, a new type of string coordinate arising for the directions that are tangential to one of the branes and normal to the other brane. In the case of our D2-, D1-brane example, this is the y-direction. For a string going from the D2-brane to the D1-brane, the string coordinate Y associated to y would be N at its beginning endpoint, since y is tangential to the D2-brane, and D at its final endpoint, since y is normal to the D1-brane. In short, for a string going from the D2-brane to the D1-brane, Y is an ND coordinate. For a string going from the D1-brane to the D2-brane, Y is a DN coordinate.

For the Dp-, Dq-brane configuration, the mixed coordinates were listed in (14.4.1). For open strings stretching from the Dp-brane to the Dq-brane, the analogously labeled string coordinates satisfy a Neumann boundary condition on the starting Dp-brane, and a Dirichlet boundary condition on the ending Dq-brane. They are ND coordinates. In summary, the string coordinates split into

$$\underbrace{X^0,\, X^1,\, \ldots\, ,X^q}_{\text{NN coordinates}} \quad \underbrace{X^{q+1},\, X^{q+2},\, \ldots,\, X^p}_{\text{ND cooordinates}} \quad \underbrace{X^{p+1},\, X^{p+2},\, \ldots,\, X^d}_{\text{DD cooordinates}} \,. \qquad (14.4.2)$$

In light-cone, we need three types of indices to label the string coordinates:

$$\underbrace{X^+, X^-, \{X^i\}}_{\text{NN}} \;\; \underbrace{\{X^r\}}_{\text{ND}} \;\; \underbrace{\{X^a\}}_{\text{DD}}, \qquad (14.4.3)$$

where

$$i = 2, \ldots, q, \quad r = q+1, \ldots, p, \quad \text{and} \quad a = p+1, \ldots, d\,. \qquad (14.4.4)$$

We think of the Dp-brane as the first brane, and the Dq-brane as the second brane. The position of the Dp-brane is specified by the coordinates $\bar{x}_1^a$. The position of the Dq-brane is specified by the coordinates $\bar{x}_2^r$ and $\bar{x}_2^a$. In our D2-, D1-brane example of Figure 14.3, the role of $\bar{x}_2^r$ is played by the y-coordinate of the D1-brane. This coordinate can be set to zero by a suitable choice of axes.

Let us begin our analysis of the ND coordinates X^r. The boundary conditions are

$$\frac{\partial X^r}{\partial \sigma}(\tau,\sigma)\Big|_{\sigma=0} = 0\,, \quad X^r(\tau,\sigma)\Big|_{\sigma=\pi} = \bar{x}_2^r\,. \qquad (14.4.5)$$

The $\bar{x}_2^r$ coordinates, as mentioned above, could have been chosen to be equal to zero by a suitable choice of axes. As opposed to the coordinate differences $\bar{x}_1^a - \bar{x}_2^a$ that define the separation between the D-branes, the $\bar{x}_2^r$ coordinates will not play any significant role. Consider now the usual expansion

$$X^r(\tau,\sigma) = \frac{1}{2}\Big(f^r(\tau+\sigma) + g^r(\tau-\sigma)\Big)\,. \qquad (14.4.6)$$

The boundary condition at $\sigma = 0$ gives us

$$f^{r\prime}(u) = g^{r\prime}(u)\,, \quad \rightarrow \quad g^r(u) = f^r(u) + c_0^r\,. \qquad (14.4.7)$$

Bearing in mind that the second boundary condition will set X^r equal to $\bar{x}_2^r$ at $\sigma = \pi$, we choose $c_0^r = 2\bar{x}_2^r$ so that

$$X^r(\tau, \sigma) = \bar{x}_2^r + \frac{1}{2}\Big(f^r(\tau+\sigma) + f^r(\tau-\sigma)\Big)\,. \qquad (14.4.8)$$

The condition at $\sigma = \pi$ gives us

$$f^r(u + 2\pi) = -f^r(u)\,. \qquad (14.4.9)$$

The function $f^r(u)$ goes into minus itself when its argument increases by 2π. To find an appropriate mode expansion, we first note that $f^r(u)$ is *periodic with period* 4π. This is a necessary but not sufficient condition for (14.4.9) to hold. Any function with period T can be expanded in terms of the basis functions

$$\Big\{\cos\Big(\frac{2\pi n u}{T}\Big), \sin\Big(\frac{2\pi n u}{T}\Big)\Big\} \quad n = 0, 1, 2, \ldots, \infty\,. \qquad (14.4.10)$$

For the case of interest $T = 4\pi$ and therefore we can write

$$f^r(u) = \sum_{n=0}^{\infty}\Big[f_n^r \cos\Big(\frac{nu}{2}\Big) + h_n^r \sin\Big(\frac{nu}{2}\Big)\Big]\,. \qquad (14.4.11)$$

We now impose the condition (14.4.9) of anti-periodicity. Indeed, computing $f^r(u + 2\pi)$:

$$\begin{aligned} f^r(u+2\pi) &= \sum_{n=0}^{\infty}\Big[f_n^r \cos\Big(\frac{nu}{2} + n\pi\Big) + h_n^r \sin\Big(\frac{nu}{2} + n\pi\Big)\Big]\,, \\ &= \sum_{n=0}^{\infty}(-1)^n\Big[f_n^r \cos\Big(\frac{nu}{2}\Big) + h_n^r \sin\Big(\frac{nu}{2}\Big)\Big]\,. \end{aligned} \qquad (14.4.12)$$

For this right hand side to be precisely equal to the negative of the right hand side in (14.4.11) we need to restrict the sums to odd n. Therefore, we have

$$f^r(u) = \sum_{n \text{ odd}}\Big[f_n^r \cos\Big(\frac{nu}{2}\Big) + h_n^r \sin\Big(\frac{nu}{2}\Big)\Big]\,. \qquad (14.4.13)$$

Finally, substituting back into (14.4.8) and relabeling the expansion coefficients, we get

$$X^r(\tau, \sigma) = \bar{x}_2^r + \sum_{n \text{ odd}}\Big[A_n^r \cos\Big(\frac{n\tau}{2}\Big) + B_n^r \sin\Big(\frac{n\tau}{2}\Big)\Big]\cos\Big(\frac{n\sigma}{2}\Big)\,. \qquad (14.4.14)$$

This is our expansion of the ND coordinates.

To proceed with the quantization we define useful oscillators. Note that in all the examples considered thus far the α_n oscillator is matched with an exponential $e^{-in\tau}$. In the present case, this suggests that the oscillators must carry fractional mode labels! Another useful guidance is the desired simplicity of $\dot{X}^r \pm X^{r\prime}$. With these pointers, we are led to write

$$X^r(\tau,\sigma) = \bar{x}_2^r + i\sqrt{2\alpha'} \sum_{n\,\in\mathbb{Z}_{\rm odd}} \frac{2}{n}\, \alpha^r_{\frac{n}{2}}\, e^{-i\frac{n}{2}\tau} \cos\Big(\frac{n\sigma}{2}\Big)\,, \qquad (14.4.15)$$

where the sum runs over both positive and negative odd integers. The factor of i in front of the sum is necessary so that the Hermiticity of X^r imposes the standard Hermiticity property:

$$\big(\alpha^r_{\frac{n}{2}}\big)^\dagger = \alpha^r_{-\frac{n}{2}}\,. \qquad (14.4.16)$$

Note that the $\bar{x}_2^r$ are constants and do not become operators. There are no zero modes in the expansion of X^r, and therefore ND coordinates carry no momentum. We also record the derivatives

$$\dot{X}^r \pm X^{r\prime} = \sqrt{2\alpha'} \sum_{n\,\in\mathbb{Z}_{\rm odd}} \alpha^r_{\frac{n}{2}}\, e^{-i\frac{n}{2}(\tau\pm\sigma)}\,, \qquad (14.4.17)$$

which are indeed of the expected form. The (non-trivial) commutation relations take the form

$$[X^r(\tau,\sigma), \dot{X}^s(\tau,\sigma')] = i(2\pi\alpha')\delta(\sigma-\sigma')\delta^{rs}\,. \qquad (14.4.18)$$

Since both the expansions (14.4.17) and the commutators (14.4.18) take standard form, we can use (12.2.11), which for the present case reads

$$\Big[\,(\dot{X}^r \pm X^{r\prime})(\tau,\sigma)\,,\,(\dot{X}^s \pm X^{s\prime})(\tau,\sigma')\,\Big] = 4\pi\alpha'\, i\eta^{rs}\frac{d}{d\sigma}\delta(\sigma-\sigma')\,. \quad (14.4.19)$$

Additionally, equation (12.2.14), with minor modifications also holds for $\sigma,\sigma' \in [0,2\pi]$:

$$\sum_{m',n'\in\mathbb{Z}_{\rm odd}} e^{-i\frac{m'}{2}(\tau+\sigma)} e^{-i\frac{n'}{2}(\tau+\sigma')}[\alpha^r_{m'}\,,\alpha^s_{n'}\,] = 2\pi\, i\eta^{rs}\frac{d}{d\sigma}\delta(\sigma-\sigma')\,. \quad (14.4.20)$$

To extract the commutators we apply the following integral operators to the left and to the right of the equation The operations are

$$\frac{1}{2\pi}\int_0^{2\pi} d\sigma\, e^{i\frac{m}{2}\sigma}\,\cdot\,\frac{1}{2\pi}\int_0^{2\pi} d\sigma'\, e^{i\frac{n}{2}\sigma'}\,, \tag{14.4.21}$$

where m and n are odd integers. The set of functions $e^{i\frac{k}{2}\sigma}$ with $k \in \mathbb{Z}_{\text{odd}}$ are all orthogonal over the interval $[0, 2\pi]$. This happens because the sum or difference of two odd integers is an even integer. The integrations in (14.4.21) therefore select a single commutator, and one can show that

$$\Big[\,\alpha^r_{\frac{m}{2}}\,,\alpha^s_{\frac{n}{2}}\,\Big] = \frac{m}{2}\,\delta^{rs}\,\delta_{m+n,0}\,. \tag{14.4.22}$$

This is the expected form of the commutation relations.

Quick Calculation 14.2. Prove equation (14.4.22).

Let's now calculate the mass-squared operator. This operator receives contributions from all the coordinates in this sector: the NN-, the ND-, and the DD-coordinates. This is clear from (14.2.1) since the original light-cone index I now runs over i, r, and a labels. Given that the linear combination of derivatives (14.4.17) take the standard form, the contribution from the ND coordinates takes a familiar form. At the classical level can be read as a minor modification of equation (14.3.12), finding

$$\begin{aligned}2p^+\,p^- = \frac{1}{\alpha'}\Big(\alpha' p^i p^i + \frac{1}{2}\,\alpha_0^a\,\alpha_0^a + \sum_{n=1}^{\infty}\Big[\alpha^i_{-n}\,\alpha^i_n + \alpha^a_{-n}\,\alpha^a_n\Big]\\ + \sum_{m\in\mathbb{Z}^+_{\text{odd}}}\alpha^r_{-\frac{m}{2}}\alpha^r_{\frac{m}{2}} - a\Big)\,.\end{aligned} \tag{14.4.23}$$

Here $\mathbb{Z}^+_{\text{odd}}$ denotes the odd positive integers. In writing this equation we have restored the subtraction constant a because an issue arises with the ordering of the oscillators. This issue can be settled using the heuristic arguments described in the discussion of (12.4.7) and of (12.4.15). Since all the oscillators for the NN and DD directions are integrally-moded, their normal-ordering constant is the same, and each of these coordinates contributes to a an amount

$$\frac{1}{2}\Big(1+2+3+4+\cdots\Big) = \frac{1}{2}\Big(-\frac{1}{12}\Big) = -\frac{1}{24}\,. \tag{14.4.24}$$

With a total of 24 transverse light-cone coordinates, if we only have NN and DD coordinates we get a normal ordering constant of (-1). The ND coordinates, however, give a different subtraction. The sum that must be rearranged in this case is

$$\frac{1}{2}\sum_{m\in\mathbb{Z}_{\rm odd}} \alpha^r_{-\frac{m}{2}}\alpha^r_{\frac{m}{2}} = \sum_{m\in\mathbb{Z}^+_{\rm odd}} \alpha^r_{-\frac{m}{2}}\alpha^r_{\frac{m}{2}} + \frac{1}{2}\sum_{m\in\mathbb{Z}^+_{\rm odd}}\Big[\alpha^r_{\frac{m}{2}}\,,\alpha^r_{-\frac{m}{2}}\Big] \tag{14.4.25}$$

The first term in the right hand side is the one that appears in (14.4.23), and the second term is the ordering contribution. Since we have $(p-q)$ ND-coordinates, the ordering constant is

$$\frac{1}{2}\sum_{m\in\mathbb{Z}^+_{\rm odd}}\Big[\alpha^r_{\frac{m}{2}}\,,\alpha^r_{-\frac{m}{2}}\Big] = \frac{1}{4}\,(p-q)\sum_{m\in\mathbb{Z}^+_{\rm odd}} m\,, \tag{14.4.26}$$

where we used (14.4.22). We need to calculate the sum of all odd integers. This is an infinite sum that can be given a concrete finite value using (14.4.24). This is done as follows:

$$\sum_{n=1}^{\infty} n = \sum_{n\in\mathbb{Z}^+_{\rm odd}} n + \sum_{n\in\mathbb{Z}^+_{\rm even}} n = \sum_{n\in\mathbb{Z}^+_{\rm odd}} n + 2\sum_{n=1}^{\infty} n\,, \tag{14.4.27}$$

and, as a result,

$$\sum_{n\in\mathbb{Z}^+_{\rm odd}} n = -\sum_{n=1}^{\infty} n = \frac{1}{12}\,. \tag{14.4.28}$$

It follows from this result that the contribution (14.4.26) to the normal ordering constant from the ND coordinates is

$$\frac{1}{4}\,(p-q)\sum_{m\in\mathbb{Z}^+_{\rm odd}} m = \frac{1}{48}(p-q)\,. \tag{14.4.29}$$

This calculation has shown that each ND coordinate contributes $+1/48$ to the normal ordering constant in $\alpha' M^2$. A DN string coordinate will contribute exactly the same constant. In summary, the normal ordering contributions to $\alpha' M^2$ for the various string coordinates are

$$\boxed{a_{NN} = a_{DD} = -\frac{1}{24}\,, \qquad a_{ND} = a_{DN} = \frac{1}{48}\,.} \tag{14.4.30}$$

Returning to the problem at hand, the total ordering constant a is given by (14.4.29) plus the contribution of the $(24-(p-q))$ coordinates that are either NN or DD:

$$a = -\frac{1}{24}\big(24-(p-q)\big) + \frac{1}{48}(p-q) = -1 + \frac{1}{16}(p-q)\,. \tag{14.4.31}$$

With this information we can now write the expression for M^2. Following the same steps as in (14.3.13) we now find

$$M^2 = \Big(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\Big)^2 + \frac{1}{\alpha'}\Big(N^\perp - 1 + \frac{1}{16}(p-q)\Big)\,, \tag{14.4.32}$$

where

$$N^\perp = \sum_{n=1}^{\infty}\sum_{i=2}^{q} n a_n^{i\,\dagger} a_n^i + \sum_{k\in\mathbb{Z}^+_{\text{odd}}}\ \sum_{r=q+1}^{p} \frac{k}{2} a_{\frac{k}{2}}^{r\dagger} a_{\frac{k}{2}}^{r} + \sum_{m=1}^{\infty}\sum_{a=p+1}^{d} m\, a_m^{a\,\dagger} a_m^a\,. \tag{14.4.33}$$

This formula incorporates all the effects we have discussed – a normal ordering constant that is shifted upwards, a contribution due to the stretched strings if the branes do not coincide, and a number operator that now includes contributions from NN, DD, and ND coordinates.

What is the state space, and what are the fields associated with the two lowest mass levels of the quantum open strings stretching between the branes? The ground states are labeled as

$$|p^+, \vec{p}\,; [12]\rangle\,, \qquad \vec{p} = (p^2, \ldots, p^q)\,. \tag{14.4.34}$$

The momentum labels of the states indicate that the quantum states give rise to fields that live in a $(q+1)$-dimensional spacetime. Roughly, they live on the world-volume of the Dq-brane, the brane of lower dimensionality. The general rule is clear – the spacetime dimensionality for the fields arising in any given sector of the state space equals the number of NN string coordinates in the sector. The state space is built by letting the three types of oscillators – $a_p^{i\dagger}$, $a_{k/2}^{r\dagger}$, and $a_m^{a\dagger}$ – act on the vacuum states.

The ground states have $N^\perp = 0$ and correspond to a single scalar field on the Dq-brane. This scalar is in general massive, but can be tachyonic or massless depending on the separation of the branes and the value of $p-q$. Assume, for simplicity, that the branes coincide. If additionally $p-q=16$, the scalar would be massless. The next-level states are of the form

$$a_{\frac{1}{2}}^{r\dagger}|p^+, \vec{p}\,; [12]\rangle\,, \quad N^\perp = 1/2\,. \tag{14.4.35}$$

These states give rise to $(p - q)$ of scalar fields, since the index r does not correspond to a world-volume direction on the Dq-brane. All other states are necessarily massive since they have $N^{\perp} \geq 1$, and this together with $p > q$ implies that $M^2 > 0$. In particular, we do not find massless gauge fields.

Problems

Problem 14.1. *A Dp-brane with orientifolds.*

This problem can be viewed as a sequel to Problem 13.6. We study here the effects of orientifolds on open strings.

The space-filling O25-plane truncates the spectrum down to the set of states invariant under a twist that acts on all string coordinates (see final comments in Problem 13.6). When we have a Dp-brane, the twist operator acts on the open string coordinates as follows

$$\Omega\, X^a(\tau,\sigma)\,\Omega^{-1} = X^a(\tau,\pi-\sigma)\,, \tag{1}$$

$$\Omega\, X^i(\tau,\sigma)\,\Omega^{-1} = X^i(\tau,\pi-\sigma)\,. \tag{2}$$

As usual, assume that $\Omega\, x_0^-\,\Omega^{-1} = x_0^-$, and $\Omega\, p^+\,\Omega^{-1} = p^+$.

(a) Give the action of the twist operator on the oscillators α_n^a and α_n^i. What is the expected twist action on α_n^-? Does it work out?

(b) Assume that the ground states $|p^+,\vec{p}\rangle$ are twist invariant. Find the states of the theory for $N^\perp \le 2$. As you will see, some massless states survive. Interpret these states along the lines of the discussion below (14.2.30).

Replace the O25-plane by an Op-plane coincident with the Dp-brane at $\bar{x}^a = 0$. Let Ω_p denote the operator for which this theory keeps only the states with $\Omega_p = +1$.

(c) How should equations (1) and (2) change when Ω is replaced by Ω_p? Give the Ω_p action on the oscillators α_n^a and α_n^i.

(d) Describe the full spectrum of the theory as a simple truncation of the Dp-brane spectrum. You will find no massless scalars in this case. What does this suggest regarding possible motions of the Dp-brane?

Problem 14.2. *String products and orientation reversing symmetries.*

Equation (14.3.22) tells how open string *sectors* combine under interactions. The same product notation can be used for strings. By

$$|A\rangle * |B\rangle\,, \tag{1}$$

we mean the string state that is obtained when a string in state $|A\rangle$ interacts with a string in state $|B\rangle$. The string product must obey the rule of sectors: the state in (1) must belong to the sector $[A] * [B]$, where $[A]$ and $[B]$ denote the sectors where string states $|A\rangle$ and $|B\rangle$ belong, respectively.

Use pictures of strings A and B to motivate the equations

$$\Omega\left(|A\rangle * |B\rangle\right) = \left(\Omega|B\rangle\right) * \left(\Omega|A\rangle\right), \tag{2}$$

$$\Omega_p\left(|A\rangle * |B\rangle\right) = \left(\Omega_p|B\rangle\right) * \left(\Omega_p|A\rangle\right). \tag{3}$$

Here Ω is string orientation reversal and Ω_p is orientifolding (orientation reversal plus reflection about a set of coordinates).

Problem 14.3. *N coincident Dp-branes and orientifolds.*

Let N coincident Dp-branes coincide with an Op-plane, all of them located at $\bar{x}^a = 0$. The orientifolding symmetry Ω_p, as usual, includes reflection of the coordinates normal to the orientifold and orientation reversal of strings. Assume that the reflection of coordinates leaves each of the Dp-branes invariant (as opposed to mapping them into each other).

(a) Explain why it is reasonable to postulate that

$$\Omega_p|p^+, \vec{p}; [ij]\rangle = |p^+, \vec{p}; [ji]\rangle .$$

What are the ground states of the theory? How many are there?

(b) Describe the full open string spectrum of the theory in terms of the spectrum of a single Dp-brane. Check that for $N = 1$ you reproduce the result of Problem 14.1, part (d).

Problem 14.4. *Separated Dp-branes and an Op-plane.*

We have learned that an orientifold acts as a kind of mirror. If we are to have D-branes that do not coincide with an orientifold, there must be mirror D-branes at the reflected points. Therefore, to analyze the theory of Dp-branes off an orientifold Op-plane we begin with N Dp-branes and N mirror Dp-branes at the reflected positions. We must then define the orientifold action on all the states of the theory of $2N$ Dp-branes. Finally, we use this action to truncate down to the invariant states, obtaining in this way the states of the orientifold theory.

Consider the situation illustrated in Figure 14.4, where we show the configuration as seen in a plane spanned by two coordinates normal to the branes and the orientifold. The N Dp branes are labelled $1, 2, \ldots, N$, and the mirror images are labeled $\bar{1}, \bar{2}, \ldots, \bar{N}$. Two strings are exhibited: one in the [24] sector and the other in the $[1\bar{1}]$ sector.

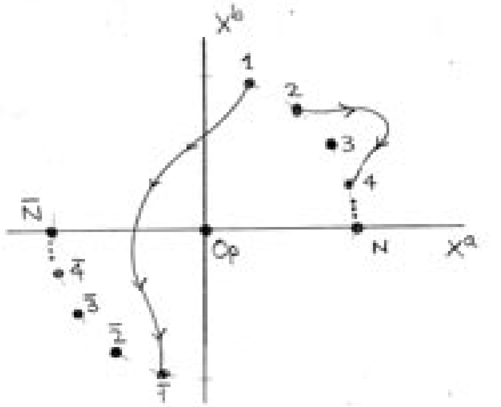

Figure 14.4: Problem 14.4. A set of N Dp-branes together with a set of N image Dp-branes. The orientifold Op plane is at the origin.

(a) Show the two strings obtained by the orientifold symmetry. Since the arguments $p^+, \vec{p}$ of the ground states are always present, let's omit them for brevity. The ground states are of four types:

$$|[ij]\rangle\,, \quad |[i\,\bar{j}]\rangle\,, \quad |[\bar{i}\,j]\rangle\,, \quad |[\bar{i}\,\bar{j}]\rangle\,. \tag{1}$$

Each class contains N^2 ground states since i and j run from 1 to N, and $\bar{i}$ and $\bar{j}$ run from $\bar{1}$ to $\bar{N}$. Define an expected action of Ω_p on the ground states in (1). Show that your choice satisfies $\Omega_p^2 = 1$ acting on the ground states.

(b) What are the possible interactions between strings in the four types of sectors built on the states (1)? Write your answers using the notation of (14.3.22).

(c) It is a fact about string interactions that the string product of ground states gives states that have a component along a ground state. Thus, for example,

$$|[i\,\bar{j}]\rangle * |[\bar{j}\,k]\rangle = |[i\,k]\rangle + \ldots\,. \tag{2}$$

Write the other possible ground state products. Test the consistency of your definition of Ω_p by acting with Ω_p on both sides of the equations giving ground state products. To act on products use equation (3) of Problem 14.2.

(d) Find the Ω_p action on the α_n^I oscillators using $\Omega_p X^I(\tau,\sigma)\Omega_p^{-1} = X^I(\tau, \pi - \sigma)$. Since the strings are stretched along or have nonzero values for the

x^a coordinates an equation of the type $\Omega_p X^a(\tau,\sigma)\Omega_p^{-1} = -X^a(\tau,\pi-\sigma)$ cannot be fully implemented. Using (14.2.13) for X^a, for example, we would need $\Omega_p \bar{x}^a \Omega_p^{-1} = -\bar{x}^a$, which cannot hold since the $\bar{x}^a$ are numbers. A legal derivation of the Ω_p action on the α_n^a oscillators can be obtained by requiring $\Omega_p \dot{X}^a(\tau,\sigma)\Omega_p^{-1} = -\dot{X}^a(\tau,\pi-\sigma)$. Verify that for any arbitrary product R of oscillators of both types

$$\Omega_p\, R\, \Omega_p^{-1} = (-1)^{N^\perp} R\,, \tag{3}$$

where $N^\perp$ is the total number of R.

(e) Describe the orientifold spectrum in terms of the spectrum of a single Dp-brane. For this consider an arbitrary product R of oscillators and build the general states

$$\sum_{ij}\Big(r_{ij} R|[ij]\rangle + r_{i\bar{j}} R|[i\bar{j}]\rangle + r_{\bar{i}j} R|[\bar{i}j]\rangle + r_{\bar{i}\bar{j}} R|[\bar{i}\bar{j}]\rangle \Big)\,, \tag{4}$$

where $r_{ij}, r_{i\bar{j}}, r_{\bar{i}j}$, and $r_{\bar{i}\bar{j}}$ are a set of four N by N matrices. Find the conditions that Ω_p invariance imposes on these matrices. There are two cases to consider, depending on the number $N^\perp$ of R. You should find that for $N^\perp$ odd there are $N(2N-1)$ linearly independent states in (4). For $N^\perp$ even there are $N(2N+1)$ linearly independent states.

Since the gauge fields arise from $N^\perp = 1$ states, there are $N(2N-1)$ of them. This is the number of entries in a $2N$ by $2N$ antisymmetric matrix. The interacting theory of such gauge fields is an $SO(2N)$ Yang-Mills gauge theory (SO stands for special orthogonal). The $SO(10)$ gauge theory, for example, can be used to build a grand unified theory of the strong and electroweak interactions.

Problem 14.5. *Separated Dp-branes and a different Op-plane.*

The brane setup here is that of Problem 14.4. In part (a) of that problem, you defined a simple action of Ω_p on the ground states $|[ij]\rangle$, $|[i\,\bar{j}]\rangle$, $|[\bar{i}\,j]\rangle$, and $|[\bar{i}\,\bar{j}]\rangle$. Find an alternative Ω_p action where some of the relations have a minus sign: $\Omega_p|[\ldots]\rangle = \pm|[\ldots]\rangle$. Test its consistency by verifying that $\Omega_p^2 = 1$ on ground states, and that the products of ground states are compatible with Ω_p action, as you tested in part (c) of Problem 14.4. Determine the spectrum in this variant orientifold theory.

The gauge fields arise from $N^\perp = 1$, and you will find that there are $N(2N+1)$ of them. The interacting theory of such gauge fields is an $USp(2N)$ Yang-Mills gauge theory (USp stands for unitary symplectic).

Problem 14.6. *DN string coordinates.*

In section 14.4 we considered strings stretching from a Dp- to a Dp-brane, focusing on coordinates X^r satisfying ND boundary conditions. Consider now strings stretching from the Dq- to the Dp-brane. For such strings, the coordinates X^r are of DN type.

(a) Write the boundary conditions satisfied by the X^r coordinates and use them to derive a mode expansion along the lines of our result (14.4.15) for a ND coordinate.

(b) Find also the equations that replace (14.4.17). Explain briefly why the mass-squared formula (14.4.32) needs no modification.

(c) If we let $\sigma \to \pi - \sigma$ in (14.4.15) we get automatically a function with DN boundary conditions. Compare with the mode expansion you found in (a), and explain why the Hermiticity properties are consistent.

Problem 14.7. *Strings in a configuration with a Dp-brane and a D25-brane.*

Consider the full state space of open string theory in a configuration with a Dp-brane and a D25-brane. Assume $1 \leq p \leq 24$. For each sector of the theory, give the M^2 operator, and examine explicitly the states arising in the two lowest levels indicating the types of fields they correspond to and where these fields live.

Problem 14.8. *A pair of intersecting D22-branes.*

We study here a configuration of two D22-branes One of them, henceforth called brane 1, is defined by $x^{25} = x^{23} = x^{22} = 0$. The other one henceforth called brane 2, is defined by $x^{24} = x^{23} = x^{22} = 0$.

Draw a picture of the brane configuration as it appears in the (x^{24}, x^{25}) plane. Construct a table, such as table 14.1, adding more columns to include all coordinates, and more rows to include all sectors. For each sector of the theory, give the M^2 operator and examine explicitly the states arising in the two lowest levels indicating the types of fields they correspond to and where these fields live.

A basic point: if x^a is a Dirichlet direction for a configuration of two intersecting D-branes, the x^a coordinates of the two D-branes must be the same. Explain why.

Chapter 15

String Charge and Electric Charge

If a point particle couples to the Maxwell field, the point particle carries electric charge. Strings couple to the Kalb-Ramond field, therefore, strings carry a new kind of charge – string charge. For a stretched string, string charge can be visualized as a current flowing along the string. For a string to end on a D-brane without violating string charge conservation, interesting effects must take place: the string endpoints must carry quantized electric charge, and electric field lines on the D-brane must carry string charge.

15.1 Fundamental string charge

As we have seen before, a point particle can carry electric charge because there is an allowed interaction coupling the particle to a Maxwell field. The world-line of the point particle is one-dimensional and the Maxwell gauge field A_μ carries one index. This matching is important: the particle trajectory has a tangent vector $dx^\mu(\tau)/d\tau$, where τ parameterizes the world-line. Having one Lorentz index, the tangent vector can be multiplied by the gauge field A_μ to form a Lorentz scalar. For a point particle of charge q, the interaction is written as

$$\frac{q}{c}\int A_\mu(x(\tau))\frac{dx^\mu(\tau)}{d\tau}\,d\tau\,. \qquad (15.1.1)$$

The complete interacting system of the charged particle and the Maxwell field is defined by the action considered in Problem 5.6:

$$S' = -mc\int_{\mathcal{P}} ds + \frac{q}{c}\int_{\mathcal{P}} A_\mu(x)dx^\mu - \frac{1}{16\pi c}\int d^D x \left(F_{\mu\nu}F^{\mu\nu}\right). \tag{15.1.2}$$

The first term on the right-hand side is the particle action, and the last term is the action for the Maxwell field.

Can a relativistic string be charged? The above argument makes it clear that Maxwell charge is naturally carried by points. Closed strings do not have special points, but open strings do. It is therefore plausible that open string endpoints may carry electric Maxwell charge. We will show later that this is indeed the case. At this moment, however, we are looking for something fundamentally different. Since electric Maxwell charge is naturally associated to points, we may wonder if there is some new kind of charge that is naturally associated to strings. For a new kind of charge, we need a new kind of gauge field. Thus, we may ask: Is there a field in string theory that is related to the string, as the Maxwell field is related to a particle? The answer is yes. The field is the Kalb-Ramond antisymmetric two-tensor $B_{\mu\nu}(= -B_{\nu\mu})$, a massless field in closed string theory.

Let us now mimic the logic that led to (15.1.1). At any point of the string trajectory we have two linearly independent tangent vectors. Indeed, with world-sheet coordinates τ and σ, the two tangent vectors can be chosen to be $\partial X^\mu/\partial\tau$ and $\partial X^\mu/\partial\sigma$. Having two tangent vectors, we can use the two-index field $B_{\mu\nu}$ to construct a Lorentz scalar:

$$\boxed{-\int d\tau d\sigma\ \frac{\partial X^\mu}{\partial\tau}\frac{\partial X^\nu}{\partial\sigma}\, B_{\mu\nu}\left(X(\tau,\sigma)\right).} \tag{15.1.3}$$

This is how the string couples to the antisymmetric Kalb-Ramond field. It is called an *electric* coupling, because it is the natural generalization of the electric coupling of a point particle to a Maxwell field. Thus we will say that the string carries *electric Kalb-Ramond charge* – a statement that we must understand in detail. The coupling in (15.1.3) is invariant under reparameterizations of the world-sheet (Problem 15.1), although interestingly, not fully so. This will have physical consequences, as we shall see shortly.

Just as (15.1.2) represents the complete dynamics of the particle and the Maxwell field, for the string, the coupling (15.1.3) must be supplemented by the string action $S_{\rm str}$ and a term giving dynamics to the $B_{\mu\nu}$ field:

$$S = S_{\rm str} - \frac{1}{2}\int d\tau d\sigma B_{\mu\nu}\left(X(\tau,\sigma)\right)\frac{\partial X^{[\mu}}{\partial \tau}\frac{\partial X^{\nu]}}{\partial \sigma} + \int d^D x\Big(-\frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big)\,, \tag{15.1.4}$$

where we have defined the antisymmetrization

$$a^{[\mu}b^{\nu]} \equiv a^\mu b^\nu - a^\nu b^\mu\,. \tag{15.1.5}$$

This antisymmetrization is responsible for the factor of 1/2 multiplying the second term in the right hand side of (15.1.4). We have antisymmetrized the factor multiplying $B_{\mu\nu}$ because it is natural to do so: since $B_{\mu\nu}$ is antisymmetric, the symmetric part of the factor cannot contribute to the product. The last term in the action uses the square of the field strength $H_{\mu\nu\rho}$ associated to $B_{\mu\nu}$. As discussed in Problem 10.6:

$$H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}\,. \tag{15.1.6}$$

Note the hybrid nature of the action (15.1.4): part of it is an integral over the string world-sheet, and part of it is an integral over all of spacetime.

In order to appreciate the nature of string charge, we reconsider Maxwell equations (3.3.23), where the electric current appears as a source of the electromagnetic field:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{4\pi}{c} j^\mu\,. \tag{15.1.7}$$

Here the electric charge density is j^0/c. Moreover, a static particle gives rise to electric charge, but zero electric current. The particle is a source for the Maxwell field. The string, on the other hand, is a source for the $B_{\mu\nu}$ field, and we will have an equation analogous to (15.1.7). This equation is the equation of motion for the $B_{\mu\nu}$ field, which we will obtain now by calculating the variation of the action (15.1.4) under a variation $\delta B_{\mu\nu}$. The variation of the last term in the action was calculated in Problem 10.6:

$$\delta\left[\int d^D x\big(-\frac{1}{6}HH\big)\right] = \int d^D x\,\delta B_{\mu\nu}(x)\,\frac{\partial H^{\mu\nu\rho}}{\partial x^\rho}\,. \tag{15.1.8}$$

To vary the second term in S we must vary $B_{\mu\nu}(x)$, but in this term the field is evaluated on the string world-sheet. This field can be rewritten as an integral over all of spacetime of $B_{\mu\nu}(x)$ times a delta function localizing the field to the world-sheet:

$$B_{\mu\nu}\left(X(\tau,\sigma)\right) = \int d^D x\ \delta^D\left(x - X(\tau,\sigma)\right) B_{\mu\nu}(x)\,. \tag{15.1.9}$$

Using this identity, the second term in S is rewritten as

$$\begin{aligned} & -\int d^D x B_{\mu\nu}(x)\,\frac{1}{2}\int d\tau d\sigma \delta^D\left(x - X(\tau,\sigma)\right)\frac{\partial X^{[\mu}}{\partial\tau}\frac{\partial X^{\nu]}}{\partial\sigma} \\ & \equiv -\int d^D x B_{\mu\nu}(x) j^{\mu\nu}(x)\,, \end{aligned} \tag{15.1.10}$$

where we have introduced the symbol $j^{\mu\nu}$ with value

$$j^{\mu\nu}(x) = \frac{1}{2}\int d\tau d\sigma\, \delta^D\left(x - X(\tau,\sigma)\right)\Big(\frac{\partial X^\mu}{\partial\tau}\frac{\partial X^\nu}{\partial\sigma} - \frac{\partial X^\nu}{\partial\tau}\frac{\partial X^\mu}{\partial\sigma}\Big)\,. \tag{15.1.11}$$

It is noteworthy that $j^{\mu\nu}$ is only supported on spacetime points that belong to the string world-sheet. Indeed, if x is not on the world-sheet, the argument of the delta function is never zero, and the integral vanishes. The object $j^{\mu\nu}$ will play the role of a current. By construction, it is antisymmetric under the exchange of its indices:

$$j^{\mu\nu} = -j^{\nu\mu}\,. \tag{15.1.12}$$

We have now done all the work needed to find the equation of motion for $B_{\mu\nu}$. Combining equations (15.1.8) and (15.1.10), the total variation of the action S is

$$\delta S = \int d^D x\, \delta B_{\mu\nu}(x)\Big(\frac{\partial H^{\mu\nu\rho}}{\partial x^\rho} - j^{\mu\nu}\Big)\,. \tag{15.1.13}$$

If this variation is to vanish for arbitrary but antisymmetric $\delta B_{\mu\nu}$, the antisymmetric part of the factor multiplying $\delta B_{\mu\nu}$ must vanish (Problem 15.2). Since the term in parenthesis is antisymmetric, it must vanish:

$$\frac{\partial H^{\mu\nu\rho}}{\partial x^\rho} = j^{\mu\nu}\,. \tag{15.1.14}$$

Quick Calculation 15.1. To test your understanding of antisymmetric variations, consider indices $i, j = 1, 2$ that run over two values, and arbitrary antisymmetric variations δB_{ij} such that $\delta B_{ij} G^{ij} = 0$. Show explicitly that the only condition you get is $G^{ij} - G^{ji} = 0$.

The similarity between (15.1.14) and (15.1.7) is quite remarkable. It suggests that $j^{\mu\nu}$ is some kind of conserved current. The vector j^μ in the right hand side of (15.1.7) is a conserved current because

$$\frac{4\pi}{c}\frac{\partial j^\mu}{\partial x^\mu} = \frac{\partial^2 F^{\mu\nu}}{\partial x^\mu \partial x^\nu} = 0\,, \tag{15.1.15}$$

on account of the antisymmetry of $F^{\mu\nu}$ and the exchange symmetry of the partial derivatives. In an exactly similar fashion, equation (15.1.14) gives

$$\frac{\partial j^{\mu\nu}}{\partial x^\mu} = \frac{\partial^2 H^{\mu\nu\rho}}{\partial x^\mu \partial x^\rho} = 0\,. \tag{15.1.16}$$

The μ index in $j^{\mu\nu}$ is tied to the conservation equation, and the ν index is free. The tensor $j^{\mu\nu}$ can thus be viewed as a set of currents labelled by the index ν. For each fixed ν, the current components are given by the various values of μ. Since the zeroth component of a current is a charge density, we have several charge densities $j^{0\nu}$. More precisely, since $j^{00} = 0$ (15.1.12), the non-vanishing charge densities are j^{0k}, with k running over spatial values. Therefore the charge densities of a string define a spatial vector:

$$\boxed{\text{Kalb-Ramond charge density is a vector } \vec{j}^{\,0} \text{ with components } j^{0k}\,.} \tag{15.1.17}$$

We will soon prove that the charge vector is tangent to the string! Consider equation (15.1.16) for $\nu = 0$:

$$\frac{\partial j^{\mu 0}}{\partial x^\mu} = -\frac{\partial j^{0k}}{\partial x^k} = 0\,, \tag{15.1.18}$$

which is the statement that the string charge density is a divergenceless vector:

$$\boxed{\nabla \cdot \vec{j}^{\,0} = 0\,.} \tag{15.1.19}$$

The vector string charge $\vec{Q}$ is naturally defined as the space integral of the string charge density

$$\vec{Q} = \int d^d x \vec{j}^{\,0}\,. \tag{15.1.20}$$

To understand string charge more concretely, let's evaluate $j^{\mu\nu}$ in the static gauge $\tau = X^0/c$. With this condition, the delta function in equation

(15.1.11) is of the form

$$\delta\left(x^0 - X^0(\tau,\sigma)\right)\delta\left(\vec{x} - \vec{X}(\tau,\sigma)\right) = \frac{1}{c}\delta(\tau - t)\ \delta\ \left(\vec{x} - \vec{X}(\tau,\sigma)\right)\,, \quad (15.1.21)$$

and we we can perform the τ-integral to find

$$j^{\mu\nu}(\vec{x},t) = \frac{1}{2c}\int d\sigma\delta\left(\vec{x} - \vec{X}(t,\sigma)\right)\left[\frac{\partial X^\mu}{\partial t}\frac{\partial X^\nu}{\partial \sigma} - \frac{\partial X^\nu}{\partial t}\frac{\partial X^\mu}{\partial \sigma}\right](t,\sigma)\,. \quad (15.1.22)$$

Clearly, at any fixed time t_0, the current $j^{\mu\nu}$ is supported on the string – the set of points $\vec{X}(t_0,\sigma)$. For j^{0k} the second term in (15.1.22) does not contribute on account of $X^0 = ct$, and we find

$$\vec{j^0}(\vec{x},t) = \frac{1}{2}\int d\sigma\delta\left(\vec{x} - \vec{X}(t,\sigma)\right)\vec{X}'(t,\sigma)\,. \quad (15.1.23)$$

The $\vec{X}'$ factor in the right-hand side of this equation tells us that at any point on the string the string charge density $\vec{j^0}$ is indeed tangent to the string; in fact, it points along the tangent defined by increasing σ. In other words, the string charge lies along the orientation of the string! The string orientation is defined to be the direction of increasing σ.

This might seem puzzling. We have emphasized that the reparameterization invariance of the string action means that a change of parameterization cannot change the physics. Changing the direction of increasing σ is a reparameterization. How can this change the charge density of a string? While the Nambu-Goto action is invariant under any reparameterization, the coupling (15.1.3) of the string to the Kalb-Ramond field *is not*. If we let $\sigma \to \pi - \sigma$ while keeping τ invariant, the measure $d\tau d\sigma$ does not change sign but $X^{\nu\prime}$ does. As a result, (15.1.3) changes sign. In fact, reparameterizations that involve a change of orientation in the world-sheet will change the sign of this term (Problem 15.1).

Open strings are therefore *oriented* curves. At any fixed time, they are fully specified by a curve in space together with the identification of the endpoint that corresponds to $\sigma = 0$ (or equivalently, the endpoint $\sigma = \pi$). While closed strings do not have endpoints, they still have orientation, also defined by the direction of increasing σ. The open and closed string theories we examined in previous chapters were theories of oriented open strings and oriented closed strings, respectively. There are, however, theories of unoriented strings. These are consistent theories obtained by truncating the state space of (oriented) string theories down to the subspace of states that are invariant under the operation of orientation reversal. We examined these theories in a series of problems beginning with Probs. 12.10 and 13.5. Interestingly, the theory of unoriented closed strings has no Kalb-Ramond field in the spectrum. This fits in nicely with our discussion here: string states of unoriented strings are expected to carry no string charge and there is no "need" for the Kalb-Ramond field.

The integral in (15.1.23) is easily evaluated for the case an an infinitely-long, static string stretched along the x^1 axis (a similar configuration was studied in section 6.7). This string is described by the equations

$$X^1(t,\sigma) = f(\sigma)\,, \quad X^2 = X^3 = \cdots = X^d = 0\,, \tag{15.1.24}$$

where $f(\sigma)$ is a function of σ whose range is from $-\infty$ to $+\infty$. The function f must be a strictly increasing or a strictly decreasing function of σ. We expect this distinction to matter since these two alternatives correspond to oppositely oriented strings. Since only X^1 has σ dependence, equation (15.1.23) implies that the only non-vanishing $j^{\mu\nu}$ component is j^{01} $(= -j^{10})$:

$$\begin{aligned} j^{01}(\vec{x},t) &= \frac{1}{2}\int d\sigma\delta\left(x^1 - X^1(\tau,\sigma)\right)\delta(x^2)\delta(x^3)\ldots\delta(x^d)\, f'(\sigma) \\ &= \frac{1}{2}\delta(x^2)\delta(x^3)\ldots\delta(x^d)\int_{-\infty}^{\infty} d\sigma\delta(x^1 - f(\sigma))f'(\sigma)\,. \end{aligned} \tag{15.1.25}$$

Letting $\sigma(x^1)$ denote the unique solution of $x^1 - f(\sigma) = 0$, we have

$$\int_{-\infty}^{\infty} d\sigma\delta(x^1 - f(\sigma))f'(\sigma) = \frac{f'(\sigma(x^1))}{|f'(\sigma(x^1))|} = \text{sgn}(f'(\sigma(x^1)))\,, \tag{15.1.26}$$

where $\text{sgn}(a)$ denotes the sign of a. Since the function f is monotonic this sign is either positive or negative for all x^1. Thus, back to $j^{01}(\vec{x},t)$,

$$j^{01}(x^1,\ldots,x^d;t) = \tfrac{1}{2}\,\text{sgn}(f')\;\delta(x^2)\ldots\delta(x^d) = \tfrac{1}{2}\,\text{sgn}(f')\delta(\vec{x}_\perp)\,, \tag{15.1.27}$$

where $\vec{x}_\perp$ is the vector whose component comprise the directions orthogonal to the string. The string charge density is localized on the string and we see explicitly the orientation dependence in the sign of f'. For an arbitrary static string the spatial string coordinates X^k are time-independent. As a result, equation (15.1.22) implies that

$$j^{ik} = 0\,, \quad \text{for a static string}\,. \tag{15.1.28}$$

For a static string only the string charge densities j^{0k} are non-vanishing.

15.2 Visualizing string charge

In classical Maxwell electromagnetism charge configurations can be of several types. We can have idealized point charges, line charges, surface charges, and continuous charge distributions. Since we have seen that the string charge is localized on the string, you may perhaps think that string charge can be imagined as some Maxwell linear charge density on the string. Not true! The proper analogy to string *charge density* is a Maxwell *current* on the string. Indeed, we saw that there is a full spatial vector worth of string charge densities that point in the direction of the string – this is just what a Maxwell current on the string would look like.

If we were to integrate the string charge density over space, the total string charge $\vec{Q}$ (15.1.20) of an infinitely-long stretched string would be infinite. This is clear from (15.1.27). You will also show in Problem 15.3 that the charge $\vec{Q}$ associated to a closed string vanishes. It is therefore useful to introduce a related notion of string charge that simply counts strings. The string charge $\mathcal{Q}$, to be defined below, is a single number and it is quantized. For point charges in electromagnetism, you can select a volume and count the number of charges you are enclosing. In string theory, you select a space surrounding a set of strings and $\mathcal{Q}$ will count the number of strings you are linking. We will see how to calculate $\mathcal{Q}$ in term of the charge densities discussed above.

The string charge density $\vec{j}^0$ behaves as an electric Maxwell current because (15.1.19) holds in general. Electric charge conservation in electromagnetism requires

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0\,. \tag{15.2.1}$$

In magnetostatics, one assumes that the electric charge density ρ is time independent, and as a result electric currents densities must be divergenceless. This just means that charge does not accummulate anywhere at any time. Divergenceless currents cannot stop. When they flow on wires either one has wires that make loops – closed strings for us – or infinite wires. We learned that $\nabla \cdot \vec{j}^0 = 0$ vanishes even if we have time-dependent string configurations. Thus electric string charge is always analogous to an electric current in magnetostatics. String charge conservation is satisfied if strings form closed loops, or if they are infinitely long.

If strings cannot just stop, how can open strings end on D-branes? We will explore this in the next two sections. We will see that strings can end on D-branes because the string *endpoints* carry electric charge for the Maxwell field living on the D-brane. The electric field lines associated to this Maxwell charge actually carry the string charge! One can view the string charge density of a string that ends on a D-brane as a current flowing on the string, that upon reaching the brane spreads out along the electric field lines.

To elaborate further on the magnetostatic analogy we will again consider a static, stretched string, and examine the Kalb-Ramond field it creates. To simplify matters we will work in four-dimensional spacetime, or, equivalently, with three spatial dimensions. Consider equation (15.1.14). There are two possibilities: either both free indices are space indices, or one is a time index and the other is a space index. In the first case we have

$$\frac{\partial H^{ik\rho}}{\partial x^\rho} = 0\,, \tag{15.2.2}$$

since j^{ik} vanishes for static strings. We satisfy this equation with the following ansatz: *all H's are time independent*, and,

$$H^{ijk} = 0\,. \tag{15.2.3}$$

The other equation to consider is

$$\frac{\partial H^{0k\rho}}{\partial x^\rho} = j^{0k}\,. \tag{15.2.4}$$

We cast this equation into the form of a Maxwell equation by introducing a vector B_m defined as

$$H^{0kl} = \epsilon^{klm} B_m\,. \tag{15.2.5}$$

Here ϵ^{ijk} is totally antisymmetric and satisfies $\epsilon^{123} = 1$. Substituting back into (15.2.4), we find

$$\epsilon^{klm}\frac{\partial B_m}{\partial x^l} = j^{0k} \quad\longrightarrow\quad (\nabla \times \vec{B})_k = j^{0k}\,. \tag{15.2.6}$$

At this stage, the relevant components of H have been encoded into a "magnetic field", and equation (15.2.6) takes the form

$$\nabla \times \vec{B} = \vec{j}^{\,0}\,. \tag{15.2.7}$$

Up to a factor of $4\pi/c$, which there is no need to insert, this is Ampere's equation for the magnetic field of a current. Note that equation (15.2.7) is just a recasting of the original equations for H; if we cannot solve it, there is no solution for H. The consistency condition for (15.2.7) is familiar from magnetostatics. Since the divergence of a curl is zero, the existence of a solution requires (once again) that $\vec{j}^0$ be divergenceless. Alternatively, given a closed contour Γ that is the boundary of a surface S the integral form of equation (15.2.7) is

$$\oint_\Gamma \vec{B} \cdot d\vec{\ell} = \int_S \vec{j}^0 \cdot d\vec{a}\,. \tag{15.2.8}$$

If the contour Γ links the string, the string will pierce through S, and must do so for any surface whose boundary is Γ. If the string ended at some point, the current $\vec{j}^0$ would as well, and the equations would be inconsistent.

Equation (15.2.8) naturally leads to the definition of the string charge $\mathcal{Q}$ announced at the beginning of this section. The string charge $\mathcal{Q}$ linked by a contour Γ is from (15.2.8)

$$\mathcal{Q} \equiv \int_\Gamma \vec{B} \cdot \vec{d\ell} = \int_S \vec{j}^0 \cdot d\vec{a} \tag{15.2.9}$$

Note the difference from the calculation of the charge enclosed by a surface in electromagnetism. There we integrated the scalar electric charge density over the volume enclosed by the surface. Strings are not enclosed but rather surrounded. This is the natural analog: electric charges do not touch the surface that encloses them; strings, even infinitely long ones, must not touch the "surface" that links them. We calculate the string number linked by a contour Γ by integrating the flux of string charge density across the surface S whose boundary is Γ. Finally, in Maxwell theory the charge can also be computed as a flux integral of the electric field on the surface enclosing the charges. The string charge is analogously computed as an integration of the Kalb-Ramond field strength H (or $\vec{B}$ in our analysis) along the contour that links the strings. In a world with three spatial dimensionas a charge is enclosed by a two-sphere S^2 and a string is linked by a circle, or a one-sphere S^1.

Quick Calculation 15.2. In a world with four spatial dimensions x^1, x^2, x^3, and x^4, a string lies along the x^1 axis. Write a couple of equations defining a sphere that links the string.

Let us calculate, for illustration, the charge carried by the string stretched along the x^1 axis that we considered in the previous section. We assume, however, that there are only three spatial dimensions so that the results of the present section apply as well. Choosing the orientation so that $f'(\sigma) > 0$, equation (15.1.27) gives

$$j^{01} = \tfrac{1}{2}\delta(y)\delta(z)\,. \tag{15.2.10}$$

Consider now a closed curve linking the string and lying on the $x = x_0$ plane. Both the area vector and and $\vec{j^0}$ point in the x direction, giving us

$$\mathcal{Q} = \int_S \vec{j^0}\cdot d\vec{a} = \int dydz\,\tfrac{1}{2}\,\delta(y)\delta(z) = \tfrac{1}{2}\,. \tag{15.2.11}$$

It follows from this result that in general $\mathcal{Q} = N/2$, where N is the number of strings linked. The $\vec{B}$ field in this example can be readily calculated. This determines the field strength H. It is also possible to write an explicit expression for the antisymmetric tensor field $B_{\mu\nu}$, as you may do in Problem 15.4.

15.3 Strings ending on D-branes

We learned in Chapter 14 that there is a Maxwell field living on the world-volume of any Dp-brane. Indeed, photon states arise from the quantization of open strings whose ends lie on the D-brane. The quantization of closed strings revealed states that arise from a Kalb-Ramond field $B_{\mu\nu}$ living over all spacetime. We have seen that the string couples electrically to the $B_{\mu\nu}$ field; the string is a source for the $B_{\mu\nu}$ field. There is therefore an obvious question: If D-branes have Maxwell fields, is there any object that carries electric charge for these fields? This puzzle is related to another one: What happens to the string charge – which as we learned can be visualized as a current – when a string ends on a D-brane? Does string charge conservation fail to hold?

Puzzles with charge conservation have led to interesting insights in the past. It led, for example, to the recognition that the displacement current was necessary to restore charge conservation in time-dependent electromagnetic processes. In the present string theory case, the solution involves the realization that the ends of the open string behave as electric point charges! They are charged under the Maxwell field living on the D-brane where the string ends. This interplay between string charge and electric charge, and between the associated $B_{\mu\nu}$ and Maxwell fields, turns out to eliminate the possible failure of charge conservation.

Current conservation is intimately related to gauge invariance. In electromagnetism, the coupling of the gauge field to a current is a term in the action taking the form

$$S_{\text{coup}} = \int d^D x A_\mu(x)\, j^\mu(x)\,. \tag{15.3.1}$$

The gauge transformations are

$$\delta A_\mu(x) = \partial_\mu \epsilon\,, \tag{15.3.2}$$

and the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is gauge invariant: $\delta F_{\mu\nu} = 0$. Generally, terms other than (15.3.1) are gauge invariant. In equation (15.1.2), for example, the coupling term is the middle term in the right-hand side. The first and last terms are manifestly gauge invariant. The gauge invariance of the coupling (15.3.1) requires

$$\delta S_{\text{coup}} = \int d^D x\ (\partial_\mu \epsilon)\, j^\mu(x) = -\int d^D x\ \epsilon\, \partial_\mu j^\mu(x)\,, \tag{15.3.3}$$

where we integrated by parts and set the boundary terms to zero by assuming that the parameter ϵ vanishes at infinity. We now see that current conservation ($\partial_\mu j^\mu = 0$) will imply gauge invariance $\delta S_{\text{coup}} = 0$.

Similar ideas hold for the couplings of the Kalb-Ramond field $B_{\mu\nu}$. The gauge transformations of $B_{\mu\nu}$ were given in Problem 10.6:

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu\,. \tag{15.3.4}$$

The totally-antisymmetric field strength $H_{\mu\nu\rho}$ (15.1.6) is invariant under these gauge transformations. As indicated in the second line of (15.1.10), the coupling of $B_{\mu\nu}$ to a current $j^{\mu\nu}(= -j^{\nu\mu})$ is of the general form

$$-\int d^D x B_{\mu\nu}(x) j^{\mu\nu}(x)\,. \tag{15.3.5}$$

Quick Calculation 15.3. Prove that the coupling term (15.3.5) is invariant under the gauge transformations (15.3.4) if $j^{\mu\nu}$ is a conserved current.

The above results indicate that we can investigate possible current nonconservation by focusing on the gauge invariance properties of the actions. Let's therefore reconsider the term in the action (15.1.4) coupling the string to the $B_{\mu\nu}$ field:

$$S_B = -\frac{1}{2}\int d\tau d\sigma \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X(\tau,\sigma))\,. \tag{15.3.6}$$

Here we have introduced two dimensional indices $\alpha, \beta = 0, 1$, and $\partial_0 = \partial/\partial\tau$ and $\partial_1 = \partial/\partial\sigma$. Also, $\epsilon^{\alpha\beta}$ is totally antisymmetric with $\epsilon^{01} = 1$. Since the gauge invariance of S_B is a little subtle, we will study a simple case first. We will first check the gauge invariance of the action coupling a point particle to the Maxwell field. The action for this coupling is

$$\frac{q}{c}\int A_\mu(x)dx^\mu\,. \tag{15.3.7}$$

Why is this invariant under (15.3.2)? Using a parameter τ ranging from $-\infty$ to $+\infty$, the variation is proportional to

$$\begin{aligned}\int_{-\infty}^{\infty} d\tau \delta A_\mu(x(\tau))\,\frac{dx^\mu}{d\tau} &= \int_{-\infty}^{\infty} d\tau \frac{\partial\epsilon(x(\tau))}{\partial x^\mu}\,\frac{dx^\mu}{d\tau} = \int_{-\infty}^{\infty} d\tau \frac{d\epsilon(x(\tau))}{d\tau} \\ &= \epsilon(x(\tau=\infty)) - \epsilon(x(\tau=0)).\end{aligned} \tag{15.3.8}$$

Since τ parameterizes time, $t(\tau \to \pm\infty) = \pm\infty$. Gauge invariance then follows if we assume that the gauge parameter vanishes in the infinite past and future: $\epsilon(\vec{x}, t = \pm\infty) = 0$.

Let's now return to our problem, the gauge invariance of the action (15.3.6). Since the arguments of $B_{\mu\nu}$ are the string coordinates, the gauge transformations take the form

$$\delta B_{\mu\nu}(X) = \frac{\partial\Lambda_\nu}{\partial X^\mu} - \frac{\partial\Lambda_\nu}{\partial X^\mu}\,, \tag{15.3.9}$$

where the arguments of Λ are also the string coordinates $X(\tau,\sigma)$. The terms multiplying $B_{\mu\nu}$ in (15.3.6) are antisymmetric in μ and ν (check it!). As a result, each term in (15.3.9) gives the same contribution to the variation:

$$\delta S_B = -\int d\tau d\sigma\, \epsilon^{\alpha\beta}\, \frac{\partial\Lambda_\nu}{\partial X^\mu}\,\partial_\alpha X^\mu \partial_\beta X^\nu = -\int d\tau d\sigma \epsilon^{\alpha\beta}\, \partial_\alpha\Lambda_\nu\, \partial_\beta X^\nu\,. \tag{15.3.10}$$

Writing out the various terms:

$$\begin{aligned}\delta S_B &= -\int d\tau d\sigma \left(\partial_\tau \Lambda_\nu \, \partial_\sigma X^\nu - \partial_\sigma \Lambda_\nu \, \partial_\tau X^\nu\right) \, , \\ &= -\int d\tau d\sigma \Big(\partial_\tau(\Lambda_\nu \partial_\sigma X^\nu) - \partial_\sigma(\Lambda_\nu \partial_\tau X^\nu)\Big),\end{aligned} \tag{15.3.11}$$

where we note that we now have a total derivative structure. The ∂_τ term gives us no trouble since we can assume this term vanishes at the endpoints of time. We cannot ignore the ∂_σ term, however, because we have string endpoints. If the string under consideration is a closed string, then there is no boundary in σ, and the total derivative gives no contribution, assuming there is no compactification. This shows the gauge invariance of S_B for the case of closed strings.

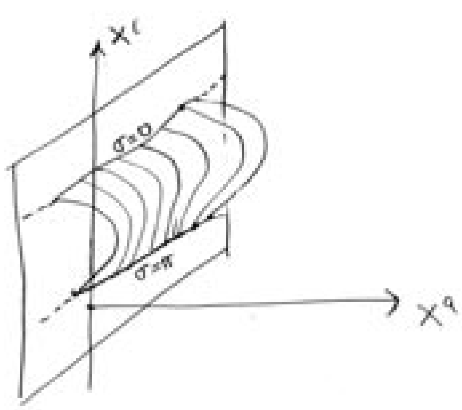

Figure 15.1: A D brane and the target space surface traced by an open string. The open string endpoints are the images of the world-sheet lines $\sigma = 0$ and $\sigma = \pi$. The coordinates X^a are transverse to the brane, and the coordinates X^m live on the brane.

For the open string, however, there is a problem with the gauge invariance of S_B. This is because the open string world-sheet has boundaries. In spacetime these boundaries must appear as lines on the world-volume of a D-brane, as shown in Figure 15.1. Let's now calculate the variation of the S_B for open strings. We will call the coordinates that live on the brane X^m

and the coordinates transverse to the brane X^a:

$$X^\mu = (X^m, X^a)\,, \qquad \mu = (m, a)\,. \tag{15.3.12}$$

If the D-brane is a Dp-brane, then $m = 0, 1, \ldots, p$. As before, we drop the ∂_τ term in (15.3.11) and we focus on

$$\delta S_B = \int d\tau d\sigma\, \partial_\sigma(\Lambda_\nu \partial_\tau X^\nu) = \int d\tau \Big[\Lambda_m \partial_\tau X^m + \Lambda_a \partial_\tau X^a \Big]_{\sigma=0}^{\sigma=\pi}. \tag{15.3.13}$$

Since the X^a are DD coordinates, $\partial_\tau X^a = 0$ at both endpoints, and the second term above gives no contribution. As a result,

$$\delta S_B = \int d\tau \Lambda_m \partial_\tau X^m \Big|_{\sigma=\pi} - \int d\tau \Lambda_m \partial_\tau X^m \Big|_{\sigma=0}. \tag{15.3.14}$$

Gauge invariance has failed because of these two boundary terms. This demonstrates the advertised problem: open strings have a problem with charge conservation at its endpoints. We must restore gauge invariance. As we have already said, the solution is that the Maxwell fields on the brane couple to the ends of the string.

So let's add a couple of terms to the string action that give electric charge to the string endpoints:

$$\boxed{S = S_B + \int d\tau\, A_m(X) \frac{dX^m}{d\tau}\Big|_{\sigma=\pi} - \int d\tau\, A_m(X) \frac{dX^m}{d\tau}\Big|_{\sigma=0}.} \tag{15.3.15}$$

Since the terms above have opposite signs, the string endpoints are oppositely charged. Conventionally, we have chosen the string to begin at the negatively charged endpoint and to end at the positively charged endpoint. The absolute values of the charges can only be determined if we give the normalization of the F^2 terms on the D-brane. In equation (15.1.2), for example, the particle charge is q only if the last term in the action takes the displayed form. We will not determine here the F^2 terms on the D-brane, which involve, among other factors, the string coupling. More briefly, and in the same notation of (15.3.14), we rewrite (15.3.15) as

$$S = S_B + \int d\tau A_m\, \partial_\tau X^m \Big|_{\sigma=\pi} - \int d\tau\, A_m \partial_\tau X^m \Big|_{\sigma=0}. \tag{15.3.16}$$

How can these terms restore gauge invariance? By letting the Maxwell field vary under the gauge transformation of the $B_{\mu\nu}$ field! This is a little strange and surprising, but without an interplay between the two types of fields we could not fix our problem of gauge invariance.

So we postulate that whenever we vary $B_{\mu\nu}$ with a gauge parameter $\Lambda_\mu = (\Lambda_m, \Lambda_a)$, we must also vary the Maxwell field A_m on the D-brane:

$$\boxed{\begin{aligned} \delta B_{\mu\nu} &= \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu , \\ \delta A_m &= -\Lambda_m \,. \end{aligned}} \tag{15.3.17}$$

If we vary A_m as stated, the variation of the last two terms in (15.3.16) cancels the variations found in (15.3.14), thus restoring gauge invariance.

Letting A vary as in (15.3.17) solves the problem at hand, but it raises some interesting questions with important implications. Besides wanting the string action to be gauge invariant, we must preserve the gauge invariance of the Maxwell action F^2. So we ask: is F_{mn} gauge invariant? It is not! Indeed

$$\delta F_{mn} = \partial_m \delta A_n - \partial_n \delta A_m = -\partial_m \Lambda_n + \partial_n \Lambda_m = -\delta B_{mn} \,, \tag{15.3.18}$$

where in the last step we recognized that the variation coincides with the gauge transformation of B_{mn}. This is significant, because it follows that there is a gauge invariant combination:

$$\delta(F_{mn} + B_{mn}) = 0 \,. \tag{15.3.19}$$

We call the new invariant quantity

$$\boxed{\mathcal{F}_{mn} \equiv F_{mn} + B_{mn} \,, \qquad \delta \mathcal{F}_{mn} = 0 \,.} \tag{15.3.20}$$

On the D-brane $\mathcal{F}_{mn}$ is the physically relevant field strength. The familiar field strength F_{mn} is not fully physical because it is not gauge invariant. Maxwell's equations will be modified by replacing F by $\mathcal{F}$. In many circumstances this will be a small modification, and for zero B, $\mathcal{F}$ equals F. The interplay between these fields helps us understand intuitively the fate of the string charge as a string ends on a D-brane. We turn to this issue now.

15.4 String charge and electric field lines

Even though a static string configuration is time-independent, we have seen that we must think of the string charge density as a kind of current flowing down the string. Suppose we have a string ending on a D-brane, as shown in Figure 15.2. The current cannot stop flowing at the string endpoint, so it must flow out *into* the D-brane. How can it do so? We know that the string endpoint is charged, so electric field lines emerge from it spreading out inside the D-brane. The field lines cannot go into the ambient space since the Maxwell field only lives on the D-brane. We will see that in fact, the electric field lines spreading inside the D-brane carry the string charge! This is so because electric field lines couple to the $B_{\mu\nu}$ field just as the fundamental string charge does.

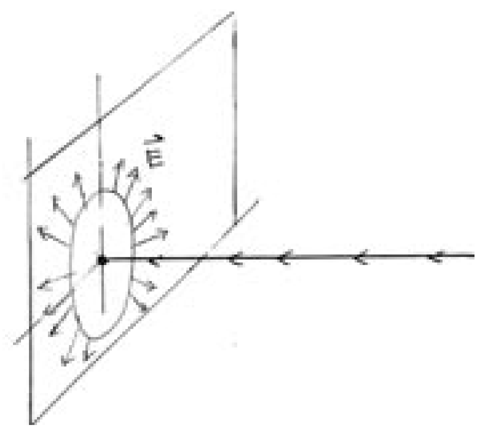

Figure 15.2: A string ending on a D-brane. The string charge carried by the string can be viewed as a "current" flowing down the string. The current is carried by the electric field lines on the D-brane world-volume.

As equation (15.1.10) indicates, string charge j^{0k} is, by definition, the quantity that couples to B_{0k}. Whatever couples to B_{0k}, appears in the right hand side of (15.1.14) as a contribution to j^{0k}. On the D-brane, the dynamics of the Maxwell fields arises from the Lagrangian density $-\frac{1}{4}\mathcal{F}^{mn}\mathcal{F}_{mn}$, the gauge invariant generalization of the Maxwell Lagrangian density. Expanding it out in terms of Maxwell and Kalb-Ramond field strengths,

$$-\frac{1}{4}\mathcal{F}^{mn}\mathcal{F}_{mn} = -\frac{1}{4}B^{mn}B_{mn} - \frac{1}{4}F^{mn}F_{mn} - \frac{1}{2}F^{mn}B_{mn}\,. \tag{15.4.1}$$

The last term above is particularly interesting. We can expand it further as

$$-\frac{1}{2}F^{mn}B_{mn} = -F^{0k}B_{0k} + \cdots\,. \tag{15.4.2}$$

The complete action for the D-brane and the string will include this $F^{0k}B_{0k}$ term. Anything that couples to B_{0k} is string charge, so F^{0k} represents string charge on the brane. But $F^{0k} = E_k$ is the electric field. Therefore, electric field lines on the D-brane carry fundamental string charge.

Problems

Problem 15.1. *Reparameterization invariance of the string/Kalb-Ramond coupling.*

Consider a world-sheet with coordinates (τ, σ) and a reparameterization that leads to coordinates $(\tau'(\tau, \sigma), \sigma'(\tau, \sigma))$. Show that the coupling (15.1.3) transforms as follows

$$\int d\tau' d\sigma' \, \frac{\partial X^\mu}{\partial \tau'} \frac{\partial X^\nu}{\partial \sigma'} \, B_{\mu\nu}(X) = \text{sgn}(\gamma) \int d\tau d\sigma \, \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} \, B_{\mu\nu}(X) \, ,$$

where $\text{sgn}(\gamma)$ denotes the sign of γ, and

$$\gamma = \frac{\partial \tau}{\partial \tau'} \frac{\partial \sigma}{\partial \sigma'} - \frac{\partial \tau}{\partial \sigma'} \frac{\partial \sigma}{\partial \tau'} \, .$$

Note that your proof requires the antisymmetry of $B_{\mu\nu}$. If $\text{sgn}(\gamma) = +1$, the reparameterization is orientation-preserving. If $\text{sgn}(\gamma) = -1$, the reparameterization is orientation-reversing. Give two examples of nontrivial orientation preserving reparametrizations and two examples of nontrivial orientation reversing reparameterizations.

Problem 15.2. *Antisymmetric variations and equations of motion.*

Let $\delta B_{\mu\nu} = -\delta B_{\mu\nu}$ be an arbitrary antisymmetric variation ($\mu, \nu = 0, 1, \ldots, d$). Show that

$$\delta B_{\mu\nu} G^{\mu\nu} = 2 \sum_{\mu > \nu}^{d} \sum_{\nu = 0}^{d} \delta B_{\mu\nu} (G^{\mu\nu} - G^{\nu\mu}) \, .$$

Use this result to prove that the condition $\delta B_{\mu\nu} G^{\mu\nu} = 0$ for all antisymmetric variations, implies that $G^{\mu\nu} - G^{\nu\mu} = 0$.

Problem 15.3. *Kalb-Ramond string charge $\vec{Q}$.*

Consider a string at some fixed time t_0 and a region $\mathcal{R}$ of space that contains a portion of this string: the string enters the region $\mathcal{R}$ at a point $\vec{x}_i$ and leaves the region $\mathcal{R}$ at a point $\vec{x}_f$. Use (15.1.23) to calculate the string charge $\vec{Q} = \int_{\mathcal{R}} d^d x \vec{j}^0$ enclosed in $\mathcal{R}$ at time t_0. Use your result to show that the total string charge $\vec{Q}$ associated to a closed string is zero.

A more abstract proof that $\vec{Q}$ is zero for any localized configuration of closed strings requires showing that $\nabla \cdot \vec{j}^0 = 0$ implies $\int d^d x \vec{j}^0 = 0$. If you have trouble showing this you may look at your favorite E&M book: the same proof is needed to show that the multipole expansion for the magnetic field of a localized current in magnetostatics has no monopole term.

Problem 15.4. *Kalb-Ramond field of a string.*

Following the discussion in section 15.2, calculate the field $H^{\mu\nu\rho}$ created by a string stretched along the x axis. Verify that all equations in (15.1.14) are satisfied by the proposed solution. Find a simple $B^{\mu\nu}$ giving rise to the field strength H. Interpret the answer in terms of a magnetostatics analog.

Problem 15.5. *Explicit checks of current conservation.*

In Problem 5.3 you constructed the current vector of a charged point particle:

$$j^\mu(\vec{x}, t) = qc \int d\tau \, \delta^D(x - x(\tau)) \, \frac{dx^\mu(\tau)}{d\tau} \, .$$

Verify directly that this is a conserved current ($\partial_\mu j^\mu = 0$). Now extend this result to the case of the string. Verify directly that the current in (15.1.11) is conserved.

Problem 15.6. *Equation of motion for a string in a Kalb-Ramond background.*

Consider the string action (6.4.2) supplemented by the coupling (15.1.3) to the Kalb-Ramond field. Perform a variation δX^μ and prove that the resulting equations of motion for the string take the form

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = -H_{\mu\nu\rho} \frac{\partial X^\nu}{\partial \tau} \frac{\partial X^\rho}{\partial \sigma} \, . \tag{1}$$

Problem 15.7. *H-field and a circular closed string*

Assume a constant, uniform H field that takes the value $H_{012} = h$, with all other components equal to zero. Assume we also have circular closed string lying on the (x^1, x^2) plane. The purpose of this problem is to show that the tension of the string and the force on the string due to H can give rise to an equilibrium radius. As we will also see, the equilibrium is unstable.

We will analyze the problem in two ways. In the first way, we use the equation of motion (1) derived in Problem 15.6, working with $X^0 = \tau$ ($c = 1$):

(a) Find simplified forms for $\mathcal{P}^\tau_\mu$ and $\mathcal{P}^\sigma_\mu$ for a static string.

(b) Check that the $\mu = 0$ component of the equation of motion is trivially satisfied. Show that the $\mu = 1$ and $\mu = 2$ components give the same result, fixing the radius R of the string at the value $R = T_0/h$.

In the second way, we evaluate the action using the simplified geometry of the problem. For this assume we have a time-dependent radius $R(t)$.

(c) Find $B_{\mu\nu}$ fields that give rise to the H field. In fact, you can find a solution where only B_{01} and/or B_{02} are nonzero.

(d) Show that the coupling term (15.1.3) for the string in question is equal to

$$\int dt\, \pi h R^2(t)\,.$$

Explain why this term represents (minus) potential energy.

(e) Consider the full action for this circular string, and use this to compute energy functional $E(R(t), \dot{R}(t))$ (your analysis of the circular string in Problem 6.3 may save you a little work).

(d) Assume $\dot{R}(t) = 0$ and plot the energy functional $E(R(t))$ for $h > 0$ and $h < 0$. Show that the equilibrium value of the radius coincides with the one obtained before, and explain why this equilibrium value is unstable.

Chapter 16

String thermodynamics and black holes

The thermodynamics of strings is governed largely by the exponential growth of the number of quantum states accessible to a string, as a function of its energy. We estimate such growth rates by counting partitions of large integers. As we increase the energy of a string, the behavior of the entropy indicates that the temperature approaches a finite constant: the Hagedorn temperature. We calculate the finite-temperature partition function for bosonic string theory. We explain how the counting of string states can be used to give a statistical mechanics derivation of the entropy of black holes. The calculations give results in qualitative agreement with the expected entropy of Schwarzchild black holes, and in quantitative agreement with the expected entropy of certain charged black holes.

16.1 A review of statistical mechanics

Our study of string thermodynamics will make use of both the microcanonical and canonical ensembles. Recall that the *microcanonical ensemble* consists of a collection of copies of a particular system A, one for each state accessible to A at a particular fixed energy E. In the *canonical ensemble* we consider the system A in thermal contact with a reservoir at a temperature T. This ensemble contains copies of system A together with the reservoir, one copy for each allowed state of the combined system. In the canonical ensemble the energy of system A varies among members of the ensemble.

Let's begin with the microcanonical ensemble. The system A is imagined to be in isolation with a fixed energy. We let $\Omega(E)$ denote the number of possible states of the system A when it has energy E. The entropy S of the system is defined in terms of the number of states as

$$S(E) = k \ln \Omega(E)\,, \tag{16.1.1}$$

where k is Boltzmann's constant. The temperature T of the system is defined in terms of derivatives of the entropy with respect to the energy:

$$\frac{1}{T} = \frac{\partial S}{\partial E}\,. \tag{16.1.2}$$

The canonical ensemble is sometimes easier to work with. Imagine a system A which has a fixed volume and which is in thermal contact with a reservoir of temperature T. This system could be a box full of strings, or it could be a box containing a single string. It is also not necessary to specify what the reservoir is. Suppose we know the quantum states $\{\alpha\}$ of the system and their associated energies $\{E_\alpha\}$. Then, the partition function Z for system A is defined as

$$Z \equiv \sum_\alpha e^{-\beta E_\alpha}\,, \quad \beta = \frac{1}{kT}\,. \tag{16.1.3}$$

The partition function is useful because it can be used to calculate interesting quantities. For instance, if system A is known to have temperature T, then, using Z, we can calculate the probability that A is in a particular quantum state. By definition, the partition function depends both on the temperature T and on the external parameters of the system. These are the parameters that determine the energy levels of the system. The systems we will consider have only one external parameter: the volume V occupied by the system. Thus we will think of Z as $Z(T, V)$, or

$$Z = Z(\beta, V)\,. \tag{16.1.4}$$

The probability P_α that the system, in contact with the reservoir of temperature T, is in the state α is

$$P_\alpha = \frac{e^{-\beta E_\alpha}}{Z}\,. \tag{16.1.5}$$

Clearly, $\sum_\alpha P_\alpha = 1$, as required by the probabilistic interpretation of P_α. We can calculate the average energy E of the system A in the ensemble by differentiation of the partition function:

$$E = \sum_\alpha P_\alpha E_\alpha = -\frac{\partial \ln Z}{\partial \beta}\,. \tag{16.1.6}$$

The pressure p of the system can also be calculated from the partition function (see Problem 16.1). It is given by

$$p = \frac{1}{\beta}\frac{\partial \ln Z}{\partial V}\,. \tag{16.1.7}$$

Another useful quantity is the Helmholtz free energy F. Its basic properties can be obtained in a few steps starting from the first law of thermodynamics. The change dE in the energy of a system whose only external parameter is the volume V is given by

$$dE = TdS - pdV\,. \tag{16.1.8}$$

Here T is the temperature of the system and p is the pressure. Moreover, TdS is the heat transferred into the system, and $(-pdV)$ is the mechanical work done *on* the system. Equation (16.1.8) implies that E should be viewed as a function $E(S,V)$ of S and V, and that

$$T = \Big(\frac{\partial E}{\partial S}\Big)_V\,, \quad p = -\Big(\frac{\partial E}{\partial V}\Big)_S\,. \tag{16.1.9}$$

We can also write the change in energy in (16.1.8) as

$$dE = d(TS) - S\,dT - p\,dV\,, \tag{16.1.10}$$

which means that

$$d(E - TS) = -S\,dT - p\,dV\,. \tag{16.1.11}$$

The free energy F is defined as

$$F \equiv E - TS\,, \tag{16.1.12}$$

and therefore we have

$$dF = -S\,dT - p\,dV\,. \tag{16.1.13}$$

We see that for processes at constant temperature, the free energy represents the amount of energy that can go into mechanical work. For a chemical reaction that releases energy, for example, the entropy of the system typically decreases. Not all of the energy released can then be used for work, only the free energy can. Since the total entropy cannot decrease, the rest of the energy goes into heat that increases the entropy of the world. It follows from (16.1.13) that F should be viewed as a function $F(T,V)$ of T and V, and,

$$S = -\Big(\frac{\partial F}{\partial T}\Big)_V \,, \quad p = -\Big(\frac{\partial F}{\partial V}\Big)_T \,. \tag{16.1.14}$$

The free energy can be calculated from the partition function, as you may review in Problem 16.1. It is given by

$$F = -kT \ln Z \,. \tag{16.1.15}$$

Our aim is to use the basic thermodynamic relations reviewed above to compute interesting properties of the string. One central computation is that of the partition function for a string. This problem is a bit complex, so we first consider simpler problems that will help us build the necessary tools.

The first result we need is a formula for the number of partitions of integers. We will obtain this mathematical result using a physical method: the analysis of the high-temperature behavior of a quantum *non-relativistic string*, call it a “quantum violin string”. With this result we calculate the entropy/energy relation for an idealized quantum *relativistic* string; a string where we ignore the momentum labels of the quantum states. The Hagedorn temperature already emerges in this context. After a discussion of the partition function for the relativistic point-particle, we assemble all of our results to compute the partition function of the relativistic string.

In the latter part of this chapter we discuss a significant success of string theory: giving a statistical-mechanics interpretation of the entropy of black holes. This entropy, first arrived at via thermodynamical considerations, arises from the degeneracy of string states that have the macroscopic properties of the black holes. The agreement between the string calculations and the thermodynamical expectation is only qualitative for the case of Schwarzchild black holes, but is quantitave for certain types of extremal black holes.

16.2 Partitions and the quantum violin string

Consider a quantum mechanical non-relativistic string with fixed endpoints: a quantum violin string. This string, studied classically in Chapter 4, has an infinite set of vibrating frequencies, all multiples of a basic frequency ω_0. Its idealization as a quantum string is a collection of simple harmonic oscillators with frequencies $\omega_0, 2\omega_0, 3\omega_0$, and so on. Each simple harmonic oscillator (SHO) has its own creation and annihilation operators, as well as its own Hamiltonian:

$$\begin{array}{llll} \mathrm{SHO}_{\ \omega_0}: & (a_1, a_1^\dagger)\,, & H_{\ \omega_0} = & \hbar\omega_0\, a_1^\dagger a_1\,, \\ \mathrm{SHO}_{2\omega_0}: & (a_2, a_2^\dagger)\,, & H_{2\omega_0} = & 2\,\hbar\omega_0\, a_2^\dagger a_2\,, \\ \mathrm{SHO}_{3\omega_0}: & (a_3, a_3^\dagger)\,, & H_{3\omega_0} = & 3\,\hbar\omega_0\, a_3^\dagger a_3\,, \\ \vdots & \vdots & \vdots & \vdots \end{array} \tag{16.2.1}$$

Here we have discarded zero-point energies, and all oscillators satisfy the conventional commutation relations

$$[a_l, a_m^\dagger] = \delta_{mn}\,. \tag{16.2.2}$$

Since the quantum string is the union of all these oscillators, the Hamiltonian for the string is

$$H = \sum_{\ell=1}^{\infty} H_{\ell\omega_0} = \hbar\omega_0 \sum_{\ell=1}^{\infty} \ell\, a_\ell^\dagger a_\ell\,. \tag{16.2.3}$$

We recognize here the number operator $\widehat{N}$:

$$\widehat{H} = \hbar\omega_0\, \widehat{N}\,, \qquad \widehat{N} = \sum_{\ell=1}^{\infty} \ell\, a_\ell^\dagger a_\ell\,. \tag{16.2.4}$$

The vacuum state of the string is a state $|\Omega\rangle$ such that

$$a_\ell|\Omega\rangle = 0\,, \quad \text{for all } \ell\,. \tag{16.2.5}$$

A quantum state $|\Psi\rangle$ of this string is obtained by letting creation operators act on the vacuum:

$$|\Psi\rangle \;=\; (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\ldots(a_l^\dagger)^{n_l}\cdots|\Omega\rangle\;. \tag{16.2.6}$$

N	list of states	$p(N)$
1	$a_1^\dagger$	1
2	$a_2^\dagger\,,\,(a_1^\dagger)^2$	2
3	$a_3^\dagger\,,\,a_2^\dagger a_1^\dagger\,,\,(a_1^\dagger)^3$	3
4	$a_4^\dagger\,,a_3^\dagger a_1^\dagger\,,(a_2^\dagger)^2\,,a_2^\dagger(a_1^\dagger)^2\,,(a_1^\dagger)^4$	5

Table 16.1: Counting states of fixed total number eigenvalue N. $p(N)$ denotes the number of partitions of the integer N.

The state is therefore specified by the set $\{n_1, n_2, n_3, \cdots\}$ of occupation numbers. The number operator acting on the state $|\Psi\rangle$ gives us

$$\widehat{N}|\Psi\rangle = N|\Psi\rangle\,, \tag{16.2.7}$$

where

$$N = n_1 + 2n_2 + 3n_3 + \ldots = \sum_{\ell=1}^{\infty} \ell n_\ell\,. \tag{16.2.8}$$

It then follows from (16.2.4) that the energy of $|\Psi\rangle$ is given by

$$\widehat{H}|\Psi\rangle = E|\Psi\rangle \quad \rightarrow \quad E = \hbar\omega_0\,N\,. \tag{16.2.9}$$

A natural counting question arises here. For a fixed positive integer N, how many states are there with $\widehat{N}$ eigenvalue equal to N? This number, denoted as $p(N)$, is so important that it has been given a name: the *partitions* of N. Before explaining the reason for this terminology, let us determine $p(N)$ for $N = 1, 2, 3$ and 4. Shown in Table 16.1 are the states with those values of N. For brevity, we show only the oscillators, omitting the vacuum state $|\Omega\rangle$ that they act on. The fourth line, for example, shows that there are five states with $\widehat{N}$ eigenvalue equal to four. Thus $p(4) = 5$.

It is appropriate to name the quantity $p(N)$ *partitions* of N. A partition of N is a set of positive integers that add up to N. The order of the elements in the set is immaterial. Thus, for example, $\{3, 2\}$ is a partition of 5, and so is $\{2, 1, 1, 1\}$. The partitions of 4 are

$$\{4\}\,, \quad \{3,1\}\,, \quad \{2,2\}\,, \quad \{2,1,1\}\,, \quad \{1,1,1,1\}\,. \tag{16.2.10}$$

The number of states with $\widehat{N}$ eigenvalue equal to N coincides with the number of partitions of N. Indeed, given a partition of N we can build a state by attaching each element of the partition as a subscript to an oscillator $a^\dagger$, and letting the resulting collection of oscillators act on the vacuum. Note that this is exactly how the states in the last line of Table 16.1 are built from the partitions of 4 given in (16.2.10). Conversely, given a state with number N, the set of subscripts of all oscillators in the state gives a partition of N.

We would like to find a formula for $p(N)$. But our analysis will not give us that much. We will derive an expression that describes $\ln p(N)$ accurately for large N. A more refined calculation gives the famous approximation for $p(N)$ found by Hardy and Ramanujan. There exist closed-form expressions for $p(N)$, but they are extremely complicated!

Our strategy will be as follows. We know that the entropy S is given as a function of the energy E by (16.1.1). For a given E, $N = E/(\hbar\omega_0)$, and $\Omega(E)$ is simply $p(N)$. Therefore

$$S(E) = k \ln\, p\left(\frac{E}{\hbar\omega_0}\right) = k \ln\, p(N)\,. \tag{16.2.11}$$

If we can find $S(E)$, then we will have found the function $p(N)$. To find $S(E)$ we will calculate the partition function Z for the quantum violin string. From Z we will find the free energy F. We will be able to evaluate the free energy explicitly only by assuming high temperature. It is then easy to find the high-energy behavior of the entropy $S(E)$. This can be used to find a large-N approximation for $p(N)$.

Let's now begin with the calculation of the partition function. We have

$$Z = \sum_\alpha \exp\left(-\frac{E_\alpha}{kT}\right) = \sum_{n_1, n_2, n_3 \ldots} \exp\left[-\frac{\hbar\omega_0}{kT}(n_1 + 2n_2 + 3n_3 + \cdots)\right]. \tag{16.2.12}$$

In writing this equation we have recognized that the set of all states is labelled by the set of occupation numbers. To sum over all states is to sum over all occupation numbers, each of which ranges from zero to infinity. We have also used the value of the energy $E = \hbar\omega_0 N$. Since the exponential of a sum can be written as a product of exponentials, the sums over the occupation numbers are all independent:

$$Z = \sum_{n_1} \exp\Big[-\frac{\hbar\omega_0}{kT} n_1\Big] \cdot \sum_{n_2} \exp\Big[-\frac{\hbar\omega_0}{kT}\, 2n_2\Big] \;\cdot\; \ldots\ldots \quad . \tag{16.2.13}$$

Therefore we have

$$Z = \prod_{l=1}^{\infty} \sum_{n_l=0}^{\infty} \exp\left(-\frac{\hbar\omega_0 l n_l}{kT}\right) . \tag{16.2.14}$$

The sum over n_ℓ is a geometric series, so we find

$$Z = \prod_{l=1}^{\infty} \left[1 - \exp\left(-\frac{\hbar w_0 l}{kT}\right)\right]^{-1} . \tag{16.2.15}$$

Finally, the free energy F is found using (16.1.15):

$$F = -kT \ln Z = kT \sum_{l=1}^{\infty} \ln\Big[1 - \exp\Big(-\frac{\hbar\omega_0 l}{kT}\Big)\Big] . \tag{16.2.16}$$

We cannot go any further unless we do some approximations. If the temperature T is high enough so that

$$\frac{\hbar\omega_0}{kT} << 1 , \tag{16.2.17}$$

then each term in the sum (16.2.16) differs very little from the previous one. This allows us to approximate the sum by an integral:

$$F \simeq kT \int_1^{\infty} dl \, \ln\Big[1 - \exp\Big(-\frac{\hbar\omega_0 l}{kT}\Big)\Big] . \tag{16.2.18}$$

The choice $l = 1$ for the lower limit of integration, as opposed to zero, will play no role. Indeed, changing variables of integration to

$$x = \frac{\hbar\omega_0}{kT} \, l , \tag{16.2.19}$$

we find

$$F \simeq \frac{(kT)^2}{\hbar\omega_0} \int_0^{\infty} dx \ln(1 - e^{-x}) . \tag{16.2.20}$$

Using the expansion

$$\ln(1-y) = -\Big(y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + \frac{1}{4}y^4 + \cdots\Big) , \tag{16.2.21}$$

which is valid for any $0 \leq y < 1$, we have

$$
\begin{aligned}
F &\simeq -\frac{(kT)^2}{\hbar\omega_0}\int_0^\infty dx\left(e^{-x}+\frac{1}{2}e^{-2x}+\frac{1}{3}e^{-3x}+\cdots\right), \\
&\simeq -\frac{(kT)^2}{\hbar\omega_0}\Big[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\Big].
\end{aligned}
\tag{16.2.22}
$$

The sum in brackets is a familiar one. It is, in fact, the zeta-function (12.4.14) with argument equal to two:

$$
\zeta(2) = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = \frac{\pi^2}{6}. \tag{16.2.23}
$$

Thus we finally obtain the high temperature limit of the free energy:

$$
F \simeq -\frac{(kT)^2}{\hbar\omega_0}\frac{\pi^2}{6} = -\frac{1}{\hbar\omega_0}\frac{\pi^2}{6}\frac{1}{\beta^2}. \tag{16.2.24}
$$

For this string the free energy has no volume dependence.

We can now calculate the entropy as a function of temperature. Using (16.1.14) we find

$$
S = -\frac{\partial F}{\partial T} = k\,\frac{\pi^2}{3}\left(\frac{kT}{\hbar\omega_0}\right). \tag{16.2.25}
$$

Since we are interested in the entropy as a function of energy, we also compute the energy. Making use of (16.1.6) we have

$$
E = -\frac{\partial \ln Z}{\partial\beta} = \frac{\partial}{\partial\beta}(\beta F) = -\frac{\pi^2}{6}\frac{1}{\hbar\omega_0}\frac{\partial}{\partial\beta}\left(\frac{1}{\beta}\right), \tag{16.2.26}
$$

which gives

$$
E = \frac{\pi^2}{6}\frac{1}{\hbar\omega_0}\frac{1}{\beta^2} = \frac{\pi^2}{6}\left(\frac{kT}{\hbar\omega_0}\right)^2\hbar\omega_0\,. \tag{16.2.27}
$$

Quick Calculation 16.1. Verify that the energy E can also be calculated from $F = E - TS$.

Combining (16.2.25) and (16.2.27) yields

$$
S(E) = k\pi\sqrt{\frac{2}{3}\frac{E}{\hbar\omega_0}} = k\,2\pi\sqrt{\frac{N}{6}}\,. \tag{16.2.28}
$$

Comparing with equation (16.2.11) we finally read

$$\ln p(N) \simeq 2\pi\sqrt{\frac{N}{6}}\,. \tag{16.2.29}$$

This was our goal, an estimate of $\ln p(N)$ for large N. Indeed, we must require large N since

$$N = \frac{E}{\hbar\omega_0} = \frac{\pi^2}{6}\left(\frac{kT}{\hbar\omega_0}\right)^2 >> 1\,, \tag{16.2.30}$$

because of our high temperature assumption (16.2.17).

The result (16.2.29) is only the leading term of the celebrated Hardy-Ramanujan asymptotic expansion of $p(N)$:

$$\boxed{p(N) \simeq \frac{1}{4N\sqrt{3}}\exp\Bigl(2\pi\sqrt{\frac{N}{6}}\,\Bigr)\,.} \tag{16.2.31}$$

This is not an exact formula either, but is an accurate estimate of $p(N)$, as opposed to our accurate estimate of the logarithm of $p(N)$. We will not give here a derivation of the Hardy-Ramanujan result. It is fun, however, to test the accuracy of the Hardy-Ramanujan expansion. In Table 16.2 we compare the values of $p(N)$, as calculated exactly, with the estimate $p_{\rm est}(N)$ provided by (16.2.31). The estimate gives an error of about one-half of a percent for $N = 10000$.

We now need a minor generalization of (16.2.31). Assume the string can vibrate in d transverse directions. Then, for each frequency $\ell\omega_0$, we must have d harmonic oscillators representing the possible polarizations of the motion. Furthermore, the associated occupation numbers need a superscript labelling the d polarizations:

$$\begin{array}{cccc} n_1^{(1)} & n_1^{(2)} & \dots & n_1^{(d)} \\ n_2^{(1)} & n_2^{(2)} & \dots & n_2^{(d)} \\ \dots & \dots & \dots & \dots \\ n_l^{(1)} & n_l^{(2)} & \dots & n_l^{(d)} \\ \dots & \dots & \dots & \dots \end{array} \tag{16.2.32}$$

In order to sum over all possible states in the new partition function Z_d, we must sum over all possible values of the occupation numbers $n_k^{(q)}$, where

N	$p(N)$	$p(N)_{\text{est}}$	$p(N)/p_{\text{est}}(N)$
5	7	8.94	0.7829
10	42	48.10	0.8731
100	190569292	199281893.25	0.9563
1000	2.406×10^{31}	2.440×10^{31}	0.9860
10000	3.617×10^{106}	3.633×10^{106}	0.9956

Table 16.2: Comparing the exact values of $p(N)$ with the estimate $p(N)_{\text{est}}$ provided by the Hardy-Ramanujan formula.

$k = 1, 2, \ldots, \infty$, and $q = 1, 2, \ldots, d$. This gives

$$Z_d = \sum_{n_k^{(1)}, \ldots, n_k^{(d)}} \exp\Big[-\frac{\hbar\omega_0}{kT} \sum_{\ell=0}^{\infty} \sum_{q=1}^{d} \ell n_\ell^{(q)}\Big] . \tag{16.2.33}$$

The sums over the various $n^{(q)}$ factorize, so,

$$Z_d = \sum_{n_k^{(1)}} \exp\Big[-\frac{\hbar\omega_0}{kT} \sum_{\ell=0}^{\infty} \ell n_\ell^{(1)}\Big] \ldots \sum_{n_k^{(d)}} \exp\Big[-\frac{\hbar\omega_0}{kT} \sum_{\ell=0}^{\infty} \ell n_\ell^{(d)}\Big] . \tag{16.2.34}$$

Each factor here is equal to the previously calculated partition function Z. We therefore have

$$Z_d = (Z)^d . \tag{16.2.35}$$

The new free energy F_d is also easy to calculate:

$$F_d = -kT \ln Z_d = -kT \, d \ln Z = F \, d . \tag{16.2.36}$$

The entropy, obtained by differentiation of the free energy, also acquires a multiplicative factor of d:

$$S_d = S \, d . \tag{16.2.37}$$

For the energy E_d, the same multiplicative factor exists on account of (16.1.6). We also note that E_d is equal to $\hbar\omega_0 N$, where N is now the total occupation number

$$E_d = E\,d = \hbar\omega_0 N\,, \qquad N = \sum_{\ell,q} \ell n_\ell^{(q)}\,. \tag{16.2.38}$$

Using our earlier result for S in (16.2.28) we now have

$$S_d = d\,(k\,2\pi)\sqrt{\frac{1}{6}\frac{E}{\hbar\omega_0}} = k\,2\pi\,\sqrt{\frac{d}{6}\frac{E\,d}{\hbar\omega_0}} = k\,2\pi\,\sqrt{\frac{Nd}{6}}\,, \tag{16.2.39}$$

where we made use of (16.2.38).

Let us call $p_d(N)$ the number of partitions of N when we have a d-fold degeneracy. This means, for example, that the partition $\{3,2,1\}$ of 6 now gives rise to many partitions written like $\{3_{p_1}, 2_{p_2}, 1_{p_3}\}$, where we include subscripts p_i that can take all possible values from one to d. A partition with different subscripts is considered a different partition. We now see that, for a given energy, with associated number N, the number of states is $p_d(N)$. Therefore $S_d = k\ln p_d(N)$, and comparing with (16.2.39) we conclude that for large N

$$\ln\, p_d(N) \simeq 2\pi\sqrt{\frac{Nd}{6}}\,. \tag{16.2.40}$$

The more accurate version of this result can be shown to be

$$p_d(N) \simeq \frac{1}{\sqrt{2}}\left(\frac{d}{24}\right)^{(d+1)/4} N^{-(d+3)/4}\exp\Big(2\pi\sqrt{\frac{Nd}{6}}\Big)\,. \tag{16.2.41}$$

You can see that for $d = 1$ this reduces to $p(N)$, as given in (16.2.31). For $d = 24$, the number of transverse light-cone directions in the bosonic string, the expression simplifies a little:

$$\boxed{p_{24}(N) \simeq \frac{1}{\sqrt{2}}\, N^{-27/4}\exp\Big(4\pi\sqrt{N}\Big)\,.} \tag{16.2.42}$$

Quick Calculation 16.2. Show that for large N

$$\frac{p_{24}(N+1)}{p_{24}(N)} \simeq \exp\Big(\frac{2\pi}{\sqrt{N}}\Big)\,. \tag{16.2.43}$$

This means that the fractional change in the number of partitions when the argument is increased by one unit goes down to zero as $N \to \infty$.

Quick Calculation 16.3. Use direct counting to confirm that $p_{24}(1) = 24$, $p_{24}(2) = 324$, $p_{24}(3) = 3200$, and $p_{24}(4) = 25650$.

Counting other types of partitions is also interesting. Consider, for example, partitions of integers into unequal parts. The possible partitions of 6 into unequal integers are

$$\{6\}\,,\{5,1\}\,,\{4,2\}\,,\{3,2,1\}\,. \tag{16.2.44}$$

We denote by $q(N)$ the number of partitions of N into unequal parts, so $q(6)$, for example, is equal to four. We can use a fermionic version of the violin string problem to determine the large N behavior of $q(N)$. The frequencies of the oscillators are not changed, but this time we demand that each occupation number can only be equal to zero or to one. Since no creation operator can be used more than once, the total number N of any state is effectively split into contributions all of whose parts are unequal. Creation operators that cannot be used more than once create fermionic excitations. We therefore say that we have fermionic oscillators. With a little abuse of languange, the numbers entering an unequal partition are called fermionic numbers. You will show in Problem 16.2 that for large N,

$$\ln q(N) \sim 2\pi\sqrt{\frac{N}{12}}\,. \tag{16.2.45}$$

We extended the earlier counting of $p(N)$ to the case where the elements of a partition can carry d labels. If the elements of an unequal partition can carry d_f labels, the number $q_{d_f}(N)$ of partitions is obtained from (16.2.45) by replacing $N \to Nd_f$. In such partitions a fermionic number can appear more than once if it uses a different label each time. This counting corresponds to a system with d_f species of fermionic oscillators.

A final generalization is useful. We consider partitions of N with d labels for the ordinary numbers, and with d_f labels for the fermionic numbers. In this case (Problem 16.4) we find that for large N

$$\boxed{\ln P(N;d,d_f) \sim 2\pi\sqrt{\frac{N}{6}\Big(d+\frac{d_f}{2}\Big)}\,.} \tag{16.2.46}$$

As an example, let's calculate $P(2;1,2)$, that is, the partitions of 2 into ordinary and fermionic numbers, with the latter having two possible labels.

The list of partitions is

$$\{2\}\,,\ \{2_1\}\,,\ \{2_2\}\,,\ \{1\,,1\,\}\,,\ \{1_1\,,1\,\}\,,\ \{1_2\,,1\,\}\,,\ \{1_1\,,1_2\,\}\,. \qquad (16.2.47)$$

The labels on the fermionic numbers are shown as subscripts. We see that $P(2;1,2) = 7$.

The general partition in (16.2.46) is useful for calculations in superstring theories. The states in these theories are built with both bosonic and fermionic creation operators. An application to a supersymmetric black hole will be considered in section 16.7.

16.3 Hagedorn temperature

Let's now return to the subject of relativistic strings. We will consider open string theory in the case where the open string states carry no spatial momentum. This will happen, for example, if the open string endpoints end on a D0-brane. With zero spatial momentum, the energy levels of the string are simply given by the rest masses of its quantum states. The mass-squared of a given state can be expressed in terms of the number operator $N^{\perp}$ (12.6.6):

$$M^2 = \frac{1}{\alpha'}(N^{\perp} - 1) \simeq \frac{N^{\perp}}{\alpha'}\,, \qquad (16.3.1)$$

in the approximation of large $N^{\perp}$. It follows that the energy $E = M$ is related to the number operator by the simple equality

$$\sqrt{N^{\perp}} = \sqrt{\alpha'}\,E\,. \qquad (16.3.2)$$

In the micro-canonical ensemble, the number of states $\Omega(E)$ equals $p_{24}(N^{\perp})$, because we have 24 transverse light-cone directions, and consequently 24 oscillator labels for each mode number. Therefore, equation (16.2.39) gives

$$S(E) = k \ln p_{24}(N^{\perp}) = k\, 2\pi \sqrt{\frac{N^{\perp}\cdot 24}{6}} = k\, 4\pi\sqrt{N^{\perp}}\,. \qquad (16.3.3)$$

Making use of the number/energy relation in (16.3.2) we find

$$\boxed{\;S = k\, 4\pi\sqrt{\alpha'}\,E\,.\;} \qquad (16.3.4)$$

This is the entropy/energy relation at high energy. An entropy proportional to the energy is unusual because it leads to a constant temperature:

$$\frac{1}{kT} = \frac{1}{k}\frac{\partial S}{\partial E} = 4\pi\sqrt{\alpha'}\,. \tag{16.3.5}$$

This temperature is called the Hagedorn temperature T_H:

$$\boxed{\frac{1}{\beta_H} = kT_H = \frac{1}{4\pi\sqrt{\alpha'}}\,.} \tag{16.3.6}$$

Here kT_H is the thermal energy associated with the Hagedorn temperature. In this high-energy approximation we are working with, we can increase arbitrarily the energy of strings and their temperature will remain fixed at the Hagedorn temperature. It is interesting to compare the energy kT_H to the rest mass of the particles found in the first massive level of the string. This corresponds to $N^\perp = 2$ in (16.3.1) and gives $E = M = 1/\sqrt{\alpha'}$. The ratio of the Hagedorn thermal energy to this rest energy is

$$\frac{kT_H}{\left(\frac{1}{\sqrt{\alpha'}}\right)} = \frac{1}{4\pi} \simeq \frac{1}{12.6}\,. \tag{16.3.7}$$

This shows that the Hagedorn thermal energy is quite small compared with the rest energy of almost any particle state of the string. This is an important result that will play a role in our later work in this chapter.

The entropy/energy relation in (16.3.4) holds also for closed strings having no spatial momentum. Recalling (13.2.14), we find

$$M^2 = \frac{2}{\alpha'}(N^\perp + \overline{N}^\perp - 2) \simeq \frac{4}{\alpha'}N^\perp\,, \tag{16.3.8}$$

since closed string states satisfy $N^\perp = \overline{N}^\perp$. It follows that the energy $E = M$ is related to the number operator as

$$2\sqrt{N^\perp} = \sqrt{\alpha'}\,E\,. \tag{16.3.9}$$

This time, the number of states $\Omega(E)$ is equal to the product of available states in the left-moving and in the right-moving sectors:

$$\Omega(E) = p_{24}\left(N^\perp\right) p_{24}\big(\overline{N}^\perp\big) = \left(p_{24}(N^\perp)\right)^2\,. \tag{16.3.10}$$

As a result, the entropy S is precisely twice that indicated in (16.3.3):

$$S(E) = k\,4\pi\,(2\sqrt{N^\perp}) = k\,4\pi\sqrt{\alpha'}E\,, \tag{16.3.11}$$

making use of (16.3.9) in the last step. We see that the Hagedorn temperature T_H is also the approximate temperature of very energetic closed strings.

16.4 Relativistic particle partition function

As a warmup to our computation of the partition function for a string, we compute here the partition function for a particle. We will work with a relativistic particle of mass m that lives in a D-dimensional spacetime, or equivalently in $d = D - 1$ space dimensions. Moreover, we assume that this particle is confined to a box of volume

$$V = L_1 L_2 \cdots L_d\,. \tag{16.4.1}$$

This box is in thermal contact with a reservoir at temperature T. The particle has an energy/momentum relation

$$E(\vec{p}) = \sqrt{\vec{p}^{\,2} + m^2}\,. \tag{16.4.2}$$

The quantum states of the particle in the box are labelled by the momenta $\vec{p}$, and therefore the partition function $Z(m^2)$ is given by

$$Z(m^2) = \sum_{\vec{p}} \exp(-\beta E(\vec{p}))\,. \tag{16.4.3}$$

In order to evaluate this partition function one must turn the sum over quantized momenta into an integral; this is where the volume dependence of Z comes in. The quantum wavefunctions with momentum $\vec{p} = \hbar\vec{k}$ have spatial dependence $\exp(i\vec{k}\cdot\vec{x})$. The periodicity of these wavefunctions in the box requires that for each spatial direction i

$$k_i L_i = 2\pi n_i\,, \qquad i = 1, 2, \cdots, d\,. \tag{16.4.4}$$

Here the n_i's are integers. Equivalently, in terms of momenta,

$$n_i = p_i\,\frac{L_i}{(2\pi\hbar)}\,. \tag{16.4.5}$$

It follows that summing over the various momenta is the same as summing over the various n_i. For an arbitrary smooth function $f[E]$ of the energy, we can thus write

$$\sum_{\vec{p}} f[E(\vec{p})] = \sum_{\vec{n}} f[E(\vec{p}(\vec{n}))] \simeq \int dn_1 dn_2 \ldots dn_d \, f[E(\vec{p}(\vec{n}))] \,, \qquad (16.4.6)$$

where the approximation by an integral is allowed because, for large boxes, the momenta change very little when a counter n_i changes by one unit. Using (16.4.5) and (16.4.1) we obtain

$$\sum_{\vec{p}} f[E(\vec{p})] \simeq V \int \frac{d^d p}{(2\pi\hbar)^d} \; f[E(\vec{p})] \,. \qquad (16.4.7)$$

This is the general prescription for dealing with sums over momenta. Applied to our case of interest (16.4.3) it gives

$$Z(m^2) = V \int \frac{d^d \vec{p}}{(2\pi\hbar)^d} \exp\Bigl(-\beta\sqrt{\vec{p}^{\,2} + m^2}\,\Bigr) \,. \qquad (16.4.8)$$

This is the integral representation of the partition function for a relativistic point particle of rest mass m. The temperature and volume arguments of Z are left implicit. Working with $\hbar = 1$, and letting $\vec{p} = m\vec{u}$, we find

$$Z(m^2) = V m^d \int \frac{d^d \vec{u}}{(2\pi)^d} \exp\left(-\beta m \sqrt{1 + \vec{u}^{\,2}}\right) \,. \qquad (16.4.9)$$

This integral is not elementary, but it can be written in terms of derivatives of the modified Bessel functions with argument βm (Problem 16.6). Rather than doing so, we will examine the integral in the domain of interest. For our string theory applications, this is the case when the thermal energy is much smaller than the rest energy of the particle. Indeed, as we saw earlier, for temperatures below the Hagedorn temperature, all but a few string states satisfy this condition. Thus we consider the situation where

$$\beta m >> 1 \,, \quad \text{low temperature} \,. \qquad (16.4.10)$$

We now claim that the leading approximation to the integral can be found by expanding the square root in (16.4.9) for $\vec{u}^{\,2}$ small. This is explained as follows. Using spherical coordinates and letting $\vec{u}^{\,2} = u^2$, we note that $d^d\vec{u} \sim$

$u^{d-1}du$ (recall the familiar cases of $d = 2, 3$). As a plain one-dimensional integral, the integrand in (16.4.9) is thus of the form

$$\text{integrand} \sim u^{d-1}\, e^{-\beta m\sqrt{1+u^2}}\,. \tag{16.4.11}$$

This integrand vanishes at $u = 0$ and $u = \infty$, and it peaks somewhere in between, giving the largest contribution to the integral. The maximum of the integrand can be found by setting the u-derivative of (16.4.11) equal to zero. This equation gives

$$\frac{d-1}{\beta m} = \frac{u^2}{\sqrt{1+u^2}}\,. \tag{16.4.12}$$

Since βm is large, the left hand side is very small, and u^2 must also be small. We can therefore neglect the u^2 in the square root and we find that the integrand is largest for

$$u^2 \simeq \frac{d-1}{\beta m} << 1\,. \tag{16.4.13}$$

We are therefore allowed to expand the square root in (16.4.9) to write

$$Z(m^2) \simeq V m^d e^{-\beta m} \int \frac{d^d\vec{u}}{(2\pi)^d} \exp\Big(-\frac{1}{2}\,\beta m\,\vec{u}^{\,2}\Big)\,. \tag{16.4.14}$$

The integral is now gaussian, and is readily evaluated

$$Z(m^2) \simeq V e^{-\beta m}\Big(\frac{m}{2\pi\beta}\Big)^{\frac{d}{2}}\,. \tag{16.4.15}$$

This is our final form for the partition function of a relativistic particle in the low-temperature limit. One can verify that this partition function is dimensionless, as it should be. Except for the additional factor $e^{-\beta m}$, this partition function coincides with the exact partition function for a non-relativistic particle. The exponential factor accounts for the contribution of the relativistic rest energy to the energy of the particle.

16.5 Single string partition function

We are now finally ready to evaluate the partition function for a single open string placed in a box of volume V. In order to calculate this, we must enumerate the quantum states of the string. The states are obtained by acting with the light-cone creation operators on the momentum eigenstates. A generic state is written as in (12.6.4):

$$|\lambda, p\rangle = \prod_{n=1}^{\infty} \prod_{I=2}^{25} (a_n^{I\dagger})^{\lambda_{n,I}} \, |p^+, \, \vec{p}_T\rangle \, , \tag{16.5.1}$$

where the notation $|\lambda, p\rangle$ emphasizes that the momentum components as well as the occupation numbers $\lambda_{n,I}$ are labels of the string states. The d components $(p^+, \vec{p}_T)$ listed in the momentum eigenstate specify the light-cone energy p^- via the on-shell condition:

$$M^2(\{\lambda_{n,I}\}) = -p^2 = 2p^+p^- - p^I p^I \, , \tag{16.5.2}$$

where

$$M^2(\{\lambda_{n,I}\}) = \frac{1}{\alpha'}(N^\perp - 1) \, , \qquad N^\perp = \sum_{n,I} n \, \lambda_{n,I} \, . \tag{16.5.3}$$

Since both the spatial momentum and the energy are determined for the above states, we can label the string states with the set $\{\lambda_{n,I}\}$ of occupation numbers and the *spatial* momentum $\vec{p}$. We then write

$$E(\{\lambda_{n,I}\}, \vec{p}) = \sqrt{M^2(\{\lambda_{n,I}\}) + \vec{p}^{\,2}} \, . \tag{16.5.4}$$

To find the partition function Z_{str} of a single string, we must sum over all states $|\lambda, p\rangle$, or equivalently over all spatial momenta $\vec{p}$ and all values of the occupation numbers $\lambda_{n,I}$:

$$Z_{\text{str}} = \sum_\alpha \exp(-\beta E_\alpha) = \sum_{\lambda_{n,I}} \sum_{\vec{p}} \exp\Big[-\beta \sqrt{M^2(\{\lambda_{n,I}\}) + \vec{p}^{\,2}}\Big] \, . \tag{16.5.5}$$

We recognize, however, that the momentum sum simply gives the partition function for a relativistic particle of mass-squared $M^2(\{\lambda_{n,I}\})$. We thus write

$$Z_{\text{str}} = \sum_{\lambda_{n,I}} Z\left(M^2(\{\lambda_{n,I}\})\right) . \tag{16.5.6}$$

Since the mass M^2 depends only on $N^\perp$, and there are $p_{24}(N^\perp)$ states with number eigenvalue $N^\perp$, the sum over occupation numbers $\{\lambda_{n,I}\}$ can be traded for a sum over $N^\perp \equiv N$:

$$Z_{\rm str} = \sum_{N=0}^{\infty} p_{24}(N)\, Z(M^2(N))\,. \tag{16.5.7}$$

So far, no approximations have been made, and the above result is exact.

In order to proceed further, we approximate this sum by turning it into an integral; this is accurate for large N. We get

$$Z_{\rm str} \simeq \int_0^\infty dN\, p_{24}(N)\, Z(M^2(N))\,. \tag{16.5.8}$$

It is customary to define a density of states $\rho(M)$ as a function of the mass M, and to use the mass as the variable of integration. This is done using the relation

$$p_{24}(N)dN = \rho(M)dM\,. \tag{16.5.9}$$

We express the left-hand side in terms of mass by using $\alpha' M^2 \simeq N$:

$$dN = 2\alpha' M dM = 2(\sqrt{\alpha'}M)\, d(\sqrt{\alpha'}M)\,. \tag{16.5.10}$$

Moreover, using (16.2.42) and (16.3.6) we find

$$p_{24}(N) \simeq \frac{1}{\sqrt{2}}(\sqrt{\alpha'}M)^{-27/2}\exp(\beta_H M)\,. \tag{16.5.11}$$

Substituting these two equations back into (16.5.9) gives us

$$\rho(M)dM = \sqrt{2}\,(\sqrt{\alpha'}M)^{-25/2}\,\exp(\beta_H M)\, d(\sqrt{\alpha'}M)\,. \tag{16.5.12}$$

Note incidentally that

$$\rho(M) \sim M^{-25/2}\exp(\beta_H M)\,, \tag{16.5.13}$$

showing that the exponential growth in the density of states is controlled by the Hagedorn temperature. As we will see shortly, the partition function does not converge for temperatures higher than the Hagedorn temperature. With (16.5.12) and (16.5.9), the partition function in (16.5.8) becomes

$$Z_{\rm str} \simeq \sqrt{2}\int_0^\infty (\sqrt{\alpha'}M)^{-25/2}\,\exp(\beta_H M)\, Z(M^2)\, d(\sqrt{\alpha'}M)\,. \tag{16.5.14}$$

It only remains to write the relativistic particle partition function (16.4.15) in terms of M and kT_H. With the help of

$$\frac{M}{2\pi\beta} = 2(\sqrt{\alpha'}M)\, kT\, kT_H\,, \qquad \beta M = 4\pi(\sqrt{\alpha'}M)\,\frac{T_H}{T}\,, \tag{16.5.15}$$

we find

$$Z(M^2) \simeq V 2^{25/2}\,(kT\, kT_H)^{25/2}\,(\sqrt{\alpha'}M)^{25/2}\exp\Bigl(-4\pi\,\sqrt{\alpha'}M\,\frac{T_H}{T}\Bigr)\,. \tag{16.5.16}$$

Substituting this result into (16.5.14) the string partition function becomes

$$Z_{\rm str} \simeq 2^{13} V\,(kT\, kT_H)^{25/2}\int_0^\infty d(\sqrt{\alpha'}M)\exp\Bigl(-4\pi\sqrt{\alpha'}M\Bigl[\frac{T_H}{T}-1\Bigr]\Bigr)\,. \tag{16.5.17}$$

Notice that the powers of M in the integrand cancelled out. Setting $x = \sqrt{\alpha'}M$, the above expression turns into

$$Z_{\rm str} \simeq 2^{13} V\,(kT\, kT_H)^{25/2}\int_0^\infty dx\,\exp\Bigl(-4\pi x\Bigl[\frac{T_H}{T}-1\Bigr]\Bigr)\,. \tag{16.5.18}$$

The integral only converges for $T < T_H$, where we have

$$\boxed{Z_{\rm str} \simeq \frac{2^{11}}{\pi} V\,(kT\, kT_H)^{25/2}\Bigl(\frac{T}{T_H - T}\Bigr)\,.} \tag{16.5.19}$$

This is our final expression for the approximate partition function of a single open string in a box of volume V, in thermal contact with a reservoir at temperature T. We made several approximations. In particular, we assumed that the largest contributions arise from large $N^\perp$ and turned the sum over $N^\perp$ into an integral. This approximation is not delicate enough to represent the contributions from massless states and from tachyon states. The partition function of a tachyon is indeed problematic: equation (16.4.8) tells us that Z is complex number if $m^2 < 0$. Since the tachyon represents an instability, and (16.5.19) ignores this complication, we can expect this result to capture roughly the physics of an open string in a theory without a tachyon. We have also assumed no string interactions. In this approximation, the string in the box cannot not break to become a set of short strings.

Despite all of these limitations, equation (16.5.19) gives us some interesting information. Using (16.1.7), and since the volume dependence of $Z_{\rm str}$ is only multiplicative, we find

$$p = \frac{1}{\beta}\frac{\partial \ln V}{\partial V} = \frac{1}{\beta V} \quad \longrightarrow \quad pV = kT\,. \tag{16.5.20}$$

This is not different from the equation of state for a non-relativistic particle in a box of volume V. Indeed, the volume dependence of the partition function is the same for a particle and for a string.

The calculation of the average energy (16.1.6) is a little more nontrivial. For this computation we need only the β-dependence of $\ln Z$. Using (16.5.19) we find

$$\ln Z_{\rm str} = -\frac{25}{2}\ln\beta - \ln(\beta-\beta_H) + \ldots\,, \tag{16.5.21}$$

where the dots represent terms without β dependence. It now follows that

$$E = -\frac{\partial \ln Z}{\partial \beta} = \frac{25}{2}\,kT + kT_H\Big(\frac{T}{T_H - T}\Big)\,. \tag{16.5.22}$$

For low temperatures the energy is approximately given by

$$E \simeq \frac{25}{2}\,kT + kT = \Big(\frac{25}{2}+1\Big)kT\,, \qquad T << T_H\,. \tag{16.5.23}$$

The first term, proportional to $d/2$, is the standard average energy of a particle in d spatial dimensions at a temperature T. The additional kT appearing in the energy is due to string effects. For temperatures smaller, but very close to the Hagedorn temperature, the energy is roughly

$$E = \frac{kT_H}{1-\frac{T}{T_H}}\,, \qquad T \approx T_H\,. \tag{16.5.24}$$

It follows that the energy grows without bound as the temperature approaches the Hagedorn temperature. This is, of course, a string effect.

16.6 Black holes and entropy

A black hole is formed when the mass of an object is increased while keeping the object of the same size, or when the size of an object is reduced while keeping its mass constant. Black holes exist in our universe. The existence of a supermassive black hole at the center of our galaxy has been established beyond reasonable doubt. Most likely, there are millions of black holes in every galaxy. They are the remnants of ordinary stars that are a few times more massive than the sun.

Black holes pose very significant theoretical challenges. In Einstein's theory of general relativity they appear as classical solutions representing matter that has collapsed down to a point with infinite density: a singularity. Although dealing with singularities is quite delicate, the real puzzles of black holes arise at the quantum level. Black holes have temperature and can radiate. They have entropy as well. String theory has had definite successes in understanding some of these properties. In this section we review basic features of black holes and use string theory to discuss the entropy of four-dimensional Schwarzchild black holes. In the following section we will examine a particular five-dimensional black hole whose entropy can be calculated exactly in string theory.

The simplest black holes are Schwarzchild black holes. These black holes are spherically symmetric, static solutions that represent the gravitational field of a point mass M. In this black hole, the point singularity is separated from the outside world by an *event horizon*. This is a two-sphere centered at the singularity, whose radius R is called the Schwarzchild radius, or the radius of the hole. If any object ventures inside the event horizon it will irrevocably fall into the singularity. Classically, nothing can escape from the region enclosed by the event horizon. The value R of the Schwarschild radius can be estimated by assuming that the total energy of any particle at the horizon is equal to zero. For a particle of mass m, this energy includes the rest energy mc^2 and the gravitational potential energy $-GMm/R$. Setting the sum of these to equal to zero, we find

$$mc^2 - \frac{GMm}{R} = 0 \quad \longrightarrow \quad R \simeq \frac{GM}{c^2}\,. \tag{16.6.1}$$

In fact, the exact answer is

$$R = \frac{2GM}{c^2}\,. \tag{16.6.2}$$

The Schwarzchild radius of the sun is about 3 km. The Schwarzchild radius of the earth is about 1cm. The Schwarzchild radius of a billion-ton asteroid is of the order of 10^{-15}m. It is possible to use Newtonian gravitation to estimate the gravitational field at the horizon:

$$|\vec{g}| = \frac{GM}{R^2} = \frac{c^4}{4GM}\,. \tag{16.6.3}$$

This gravitational field becomes small for very massive black holes.

Quick Calculation 16.4. Show that an object of uniform mass density ρ forms a black hole if its radius is larger than $c/\sqrt{8\pi G\rho/3}$.

If we believe that the second law of thermodynamics holds generally, the existence of black holes leads to some surprising conclusions. Assume that certain amount of hot gas falls into a black hole forming a new black hole with a slightly higher mass. Since the total entropy of the system made by the gas and the black hole cannot decrease, the new black hole must have increased its entropy by at least the amount of entropy carried by the gas. We are thus led to believe that black holes must have entropy. You know that a system has entropy when there are many microscopic states of the system that are consistent with its macroscopic properties. On the other hand, if the black hole represents a point mass singularity at the origin, it is hard to see what are the microstates that give rise to the entropy.

Black holes are also assigned a temperature T. This temperature, in fact, behaves as the gravitational field at the horizon: it is inversely proportional to the mass of the hole. In natural units, $kT = 1/(8\pi M)$, so inserting back the factors of $\hbar, c$, and G (Problem 3.7), we find

$$kT = \frac{\hbar c^3}{8\pi GM}\,. \tag{16.6.4}$$

This equation allows us to calculate the entropy of the black hole using $E = Mc^2$ for the energy of the black hole and the first law of thermodynamics $dE = TdS$:

$$dE = c^2 dM = TdS = \frac{\hbar c^3}{8\pi GM}\,\frac{1}{k}dS\,. \tag{16.6.5}$$

A little rearrangement yields:

$$\frac{1}{k}\,dS = \frac{4\pi G}{\hbar c}dM^2\,. \tag{16.6.6}$$

Integrating this equation, and assuming that the entropy of a zero-mass black hole is zero, we find

$$\frac{S}{k} = \frac{4\pi G}{\hbar c}\,M^2\,. \tag{16.6.7}$$

The entropy of the black hole is proportional to the *square* of its mass. A useful alternative expression for the entropy uses the area A of the event horizon. With $A = 4\pi R^2$ and R given in (16.6.2), one readily obtains

$$\frac{S}{k} = \frac{1}{4}\frac{c^3}{\hbar G}A = \frac{A}{4\ell_P^2}\,, \tag{16.6.8}$$

where ℓ_P is the Planck length. The last right-hand side in this equation has a simple interpretation: the entropy is one-fourth of the area of the horizon expressed in units of Planck-length squared. Since ℓ_P^2 is a remarkably small area, the entropy of any astrophysical-size black hole is extremely large. The entropy of a black hole is roughly reproduced if one imagines having a degree of freedom with a finite number of states for each horizon element of area ℓ_P^2. String theory provides candidate degrees of freedom for black holes, but they do not relate directly to the horizon area.

In string theory, we attempt to relate a stationary Schwarzchild black hole to a string with a high degree of excitation but zero momentum. In the microcanonical ensemble, a string state with energy E has an entropy (16.3.4). This is true both for open and for closed strings (see (16.3.11)). Identifying $E = M$ and working henceforth with $\hbar = c = 1$, we have

$$\frac{S_{\text{str}}}{k} = 4\pi\sqrt{\alpha'}\,M\,, \tag{16.6.9}$$

where we have added the subscript 'str' to refer to the entropy of the string. This result should be compared to the black-hole entropy (16.6.7):

$$\frac{S_{\text{bh}}}{k} = 4\pi G\,M^2\,. \tag{16.6.10}$$

The disagreement appears to be clear: the entropy of a black hole goes like the mass squared, while the entropy of a string goes like the mass. We will soon show, however, that the apparent disagreement was to be expected. Properly understood, there is a surprising agreement between these equations. The linear dependence of the string entropy on the mass M of the string is not surprising. Entropy is an extensive quantity, and for a string, the mass M is roughly proportional to its length L. The black hole entropy, on the other hand, exhibits a surprising feature. It is not proportional to the volume enclosed by the event horizon, but rather, to the area of the horizon. This failure of extensivity is a feature of gravitational physics.

Before considering the relation between equations (16.6.9) and (16.6.10), let's give a heuristic derivation of the string entropy. For this, we consider a string of mass M and estimate its length L to be roughly given by

$$M \sim T_0 L \sim \frac{1}{\alpha'} L \,, \tag{16.6.11}$$

where $T_0 \sim 1/\alpha'$ is the string tension. We now imagine the string built by joining together string bits, each of which is of length $\ell_s = \sqrt{\alpha'}$. Assume that each time we add a bit, it can point in any of n possible directions. The number n may be equal to the number of spatial dimensions, but since our arguments are rough, we will not attempt to be specific. Since the number of string bits is $L/\sqrt{\alpha'}$, the number of ways Ω that we can build this string is roughly

$$\Omega \sim n^{L/\sqrt{\alpha'}} \sim n^{M\sqrt{\alpha'}} \sim e^{M\sqrt{\alpha'}\ln n} \,. \tag{16.6.12}$$

The entropy of the string is obtained by taking the logarithm of Ω:

$$\frac{S_{\rm str}}{k} \sim M\sqrt{\alpha'} \sim M\,\ell_s \,, \tag{16.6.13}$$

where we discarded the $\ln n$ factor, in keeping with the accuracy of the estimate. This result is consistent with the expression given in (16.6.10).

The reason equations (16.6.9) and (16.6.10) disagree is that the black hole entropy $S_{\rm bh}$ was calculated in a regime where interactions are necessary, while the string entropy $S_{\rm str}$ was calculated for free strings. We did not have the right to expect agreement, unless for some reason, interactions did not affect the calculation of the entropy of strings. There is no such reason in the theory of bosonic strings.

Interactions are necessary in the black hole entropy calculation because Newton's constant G vanishes if the string coupling constant g is set to zero. Indeed, we recall (13.4.6), which states that

$$G \sim g^2\,\alpha' = g^2\ell_s^2 \,. \tag{16.6.14}$$

The black hole entropy and its radius are then given as

$$\begin{aligned} \frac{S_{\rm bh}}{k} &\sim GM^2 \sim g^2\ell_s^2 M^2 \,, \\ R &\sim GM \;\; \sim g^2\ell_s^2 M \,. \end{aligned} \tag{16.6.15}$$

While they incorporate the string coupling dependence via Newton's constant, the above results use classical general relativity, where, for example, the concept of a horizon makes sense. We are allowed to neglect string theory corrections to general relativity as long as black holes are larger than the string length.

Consider now a large black hole with entropy S_0, mass M_0 and radius $R_0 >> \ell_s$. Fix also the string coupling at some finite value g_0. Equations (16.6.15) then give us

$$\begin{aligned} \frac{S_0}{k} &\sim g_0^2\, \ell_s^2\, M_0^2\,, \\ R_0 &\sim g_0^2\, \ell_s^2\, M_0\,. \end{aligned} \tag{16.6.16}$$

Since the calculation of the string entropy is valid for zero, or possibly small string coupling, imagine now the process of dialing down the value of the string coupling. This is done by changing the expectation value of the dilaton, as explained in section 13.4. It is reasonable to assume that this process can be carried out reversibly, so we can expect the black hole entropy to remain unchanged. On the other hand, as we dial down the coupling g the mass of the black hole increases like $1/g$ to keep the entropy constant in (16.6.15). The mass is not increasing, however, if measured in units of Planck mass, since $G \sim 1/m_P^2$. The radius R of the black hole decreases, as it follows from the second relation in (16.6.15) bearing in mind that $M \sim 1/g$.

Let g_*, R_*, and M_* denote the final values of the string coupling, black hole radius, and black hole mass, respectively. The constancy of the entropy, and the formula for the radius give

$$\begin{aligned} \frac{S_0}{k} &\sim g_0^2\, \ell_s^2\, M_0^2 = g_*^2\, \ell_s^2 M_*^2\,, \\ R_* &\sim g_*^2\, \ell_s^2\, M_s\,. \end{aligned} \tag{16.6.17}$$

We do not expect these results to hold when the black hole becomes smaller that the string length, so let's fix $R_* = \ell_s$ as the minimum radius for which equations (16.6.17) can be trusted. The condition $R_* = \ell_s$ tells us that

$$g_*^2\, \ell_s^2\, M_s \sim \ell_s \quad \longrightarrow \quad M_* \sim \frac{1}{g_*^2 \ell_s}\,. \tag{16.6.18}$$

Back into the expression for the entropy S_0, we find

$$\frac{S_0}{k} \sim \frac{1}{g_*^2}\,. \tag{16.6.19}$$

The coupling g_* is clearly very small since S_0 was assumed to be very large. At such weak coupling we can reasonably trust the free string theory expression (16.6.13) for the entropy. Since the black hole we are comparing with has mass M_*, we consider a string of mass M_*. The entropy is then given by

$$\frac{S_{\rm str}}{k} \sim M_*\,\ell_s \sim \left(\frac{1}{g_*^2\ell_s}\right)\ell_s \sim \frac{1}{g_*^2}\,, \qquad (16.6.20)$$

where we made use of (16.6.18). Comparing with S_0 in (16.6.19), we see that $S_{\rm str} \sim S_0$. This agreement is evidence for the hypothesis that a Schwarzchild black hole is the strong coupling version of a string with a very high degree of excitation. It is far from a proof, however. As you have seen, we have only written approximate relations, and we have made a series of assumptions about the ranges of validity of certain results. A proof remains to be found at this time. Nevertheless, there is additional circumstantial evidence that this picture is at least roughly correct. It is possible to estimate the "size" of a string using the picture of string bits and assuming that the string is a random walk. One can then show that for any fixed coupling g there is a mass beyond which any excited string state is smaller than its Schwarzchild radius (Problem 16.8). This suggests that very heavy string states will form black holes.

16.7 Counting states of a black hole

Our computation of the entropy of strings can be done in the limit when we neglect the effects of interactions. Since a black hole can only exist once interactions are turned on, an exact computation of the entropy of a black hole in string theory requires that the counting of states done with string coupling $g = 0$ remain valid when $g \neq 0$.

For the Schwarzchild black holes considered in the previous section this does not happen. As a result, we could only confirm qualitative agreement over a narrow range of couplings where the gravity computation and the free string theory computation could both hold. In this section we wish to consider a particular five-dimensional black hole that appears in *superstring* theory. In this black hole, as we will explain below, the counting of states at zero string coupling will remain valid when the coupling becomes non-zero. It is the simplest known black hole with this property. Four-dimensional black holes with the same property are known, but are slightly more complicated. This is why we focus here on the five-dimensional black hole.

Such remarkable property is due to supersymmetry, a symmetry that relates bosons to fermions. As long as this symmetry is present, certain quantities can be calculated at zero coupling, and the results are valid for all values of the coupling. Superstring theories living in ten-dimensional Minkowski spacetime have supersymmetry. It is a challenge to compactify spacetime and preserve supersymmetry, but this happens if we curl up dimensions into circles. If we now include a black hole solution in the compactified spacetime, supersymmetry can be lost. The black hole we are interested in is special: some supersymmetry survives.

The starting point is type IIB superstring theory, a ten-dimensional theory of closed strings. One can search for black hole solutions in the regime where the string theory is well-approximated by a field theory of gravity, Kalb-Ramond fields, and other fields, including fermions. Such a theory is called type IIB supergravity. The black hole in question is obtained after curling up five of the spatial dimensions into circles. These curled up dimensions are x^5, x^6, x^7, x^8, and x^9. The black hole is a spherically symmetric configuration in the un-compactified effective spacetime M^5 defined by the coordinates x^0, x^1, x^2, x^3, and x^4. We cannot discuss here the full construction of the black hole, so we will simply summarize the results that are obtained:

(1) The black hole carries three different electric charges with respect to three Maxwell-like gauge fields that live on M^5. These charges are denoted by the integers

$$Q_1,\ Q_5,\ \text{and}\ \ N\,. \tag{16.7.1}$$

A specific black-hole is obtained by choosing these three integers.

(2) The black-holes is *extremal*: it has the minimal mass that is compatible with its charges. It does not radiate, since radiation would reduce its mass without the necessary change of charge. The black hole has zero temperature. In addition, its presence preserves a large part of the original supersymmetry of the IIB theory in ten-dimensional Minkowski space.

(3) The black-hole horizon is a three-sphere with finite volume A_H. The thermodynamically expected black hole entropy S_{bh} is calculated using

the five-dimensional analog of (16.6.8):

$$\frac{S_{\rm bh}}{k} = \frac{A_H}{4G^{(5)}} = 2\pi\sqrt{NQ_1Q_5}\,. \tag{16.7.2}$$

Here $G^{(5)}$ is the five-dimensional Newton constant and we have set $\hbar = c = 1$. Interestingly, the entropy only depends on the charges carried by the black hole, and not on other parameters, like the string coupling, or the size of the circles used for the compactification.

The goal of a string theory computation is to reproduce the entropy (16.7.2) by a counting of states. String theory must explain why this black hole can be constructed in many possible ways. As before, we know how to count states in non-interacting string theory. This time, however, the black hole respects supersymmetry and this guarantees that the zero-coupling counting holds for non-zero coupling.

At zero coupling the black hole is constructed by considering type IIB superstring theory, with the five coordinates $x^5, \ldots, x^9$ curled up into circles. The charges Q_1 and Q_5 are generated by wrapping a number Q_1 of D1-branes along the circle x^5, and a number Q_5 of D5-branes around the five circles. Since a D5-brane has five spatial dimensions, the D5-branes wrap completely around the compact extra dimensions. How does this look to the five-dimensional observer in M^5? Since all directions along M^5 are Dirichlet for the D5-branes, the D5-branes have fixed positions on M^5. They appear as a collection of points. The same is true for the D1-branes. In the configuration we are trying to build, we require that all these points coincide. Thus all D-branes are coincident, and are seen by the observer as a single point in M^5. This point is the center of the would-be black hole that forms when the coupling is turned on. So far, this configuration of D-branes cannot be built in different ways preserving supersymmetry. A few discrete choices are possible, we can choose, for example, another coordinate to wrap all of the D1-branes. But any constant number, independent of the charges will not help us get the correct entropy. So, where does the entropy come from?

We recall that the macroscopic black hole had an additional charge N. What does it correspond to in the brane construction? It is a momentum quantum number. The momentum around the circle x^5 must equal

$$p^5 = \frac{N}{R}\,, \tag{16.7.3}$$

where R is the radius of the circle. This momentum cannot be carried by the D-branes since they are translationally invariant along the x^5-direction. The momentum is carried by open strings attached to the D-branes! We can now see how it is possible to get many states: there are many kinds of strings stretching between the Q_1 D1-branes and the Q_5 D5-branes. We have (1,1) strings going from D1-branes to D1-branes. We have (5,5) strings going from D5-branes to D5-branes. Finally, there are (1,5) and (5,1) strings, going from D1-branes to D5-branes and viceversa, respectively. Moreover, the total momentum quantum number N can be split between many open strings. Supersymmetry, however, makes one extra demand: all of the open strings must carry momentum in the same direction along x^5.

To proceed further, we need some known facts about the combined system of coincident D1- and D5-branes:

(1) The D1/D5 brane system is a bound-state system. Open strings of type (1,1) and (5,5) become massive and do not become excited in the configuration we are interested in. These strings can be dropped from the counting.

(2) The total number of ground states of a (1,5) string and the oppositely-oriented (5,1) string is eight: four bosonic ground states and four fermionic ground states.

(3) The Q_1 D1-branes may join to form a single D1-brane wrapped Q_1 times around the circle. Similarly, the Q_5 D5-branes can join to form a single D5-brane wrapped Q_5 times around the circle. If this happens, the charges are not changed.

Bearing this information in mind, we see that the momentum number N must be split among open strings that go in between D1-branes and D5-branes. We need a partition of N, but of which kind? Let's assume, for the time being that $N \gg Q_1Q_5$ and do a preliminary counting that will work but is not generally valid.

We have to partition N and for each element of a partition we have to tell what kind of state is carrying such momentum quantum number. There are Q_1Q_5 ways of picking a D1-brane and a D5-brane. But then, four additional ways to pick a bosonic excitation or, alternatively, four ways to pick a fermionic excitation (see (2)). As a result, we have $d = 4Q_1Q_5$ bosonic labels and $d_f = 4Q_1Q_5$ fermionic labels. Making use of (16.2.46), the entropy is then

$$\frac{S_{\rm str}}{k} = \ln P\Big(N;\, 4Q_1Q_5,\, 4Q_1Q_5\Big) \sim 2\pi\sqrt{\frac{N}{6}(4Q_1Q_2)\frac{3}{2}} = 2\pi\sqrt{NQ_1Q_5}\,, \tag{16.7.4}$$

in perfect agreement with (16.7.2). This is very nice, but not general enough. The restriction $N >> Q_1Q_5$ is needed because in (16.2.46) N must be much larger than both d and d_f. It can be shown that if N, Q_1 and Q_5 all grow large simultaneously, $\ln P$ actually fails to give the expected entropy. This means that we have not quite yet identified the general counting that gives the entropy.

The clue is given in item (3) of the list above. Imagine the D1-brane wrapped Q_1 times around the circle x^5. Consider then, a (1,1) string moving along the D1-brane. How is the momentum of the string quantized? For such string, the circle has effectively become Q_1 times longer: $(2\pi R)Q_1$ is the distance the string must travel to return to its original starting point on the D1-brane. Accordingly, the string momentum is quantized in units of $1/(Q_1R)$. This is true with one proviso. The individual open strings can have their momentum quantized with this finer unit, but the total momentum of all the open strings must still be quantized in units of $1/R$. This is because the system comprised by the D1-brane and the attached open strings must be invariant under a $2\pi R$-long translation along the circle. As a result, the total momentum of the system must be quantized in units of $1/R$. Since the D1-brane has no momentum, the claim follows.

We must focus, however, on the strings stretching between D1- and D5-branes. Imagine now that the D5-branes are also wrapped. For simplicity, assume that Q_1 and Q_5 are relatively prime (we will relax this assumption shortly). Consider now a (1,5) string. How many times must it go around the circle so that *both* of its endpoints return to their original positions? After Q_1 turns the first endpoint does, but not the second. After Q_5 turns the second endpoint returns to its starting point, but the first does not. Being relatively prime, it takes Q_1Q_5 turns to have both endpoints return to their original positions on the respective branes. As a result, the momentum of (1,5) and (5,1) strings is quantized with the even finer unit of $1/(Q_1Q_5R)$! This can be arranged to be approximately true even if Q_1 and Q_5 are not relatively prime. Take, for example $Q_1 = Q_5 = 100$. We can take the D1-brane and split off one turn, to get a system with $Q_1' = 99$, plus one extra D1-brane.

Since Q_1' and Q_5 are relatively prime, the momentum of most open strings is then quantized in units of $1/(Q_1'Q_5R)$, which is approximately equal to $1/(Q_1Q_5R)$. In general, for large Q_1 and Q_5 we can find relatively prime numbers $Q_1' < Q_1$ and $Q_5' < Q_5$, such that $Q_1' \sim Q_1$ and $Q_2' \sim Q_2$.

With this finer unit of quantization, the total momentum in (16.7.3) is suggestively written as

$$p^5 = \frac{NQ_1Q_5}{Q_1Q_5R}\,, \tag{16.7.5}$$

This time we must partition the quantum number NQ_1Q_5. Since we just have one long D1-brane and one D5-brane, there is just one kind of string stretching across the branes. Therefore, the labels on the elements of a partition are either four bosonic ones, or four fermionic ones: $d = d_f = 4$. As a result, the entropy is given by

$$\frac{S_{\text{str}}}{k} = \ln P\Big(NQ_1Q_5;\, 4,\, 4\Big) \sim 2\pi\sqrt{\frac{NQ_1Q_5}{6}(4)\frac{3}{2}} = 2\pi\sqrt{NQ_1Q_5}\,. \tag{16.7.6}$$

The agreement with the black hole entropy is now complete and holds generally.

Giving a statistical mechanics derivation of the black hole entropy is a significant accomplishment of string theory. After all, black holes do exist, and they have entropy. Much work remains to be done in string theory to understand fully black holes. As we have seen, Schwarzchild black holes are not under such precise control. Moreover, there are puzzles associated to the fate of the information that falls into a black hole.

String theory gives a clear picture of the zero-coupling degrees of freedom of the would-be black hole. Moreover, we know that the counting continues to hold for non-zero coupling. Nevertheless we do not know how these degrees of freedom look by the time the black hole is formed. Thus mysteries remain.

Problems

Problem 16.1. *Review of statistical mechanics.*

(a) Prove equation (16.1.7). Hint: The energy levels $E_\alpha(V)$ of the system depend on the volume. As the volume changes quasi-statically, the change of mean energy is calculated using the equilibrium distribution of states. The change of mean energy can be interpreted as due to work against the pressure.

(b) Prove equation (16.1.15). Hint: Consider the differential $d \ln Z(T, V)$.

Problem 16.2. *Fermionic violin string and counting unequal partitions.*

Consider a system of simple harmonic oscillators of frequencies $\omega_0, 2\omega_0, \ldots$ identical to those of the bosonic violin string oscillators of section 16.2. This time, however, each occupation number n_ℓ can only take the values 0 or 1. Oscillators with such property are said to be fermionic oscillators.

(a) Calculate the free energy of such string in the high temperature limit. The answer involves the sum

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots .$$

You can calculate this sum using the result in (16.2.23).

(b) Let $q(N)$ denote the number of partitions of N into *unequal* pieces. Use your result in (a) above to show that the large N expansion for $\ln q(N)$ is given by (16.2.45).

(c) Now assume this string is relativistic, with the energy related to the mode number as in (16.3.2): $\sqrt{N} = \sqrt{\alpha'} E$. What would be the "Hagedorn" temperature for a string with very high energy?

Problem 16.3. *Generating functions for partitions.*

A particularly simple infinite product provides a generating function for the partitions $p(n)$:

$$\prod_{n=1}^{\infty} \frac{1}{(1 - x^n)} = \sum_{n=0}^{\infty} p(n) x^n .$$

Here $p(0) \equiv 1$. To evaluate the left-hand side, each factor is expanded as an infinite Taylor series around $x = 0$. Test this formula for $n \leq 4$ and explain (in words) why it works in general. Find a generating function for unequal partitions $q(n)$ and test it for low values of n.

Problem 16.4. *Generalized counting of partitions.*

Prove the formula (16.2.46) for the counting of partitions $P(N; d, d_f)$ of N into ordinary integers with d labels and fermionic integers with d_f labels. Calling Z the partition function of ordinary oscillators, and Z_f the partition function of Problem 16.2, begin your derivation by explaining why the partition function Z_T for the composite system of bosonic and fermionic labelled oscillators is given by

$$Z_T = (Z)^d (Z_f)^{d_f} .$$

Problem 16.5. *Open superstring Hagedorn temperature*

Consider the supersymmetric open superstring theory described in section 13.5.

(a) Show that the total number of states (NS and R sectors) with number $N^\perp$ is $16P(N^\perp; 8, 8)$. Hints: One of the two sectors is easier to count, then use supersymmetry.

(b) Following the method of section 16.3 calculate the Hagedorn temperature for an open superstring. Show that it is a factor of $\sqrt{2}$ larger than the Hagedorn temperature of the bosonic string.

Problem 16.6. *Partition function of the relativistic particle*

Evaluate exactly the partition function (16.4.9) for the relativistic point particle in terms of (derivatives of) modified Bessel functions making use of the integral definition

$$K_\nu(z) = \frac{\sqrt{\pi}\left(\frac{1}{2}z\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty e^{-z\cosh t} \sinh^{2\nu} t \, dt .$$

Use the following asymptotic expansion, valid for large z

$$K'_\nu(z) \sim -\sqrt{\frac{\pi}{2z}}\left[1 + \frac{4\nu^2 + 3}{8z} + \cdots\right],$$

to confirm our low temperature result in (16.4.15). Calculate the first nontrivial correction to this result.

Problem 16.7. *Corrections to temperature/energy relation in the idealized string.*

We found the Hagedorn temperature in the idealized string model by computing the entropy/energy relation in the high energy approximation where $\ln p_{24}(N) \sim 4\pi\sqrt{N}$. Use the more accurate expression for the partitions $p_{24}(N)$ as given in (16.2.42) to find the corrections to the temperature/energy relation. You will find the surprising result that as the energy goes to infinity the temperature goes to T_H from above! Plot $T(E)$ and calculate the specific heat C in the large energy regime.

Problem 16.8. *Estimating the size of a string state.*

We used the heuristic picture of a string made out of string bits to estimate correctly the entropy (16.6.13) of a string. We now want to use this picture to estimate the size of a string state. Assume that each string bit can point randomly in any of d orthogonal directions. The string can then be viewed as a random walk with a number of steps equal to the number of bits.

(a) Use the random-walk formula for the average value of the square of the displacement to show that the "size" R_{str} of a string of mass M is

$$R_{\text{str}}(M) \sim M^{1/2}\, \ell_s^{3/2} \sim N^{1/4} \ell_s \,, \tag{1}$$

where N is the number eigenvalue associated to the mass M. Note that the size grows like the square root of the mass, while the length of the string grows like the mass.

(b) Show that the size of a string becomes smaller than its Schwarzchild radius if its mass exceeds $\overline{M}$, where

$$\overline{M} \sim \frac{1}{g^4 \ell_s} \sim \frac{m_P}{g^3} \,. \tag{2}$$

Give a rough estimate of $\overline{M}$ in kg. when $g \sim 0.01$. What is the corresponding value of N?

(c) Consider a black hole of a million solar masses. Assume $g = 0.01$ and calculate the value of N and the size of the string that models this black hole. Compare the string size to the Schwarzchild radius.

The random walk model of string states applies to strings with little or no angular momentum (recall that the size of a rigidly rotating open string is proportional to the mass). In this model the string is much smaller than its length. The effect of string interactions appears to reduce further the size of the strings.