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Dynamics of extended bodies in general relativity
II. Moments of the charge-current vector†

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The problem is considered of defining multipole moments for a tensor field given on a curved spacetime, with the aim of applying this to the energy-momentum tensor and charge-current vector of an extended body. Consequently, it is assumed that the support of the tensor field is bounded in spacelike directions. A definition is proposed for 'a set of multipole moments' of such a tensor field relative to an arbitrary bitensor propagator. This definition is not fully determinate, but any such set of moments completely determines the original tensor field. By imposing additional conditions on the moments in two different ways, two uniquely determined sets of moments are obtained for a vector field J^{α}. The first set, the *complete moments*, always exists and agrees with moments defined less explicitly by Mathisson. If $\nabla_{\alpha} J^{\alpha} = 0$, as is the case for the charge-current vector, these moments are interrelated by an infinite set of corresponding restrictions. The second set, the *reduced moments*, exists if and only if $\nabla_{\alpha} J^{\alpha} = 0$. These avoid such an infinite set of interrelations, there being instead only one such restriction, the constancy of the total charge of the body. The energy-momentum tensor will be treated in a subsequent paper.

1. Introduction

This is the second of a series of papers which will develop a theory of the dynamics of extended bodies in the general theory of relativity, motions being considered under the influence of both gravitational and electromagnetic fields. In the first paper, Dixon (1969), hereafter referred to as I, definitions were proposed for the total momentum and angular momentum of such a body. These were supported by an analysis of their properties for a test body in a de Sitter universe, and were used to define the centre of mass of the body. The existence and uniqueness of the centre of mass then followed from some results of Beiglböck (1967).

The fundamental description of such a body in the general theory of relativity is by a symmetric energy-momentum tensor $T^{\alpha\beta}$ and a charge-current vector J^{α}. As a consequence of the Einstein–Maxwell field equations, these satisfy the 'generalized conservation equations'

$$\nabla_{\beta} T^{\alpha\beta} = -F^{\alpha\beta} J_{\beta} \tag{1.1}$$

and

$$\nabla_{\alpha} J^{\alpha} = 0, \tag{1.2}$$

where $F^{\alpha\beta}$ is the electromagnetic field tensor. But to be able to state equations of motion for the mass centre of the body in a convenient form, it is necessary to use instead an alternative description by a set of multipole moments of these fundamental tensors. This was first attempted by Mathisson, who in 1937 gave equations

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of motion for a body in a purely gravitational field on the assumption that only the monopole and dipole moments of the body need be retained. His method could in principle be extended to the case where any finite number of moments were retained, but in practice the algebra required would increase enormously at each step. He called his fundamental result, from which the equations of motion were derived, the *variational equation of dynamics*, and a simplified proof of it was given later by Bielecki, Mathisson & Weyssenhoff (1939). In 1940 he gave an extension of the method to allow for external forces, such as electromagnetism, but this extension was restricted to special relativity.

Tulczyjew (1959) simplified Mathisson's procedure of 1937 by introducing a singular energy-momentum tensor constructed out of the moments, whose conservation is equivalent to Mathisson's variational equation. An alternative approach leading to the same results has been given by Papapetrou (1951). This has been extended to include an electromagnetic field by Urich & Papapetrou (1955) and put into a more covariant form by Dixon (1964). A more detailed summary of these theories is given in Dixon (1967).

In all these theories the 'conservation equations' (1.1) and (1.2) imply an infinite number of relations between the moments. This suggests that a reduced set of moments may exist which determines the complete set, but which is interrelated by at most a finite number of equations. It was shown by Dixon (1967) that this is so for a body moving in an electromagnetic field in special relativity. The reduced moments obtained there included the total charge, momentum and angular momentum (spin) of the body, the only relations between them imposed by (1.1) and (1.2) being the constancy of the total charge and equations of motion for the momentum and spin. These equations of motion were stated in an exact form, without assuming analyticity of the electromagnetic field, and an approximation procedure was developed based on the smallness of the body compared with a typical length scale for the external field.

The present paper begins the task of extending these results to the general relativistic case, and at the same time improving on some unsatisfactory features. We here consider in detail the multipole moment structure of a charge-current vector J^α satisfying (1.2), leaving the structure of $T^{\alpha\beta}$ and the actual derivation of the equations of motion to the next paper. Having once developed the techniques needed to study this simpler case of J^α, their application to the more complicated case of $T^{\alpha\beta}$ will be relatively straightforward.

We shall begin by considering the moment structure of a scalar function, and gradually introduce the complications that occur for a conserved J^α. Section 2 derives the moments of a scalar function, building up gradually from Euclidean three-dimensional space, through Minkowskian spacetime to a general curved spacetime. The uniqueness problem for these moments is then discussed in §3. Section 4 extends these results to an arbitrary tensor field, thus obtaining its 'complete set of moments'. Section 5 is concerned with the extraction of the 'reduced set of moments' discussed above for a vector field satisfying (1.2). On the assumption of the existence of such moments, a proof is given there of their uniqueness, in the course of which an explicit formula is obtained for them in terms of J^α. With the use of this, the existence proof becomes relatively simple, and is given in § 6. Section 7 shows how these results may be applied to the theory of point particles, and in so doing clarifies the interpretation of some of these results. Section 8 gives a summary and discussion of the results. This summary is largely self-contained, with only a few references to definitions given in earlier sections, and may be read as a continuation of this Introduction to see in more detail what will be proved below.

We shall need to make extensive use of bitensors, and in particular of the world function of Synge (1960) and De Witt & Brehme (1960). This and other notation and conventions is summarized in the appendix, and will be used without further explicit reference. It will also be found convenient to use the terminology of the theory of fibre bundles, but this will be mainly for descriptive purposes. Our work will use very few theorems from that theory. This terminology is explained in Kobayashi & Nomizu (1963) and many other standard texts.

2. Moments of a scalar function

First consider a continuous scalar function $f(\boldsymbol{x})$ of compact support on a three-dimensional Euclidean space. Its multipole moments with respect to the origin are defined by

$$F^{a_1\ldots a_n} := \int x^{a_1}\ldots x^{a_n} f(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} \quad \text{for} \quad n \geqslant 0, \tag{2.1}$$

where Latin indices run from 1 to 3. They form a denumerable set of parameters from which the original function f may be reconstructed. To do so, we simply note that the Fourier transform $\tilde{f}(\boldsymbol{k})$ of $f(\boldsymbol{x})$, defined by

$$\tilde{f}(\boldsymbol{k}) := \int f(\boldsymbol{x}) \exp{(\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{x})}\,\mathrm{d}^3\boldsymbol{x}, \tag{2.2}$$

may be evaluated in terms of these moments by

$$\tilde{f}(\boldsymbol{k}) = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} k_{a_1}\ldots k_{a_n} F^{a_1\ldots a_n}, \tag{2.3}$$

convergent for all vectors $\boldsymbol{k}$. The original function may then be reconstructed using the Fourier inversion theorem:

$$f(\boldsymbol{x}) = (2\pi)^{-3} \int \tilde{f}(\boldsymbol{k}) \exp{(-\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{x})}\,\mathrm{d}^3\boldsymbol{k}. \tag{2·4}$$

This technique is well known in statistics, where $\tilde{f}(\boldsymbol{k})$ is called the modified moment generating function of f.

Now move to the four-dimensional spacetime $\mathscr{E}$ of special relativity, and consider a continuous scalar field $f(x)$ whose support is a world tube W which extends to past and future infinity but whose section by any spacelike hyperplane is bounded. These are the conditions satisfied by the world tube traced out by any finite body.

Let L be the timelike world line of an observer, not necessarily in uniform motion, and let $z^\alpha(s)$ be its parametric form, where s is the proper time along it. At each instant s, the instantaneous rest space of the observer is the hyperplane $\Sigma(s)$ through $z^\alpha(s)$ orthogonal to L. We wish to construct a set of multipole moments, as functions of s, which for each s depend only on the restriction of f to the corresponding $\Sigma(s)$, and from which f may be reconstructed similarly to the above case. We could simply apply the above analysis to the restriction of f to each $\Sigma(s)$. However, it will be more convenient to adopt a rather more covariant approach, by seeking a set of moments from which the Fourier transform $\tilde{f}(k)$ of $f(x)$ itself may be evaluated, instead of considering separately its value on each $\Sigma(s)$. Now our restrictions on the support of f are not sufficient to ensure the existence of $\tilde{f}$ in the usual sense, but it always exists as a generalized function. Following the terminology of Gel'fand & Shilov (1964), let K be the space of all C^∞ scalar functions on $\mathscr{E}$ of compact support, and Z be the space of all slowly increasing analytic functions on $\mathscr{E}$. Then the operation $\sim$ of Fourier transformation provides a bijection of K onto Z, with

$$\phi(x) \to \tilde{\phi}(k) := \int \phi(x) \exp(\mathrm{i}k.x)\,\mathrm{d}^4x, \tag{2.5}$$

where the dot denotes the Minkowskian scalar product. If we now consider f as a continuous linear functional on K whose value at $\phi \in K$ is $\langle f, \phi\rangle$, then $\tilde{f}$ is the generalized function on Z defined by

$$\langle \tilde{f}, \tilde{\phi}\rangle = (2\pi)^4 \langle f, \phi\rangle \tag{2.6}$$

for all $\phi \in K$. The notation $\langle\,\rangle$ is defined by equation (A 19) of the Appendix.

Let w^α be any vector field such that the infinitesimal displacement of every point by $w^\alpha\,\mathrm{d}s$ maps $\Sigma(s)$ into $\Sigma(s+\mathrm{d}s)$ for each s. Then the integration in $\langle f, \phi\rangle$ may be decomposed as

$$\langle f, \phi\rangle = \int \mathrm{d}s \int_{\Sigma(s)} f\phi w^\alpha\,\mathrm{d}\Sigma_\alpha. \tag{2.7}$$

But on using the inverse of (2.5), we see that

$$\int_{\Sigma(s)} f\phi w^\alpha\,\mathrm{d}\Sigma_\alpha = (2\pi)^{-4} \int \mathrm{d}^4k \tilde{F}(s,k)\,\tilde{\phi}(k) \exp[-\mathrm{i}k.z(s)], \tag{2.8}$$

where

$$\tilde{F}(s,k) := \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{\lambda_1} \dots k_{\lambda_n} F^{\lambda_1 \dots \lambda_n}(s), \tag{2.9}$$

$$F^{\lambda_1\dots\lambda_n}(s) := \int_{\Sigma(s)} r^{\lambda_1} \dots r^{\lambda_n} f w^\alpha\,\mathrm{d}\Sigma_\alpha \quad \text{for} \quad n \geqslant 0, \tag{2.10}$$

and $r^\lambda := x^\lambda - z^\lambda(s)$. Putting this into (2.7) and using (2.6) then gives

$$\langle \tilde{f}, \tilde{\phi}\rangle = \int \mathrm{d}s \int \mathrm{d}^4k\, \tilde{F}(s,\,k) \exp[-\mathrm{i}k.z(s)]\,\tilde{\phi}(k). \tag{2.11}$$

The $F^{\dots}$'s are the required moments, and this expresses $\tilde{f}$, as a functional on Z, in terms of them. We note that the $F^{\dots}$'s are totally symmetric, and that consequently they are completely determined if the function $\tilde{F}(s, k)$ is known. This latter function, which is analytic in k^λ, we call the *moment generating function* of f.

If we could interchange the order of integration in (2.11), this would show that

$$\tilde{f}(k) = \int \mathrm{d}s\, \tilde{F}^*(s, k) \exp[\mathrm{i}k \,.\, z(s)], \tag{2.12}$$

where * denotes complex conjugation. This would be a valid operation if f had compact support, giving the usual Fourier transform as defined by (2.5). But if the support of f extends to infinity in some direction, as we are supposing, then in general we may not so interchange the order of integration. A manifestation of this is that in such a case the integral (2.12) over s will be divergent. Equations (2.11) and (2.9) together thus form the closest analogue which we can have to (2.3) for the three-dimensional case, with f then determined as a functional by (2.6). Since f is continuous, the functional $\langle f, \phi \rangle$ does completely determine f itself, and hence although we lack an *explicit* formula for f, it is completely determined by its moments (2.10).

If we had directly applied the definition (2.1) to the restriction of f to $\Sigma(s)$, the moments would have been given by (2.10) with w^α replaced by $v^\alpha := \mathrm{d}z^\alpha/\mathrm{d}s$, from which we see that the two procedures agree if and only if L is straight, i.e. for an inertial observer. The value of f on $\Sigma(s)$ would have been expressible explicitly in terms of the moments at that value of s, but manipulation of the result, especially evaluation of derivatives of f in terms of the moments, would have been very complicated. The choice (2.10) loses us a direct expression for the value of f at a point, giving only the functional $\langle f, \phi \rangle$, but gains considerably in ease of manipulation, since almost all the usual rules for manipulating ordinary Fourier transforms hold also when the transforms are generalized functions. For example,

$$\widetilde{\partial_\alpha f}(k) = -\mathrm{i}k_\alpha \tilde{f}(k), \quad \text{showing that} \quad \langle \partial_\alpha f, \phi \rangle = (2\pi)^{-4} \langle \tilde{f}, \mathrm{i}k_\alpha \tilde{\phi} \rangle,$$

which is evaluable immediately from (2.11).

Before attempting to extend this to a curved spacetime, it is useful first to write (2.11) slightly differently. If $\tilde{\phi}(z, k)$ is the Fourier transform of $\phi(x)$ referred to z^λ as origin:

$$\left.\begin{aligned} \tilde{\phi}(z, k) &:= \int \phi(x) \exp[\mathrm{i}k \,.\, (x - z)]\, \mathrm{d}^4 x \\ &= \tilde{\phi}(k) \exp(-\mathrm{i}k \,.\, z), \end{aligned}\right\} \tag{2.13}$$

then (2.11) and (2.6) give

$$\langle \tilde{f}, \tilde{\phi} \rangle = (2\pi)^{-4} \int \mathrm{d}s \int \mathrm{d}^4 k\, \tilde{F}(s, k)\, \tilde{\phi}(z(s), k). \tag{2.14}$$

We shall now see how this result may be taken over to the curved spacetime case almost without alteration, provided that the quantities appearing in it are suitably interpreted.

First, consider the definition (2.13) of the Fourier transform. Since z is taken as the origin, the vector k^λ is most naturally considered as a tangent vector at z, i.e.

as an element of the tangent space $T_z(\mathscr{E})$ to $\mathscr{E}$ at z. Hence to make the scalar product $k.(x-z)$ meaningful, we must also treat $x^\lambda - z^\lambda$ as an element of T_z. It is that element of T_z which is mapped into x by the exponential map Exp_z of T_z to $\mathscr{E}$. The integral in (2.13) is thus to be considered as taken over the tangent space T_z, the function ϕ on $\mathscr{E}$ being identified with the function $\phi \circ \operatorname{Exp}_z$ on T_z. If we now let z vary over $\mathscr{E}$, $\tilde{\phi}(z,k)$ becomes a scalar field defined on the tangent bundle $T(\mathscr{E})$.

This shows that if we are dealing with an arbitrary pseudo-Riemannian manifold $\mathscr{M}$, the Fourier transform relates scalar functions on the tangent bundle $T(\mathscr{M})$ rather than on $\mathscr{M}$ itself. We shall continue to denote the operation by $\sim$, now defined as follows. Let Φ be a scalar function on $T(\mathscr{M})$ and let $z \in \mathscr{M}$ and $k, X \in T_z(\mathscr{M})$. Suppose that Φ has compact support on $T_z(\mathscr{M})$ for each z. Then we put

$$\tilde{\Phi}(z,k) := \int_{T_z} \Phi(z,X) \exp(\mathrm{i}k.X)\,\mathrm{D}X, \tag{2.15}$$

where $\mathrm{D}X$ is the scalar volume element on $T_z(\mathscr{M})$. Conversely

$$\Phi(z,X) = (2\pi)^{-n} \int_{T_z} \tilde{\Phi}(z,k) \exp(-\mathrm{i}k.X)\,\mathrm{D}k \tag{2.16}$$

if $n = \dim \mathscr{M}$. We shall only be concerned with the case $n = 4$.

With this notation, the function $\tilde{\phi}(z,k)$ is just the Fourier transform of the function $\Phi := \phi \circ \operatorname{Exp}$, where Exp is the exponential map of $T(\mathscr{E})$ to $\mathscr{E}$. Bearing this in mind, we now try to find an analogue of (2.14) for a curved spacetime $\mathscr{M}$. K will now be taken as the space of C^∞ scalar functions on $\mathscr{M}$ of compact support, but Z has no immediate analogue on $\mathscr{M}$. Again suppose that f is a continuous scalar function on $\mathscr{M}$ whose support W extends to past and future infinity but which is bounded in spacelike directions, and take L, parametrized as $z^\lambda(s)$ by the proper time s along it, to be a timelike world line representing an observer. We shall now take $\Sigma(s)$ to be an arbitrary spacelike hypersurface through $z(s)$ which depends continuously on s. Although we shall later choose a particular family $\Sigma(s)$ of hypersurfaces, for the present this is not necessary. Then if, as before, w^α is a vector field such that displacement of every point by $w^\alpha \mathrm{d}s$ maps $\Sigma(s)$ into $\Sigma(s+\mathrm{d}s)$ for each s, we have, as in (2.7),

$$\langle f, \phi \rangle = \int \mathrm{d}s \int_{\Sigma(s)} f\phi \sqrt{(-g)}\, w^\alpha \, \mathrm{d}\Sigma_\alpha \tag{2.17}$$

for all $\phi \in K$.

In general, the exponential map is not defined on the whole of $T(\mathscr{M})$. But consider $\mathscr{M}$ as a submanifold of $T(\mathscr{M})$ by identifying each point $z \in \mathscr{M}$ with the zero vector of $T_z(\mathscr{M})$. Then there does exist a neighbourhood N of $\mathscr{M}$ in $T(\mathscr{M})$ such that $\operatorname{Exp}: N \to \mathscr{M}$ not only exists but has class C^∞, as is shown, for example, by Kobayashi & Nomizu (1963). Let us write $N_z := N \cap T_z(\mathscr{M})$ and $U_z := \operatorname{Exp}(N_z)$. If $\mathscr{M}$ is taken as the flat spacetime $\mathscr{E}$, then N_z and U_z are the whole of $T_z(\mathscr{E})$ and $\mathscr{E}$ respectively. Now we have observed above that the function $\tilde{\phi}(z,k)$ occurring in (2.14) is the Fourier transform of the function $\Phi := \phi \circ \operatorname{Exp}$ on $T(\mathscr{E})$, which suggests that we also need to use (2.16) with $\Phi := \phi \circ \operatorname{Exp}$ in deriving the analogous result on $\mathscr{M}$. But the restricted domain in $T(\mathscr{M})$ on which Exp is defined causes a difficulty which was not present in flat spacetime. To avoid it, we further note that the integral over each $\Sigma(s)$ in (2.17) involves only the restriction of ϕ to $\Sigma(s) \cap W$. Thus provided $\Sigma(s) \cap W$ lies for each s in the corresponding open set $U_{z(s)}$, a knowledge of the function $\phi \circ \mathrm{Exp}$ on $N_{z(s)}$ is sufficient to evaluate the corresponding integral over $\Sigma(s)$.

Making this assumption, we can now proceed as follows. Put

$$\Sigma^* := \bigcup_s (\mathrm{Exp}_{z(s)})^{-1} (\Sigma(s) \cap W), \tag{2.18}$$

and let Φ be any C^∞ scalar function of compact support on $T(\mathscr{M})$ which agrees with $\phi \circ \mathrm{Exp}$ on some open set containing Σ^*. We impose this agreement in the neighbourhood of Σ^*, and not merely on Σ^* itself, so that their derivatives will also agree on Σ^*. Although this is not necessary for present purposes, it will be useful later. Introduce the world function biscalar $\sigma(z,x)$, and in accordance with the notation and conventions given in the appendix, denote its derivative at z by σ^λ. Then for all $x \in U_z$, we have

$$X^\lambda = -\sigma^\lambda(z,x) \quad \text{if} \quad x = \mathrm{Exp}_z X. \tag{2.19}$$

Thus on using (2.16), we obtain

$$\phi(x) = (2\pi)^{-4} \int \tilde{\Phi}(z(s),k) \exp(\mathrm{i}k\,.\,\sigma)\, \mathrm{D}k \tag{2.20}$$

for all x in some neighbourhood of $\Sigma(s) \cap W$, where $k\,.\,\sigma := k_\lambda \sigma^\lambda(z(s),x)$ and the integral is over $T_{z(s)}$. This gives immediately

$$\int_{\Sigma(s)} f\phi\sqrt{(-g)}\, w^\alpha \,\mathrm{d}\Sigma_\alpha = (2\pi)^{-4} \int \mathrm{D}k\, \tilde{F}(s,k)\, \tilde{\Phi}(z(s),k), \tag{2.21}$$

where the moment generating function $\tilde{F}$ is still given by (2.9), but now with

$$F^{\lambda_1 \ldots \lambda_n}(s) := (-1)^n \int_{\Sigma(s)} \sigma^{\lambda_1} \ldots \sigma^{\lambda_n} f\sqrt{(-g)}\, w^\alpha \,\mathrm{d}\Sigma_\alpha \quad \text{for} \quad n \geqslant 0. \tag{2.22}$$

By integration over s we get the functional $\langle f, \phi \rangle$ expressed in terms of the moments by

$$\langle f, \phi \rangle = (2\pi)^{-4} \int \mathrm{d}s \int \mathrm{D}k\, \tilde{F}(s,k)\, \tilde{\Phi}(z(s),k). \tag{2.23}$$

This is the required generalization of (2.14).

The indices λ on σ^λ and $F^{\lambda_1 \ldots \lambda_n}$ now signify by our conventions that they are tensor indices at z. This convention was not consistently used in flat spacetime, as it is permissible there to add vectors at different points, since global parallelism is well defined. But when using bitensors in a curved spacetime, it is essential to remember it. The factor $(-1)^n$ in the definition (2.22) is to preserve agreement with (2.10), as in flat spacetime $\sigma^\lambda = -r^\lambda$.

Two points should be noted about the validity of the above formulae. In the proof of (2.21) for a *fixed* value of s, we need only assume that $\phi \circ \mathrm{Exp}$ and Φ agree on $(\mathrm{Exp}_{z(s)})^{-1} (\Sigma(s) \cap W)$. To complete the deduction of (2.23) we need this agreement on the whole of Σ^*. Both of these are assured by the assumption made above that $\phi \circ \mathrm{Exp}$ and Φ agree in some neighbourhood of Σ^*. While the distinction between these various conditions on Φ seems very subtle at present, it will be seen to be very important when we come, in the next section, to consider uniqueness, i.e. how far the moments are themselves determined by requiring either (2.21) or (2.23) to be valid. This, in its turn, will be crucial to the entire development of the theory. So careful attention must be paid to these conditions of validity. Of the three listed above, the final one, giving the strongest restriction on Φ, is the most important for our purposes. The other two will play a helpful role in the intermediate development in § 3, but will not be used again after that.

The only geometric assumptions made in the above derivation are that the support W of f is bounded in spacelike directions, and that for each s, $\Sigma(s) \cap W \subset U_{z(s)}$. So that we are free to choose the hypersurfaces $\Sigma(s)$ in any convenient way, let us impose a restriction on W which ensures that these may be satisfied, for suitable choice of L, for every choice of $\Sigma(s)$: assume that the intersection of W with an arbitrary spacelike hypersurface lies in some convex neighbourhood U of $\mathscr{M}$ whose closure is compact. The existence of such neighbourhoods has been proved by Whitehead (1932). The world line L may then be taken as any line meeting each $\Sigma(s)$ within the corresponding open set U. This includes any reasonable choice of world line which is supposed to represent the motion of the body, and in particular the world line of the mass centre as defined in I.

Now consider what benefits can be obtained by a special choice of $\Sigma(s)$. For any choice of $\Sigma(s)$ the moments (2.22) are totally symmetric, as were the flat spacetime moments (2.10). But since in (2.10) the hyperplane $\Sigma(s)$ is orthogonal to $v^\lambda(s)$, we have $v_\lambda r^\lambda = 0$, and so additionally the flat spacetime moments are orthogonal to v^λ on each index. To reproduce this condition with the moments (2.22) requires $\Sigma(s)$ to be chosen so that

$$v_\lambda(s)\, \sigma^\lambda(z(s), x) = 0 \quad \text{if} \quad x \in \Sigma(s), \tag{2.24}$$

i.e. $\Sigma(s)$ must be taken as the hypersurface generated by all geodesics through $z(s)$ orthogonal to $v^\lambda(s)$. We shall later find it convenient to generalize (2.24) somewhat by replacing $v^\lambda(s)$ by an *arbitrary* continuous field $n^\lambda(s)$ of timelike unit vectors along L. This will enable us to take L as the world line L_0 of the mass centre and $n^\lambda(s)$ as parallel to the total momentum vector p^λ of the body, as was done in § 6 of I in defining the mass centre. It was seen there that in general p^λ is not tangential to L_0. For the present we shall leave L and $n^\lambda(s)$ as arbitrary, and take $\Sigma(s)$ as generated by all geodesics through $z(s)$ orthogonal to $n^\lambda(s)$. The moments (2.22) then satisfy the symmetry and orthogonality conditions

$$F^{\lambda_1 \ldots \lambda_n} = F^{(\lambda_1 \ldots \lambda_n)} \tag{2.25}$$

and
$$n_{\lambda_1} F^{\lambda_1 \ldots \lambda_n} = 0 \tag{2.26}$$
for $n \geqslant 1$.

Finally, note that even if we choose $n^\lambda = v^\lambda$ and consider flat spacetime, equation (2.23) is still more general than (2.14). For given $\phi \in K$, the transform $\tilde{\phi}(z, k)$ in (2.14) is completely determined by (2.13), whereas the $\tilde{\Phi}$ of (2.23) is restricted but not completely determined by the conditions placed above on Φ. Actually, those conditions do not allow $\tilde{\phi}(z, k)$ as a possible choice for $\tilde{\Phi}(z, k)$, as Φ is required to have compact support on $T(\mathscr{E})$, which $\phi \circ \mathrm{Exp}$ does not have. This is an irrelevant difference, as we could equally well have merely required Φ to have compact support on each tangent space $T_z(\mathscr{E})$. But the generalization itself is not the minor one which it at first seems to be; we shall see in the next section that it plays an important part in the problem of uniqueness of the moments.

3. The uniqueness problem

If $\tilde{F}$ is defined by (2.9) with the aid of the moments (2.22), then it satisfies (2.23) if Φ, of class C^∞ and compact support, agrees with $\phi \circ \mathrm{Exp}$ in the neighbourhood of Σ^*. Furthermore, for any choice of $\Sigma(s)$ these moments satisfy (2.25), and for a suitable choice of $\Sigma(s)$ they also satisfy (2.26). It will later be important for us to know whether these are sufficient properties of the moments to characterize them completely, and it is to this question that we now turn. Any alternative sets of moments considered must be such that the series (2.9) defining $\tilde{F}(s, k)$ converges for all k^λ, in order for equation (2.23) to be meaningful. In addition, we shall assume that the k-space integral in (2.23) is absolutely convergent.

We first note that in order even to state the question, it is necessary to be given the family $\Sigma(s)$ of hypersurfaces, as this enters into the conditions which Φ must satisfy if (2.23) is to be valid. So even though equation (2.23) superficially appears to make no mention of the $\Sigma(s)$'s, they do enter implicitly and so can be assumed known in discussing the uniqueness. Next, observe that if the cross sections $\Sigma(s) \cap W$ are not disjoint in the interior of W, then we cannot expect uniqueness. To see this, let V be an open set in W whose points lie on more than one cross-section. We shall consider only the simplest possibility, in which there exists a value s_0 of s such that all points in V lie on both a cross-section with $s < s_0$ and one with $s > s_0$, but where $\Sigma(s_0)$ itself does not intersect V. Let f^* be any continuous function whose support lies in V, and replace f by $f + f^*$ in (2.22) in defining the moments with $s < s_0$, and by $f - f^*$ in defining the moments with $s > s_0$. Then (2.21) will also hold with the corresponding substitutions, from which (2.23) will follow on integration over s, as the contributions from f^* will cancel identically. The moments so defined will still satisfy (2.25), and (2.26) if applicable, thus demonstrating the non-uniqueness claimed above.

The trouble which arises with the non-disjoint cross-sections stems from the irreversibility of the transition from (2.21) to (2.23); the latter was valid while the former was not. The next point to show is that (2.21) and (2.25), with $\tilde{F}$ defined by (2.9), together *do* imply that the $F^{\cdots}$'s are given by (2.22). This is on the assumption that (2.21) is valid, for each fixed s, if $\phi \circ \mathrm{Exp}$ and Φ agree on $(\mathrm{Exp}_{z(s)})^{-1}(\Sigma(s) \cap W)$, this being the most general condition under which it was proved above. We shall then only be left to investigate whether (2.21) follows from (2.23) for disjoint cross-sections. To prove this implication, note that for each s, (2.21) determines the value of its right-hand side for an arbitrary C^∞ function Φ of compact support on $T_{z(s)}$; we merely have to choose ϕ suitably, which can always be done as Exp is a diffeomorphism on $N_{z(s)}$. Note that for a given Φ, a different ϕ may be needed for different values of s. Since the tangent space $T_{z(s)}$ is flat, the allowed functions $\tilde{\Phi}$ are thus the elements of the function space Z on $T_{z(s)}$, so that for each s, (2.21) determines $\tilde{F}$ as a functional on Z. But $\tilde{F}$ is a continuous function of k^λ, so that this functional is a regular one. If we use the absolute convergence of the integral, a lemma of Gel'fand & Shilov (1968, p. 236), now shows that $\tilde{F}(s,k)$ is uniquely determined by this functional. Because of the imposed symmetry (2.25), the $F^{\cdots}$'s are then uniquely determined by (2.9) as the coefficients in the power series expansion of $\tilde{F}$. This proves their uniqueness; we have already shown that the moments (2.22) satisfy both (2.21) and (2.25), which completes the proof.

For the remainder of this discussion, we shall restrict ourselves to a rather more stringent condition on the cross-sections than that they be disjoint within W, as the general disjoint case does not seem to be easily treatable. We shall assume that Σ^* provides a C^∞ imbedding of W into $T(\mathscr{M})$, or equivalently that the world line L is of class C^∞, and that the scalar function $\tau(x)$ defined by

$$\tau(x) = s \quad \text{if} \quad x \in \Sigma(s) \tag{3.1}$$

is of class C^∞ in W.

Remember now that although we *wished* to prove (2.23) only for those functions Φ which agree with $\phi \circ \mathrm{Exp}$ in the neighbourhood of Σ^*, we *actually* proved it under the weaker restriction on Φ of agreement only *on* Σ^*. The stronger requirement on Φ, which gives a weaker range of validity to (2.23), creates an additional difficulty in uniqueness considerations, a difficulty which has essentially the same origin as the advantages which will be gained from it later. This will become clear in the next section, when we consider expressing the derivative of f in terms of its moments. It is, as a result, convenient first to investigate the consequences of the stronger validity condition and then to show how the case of the weaker condition may be reduced to that of the stronger one. So for the moment assume that (2.23) holds for all Φ agreeing with $\phi \circ \mathrm{Exp}$ *on* Σ^*. We show that under the conditions on L and $\Sigma(s)$ stated above, this then implies (2.21) with its most general range of validity.

In the above derivation of (2.22) from (2.21), we saw that given an arbitrary Φ, to each s there corresponds a ϕ such that $\phi \circ \mathrm{Exp}$ and Φ agree on $(\mathrm{Exp}_{z(s)})^{-1}(\Sigma(s) \cap W)$. The differentiability conditions imposed above on Σ^* now ensure that ϕ may be chosen to be independent of s, producing this agreement on the whole of Σ^*. However, in general it is still not possible to obtain this agreement in any neighbourhood of Σ^*, which fact has important consequences, as will be seen below. For agreement of $\phi \circ \mathrm{Exp}$ with Φ on Σ^* requires only that

$$\phi(x) = \Phi(z, X) \quad \text{for} \quad x \in W, \tag{3.2}$$

where $z = z(\tau(x))$ and $x = \mathrm{Exp}_z X$. A ϕ of compact support can always be chosen which satisfies this, and the postulated differentiability of the functions $z(s)$ and $\tau(x)$ ensures that this is compatible with ϕ having class C^∞. But ϕ is thus uniquely determined within W by the restriction of Φ to Σ^*, leaving no further freedom by which the agreement might be further extended.

Choose a ϕ satisfying (3.2). Let $\theta(s)$ be an arbitrary C^∞ function of s, and define

$$'\Phi(z,X) := \theta(\tau(z))\,\Phi(z,X) \quad \text{for all} \quad (z,X)\in T(\mathscr{M}). \tag{3.3}$$

This, too, has class C^∞ and compact support, and with (3.2) it implies

$$\theta(\tau(x))\,\phi(x) = {'\Phi}(z,X) \quad \text{for} \quad x\in W \quad \text{and} \quad z = z(\tau(x)). \tag{3.4}$$

So (2.23) remains valid if we replace Φ by $'\Phi$ and $\phi(x)$ by $\theta(\tau(x))\,\phi(x)$, which on using (2.17) yields

$$\int \mathrm{d}s\,\theta(s)\left\{\int_{\Sigma(s)} f\phi w^\alpha \sqrt{(-g)}\,\mathrm{d}\Sigma_\alpha - (2\pi)^{-4}\int \mathrm{D}k\,\tilde{\Phi}\,(z(s),k)\,\tilde{F}\,(s,k)\right\} = 0. \tag{3.5}$$

This can only hold for all C^∞ functions $\theta(s)$ if the quantity enclosed in curly brackets vanishes. This proves (2.21) provided ϕ and Φ are related as in (3.2). But for given s, equations (2.21) involves ϕ only through its restriction to $\Sigma(s)\cap W$. It must thus also hold if $\phi\circ\mathrm{Exp}$ and Φ agree only on $(\mathrm{Exp}_{z(s)})^{-1}(\Sigma(s)\cap W)$, which completes the required proof.

By combining the above results, we see that if agreement of Φ with $\phi\circ\mathrm{Exp}$ is required only *on* Σ^*, then (2.23) and (2.25) together are sufficient to uniquely characterize the moments. In particular, if $\Sigma(s)$ is generated by geodesics through $z(s)$ orthogonal to some timelike vector $n^\lambda(s)$, then (2.26) is a consequence of (2.23) and (2.25). But if agreement is required in the neighbourhood of Σ^*, this uniqueness no longer holds. To produce a counter-example, consider equation (2.23) in the flat spacetime $\mathscr{E}$, and as in (2.14) take $\Sigma(s)$ as the hyperplane through $z(s)$ orthogonal to L. Provided the resulting cross-sections of W are disjoint, the C^∞ nature of $\tau(x)$ in W is ensured if L has class C^∞. Now as $\sigma^\lambda(z,x) = z^\lambda - x^\lambda$ in $\mathscr{E}$, if we put $v^\lambda := \mathrm{d}z^\lambda/\mathrm{d}s$ and differentiate equation (2.20) with respect to s, we get

$$\int\left\{\frac{\partial}{\partial s}\tilde{\Phi} + \mathrm{i}k\,.\,v\tilde{\Phi}\right\}\exp(\mathrm{i}k\,.\,\sigma)\,\mathrm{D}k = 0 \quad \text{for} \quad x\in\Sigma(s)\cap W. \tag{3.6}$$

This would not hold if (2.20) had only been valid for x *on* $\Sigma(s)\cap W$. Let $h(x)$ be any C^∞ scalar function on $\mathscr{E}$ whose support lies in W. Then the product of h with the left hand side of (3.6) vanishes throughout $\Sigma(s)$. So if we multiply this product by $w^\alpha \mathrm{d}\Sigma_\alpha$ and integrate over $\Sigma(s)$, the result must vanish. But if we define moments $H^{\cdots}$ of h by (2.22) with f replaced by h, this integral can be put in the form

$$(2\pi)^4\frac{\mathrm{d}}{\mathrm{d}s}\int_{\Sigma(s)}\phi h\sqrt{(-g)}\,w^\alpha\,\mathrm{d}\Sigma_\alpha - \int \mathrm{D}k\,\tilde{\Phi}\sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}k_{\lambda_1}\dots k_{\lambda_n}{}^*H^{\lambda_1\dots\lambda_n}(s), \tag{3.7}$$

where ${}^*H := \mathrm{d}H/\mathrm{d}s$ and

$${}^*H^{\lambda_1\dots\lambda_n} := \mathrm{d}H^{\lambda_1\dots\lambda_n}/\mathrm{d}s + nv^{(\lambda_1}H^{\lambda_2\dots\lambda_n)} \quad \text{for} \quad n\geqslant 1. \tag{3.8}$$

As ϕ vanishes on $\Sigma(s)$ if $|s|$ is sufficiently large, the integral over all s of the first term of (3.7) vanishes. As the whole expression (3.7) is zero, this shows that

$$\int \mathrm{d}s \int \mathrm{D}k\, \tilde{\Phi} \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{\lambda_1} \dots k_{\lambda_n} {}^*H^{\lambda_1 \dots \lambda_n} = 0 \tag{3.9}$$

for all functions $\tilde{\Phi}$ admissible in (2.23). Hence (2.23) will remain valid if we replace $F^{\lambda_1 \dots \lambda_n}(s)$ by

$$F^{\lambda_1 \dots \lambda_n}(s) + a^* H^{\lambda_1 \dots \lambda_n}(s) \tag{3.10}$$

for an arbitrary constant a, so verifying the non-uniqueness of the moments.

It now follows that (2.26) is no longer a consequence of (2.23) and (2.25) even if the $\Sigma(s)$'s are suitably chosen. We show next that if we *impose* (2.26) as an additional condition, then not only is uniqueness of the moments recovered, but the cross-sections are also uniquely determined, i.e. (2.23) with any other choice of $\Sigma(s)$ within W is inconsistent with (2.25) and (2.26). Again, for this choice of $\Sigma(s)$ the C^∞ nature of $\tau(x)$ is ensured, if the cross sections are disjoint, by requiring L and $n^\lambda(s)$ both to have class C^∞.

It is convenient to use, in place of the coordinate basis of each $T_{z(s)}$, an orthonormal tetrad basis $\underset{\kappa}{e}{}^\lambda(s)$, $\kappa = 0, 1, 2, 3$, in which $\underset{0}{e}{}^\lambda(s) = n^\lambda(s)$. Then $\mathrm{D}k = \mathrm{d}k^0 \dots \mathrm{d}k^3$, and (2.26) with (2.9) shows that $\tilde{F}(s, k)$ is independent of k^0. It may thus be written as $\bar{F}(s, \boldsymbol{k})$, say, where $\boldsymbol{k} := (k^1, k^2, k^3)$. But (2.15) implies

$$\left.\begin{aligned} \bar{\Phi}(s, \boldsymbol{k}) &:= \int_{X^0=0} \Phi(z(s), X) \exp(-\mathrm{i}\boldsymbol{k} \,.\, \boldsymbol{X})\, \mathrm{d}^3\boldsymbol{X} \\ &= (2\pi)^{-1} \int \tilde{\Phi}(z(s), k)\, \mathrm{d}k^0, \end{aligned}\right\} \tag{3.11}$$

where $\boldsymbol{X} := (X^1, X^2, X^3)$. The use of this enables the k^0-integration in (2.23) to be performed explicitly, to give

$$\langle f, \phi \rangle = (2\pi)^{-3} \int \mathrm{d}s \int \mathrm{d}^3\boldsymbol{k}\, \bar{\Phi}(s, \boldsymbol{k})\, \bar{F}(s, \boldsymbol{k}). \tag{3.12}$$

Now equation (3.11) shows that $\bar{\Phi}(s, \boldsymbol{k})$ depends only on the restriction of $\Phi(z(s), X)$ to the hyperplane $X^0 = 0$ of $T_{z(s)}$. The image of this hyperplane under $\mathrm{Exp}_{z(s)}$ is the hypersurface in $\mathscr{M}$ generated by all geodesics through $z(s)$ orthogonal to $n^\lambda(s)$. This must thus agree with $\Sigma(s)$ within W, as otherwise $\bar{\Phi}$ would not completely determine ϕ within W. Equation (3.12) could not then be valid, as its left-hand side is not determined unless ϕ is known throughout W. This proves, as required, that the cross-sections of W to be used in (2.23) are uniquely determined by (2.26). Moreover, not only does the right-hand side of (3.12), and thus also of (2.23), only involve the restriction of Φ to the hyperplanes $X^0 = 0$; we also know by hypothesis that it depends on Φ only through its value in the neighbourhood of Σ^*. Combining these shows that it can only depend on the restriction of Φ to Σ^*. Hence (2.23) must hold also under the stronger validity criterion discussed above, for which uniqueness of the moments has already been proved.

We have thus shown that if we require only the weaker range of validity for (2.23), then the addition of both (2.25) and (2.26) is sufficient to completely characterize the moments as those given by (2.22), but that (2.25) alone is not sufficient to do so. The discussion in the paragraph immediately preceding (2.13) indicates that it is (2.23), rather than (2.22), which will be important in studying the properties of the moments, and this is indeed true. Consequently, it is convenient to consider the logical order of development as being the reverse of that which we have followed. Equation (2.23) (with the weaker range of validity) should be taken as defining what is meant by 'a set of multipole moments for f'. This should be followed by a study of the uniqueness problem, culminating in the proof that if (2.25) and (2.26) are additionally imposed, then the moments are uniquely determined and are given by (2.22). Such a development would bring out two important points. First, that there may be several different sets of multipole moments, all satisfying (2.23) but corresponding to the imposition of additional conditions different from (2.25) and (2.26). Each such set might be the best for the study of some particular problem. And secondly that the explicit expressions for the moments, such as (2.22), are of secondary importance, so that it is quite irrelevant whether these expressions are simple or complicated; the importance is in (2.23) and the additional conditions imposed. Although actually only the one set of moments seems important for a scalar function, when we consider the conserved vector field J^α of (1.2) we *shall* find two such sets. One of these is the 'complete set of moments', which exists for any vector field and is obtained in the following section. The other is the 'reduced set of moments' discussed in the introduction, which only exists when (1.2) is satisfied. While the complete set may be obtained quite simply from our results for a scalar field, it will be necessary to follow the reverse order outlined here to obtain the reduced set.

Finally, observe how the generalization discussed in the final paragraph of the preceding section plays an important role in ensuring uniqueness. If $\tilde{\phi}(z,k)$ is the only allowed transform of $\phi(x)$, then the proof of non-uniqueness given above for the case of non-disjoint cross-sections of W is valid if the hyperplanes $\Sigma(s)$ intersect *anywhere*, not necessarily within W. This always occurs unless the $\Sigma(s)$'s are all parallel. In particular, if only $\tilde{\phi}$ is allowed *and* $n^\lambda = v^\lambda$, uniqueness holds if and only if L is a straight line.

4. Extension to tensor fields

Let us now consider how the above work may be extended to a tensor field on $\mathscr{M}$. We shall first consider a contravariant vector field f^α, and then generalize from that to an arbitrary tensor field. Again assume that f^α is continuous and that its support satisfies the conditions imposed in § 2.

In the case of flat spacetime the extension is trivial, as we may simply consider each component separately as a scalar function when referred to a Minkowskian coordinate system. We used this in § 2 when we showed that $\partial_\alpha f$ could be expressed in terms of the moments of f by using

$$\langle \partial_\alpha f, \phi\rangle = (2\pi)^{-4}\langle \tilde{f}, \mathrm{i}k_\alpha \tilde{\phi}\rangle.$$

But in a general spacetime, or in a curvilinear coordinate system in flat spacetime, it is not a covariant procedure to integrate a component of a vector field. The simplest method of bypassing this difficulty is to introduce a non-singular bitensor field, $Z^{\lambda}_{.\alpha}(z, x)$ say, which may be used to transport $f^\alpha(x)$ throughout the region of integration to one fixed point z before performing the integration. For present purposes $Z^{\lambda}_{.\alpha}$ may be chosen arbitrarily. In practice the choice will be made to simplify whatever manipulation we wish to perform with the resulting moments, as we shall see later. Possible examples are the bitensor of parallel propagation $\bar{g}^{\lambda}_{.\alpha}$ and the second derivative of the world function $\sigma^{\lambda}_{.\alpha}$.

In this way we may define moments of f^α by analogy with (2.22) thus:

$$F^{\lambda_1\ldots\lambda_n\mu}(s) := (-1)^n \int_{\Sigma(s)} \sigma^{\lambda_1}\ldots\sigma^{\lambda_n} Z^{\mu}_{.\alpha} f^\alpha \surd(-g)\, w^\beta \,\mathrm{d}\Sigma_\beta \quad \text{for} \quad n \geqslant 0, \tag{4.1}$$

where the arguments of $Z^{\mu}_{.\alpha}$ are $(z(s), x)$, so that the $F^{\cdots}$'s are tensors at $z(s)$. As in (2.9) we may then form a moment generating function $\tilde{F}^\mu(s, k)$ by putting

$$\tilde{F}^\mu(s,k) := \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{\lambda_1}\ldots k_{\lambda_n} F^{\lambda_1\ldots\lambda_n\mu}(s). \tag{4.2}$$

To obtain the required analogue of (2.23), we must treat f^α as a functional on the space K_1 of C^∞ vector fields of compact support on $\mathscr{M}$. Bearing in mind (4.1), we see that $\langle f^\alpha, \phi_\alpha\rangle$ can be written as

$$\langle f^\alpha, \phi_\alpha\rangle = \int \mathrm{d}s \int_{\Sigma(s)} (Z^{\lambda}_{.\alpha} f^\alpha)\,(Z^{-1\beta}_{\ \ .\lambda}\phi_\beta)\,\surd(-g)\, w^\gamma \,\mathrm{d}\Sigma_\gamma \tag{4.3}$$

for $\phi_\alpha \in K_1$, where $Z^{-1\beta}_{\ \ .\lambda}$ is the inverse of $Z^{\lambda}_{.\alpha}$. We may now apply (2.21) separately to each term of the summation over λ. When ϕ is taken as $Z^{-1\beta}_{\ \ .\lambda}\phi_\beta$, let the corresponding function Φ on $T(\mathscr{M})$ be denoted by Φ_λ. Then we get

$$\int_{\Sigma(s)} f^\alpha \phi_\alpha \surd(-g)\, w^\beta \,\mathrm{d}\Sigma_\beta = (2\pi)^{-4}\int \mathrm{D}k\, \tilde{F}^\lambda(s,k)\, \tilde{\Phi}_\lambda(z(s), k), \tag{4.4}$$

from which by integration over s,

$$\langle f^\alpha, \phi_\alpha\rangle = (2\pi)^{-4}\int \mathrm{d}s \int \mathrm{D}k\, \tilde{F}^\lambda(s,k)\, \tilde{\Phi}_\lambda(z(s), k). \tag{4.5}$$

This is the required analogue of (2.23) but the conditions so imposed on the function Φ_λ are stated in a form which looks very dependent on the coordinate system used. To restate them in a form which looks more clearly covariant, we must look closely at the nature of this function. Its domain is the tangent bundle $T(\mathscr{M})$, but the restrictions placed on it only determine it in some neighbourhood of Σ^*. However, in this neighbourhood the four functions $\Phi_\lambda(z, X)$ will transform as the components of a covariant vector at z if the coordinate system on $\mathscr{M}$ is changed. It is thus consistent to regard Φ_λ as a function on $T(\mathscr{M})$ whose value at any point of $T_z(\mathscr{M})$ is a covariant vector at z, i.e. an element of the cotangent space $T^*_z(\mathscr{M})$. To make meaningful its differentiability, we must join together these cotangent spaces to form the cotangent bundle $T^*(\mathscr{M})$; Φ_λ is then a mapping of $T(\mathscr{M})$ to $T^*(\mathscr{M})$ whose differentiability, as a mapping between differentiable manifolds, is well defined. We shall also insist that any $p \in T(\mathscr{M})$ and its image $\Phi(p) \in T^*(\mathscr{M})$ lie in fibres above the same point of $\mathscr{M}$; this certainly holds in the neighbourhood of Σ^*, as we have already seen, and it is convenient to require it in general. We shall describe this by saying that Φ_λ is *fibre-transitive*,† as I do not know of a standard name for this property. In this language, Φ_λ is an arbitrary C^∞ fibre-transitive map of compact support of $T(\mathscr{M})$ into $T^*(\mathscr{M})$ satisfying

$$\Phi_\lambda(z, X) = Z^{-1\alpha}{}_{\lambda}\, \phi_\alpha(x), \quad \text{where} \quad x = \operatorname{Exp}_z X, \tag{4.6}$$

for all (z, X) in some neighbourhood of Σ^*. There is no difficulty in defining the Fourier transform $\tilde{\Phi}_\lambda$ by

$$\tilde{\Phi}_\lambda(z, k) := \int_{T_z} \Phi_\lambda(z, X) \exp(\mathrm{i}k.X)\, \mathrm{D}X, \tag{4.7}$$

exactly as in (2.15), as the integral is over a flat tangent space. The resulting $\tilde{\Phi}_\lambda$ is again a C^∞ fibre-transitive map of $T(\mathscr{M})$ into $T^*(\mathscr{M})$.

The conditions (2.25) and (2.26) now become

$$F^{\lambda_1 \ldots \lambda_n \mu} = F^{(\lambda_1 \ldots \lambda_n)\mu} \tag{4.8}$$

and

$$n_{\lambda_1} F^{\lambda_1 \ldots \lambda_n \mu} = 0 \tag{4.9}$$

for $n \geqslant 1$, with (4.8) always valid while (4.9) holds for the appropriate choice of $\Sigma(s)$. The uniqueness results proved in §3 carry over without alteration to the present case.

We see that, in contrast to the scalar case, the moments $F^{\lambda_1 \ldots \lambda_n \mu}$ for $n \geqslant 1$, and also the orthogonality condition (4.9) for $n \geqslant 2$, do not have an irreducible tensor symmetry. Equation (4.9) can be decomposed into the two equations

$$n_{\lambda_1} F^{\lambda_1 \ldots \lambda_{n-1}[\lambda_n \mu]} = 0 \tag{4.10}$$

and

$$n_{\lambda_1} F^{\lambda_1 (\lambda_2 \ldots \lambda_n \mu)} = 0 \tag{4.11}$$

for $n \geqslant 2$, while the moments can be separated into two components

$$F_1^{\lambda_1 \ldots \lambda_n \mu} := F^{(\lambda_1 \ldots \lambda_n \mu)} \tag{4.12}$$

and

$$F_2^{\lambda_1 \ldots \lambda_n \mu} := F^{\lambda_1 \ldots \lambda_{n-1}[\lambda_n \mu]} \tag{4.13}$$

for $n \geqslant 1$. We may express this in group-theoretic terms using the theory of the symmetric group as expressed, for example, by Weyl (1946). If $(n_1, n_2, \ldots, n_r)$ is a partition of n into r parts, with $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r$, let $[n_1, \ldots, n_r]$ denote the irreducible representation of the symmetric group on n symbols whose Young diagram represents this partition of n. Then the above decompositions correspond to the reduction

$$[n] \otimes [1] = [n, 1] \oplus [n+1]$$

† See note added in proof on page 546.

of the direct product representation. We see that (4.10) is a condition only on the $F_2^{\cdot\cdot}$'s, although (4.11) involves both the $F_1^{\cdot\cdot}$'s and the $F_2^{\cdot\cdot}$'s. The importance of this decomposition will become clear in the next section, when we come to define the reduced set of moments for a conserved vector field.

The extension to a general contravariant tensor field is now clear; we simply use one propagator $Z^{\lambda}_{.\alpha}$ as above for each tensor index. If we consider tensors with covariant indices, however, a different propagator is required, ${}^*Z_{\lambda}^{.\alpha}$ say, replacing $Z^{\lambda}_{.\alpha}$ throughout the above argument. The moments of a mixed tensor $f^{\alpha}_{.\beta}$ will then be given by

$$F^{\lambda_1\dots\lambda_n\mu}{}_{\nu}(s) := (-1)^n \int_{\Sigma(s)} \sigma^{\lambda_1}\dots\sigma^{\lambda_n} Z^{\mu}_{.\alpha}\, {}^*Z_{\nu}^{.\beta} f^{\alpha}_{.\beta} \surd(-g)\, w^{\gamma}\, \mathrm{d}\Sigma_{\gamma}. \tag{4.14}$$

There are two natural choices which may be made for ${}^*Z_{\lambda}^{.\alpha}$. If we wish the moments (4.14) of $f^{\alpha}_{.\beta}$ to be obtained from those of $f^{\alpha\beta}$ by simply lowering their final index as a tensor index at $z(s)$, we must choose

$${}^*Z_{\lambda}^{.\alpha} = g^{\alpha\beta}(x)\, g_{\lambda\mu}(z)\, Z^{\mu}_{.\beta}, \tag{4.15}$$

i.e. simply raise and lower the indices on $Z^{\mu}_{.\beta}$ with the metric at the appropriate point. But if we wish the moments of the scalar field $f^{\alpha}_{.\alpha}$, defined as in (2.22), to be given simply by contracting the last two indices μ and ν of the moments (4.14), then we must choose

$${}^*Z_{\lambda}^{.\alpha} = Z^{-1\alpha}_{\ \ .\lambda}. \tag{4.16}$$

These two desirable properties are thus in general incompatible, and hence, as for the choice of $Z^{\lambda}_{.\alpha}$ itself, we must choose whichever is most convenient for the application with which we are concerned. In a sense the second property is the more fundamental, as it does not involve the metric tensor. Whatever choice is made for $Z^{\lambda}_{.\alpha}$ and ${}^*Z_{\lambda}^{.\alpha}$, if we are considering a tensor field of type (r, s), i.e. of contravariant degree r and covariant degree s, the corresponding function $\Phi_{\lambda\dots}{}^{\mu\dots}$ will be a C^{∞} fibre-transitive map of compact support of $T(\mathscr{M})$ into the tensor bundle $T^s_r(\mathscr{M})$ related to the corresponding $\phi_{\dots}{}^{\dots}$ by

$$\Phi_{\lambda\dots}{}^{\mu\dots}(z, X) = Z^{-1\alpha}_{\ \ .\lambda}\dots {}^*Z^{-1}{}_{\beta}^{.\mu}\dots \phi_{\alpha\dots}{}^{\beta\dots}(x), \tag{4.17}$$

where $x = \mathrm{Exp}_z X$.

We next turn to this choice of $Z^{\lambda}_{.\alpha}$. In § 2 we saw that in flat spacetime the moments of f could be used to express $\langle \partial_{\alpha} f, \phi\rangle$ by simply replacing $\tilde{\phi}$ in (2.14) by $\mathrm{i}k_{\alpha}\tilde{\phi}$. Let us now try to do similarly in a curved spacetime. It is more convenient to treat the related problem of a vector f^{α} rather than a scalar f, and to evaluate its scalar divergence $\nabla_{\alpha} f^{\alpha}$, as we are then dealing throughout with a scalar test function ϕ. We start from

$$\langle \nabla_{\alpha} f^{\alpha}, \phi\rangle = -\langle f^{\alpha}, \nabla_{\alpha}\phi\rangle, \tag{4.18}$$

and so to be able to use (4.5) we seek a function $\Phi_{\lambda}(z, X)$ satisfying (4.6) with

$$\phi_{\alpha} := \nabla_{\alpha}\phi.$$

Let $\Phi(z, X)$ be any C^{∞} function of compact support on $T(\mathscr{M})$ agreeing with $\phi \circ \mathrm{Exp}$ in some neighbourhood V of Σ^*. Then with the notation of (2.19),

$$\Phi(z, X) = \phi(x) \quad \text{if} \quad (z, X) \in V, \tag{4.19}$$

and also
$$\partial X^\lambda/\partial x^\alpha = -\sigma^\lambda_{.\alpha}. \tag{4.20}$$

So if we define, as in I, the bitensor $H^\alpha_{.\lambda}$ by
$$H^\alpha_{.\lambda} = -\sigma^{-1\alpha}_{.\lambda}, \tag{4.21}$$
then
$$\partial x^\alpha/\partial X^\lambda = H^\alpha_{.\lambda}. \tag{4.22}$$

Differentiation of (4.19) thus gives
$$\partial \Phi(z, X)/\partial X^\lambda = H^\alpha_{.\lambda}\phi_\alpha(x) \quad \text{for} \quad (z, X) \in V. \tag{4.23}$$

Hence if we choose
$$Z^\lambda_{.\alpha} = -\sigma^\lambda_{.\alpha}, \tag{4.24}$$
we may take $\Phi_\lambda = \partial\Phi/\partial X^\lambda$ in (4.6), and then $\tilde{\Phi}_\lambda(z, k) = -\mathrm{i}k_\lambda \tilde{\Phi}(z, k)$. Equations (4.18) and (4.5) now give
$$\langle \nabla_\alpha f^\alpha, \phi\rangle = -(2\pi)^{-4}\int \mathrm{d}s \int \mathrm{D}k\, \tilde{\Phi}(z(s), k) \sum_{n=1}^{\infty} \frac{(-\mathrm{i})^n}{(n-1)!} k_{\lambda_1}\ldots k_{\lambda_n} F^{(\lambda_1\ldots\lambda_n)}(s). \tag{4.25}$$

This shows, with (4.12), that it is only F^λ and the $F^{\cdot\cdot}_{\mathrm{I}}$ components of the higher moments which are needed to evaluate $\nabla_\alpha f^\alpha$. Comparison with (2.23) and (2.9) shows that if we put
$$*F = 0, \quad *F^\lambda = -F^\lambda, \quad \text{and} \quad *F^{\lambda_1\ldots\lambda_n} = -nF^{\lambda_1\ldots\lambda_n}_{\mathrm{I}} \quad \text{for} \quad n \geqslant 2, \tag{4.26}$$
then the $*F^{\cdots}$'s form a set of moments for the scalar function $\nabla_\alpha f^\alpha$, in the sense of obeying (2.23), and which satisfy (2.25) but not the final condition (2.26) needed for the uniqueness theorem to hold. Note that (4.25) only holds if $\Phi = \phi \circ \mathrm{Exp}$ in the neighbourhood of Σ^*, and that its derivation depends on (4.5) but not on (4.9) or, equivalently, the specific choice (4.1) for the moments.

Because of the simplicity of (4.25), we shall find the choice (4.24) for $Z^\lambda_{.\alpha}$ a particularly useful one. If we combine it with (4.16) for $*Z^{.\alpha}_\lambda$, we can write the ungainly condition (4.17) in a neat form by borrowing some further notation from differential geometry. As before, this may be found in Kobayashi & Nomizu (1963). If $\mathscr{N}$ and $\mathscr{M}$ are two differentiable manifolds of the same dimension, any diffeomorphism $f: \mathscr{N} \to \mathscr{M}$ induces an isomorphism $f^*: T^*_x(\mathscr{M}) \to T^*_y(\mathscr{N})$ on the cotangent spaces at corresponding points, where $y \in \mathscr{N}$ and $x = f(y)$. In terms of local coordinates, the image $f^*(A) = B$, where $A \in T^*_x(\mathscr{M})$, is given by
$$B_\alpha(y) = \frac{\partial x^\beta}{\partial y^\alpha} A_\beta(x). \tag{4.27}$$

The map f^* moreover may be uniquely extended to an algebra isomorphism of the corresponding tensor algebras at $x \in \mathscr{M}$ and $y \in \mathscr{N}$. This extension may also be denoted by f^*, and then if $A \in (T^s_r)_x(\mathscr{M})$ and $B = f^*(A)$, in local coordinates
$$B_{\alpha\ldots}{}^{\gamma\ldots}(y) = \frac{\partial x^\beta}{\partial y^\alpha}\cdots\frac{\partial y^\gamma}{\partial x^\delta}\cdots A_{\beta\ldots}{}^{\delta\ldots}(x). \tag{4.28}$$

Now apply this with $\mathscr{N} = T_z(\mathscr{M})$ and $f = \mathrm{Exp}_z$. Comparison with (4.17) shows that with choice of $Z^\lambda_{.\alpha}$ and $*Z^{.\alpha}_\lambda$ given by (4.24) and (4.16), it may be written simply as
$$\Phi(z, X) = (\mathrm{Exp}_z)^*\phi(x), \tag{4.29}$$

where $x = \mathrm{Exp}_z X$ and the tensor indices have been suppressed. In this form it appears as the most natural generalization of the relation $\Phi = \phi \circ \mathrm{Exp}$ which holds in the scalar case. We shall make this choice of propagators throughout the remainder of the present paper, except in § 7, where the choice $Z^{\lambda}_{.\alpha} = \bar{g}^{\lambda}_{.\alpha}$, the parallel propagator, will be discussed briefly.

5. A conserved vector field

We shall be particularly concerned with the case of a vector field J^{α} which satisfies the conservation equation $\nabla_{\alpha} J^{\alpha} = 0$, in particular the electromagnetic charge-current vector. For such a vector field, the interrelations between the moments of any set satisfying (4.5), which are equivalent to this restriction on J^{α}, are given implicitly by the vanishing of the right hand side of (4.25) for all admissible transforms $\tilde{\Phi}(z, k)$. This is essentially an exact form of Mathisson's 'variational equation of dynamics', applied to a vector field rather than a tensor field. Unfortunately, as the $^{*}F^{\cdots}$'s do not satisfy all the conditions needed for the validity of our uniqueness theorem, we are not able to deduce from this that the $^{*}F^{\cdots}$'s themselves all vanish. It is, in fact, very difficult to extract from (4.25) *any* explicit information about these interrelations unless only a finite number of these $^{*}F^{\cdots}$'s are non-zero.

It is essentially this difficulty which caused Mathisson (1937) and the other authors discussed in the Introduction to use an approximation in which only a finite number of the moments were considered, the rest being taken as zero. The vanishing of (4.25) then becomes a manageable condition, which was solved explicitly. Mathisson's moments actually correspond to (4.1) with the choice $Z^{\lambda}_{.\alpha} = \bar{g}^{\lambda}_{.\alpha}$ for propagator and (2.24) (as written, with v^{λ}) for the $\Sigma(s)$'s, although they are not given in an explicit form in his work. This is briefly discussed further in § 7.

The discussion at the end of § 3, however, suggests an alternative possibility which would enable an exact treatment to be given. If we weaken the condition (4.9), we may introduce sufficient flexibility into the choice of moments that it is possible to impose

$$F_{1}^{\lambda_1 \cdots \lambda_n} = 0 \quad \text{for} \quad n \geqslant 2, \tag{5.1}$$

and so reduce the sum in (4.25) to a single term which is easily treated. Since (4.9) actually decomposes into the two components (4.10) and (4.11), the former only involving the $F_{2}^{\cdot\cdot}$ components of the moments, the most natural weakening of (4.9) to consider is to drop (4.11), which involves the $F_{1}^{\cdot\cdot}$'s, but to retain (4.10). In the present section we shall write J^{α} instead of f^{α} for the vector field considered, it being assumed to satisfy (1.2), and $m^{\lambda_1 \cdots \lambda_n \mu}$ instead of $F^{\lambda_1 \cdots \lambda_n \mu}$ for the hypothesized set of moments satisfying the above conditions. The corresponding moment generating function, defined as in (4.2), will be denoted by $\tilde{m}^{\mu}(s, k)$. Written out explicitly, the symmetry conditions for the $m^{\cdots}$'s are

$$m^{\lambda_1 \cdots \lambda_n \mu} = m^{(\lambda_1 \cdots \lambda_n)\mu} \quad \text{and} \quad m^{(\lambda_1 \cdots \lambda_n \mu)} = 0 \quad \text{for} \quad n \geqslant 1 \tag{5.2}$$

with the orthogonality condition

$$n_{\lambda_1} m^{\lambda_1 \cdots \lambda_{n-1} [\lambda_n \mu]} = 0 \quad \text{for} \quad n \geqslant 2. \tag{5.3}$$

The form taken by (4.5) is now

$$\langle J^{\alpha}, \phi_{\alpha}\rangle = (2\pi)^{-4}\int \mathrm{d}s \int \mathrm{D}k\, \tilde{m}^{\lambda}(s,k)\, \tilde{\Phi}_{\lambda}(z(s),k). \tag{5.4}$$

Such a set of moments was found by Dixon (1967) for flat spacetime and with $n^{\lambda} = v^{\lambda}$, suggesting that it is not unreasonable to look for such a set in this more general case. However, the method used there does not naturally adapt to the case of a general timelike n^{λ} or to the general relativistic case, neither was the uniqueness proved there. We shall tackle the problem by first investigating the properties the moments would have if they existed. This will lead to a constructional proof of their uniqueness, after which the existence proof will be relatively straightforward. These $m^{\cdots}$'s form the 'reduced set of moments' of J^{α} discussed above. As in §3, we shall assume that the k-space integral in (5.4) is absolutely convergent for each s.

We begin by putting the right-hand side of (5.4) in a form analogous to that of (3.12). As in the derivation of (3.12), use an orthonormal tetrad basis $\underset{\kappa}{e}^{\lambda}(s)$ of $T_{z(s)}$ in which $\underset{0}{e}^{\lambda}(s) = n^{\lambda}(s)$. It follows from (5.2) and (5.3) that

$$n_{\lambda_1} n_{\lambda_2} m^{\lambda_1 \ldots \lambda_n \mu} = 0 \quad \text{for} \quad n \geqslant 2. \tag{5.5}$$

The moment generating function $\tilde{m}^{\lambda}(s,k)$ can thus be written as

$$\tilde{m}^{\lambda}(s,k) = \bar{A}^{\lambda}(s,\boldsymbol{k}) + k_0 \frac{\partial}{\partial k_{\lambda}} \bar{B}(s,\boldsymbol{k}), \tag{5.6}$$

where
$$\bar{A}^{\lambda}(s,\boldsymbol{k}) := m^{\lambda} + \sum_{n=1}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{a_1} \ldots k_{a_n} m^{a_1 \ldots a_n \lambda} \tag{5.7}$$

and
$$\bar{B}(s,\boldsymbol{k}) := \sum_{n=1}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{a_1} \ldots k_{a_n} m^{0 a_1 \ldots a_n}. \tag{5.8}$$

These $\bar{A}^{\lambda}$ and $\bar{B}$ are the analogues of $\bar{F}(s,\boldsymbol{k})$ of §3. Using (5.6) to perform the k^0-integration, we now get

$$(2\pi)^{-1}\int \mathrm{D}k \tilde{m}^{\lambda}\tilde{\Phi}_{\lambda} = \int \mathrm{d}^3\boldsymbol{k} \left(\bar{\Phi}_{\lambda}\bar{A}^{\lambda} + \bar{\Psi}_a \frac{\partial}{\partial k_a}\bar{B}\right), \tag{5.9}$$

where
$$\bar{\Phi}_{\lambda}(s,\boldsymbol{k}) := \int_{X^0=0} \Phi_{\lambda}(z(s),X) \exp(-\mathrm{i}\boldsymbol{k}.\boldsymbol{X})\, \mathrm{d}^3\boldsymbol{X} \tag{5.10}$$

and
$$\bar{\Psi}_a(s,\boldsymbol{k}) := \mathrm{i}\int_{X^0=0} \frac{\partial}{\partial X^0} \Phi_a(z(s),X) \exp(-\mathrm{i}\boldsymbol{k}.\boldsymbol{X})\, \mathrm{d}^3\boldsymbol{X}. \tag{5.11}$$

An integration by parts now puts the right-hand side of (5.9) in the desired form

$$\int \mathrm{d}^3\boldsymbol{k} \left[\bar{A}^{\lambda}\bar{\Phi}_{\lambda} - \bar{B}\frac{\partial}{\partial k_a}\bar{\Psi}_a\right], \tag{5.12}$$

as the vanishing of the integrated term follows from the assumed absolute convergence of the k-space integral in (5.4). Since (5.11) implies

$$\frac{\partial}{\partial k_a}\bar{\Psi}_a = -\int_{X^0=0} X^a \frac{\partial}{\partial X^0}\Phi_a \exp(-\mathrm{i}\boldsymbol{k}.\boldsymbol{X})\, \mathrm{d}^3\boldsymbol{X}, \tag{5.13}$$

we see that the left-hand side of (5.9) depends on $\Phi_\lambda\,(z(s), X)$ only as far as the value of

$$\Phi_\lambda \quad \text{and} \quad \frac{\partial}{\partial X^0}(X^\lambda \Phi_\lambda) \quad \text{on} \quad X^0 = 0. \tag{5.14}$$

As in §3, this can now be used to prove uniqueness of the choice of hypersurfaces $\Sigma(s)$.

Our next step must be to evaluate the right-hand side of (5.4) if Φ_λ is replaced by an *arbitrary* C^∞ fibre-transitive map $E_\lambda(z, X)$ of compact support, of $T(\mathscr{M})$ to $T^*(\mathscr{M})$. The corresponding result in §3, for the right-hand side of (2.23), was trivial, as we were able to deduce that (2.23) itself was valid for arbitrary Φ of compact support on $T(\mathscr{M})$, provided that ϕ was chosen suitably. But the occurrence of an X^0-derivative in (5.14) prevents us from making the analogous deduction for (5.4).

Consider E_λ as given, and let $\Omega(z, X)$ be an arbitrary C^∞ scalar function on $T(\mathscr{M})$ of compact support. Put

$$C_\lambda(z, X) := E_\lambda(z, X) + \frac{\partial \Omega}{\partial X^\lambda}, \tag{5.15}$$

so that the corresponding Fourier transforms satisfy

$$\tilde{C}_\lambda(z, k) = \tilde{E}_\lambda - \mathrm{i}k_\lambda \tilde{\Omega}. \tag{5.16}$$

Using (5.2), we thus get

$$\int \mathrm{D}k \tilde{E}_\lambda \tilde{m}^\lambda = \int \mathrm{D}k\, \tilde{C}_\lambda \tilde{m}^\lambda - (2\pi)^4 m^\lambda(s) \left[\frac{\partial \Omega}{\partial X^\lambda}\right]_{X^\mu=0}, \tag{5.17}$$

the integrals being over $T_{z(s)}(\mathscr{M})$. This shows that if the right-hand side of (5.4) can be evaluated with C_λ replacing Φ_λ, then it can be evaluated with E_λ replacing Φ_λ. But we know that the right-hand side of (5.4) depends on the values taken by Φ_λ only to the limited extent given in (5.14). With this knowledge we try to choose the Ω of (5.15) in such a way that (5.4) itself can be used to perform this evaluation for C_λ. More explicitly, we seek an Ω for which there exists a C^∞ function $\phi_\alpha(x)$ on $\mathscr{M}$ of compact support, and a Φ_λ related to it as in (4.6), such that for each s

$$\Phi_\lambda - C_\lambda = 0 \quad \text{if} \quad n_\mu X^\mu = 0 \tag{5.18}$$

and

$$n^\mu \frac{\partial}{\partial X^\mu}[X^\lambda(\Phi_\lambda - C_\lambda)] = 0 \quad \text{if} \quad n_\mu X^\mu = 0, \tag{5.19}$$

where the arguments of the functions are $(z(s), X)$. [We are here translating (5.14) back into a general coordinate system.] If such functions can be found, then the dependence expressed by (5.14) gives

$$\int \mathrm{D}k \tilde{m}^\lambda \tilde{\Phi}_\lambda = \int \mathrm{D}k \tilde{m}^\lambda \tilde{C}_\lambda, \tag{5.20}$$

and hence by (5.4) and (5.17)

$$(2\pi)^{-4} \int \mathrm{d}s \int \mathrm{D}k \tilde{m}^\lambda \tilde{E}_\lambda = \langle J^\alpha, \phi_\alpha \rangle - \int m^\lambda \left(\frac{\partial \Omega}{\partial X^\lambda}\right)_{X^\mu=0} \mathrm{d}s. \tag{5.21}$$

As in § 3, we shall assume that Σ^* satisfies the condition expressed by (3.1). Then, as in the discussion concerning equation (3.2), it follows from (5.18) and (4.6) that the restriction of ϕ_α to W is uniquely determined by C_λ. To express this explicitly, first define a function $c_\alpha(s,x)$, which for each s is defined only for x in the neighbourhood of $\Sigma(s) \cap W$, by

$$c_\alpha(s, \mathrm{Exp}_{z(s)} X) = -\sigma_{\alpha}^{\cdot\lambda} C_\lambda(z(s), X). \tag{5.22}$$

Then
$$\phi_\alpha(x) = c_\alpha(\tau(x), x), \quad \text{for all} \quad x \in W. \tag{5.23}$$

We must now check whether or not (5.19) is satisfied. Equation (5.22) gives

$$X^\lambda C_\lambda(z(s), X) = \sigma^\alpha(z(s), x)\, c_\alpha(s, x), \tag{5.24}$$

if $x := \mathrm{Exp}_{z(s)} X$ is in the neighbourhood of $\Sigma(s) \cap W$, while (5.23) with (4.6) and (4.24) gives

$$X^\lambda \Phi_\lambda(z(s), X) = \sigma^\alpha(z(s), x)\, \phi_\alpha(x) = \sigma^\alpha(z(s), x)\, c_\alpha(\tau(x), x) \tag{5.25}$$

for such points x. Hence for $x \in \Sigma(s) \cap W$, on using (4.22) we get

$$n^\mu \frac{\partial}{\partial X^\mu}[X^\lambda(\Phi_\lambda - C_\lambda)] = \sigma^\alpha \left(\frac{\partial}{\partial s} c_\alpha\right) n^\lambda H^{\beta}_{\cdot\lambda} \partial_\beta \tau. \tag{5.26}$$

Since (4.6) places no restriction on $\Phi_\lambda(z, X)$ outside the neighbourhood of Σ^*, this shows that a ϕ_α with the desired properties exists if and only if

$$\sigma^\alpha(z(s), x) \frac{\partial}{\partial s} c_\alpha(s, x) = 0 \quad \text{for} \quad x \in \Sigma(s) \cap W. \tag{5.27}$$

We now show that, given E_λ, the scalar function Ω may always be chosen so that (5.27) is satisfied. First define, in analogy with (5.22), $e_\alpha(s,x)$ and $\omega(s,x)$ by

$$e_\alpha(s, x) = -\sigma_{\alpha}^{\cdot\lambda} E_\lambda(z(s), X) \quad \text{and} \quad \omega(s, x) = \Omega(z(s), X) \tag{5.28}$$

for $x := \mathrm{Exp}_{z(s)} X$ in the neighbourhood of $\Sigma(s) \cap W$. Then (5.15) with (4.22) and (5.22) shows that a possible $c_\alpha(s,x)$ is

$$c_\alpha := e_\alpha + \partial_\alpha \omega, \tag{5.29}$$

which puts (5.27) in the form

$$\sigma^\alpha \partial_\alpha \left(\frac{\partial \omega}{\partial s}\right) = -\sigma^\alpha \frac{\partial}{\partial s} e_\alpha \quad \text{for all} \quad x \in \Sigma(s) \cap W. \tag{5.30}$$

Now if $x(s,u)$ is the parametric form of a geodesic in $\Sigma(s)$ through $z(s)$, with u an affine parameter and $x(s,0) = z(s)$, then

$$\sigma^\alpha(z(s), x(s,u)) = u\, \partial x^\alpha / \partial u. \tag{5.31}$$

Hence (5.30) shows that along every such geodesic,

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\partial \omega}{\partial s}\right) = -u^{-1} \sigma^\alpha \frac{\partial}{\partial s} e_\alpha \tag{5.32}$$

and by integration of this along all such geodesics, a solution of (5.30) may be found for $\partial\omega/\partial s$ throughout $\Sigma(s) \cap W$ which corresponds to an arbitrary initial value $a(s)$, say, at $z(s)$. An $\omega(s, x)$ may now be chosen, defined, for each s, in the neighbourhood of $\Sigma(s) \cap W$, which not only satisfies this but which also takes arbitrarily prescribed values on $\Sigma(s) \cap W$. Thus if $\nu(x)$ is an arbitrary C^∞ function on $\mathscr{M}$, we may impose

$$\omega(\tau(x), x) = \nu(x) \quad \text{for all} \quad x \in W, \tag{5.33}$$

which implies

$$\partial_\alpha \omega = \partial_\alpha \nu - \frac{\partial\omega}{\partial s} \partial_\alpha \tau \quad \text{for} \quad x \in \Sigma(s) \cap W. \tag{5.34}$$

Since E_λ is of compact support on $T(\mathscr{M})$, we see from (5.32) and (5.33) that if $a(s)$ and $\nu(x)$ are also chosen to have compact support, in s and on $\mathscr{M}$ respectively, then $\omega(s, x)$ may be chosen to vanish identically for sufficiently large $|s|$. Hence an Ω exists, related to this ω as in (5.28), which also has compact support on $T(\mathscr{M})$, thus proving the claim made above.

Using (5.23), (5.29), (5.32) and (5.34) we now get that

$$\phi_\alpha(x) = e_\alpha(\tau(x), x) + \partial_\alpha \nu - [\psi(x) + a(\tau(x))] \partial_\alpha \tau \quad \text{for all} \quad x \in W, \tag{5.35}$$

where $\psi(x)$ is defined by

$$\psi(z(s)) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}u} \psi(x(s, u)) = -u^{-1} \sigma^\alpha \frac{\partial}{\partial s} e_\alpha, \tag{5.36}$$

the latter holding along all geodesics $x(s, u)$ in $\Sigma(s) \cap W$ through $z(s) = x(s, 0)$. We next substitute (5.35) into (5.21). In doing so, recall that by the definition of w^α, $w^\alpha \partial_\alpha \tau = 1$, and by the definition of τ, $\partial_\alpha \tau$ is everywhere normal to $\Sigma(\tau(x))$. Hence $(\partial_\alpha \tau) w^\beta \mathrm{d}\Sigma_\beta = \mathrm{d}\Sigma_\alpha$. Using this and (1.2), we get (5.21) in the form

$$\begin{aligned}(2\pi)^{-4} \int \mathrm{d}s \int \mathrm{D}k\, \tilde{m}^\lambda \tilde{E}_\lambda = \int \mathrm{d}s \int \mathfrak{J}^\alpha e_\alpha(s, x)\, w^\beta \,\mathrm{d}\Sigma_\beta - \int \mathrm{d}s \int \psi \mathfrak{J}^\alpha \,\mathrm{d}\Sigma_\alpha \\ + \int a(s) \left[(m^\lambda n_\lambda)(v^\mu n_\mu)^{-1} - \int \mathfrak{J}^\alpha \,\mathrm{d}\Sigma_\alpha\right] \mathrm{d}s \\ - \int m^\lambda [\partial_\lambda \nu]_{z(s)} \mathrm{d}s,\end{aligned} \tag{5.37}$$

where $\mathfrak{J}^\alpha := \sqrt{(-g)}\, J^\alpha$. Now $a(s)$ and $\nu(x)$ are arbitrary, and hence this can only be valid for all possible choices of these functions if the last two terms vanish identically, giving

$$(2\pi)^{-4} \int \mathrm{d}s \int \mathrm{D}k\, \tilde{m}^\lambda \tilde{E}_\lambda = \int \mathrm{d}s \int \mathfrak{J}^\alpha e_\alpha(s, x)\, w^\beta \,\mathrm{d}\Sigma_\beta - \int \mathrm{d}s \int \psi \mathfrak{J}^\alpha \,\mathrm{d}\Sigma_\alpha, \tag{5.38}$$

$$m^\lambda n_\lambda = v^\lambda n_\lambda \int \mathfrak{J}^\alpha \,\mathrm{d}\Sigma_\alpha \tag{5.39}$$

and

$$\int m^\lambda [\partial_\lambda \nu]_{z(s)} \mathrm{d}s = 0. \tag{5.40}$$

Equation (5.38) is the desired extension of (5.4). If $e_\alpha(s, x)$ is independent of s, (5.36) shows that $\psi = 0$ throughout W, and hence (5.38) simply reduces to (5.4).

Equations (5.39) and (5.40) now enable us to evaluate $m^\lambda(s)$. First consider (5.40). Since ν is arbitrary, on L the value of ν, and of its derivative in the tangent plane orthogonal to L, may both be prescribed arbitrarily. So if $\rho(s)$ and $b_\kappa(s)$ are respectively scalar and vector fields of class C^∞ on L of compact support, $\nu(x)$ may be chosen such that

$$\nu(z(s)) = \rho(s) \quad \text{and} \quad (A^\lambda_\kappa - v_\kappa v^\lambda)(\partial_\lambda \nu - b_\lambda) = 0 \quad \text{on} \quad L, \tag{5.41}$$

where A^λ_κ is the unit tensor. Putting this into (5.40) gives

$$\int \left[v_\lambda m^\lambda \frac{\mathrm{d}\rho}{\mathrm{d}s} + (A^\lambda_\kappa - v_\kappa v^\lambda)\, b_\lambda m^\kappa \right] \mathrm{d}s = 0. \tag{5.42}$$

As b_λ is arbitrary, this implies

$$m^\lambda = q v^\lambda \tag{5.43}$$

for some scalar $q(s)$. Now (5.42) reduces to

$$\int q \frac{\mathrm{d}\rho}{\mathrm{d}s}\, \mathrm{d}s = 0,$$

from which

$$\mathrm{d}q/\mathrm{d}s = 0. \tag{5.44}$$

Substituting (5.43) into (5.39) gives

$$q = \int_{\Sigma(s)} \mathfrak{J}^\alpha \, \mathrm{d}\Sigma_\alpha, \tag{5.45}$$

showing that q is simply the total charge of the body. Equation (5.44) is thus simply the law of conservation of charge, while (5.43) shows that the reduced monopole moment m^λ depends purely on the charge of the body, as v^λ is fixed by the choice of baseline L. This should be contrasted with the full monopole moment vector defined by (4.1).

We may now continue by treating (5.38) in the manner of § 3. Let $\theta(s)$ be an arbitrary C^∞ function of s, and put

$$'E_\lambda(z, X) := \theta(\tau(z))\, E_\lambda(z, X). \tag{5.46}$$

Then by (5.28) we may take the corresponding $'e_\alpha(s, x)$ as

$$'e_\alpha(s, x) := \theta(s)\, e_\alpha(s, x). \tag{5.47}$$

The equation for $'\psi(x)$ corresponding to (5.36) will then be

$$\frac{\mathrm{d}}{\mathrm{d}u}\, '\psi = -u^{-1}\theta\sigma^\alpha \frac{\partial}{\partial s} e_\alpha - u^{-1}(\sigma^\alpha e_\alpha)\frac{\mathrm{d}\theta}{\mathrm{d}s}. \tag{5.48}$$

Thus if $X^\lambda E_\lambda = 0$ when $X^\lambda n_\lambda = 0$, so that $\sigma^\alpha e_\alpha = 0$ on $\Sigma(s) \cap W$, we may choose $'\psi(x) = \theta(\tau(x))\, \psi(x)$ in W. Applying (5.38) to $'E_\lambda$ and using the arbitrariness of θ then shows that

$$(2\pi)^{-4} \int \mathrm{D}k\, \tilde{m}^\lambda \tilde{E}_\lambda = \int \mathfrak{J}^\alpha e_\alpha w^\beta \, \mathrm{d}\Sigma_\beta - \int \psi \mathfrak{J}^\alpha \, \mathrm{d}\Sigma_\alpha. \tag{5.49}$$

For this case, the value of $\psi(x)$ on a particular $\Sigma(s)$ must only depend on $e_\alpha(s, x)$ for that same value of s. To put (5.36) in a form where this is clearly exhibited, define

$$\rho(s, x) := \sigma^\alpha(z(s), x)\, e_\alpha(s, x). \tag{5.50}$$

Then the vanishing of $\sigma^\alpha e_\alpha$ on $\Sigma(s) \cap W$ implies $\rho(\tau(x), x) = 0$ for all $x \in W$, which by differentiation yields $\partial\rho/\partial s = -w^\alpha \partial_\alpha \rho$. With the use of this, (5.36) gives

$$\frac{\mathrm{d}\psi}{\mathrm{d}u} = u^{-1}[v^\lambda \sigma^{\alpha}_{;\lambda} e_\alpha + w^\alpha \partial_\alpha(\sigma^\beta e_\beta)], \tag{5.51}$$

in which the right-hand side no longer contains any derivatives with respect to s.

We now use (5.49) to evaluate $m^{\lambda_1 \ldots \lambda_n \mu}$ for $n \geqslant 1$, the value for $n = 0$ having already been found. Let $D_2(\mathscr{M})$ denote the bundle of 2-forms (skew covariant second rank tensors) on $\mathscr{M}$. Choose an arbitrary C^∞ fibre-transitive map $E_{\kappa\lambda}(z, X)$, of compact support, of $T(\mathscr{M})$ to $D_2(\mathscr{M})$. Then if we define

$$E_\lambda(z, X) := X^\mu E_{\mu\lambda}(z, X), \tag{5.52}$$

it satisfies $X^\lambda E_\lambda = 0$ throughout $T(\mathscr{M})$. Hence (5.49) is valid for this E_λ, and in addition it is possible to evaluate the corresponding ψ explicitly. To do so, first use (5.52) and (5.28) to put (5.51) in the form

$$\frac{\mathrm{d}\psi}{\mathrm{d}u} = -u^{-1} v^\kappa \sigma^{\alpha}_{;\kappa} \sigma^{;\lambda}_{\alpha} \sigma^\mu E_{\lambda\mu}. \tag{5.53}$$

Let $x(u)$ be the geodesic along which the integration is being performed, with $x(0) = z(s)$. Put also $y := x(1)$. Then if we express $E_{\lambda\mu}$ in terms of its Fourier transform $\tilde{E}_{\lambda\mu}$, (5.53) becomes

$$\frac{\mathrm{d}\psi}{\mathrm{d}u} = -(2\pi)^{-4} v^\kappa \sigma^{\alpha}_{;\kappa} \sigma^{;\lambda}_{\alpha} \sigma^\mu(z, y) \int \mathrm{D}k\, \tilde{E}_{\lambda\mu} \exp[\mathrm{i}uk \,.\, \sigma(z, y)] \tag{5.54}$$

on using $\sigma^\mu(z, x) = u\sigma^\mu(z, y)$. Now define

$$\underset{n}{\Theta}{}^{\kappa\lambda}(z, y) := (n+1) \int_0^1 \sigma^{\kappa}_{;\alpha} \sigma^{\alpha\lambda} u^n \,\mathrm{d}u \quad \text{for} \quad n \geqslant 0, \tag{5.55}$$

where the arguments of the σ's are $(z(s), x(u))$, and use it to integrate (5.54) from $u = 0$ to $u = 1$. If we replace y by x in the result, we get

$$\psi(x) = -(2\pi)^{-4} v_\kappa \int \mathrm{D}k\, \tilde{E}_{\lambda\mu} \sigma^\mu \sum_{n=0}^{\infty} \frac{(\mathrm{i}k \,.\, \sigma)^n}{(n+1)!} \underset{n}{\Theta}{}^{\kappa\lambda} \quad \text{for} \quad x \in W, \tag{5.56}$$

as required, the arguments of all bitensors being now $(z(\tau(x)), x)$.

Note that the bitensor $\underset{n}{\Theta}{}^{\kappa\lambda}(z, y)$ may be defined by (5.55) for *every* pair of points (z, y) which may be joined by a geodesic, and not only for $y \in \Sigma(s)$ and $z = z(s)$. Its significance will be discussed further in § 7. In flat spacetime it reduces to $g^{\kappa\lambda}$ for every n, which is the reason for putting the factor $(n+1)$ in its definition.

Using (5.56), we now see that

$$\int \psi \mathfrak{J}^\alpha \mathrm{d}\Sigma_\alpha = -(2\pi)^{-4} v_\kappa \int \mathrm{D}k\, \tilde{E}_{\mu\nu} \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{(n+1)!} k_{\lambda_1} \ldots k_{\lambda_n} q^{\lambda_1 \ldots \lambda_n \mu\nu\kappa}(s), \tag{5.57}$$

where $$q^{\lambda_1 \ldots \lambda_n \mu\nu}(s) := (-1)^n \int_{\Sigma(s)} \sigma^{\lambda_1} \ldots \sigma^{\lambda_n} \underset{n-1}{\Theta}{}^{\mu\nu} \mathfrak{J}^\alpha \mathrm{d}\Sigma_\alpha \quad \text{for} \quad n \geqslant 1. \tag{5.58}$$

But we also have, on using (5.52) and (5.28), that

$$\int \mathfrak{J}^{\alpha} e_{\alpha} w^{\beta} \,\mathrm{d}\Sigma_{\beta} = (2\pi)^{-4} \int \mathrm{D}k\, \tilde{E}_{\mu\nu} \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{\lambda_1} \ldots k_{\lambda_n} j^{\lambda_1 \ldots \lambda_n \mu\nu}(s), \tag{5.59}$$

where $$j^{\lambda_1 \ldots \lambda_n \mu}(s) := (-1)^{n+1} \int_{\Sigma(s)} \sigma^{\lambda_1} \ldots \sigma^{\lambda_n} \sigma^{\mu}_{.\alpha} \mathfrak{J}^{\alpha} w^{\beta} \,\mathrm{d}\Sigma_{\beta} \quad \text{for} \quad n \geqslant 0. \tag{5.60}$$

so (5.49) gives

$$\int \mathrm{D}k\, \tilde{m}^{\lambda} \tilde{E}_{\lambda} = \int \mathrm{D}k\, \tilde{E}_{\mu\nu} \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{\lambda_1} \ldots k_{\lambda_n} Q^{\lambda_1 \ldots \lambda_n \mu\nu}(s), \tag{5.61}$$

where $$Q^{\lambda_1 \ldots \lambda_n \mu\nu} := j^{\lambda_1 \ldots \lambda_n [\mu\nu]} + (n+1)^{-1} q^{\lambda_1 \ldots \lambda_n [\mu\nu]\kappa} v_{\kappa} \quad \text{for} \quad n \geqslant 0. \tag{5.62}$$

But (5.52) also implies $$\tilde{E}_{\lambda}(z,k) = \mathrm{i} \frac{\partial}{\partial k_{\mu}} \tilde{E}_{\lambda\mu}(z,k), \tag{5.63}$$

so that $$\begin{aligned} \int \mathrm{D}k\, \tilde{m}^{\lambda} \tilde{E}_{\lambda} &= -\mathrm{i} \int \mathrm{D}k\, \tilde{E}_{\lambda\mu} \frac{\partial}{\partial k_{\mu}} \tilde{m}^{\lambda} \\ &= \int \mathrm{D}k\, \tilde{E}_{\mu\nu} \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{\lambda_1} \ldots k_{\lambda_n} m^{\lambda_1 \ldots \lambda_n [\mu\nu]}. \end{aligned} \tag{5.64}$$

The right-hand sides of (5.61) and (5.64) are thus equal for arbitrary $E_{\mu\nu}$, and so the arguments used in § 3 enable us to deduce

$$m^{\lambda_1 \ldots \lambda_n [\mu\nu]} = Q^{\lambda_1 \ldots \lambda_n \mu\nu} \quad \text{for} \quad n \geqslant 0. \tag{5.65}$$

With the use of (5.2), this implies

$$m^{\lambda_1 \ldots \lambda_n \mu} = \frac{2n}{n+1} Q^{(\lambda_1 \ldots \lambda_n)\mu} \quad \text{for} \quad n \geqslant 1. \tag{5.66}$$

We have thus proved that the $m^{\cdots}$'s, if they exist, must be given by (5.66), which with (5.62), (5.60) and (5.58) provides an explicit formula for them as integrals of $\mathfrak{J}^{\alpha}$ over the cross-sections $\Sigma(s)$ of the body. In particular, this proves their uniqueness. Note that the $j^{\cdots}$'s defined by (5.60) form the 'complete set of moments' for J^{α} corresponding to the choice (4.24) for $Z^{\lambda}_{.\alpha}$, as is seen by comparison with (4.1). The $j^{\cdots}$'s alone are thus sufficient to determine J^{α}, and hence also the $m^{\cdots}$'s. But there is no simple expression for the $m^{\cdots}$'s in terms of the $j^{\cdots}$'s alone, which is the reason why the extraction of explicit information from (4.25) is so difficult for the complete set of moments. The relationship between the $j^{\cdots}$'s and $m^{\cdots}$'s is discussed in more detail in Dixon (1967). In place of the infinite set of interrelations between the complete set of moments implied by the conservation equation, the only consequent restriction on the reduced moments is that m^{λ} has the form (5.43), subject to (5.44). This will be proved in § 8.

6. The existence proof

We still have to prove that the $m^{\cdots}$'s just obtained do actually have all the properties required of them. These properties are the symmetry and orthogonality

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conditions (5.2) and (5.3), the validity of (5.4), and the absolute convergence of the k-space integral in (5.4). First, note that (5.62), (5.60) and (5.58) imply

$$\left.\begin{aligned} &Q^{\lambda_1\dots\lambda_n\mu\nu} = Q^{(\lambda_1\dots\lambda_n)[\mu\nu]} \quad \text{for} \quad n \geqslant 0 \\ \text{and} \qquad &Q^{\lambda_1\dots\lambda_{n-1}[\lambda_n\mu\nu]} = 0 \quad \text{for} \quad n \geqslant 1. \end{aligned}\right\} \tag{6.1}$$

Moreover, by construction $n_\lambda(s)\,\sigma^\lambda(z(s),x) = 0$ if $x \in \Sigma(s)$, which gives also

$$n_{\lambda_1} Q^{\lambda_1\dots\lambda_n\mu\nu} = 0 \quad \text{if} \quad n \geqslant 1. \tag{6.2}$$

Together (5.66) and (6.1) imply both (5.2), as required, and also (5.65). This latter equation may be combined with (6.2) to verify (5.3). The symmetry and orthogonality conditions are thus verified. Incidentally, equations (5.65) and (5.66) show that for $n \geqslant 1$, $m^{\lambda_1\dots\lambda_n\mu}$ and $Q^{\lambda_1\dots\lambda_{n-1}\mu\nu}$ are equivalent ways of describing the same quantity. Their symmetries, (5.2) and (6.1) respectively, both correspond to the partition $[n, 1]$ of $(n+1)$, depending on whether the symmetrization or the antisymmetrization operations of the corresponding Young diagram are performed first.

Next, note that by following the techniques used in §3 of Dixon (1967), and in particular the derivation of equations (3.34) and (3.35) of that paper, we may substitute our explicit formulae for the $m^{\dots}$'s into the definition of $\tilde{m}^\lambda$ and perform the summations. The resulting expression is rather complicated, and as it has no interest in itself, we do not give it here. From it, we first deduce that there exist continuous (non-tensorial) functions $A^\lambda(s)$ and $B^{\lambda\mu}(s)$ such that

$$|\tilde{m}^\lambda(s,k)| \leqslant A^\lambda(s) + \sum_\mu B^{\lambda\mu}(s)|k_\mu| \tag{6.3}$$

for all k^λ. Here, $|\tilde{m}^\lambda|$ and $|k_\mu|$ denote respectively the moduli of the complex number $\tilde{m}^\lambda$ and real number k_μ for fixed values of λ and μ, and not the magnitudes of the corresponding vectors. This immediately implies the required absolute convergence of the k-space integrals in (5.4). Secondly, we can deduce from this resulting expression for $\tilde{m}^\lambda$ that with the notation of equation (5.38),

$$\int \mathrm{D}k\, \tilde{m}^\lambda \tilde{E}_\lambda$$

depends on $E_\lambda(z, X)$ only through the value of $e_\alpha(s,x)$ in the neighbourhood of $\Sigma(s) \cap W$. This will be needed later.

Strictly, we must replace $\Sigma(s) \cap W$ in this last statement by that set, $W(s)$ say, obtained by filling in any holes in $\Sigma(s) \cap W$. More precisely, $W(s)$ is the union of all geodesic segments joining $z(s)$ to any point of $\Sigma(s) \cap W$. For bodies of reasonable shape, however, $W(s)$ will coincide with $\Sigma(s) \cap W$. If this is not the case, then to make the conditions of our uniqueness and existence theorems agree, we should in the statements of these theorems replace W by $\bigcup_s W(s)$. We have not done this in the above work as it would be one more complication to an already complicated theory, which would add nothing of significance other than mere complexity. With this alteration, the previous proofs go through with only trivial modifications.

It remains only to show that the moments $m^{\dots}$ defined by (5.66) do indeed satisfy (5.4). We do this by proving that the main stages of the preceding argument are each reversible. These steps are from (5.4) to (5.38), thence to (5.49) and finally to (5.66). Now the derivation of (5.66) ensures the validity of (5.49) for all E_λ of the form (5.52), whilst the truth of (5.38) for all E_λ trivially implies (5.4) by specialization. The difficult step to reverse is from (5.38) to (5.49). The latter equation trivially implies the former for all E_λ for which it itself is valid, but we have so far only ensured this for E_λ of the form (5.52). In this section we shall show that this is sufficient to imply the truth of (5.38) for all E_λ *provided* m^λ is given by (5.43) with (5.45). Note that the value of m^λ does not enter into either the transition from (5.66) to (5.49) or from (5.38) to (5.4).

The proof falls into three distinct parts. We first show that if $E_\lambda(z, X)$ has compact support and satisfies

$$X^\lambda E_\lambda = 0 \tag{6.4}$$

throughout $T(\mathscr{M})$, then it has the form (5.52) for some $E_{\lambda\mu} = E_{[\lambda\mu]}$ of compact support. Hence (5.38) is valid for all E_λ satisfying (6.4). Its validity is then extended to a general E_λ in two steps, first by using a transformation of the form (5.15) and then by the result just proved on the restricted dependence of the left-hand side of (5.38) on E_λ.

First, note that differentiation of (6.4) implies

$$E_\lambda(z, X) = -X^\mu \frac{\partial}{\partial X^\lambda} E_\mu(z, X). \tag{6.5}$$

But also

$$\frac{\mathrm{d}}{\mathrm{d}u} E_\lambda(z, uX) = X^\mu \left(\frac{\partial}{\partial X^\mu} E_\lambda\right)_{(z,uX)} \tag{6.6}$$

if u is a real scalar parameter, which together with (6.5) gives

$$\frac{\mathrm{d}}{\mathrm{d}u}(uE_\lambda(z, uX)) = 2uX^\mu \left[\frac{\partial}{\partial X^{[\mu}} E_{\lambda]}\right]_{(z,uX)}. \tag{6.7}$$

Hence if

$$'E_{\lambda\mu}(z, X) := 2\int_0^1 u \left[\frac{\partial}{\partial X^{[\lambda}} E_{\mu]}\right]_{(z,uX)} \mathrm{d}u, \tag{6.8}$$

we have $E_\lambda = X^\mu\, 'E_{\mu\lambda}$. Now in general, $'E_{\lambda\mu}$ will not have compact support, and so cannot be taken as the $E_{\lambda\mu}$ of (5.52). But let $\theta(z, X)$ be any scalar function on $T(\mathscr{M})$ of class C^∞ and compact support and which equals unity throughout the support of E_λ. Then $E_{\lambda\mu} := \theta' E_{\lambda\mu}$ will also satisfy (5.52), and in addition does have compact support. This completes the first part of the proof.

Turning now to the second step, let E_λ still satisfy (6.4), as above, and let Ω be an arbitrary C^∞ function on $\mathscr{M}$ of compact support. Define $C_\lambda(z, X)$, $c_\alpha(s, x)$, $e_\alpha(s, x)$ and $\omega(s, x)$ as in (5.15), (5.22) and (5.28). We have proved that (5.38) holds for this E_λ and e_α. We now deduce that it still holds if E_λ and e_α are replaced by C_λ and c_α.

Writing out (5.38) for this case, we have

$$(2\pi)^{-4}\int \mathrm{d}s \int \mathrm{D}k\, \tilde{E}_\lambda \tilde{m}^\lambda = \int \mathrm{d}s \int \mathfrak{T}^\alpha e_\alpha w^\beta \,\mathrm{d}\Sigma_\beta - \int \mathrm{d}s \int \psi \mathfrak{T}^\alpha \,\mathrm{d}\Sigma_\alpha, \tag{6.9}$$

where $\psi(x)$ is defined within W by

$$\psi(z(s)) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}u}\psi(x(s,u)) = u^{-1}v^{\kappa}\sigma^{\alpha}_{\kappa}e_{\alpha}, \tag{6.10}$$

the latter holding along all geodesics $x(s,u)$ in $\Sigma(s) \cap W$ with $x(s,0) = z(s)$. But (5.17) holds as before, and also by (5.43) and (5.28),

$$m^{\lambda}\left[\frac{\partial \Omega}{\partial X^{\lambda}}\right]_{X^{\mu}=0} = q\left[\frac{\mathrm{d}\eta}{\mathrm{d}s} - \frac{\partial}{\partial s}\omega(s,x)\bigg|_{x=z(s)}\right], \tag{6.11}$$

where
$$\eta(s) := \omega(s, z(s)). \tag{6.12}$$

Hence
$$(2\pi)^{-4}\int \mathrm{d}s\int \mathrm{D}k\,\tilde{C}_{\lambda}\tilde{m}^{\lambda} = (2\pi)^{-4}\int \mathrm{d}s\int \mathrm{D}k\,\tilde{E}_{\lambda}\tilde{m}^{\lambda} - \int q\left[\frac{\partial \omega}{\partial s}\right]_{(s,z(s))}\mathrm{d}s. \tag{6.13}$$

Also (5.29) still holds, and from this and (1.2) by following the method of derivation of (5.37), we may deduce that

$$\int \mathrm{d}s\int \mathfrak{J}^{\alpha}e_{\alpha}w^{\beta}\,\mathrm{d}\Sigma_{\beta} = \int \mathrm{d}s\int \mathfrak{J}^{\alpha}c_{\alpha}w^{\beta}\,\mathrm{d}\Sigma_{\beta} + \int \mathrm{d}s\int \frac{\partial\omega}{\partial s}\mathfrak{J}^{\alpha}\,\mathrm{d}\Sigma_{\alpha}. \tag{6.14}$$

Combining (6.9), (6.13) and (6.14) thus gives

$$(2\pi)^{-4}\int \mathrm{d}s\int \mathrm{D}k\,\tilde{C}_{\lambda}\tilde{m}^{\lambda} = \int \mathrm{d}s\int \mathfrak{J}^{\alpha}c_{\alpha}w^{\beta}\,\mathrm{d}\Sigma_{\beta} - \int \mathrm{d}s\int \chi\mathfrak{J}^{\alpha}\,\mathrm{d}\Sigma_{\alpha}, \tag{6.15}$$

where
$$\chi(x) = \psi(x) - \frac{\partial\omega}{\partial s}\bigg|_{(s,x)} + \frac{\partial\omega}{\partial s}\bigg|_{(s,z(s))} \quad \text{for} \quad x \in \Sigma(s) \cap W. \tag{6.16}$$

This has the same form as (5.38), and verifies that equation for C_{λ} and c_{α} provided that the corresponding value of ψ, defined by (5.36), agrees with the χ defined by (6.16). Now (6.16) and (6.10) immediately imply $\chi(z(s)) = 0$, showing that the first of conditions (5.36) is satisfied. Also, (6.4) and (5.28) together give

$$\sigma^{\alpha}(z(s),x)\,e_{\alpha}(s,x) = 0. \tag{6.17}$$

We have already seen that (5.29) is satisfied, both this and (6.17) holding in some neighbourhood of $\Sigma(s) \cap W$. Differentiate (6.17) with respect to s and use (5.29) to show that

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\partial\omega}{\partial s}\right) = u^{-1}\sigma^{\alpha}\frac{\partial}{\partial s}c_{\alpha} + u^{-1}v^{\lambda}\sigma^{\alpha}_{\ \lambda}e_{\alpha} \tag{6.18}$$

along all geodesics $x(u)$ in this neighbourhood for which $x(0) = z(s)$. If this geodesic actually lies in $\Sigma(s) \cap W$, we may use (6.18) to differentiate (6.16) along the geodesic. On noting that its last term is constant along this line, and using (6.10), we then get

$$\mathrm{d}\chi/\mathrm{d}u = -u^{-1}\sigma^{\alpha}\,\partial c_{\alpha}/\partial s. \tag{6.19}$$

This agrees with the second of conditions (5.36) on ψ, and completes the proof of (5.38) for C_{λ} and c_{α}.

Now (5.29) and (6.17) together show that

$$\mathrm{d}\omega(s, x(u))/\mathrm{d}u = u^{-1}\sigma^{\alpha}c_{\alpha} \tag{6.20}$$

with the same notation and range of validity as (6.18). Suppose now that $C_{\lambda}(z, X)$ is given as an *arbitrary* C^{∞} fibre-transitive map with compact support of $T(\mathscr{M})$ to $T^*(\mathscr{M})$, and let c_{α} be related to it as in (5.22). Let $\eta(s)$ be an arbitrary C^{∞} function on L of compact support. Then using $\eta(s)$ to give the appropriate initial value, by requiring (6.12) to be satisfied, we may integrate (6.20) along all geodesics through $z(s)$ to give ω throughout the domain of definition of c_{α}. Defining $e_{\alpha} := c_{\alpha} - \partial_{\alpha}\omega$, it will then satisfy (6.17). Moreover, as C_{λ} has compact support, then c_{α}, ω and e_{α} will all vanish identically for sufficiently large $|s|$. Hence we may choose an E_{λ} and Ω of compact support which are related to e_{λ} and ω as in (5.28) and such that E_{λ} satisfies (6.4). Define now $'C_{\lambda} := E_{\lambda} + \partial\Omega/\partial X^{\lambda}$. Then $'C_{\lambda}$ and c_{α} are also related as in (5.22), and by our above result (5.38) is valid, for the appropriate ψ, with these functions replacing E_{λ} and e_{α}. But by our earlier results on the dependence of the left hand side of (5.38) on E_{λ}, we get that

$$\int \mathrm{D}k\, \tilde{m}^{\lambda}\tilde{C}_{\lambda} = \int \mathrm{D}k\, \tilde{m}^{\lambda}\,'\tilde{C}_{\lambda}. \tag{6.21}$$

Hence (5.38) is valid with C_{λ} and c_{α} replacing E_{λ} and e_{α}. As C_{λ} was arbitrary, this completes the proof of the general validity of (5.38).

7. Interpretation

Having completed the existence and uniqueness proofs for both the complete and the reduced sets of moments, we shall look now in more detail at the basic formula (4.5). For the present, consider a general choice of $Z^{\lambda}_{.\alpha}$ in the relation (4.6) connecting ϕ_{α} with Φ_{λ}, and take $\tilde{F}^{\mu}$ as given by (4.2) but with no particular symmetry and orthogonality conditions imposed on the moments.

It was shown in Dixon (1967) that it is not permissible to perform the k-space integration in (4.5) term by term on the series (4.2) for $\tilde{F}^{\mu}$. However, we may integrate any finite number of terms of the series without affecting its validity. To do so, we first deduce from (4.6) and (4.7) that

$$Z^{-1\alpha}_{.\lambda}\phi_{\alpha}(x) = (2\pi)^{-4}\int \tilde{\Phi}_{\lambda}(z(s), k)\exp(\mathrm{i}k.\sigma)\,\mathrm{D}k \tag{7.1}$$

for all x in some neighbourhood of $\Sigma(s) \cap W$. Form the n-fold symmetrized covariant derivative of this at x and take the limit $x \to z$. Using equation (A 12) and the notation $\langle\rangle$ for coincidence limits of bitensors, both given in the appendix, we get

$$(-\mathrm{i})^{n}\int k_{\lambda_1}\ldots k_{\lambda_n}\tilde{\Phi}_{\mu}(z, k)\,\mathrm{D}k = (2\pi)^{4}\,\delta^{\alpha_1\ldots\alpha_n}_{\lambda_1\ldots\lambda_n}\langle\nabla_{(\alpha_1\ldots\alpha_n)}(Z^{-1\beta}_{.\mu}\phi_{\beta})\rangle. \tag{7.2}$$

Before turning to the choice (4.24) for $Z^{\lambda}_{.\alpha}$, which is our main interest, consider first the alternative choice

$$Z^{\lambda}_{.\alpha} = \bar{g}^{\lambda}_{.\alpha}, \tag{7.3}$$

$\bar{g}^{\lambda}_{.\alpha}$ being the parallel propagator. On using (A 13) of the appendix, we see that then the right-hand side of (7.2) reduces simply to $(2\pi)^4 \nabla_{(\lambda_1 \ldots \lambda_n)} \phi_\mu$, which enables equation (4.5) to be written as

$$\langle f^\alpha, \phi_\alpha \rangle = \int \mathrm{d}s \left\{ \sum_{n=0}^{N} \frac{1}{n!} F^{\lambda_1 \ldots \lambda_n \mu} \nabla_{(\lambda_1 \ldots \lambda_n)} \phi_\mu \right. \\ \left. + (2\pi)^{-4} \int \mathrm{D}k \, \tilde{\Phi}_\mu \sum_{n=N+1}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{\lambda_1} \ldots k_{\lambda_n} F^{\lambda_1 \ldots \lambda_n \mu} \right\} \qquad (7.4)$$

for arbitrary N.

The contribution from the first N terms alone can be written as a distribution of support L, with the use of the Dirac δ-function:

$$f^\mu = \sum_{n=0}^{N} \frac{(-1)^n}{n!} \nabla_{(\lambda_1 \ldots \lambda_n)} \int \mathrm{d}s \, F^{\lambda_1 \ldots \lambda_n \mu}(s) \, \delta(x - z(s)) \\ + \text{contributions from higher moments.} \qquad (7.5)$$

We see that the 2^n-pole moment contributes to f^μ a term involving the nth derivative of a δ-function, in accordance with the usual notions of multipole sources. If (7.5) is truncated after a finite number of terms, and combined with the symmetry and orthogonality conditions (4.8) and (4.9), it is precisely the vector equivalent of the form used by Tulczyjew (1959) for $T^{\alpha\beta}$ in his version of Mathisson's procedure. This shows that the moments used by Mathisson are given by the generalization to a second rank tensor of (4.1), with (7.3) as propagator. Because of the simple form given by (7.4) for the integrated terms, if we are not interested in the divergence of the vector field then (7.3) is probably the most convenient choice of $Z^{\lambda}_{.\alpha}$. So far we have defined the complete set of moments of a vector field as satisfying (4.8) and (4.9), and thus given by (4.1), but without any specific choice of $Z^{\lambda}_{.\alpha}$. For these reasons we now complete the definition by adding (7.3). Note that for this $Z^{\lambda}_{.\alpha}$ both expressions (4.15) and (4.16) give $^*Z^{\cdot\alpha}_{\lambda} = \bar{g}^{\cdot\alpha}_{\lambda}$, which is another advantage of this choice.

If we turn now to the case $Z^{\lambda}_{.\alpha} = -\sigma^{\lambda}_{.\alpha}$, we cannot write the general result in as simple a form as (7.4), and so we shall restrict ourselves to the case $N = 2$. We shall also revert to the notation J^α and $m^{\cdots}$ as in §§5 and 6, as for this choice of $Z^{\lambda}_{.\alpha}$ our prime interest is with the symmetry and orthogonality conditions (5.2) and (5.3). Using (7.2) together with (A 17) and (A 18), we then get for this case that

$$\langle J^\alpha, \phi_\alpha \rangle = \int \mathrm{d}s \, \{ m^\mu \phi_\mu + m^{\lambda\mu} \nabla_\lambda \phi_\mu + \tfrac{1}{2} m^{\kappa\lambda\mu} [\nabla_{(\kappa\lambda)} \phi_\mu - \tfrac{1}{3} R_{\mu(\kappa\lambda)}{}^{\nu} \phi_\nu] \} \\ + \text{contributions from higher moments.} \qquad (7.6)$$

Further explicit terms may easily be obtained, the general term having the same form as in (7.4) but with the addition to $\nabla_{(\lambda_1 \ldots \lambda_n)} \phi_\mu$ of terms involving the curvature tensor. Again, we may write the integrated terms as a distribution as in (7.5), obtaining

$$J^\mu = \int \mathrm{d}s \, [m^\mu - \tfrac{1}{6} m^{\kappa\lambda\nu} R_{\nu(\kappa\lambda)}{}^{\mu}] \, \delta(x - z) - \nabla_\lambda \int \mathrm{d}s \, m^{\lambda\mu} \delta(x - z) \\ + \tfrac{1}{2} \nabla_{(\kappa\lambda)} \int \mathrm{d}s \, m^{\kappa\lambda\mu} \delta(x - z) - \ldots. \qquad (7.7)$$

The first term of this series, which would normally be called the monopole term, is thus seen to involve not only the monopole moment vector m^μ but also the quadrupole $m^{\kappa\lambda\mu}$. In general, the 2^n-pole moment tensor will be found to contribute not only to the term involving the nth derivative of $\delta(x-z)$, but also to all lower terms in a form which vanishes when $R_{\kappa\lambda\mu\nu} = 0$. The significance of these extra terms becomes clear when we evaluate $\nabla_\mu J^\mu$. This can be done directly from (7.7), but it is easier to put $\phi_\alpha = \partial_\alpha \phi$ in (7.6) and translate the result back into the notation of δ-functions. On noting that then

$$\nabla_{(\kappa\lambda)}\phi_\mu = \nabla_{(\kappa\lambda\mu)}\phi + \tfrac{1}{3}R_{\mu(\kappa\lambda)\nu}\phi^\nu, \tag{7.8}$$

we get
$$\nabla_\mu J^\mu = \nabla_\mu \int \mathrm{d}s\, m^\mu \delta(x-z) - \nabla_{(\lambda\mu)} \int \mathrm{d}s\, m^{\lambda\mu}\delta(x-z)$$
$$+ \tfrac{1}{2}\nabla_{(\kappa\lambda\mu)} \int \mathrm{d}s\, m^{\kappa\lambda\mu}\delta(x-z) - \ldots, \tag{7.9}$$

the terms involving the curvature tensor cancelling identically. This happens to all multipole orders, and is nothing more than a restatement of (4.25), but it provides an alternative way of looking at (4.25) which may help to clarify its significance. The symmetry properties (5.2) show that all terms in (7.9) except the first vanish identically, while on using $m^\mu = qv^\mu$, with q constant, we get the vanishing of the first term. Hence (7.7), suitably extended, gives us a canonical form for a conserved singular J^μ of any multipole order in which $\nabla_\mu J^\mu = 0$ is identically satisfied. Although our existence and uniqueness proofs are not strictly valid for a singular J^α, it can be shown by other means that if a singular J^α is given satisfying $\nabla_\alpha J^\alpha = 0$, then there does exist a unique set of $m^{\cdots}$'s satisfying both (7.7), extended to the appropriate multipole order, and also (5.2) and (5.3), and that then $m^\lambda = qv^\lambda$ for some constant q. This shows the relation between our formalism for extended bodies and the theory of point particles described by a singular J^α and $T^{\alpha\beta}$.

The choice of $-\sigma^\lambda_{.\alpha}$ rather than $\bar{g}^\lambda_{.\alpha}$ for $Z^\lambda_{.\alpha}$ is thus seen to insert into the integrated form (7.7) precisely those terms involving the curvature tensor which are needed to cancel the similar terms arising from the commutation of covariant derivatives when one evaluates the divergence $\nabla_\mu J^\mu$. The bitensor $\sigma^\lambda_{.\alpha}$ occurs in two different ways in the corresponding explicit expressions for the $m^{\cdots}$'s given by (5.66). It occurs through (5.60) as a propagator in the same manner in which $Z^\lambda_{.\alpha}$ occurs in the complete moments (4.1), but it also occurs through (5.55), the definition of $\underset{n}{\Theta}{}^{\kappa\lambda}(z,x)$, in the combination

$$G^{\kappa\lambda} := \sigma^\kappa_{.\alpha}\sigma^{\alpha\lambda}. \tag{7.10}$$

In this form $G^{\kappa\lambda}$ appears as a bitensor with arguments (z,x), having scalar character at x and being a symmetric tensor at z. In the derivation of the equations of motion in the next paper it will play a more important part than it does in the above work, but there it will be convenient to regard it as defined on $T(\mathscr{M})$, its arguments being (z,X), where $x = \mathrm{Exp}_z X$. It is then a fibre-transitive map from $T(\mathscr{M})$ into the bundle of symmetric second rank contravariant tensors on $\mathscr{M}$. Using this description and the notation of (4.29), the definition (7.10) takes on the more elegant form

$$G(z, X) := (\mathrm{Exp}_z)^* g(x), \tag{7.11}$$

where indices have been suppressed and $g(x)$ denotes the contravariant form of the metric tensor at x. It is thus a lifting of the metric tensor into the tangent bundle. The difference $G^{\kappa\lambda}(z, X) - g^{\kappa\lambda}(z)$ can be regarded as a gravitational potential tensor of x relative to z, and it is in this role that it will appear in the equations of motion. The bitensors $\underset{n}{\Theta}{}^{\kappa\lambda}(z, x) - g^{\kappa\lambda}(z)$ may thus be considered as moments of this gravitational potential.

8. Summary and discussion

We have now completed the proofs of the main results of this paper. As the account given above has been a gradual build-up of the theory from simpler examples, it is easy to lose track of exactly what assumptions have been made and what has been proved. So we give here a fairly self-contained statement of the principal results. The object studied is a continuous contravariant vector field on a spacetime $\mathscr{M}$, the intersection of whose support W with an arbitrary spacelike hypersurface lies in some convex neighbourhood in $\mathscr{M}$, whose closure is compact.

To state the results, we must first set up the following geometrical construction. Take a family $\Sigma(s)$ of spacelike hypersurfaces such that the cross sections $\Sigma(s) \cap W$ of W are disjoint and together fill W. Let L be a C^∞ timelike world line whose intersection $z(s)$ with each $\Sigma(s)$ lies within the corresponding convex neighbourhood containing $\Sigma(s) \cap W$. Without loss of generality the parameter s will be assumed to be the proper time along L. Define the subset Σ^* of the tangent bundle $T(\mathscr{M})$ of $\mathscr{M}$ by

$$\Sigma^* := \bigcup_s (\mathrm{Exp}_{z(s)})^{-1} (\Sigma(s) \cap W), \tag{8.1}$$

where $\mathrm{Exp}_{z(s)}$ is the exponential map of $T_{z(s)}(\mathscr{M})$ to $\mathscr{M}$, and assume it to be a C^∞ submanifold of $T(\mathscr{M})$. Let $n^\lambda(s)$ be a C^∞ field of timelike unit vectors along L, and let $Z^\lambda_{\cdot\alpha}$ be any non-singular bitensor field defined for all point pairs $(z, \mathrm{Exp}_z X)$, where (z, X) runs through some neighbourhood of Σ^*.

In the statements of the results given below, two sets of notation will be used. When the vector field is denoted by f^α (resp. J^α), the corresponding moments will be denoted by $F^{\lambda_1 \dots \lambda_n \mu}$ (resp. $m^{\lambda_1 \dots \lambda_n \mu}$). In the following paragraph we shall use the former set of notation, but it is intended to apply equally to both cases.

The vector field f^α is next represented as a functional on the set K_1 of all C^∞ covariant vector fields on $\mathscr{M}$ of compact support, its value at $\phi_\alpha \in K_1$ being

$$\langle f^\alpha, \phi_\alpha \rangle := \int f^\alpha \phi_\alpha \sqrt{(-g)}\, \mathrm{d}^4 x. \tag{8.2}$$

Using terminology of the theory of fibre bundles as given in §4, let $\Phi_\lambda(z, X)$ be any C^∞ fibre-transitive map with compact support, of $T(\mathscr{M})$ into the cotangent bundle $T^*(\mathscr{M})$, such that

$$\Phi_\lambda(z, X) = Z^{-1\alpha}_{\cdot\lambda} \phi_\alpha(x), \tag{8.3}$$

where $x = \mathrm{Exp}_z X$, for all (z, X) in some neighbourhood of Σ^*. The Fourier transform of Φ_λ is denoted by $\tilde{\Phi}_\lambda$, and defined by (4.7). Relative to the propagator $Z^\lambda_{.\alpha}$, the set of tensor fields $F^{\lambda_1\ldots\lambda_n\mu}(s)$ on L is said to form a *set of multipole moments* for f^α if

$$\langle f^\alpha, \phi_\alpha \rangle = (2\pi)^{-4} \int \mathrm{d}s \int \mathrm{D}k\, \tilde{\Phi}_\mu(z(s), k) \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} k_{\lambda_1}\ldots k_{\lambda_n} F^{\lambda_1\ldots\lambda_n\mu}(s) \tag{8.4}$$

for all ϕ_α and Φ_λ satisfying the above restrictions, and if the k-space integral in this formula is absolutely convergent for each s. This k-space integral is over the tangent space $T_{z(s)}(\mathscr{M})$, $\mathrm{D}k$ being the scalar volume element on that space. Then we have proved the following two results.

(I) *The complete set of moments for a vector field f^α*

Suppose we seek a set of multipole moments $F^{\lambda_1\ldots\lambda_n\mu}(s)$ for f^α which satisfy

$$F^{\lambda_1\ldots\lambda_n\mu} = F^{(\lambda_1\ldots\lambda_n)\mu} \quad \text{and} \quad n_{\lambda_1} F^{\lambda_1\ldots\lambda_n\mu} = 0 \quad \text{for} \quad n \geqslant 1 \tag{8.5}$$

in addition to (8.4). Then such moments exist if and only if $\Sigma(s)$ is, within W, the hypersurface generated by all geodesics through $z(s)$ orthogonal to $n^\lambda(s)$. If this is so, then they are unique and are given by

$$F^{\lambda_1\ldots\lambda_n\mu}(s) = (-1)^n \int_{\Sigma(s)} \sigma^{\lambda_1}\ldots\sigma^{\lambda_n} Z^\mu_{.\alpha} f^\alpha \sqrt{(-g)}\, w^\beta \mathrm{d}\Sigma_\beta \quad \text{for} \quad n \geqslant 0. \tag{8.6}$$

Here, w^α is any vector field such that displacement of every point by $w^\alpha \mathrm{d}s$ drags $\Sigma(s)$ into $\Sigma(s+\mathrm{d}s)$ for each s. Equation (8.4) is then true for a larger class of Φ_λ's than actually imposed above, namely for those satisfying (8.3) for all (z, X) *on* Σ^*, rather than in some neighbourhoood of Σ^*. We may perform the k-space integration term by term in (8.4) on any *finite* number of terms in the series, and may express the contribution to f^α from the integrated terms as a distribution with support L. For the particular choice $Z^\lambda_{.\alpha} = \bar{g}^\lambda_{.\alpha}$, the parallel propagator, this distribution has the particularly simple form

$$f^\mu = \sum_{n=0}^{N} \frac{(-1)^n}{n!} \nabla_{(\lambda_1\ldots\lambda_n)} \int \mathrm{d}s\, F^{\lambda_1\ldots\lambda_n\mu}(s)\, \delta(x - z(s))$$
$$+ \text{contributions from higher moments.} \tag{8.7}$$

With this choice of $Z^\lambda_{.\alpha}$, we call the $F^{\cdots}$'s the *complete set of moments* for f^α. These results may easily be extended to a tensor field of arbitrary rank. Applied to the energy-momentum tensor $T^{\alpha\beta}$ with n^λ taken as tangent to L, we get the moments used by Mathisson (1937) and Tulczyjew (1959).

(II) *The reduced set of moments for a conserved vector field J^α*

If J^α has class C^1 and satisfies $\qquad \nabla_\alpha J^\alpha = 0, \qquad (8.8)$

the moments of the complete set will have corresponding interrelations. These form an infinite set, none of which can be obtained exactly in an explicit form. But for such a J^α we may consider a *reduced set of moments*, satisfying

$$\left.\begin{aligned} m^{\lambda_1\ldots\lambda_n\mu} = m^{(\lambda_1\ldots\lambda_n)\mu} \quad \text{and} \quad m^{(\lambda_1\ldots\lambda_n\mu)} = 0 \quad \text{for} \quad n \geqslant 1,\\ \text{and} \qquad n_{\lambda_1} m^{\lambda_1\ldots\lambda_{n-1}[\lambda_n\mu]} = 0 \quad \text{for} \quad n \geqslant 2 \end{aligned}\right\} \tag{8.9}$$

in place of (8.5). These are defined only for $Z^\lambda_{.\alpha} = -\sigma^\lambda_{.\alpha}$, and again, they exist if and only if $\Sigma(s)$ is as given in case I. They are then unique and are given by

$$\left.\begin{aligned} m^\lambda(s) = qv^\lambda(s),\\ \text{and} \qquad m^{\lambda_1\ldots\lambda_n\mu} = (2n/(n+1))\, Q^{(\lambda_1\ldots\lambda_n)\mu} \quad \text{for} \quad n \geqslant 1, \end{aligned}\right\} \tag{8.10}$$

where
$$Q^{\lambda_1\ldots\lambda_n\mu\nu} = j^{\lambda_1\ldots\lambda_n[\mu\nu]} + (n+1)^{-1} q^{\lambda_1\ldots\lambda_n[\mu\nu]\kappa} v_\kappa \quad \text{for} \quad n \geqslant 2, \tag{8.11}$$

and $v^\lambda := \mathrm{d}z^\lambda/\mathrm{d}s$. The $j^{\cdots}$'s and $q^{\cdots}$'s are given as explicit integrals by

$$j^{\lambda_1\ldots\lambda_n\mu}(s) = (-1)^{n+1} \int_{\Sigma(s)} \sigma^{\lambda_1}\ldots\sigma^{\lambda_n} \sigma^\mu_{.\alpha} \mathfrak{J}^\alpha w^\beta \,\mathrm{d}\Sigma_\beta \quad \text{for} \quad n \geqslant 0, \tag{8.12}$$

$$q^{\lambda_1\ldots\lambda_n\mu\nu}(s) = (-1)^n \int_{\Sigma(s)} \sigma^{\lambda_1}\ldots\sigma^{\lambda_n} \underset{n-1}{\Theta^{\mu\nu}} \mathfrak{J}^\alpha \,\mathrm{d}\Sigma_\alpha \quad \text{for} \quad n \geqslant 1, \tag{8.13}$$

and
$$q = \int_{\Sigma(s)} \mathfrak{J}^\alpha \,\mathrm{d}\Sigma_\alpha. \tag{8.14}$$

Here, w^α is as in case I, and the $\underset{n}{\Theta^{\mu\nu}}(z,x)$, $n \geqslant 0$, are bitensor fields on $\mathscr{M}$ defined by (5.55) for every pair of points which may be joined by a geodesic. In a flat spacetime, $\underset{n}{\Theta^{\mu\nu}} = g^{\mu\nu}$ for all n. We see that q is just the total charge of the body. One should note that in case II equation (8.4) does *not* hold for the more general class of Φ_λ for which it holds in case I, and that for case II the choice of the propagator $Z^\lambda_{.\alpha}$ is virtually forced on us through the arguments leading up to equation (4.25).

The importance of the reduced set of moments is that the only restriction on the $m^{\cdots}$'s implied by (8.8) is $\mathrm{d}q/\mathrm{d}s = 0$, the constancy of the total charge. So these reduced moments eliminate the redundancy in the complete set caused by the interrelations induced by (8.8). To prove this, we see from (4.25) that if $m^{\lambda_1\ldots\lambda_n\mu}$ is a set of multipole moments for J^α relative to the propagator $Z^\lambda_{.\alpha} = -\sigma^\lambda_{.\alpha}$, and if they satisfy (8.9), then

$$\langle \nabla_\alpha J^\alpha, \phi \rangle = (2\pi)^{-4}\,\mathrm{i} \int \mathrm{d}s \int \mathrm{D}k\, \tilde{\Phi}(z(s),k)\, k_\lambda m^\lambda \tag{8.15}$$

for all $\phi \in K$. Using the analogue of (7.2) for the scalar case, this reduces to

$$\langle \nabla_\alpha J^\alpha, \phi \rangle = \int \mathrm{d}s\, m^\lambda \partial_\lambda \phi. \tag{8.16}$$

Now the left-hand side of this vanishes for all $\phi \in K$ if and only if $\nabla_\alpha J^\alpha = 0$, while the vanishing of the right-hand side gives (5.40), from which we deduced (5.43) and (5.44). Hence $\nabla_\alpha J^\alpha = 0$ if and only if $m^\lambda = qv^\lambda$, with $\mathrm{d}q/\mathrm{d}s = 0$, as required, and in § 5 we proved that q is then the total charge of the body.

The present paper has given a detailed analysis of the moment structure of a vector field, proving both the existence and uniqueness of the reduced set of moments for the charge-current vector. In the next paper we shall treat the energy-momentum tensor $T^{\alpha\beta}$, but because of the increased complexity of the problem we shall not give as complete a theory of its moments. Much of the complication of the present paper arises from the need to state formulae such as (8.4) in a form from which both existence and uniqueness follows. If one merely wished to show that the moments defined by (8.10) through (8.14) had many useful properties such as satisfying (4.25), a much simpler derivation could be given. The penalty paid for this simplification is a lack of knowledge as to how far the properties derived actually serve to characterize the moments as unique. For the case of $T^{\alpha\beta}$, the calculation is sufficiently intricate that we shall only give it in such a simplified form. To provide a guide to the procedure, a brief outline will also be given of the corresponding treatment of J^{α}. The work of the present paper convinces the author that the results for $T^{\alpha\beta}$ can be put in a form from which uniqueness follows, but it is felt that little would be gained by so doing. The detailed study of J^{α} probably illustrates in a simpler way all the aspects of the problem that would arise if $T^{\alpha\beta}$ were similarly treated. We shall find that just as $\mathrm{d}q/\mathrm{d}s = 0$ is the only consequence of (1.2) for the reduced moments of J^{α}, so the equations of motion for the momentum and angular momentum of the body appear as the only restrictions on the reduced moments of $T^{\alpha\beta}$ implied by (1.1).

Finally, for comparison with the equations of motion stated in § 7 of I, it should be remarked that the $Q^{\cdots}$'s used there differ by a factor from the $Q^{\cdots}$'s defined by (8.11). This minor change from our present notation is for convenience in later developments. The relation is that $\chi Q^{\lambda_1 \cdots \lambda_n \mu\nu}$ of I must be identified with $Q^{\lambda_1 \cdots \lambda_n \mu\nu}$ of the present paper.

Appendix. On notation and conventions, and the theory of bitensors

Spacetime is considered as a four-dimensional pseudo-Riemannian manifold of class C^{∞}, with metric tensor $g_{\alpha\beta}$ of signature -2, with respect to which covariant differentiation is denoted by ∇_{α}, partial differentiation being denoted by ∂_{α}. In repeated differentiation, the kernel ∇ or ∂ is only written once, e.g. $\nabla_{\alpha\beta} A_{\gamma} = \nabla_{\alpha} \nabla_{\beta} A_{\gamma}$. The scalar product $g_{\alpha\beta} a^{\alpha} b^{\beta}$ of two vectors is frequently denoted by $a \, . \, b$. Symmetrization and antisymmetrization of indices is denoted by ( ) and [ ] respectively, indices to be omitted from these operations being enclosed between vertical lines, e.g.

$$A_{[\alpha|\beta|\gamma]} = \tfrac{1}{2}(A_{\alpha\beta\gamma} - A_{\gamma\beta\alpha}). \tag{A 1}$$

The sign of the curvature tensor is such that the Ricci identity for a covariant vector A_{α} is

$$\nabla_{[\alpha\beta]} A_{\gamma} = -\tfrac{1}{2} R_{\alpha\beta\gamma}{}^{\delta} A_{\delta}. \tag{A 2}$$

The electromagnetic field tensor $F_{\alpha\beta}$ is taken such that in flat spacetime, with Minkowskian coordinates with metric tensor $g_{\alpha\beta} = \operatorname{diag}(1, -1, -1, -1)$, the electric and magnetic field vectors in the 3-spaces $x^0 = \text{const.}$ are given by

$$\boldsymbol{E} = (F^{01}, F^{02}, F^{03}) \quad \text{and} \quad \boldsymbol{H} = (F^{23}, F^{31}, F^{12}). \tag{A 3}$$

Here, as elsewhere, 3-vectors are denoted by bold face type. Whenever a flat spacetime is considered, it will be assumed that such Minkowskian coordinates are being used unless the contrary is explicitly stated. The charge-current vector J^α is such that $J^\alpha = \rho v^\alpha$ for a charge distribution with charge density ρ and velocity v^α. These conventions largely follow those of Schouten (1954).

For the theory of bitensors, we follow closely the notation of DeWitt & Brehme (1960). As these quantities have tensor character at more than one point, it is necessary to indicate the point associated with each index. Unless otherwise stated, we shall always label the points involved as x and z, and use $\alpha, \beta, \ldots$ as indices at x and κ, $\lambda, \ldots$ at z. With this convention we may unambiguously write A^α and A^λ to denote the value of a vector field A^α at x and z respectively, and may similarly omit the arguments in any bitensor unless it has scalar character at one of its points.

Two particularly important bitensors are the world function biscalar $\sigma(x,z)$ and the parallel propagator $\bar{g}^{\lambda}_{\cdot\alpha}(z,x)$, defined as follows. Let $x(u)$ be the parametric form of a geodesic joining $z = x(u_1)$ and $y = x(u_2)$, with u an affine parameter along it. Then

$$\sigma(z,y) := \tfrac{1}{2}(u_2 - u_1)\int_{u_1}^{u_2} g_{\alpha\beta}(x(u))\frac{\mathrm{d}x^\alpha}{\mathrm{d}u}\frac{\mathrm{d}x^\beta}{\mathrm{d}u}\,\mathrm{d}u, \tag{A 4}$$

where, as elsewhere, a colon placed before an equals sign indicates that the equation is to be considered as defining the quantity to the left of the colon. If θ is an index at y, $\bar{g}^{\theta}_{\cdot\lambda}$ is defined by requiring that if A^λ be any vector at z, then $A^\theta := \bar{g}^{\theta}_{\cdot\lambda}A^\lambda$ is the vector at y obtained by propagating A^λ to y parallelly along the geodesic $x(u)$. We shall use σ and $\bar{g}^{\lambda}_{\cdot\alpha}$ only in a region of spacetime where there exists a unique geodesic joining every pair of points. This ensures that they are both well defined and single valued. Covariant derivatives of $\sigma(x,z)$ will be denoted simply by appropriate suffixes, thus

$$\sigma_{\alpha\beta\kappa}(x,z) := \nabla_\kappa \nabla_{\beta\alpha}\sigma(x,z), \tag{A 5}$$

where, in accordance with our convention on indices, $\nabla_{\beta\alpha}$ acts at x and ∇_κ at z. Note that for any bitensor, covariant derivatives at x commute with those at z. It is shown by Synge (1960) (see also I) that with the notation of (A 4),

$$\left.\begin{aligned} \sigma^\lambda(z,x(u)) &= -(u-u_1)\,(\mathrm{d}x^\lambda/\mathrm{d}u)_{u=u_1} \\ \text{and}\qquad \sigma^\alpha(z,x(u)) &= (u-u_1)\,(\mathrm{d}x^\alpha/\mathrm{d}u). \end{aligned}\right\} \tag{A 6}$$

This shows that $-\sigma^\lambda(z,x)$ is a vector at z tangent to the geodesic joining z to x and whose length is equal to that of this geodesic. Hence it is a natural generalization of the position vector of x relative to z, and reduces to this position vector in flat spacetime. It also follows from this that

$$2\sigma = \sigma_\kappa \sigma^\kappa = \sigma_\alpha \sigma^\alpha \tag{A 7}$$

and on recalling the defining property of $\bar{g}^{\lambda}_{\cdot\alpha}$, that

$$\sigma^\beta \nabla_\beta \bar{g}^{\lambda}_{\cdot\alpha} = 0, \tag{A 8}$$

all arguments being (x,z).

Let us denote the coincidence limit $x \to z$ of a bitensor by $\langle\ \rangle$. Then we have the results, given by Synge (1960), that

$$\left.\begin{aligned} \langle \sigma_\kappa \rangle = \langle \sigma_\alpha \rangle = 0, \qquad & \langle \sigma_{\kappa\lambda} \rangle = g_{\kappa\lambda}, \\ \langle \sigma_{\kappa\alpha} \rangle = -\delta^\lambda_\alpha g_{\kappa\lambda} \quad \text{and} \quad & \langle \sigma_{\alpha\beta} \rangle = \delta^{\kappa\lambda}_{\alpha\beta} g_{\kappa\lambda}. \end{aligned}\right\} \tag{A 9}$$

In the latter two equations, the Kronecker delta is needed because of our conventions on indices. As the coincidence limit is a tensor at z, we can only use indices $\kappa, \lambda, \ldots$ to state its value, and so $\delta^\lambda_\alpha g_{\kappa\lambda}$ must be written instead of the more natural $g_{\kappa\alpha}(z)$. The Kronecker delta is being used purely as a substitution operator, in accordance with the usage of Schouten (1954), and is thus distinguished from the unit tensor, which is denoted by A^α_β. We write the kernel letter only once in products of Kronecker deltas, thus $\delta^{\kappa\lambda}_{\alpha\beta} := \delta^\kappa_\alpha \delta^\lambda_\beta$.

We can now deduce from (A 7) and (A 8) some very useful results on coincidence limits of derivatives of σ and $\bar{g}^{\lambda}_{\cdot\alpha}$. Our first result is that

$$\langle \sigma_{\beta(\alpha_1 \ldots \alpha_n)} \rangle = 0 \quad \text{if} \quad n \geqslant 2. \tag{A 10}$$

To prove it, first deduce from (A 7) that

$$\sigma^\beta = \sigma^{\beta\gamma} \sigma_\gamma. \tag{A 11}$$

Now assuming (A 10) for $2 \leqslant n \leqslant (N-1)$, take the N-fold symmetrized covariant derivative at x of (A 11) and let $x \to z$ in the result. Using (A 9) and the induction hypothesis, we immediately obtain (A 10) for $n = N$. As taking $N = 2$ also proves the result for $n = 2$, by induction the result holds for all n.

We next prove that

$$\langle \sigma_{\kappa(\alpha_1 \ldots \alpha_n)} \rangle = 0 \quad \text{if} \quad n \geqslant 2 \tag{A 12}$$

and

$$\langle \nabla_{(\alpha_1 \ldots \alpha_n)} \bar{g}^{\kappa}_{\cdot\beta} \rangle = 0 \quad \text{if} \quad n \geqslant 1. \tag{A 13}$$

Again we need the coincidence limit of the n-fold symmetrized covariant derivative at x of certain equations. Applying these operations to the first of equations (A 7) and using (A 9) and (A 10) yields the induction step needed to prove (A 12). Applying them to (A 8) and using (A 9) and (A 10) yields (A 13) directly, without needing an inductive proof. Finally, (A 10) may be extended to show that

$$\langle \mathscr{S} \sigma_{\alpha_1 \ldots \alpha_n} \rangle = 0 \quad \text{if} \quad n \geqslant 3, \tag{A 14}$$

where $\mathscr{S}$ is an operator symmetrizing *any* $(n-1)$ of the n derivative indices on σ. This is proved by induction, using the Ricci identity to commute the index β in (A 10) to the right, one step at a time. Using the Ricci identity we may also obtain from (A 10) and (A 13) the coincidence limits of the corresponding unsymmetrized derivatives. The corresponding limits when some of the derivatives are at z rather than at x may then be obtained using a result of Synge (1960), namely that

$$\langle \nabla_\kappa A \rangle = \nabla_\kappa \langle A \rangle - \delta^\alpha_\kappa \langle \nabla_\alpha A \rangle, \tag{A 15}$$

where A is any bitensor, with indices suppressed. In this way we may find the coincidence limit of any derivative of σ or $\bar{g}^{\lambda}_{\cdot\alpha}$.

In §7 we needed the coincidence limits of symmetrized derivatives of $H^{\alpha}_{.\lambda} := -\sigma^{-1\alpha}_{\ \ .\lambda}$. These may be obtained from a simple recurrence relation. Either by direct calculation or, more simply, from the results of §3 of I, we get

$$\sigma^{\gamma}\sigma^{\delta}\nabla_{\gamma\delta}H_{\alpha\lambda} = -\sigma^{\gamma}\sigma^{\delta}R_{\dot{\alpha}\gamma\dot{\delta}}{}^{\beta}H_{\beta\lambda} - 2\sigma^{\gamma}\nabla_{\gamma}H_{\alpha\lambda}. \tag{A 16}$$

Following the same procedure as in the derivation of (A 13), this yields

$$\langle\nabla_{(\alpha_1\ldots\alpha_n)}H_{\beta\lambda}\rangle = -\frac{n-1}{n+1}\langle\nabla_{(\alpha_1\cdots\alpha_{n-2}}\{R_{|\beta|\alpha_{n-1}\alpha_n)}{}^{\gamma}H_{\gamma\lambda}\}\rangle \text{ for } n \geqslant 2. \tag{A 17}$$

Since we may easily prove that

$$\langle H_{\beta\lambda}\rangle = \delta^{\kappa}_{\beta}g_{\kappa\lambda} \quad \text{and} \quad \langle\nabla_{\alpha}H_{\beta\lambda}\rangle = 0, \tag{A 18}$$

all higher symmetrized derivatives may be obtained by iteration of (A 17). If needed, the unsymmetrized derivatives, and those involving derivatives at z, can then be obtained by the method indicated above.

Diamond brackets $\langle\ \rangle$ will also be used to denote the scalar product of two tensor fields on $\mathscr{M}$. Let $A^{\alpha\cdots}{}_{\beta\ldots}$ and $B_{\alpha\ldots}{}^{\beta\cdots}$ be complex tensor fields on $\mathscr{M}$ of type (r, s) and (s, r) respectively, where type (r, s) means that the tensor has r contravariant and s covariant indices. Then we write

$$\langle A^{\alpha\cdots}{}_{\beta\ldots}, B_{\alpha\ldots}{}^{\beta\cdots}\rangle := \int (A^{\alpha\cdots}{}_{\beta\ldots})^* B_{\alpha\ldots}{}^{\beta\cdots}\sqrt{(-g)}\,\mathrm{d}^4x, \tag{A 19}$$

where * denotes complex conjugation. Another convenient abbreviation will be used in writing integrals over a tangent space to $\mathscr{M}$. If $T_z(\mathscr{M})$ is the tangent space to $\mathscr{M}$ at $z \in \mathscr{M}$, and the position vector X^{λ} in $T_z(\mathscr{M})$ is referred to the natural basis associated with the local coordinate system at z, then the scalar volume element on $T_z(\mathscr{M})$ will be denoted by

$$\mathrm{D}X := \sqrt{\{-g(z)\}}\,\mathrm{d}X^0\ldots\mathrm{d}X^3.$$

If f and g are any two functions, the composition function $x \to f(g(x))$ will be denoted by $f \circ g$, whenever this composition is meaningful.

Note added in proof 8 *September* 1970: If $\alpha : \mathscr{A} \to \mathscr{M}$ and $\beta : \mathscr{B} \to \mathscr{M}$ are fibre bundles over $\mathscr{M}$, a map $\Phi : \mathscr{A} \to \mathscr{B}$ was defined in §4 to be *fibre-transitive* if $\alpha = \beta \circ \Phi$. It has since come to my notice that in the theory of fibre bundles, such a map is essentially just a cross-section of the bundle $\alpha^{-1}\mathscr{B}$ over $\mathscr{A}$ induced from $\mathscr{B}$ by the projection α.

References

Beiglböck, W. 1967 *Commun. Math. Phys.* **5**, 106–130.
Bielecki, A., Mathisson, M. & Weyssenhoff, J. W. 1939 *Bull. Int. Acad. Pol. Sci. Lett., Cl. Sci. Math. Nat. Sér.* A: *Sci. Math.* pp. 22–28 (1939).
DeWitt, B. S. & Brehme, R. W. 1960 *Ann. Phys. (N.Y.)* **9**, 220–259.
Dixon, W. G. 1964 *Nuovo Cim.* **34**, 317–339.
Dixon, W. G. 1967 *J. Math. Phys. (N.Y.)* **8**, 1591–1605.
Dixon, W. G. 1970 *Proc. Roy. Soc. Lond.* A **314**, 499–527.

Gel'fand, I. M. & Shilov, G. E. 1964 *Generalized functions*, vol. 1. New York: Academic Press.
Gel'fand, I. M. & Shilov, G. E. 1968 *Generalized functions*, vol. 2. New York: Academic Press.
Kobayashi, S. & Nomizu, K. 1963 *Foundations of differential geometry*. New York: Wiley.
Mathisson, M. 1937 *Acta Phys. Polon.* **6**, 163–200.
Mathisson, M. 1940 *Proc. Camb. Phil. Soc.* **36**, 331–350.
Papapetrou, A. 1951 *Proc. Roy. Soc. Lond.* A **209**, 248–258.
Schouten, J. A. 1954 *Ricci calculus* (2nd edn). Berlin: Springer-Verlag.
Synge, J. L. 1960 *Relativity: the general theory*. Amsterdam: North-Holland Publ. Co.
Tulczyjew, W. 1959 *Acta Phys. Polon.* **18**, 393–409.
Urich, W. & Papapetrou, A. 1955 *Z. Naturforsch.* **10**a, 109–117.
Weyl, H. 1946 *Classical groups*. Princeton University Press.
Whitehead, J. H. C. 1932 *Quart. J. Math.* **3**, 33–42.