Kinematic Hopf algebra for amplitudes and form factors

Gang Chen~~1~~,[1,*] Guanda Lin~~1,2~~,[1,2,†] and Congkao Wen~~1~~[3,‡]

[1] *Centre for Theoretical Physics, Department of Physics and Astronomy, Queen Mary University of London, Mile End Road, London E1 4NS, United Kingdom*

[2] *CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing, 100190, China*

[3] ~~1~~ *Centre for Theoretical Physics, Department of Physics and Astronomy, Queen Mary University of London, Mile End Road, London E1 4NS, United Kingdom*

We propose a kinematic algebra for the Bern-Carrasco-Johansson (BCJ) numerators of tree-level amplitudes and form factors in Yang-Mills theory coupled with ~~bi-adjoint~~ biadjoint scalars. The algebraic generators of the algebra contain two parts: the first part is simply the ~~flavour~~ flavor factor of the ~~bi-adjoint~~ biadjoint scalars, and the second part that maps to ~~non-trivial~~ nontrivial kinematic structures of the BCJ numerators obeys extended ~~quasi-shuffle~~ quasishuffle fusion products. The underlying kinematic algebra allows us to present closed forms for the BCJ numerators with any number of gluons and two or more scalars for both on-shell amplitudes and form factors that involve an off-shell operator. The BCJ numerators constructed in this way are manifestly gauge invariant and obey many novel relations that are inherited from the kinematic algebra.

I. INTRODUCTION

Gauge and gravity theories play a central role in our understanding of physical phenomena. The double copy relation [1–3], which was inspired by Kawai-Lewellen-Tye relations [4] in string theory, reveals deep relations between them. The most critical step of the Bern-Carrasco-Johansson (BCJ) double copy prescription [1–3] is to ~~realise the colour-kinematics~~ realize the color-kinematics duality for gauge theory amplitudes, where the kinematic numerators (also called the BCJ numerators) satisfy the same Jacobi relations as the ~~colour~~ color factors. The ~~colour-kinematics~~ color-kinematics duality discloses the delicate perturbative structures of amplitudes in a large number of gauge theo-

*g.chen@qmul.ac.uk

†linguandak@pku.edu.cn

‡c.wen@qmul.ac.uk

ries [5–15], effective theories [16–25, 25, 26], and can be also extended to form factors [27–33]. It has led to remarkable insights and tremendous progress in the comprehension of amplitudes in both gauge theory and gravity.

An important approach to ~~study the colour-kinematics~~ studying the color-kinematics duality is to consider the underlying algebraic structures. Different versions of kinematic algebras have been ~~realised~~ realized in a variety of arenas~~, ~~ e.g., the self-dual Yang-Mills (YM) [34], ~~non-linear sigma model~~the nonlinear sigma model [17], ~~maximally-helicity-violating~~ maximally helicity-violating (MHV) and next-to-MHV sectors of YM theory [35, 36], and Chern-Simons theory [37].

It was recently found [38, 39] that in the heavy-mass effective theory (HEFT)~~a quasi-shuffle Hopf algebra~~, a quasishuffle Hopf algebra [40–44] perfectly depicts the ~~colour-kinematics~~ color-kinematics duality structure in the theory. There are three important ingredients in that algebra: ~~a)~~ (a) the generators as heavy source currents, ~~b)~~ (b) the fusion product merging two lower-point currents~~and c)~~ , and (c) the mapping rule turning the abstract algebraic element to concrete expressions. Compact closed expressions of the BCJ numerators for amplitudes of gluons coupled with two heavy particles (as well as pure gluon amplitudes after taking the decoupling limit) were obtained. However, there are some restrictions in this prescription: the number of massive particles has to be two, the heavy-mass limit is required, and the physical meaning of the currents and the fusion product is unclear. To use the kinematic Hopf algebra to study general amplitudes, we need to circumvent these restrictions.

In this ~~letter~~paper, we present an extended version of the kinematic Hopf algebra which leads to closed-form expressions for amplitudes of any number of scalars without the requirement of the heavy limit. More importantly, the physical meaning of the new algebra is transparent: the generators are physical states, and the fusion product corresponds to interaction vertices. This understanding results in a universal description of both scattering amplitudes and form factors, where the latter involve certain off-shell gauge-invariant operators.

In particular, we will consider the YM-scalar theory with a ~~bi-adjoint~~ biadjoint ϕ^3 interaction. This particular theory has played a vital role in the study of ~~colour-kinematics~~ color-kinematics duality and double copy [8, 11, 45–47] (see further comments in the discussion section). The scalars have an identical mass m [48] and bear ~~colour and flavour~~ color and flavor indices, denoted as I and a, respectively. We will consider both on-shell amplitudes and form factors with the operator $\mathrm{Tr}(\phi^2) = \sum_{a,I}(\phi^{a,I})^2$ [49]. Unless otherwise stated, amplitudes/form factors in this paper refer to the ~~colour-ordered~~ color-ordered amplitudes/form factors and carry ~~flavour~~ flavor indices. Also, we only focus on the ~~single trace~~ single-trace ones in the ~~flavour~~ flavor sector of the ~~bi-adjoint~~ biadjoint scalars, from which one may obtain ~~multi-trace~~ multitrace amplitudes using the transmutation operators in Ref. [50].

II. GENERAL FRAMEWORK FROM KINEMATIC HOPF ALGEBRA

This section provides a systematic approach constructing amplitudes and form factors via the kinematic Hopf algebra. Unlike the usual Feynman diagram computations, the resulting expressions are manifestly gauge invariant and extremely compact; furthermore, they obey the ~~colour-kinematics~~ color-kinematics duality. The main ingredients for our approach are *algebra generators, fusion products*~~*and the*~~, and the *evaluation map*, which we will describe below.

The first ingredient is the algebraic generator for single-particle external states. There are two types of single-particle generators:

$$\mathsf{K}_i = \begin{cases} T^{(i)}_{(i)} & \text{for gluons} \\ T^{(i)} t^{a_i} & \text{for scalars} \end{cases}, \tag{1}$$

where the T represents the kinematic part and t^{a_i} denotes the ~~flavour~~ flavor group generator for the scalars.

Then we combine these single-particle generators together via the fusion product. For now, let us consider the simplest examples of (i) the fusion of a single scalar state and a gluon state becoming a two-particle state; and (ii) the subsequent fusion of such a ~~two-particle state~~ two-particle-state fusion with a gluon state into some three-particle states such as

$$\begin{aligned} \mathsf{K}_1 \star \mathsf{K}_2 &= T^{(1)} t^{a_1} \star T^{(2)}_{(2)} = T^{(12)}_{(2)} t^{a_1}, \\ (\mathsf{K}_1 \star \mathsf{K}_2) \star \mathsf{K}_3 &= T^{(12)}_{(2)} t^{a_1} \star T^{(3)}_{(3)} \\ &= (-T^{(123)}_{(23)} + T^{(123)}_{(2),(3)} + T^{(123)}_{(3),(2)}) t^{a_1}, \end{aligned} \tag{2}$$

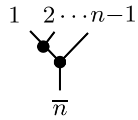
where on the ~~RHS~~ rhs of these equations, we have the ~~multi-particle~~ multiparticle generator

$$T^{(\alpha)}_{(\tau_1),\cdots,(\tau_r)} t^{a_i} \cdots t^{a_j}, \tag{3}$$

in which the superscript α denotes the order of the particles in performing the fusion product, while the subscript denotes the partition of the gluons, and the product of t^{a_i} from ~~eq~~Eq. (1) composes the ~~flavour~~ flavor structure. Notably, the fusion product is associative: $X \star (Y \star Z) = (X \star Y) \star Z$ for arbitrary generators X, Y, Z.

The last piece of the construction is the evaluation map $\langle \bullet \rangle$, which is a linear map from an algebra generator to a gauge-invariant expression appearing in actual amplitudes. More details on the fusion product and the explicit expressions for the evaluation map will be given later.

We now show how to use the three ingredients above to obtain tree-level amplitudes and form factors. This can be achieved by giving the fusion product a physical meaning. The interaction vertices in the Lagrangian usually involve ~~commutator~~ commutators of fields. In our algebraic language, such commutators are exactly the commutators of the fusion products, $[X, Y] = X \star Y - Y \star X$. More concretely, we express the amplitude as a sum of cubic graphs, and we regard each cubic graph as a nested commutator, given the above correspondence between interaction vertices and commutators. As an example, for the cubic graph



, (4)

we can interpret the vertices in the graph with commutators of the generators. The commutators are performed in an ordering $1, 2, \cdots, n{-}1$ and lead to the corresponding nested commutator

$$\widehat{\mathcal{N}}([1, 2, \cdots, n{-}1]) = [\cdots[\mathsf{K}_1, \mathsf{K}_2], \cdots, \mathsf{K}_{n-1}]\,. \tag{5}$$

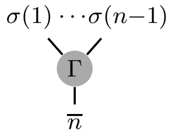
Then, the contribution to amplitude is obtained by taking the evaluation map and combining with the propagators

$$\frac{\langle \widehat{\mathcal{N}}([1, 2, \cdots, n{-}1]) \rangle}{d_{[1,2,\ldots,n-1]}}, \tag{6}$$

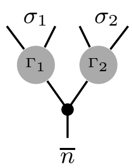
where $d_{[1,2,\ldots,n-1]}$ denotes the product of propagators that is associated with the graph.

Note that for convenience, we set the n~~-th~~ th particle to be a scalar, ~~labelled~~ labeled by $\overline{n}$ (~~characterised~~ characterized by the bar). And in the algebraic construction above, we do not need the generator K_n, since p_n can be removed by the momentum conservation. For a generic cubic diagram, the contribution takes a similar form ~~as eq~~to Eq. (6), except that the commutator structure is determined by the specific cubic graph. As a result, the amplitude is expressed as

$$\mathcal{A}(\sigma, \overline{n}) = \sum_{\Gamma \in \mathsf{R}_\sigma} \frac{\langle \widehat{\mathcal{N}}(\Gamma) \rangle}{d_\Gamma}$$



$$= \sum_{\Gamma \in R_\sigma^{(2)}} \frac{\langle [\widehat{\mathcal{N}}(\Gamma_a), \widehat{\mathcal{N}}(\Gamma_b)] \rangle}{d_{\Gamma_a} d_{\Gamma_b}}$$

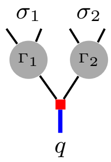


, (7)

where R_σ represents the cubic diagrams that respect the ordering σ~~,~~; the $\mathsf{R}_\sigma^{(2)}$ denotes all the inequivalent graphs with ~~colour~~ color ordering σ and two components as cubic graphs Γ_1, Γ_2; and d_Γ denotes the products of the propagators associated with a given cubic graph Γ. Importantly, by construction the numerators $\langle \widehat{\mathcal{N}}(\Gamma) \rangle$ obey Jacobi relations, and they are precisely the BCJ numerator of the graph Γ.

We will then consider the ~~colour-kinematics~~ color-kinematics duality in form factors [51]. Let us first present the construction~~then~~, then briefly explain the significance~~shortly~~. Graphically, the difference is to replace the interaction vertex involving the n~~-th scalar in eq~~th scalar in Eq. (7) with a ~~colourless~~ colorless operator $\mathrm{Tr}(\phi^2)$. Here, we assign a fusion product rather than a commutator to the operator. The form factor also has a novel representation that is very analogous to ~~eq~~Eq. (7)~~,~~:

$$\mathcal{F}_{\mathrm{Tr}(\phi^2)}(\sigma) = \sum_{\Gamma\in \mathsf{R}^{(2)}_{\sigma}} \frac{\langle \widehat{\mathcal{N}}(\Gamma_1)\star\widehat{\mathcal{N}}(\Gamma_2)\rangle}{d_{\Gamma_1} d_{\Gamma_2}}$$



$$. \tag{8}$$

Finally, d_{Γ_i} $(i = 1, 2)$ is the product of propagators in each cubic graph Γ_i, including the propagator connecting Γ_i with the operator (i.e. the red-box vertex).

Importantly, when comparing ~~eq~~Eqs. (7) and (8), we have

	~~operator~~ Operator vertex	~~cubic~~ Cubic vertex
~~colour~~ Color factor	~~single~~ Single trace	~~structure~~ Structure constant
~~algebraic~~ Algebraic rule	$X \star Y$	$[X, Y]$

Note that the structure constant is essentially a commutator. Then it is understandable to establish the equivalence between the algebraic rule and the physical ~~colour~~ color structure. See more evidence in the Supplementary Material, including the extension to form factors of operators like $\mathrm{tr}(\phi^h)$ with $h > 2$.

In the above, we have sketched the algebraic framework and how to obtain the physical observables from it. In the next section, we will spell out the details of the construction.

III. EXPLICIT ~~REALISATION~~REALIZATION: FUSION PRODUCT AND MAPPING RULE

We first explain the fusion-product rule at length, which is a ~~non-abelian generalisation~~ non-Abelian generalization of the previous ~~quasi-shuffle product~~quasishuffle product [38].

As given in ~~eq~~Eq. (3), the generators are in general products of the kinematic part and the ~~flavour~~ flavor part. These two parts are commutative and can be treated separately in the fusion product: (i) The fusion products of the ~~flavour~~ flavor part are simply the product of the standard Lie algebra generators, which is generally not an ~~abelian product;~~ Abelian product. (ii) The kinematic part obeys the ~~non-abelian quasi-shuffle~~ non-Abelian quasishuffle product

$$T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \star T^{(\beta)}_{(\omega_1),\ldots,(\omega_s)}$$

$$= \sum_{\substack{\pi|_\tau=\{(\tau_1),\ldots,(\tau_r)\}\\ \pi|_\omega=\{(\omega_1),\ldots,(\omega_s)\}}} (-1)^{t-r-s} T^{(\alpha\beta)}_{(\pi_1),\ldots,(\pi_t)}, \tag{9}$$

where $\pi|_\tau$ (or $\pi|_\omega$) means a restriction to the elements of π in τ (or ω)~~, ~~, e.g.~~,~~, $\{(235),(4),(678)\}|_{\{2,3,4,8\}} = \{(23),(4),(8)\}$. For example,

$$\begin{aligned} T^{(12)}_{(1),(2)} \star T^{(345)}_{(34)} = & -T^{(12345)}_{(1),(234)} - T^{(12345)}_{(134),(2)} + T^{(12345)}_{(1),(2),(34)} \\ & + T^{(12345)}_{(1),(34),(2)} + T^{(12345)}_{(34),(1),(2)}. \end{aligned} \tag{10}$$

Compared with the fusion product rules for the amplitudes in HEFT [38], we see that ~~eq~~Eq. (9) has a similar basic form but contains a new superscript, also marking its ~~non-abelian~~ non-Abelian nature.

Equipped with the above rules, we calculate the following fusion product of single-particle generators in an ordering α, which are ubiquitous when expanding the commutators like ~~eq~~in Eq. (7):

$$\begin{aligned} \widehat{\mathcal{N}}(\alpha) &\equiv \mathsf{K}_{\alpha(1)} \star \mathsf{K}_{\alpha(2)} \star \ldots \star \mathsf{K}_{\alpha(|\alpha|)} \\ &= t^\eta \sum_{r=1}^{|\alpha|-|\eta|} \sum_{\tau \in \mathbf{P}^{(r)}_{\{\tau\}}} (-1)^{|\alpha|-|\eta|-r} T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)}, \end{aligned} \tag{11}$$

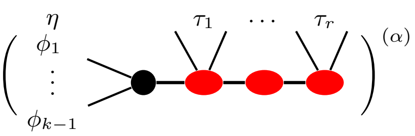
where t^η is the product of ~~flavour~~ flavor group generators, and $\mathbf{P}^{(r)}_{\{\tau\}}$ represents all the ordered partitions dividing the gluon ordering $\{\tau\}$ into r sets. The total number of terms is the Fubini number $\mathsf{F}_{|\alpha|-|\eta|}$. Let us consider a simple example for illustration:

$$\begin{aligned} \mathsf{K}_1 \star \mathsf{K}_2 \star \mathsf{K}_3 \star \mathsf{K}_4 &= t^{a_1} t^{a_4}\, T^{(1)} \star T^{(2)}_{(2)} \star T^{(3)}_{(3)} \star T^{(4)} \\ &= t^{a_1} t^{a_4} \left(T^{(1234)}_{(2),(3)} + T^{(1234)}_{(3),(2)} - T^{(1234)}_{(23)} \right). \end{aligned} \tag{12}$$

The next ingredient of our construction is the map from abstract algebraic generators to functions of physical kinematics and ~~flavour variables~~flavor variables,

$$t^\eta T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \xrightarrow[\text{map}]{\langle\bullet\rangle} \begin{cases} \mathrm{tr}(t^\eta t^{a_n}) \langle T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \rangle_m & \text{amplitude} \\ \mathrm{tr}(t^\eta)\ \ \langle T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \rangle_q & \text{form factor} \end{cases}, \tag{13}$$

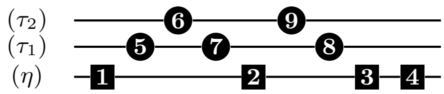
in which $\langle T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \rangle_m$ is defined as follows:

$\langle T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \rangle_m =$ 

$$= \frac{2^r \prod_{i=1}^{r}\left(p_{\Theta_L^{(\alpha)}(\tau_i)}\cdot F_{\tau_i}\cdot p_{\Theta_R^{(\alpha)}(\tau_i)}\right)}{(p_\eta^2-m^2)(p_{\eta\tau_1}^2-m^2)\cdots(p_{\eta\tau_1\cdots\tau_{r-1}}^2-m^2)}, \tag{14}$$

where $p_X = \sum_{i\in X} p_i$ and F_{τ_i} represents the product of ~~linearised~~ linearized field strengths $F_j^{\mu\nu} = p_j^\mu \varepsilon_j^\nu - \varepsilon_j^\mu p_j^\nu$ for all $j \in \tau_i$. Again, the dependence of the n~~-th~~ th scalar has been removed via the momentum conservation. Here we have assumed $k > 2$. The special case $k = 2$ (i.e., the amplitudes with two scalars) is discussed in the Supplemental ~~material~~Material, for which the evaluation map requires a minor modification. For form factors, the mapping rule is identical, except that m^2 in the denominator is replaced by q^2, the momentum square of the off-shell operator: $\langle T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)}\rangle_q = \langle T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)}\rangle_m\big|_{m^2\to q^2}$.

To further clarify $\Theta^{(\alpha)}_{L,R}$, it is illustrative to introduce the "musical diagram" as the ~~follow steps. 1.~~ following steps: (i) we embed the scalars (denoted as η) as well as the partitions of gluons τ_1 to τ_r into different levels: η lives on the bottom line, τ_1 is above it, then τ_2, until τ_r. ~~2.~~ (ii) we require that when projecting the elements in all the levels onto the bottom line, the ordering should be exactly α~~— the colour —~~ the color ordering of all the external particles. These requirements uniquely fix the relative positions in both the vertical and the horizontal directions in the "musical diagram~~".~~." $\Theta^{(\alpha)}_{L,R}(\tau_i)$ are just collections of all the ~~left-lower~~lower-left/~~right-lower~~ lower-right indices of τ_i in the musical diagram. As an example, we consider $T^{(156729834)}_{(578),(69)}$ with the corresponding musical diagram



, (15)

where we denote gluons by discs and scalars with boxes. Then we have, e.g., $p_{\Theta_L^{(\alpha)}(\tau_2)} = p_1 + p_5 \equiv p_{15}$ and $p_{\Theta_R^{(\alpha)}(\tau_2)} = p_{348}$, so that

$$\langle T^{(156729834)}_{(578),(69)}\rangle_m = \frac{4 p_1\cdot F_{578}\cdot p_{34}\ p_{15}\cdot F_{69}\cdot p_{348}}{(p_{1234}^2-m^2)(p_{1234578}^2-m^2)}. \tag{16}$$

As a corollary, if $\Theta_L^{(\alpha)}(\tau_i)$ or $\Theta_R^{(\alpha)}(\tau_i)$ is empty, we then have $p_{\Theta^{(\alpha)}_{L,R}}$=0, which leads to the vanishing condition

$$\langle T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)}\rangle_m = \cancel{0}0. \tag{17}$$

This is the case if α starts or ends with gluons.

Given these explanations, we are now ready to spell out a few examples to illustrate the algebraic construction above. The first example is a four-point amplitude:

$$\mathcal{A}(\bar{1},\bar{2},3,\bar{4}) = \frac{\langle[[\mathsf{K}_1,\mathsf{K}_2],\mathsf{K}_3]\rangle}{p_{12}^2-m^2} + \frac{\langle[\mathsf{K}_1,[\mathsf{K}_2,\mathsf{K}_3]]\rangle}{p_{23}^2-m^2}\cancel{;}. \tag{18}$$

~~here~~ Here and in the following, we denote scalars by $\bar{i}$. So in the above case, particles ~~1, 2, 4 are scalars~~1, 2, and 4 are scalars, and 3 is a gluon. Expanding the commutators and using the fusion rules together with the mapping rules, we arrive at

$$\mathcal{A}(\bar{1},\bar{2},3,\bar{4}) = \frac{\langle T^{(231)}_{(3)}\rangle_m \mathrm{tr}([t^{a_1}t^{a_2}]t^{a_4})}{p_{23}^2-m^2} = \frac{2p_2\cdot F_3\cdot p_1}{(p_{12}^2-m^2)(p_{23}^2-m^2)} f^{a_1a_2a_4}. \tag{19}$$

The final expression agrees with the correct amplitude. In this example, only the second term in ~~eq~~Eq. (18) contributes, because the BCJ numerator for the first one vanishes as a consequence of ~~eq~~Eq. (17). The second example is a three-point form factor

$$\mathcal{F}(\bar{1},3,\bar{2}) = \frac{\langle [\mathsf{K}_1,\mathsf{K}_3]\star\mathsf{K}_2\rangle}{p_{13}^2-m^2} + \frac{\langle \mathsf{K}_1\star[\mathsf{K}_3,\mathsf{K}_2]\rangle}{p_{23}^2-m^2}, \tag{20}$$

which can be simplified to

$$\mathcal{F}(\bar{1},3,\bar{2}) = \left(\frac{\delta^{a_1a_2}}{p_{13}^2-m^2} + \frac{\delta^{a_1a_2}}{p_{23}^2-m^2}\right)\frac{2p_2\cdot F_3\cdot p_1}{p_{12}^2-q^2}. \tag{21}$$

The expression agrees with known results [51]. We have checked our proposal up to all seven-point amplitudes and six-point form factors, as well as ~~eight and night-point~~ eight- and nine-point ones with two or three scalars. More examples and computation details are given in ~~section~~ Sec. C of the Supplemental ~~material and a Mathematica~~ Material and a *Mathematica* notebook, which can be found at Ref. [52].

IV. NOVEL PROPERTIES OF BCJ NUMERATORS

The algebraic construction~~, in particular~~ — in particular, the map $\langle\bullet\rangle$~~,~~ — has advantages more than just giving gauge-invariant and ~~duality satisfying~~ duality-satisfying numerators. Other interesting properties are presented below.

We first start from the following symmetry properties of the map:

1. ~~exchange~~ Exchange symmetry: The exchange symmetry for the indices of adjacent scalars i, j:

$$\langle T^{(\ldots ij\ldots)}_{(\tau_1),\ldots,(\tau_r)}\rangle_m = \langle T^{(\ldots ji\ldots)}_{(\tau_1),\ldots,(\tau_r)}\rangle_m. \tag{22}$$

2. ~~the "antipode~~"Antipode" symmetry: The antipode symmetry reverses the ordering of particles:

$$\langle T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)}\rangle_m = (-1)^{|\tau|}\langle T^{(\alpha^{-1})}_{(\tau_1^{-1}),\ldots,(\tau_r^{-1})}\rangle_m, \tag{23}$$

where α^{-1} means reversing all the elements in α and the same for τ_i^{-1}, and $|\tau|$ denotes the total number of gluons.

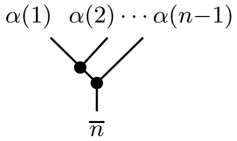
Stemming from these symmetries, we have the following three properties of numerators~~.~~:

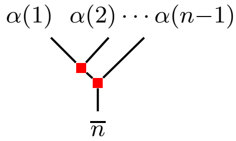
~~(1), we~~ 1. We find that the prenumerator, defined as the map of ~~eq~~Eq. (7), is invariant under the antipode action

$$\langle \widehat{\mathcal{N}}(12\ldots n{-}1)\rangle\Big|_{t^a\to\mathbb{I}} = \langle S(\widehat{\mathcal{N}}(12\ldots n{-}1))\rangle\Big|_{t^a\to\mathbb{I}}, \tag{24}$$

where S is the antipode as an ~~anti-homomorphism~~ antihomomorphism $S(X\star Y)=S(Y)\star S(X)$. The antipode acts on the generators as $S(T^{(i)})=T^{(i)}, S(T^{(j)}_{(j)})=-T^{(j)}_{(j)}$. Then Eq. (24) follows from Eq. (23). More details on the antipode can be found in Ref. [53] and ~~section~~ in Sec. B of the Supplemental ~~material~~Material.

~~(2), there is a non-trivial~~ 2. There is a nontrivial relation between the numerator of the cubic graph corresponding to the left-nested commutator and the corresponding fusion product,





$$\langle\widehat{\mathcal{N}}([\alpha])\rangle \qquad\qquad \langle\widehat{\mathcal{N}}(\alpha)\rangle, \tag{25}$$

which ~~is~~ are known as the BCJ numerator and ~~pre-numerator respectively~~prenumerator, respectively [54]. The ~~flavour~~ flavor factors of the two numerators are $\mathrm{tr}([t^\eta]t^{a_n})$ and $\mathrm{tr}(t^\eta t^{a_n})$, respectively, where t^η denotes the product of the ~~flavour~~ flavor generators t^{a_i} for $i\in\eta$, and $[t^\eta]$ represents the nest commutator of these generators. Magically, the kinematic part of the BCJ numerator and the ~~pre-numerator are identical~~prenumerator are identical:

$$\langle\widehat{\mathcal{N}}([\alpha])\rangle\big|_{f^{abc}\to 1} = \langle\widehat{\mathcal{N}}(\alpha)\rangle\big|_{t^a\to\mathbb{I}}. \tag{26}$$

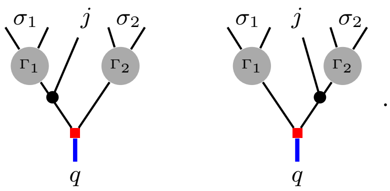
A simple example is

$$\begin{aligned}\langle\mathcal{N}(\overline{1},\overline{2},3,\overline{4})\rangle &= \langle \mathsf{K}_1\star\mathsf{K}_2\star\mathsf{K}_3\star\mathsf{K}_4\rangle\\ &= \frac{2p_{12}\cdot F_3\cdot p_4}{p_{124}^2-m^2}\mathrm{tr}(t^{a_1}t^{a_2}t^{a_4}t^{a_5}),\end{aligned} \tag{27}$$

$$\begin{aligned}\langle\mathcal{N}([\overline{1},\overline{2},3,\overline{4}])\rangle &= \langle[\mathsf{K}_1,\mathsf{K}_2]\star\mathsf{K}_3\star\mathsf{K}_4+\mathsf{K}_4\star\mathsf{K}_3\star[\mathsf{K}_1,\mathsf{K}_2]\rangle\\ &= \frac{2p_{12}\cdot F_3\cdot p_4}{p_{124}^2-m^2}\mathrm{tr}([[t^{a_1},t^{a_2}],t^{a_4}]t^{a_5}).\end{aligned} \tag{28}$$

~~(3), for~~ 3. For form factors of $\mathrm{Tr}(\phi^2)$, another relation arises [55]:

$$\langle\widehat{\mathcal{N}}([\Gamma_1,j])\star\widehat{\mathcal{N}}(\Gamma_2)\rangle = \langle\widehat{\mathcal{N}}(\Gamma_1)\star\widehat{\mathcal{N}}([j,\Gamma_2])\rangle$$



. (29)

Here, the j line can be either a gluon or scalar. When the j line is a gluon, the identity is manifest according to the vanishing condition ~~eq~~in Eq. (17). If the j line is a scalar, the identity becomes highly ~~non-trivial,~~ nontrivial; e.g., $\langle\widehat{\mathcal{N}}([\overline{1},2])\star\widehat{\mathcal{N}}([\overline{3},\overline{4}])-\widehat{\mathcal{N}}([[\overline{1},2],\overline{3}])\star\widehat{\mathcal{N}}(\overline{4})\rangle$ is evaluated as

$$-\langle\mathsf{K}_1\star\mathsf{K}_2\star\mathsf{K}_4\star\mathsf{K}_3\rangle+\langle\mathsf{K}_3\star[\mathsf{K}_1,\mathsf{K}_2]\star\mathsf{K}_4\rangle=\mathrm{tr}(t^{a_1}t^{a_4}t^{a_3})\Big(-\frac{p_1\cdot F_2\cdot p_{34}}{p_{134}^2-q^2}+\frac{p_{13}\cdot F_2\cdot p_4}{p_{134}^2-q^2}-\frac{p_3\cdot F_2\cdot p_{14}}{p_{134}^2-q^2}\Big)=0, \tag{30}$$

which implies ~~,~~

$$\langle\widehat{\mathcal{N}}([\overline{1},2])\star\widehat{\mathcal{N}}([\overline{3},\overline{4}])\rangle=\langle\widehat{\mathcal{N}}([[\overline{1},2],\overline{3}])\star\widehat{\mathcal{N}}(\overline{4})\rangle. \tag{31}$$

V. CONCLUSIONS AND DISCUSSIONS

In this ~~letter~~paper, we proposed a kinematic algebra for the BCJ numerators in YMS+ϕ^3 theory. The underlying algebraic structures lead to extremely compact expressions for the BCJ numerators ~~both in~~ in both amplitudes and form factors, and they reveal intriguing relations among them. Besides manifestly obeying the Jacobi identities, the numerators constructed in this way also enjoy many other remarkable properties such as crossing symmetry, manifest gauge invariance, and antipode symmetry.

The amplitudes and BCJ numerator in the YMS+ϕ^3 theory ~~has~~ have important application to constructing the gravitational amplitudes via ~~double copy~~ double-copying and studying the gravitational physics. For example, when double-copied with pure YM amplitudes, Einstein-Yang-Mills and Einstein-Maxwell amplitudes can be obtained, which are useful in the case of gravitational scattering of ~~photon from black hole~~ photons from a black hole [56–58]. Moreover, when double-copied with the amplitudes of (massive) spinning particles coupled to gluons, the resulting amplitudes are involved in the study of black hole scattering with spin effects [59]~~, see~~ ; see Refs. [60–76].

One more application is as follows~~. The colour-kinematic duality and double copy~~ : The color-kinematic duality and double-copying have also been studied in some effective theories with higher-dimensional interactions [21–24, 26]. As a step ~~towards such~~ in such a direction, one may consider form factors with the insertion of higher-dimensional operators. In the Supplemental ~~material~~Material, we show that the algebraic construction also works directly for form factors with such operators. Interestingly, novel relations beyond the Jacobi identity are deduced naturally from the kinematic algebra.

We now give some outlooks. First, at tree-level, one can extend the applicable scope of the Hopf algebra to more general theories; giving a proof also deserves considerations. Second, a feasible direction is to explore the kinematic algebra at the level of loop integrands. The physical picture of the fusion products (especially when involving the internal lines) suggests that they can be readily ~~generalised~~ generalized to off-shell particles. Third, it would be fascinating to find connections between our construction and other approaches in the literature, such as the Lagrangian and geometric understanding of the ~~colour-kinematics duality~~color-kinematics duality [17, 77–91], especially regarding the close relation between ~~quasi-shuffle~~ quasishuffle algebra and the permutohedron geometry [92, 93] (see also Refs. [94–98]).

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[49] The form factors are defined as

$$\mathcal{F}(1,2,\ldots,n) = \int d^D x\, e^{-iq\cdot x} \langle 1\, 2 \ldots n|\, \mathrm{Tr}(\phi^2)|0\rangle,$$

where external on-shell states are ~~labelled~~ labeled by $1,2,\ldots,n$, which can be gluons or scalars, and the operator $\mathrm{Tr}(\phi^2)$ carries an off-shell momentum $q = \sum_i p_i$.

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