Chapter 1

Introduction

1.1 Signals

A **signal** is a function of one or more variables that conveys information about some (usually physical) phenomenon.

1.1.1 Dimensionality of Signals

Signals can be classified based on the number of independent variables with which they are associated. A signal that is a function of only one variable is said to be **one dimensional** (1D). Similarly, a signal that is a function of two or more variables is said to be **multidimensional**. Human speech is an example of a 1D signal. In this case, we have a signal associated with fluctuations in air pressure as a function of time. An example of a 2D signal is a monochromatic image. In this case, we have a signal that corresponds to a measure of light intensity as a function of horizontal and vertical displacement.

1.1.2 Continuous-Time and Discrete-Time Signals

A signal can also be classified on the basis of whether it is a function of continuous or discrete variables. A signal that is a function of continuous variables (e.g., a real variable) is said to be **continuous time**. Similarly, a signal that is a function of discrete variables (e.g., an integer variable) is said to be **discrete time**. Although the independent variable need not represent time, for matters of convenience, much of the terminology is chosen as if this were so. For example, a digital image (which consists of a rectangular array of pixels) would be referred to as a discrete-time signal, even though the independent variables (i.e., horizontal and vertical position) do not actually correspond to time.

If a signal is a function of discrete variables (i.e., discrete-time) and the value of the function itself is also discrete, the signal is said to be **digital**. Similarly, if a signal is a function of continuous variables, and the value of the function itself is also continuous, the signal is said to be **analog**.

Many phenomena in our physical world can be described in terms of continuous-time signals. Some examples of continuous-time signals include: voltage or current waveforms in an electronic circuit; electrocardiograms, speech, and music recordings; position, velocity, and acceleration of a moving body; forces and torques in a mechanical system; and flow rates of liquids or gases in a chemical process. Any signals processed by digital computers (or other digital devices) are discrete-time in nature. Some examples of discrete-time signals include digital video, digital photographs, and digital audio data.

A discrete-time signal may be inherently discrete or correspond to a sampled version of a continuous-time signal. An example of the former would be a signal corresponding to the Dow Jones Industrial Average stock market index (which is only defined on daily intervals), while an example of the latter would be the sampled version of a (continuous-time) speech signal.

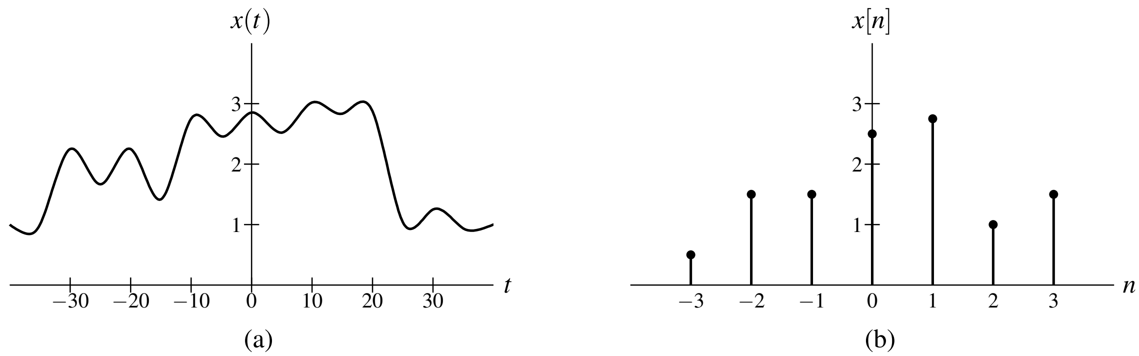


Figure 1.1: Graphical representations of (a) continuous-time and (b) discrete-time signals.

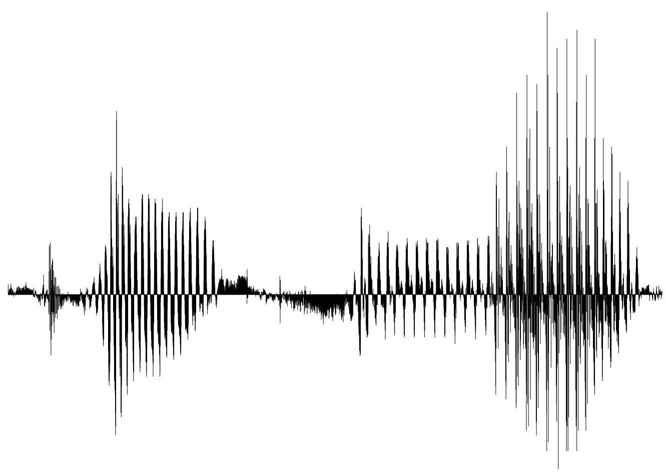


Figure 1.2: Segment of digitized human speech.

1.1.3 Notation and Graphical Representation of Signals

As a matter of notation, we typically write continuous-time signals with their independent variables enclosed in parentheses. For example, a continuous-time signal x with the real-valued independent variable t would be denoted $x(t)$. Discrete-time signals are written with their independent variables enclosed in brackets. For example, a discrete-time signal x with the integer-valued independent variable n would be denoted $x[n]$. This use of parentheses and brackets is a convention followed by much of the engineering literature. In the case of discrete-time signals, we sometimes refer to the signal as a **sequence**. Figure 1.1 shows how continuous-time and discrete-time signals are represented graphically.

1.1.4 Examples of Signals

A number of examples of signals have been suggested previously. Here, we provide some graphical representations of signals for illustrative purposes. Figure 1.2 depicts a digitized speech signal. Figure 1.3 shows an example of a monochromatic image. In this case, the signal represents light intensity as a function of two variables (i.e., horizontal and vertical position).

Figure 1.3: A monochromatic image.

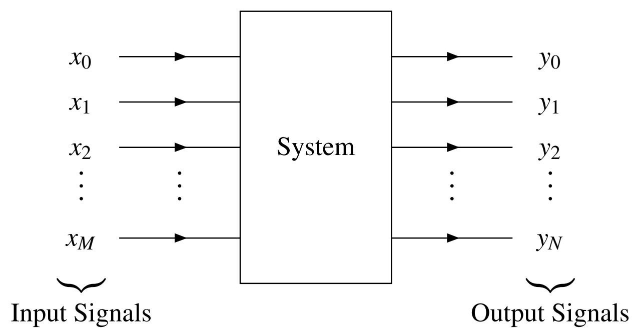


Figure 1.4: System with one or more inputs and one or more outputs.

1.2 Systems

A **system** is an entity that processes one or more input signals in order to produce one or more output signals, as shown in Figure 1.4. Such an entity is represented mathematically by a system of one or more equations.

In a communication system, the input might represent the message to be sent, and the output might represent the received message. In a robotics system, the input might represent the desired position of the end effector (e.g., gripper), while the output could represent the actual position.

1.2.1 Classification of Systems

A system can be classified based on the number of inputs and outputs it employs. A system with only one input is described as **single input**, while a system with multiple inputs is described as **multi-input**. Similarly, a system with only one output is said to be **single output**, while a system with multiple outputs is said to be **multi-output**. Two commonly occurring types of systems are single-input single-output (SISO) and multi-input multi-output (MIMO).

A system can also be classified based on the types of signals with which it interacts. A system that deals with continuous-time signals is called a **continuous-time system**. Similarly, a system that deals with discrete-time signals is said to be a **discrete-time system**. A system that handles both continuous- and discrete-time signals, is sometimes

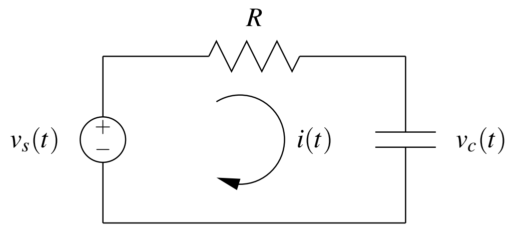


Figure 1.5: A simple RC network.

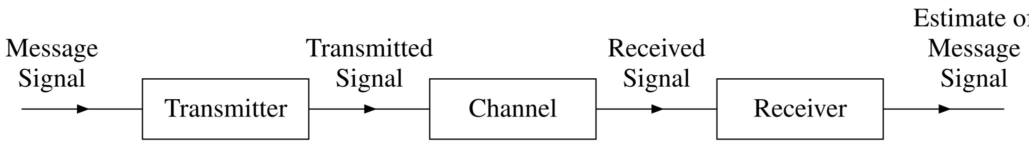


Figure 1.6: Communication system.

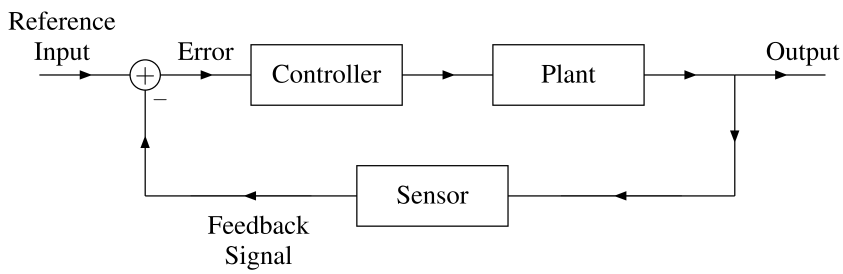


Figure 1.7: Feedback control system.

referred to as a **hybrid system** (or sampled-data system). Similarly, systems that deal with digital signals are referred to as **digital**, while systems that handle analog signals are referred to as **analog**. If a system interacts with 1D signals the system is referred to as **1D**. Likewise, if a system handles multidimensional signals, the system is said to be **multidimensional**.

1.2.2 Examples of Systems

Systems can manipulate signals in many different ways and serve many useful purposes. Sometimes systems serve to extract information from their input signals. For example, in case of speech signals, systems can be used in order to perform speaker identification or voice recognition. A system might analyze electrocardiogram signals in order to detect heart abnormalities. Amplification and noise reduction are other functionalities that systems could offer.

One very basic system is the RC network shown in Figure 1.5. Here, the input would be the source voltage $v_s(t)$ and the output would be the capacitor voltage $v_c(t)$.

Consider the communication system shown in Figure 1.6. This system takes a message at one location and reproduces this message at another location. In this case, the system input is the message to be sent, and the output is the estimate of the original message. Usually, we want the message reproduced at the receiver to be as close as possible to the original message sent by the transmitter.

A system of the general form shown in Figure 1.7 frequently appears in control applications. Often, in such applications, we would like an output to track some reference input as closely as possible. Consider, for example, a robotics application. The reference input might represent the desired position of the end effector, while the output represents the actual position.

1.3 Continuous-Time Signals and Systems

In the remainder of this course, we will focus our attention primarily on the study of 1D continuous-time signals and systems. Moreover, we will mostly concern ourselves with SISO (i.e., single-input single-output) systems. The discrete-time and multidimensional cases will be treated in other courses.

1.4 Why Study Signals and Systems?

As can be seen from the earlier examples, there are many practical situations in which we need to develop systems that manipulate/process signals. In order to do this, we need a formal mathematical framework for the study of such systems. The goal of this course is to provide the student with such a framework.

Chapter 2

Continuous-Time Signals and Systems

2.1 Transformations of the Independent Variable

An important concept in the study of signals and systems is the transformation of a signal. Here, we introduce several elementary signal transformations. Each of these transformations involves a simple modification of the independent variable.

2.1.1 Time Reversal

The first type of signal transformation that we shall consider is known as **time reversal**. Time reversal maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(-t). \tag{2.1}$$

In other words, the output signal $y(t)$ is formed by replacing t with $-t$ in the input signal $x(t)$. Geometrically, the output signal $y(t)$ is a reflection of the input signal $x(t)$ about the (vertical) line $t = 0$.

To illustrate the effects of time reversal, an example is provided in Figure 2.1. Suppose that we have the signal in Figure 2.1(a). Then, time reversal will yield the signal in Figure 2.1(b).

2.1.2 Time Scaling

Another type of signal transformation is called **time scaling**. Time scaling maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(at), \tag{2.2}$$

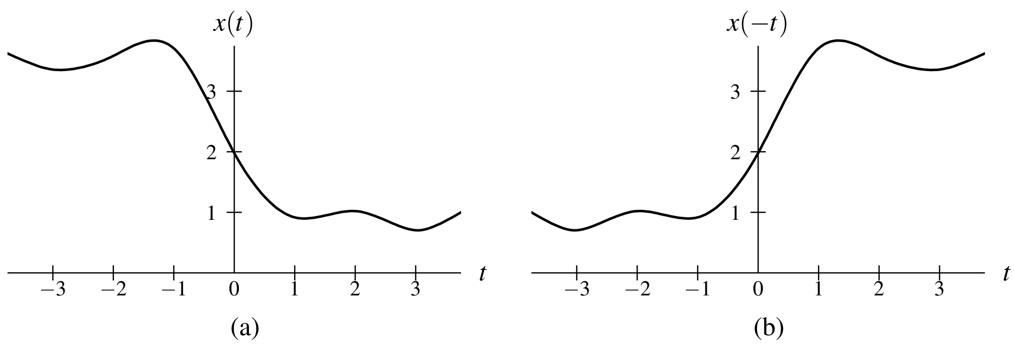


Figure 2.1: Example of time reversal.

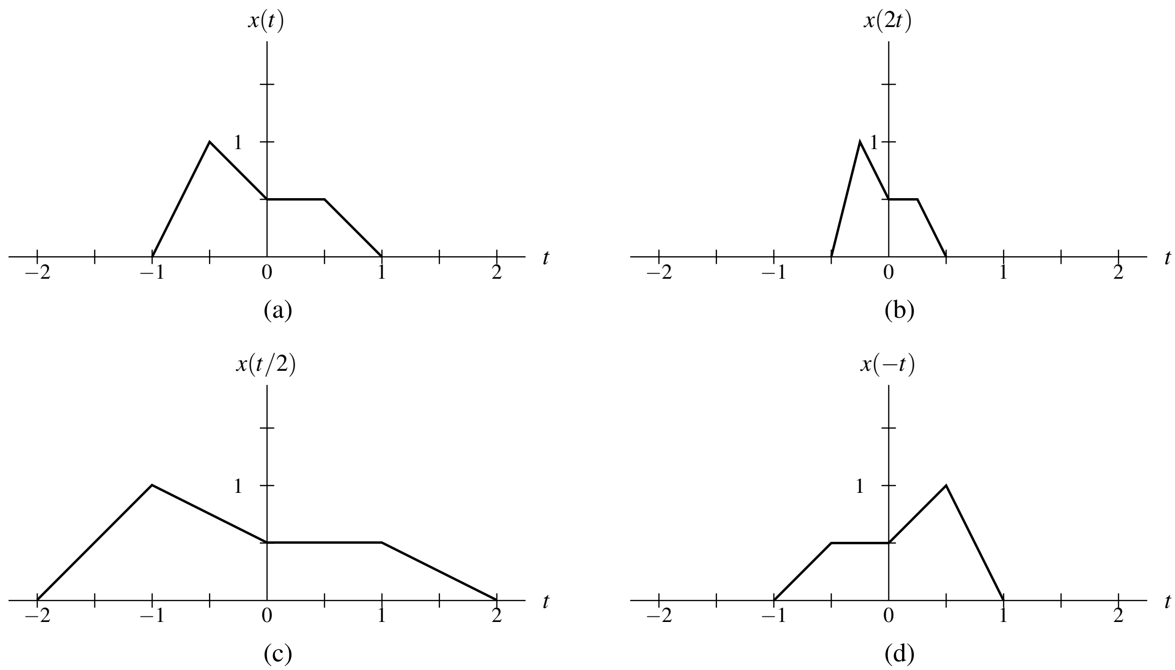


Figure 2.2: Example of time scaling.

where a is a scaling constant. In other words, the output signal $y(t)$ is formed by replacing t with at in the original signal $x(t)$. Geometrically, this transformation is associated with a compression/expansion along the time axis and/or reflection about the (vertical) line $t = 0$. If $|a| < 1$, the signal is expanded (i.e., stretched) along the time axis. If $|a| > 1$, the signal is instead compressed. If $|a| = 1$, the signal is neither expanded nor compressed. Lastly, if $a < 0$, the signal is reflected about the vertical line $t = 0$. Observe that time reversal is simply a special case of time scaling with $a = -1$.

To demonstrate the behavior of time scaling, we provide an example in Figure 2.2. Suppose that we have the signal $x(t)$ as shown in Figure 2.2(a). Then, using time scaling, we can obtain the signals in Figures 2.2(b), (c), and (d).

2.1.3 Time Shifting

The next type of signal transformation that we consider is called **time shifting**. Time shifting maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(t-b), \tag{2.3}$$

where b is a shifting constant. In other words, the output signal $y(t)$ is formed by replacing t by $t-b$ in input signal $x(t)$. Geometrically, this operation shifts the signal (to the left or right) along the time axis. If b is positive, $y(t)$ is shifted to the right relative to $x(t)$ (i.e., delayed in time). If b is negative, $y(t)$ is shifted to the left relative to $x(t)$ (i.e., advanced in time).

The effects of time shifting are illustrated in Figure 2.3. Suppose that we have the input signal $x(t)$ as shown in Figure 2.3(a). Then, the signals in Figures 2.3(b) and (c) can be obtained through time shifting.

2.1.4 Combining Time Scaling and Time Shifting

In the preceding sections, we introduced the time reversal, time scaling, and time shifting transformations. Moreover, we observed that time reversal is simply a special case of time scaling. Now, we introduce a more general transformation that combines the effects of time scaling, time shifting, and time reversal. This new transformation maps the

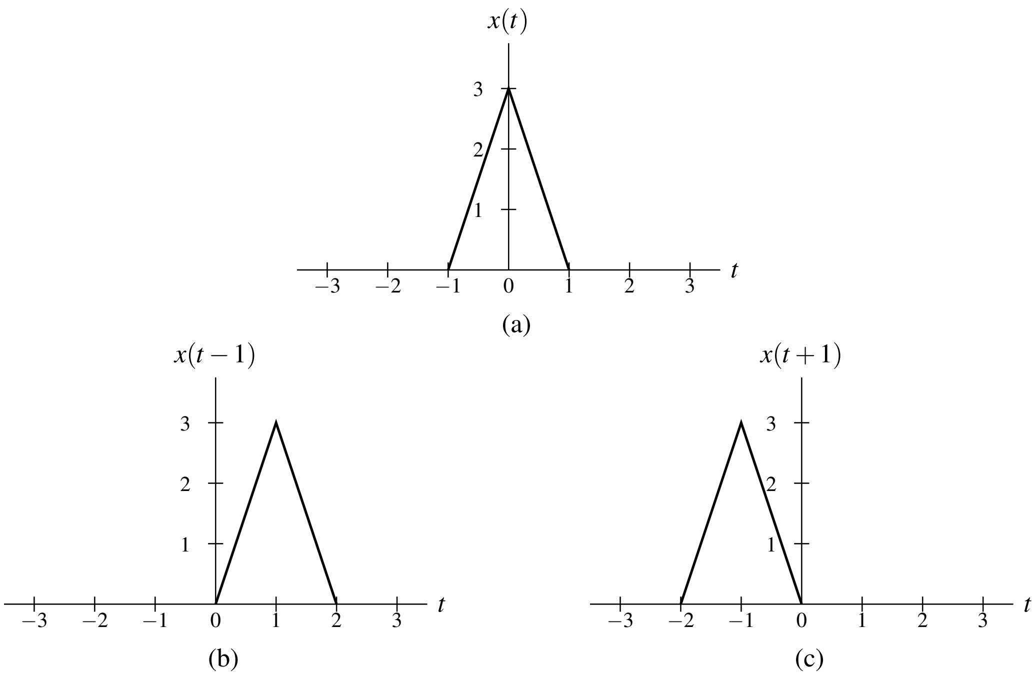


Figure 2.3: Example of time shifting.

input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(at - b), \tag{2.4}$$

where a is a scaling constant and b is a shifting constant. In other words, the output signal $y(t)$ is formed by replacing t with $at - b$ in the input signal $x(t)$. One can show that such a transformation is equivalent to first time shifting $x(t)$ by b, and then time scaling the resulting signal by a. Geometrically, this transformation preserves the shape of $x(t)$ except for a possible expansion/compression along the time axis and/or a reflection about the vertical line $t = 0$. If $|a| < 1$, the signal is stretched along the time axis. If $|a| > 1$, the signal is instead compressed. If $a < 0$, the signal is reflected about the vertical line $t = 0$.

The above transformation has two distinct but equivalent interpretations. That is, it is equivalent to each of the following:

1. first, time shifting $x(t)$ by b, and then time scaling the result by a.
2. first, time scaling $x(t)$ by a, and then time shifting the result by b/a.

One can easily confirm that both of the above equivalences hold, as shown below.

Consider the first case. First, we time shift $x(t)$ by b. Let us denote the result of this operation as $v(t)$, so that $v(t) = x(t - b)$. Next, we time scale $v(t)$ by a. This yields the result $v(at)$. From the definition of $v(t)$, however, we have $v(at) = x(at - b) = y(t)$.

Consider the second case. First, we time scale $x(t)$ by a. Let us denote the result of this operation as $v(t)$, so that $v(t) = x(at)$. Next, we time shift $v(t)$ by b/a. This yields the result $v(t - \frac{b}{a})$. From the definition of $v(t)$, however, we can write $v(t - \frac{b}{a}) = x\left(a(t - \frac{b}{a})\right) = x(at - b) = y(t)$.

Example 2.1. To illustrate the equivalent interpretations, we consider a simple example. Consider the signal $x(t)$ shown in Figure 2.4(a). Let us now determine the transformed signal $y(t) = x(at - b)$ where $a = 2$ and $b = 1$.

Solution. First, we consider the shift-then-scale method. In this case, we first shift the signal $x(t)$ by b (i.e., 1). This yields the signal in Figure 2.4(b). Then, we scale this new signal by a (i.e., 2) in order to obtain $y(t)$ as shown in Figure 2.4(c).

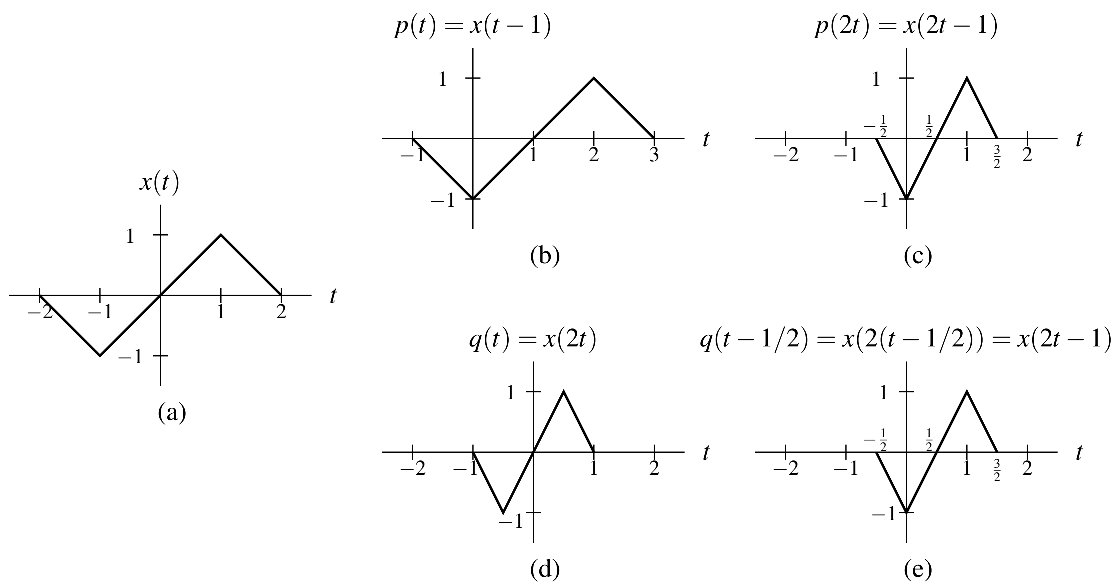


Figure 2.4: Two different interpretations of combined shifting and scaling transformation. (a) Original signal. Results obtained by shifting followed by scaling: (b) intermediate result and (c) final result. Results obtained by scaling followed by shifting: (d) intermediate result and (e) final result.

Second, we consider the scale-then-shift method. In this case, we first scale the signal $x(t)$ by a (i.e., 2). This yields the signal in Figure 2.4(d). Then, we shift this new signal by $\frac{b}{a}$ (i.e., $\frac{1}{2}$) in order to obtain $y(t)$ as shown in Figure 2.4(e).

□

2.2 Transformations of the Dependent Variable

In the preceding sections, we examined several transformations of the independent variable. Now, we consider similar transformations of the dependent variable.

2.2.1 Amplitude Scaling

The first transformation that we consider is referred to as **amplitude scaling**. Amplitude scaling maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = ax(t), \tag{2.5}$$

where a is a scaling constant. Geometrically, the output signal $y(t)$ is expanded/compressed in amplitude and/or reflected about the time axis. An amplifier is an example of a device that performs amplitude scaling.

To illustrate the effects of amplitude scaling, an example is given in Figure 2.5. Suppose that we have the signal $x(t)$ as shown in Figure 2.5(a). Then, amplitude scaling can be used to obtain the signals in Figures 2.5(b), (c), and (d).

2.2.2 Amplitude Shifting

The next transformation that we consider is referred to as **amplitude shifting**. Amplitude shifting maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(t) + b, \tag{2.6}$$

where b is a shifting constant. Geometrically, amplitude shifting adds a vertical displacement to $x(t)$.

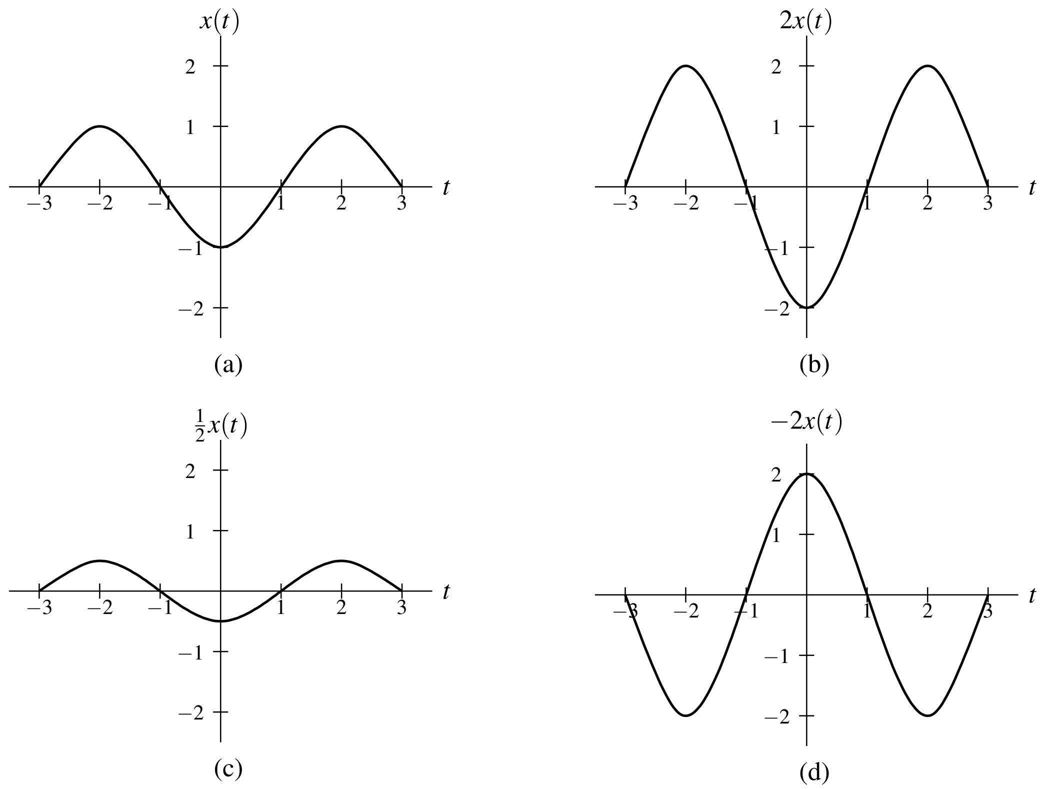


Figure 2.5: Example of amplitude scaling.

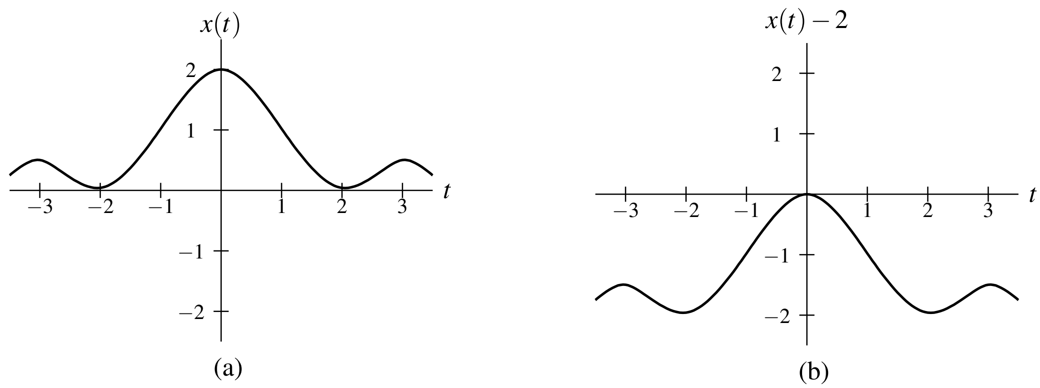


Figure 2.6: Example of amplitude shifting.

The effects of amplitude shifting are illustrated by Figure 2.6. The original signal $x(t)$ is given in Figure 2.6(a), and an amplitude-shifted version in Figure 2.6(b).

2.2.3 Combining Amplitude Scaling and Shifting

In the previous sections, we considered the amplitude-scaling and amplitude-shifting transformations. We can define a new transformation that combines the effects amplitude-scaling and amplitude-shifting. This transformation maps the input signal $x(t)$ to the output signal $y(t)$, as given by

$$y(t) = ax(t) + b, \tag{2.7}$$

where a is a scaling constant and b is a shifting constant. One can show that this transformation is equivalent to first scaling $x(t)$ by a and then shifting the resulting signal by b.

2.3 Signal Properties

Signals can possess a number of interesting properties. In what follows, we define several such properties. These properties are frequently useful in the analysis of signals and systems.

2.3.1 Even and Odd Signals

A signal $x(t)$ is said to be **even** if it satisfies

$$x(t) = x(-t) \quad \text{for all } t. \tag{2.8}$$

Geometrically, an even signal is symmetric about the vertical line $t = 0$. Some common examples of even signals include the cosine, absolute value, and square functions. Another example of an even signal is given in Figure 2.7.

A signal $x(t)$ is said to be **odd** if it satisfies

$$x(t) = -x(-t) \quad \text{for all } t. \tag{2.9}$$

Geometrically, an odd signal is antisymmetric about $t = 0$. In other words, the signal has symmetry with respect to the origin. One can easily show that an odd signal $x(t)$ must be such that $x(0) = 0$. Some common examples of odd signals include the sine, signum, and cube (i.e., $x(t) = t^3$) functions. Another example of an odd signal is given in Figure 2.8.

Any signal $x(t)$ can be represented as the sum of the form

$$x(t) = x_e(t) + x_o(t) \tag{2.10}$$

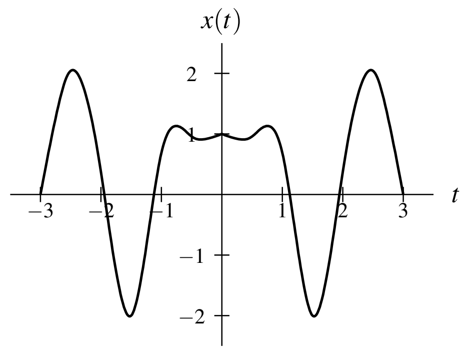


Figure 2.7: Example of an even signal.

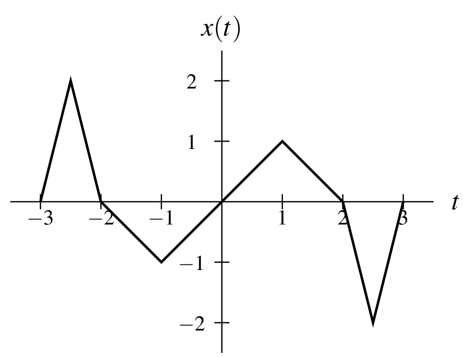


Figure 2.8: Example of an odd signal.

where $x_e(t)$ and $x_o(t)$ are even and odd, respectively, and given by

$$x_e(t) = \tfrac{1}{2}\left(x(t) + x(-t)\right) \quad \text{and} \tag{2.11}$$

$$x_o(t) = \tfrac{1}{2}\left(x(t) - x(-t)\right). \tag{2.12}$$

Often, we denote the even and odd parts of $x(t)$ as $\text{Even}\{x(t)\}$ and $\text{Odd}\{x(t)\}$, respectively. From (2.11) and (2.12), we can easily confirm that $x_e(t) + x_o(t) = x(t)$ as follows:

$$\begin{aligned} x_e(t) + x_o(t) &= \tfrac{1}{2}(x(t) + x(-t)) + \tfrac{1}{2}(x(t) - x(-t)) \\ &= \tfrac{1}{2}x(t) + \tfrac{1}{2}x(-t) + \tfrac{1}{2}x(t) - \tfrac{1}{2}x(-t) \\ &= x(t). \end{aligned}$$

Furthermore, we can easily verify that $x_e(t)$ is even and $x_o(t)$ is odd. From the definition of $x_e(t)$ in (2.11), we can write:

$$\begin{aligned} x_e(-t) &= \tfrac{1}{2}(x(-t) + x(-(-t))) \\ &= \tfrac{1}{2}(x(t) + x(-t)) \\ &= x_e(t). \end{aligned}$$

Thus, $x_e(t)$ is even. From the definition of $x_o(t)$ in (2.12), we can write:

$$\begin{aligned} x_o(-t) &= \tfrac{1}{2}(x(-t) - x(t)) \\ &= \tfrac{1}{2}(-x(t) + x(-t)) \\ &= -x_o(t). \end{aligned}$$

Thus, $x_o(t)$ is odd.

Sums involving even and odd signals have the following properties:

- The sum of two even signals is even.
- The sum of two odd signals is odd.
- The sum of an even signal and odd signal is neither even nor odd.

Products involving even and odd signals have the following properties:

- The product of two even signals is even.
- The product of two odd signals is even.
- The product of an even signal and an odd signal is odd.

2.3.2 Periodic Signals

A signal $x(t)$ is said to be **periodic** if it satisfies

$$x(t) = x(t+T), \quad \text{for all } t \text{ and some constant } T,\ T > 0. \tag{2.13}$$

The quantity T is referred to as the **period** of the signal. Two quantities closely related to the period are the **frequency** and **angular frequency**, denoted as f and ω, respectively. These quantities are defined as

$$f = \tfrac{1}{T} \quad \text{and} \quad \omega = 2\pi f = \tfrac{2\pi}{T}.$$

A signal that is not periodic is said to be **aperiodic**.

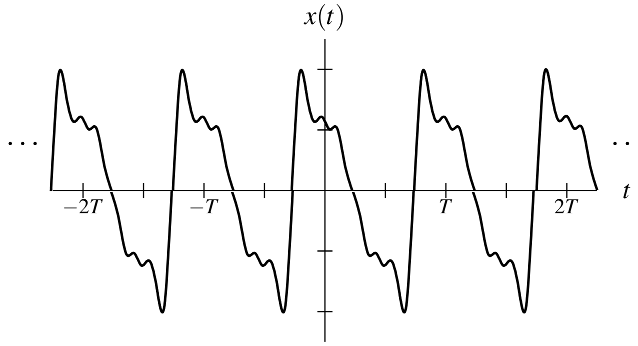


Figure 2.9: Example of a periodic signal.

Examples of periodic signals include the cosine and sine functions. Another example of a periodic signal is shown in Figure 2.9.

A signal that satisfies (2.13) must also satisfy

$$x(t) = x(t+NT) \tag{2.14}$$

for any integer value N. Therefore, a signal that is periodic with period T is also periodic with period NT. In most cases, we are interested in the smallest (positive) value of T for which (2.13) is satisfied. We refer to this value as the **fundamental period** of the signal.

Theorem 2.1 (Sum of periodic functions)**.** *Suppose that $x_1(t)$ and $x_2(t)$ are periodic signals with fundamental periods T_1 and T_2, respectively. Then, the sum $y(t) = x_1(t)+x_2(t)$ is a periodic signal if and only if the ratio T_1/T_2 is a rational number (i.e., the quotient of two integers). Suppose that $T_1/T_2 = q/r$ where q and r are integers and coprime (i.e., have no common factors), then the fundamental period of $y(t)$ is rT_1 (or equivalently, qT_2, since $rT_1 = qT_2$). (Note that rT_1 is simply the least common multiple of T_1 and T_2.)*

In passing, we note that the above result can be extended to the more general case of the sum of N periodic signals. The sum of N periodic signals $x_1(t), x_2(t), \ldots, x_N(t)$ with periods $T_1, T_2, \ldots, T_N$, respectively, is periodic if and only if the ratios of the periods are rational numbers (i.e., T_1/T_k is rational for $k = 2,3,\ldots,N$). If the sum is periodic, then the fundamental period is simply the least common multiple of $\{T_1, T_2, \ldots, T_N\}$. (Note that the **least common multiple** of the set of positive real numbers $\{T_1, T_2, \ldots, T_N\}$ is the smallest positive real number T that is an integer multiple of each T_k for $k = 1,2,\ldots,N$.)

Example 2.2. Let $x_1(t) = \sin \pi t$ and $x_2(t) = \sin t$. Determine whether the signal $y(t) = x_1(t)+x_2(t)$ is periodic.

Solution. Denote the fundamental periods of $x_1(t)$ and $x_2(t)$ as T_1 and T_2, respectively. We then have $T_1 = 2\pi/\pi = 2$ and $T_2 = 2\pi/1 = 2\pi$. (Recall that the fundamental period of $\sin \alpha t$ is $2\pi/|\alpha|$.) Consider the quotient $T_1/T_2 = 2/(2\pi) = 1/\pi$. Since π is an irrational number, this quotient is not rational. Therefore, $y(t)$ is not periodic. ☐

Example 2.3. Let $x_1(t) = \cos(2\pi t + \frac{\pi}{4})$ and $x_2(t) = \sin(7\pi t)$. Determine whether the signal $y(t) = x_1(t)+x_2(t)$ is periodic.

Solution. Let T_1 and T_2 denote the fundamental periods of $x_1(t)$ and $x_2(t)$, respectively. Thus, we have $T_1 = \frac{2\pi}{2\pi} = 1$ and $T_2 = \frac{2\pi}{7\pi} = 2/7$. We can express T_1/T_2 as the rational number $7/2$. Therefore, $y(t)$ is periodic. Furthermore, the period is $2T_1 = 7T_2 = 2$ (since 2 and 7 are coprime). ☐

Example 2.4. Let $x_1(t) = \cos(6\pi t)$ and $x_2(t) = \sin(30\pi t)$. Determine whether the signal $y(t) = x_1(t)+x_2(t)$ is periodic.

Solution. Let T_1 and T_2 denote the fundamental periods of $x_1(t)$ and $x_2(t)$, respectively. Thus, we have $T_1 = \frac{2\pi}{6\pi} = \frac{1}{3}$ and $T_2 = \frac{2\pi}{30\pi} = \frac{1}{15}$. We can express T_1/T_2 as the rational number $5/1$ (where 5 and 1 are coprime). Therefore, $y(t)$ is periodic. Furthermore, the period is $T = 1T_1 = 5T_2 = \frac{1}{3}$ (since 5 and 1 are coprime). ☐

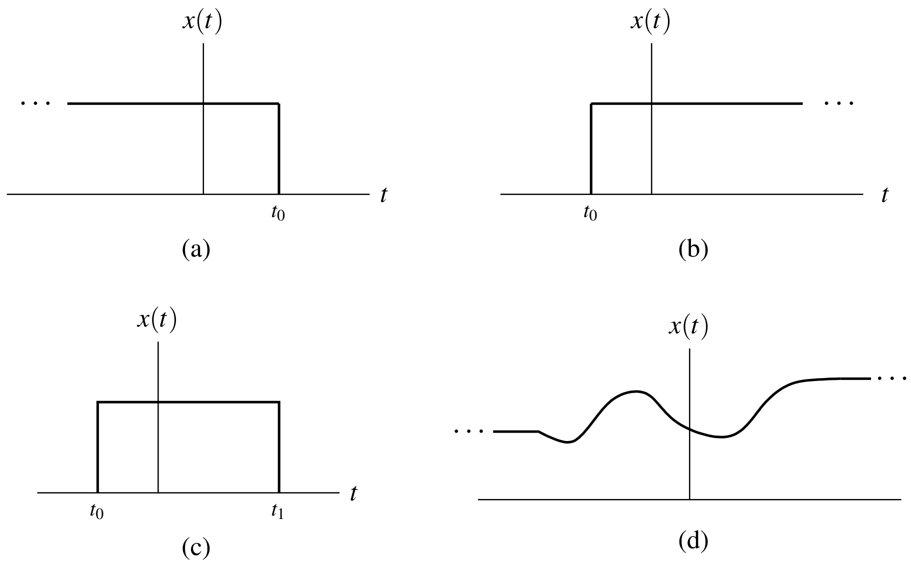


Figure 2.10: Examples of (a) left-sided, (b) right-sided, (c) finite-duration, and (d) two-sided signals.

2.3.3 Support of Signals

We can classify signals based on the interval over which their function value is nonzero. (This is sometimes referred to as the support of a signal.)

A signal $x(t)$ is said to be **left sided** if, for some finite constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t > t_0.$$

In other words, the signal is only potentially nonzero to the left of some point. A signal $x(t)$ is said to be **right sided** if, for some finite constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t < t_0.$$

In other words, the signal is only potentially nonzero to the right of some point. A signal that is both left sided and right sided is said to be **time limited** or **finite duration**. A signal that is neither left sided nor right sided is said to be **two sided**. Examples of left-sided, right-sided, finite-duration, and two-sided signals are shown in Figure 2.10.

A signal $x(t)$ is said to be **causal** if

$$x(t) = 0 \quad \text{for all } t < 0.$$

A causal signal is a special case of a right-sided signal. Similarly, a signal $x(t)$ is said to be **anticausal** if

$$x(t) = 0 \quad \text{for all } t > 0.$$

An anticausal signal is a special case of a left-sided signal.

2.3.4 Signal Energy and Power

The **energy** E contained in the signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

As a matter of terminology, a signal $x(t)$ with finite energy is said to be an **energy signal**.

The **average power** P contained in the signal $x(t)$ is given by

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt.$$

As a matter of terminology, a signal $x(t)$ with (nonzero) finite average power is said to be a **power signal**.

2.3.5 Examples

Example 2.5. Suppose that we have a signal $x(t)$ with the following properties:

$$x(t-3) \text{ is causal, and}$$
$$x(t) \text{ is odd.}$$

Determine for what values of t the signal $x(t)$ must be zero.

Solution. Since $x(t-3)$ is causal, we know that $x(t-3) = 0$ for $t < 0$. This implies that

$$x(t) = 0 \quad \text{for } t < -3. \tag{2.15}$$

Since $x(t)$ is odd, we know that

$$x(t) = -x(-t) \quad \text{for all } t. \tag{2.16}$$

From (2.15) and (2.16), we have $-x(-t) = 0$ for $t < -3$ which implies that

$$x(t) = 0 \quad \text{for } t > 3. \tag{2.17}$$

Substituting $t = 0$ into (2.16) yields $x(0) = -x(0)$ which implies that

$$x(0) = 0. \tag{2.18}$$

Combining (2.15), (2.17), and (2.18), we conclude that $x(t)$ must be zero for

$$t < -3, \quad t > 3, \quad \text{or} \quad t = 0.$$

□

Example 2.6. Suppose that we have a signal $x(t)$ with the following properties:

$$x(t-3) \text{ is even,}$$
$$x(t) = t+5 \text{ for } -5 \le t \le -3,$$
$$x(t-1) \text{ is anticausal, and}$$
$$x(t-5) \text{ is causal.}$$

Find $x(t)$ for all t.

Solution. Since $x(t-1)$ is anticausal, we know that

$$x(t-1) = 0 \quad \text{for } t > 0.$$

This implies that

$$x(t) = 0 \quad \text{for } t > -1. \tag{2.19}$$

Since $x(t-5)$ is causal, we know that

$$x(t-5) = 0 \quad \text{for } t < 0.$$

This implies that

$$x(t) = 0 \text{ for } t < -5. \tag{2.20}$$

We are given that

$$x(t) = t+5 \text{ for } -5 \leq t \leq -3. \tag{2.21}$$

Combining (2.19), (2.20), and (2.21), we have

$$x(t) = \begin{cases} 0 & \text{for } t < -5 \\ t+5 & \text{for } -5 \leq t \leq -3 \\ 0 & \text{for } t > -1. \end{cases} \tag{2.22}$$

So, we only need to determine $x(t)$ for $-3 \leq t \leq -1$. Since $x(t-3)$ is even, we know that

$$x(t-3) = x(-t-3) \text{ for all } t.$$

This implies that

$$x(t) = x(-t-6) \text{ for all } t. \tag{2.23}$$

Combining (2.21) and (2.23), we have

$$\begin{aligned} x(t) &= x(-t-6) \\ &= (-t-6)+5 \text{ for } -5 \leq -t-6 \leq -3 \\ &= -t-1 \text{ for } 1 \leq -t \leq 3 \\ &= -t-1 \text{ for } -3 \leq t \leq -1. \end{aligned}$$

Combining this with our earlier result (2.22), we have

$$x(t) = \begin{cases} 0 & \text{for } t < -5 \\ t+5 & \text{for } -5 \leq t < -3 \\ -t-1 & \text{for } -3 \leq t < -1 \\ 0 & \text{for } t \geq -1. \end{cases}$$

□

2.4 Elementary Signals

A number of elementary signals are particularly useful in the study of signals and systems. In what follows, we introduce some of the more beneficial ones for our purposes.

2.4.1 Real Sinusoidal Signals

One important class of signals is the real sinusoids. A **real sinusoidal** signal $x(t)$ has the general form

$$x(t) = A\cos(\omega t + \theta), \tag{2.24}$$

where A, ω, and θ are real constants. Such a signal is periodic with period $T = \frac{2\pi}{|\omega|}$, and has a plot resembling that shown in Figure 2.11.

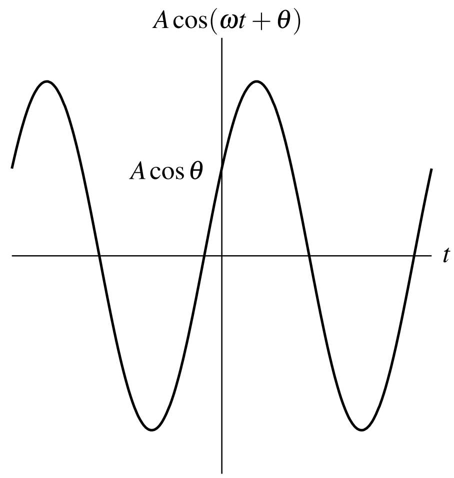


Figure 2.11: Real sinusoidal signal.

2.4.2 Complex Exponential Signals

Another important class of signals is the complex exponentials signals. A **complex exponential** signal $x(t)$ has the general form

$$x(t) = Ae^{\lambda t}, \tag{2.25}$$

where A and λ are complex constants. Complex exponentials are of fundamental importance to systems theory, and also provide a convenient means for representing a number of other classes of signals. A complex exponential can exhibit one of a number of distinctive modes of behavior, depending on the values of its parameters A and λ. In what follows, we examine some special cases of complex exponentials, in addition to the general case.

Real Exponential Signals

The first special case of the complex exponential signals to be considered is the **real exponential** signals. In this case, we restrict A and λ in (2.25) to be real. A real exponential can exhibit one of three distinct modes of behavior, depending on the value of λ, as illustrated in Figure 2.12. If $\lambda > 0$, the signal $x(t)$ increases exponentially as time increases (i.e., a growing exponential). If $\lambda < 0$, the signal $x(t)$ decreases exponentially as time increases (i.e., a decaying or damped exponential). If $\lambda = 0$, the signal $x(t)$ simply equals the constant A.

Complex Sinusoidal Signals

The second special case of the complex exponential signals that we shall consider is the **complex sinusoidal** signals. In this case, the parameters in (2.25) are such that A is complex and λ is purely imaginary (i.e., $\operatorname{Re}\{\lambda\} = 0$). For convenience, let us re-express A in polar form and λ in Cartesian form as follows:

$$A = |A|\, e^{j\theta} \quad \text{and} \quad \lambda = j\omega,$$

where θ and ω are real constants. Using Euler's relation (A.6), we can rewrite (2.25) as

$$\begin{aligned} x(t) &= Ae^{\lambda t} \\ &= |A|\, e^{j\theta} e^{j\omega t} \\ &= |A|\, e^{j(\omega t+\theta)} \\ &= |A|\cos(\omega t+\theta) + j\,|A|\sin(\omega t+\theta). \end{aligned}$$

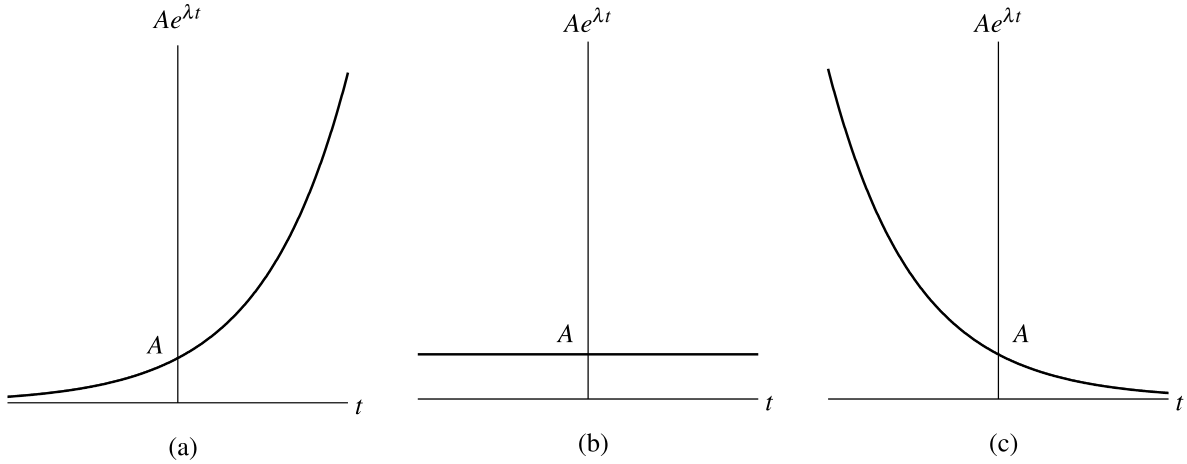


Figure 2.12: Real exponential signal for (a) $\lambda > 0$, (b) $\lambda = 0$, and (c) $\lambda < 0$.

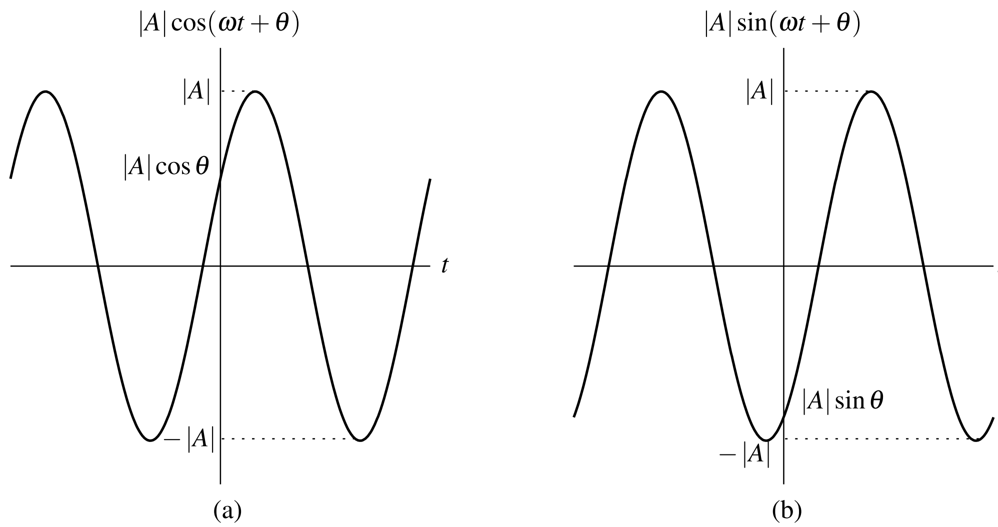


Figure 2.13: Complex sinusoidal signal. (a) Real and (b) imaginary parts.

From the above equation, we can see that $x(t)$ is periodic with period $T = \frac{2\pi}{|\omega|}$. Similarly, the real and imaginary parts of $x(t)$ are also periodic with the same period. To illustrate the form of a complex sinusoid, we plot its real and imaginary parts in Figure 2.13. The real and imaginary parts are the same except for a phase difference.

General Complex Exponential Signals

Lastly, we consider general complex exponential signals. That is, we consider the general case of (2.25) where A and λ are both complex. For convenience, let us re-express A in polar form and λ in Cartesian form as

$$A = |A|e^{j\theta} \quad \text{and} \quad \lambda = \sigma + j\omega,$$

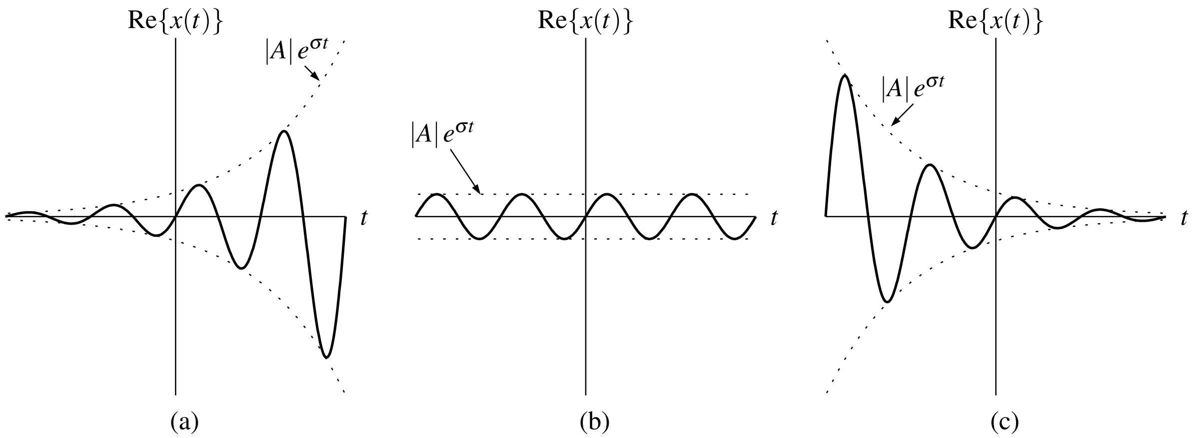


Figure 2.14: Real part of a general complex exponential for (a) $\sigma > 0$, (b) $\sigma = 0$, and (c) $\sigma < 0$.

where θ, σ, and ω are real constants. Substituting these expressions for A and λ into (2.25), we obtain

$$\begin{aligned} x(t) &= Ae^{\lambda t} \\ &= |A| e^{j\theta} e^{(\sigma + j\omega)t} \\ &= |A| e^{\sigma t} e^{j(\omega t + \theta)} \\ &= |A| e^{\sigma t} \cos(\omega t + \theta) + j |A| e^{\sigma t} \sin(\omega t + \theta). \end{aligned}$$

We can see that $\mathrm{Re}\{x(t)\}$ and $\mathrm{Im}\{x(t)\}$ have a similar form. Each is the product of a real exponential and real sinusoidal function. One of three distinct modes of behavior is exhibited by $x(t)$, depending on the value of σ. If $\sigma = 0$, $\mathrm{Re}\{x(t)\}$ and $\mathrm{Im}\{x(t)\}$ are real sinusoids. If $\sigma > 0$, $\mathrm{Re}\{x(t)\}$ and $\mathrm{Im}\{x(t)\}$ are each the product of a real sinusoid and a growing real exponential. If $\sigma < 0$, $\mathrm{Re}\{x(t)\}$ and $\mathrm{Im}\{x(t)\}$ are each the product of a real sinusoid and a decaying real exponential. These three cases are illustrated for $\mathrm{Re}\{x(t)\}$ in Figure 2.14.

2.4.3 Relationship Between Complex Exponential and Real Sinusoidal Signals

A real sinusoid can be expressed as the sum of two complex sinusoids using the identity

$$A\cos(\omega t + \theta) = \frac{A}{2}\left(e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)}\right) \quad \text{and} \tag{2.26}$$

$$A\sin(\omega t + \theta) = \frac{A}{2j}\left(e^{j(\omega t + \theta)} - e^{-j(\omega t + \theta)}\right). \tag{2.27}$$

(This result follows from Euler's relation and is simply a restatement of (A.7).)

2.4.4 Unit-Step Function

Another elementary signal often used in systems theory is the unit-step function. The **unit-step function**, denoted $u(t)$, is defined as

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0. \end{cases} \tag{2.28}$$

A plot of this function is given in Figure 2.15. Clearly, the unit-step function is discontinuous at $t = 0$. Unfortunately, there is very little agreement in the existing literature as to how to define $u(t)$ at $t = 0$. The most commonly used values are 0, 1, and 1/2. In our treatment of the subject, we simply choose to leave the value of $u(t)$ unspecified at $t = 0$ with the implicit assumption that the value (whatever it happens to be) is finite. As it turns out, for most practical purposes, it does not matter how the unit-step function is defined at the origin (as long as its value is finite). (The unit-step function is also sometimes referred to as the **Heaviside function**.)

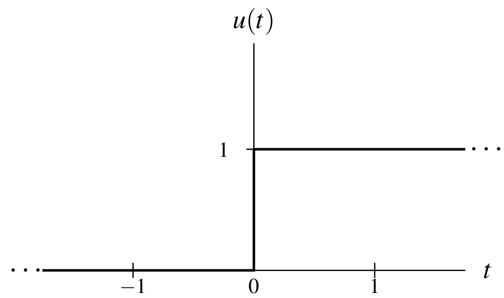


Figure 2.15: Unit-step function.

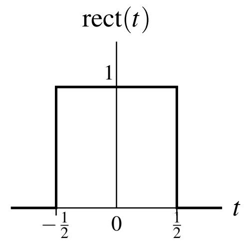


Figure 2.16: Unit rectangular pulse.

2.4.5 Unit Rectangular Pulse

Another useful signal is the unit rectangular pulse. The **unit rectangular pulse** function is denoted as $\operatorname{rect}(t)$ and is given by

$$\operatorname{rect}(t) = \begin{cases} 1 & \text{for } |t| < \frac{1}{2} \\ 0 & \text{for } |t| > \frac{1}{2}. \end{cases}$$

A plot of this signal is shown in Figure 2.16.

Example 2.7 (Extracting part of a function with a rectangular pulse)**.** Use the unit rectangular pulse to extract one period of the waveform $x(t)$ shown in Figure 2.17(a).

Solution. Let us choose to extract the period of $x(t)$ for $-\frac{T}{2} < t \leq \frac{T}{2}$. In order to extract this period, we want to multiply $x(t)$ by a function that is one over this interval and zero elsewhere. Such a function is simply $\operatorname{rect}(t/T)$ as shown in Figure 2.17(b). Multiplying $\operatorname{rect}(t/T)$ and $x(t)$ results in the function shown in Figure 2.17(c). □

2.4.6 Unit Triangular Pulse

Another useful elementary function is the **unit triangular pulse** function, which is denoted as $\operatorname{tri}(t)$ and defined as

$$\operatorname{tri}(t) = \begin{cases} 1 - 2|t| & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

A plot of this function is given in Figure 2.18.

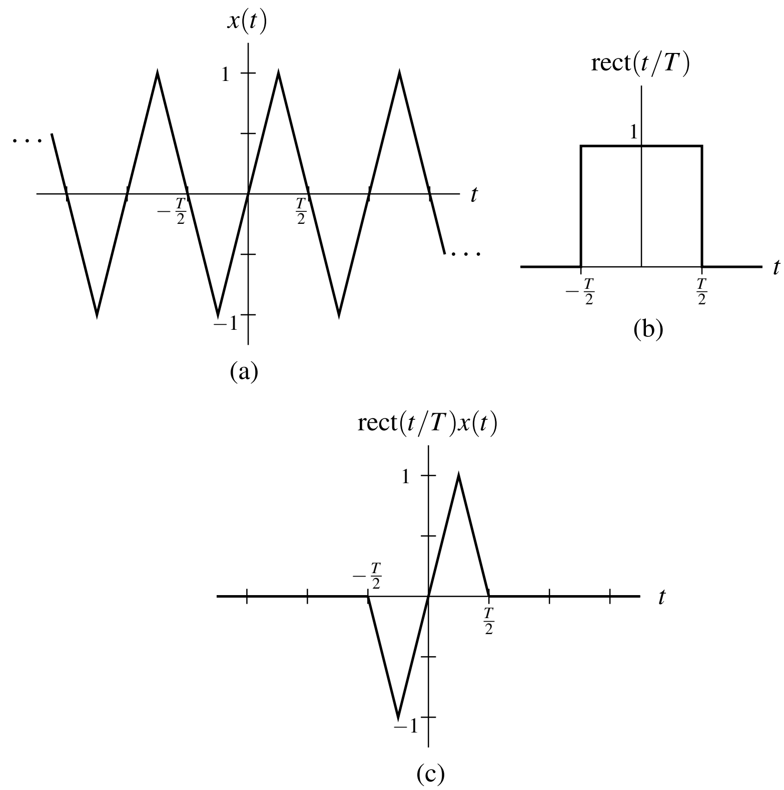


Figure 2.17: Using the unit rectangular pulse to extract one period of a periodic waveform.

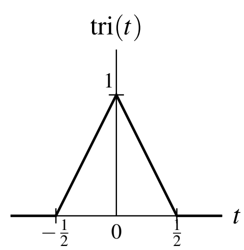


Figure 2.18: Unit triangular pulse.

2.4.7 Cardinal Sine Function

In the study of signals and systems, a function of the form

$$x(t) = \frac{\sin t}{t}$$

frequently appears. Therefore, as a matter of convenience, we give this particular function a special name, the **cardinal sine** or sinc function. That is, we define

$$\operatorname{sinc} t \triangleq \frac{\sin t}{t}.$$

(Note that the symbol "$\triangleq$" simply means "is defined as".) By using l'Hopital's rule, one can confirm that $\operatorname{sinc} t$ is well defined for $t = 0$. That is, $\operatorname{sinc} 0 = 1$. In passing, we note that the name "sinc" is simply a contraction of the function's full Latin name "sinus cardinalis" (cardinal sine).

2.4.8 Unit-Impulse Function

In systems theory, one elementary signal of fundamental importance is the unit-impulse function. We denote this function as $\delta(t)$. Instead of defining this function explicitly, it is defined in terms of its properties. In particular, the **unit-impulse function** is defined by the following two properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and} \tag{2.29a}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1. \tag{2.29b}$$

From these properties, we can see that the function is zero everywhere, except at $t = 0$ where it is undefined. Indeed, this is an unusual function. Although it is zero everywhere except at a single point, it has a nonzero integral. Technically, the unit-impulse function is not a function in the ordinary sense. Rather, it is what is known as a **generalized function**. Sometimes the unit-impulse function is also referred to as the **Dirac delta function** or **delta function**.

Graphically, we represent the impulse function as shown in Figure 2.19. Since the function assumes an infinite value at $t = 0$, we cannot plot the true value of the function. Instead, we use a vertical arrow to represent this infinite spike in the value of the signal. To show the strength of the impulse, its weight is also indicated. In Figure 2.20, we plot a scaled and shifted version of the unit impulse function.

We can view the unit-impulse function as a limiting case involving a rectangular pulse. More specifically, let us define the following rectangular pulse function

$$g(t) = \begin{cases} 1/\varepsilon & \text{for } |t| < \varepsilon/2 \\ 0 & \text{otherwise} \end{cases}$$

This function is plotted in Figure 2.21. Clearly, the area under the curve is unity for any choice of ε. The unit impulse function $\delta(t)$ is obtained by taking the following limit:

$$\delta(t) = \lim_{\varepsilon \to 0} g(t).$$

Thus, the unit-impulse function can be viewed as a limiting case of a rectangular pulse where the pulse width becomes infinitesimally small and the pulse height becomes infinitely large in such a way that the integral of the resulting function remains unity.

Informally, one can also think of the unit-impulse function $\delta(t)$ as the derivative of the unit-step function $u(t)$. Strictly speaking, however, the derivative of $u(t)$ does not exist in the ordinary sense, since $u(t)$ is discontinuous at $t = 0$. To be more precise, $\delta(t)$ is what is called the **generalized derivative** of $u(t)$. The generalized derivative is essentially an extension of the notion of (ordinary) derivative, which can be well defined even for functions with discontinuities.

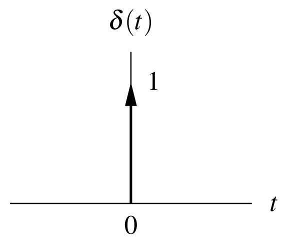


Figure 2.19: Unit impulse function.

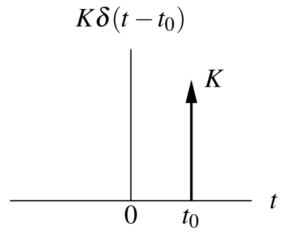


Figure 2.20: Scaled and shifted unit impulse function.

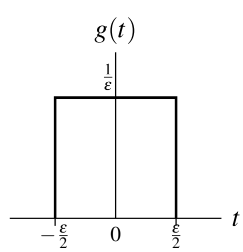


Figure 2.21: Unit-impulse function as limit of rectangular pulse.

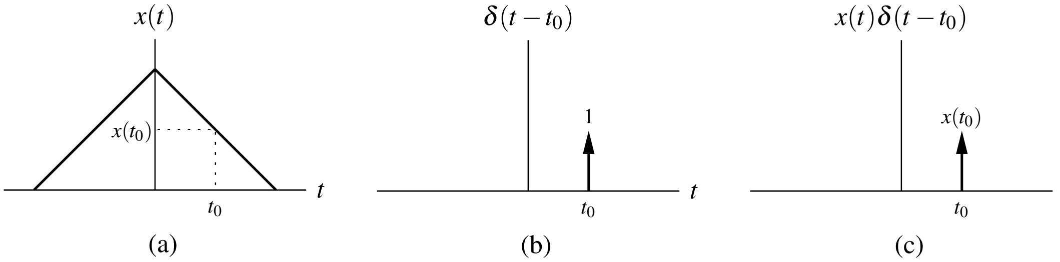


Figure 2.22: Graphical interpretation of equivalence property.

The unit-impulse function has two important properties that follow from its operational definition in (2.29). The first is sometimes referred to as the **equivalence property**. This property states that for any signal $x(t)$ that is continuous at $t = t_0$,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0). \tag{2.30}$$

This property is illustrated graphically in Figure 2.22. This result can be used to help derive the second property of interest. From (2.29b), we can write

$$\int_{-\infty}^{\infty} \delta(t-t_0)dt = 1.$$

Multiplying both sides of the preceding equation by $x(t_0)$ yields

$$\int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)dt = x(t_0).$$

Then, by using the equivalence property in (2.30), we can write

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0). \tag{2.31}$$

This result (i.e., (2.31)) is known as the **sifting property** of the unit-impulse function. As we shall see later, this property is of great importance. In passing, we also note two other identities involving the unit-impulse function which are occasionally useful:

$$\delta(t) = \delta(-t) \quad \text{and}$$
$$\delta(at) = \tfrac{1}{|a|}\delta(t),$$

where a is a nonzero real constant. (Lastly, note that it follows from the definition of $\delta(t)$ that integrating the function over any interval not containing the origin will result in the value of zero.)

Example 2.8 (Sifting property example)**.** Evaluate the integral

$$\int_{-\infty}^{\infty} [\sin t]\delta(t-\pi/4)dt.$$

Solution. Using the sifting property of the unit impulse function, we can write:

$$\begin{aligned}\int_{-\infty}^{\infty} [\sin t]\delta(t-\pi/4)dt &= \sin \pi/4 \\ &= \tfrac{1}{\sqrt{2}}.\end{aligned}$$

□

Example 2.9 (Sifting property example)**.** Evaluate the integral

$$\int_{-\infty}^{\infty} [\sin 2\pi t]\delta(4t-1)dt.$$

Solution. First, we observe that the integral to be evaluated does not quite have the same form as (2.31). So, we need to perform a change of variable. Let $\tau = 4t$ so that $t = \tau/4$ and $dt = d\tau/4$. Performing the change of variable, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}[\sin 2\pi t]\delta(4t-1)dt &= \int_{-\infty}^{\infty}\tfrac{1}{4}[\sin 2\pi\tau/4]\delta(\tau-1)d\tau \\ &= \int_{-\infty}^{\infty}\left[\tfrac{1}{4}\sin\pi\tau/2\right]\delta(\tau-1)d\tau.\end{aligned}$$

Now the integral has the desired form, and we can use the sifting property of the unit-impulse function to write

$$\begin{aligned}\int_{-\infty}^{\infty}[\sin 2\pi t]\delta(4t-1)dt &= \left[\tfrac{1}{4}\sin\pi\tau/2\right]\big|_{\tau=1} \\ &= \tfrac{1}{4}\sin\pi/2 \\ &= \tfrac{1}{4}.\end{aligned}$$

□

Example 2.10. Evaluate the integral $\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau$.

Solution. Using the equivalence property of the delta function given by (2.30), we can write

$$\begin{aligned}\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau &= \int_{-\infty}^{t}(2^2+1)\delta(\tau-2)d\tau \\ &= 5\int_{-\infty}^{t}\delta(\tau-2)d\tau.\end{aligned}$$

Using the defining properties of the delta function given by (2.29), we have that

$$\int_{-\infty}^{t}\delta(\tau-2)d\tau = \begin{cases}1 & \text{for } t>2 \\ 0 & \text{for } t<2.\end{cases}$$

Therefore, we conclude that

$$\begin{aligned}\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau &= \begin{cases}5 & \text{for } t>2 \\ 0 & \text{for } t<2\end{cases} \\ &= 5u(t-2).\end{aligned}$$

□

2.5 Signal Representation Using Elementary Signals

In the earlier sections, we introduced a number of elementary signals. Often in signal analysis, it is convenient to represent arbitrary signals in terms of elementary signals. Here, we consider how the unit-step function can be exploited in order to obtain alternative representations of signals.

Example 2.11 (Unit rectangular pulse)**.** Express the function $y(t) = \text{rect}(t)$ in terms of unit-step functions. A plot of $y(t)$ is given in Figure 2.23(c).

Solution. We observe that the unit rectangular pulse is simply the difference of two time-shifted unit-step functions. In particular, we have that

$$y(t) = u(t+\tfrac{1}{2}) - u(t-\tfrac{1}{2}).$$

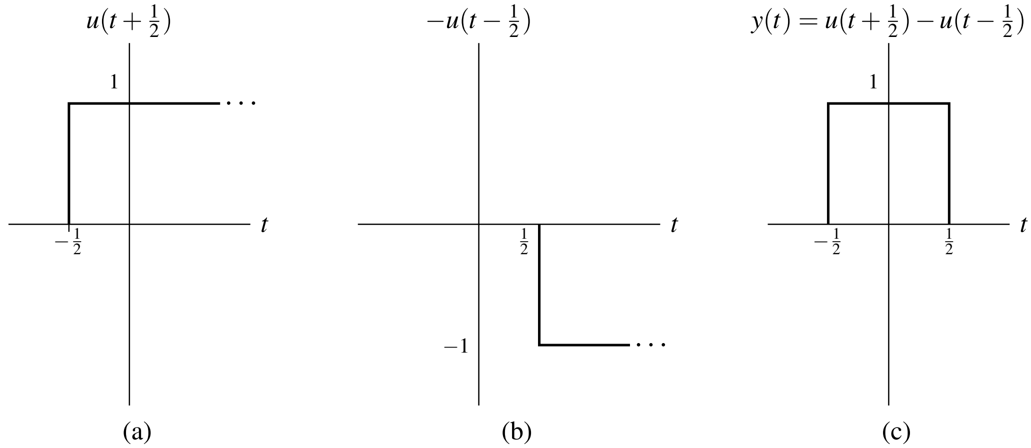


Figure 2.23: Representing the unit rectangular pulse using unit-step functions. (a) A shifted unit-step function, (b) another shifted unit-step function, and (c) their sum (which is the rectangular pulse).

Graphically, we have the scenario depicted in Figure 2.23.

In passing, we note that the above result can be generalized. Suppose that we have a rectangular pulse $x(t)$ of height 1 with a rising edge at $t = a$ and falling edge at $t = b$. One can show that

$$\begin{aligned} x(t) &= u(t-a) - u(t-b) \\ &= \begin{cases} 1 & \text{for } a < t < b \\ 0 & \text{for } t < a \text{ or } t > b. \end{cases} \end{aligned}$$

□

Example 2.12 (Piecewise linear function)**.** Using unit-step functions, find an equivalent representation of the following function:

$$x(t) = \begin{cases} t & \text{for } 0 \le t < 1 \\ 1 & \text{for } 1 \le t < 2 \\ 3-t & \text{for } 2 \le t < 3 \\ 0 & \text{otherwise.} \end{cases}$$

A plot of $x(t)$ can be found in Figure 2.24(a).

Solution. We consider each segment of the piecewise linear function separately. The first segment (i.e., for $0 \le t < 1$) can be expressed as

$$v_1(t) = t[u(t) - u(t-1)].$$

This function is plotted in Figure 2.24(b). The second segment (i.e., for $1 \le t < 2$) can be expressed as

$$v_2(t) = u(t-1) - u(t-2).$$

This function is plotted in Figure 2.24(c). The third segment (i.e., for $2 \le t < 3$) can be expressed as

$$v_3(t) = (3-t)[u(t-2) - u(t-3)].$$

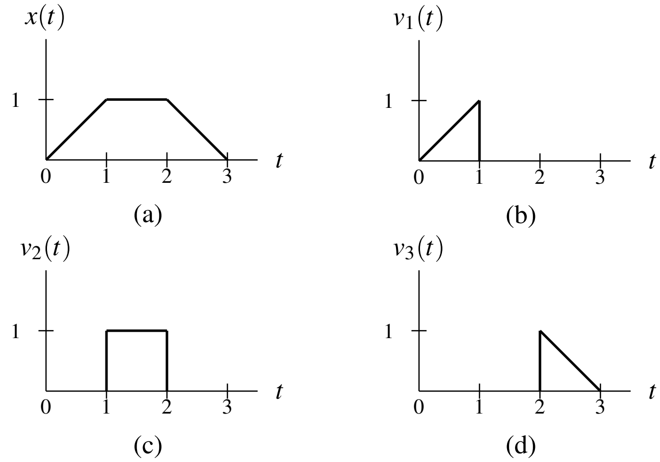


Figure 2.24: Representing piecewise linear function using unit-step functions.

This function is plotted in Figure 2.24(d). Now, we observe that $x(t)$ is simply the sum of the three preceding functions. That is, we have

$$\begin{aligned} x(t) &= v_1(t) + v_2(t) + v_3(t) \\ &= t[u(t) - u(t-1)] + [u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)] \\ &= tu(t) + (1-t)u(t-1) + (3-t-1)u(t-2) + (t-3)u(t-3) \\ &= tu(t) + (1-t)u(t-1) + (2-t)u(t-2) + (t-3)u(t-3). \end{aligned}$$

Thus, we have found an alternative representation of $x(t)$ that uses unit-step functions. □

Example 2.13 (Piecewise polynomial function)**.** Find an alternative representation of the following function by using unit-step functions:

$$x(t) = \begin{cases} 1 & \text{for } 0 \le t < 1 \\ (t-2)^2 & \text{for } 1 \le t < 3 \\ 4-t & \text{for } 3 \le t < 4 \\ 0 & \text{otherwise.} \end{cases}$$

A plot of $x(t)$ is shown in Figure 2.25(a).

Solution. We consider each segment of the piecewise polynomial function separately. The first segment (i.e., for $0 \le t < 1$) can be written as

$$v_1(t) = u(t) - u(t-1).$$

This function is plotted in Figure 2.25(b). The second segment (i.e., for $1 \le t < 3$) can be written as

$$v_2(t) = (t-2)^2[u(t-1) - u(t-3)] = (t^2 - 4t + 4)[u(t-1) - u(t-3)].$$

This function is plotted in Figure 2.25(c). The third segment (i.e., for $3 \le t < 4$) can be written as

$$v_3(t) = (4-t)[u(t-3) - u(t-4)].$$

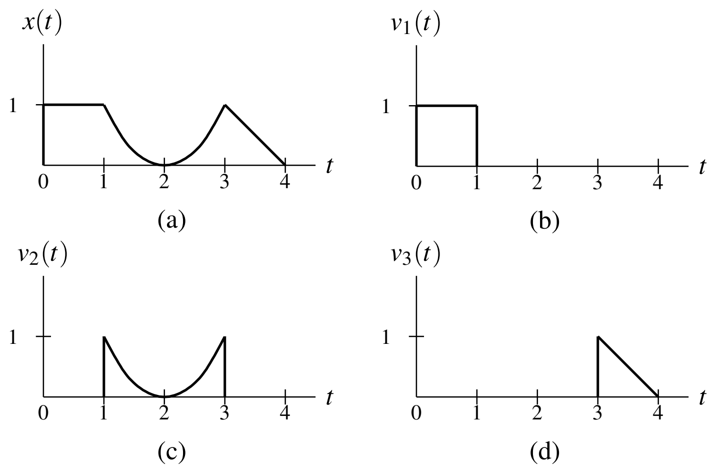


Figure 2.25: Representing piecewise polynomial function using unit-step functions.

This function is plotted in Figure 2.25(d). Now, we observe that $x(t)$ is obtained by summing the three preceding functions as follows:

$$\begin{aligned}
x(t) &= v_1(t) + v_2(t) + v_3(t) \\
&= [u(t) - u(t-1)] + (t^2 - 4t + 4)[u(t-1) - u(t-3)] + (4-t)[u(t-3) - u(t-4)] \\
&= u(t) + (t^2 - 4t + 4 - 1)u(t-1) + (4 - t - [t^2 - 4t + 4])u(t-3) - (4-t)u(t-4) \\
&= u(t) + (t^2 - 4t + 3)u(t-1) + (-t^2 + 3t)u(t-3) + (t-4)u(t-4).
\end{aligned}$$

Thus, we have found an alternative representation of $x(t)$ that utilizes unit-step functions. □

Example 2.14 (Periodic function)**.** Consider the periodic function $x(t)$ shown in Figure 2.26(a). Find a representation of $x(t)$ that utilizes unit-step functions.

Solution. We begin by finding an expression for a single period of $x(t)$. Let us denote this expression as $v(t)$. We can then write:

$$v(t) = u(t + \tfrac{1}{2}) - u(t - \tfrac{1}{2}).$$

This function is plotted in Figure 2.26(b). In order to obtain the periodic function $x(t)$, we must repeat $v(t)$ every two units (since the period of $x(t)$ is two). This can be accomplished by adding an infinite number of shifted copies of $v(t)$ as follows:

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} v(t - 2k) \\
&= \sum_{k=-\infty}^{\infty} [u(t + \tfrac{1}{2} - 2k) - u(t - \tfrac{1}{2} - 2k)].
\end{aligned}$$

Thus, we have found a representation of the periodic function $x(t)$ that makes use of unit-step functions. □

2.6 Continuous-Time Systems

Suppose that we have a system with input $x(t)$ and output $y(t)$. Such a system can be described mathematically by the equation

$$y(t) = \mathscr{T}\{x(t)\}, \tag{2.32}$$

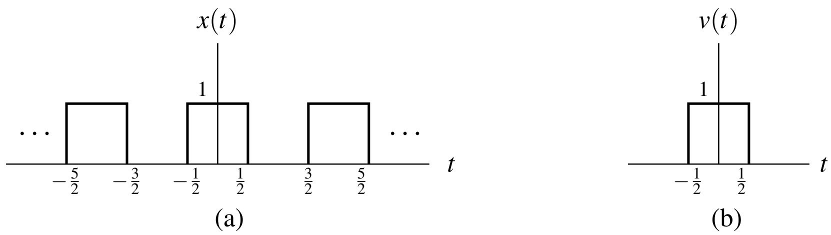


Figure 2.26: Representing periodic function using unit-step functions.

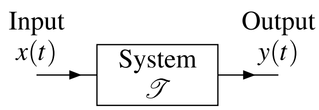


Figure 2.27: Block diagram of system.

where $\mathscr{T}$ denotes an operator (i.e., transformation). The operator $\mathscr{T}$ simply maps the input signal $x(t)$ to the output signal $y(t)$. Such an operator might be associated with a system of differential equations, for example.

Alternatively, we sometimes express the relationship (2.32) using the notation

$$x(t) \xrightarrow{\mathscr{T}} y(t).$$

Furthermore, if clear from the context, the operator $\mathscr{T}$ is often omitted, yielding the abbreviated notation

$$x(t) \to y(t).$$

Note that the symbols "$\to$" and "=" have very different meanings. For example, the notation $x(t) \to y(t)$ does not in any way imply that $x(t) = y(t)$. The symbol "$\to$" should be read as "produces" (not as "equals"). That is, "$x(t) \to y(t)$" should be read as "the input $x(t)$ produces the output $y(t)$".

2.6.1 Block Diagram Representation

Suppose that we have a system defined by the operator $\mathscr{T}$ and having the input $x(t)$ and output $y(t)$. Often, we represent such a system using a block diagram as shown in Figure 2.27.

2.6.2 Interconnection of Systems

Systems may be interconnected in a number of ways. Two basic types of connections are as shown in Figure 2.28. The first type of connection, as shown in Figure 2.28(a), is known as a **series** or **cascade** connection. In this case, the overall system is defined by

$$y(t) = \mathscr{T}_2\left\{\mathscr{T}_1\{x(t)\}\right\}. \tag{2.33}$$

The second type of connection, as shown in Figure 2.28(b), is known as a **parallel** connection. In this case, the overall system is defined by

$$y(t) = \mathscr{T}_1\{x(t)\} + \mathscr{T}_2\{x(t)\}. \tag{2.34}$$

The system equations in (2.33) and (2.34) cannot be simplified further unless the definitions of the operators $\mathscr{T}_1$ and $\mathscr{T}_2$ are known.

2.7 Properties of Continuous-Time Systems

In what follows, we will define a number of important properties that a system may possess. These properties are useful in classifying systems, as well as characterizing their behavior.

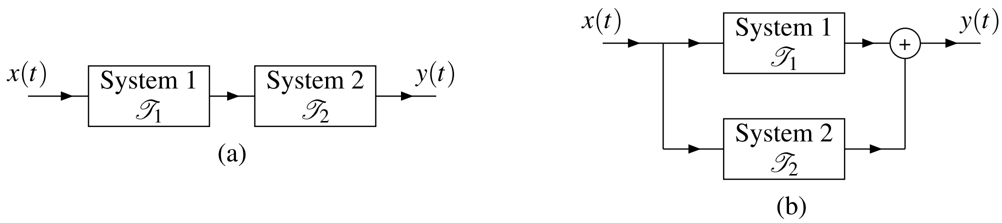


Figure 2.28: Interconnection of systems. (a) Series and (b) parallel interconnections.

2.7.1 Memory

A system is said to have **memory** if its output $y(t)$ at any arbitrary time t_0 depends on the value of its input $x(t)$ at any time other than $t = t_0$. If a system does not have memory, it is said to be **memoryless**.

Example 2.15. A system with input $x(t)$ and output $y(t)$ is characterized by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Such a system is commonly referred to as an integrator. Determine whether this system has memory.

Solution. Consider the calculation of $y(t)$ for any arbitrary time $t = t_0$. For $t = t_0$, $y(t)$ depends on $x(t)$ for $-\infty < t \leq t_0$. Thus, $y(t)|_{t=t_0}$ is dependent on $x(t)$ for some $t \neq t_0$. Therefore, this system has memory. □

Example 2.16. A system with input $x(t)$ and output $y(t)$ is characterized by the equation

$$y(t) = Ax(t),$$

where A is a real constant. This system is known as an ideal amplifier. Determine whether this system has memory.

Solution. Consider the calculation of $y(t)$ for any arbitrary time $t = t_0$. The quantity $y(t_0)$ depends on $x(t)$ only for $t = t_0$. Therefore, the system is memoryless. □

2.7.2 Causality

A system is said to be **causal** if its output $y(t)$ at any arbitrary time t_0 depends only on the values of its input $x(t)$ for $t \leq t_0$.

If the independent variable represents time, a system must be causal in order to be physically realizable. Noncausal systems can sometimes be useful in practice, however, as the independent variable need not always represent time.

Example 2.17. A system with input $x(t)$ and output $y(t)$ is described by

$$y(t) = x(-t).$$

Determine whether this system is causal.

Solution. For $t < 0$, $y(t)$ depends on $x(t)$ for $t > 0$. So, for example, if $t = -1$, $y(t)$ depends on $x(1)$. In this case, we have that $y(-1)$ depends on $x(t)$ for some $t > -1$ (e.g., for $t = 1$). Therefore, the system is noncausal. □

Example 2.18. A system with input $x(t)$ and output $y(t)$ is characterized by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Determine whether this system is causal.

Solution. Consider the calculation of $y(t_0)$ for arbitrary t_0. We can see that $y(t_0)$ depends only on $x(t)$ for $-\infty < t \leq t_0$. Therefore, this system is causal. □

2.7.3 Invertibility

A system is said to be **invertible** if its input $x(t)$ can always be uniquely determined from its output $y(t)$. From this definition, it follows that an invertible system will always produce distinct outputs from any two distinct inputs.

If a system is invertible, this is most easily demonstrated by finding the inverse system. If a system is not invertible, often the easiest way to prove this is to show that two distinct inputs result in identical outputs.

Example 2.19. A system with input $x(t)$ and output $y(t)$ is described by the equation

$$y(t) = x(t - t_0),$$

where t_0 is a real constant. Determine whether this system is invertible.

Solution. Let $\tau \triangleq t - t_0$ which implies that $t = \tau + t_0$. Using these two relationships, we can rewrite (2.19) as

$$y(\tau + t_0) = x(\tau).$$

Thus, we have solved for $x(t)$ in terms of $y(t)$. Therefore, the system is invertible. □

Example 2.20. A system with input $x(t)$ and output $y(t)$ is defined by the equation

$$y(t) = \sin[x(t)].$$

Determine whether this system is invertible.

Solution. Consider an input of the form $x(t) = 2\pi k$ where k can assume any integer value. The response to such an input is given by

$$y(t) = \sin[x(t)] = \sin 2\pi k = 0.$$

Thus, we have found an infinite number of distinct inputs (i.e., $x(t) = 2\pi k$ for $k = 0, \pm 1, \pm 2, \ldots$) that all result in the same output. Therefore, the system is not invertible. □

2.7.4 Stability

Although stability can be defined in numerous ways, in systems theory, we are often most interested in bounded-input bounded-output (BIBO) stability.

A system having the input $x(t)$ and output $y(t)$ is **BIBO stable** if, for any $x(t)$, $|x(t)| \le A < \infty$ for all t implies that $|y(t)| \le B < \infty$ for all t. In other words, a system is BIBO stable if a bounded input always produces a bounded output.

To prove that a system is BIBO stable, we must show that every bounded input leads to a bounded output. To show that a system is not BIBO stable, we simply need to find one counterexample (i.e., a single bounded input that leads to an unbounded output).

Example 2.21. A system is characterized by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

where $x(t)$ and $y(t)$ denote the system input and output, respectively. Determine whether this system is BIBO stable.

Solution. Suppose now that we choose the input to be the unit-step function. That is, suppose that $x(t) = u(t)$. Clearly, $u(t)$ is a bounded function for all t (i.e., $|u(t)| \le 1$ for all t). We can calculate the system response to this input as follows:

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} u(\tau)d\tau \\ &= \int_{0}^{t} d\tau \\ &= [\tau]|_0^t \\ &= t. \end{aligned}$$

From this result, however, we can see that as $t \to \infty$, $y(t) \to \infty$. Thus, the output becomes unbounded for arbitrarily large t, in spite of the input being bounded. Therefore, the system is unstable. □

Example 2.22. A system is described by the equation

$$y(t) = x^2(t),$$

where $x(t)$ and $y(t)$ denote the system input and output, respectively. Determine whether this system is BIBO stable.

Solution. If $|x(t)| \leq A < \infty$, then from the above equation it follows that $|y(t)| = \left|x^2(t)\right| \leq A^2$. Thus, if the input is bounded, the output is also bounded. Therefore, the system is stable. □

2.7.5 Time Invariance

Let $y(t)$ denote the response of a system to the input $x(t)$, and let t_0 denote a shift constant. If, for any choice of $x(t)$ and t_0, the input $x(t-t_0)$ produces the output $y(t-t_0)$, the system is said to be **time invariant**. In other words, a system is time invariant, if a time shift (i.e., advance or delay) in the input signal results in an identical time shift in the output signal.

Example 2.23. A system with input $x(t)$ and output $y(t)$ is characterized by the equation

$$y(t) = tx(t).$$

Determine whether this system is time invariant.

Solution. Let $y_1(t)$ and $y_2(t)$ denote the responses of the system to the inputs $x_1(t)$ and $x_2(t) = x_1(t-t_0)$, respectively, where t_0 is a real constant. We can easily deduce that

$$y_1(t) = tx_1(t) \quad \text{and}$$
$$y_2(t) = tx_2(t) = tx_1(t-t_0).$$

So, we have that

$$y_1(t-t_0) = (t-t_0)x_1(t-t_0).$$

Since $y_2(t) \neq y_1(t-t_0)$, the system is not time invariant (i.e., the system is time varying). □

Example 2.24. A system with input $x(t)$ and output $y(t)$ is characterized by the equation

$$y(t) = \sin[x(t)].$$

Determine whether this system is time invariant.

Solution. Let $y_1(t)$ and $y_2(t)$ denote the responses of the system to the inputs $x_1(t)$ and $x_2(t) = x_1(t-t_0)$, respectively, where t_0 is a real constant. We can easily deduce that

$$y_1(t) = \sin[x_1(t)] \quad \text{and}$$
$$y_2(t) = \sin x_2(t) = \sin[x_1(t-t_0)].$$

So, we have that

$$y_1(t-t_0) = \sin[x_1(t-t_0)].$$

Since $y_2(t) = y_1(t-t_0)$, the system is time invariant. □

2.7.6 Linearity

Let $y_1(t)$ and $y_2(t)$ denote the responses of a system to the inputs $x_1(t)$ and $x_2(t)$, respectively. If, for any choice of $x_1(t)$ and $x_2(t)$, the response to the input $x_1(t)+x_2(t)$ is $y_1(t)+y_2(t)$, the system is said to possess the **additivity** property.

Let $y(t)$ denote the response of a system to the input $x(t)$, and let a denote a complex constant. If, for any choice of $x(t)$ and a, the response to the input $ax(t)$ is $ay(t)$, the system is said to possess the **homogeneity** property.

If a system possesses both the additivity and homogeneity properties, it is said to be **linear**. Otherwise, it is said to be **nonlinear**.

The two linearity conditions (i.e., additivity and homogeneity) can be combined into a single condition known as superposition. Let $y_1(t)$ and $y_2(t)$ denote the responses of a system to the inputs $x_1(t)$ and $x_2(t)$, respectively, and let a and b denote complex constants. If, for any choice of $x_1(t)$, $x_2(t)$, a, and b, the input $ax_1(t)+bx_2(t)$ produces the response $ay_1(t)+by_2(t)$, the system is said to possess the **superposition** property.

To show that a system is linear, we can show that it possesses both the additivity and homogeneity properties, or we can simply show that the superposition property holds.

Example 2.25. A system with input $x(t)$ and output $y(t)$ is characterized by the equation

$$y(t) = tx(t).$$

Determine whether this system is linear.

Solution. Let $y_1(t)$ and $y_2(t)$ denote the responses of the system to the inputs $x_1(t)$ and $x_2(t)$, respectively. Let $y_3(t)$ denote the response to the input $ax_1(t)+bx_2(t)$ where a and b are complex constants. From the definition of the system, we can write

$$\begin{gathered} y_1(t) = tx_1(t), \\ y_2(t) = tx_2(t), \quad \text{and} \\ y_3(t) = t(ax_1(t)+bx_2(t)) = atx_1(t)+btx_2(t) = ay_1(t)+by_2(t). \end{gathered}$$

Since $y_3(t) = ay_1(t)+by_2(t)$, the superposition property holds and the system is linear. □

Example 2.26. A system is defined by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

where $x(t)$ and $y(t)$ denote the system input and output, respectively. Determine whether this system is additive and/or homogeneous. Determine whether this system is linear.

Solution. First, we consider the additivity property. Let $y_1(t)$, $y_2(t)$, and $y_3(t)$ denote the system responses to the inputs $x_1(t)$, $x_2(t)$, and $x_1(t)+x_2(t)$, respectively. Thus, we have

$$\begin{gathered} y_1(t) = \int_{-\infty}^{t} x_1(\tau)d\tau, \\ y_2(t) = \int_{-\infty}^{t} x_2(\tau)d\tau, \quad \text{and} \end{gathered}$$

$$\begin{aligned} y_3(t) &= \int_{-\infty}^{t} [x_1(\tau)+x_2(\tau)]d\tau \\ &= \int_{-\infty}^{t} x_1(\tau)d\tau + \int_{-\infty}^{t} x_2(\tau)d\tau \\ &= y_1(t)+y_2(t). \end{aligned}$$

Since $y_3(t) = y_1(t) + y_2(t)$, the system possesses the additivity property.

Second, we consider the homogeneity property. Let $y_1(t)$ and $y_2(t)$ denote the system responses to the inputs $x_1(t)$ and $ax_1(t)$ where a is a complex constant. Thus, we can write

$$y_1(t) = \int_{-\infty}^{t} x_1(\tau)d\tau \quad \text{and}$$

$$\begin{aligned} y_2(t) &= \int_{-\infty}^{t} ax_1(\tau)d\tau \\ &= a\int_{-\infty}^{t} x_1(\tau)d\tau \\ &= ay_1(t). \end{aligned}$$

Since $y_2(t) = ay_1(t)$, the system has the homogeneity property.

Lastly, we consider the linearity property. The system is linear since it has both the additivity and homogeneity properties. □

Example 2.27. A system with input $x(t)$ and output $y(t)$ (where $x(t)$ and $y(t)$ are complex valued) is defined by the equation

$$y(t) = \operatorname{Re}\{x(t)\}.$$

Determine whether this system is additive and/or homogeneous. Determine whether this system is linear.

Solution. First, we check if the additivity property is satisfied. Let $y_1(t)$, $y_2(t)$, and $y_3(t)$ denote the system responses to the inputs $x_1(t)$, $x_2(t)$, and $x_1(t)+x_2(t)$, respectively. Thus, we have

$$\begin{aligned} y_1(t) &= \operatorname{Re}\{x_1(t)\}, \\ y_2(t) &= \operatorname{Re}\{x_2(t)\}, \quad \text{and} \end{aligned}$$

$$\begin{aligned} y_3(t) &= \operatorname{Re}\{x_1(t)+x_2(t)\} \\ &= \operatorname{Re}\{x_1(t)\} + \operatorname{Re}\{x_2(t)\} \\ &= y_1(t)+y_2(t). \end{aligned}$$

Since $y_3(t) = y_1(t)+y_2(t)$, the system has the additivity property.

Second, we check if the homogeneity property is satisfied. Let $y_1(t)$ and $y_2(t)$ denote the system responses to the inputs $x_1(t)$ and $ax_1(t)$ where a is a complex constant. Thus, we have

$$\begin{aligned} y_1(t) &= \operatorname{Re} x_1(t), \\ y_2(t) &= \operatorname{Re}\{ax_1(t)\}, \quad \text{and} \\ ay_1(t) &= a\operatorname{Re}\{x_1(t)\}. \end{aligned}$$

In order for this system to possess the homogeneity property, $ay_1(t) = y_2(t)$ must hold for any complex a. Suppose that $a = j$. In this case, we have

$$\begin{aligned} y_2(t) &= \operatorname{Re}\{jx_1(t)\} \\ &= \operatorname{Re}\{j[\operatorname{Re}\{x_1(t)\} + j\operatorname{Im}\{x_1(t)\}]\} \\ &= \operatorname{Re}\{-\operatorname{Im}\{x_1(t)\} + j\operatorname{Re}\{x_1(t)\}\} \\ &= -\operatorname{Im}\{x_1(t)\}, \end{aligned}$$

and

$$ay_1(t) = j\operatorname{Re}\{x_1(t)\}.$$

Thus, the quantities $y_2(t)$ and $ay_1(t)$ are clearly not equal. Therefore, the system does not possess the homogeneity property.

Lastly, we consider the linearity property. Since the system does not possess both the additivity and homogeneity properties, it is not linear. □

2.7.7 Examples

Example 2.28. Suppose that we have the system with the input $x(t)$ and output $y(t)$ given by

$$y(t) = \operatorname{Odd}\{x(t)\} = \tfrac{1}{2}(x(t) - x(-t)).$$

Determine whether the system has each of the following properties: (a) memory, (b) causal, (c) invertible, (d) BIBO stable, (e) time invariant, (f) linear.

Solution. (a) MEMORY. For any $x(t)$ and any real t_0, we have that $y(t)|_{t=t_0}$ depends on $x(t)$ for $t = t_0$ and $t = -t_0$. Since $y(t)|_{t=t_0}$ depends on $x(t)$ for $t \neq t_0$, the system has memory (i.e., the system is not memoryless).

(b) CAUSALITY. For any $x(t)$ and any real constant t_0, we have that $y(t)|_{t=t_0}$ depends only on $x(t)$ for $t = t_0$ and $t = -t_0$. Suppose that $t_0 = -1$. In this case, we have that $y(t)|_{t=t_0}$ (i.e., $y(-1)$) depends on $x(t)$ for $t = 1$ but $t = 1 > t_0$. Therefore, the system is not causal.

(c) INVERTIBILITY. Consider the response $y(t)$ of the system to an input $x(t)$ of the form

$$x(t) = \alpha$$

where α is a real constant. We have that

$$\begin{aligned} y(t) &= \tfrac{1}{2}[x(t) - x(-t)] \\ &= \tfrac{1}{2}(\alpha - \alpha) \\ &= 0. \end{aligned}$$

Therefore, any constant input yields the same zero output. This, however, implies that distinct inputs can yield identical outputs. Therefore, the system is not invertible.

(d) STABILITY. Suppose that $x(t)$ is bounded. Then, $x(-t)$ is also bounded. Since the difference of two bounded functions is bounded, $x(t) - x(-t)$ is bounded. Multiplication of a bounded function by a finite constant yields a bounded result. So, the function $\frac{1}{2}[x(t) - x(-t)]$ is bounded. Thus, $y(t)$ is bounded. Since a bounded input must yield a bounded output, the system is BIBO stable.

(e) TIME INVARIANCE. Suppose that

$$\begin{aligned} x_1(t) &\to y_1(t) \quad \text{and} \\ x_2(t) = x_1(t - t_0) &\to y_2(t). \end{aligned}$$

The system is time invariant if, for any $x_1(t)$ and any real constant t_0, $y_2(t) = y_1(t - t_0)$. From the definition of the system, we have

$$y_1(t) = \tfrac{1}{2}[x_1(t) - x_1(-t)],$$

$$\begin{aligned} y_2(t) &= \tfrac{1}{2}[x_2(t) - x_2(-t)] \\ &= \tfrac{1}{2}[x_1(t - t_0) - x_1(-t - t_0)], \quad \text{and} \end{aligned}$$

$$\begin{aligned} y_1(t - t_0) &= \tfrac{1}{2}[x_1(t - t_0) - x_1(-(t - t_0))] \\ &= \tfrac{1}{2}[x_1(t - t_0) - x_1(t_0 - t)]. \end{aligned}$$

Since $y_2(t) \neq y_1(t - t_0)$, the system is not time invariant.

(f) LINEARITY. Suppose that

$$\begin{aligned} x_1(t) &\to y_1(t), \\ x_2(t) &\to y_2(t), \quad \text{and} \\ a_1x_1(t) + a_2x_2(t) &\to y_3(t). \end{aligned}$$

The system is linear if, for any $x_1(t)$ and $x_2(t)$ and any complex constants a_1 and a_2, $y_3(t) = a_1y_1(t) + a_2y_2(t)$. From the definition of the system, we have

$$y_1(t) = \tfrac{1}{2}[x_1(t) - x_1(-t)],$$

$$y_2(t) = \tfrac{1}{2}[x_2(t) - x_2(-t)], \quad \text{and}$$

$$\begin{aligned} y_3(t) &= \tfrac{1}{2}[a_1x_1(t) + a_2x_2(t) - [a_1x_1(-t) + a_2x_2(-t)]] \\ &= \tfrac{1}{2}[a_1x_1(t) - a_1x_1(-t) + a_2x_2(t) - a_2x_2(-t)] \\ &= a_1[\tfrac{1}{2}(x_1(t) - x_1(-t))] + a_2[\tfrac{1}{2}(x_2(t) - x_2(-t))] \\ &= a_1y_1(t) + a_2y_2(t). \end{aligned}$$

Since $y_3(t) = a_1y_1(t) + a_2y_2(t)$, the system is linear. □

Example 2.29. Suppose that we have the system with input $x(t)$ and output $y(t)$ given by

$$y(t) = 3x(3t + 3).$$

Determine whether the system has each of the following properties: (a) memory, (b) causal, (c) invertible, (d) BIBO stable, (e) time invariant, (f) linear.

Solution. (a) MEMORY. For any $x(t)$ and any real t_0, we have that $y(t)|_{t=t_0}$ depends on $x(t)$ for $t = 3t_0 + 3$. Since $y(t)|_{t=t_0}$ depends on $x(t)$ for $t \neq t_0$, the system has memory (i.e., the system is not memoryless).

(b) CAUSALITY. For any $x(t)$ and any real constant t_0, we have that $y(t)|_{t=t_0}$ depends only on $x(t)$ for $t = 3t_0 + 3$. Suppose that $t_0 = 0$. In this case, $y(t)|_{t=t_0}$ depends on $x(t)$ for $t = 3$, but $t = 3 > t_0$. Therefore, the system is not causal.

(c) INVERTIBILITY. From the definition of the system, we can write

$$\begin{aligned} y(t) = 3x(3t + 3) &\Rightarrow y(\tfrac{1}{3}\tau - 1) = 3x(\tau) \\ &\Rightarrow x(\tau) = \tfrac{1}{3}y(\tfrac{1}{3}\tau - 1). \end{aligned}$$

Thus, we have just solved for $x(t)$ in terms of $y(t)$. Therefore, an inverse system exists. Consequently, the system is invertible.

(d) STABILITY. Suppose that $x(t)$ is bounded such that $|x(t)| \leq A < \infty$ (for all t). Then, $|x(3t+3)| \leq A$. Furthermore, we have that $|3x(3t+3)| = 3\,|x(3t+3)|$. So, $|3x(3t+3)| \leq 3A$. Therefore, $|y(t)| \leq 3A$ (i.e., $y(t)$ is bounded). Consequently, the system is BIBO stable.

(e) TIME INVARIANCE. Suppose that

$$x_1(t) \to y_1(t) \quad \text{and}$$
$$x_2(t) = x_1(t - t_0) \to y_2(t).$$

The system is time invariant if, for any $x_1(t)$ and any real constant t_0, $y_2(t) = y_1(t - t_0)$. From the definition of the system, we have

$$y_1(t) = 3x_1(3t + 3),$$

$$\begin{aligned} y_2(t) &= 3x_2(3t + 3) \\ &= 3x_1(3t + 3 - t_0), \quad \text{and} \end{aligned}$$

$$\begin{aligned} y_1(t - t_0) &= 3x_1(3(t - t_0) + 3) \\ &= 3x_1(3t - 3t_0 + 3). \end{aligned}$$

Since $y_2(t) \neq y_1(t-t_0)$, the system is not time invariant.

(f) LINEARITY. Suppose that

$$\begin{aligned} x_1(t) &\to y_1(t), \\ x_2(t) &\to y_2(t), \quad \text{and} \\ a_1x_1(t)+a_2x_2(t) &\to y_3(t). \end{aligned}$$

The system is linear if, for any $x_1(t)$ and $x_2(t)$, and any complex constants a_1 and a_2, $y_3(t) = a_1y_1(t) + a_2y_2(t)$. From the definition of the system, we have

$$y_1(t) = 3x_1(3t+3),$$

$$y_2(t) = 3x_2(3t+3), \quad \text{and}$$

$$\begin{aligned} y_3(t) &= 3[a_1x_1(3t+3)+a_2x_2(3t+3)] \\ &= a_1[3x_1(3t+3)]+a_2[3x_2(3t+3)] \\ &= a_1y_1(t)+a_2y_2(t). \end{aligned}$$

Since $y_3(t) = a_1y_1(t) + a_2y_2(t)$, the system is linear. $\square$

2.8 Problems

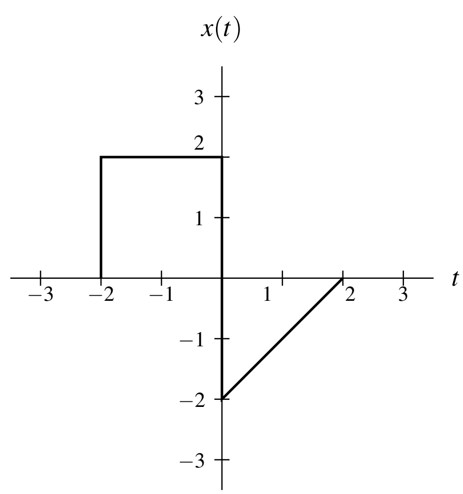
2.1 Identify the time and/or amplitude transformations that must be applied to the signal $x(t)$ in order to obtain each of the signals specified below. Choose the transformations such that time shifting precedes time scaling and amplitude scaling precedes amplitude shifting. Be sure to clearly indicate the order in which the transformations are to be applied.
(a) $y(t) = x(2t-1)$;
(b) $y(t) = x(\frac{1}{2}t+1)$;
(c) $y(t) = 2x(-\frac{1}{2}t+1)+3$;
(d) $y(t) = -\frac{1}{2}x(-t+1)-1$; and
(e) $y(t) = -3x(2[t-1])-1$.

2.2 Suppose that we have two signals $x(t)$ and $y(t)$ related as

$$y(t) = x(at-b),$$

where a and b are real constants and $a \neq 0$.
(a) Show that $y(t)$ can be formed by first time shifting $x(t)$ by b and then time scaling the result by a.
(b) Show that $y(t)$ can also be formed by first time scaling $x(t)$ by a and then time shifting the result by $\frac{b}{a}$.

2.3 Given the signal $x(t)$ shown in the figure below, plot and label each of the following signals:
(a) $x(t-1)$;
(b) $x(2t)$;
(c) $x(-t)$;
(d) $x(2t+1)$; and
(e) $\frac{1}{4}x(-\frac{1}{2}t+1)-\frac{1}{2}$.



2.4 Determine whether each of the following functions is even, odd, or neither even nor odd:
(a) $x(t) = t^3$;
(b) $x(t) = t^3\,|t|$;
(c) $x(t) = \left|t^3\right|$;
(d) $x(t) = (\cos 2\pi t)(\sin 2\pi t)$;
(e) $x(t) = e^{j2\pi t}$; and
(f) $x(t) = \frac{1}{2}[e^t + e^{-t}]$.

2.5 Prove each of the following assertions:
(a) The sum of two even signals is even.

(b) The sum of two odd signals is odd.
(c) The sum of an even signal and an odd signal is neither even nor odd.
(d) The product of two even signals is even.
(e) The product of two odd signals is even.
(f) The product of an even signal and an odd signal is odd.

2.6 Show that, if $x(t)$ is an odd signal, then

$$\int_{-A}^{A} x(t)dt = 0,$$

where A is a positive real constant.

2.7 Show that, for any signal $x(t)$,

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt,$$

where $x_e(t)$ and $x_o(t)$ denote the even and odd parts of $x(t)$, respectively.

2.8 Suppose $h(t)$ is a causal signal and has the even part $h_e(t)$ given by

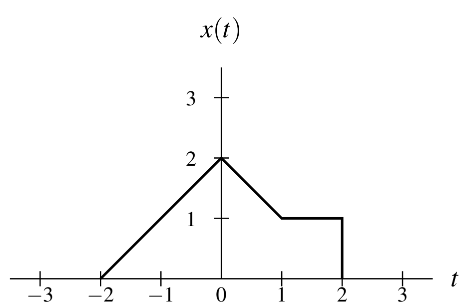
$$h_e(t) = t[u(t) - u(t-1)] + u(t-1) \quad \text{for } t > 0.$$

Find $h(t)$ for all t.

2.9 Determine whether each of the signals given below is periodic. If the signal is periodic, find its fundamental period.
(a) $x(t) = \cos 2\pi t + \sin 5t$;
(b) $x(t) = [\cos(4t - \frac{\pi}{3})]^2$;
(c) $x(t) = e^{j2\pi t} + e^{j3\pi t}$; and
(d) $x(t) = 1 + \cos 2t + e^{j5t}$.

2.10 Evaluate the following integrals:
(a) $\int_{-\infty}^{\infty} \sin\left(2t + \frac{\pi}{4}\right) \delta(t) dt$;
(b) $\int_{-\infty}^{t} [\cos\tau] \delta(\tau + \pi) d\tau$;
(c) $\int_{-\infty}^{\infty} x(t)\delta(at - b)dt$ where a and b are real constants and $a \neq 0$;
(d) $\int_0^2 e^{j2t} \delta(t-1)dt$; and
(e) $\int_{-\infty}^{t} \delta(\tau)d\tau$.

2.11 Suppose that we have the signal $x(t)$ shown in the figure below. Use unit-step functions to find a single expression for $x(t)$ that is valid for all t.



2.12 Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the following equations is linear:
(a) $y(t) = \int_{t-1}^{t+1} x(\tau)d\tau$;
(b) $y(t) = e^{x(t)}$;
(c) $y(t) = \text{Even}\{x(t)\}$; and
(d) $y(t) = x^2(t)$.

2.13 Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the following equations is time invariant:
(a) $y(t) = \frac{d}{dt}x(t)$;
(b) $y(t) = \text{Even}\{x(t)\}$;
(c) $y(t) = \int_{t}^{t+1} x(\tau-\alpha)d\tau$ where α is a constant;
(d) $y(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$;
(e) $y(t) = x(-t)$; and
(f) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$.

2.14 Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the following equations is causal and/or memoryless:
(a) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(b) $y(t) = \text{Odd}\{x(t)\}$;
(c) $y(t) = x(t-1)+1$;
(d) $y(t) = \int_{t}^{\infty} x(\tau)d\tau$; and
(e) $y(t) = \int_{-\infty}^{t} x(\tau)\delta(\tau)d\tau$.

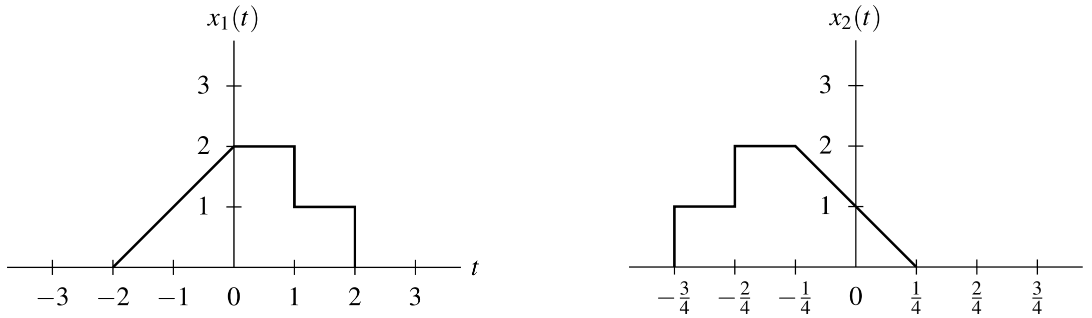
2.15 Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the equations given below is invertible. If the system is invertible, specify its inverse.
(a) $y(t) = x(at-b)$ where a and b are real constants and $a \neq 0$;
(b) $y(t) = e^{x(t)}$;
(c) $y(t) = \text{Even}\{x(t)\} - \text{Odd}\{x(t)\}$; and
(d) $y(t) = \frac{d}{dt}x(t)$.

2.16 Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the equations given below is BIBO stable.
(a) $y(t) = \int_{t}^{t+1} x(\tau)d\tau$;
(b) $y(t) = \frac{1}{2}x^2(t) + x(t)$; and
(c) $y(t) = 1/x(t)$.
[Hint for part (a): For any function $f(x)$, $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\,dx$.]

2.17 Show that if a system with input $x(t)$ and output $y(t)$ is either additive or homogeneous, it has the property that if $x(t)$ is identically zero (i.e., $x(t) = 0$ for all t), then $y(t)$ is identically zero (i.e., $y(t) = 0$ for all t).

2.18 Suppose that we have a signal $x(t)$ with the derivative $y(t) = \frac{d}{dt}x(t)$.
(a) Show that if $x(t)$ is even then $y(t)$ is odd.
(b) Show that if $x(t)$ is odd then $y(t)$ is even.

2.19 Given the signals $x_1(t)$ and $x_2(t)$ shown in the figures below, express $x_2(t)$ in terms of $x_1(t)$.



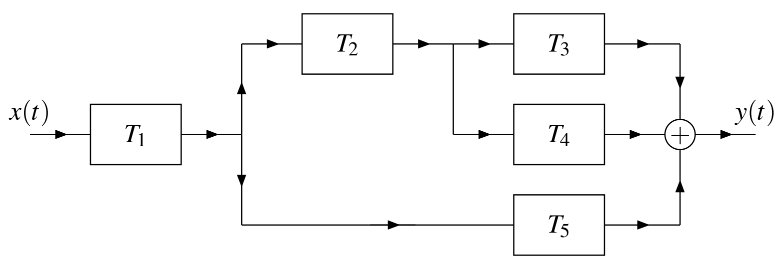
(a) (b)

2.20 Given the signal

$$x(t) = u(t+2) + u(t+1) + u(t) - 2u(t-1) - u(t-2),$$

find and sketch $y(t) = x(-4t-1)$.

2.21 For the system shown in the figure below, express the output $y(t)$ in terms of the input $x(t)$ and the transformations $T_1, T_2, \ldots, T_5$.



Chapter 3

Continuous-Time Linear Time-Invariant Systems

3.1 Introduction

In the previous chapter, we identified a number of properties that a system may possess. Two of these properties were the linearity and time invariance properties. In this chapter, we focus our attention exclusively on systems with both of these properties. Such systems are referred to as **linear time-invariant** (LTI) systems.

3.2 Continuous-Time Convolution

In the context of LTI systems, we often find an operation known as convolution to be particularly useful. The **convolution** of the functions $x(t)$ and $h(t)$ is denoted as $x(t) * h(t)$ and defined as

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \tag{3.1}$$

Throughout the remainder of these notes, the asterisk (or star) symbol (i.e., "$*$") will be used to denote convolution, not multiplication. It is important to make a distinction between convolution and multiplication, since these two operations are quite different and do not generally yield the same result.

One must also be careful when using the abbreviated (i.e., star notation) for convolution, since this notation can sometimes behave in counter-intuitive ways. For example, the expressions $x(t-t_0) * h(t-t_0)$ and $x(\tau) * h(\tau)|_{\tau=t-t_0}$, where t_0 is a nonzero constant, have very different meanings and are almost never equal.

Since the convolution operation is used extensively in system theory, we need some practical means for evaluating a convolution integral. Suppose that, for the given functions $x(t)$ and $h(t)$, we wish to compute

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Of course, we could naively attempt to compute $y(t)$ by evaluating a separate convolution integral for each possible value of t. This approach, however, is not feasible, as t can assume an infinite number of values, and therefore, an infinite number of integrals would need to be evaluated. Instead, we consider a slightly different approach. Let us redefine the integrand in terms of the intermediate function $w_t(\tau)$ where

$$w_t(\tau) = x(\tau)h(t-\tau).$$

(Note that $w_t(\tau)$ is implicitly a function of t.) This means that we need to compute

$$y(t) = \int_{-\infty}^{\infty} w_t(\tau)d\tau.$$

Now, we observe that, for most functions $x(t)$ and $h(t)$ of practical interest, the form of $w_t(\tau)$ typically remains fixed over particular ranges of t. Thus, we can compute the integral $y(t)$ by first identifying each of the distinct expressions for $w_t(\tau)$ and the range over which each expression is valid. Then, for each range, we evaluate a convolution integral. In this way, we typically only need to compute a small number of integrals instead of the infinite number required with the naive approach suggested above.

The above discussion leads us to propose the following general approach for performing the convolution operation:

1. Plot $x(\tau)$ and $h(t-\tau)$ as a function of τ.
2. Initially, consider an arbitrarily large negative value for t. This will result in $h(t-\tau)$ being shifted very far to the left on the time axis.
3. Write the mathematical expression for $w_t(\tau)$.
4. Increase t gradually until the expression for $w_t(\tau)$ changes form. Record the interval over which the expression for $w_t(\tau)$ was valid.
5. Repeat steps 3 and 4 until t is an arbitrarily large positive value. This corresponds to $h(t-\tau)$ being shifted very far to the right on the time axis.
6. For each of the intervals identified above, integrate $w_t(\tau)$ in order to find an expression for $y(t)$. This will yield an expression for $y(t)$ for each interval.
7. The results for the various intervals can be combined in order to obtain an expression for $y(t)$ that is valid for all t.

Example 3.1. Compute the convolution $y(t) = x(t) * h(t)$ where

$$x(t) = \begin{cases} -1 & \text{for } -1 \leq t < 0 \\ 1 & \text{for } 0 \leq t < 1 \\ 0 & \text{otherwise,} \end{cases} \quad \text{and}$$

$$h(t) = e^{-t}u(t).$$

Solution. We begin by plotting the signals $x(\tau)$ and $h(\tau)$ as shown in Figures 3.1(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 3.1(c). Second, we time-shift the resulting signal by t to obtain $h(t-\tau)$ as shown in Figure 3.1(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of $x(\tau)$ and $h(\tau)$, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 3.1(e) to (h).

First, we consider the case of $t < -1$. From Figure 3.1(e), we can see that

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{3.2}$$

Second, we consider the case of $-1 \leq t < 0$. From Figure 3.1(f), we can see that

$$\begin{aligned} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{t} -e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{t} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{t} \\ &= -e^{-t}[e^{t}-e^{-1}] \\ &= e^{-t-1}-1. \end{aligned} \tag{3.3}$$

Third, we consider the case of $0 \leq t < 1$. From Figure 3.1(g), we can see that

$$\begin{aligned}
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{t} e^{\tau-t}d\tau \\
&= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{t} e^{\tau}d\tau \\
&= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{t} \\
&= -e^{-t}[1-e^{-1}] + e^{-t}[e^{t}-1] \\
&= e^{-t}[e^{-1}-1+e^{t}-1] \\
&= 1+(e^{-1}-2)e^{-t}.
\end{aligned} \tag{3.4}$$

Fourth, we consider the case of $t \geq 1$. From Figure 3.1(h), we can see that

$$\begin{aligned}
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{1} e^{\tau-t}d\tau \\
&= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{1} e^{\tau}d\tau \\
&= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{1} \\
&= e^{-t}[e^{-1}-1+e-1] \\
&= (e-2+e^{-1})e^{-t}.
\end{aligned} \tag{3.5}$$

Combining the results of (3.2), (3.3), (3.4), and (3.5), we have that

$$x(t)*h(t) = \begin{cases} 0 & \text{for } t < -1 \\ e^{-t-1}-1 & \text{for } -1 \leq t < 0 \\ (e^{-1}-2)e^{-t}+1 & \text{for } 0 \leq t < 1 \\ (e-2+e^{-1})e^{-t} & \text{for } 1 \leq t. \end{cases}$$

The convolution result $x(t)*h(t)$ is plotted in Figure 3.1(i). □

Example 3.2. Compute the convolution $y(t) = x(t)*h(t)$ where

$$x(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1 \\ 0 & \text{otherwise,} \end{cases} \quad \text{and}$$

$$h(t) = \begin{cases} t & \text{for } 0 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Solution. We begin by plotting the signals $x(\tau)$ and $h(\tau)$ as shown in Figures 3.2(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 3.2(c). Second, we time-shift the resulting signal by t to obtain $h(t-\tau)$ as shown in Figure 3.2(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of $x(\tau)$ and $h(\tau)$, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 3.2(e) to (h).

First, we consider the case of $t < 0$. From Figure 3.2(e), we can see that

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{3.6}$$

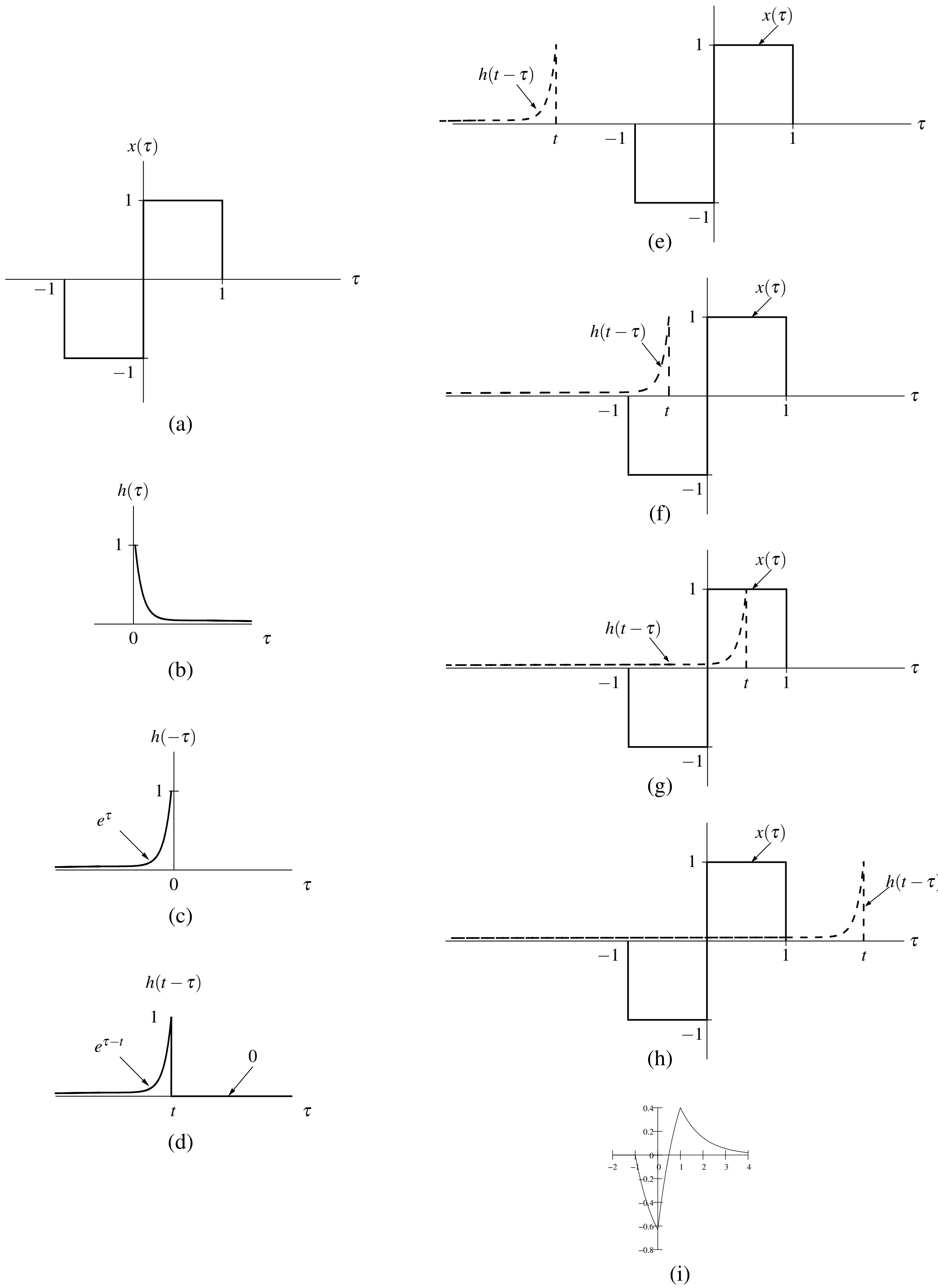


Figure 3.1: Evaluation of the convolution integral. The (a) input signal $x(\tau)$, (b) impulse response $h(\tau)$, (c) time-reversed impulse response $h(-\tau)$, and (d) impulse response after time-reversal and time-shifting $h(t-\tau)$. The functions associated with the product in the convolution integral for (e) $t < -1$, (f) $-1 \leq t < 0$, (g) $0 \leq t < 1$, and (h) $t \geq 1$, (i) The convolution result $x(t) * h(t)$.

Second, we consider the case of $0 \leq t < 1$. From Figure 3.2(f), we can see that

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{0}^{t}(t-\tau)d\tau \\ &= [t\tau - \tfrac{1}{2}\tau^2]|_0^t \\ &= t^2 - \tfrac{1}{2}t^2 \\ &= \tfrac{1}{2}t^2. \end{aligned} \tag{3.7}$$

Third, we consider the case of $1 \leq t < 2$. From Figure 3.2(g), we can see that

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{1}(t-\tau)d\tau \\ &= [t\tau - \tfrac{1}{2}\tau^2]_{t-1}^{1} \\ &= t - \tfrac{1}{2}(1)^2 - [t(t-1) - \tfrac{1}{2}(t-1)^2] \\ &= t - \tfrac{1}{2} - [t^2 - t - \tfrac{1}{2}(t^2 - 2t + 1)] \\ &= -\tfrac{1}{2}t^2 + t. \end{aligned} \tag{3.8}$$

Fourth, we consider the case of $t \geq 2$. From Figure 3.2(h), we can see that

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{3.9}$$

Combining the results of (3.6), (3.7), (3.8), and (3.9), we have that

$$x(t) * h(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{2}t^2 & \text{for } 0 \leq t < 1 \\ -\frac{1}{2}t^2 + t & \text{for } 1 \leq t < 2 \\ 0 & \text{for } t \geq 2. \end{cases}$$

The convolution result $x(t) * h(t)$ is plotted in Figure 3.2(i). □

Example 3.3. Compute the quantity $y(t) = x(t) * h(t)$, where

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } 0 \leq t < 1 \\ -t+2 & \text{for } 1 \leq t < 2 \\ 0 & \text{for } t > 2, \end{cases} \quad \text{and}$$

$$h(t) = u(t) - u(t-1).$$

Solution. Due to the somewhat ugly nature of the expressions for $x(t)$ and $h(t)$, this problem can be more easily solved if we use the graphical interpretation of convolution to guide us. We begin by plotting the signals $x(\tau)$ and $h(\tau)$, as shown in Figures 3.3(a) and (b), respectively.

Next, we need to determine $h(t-\tau)$, the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 3.3(c). Second, we time-shift the resulting signal by t to obtain $h(t-\tau)$ as shown in Figure 3.3(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of $x(\tau)$ and $h(\tau)$, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 3.3(e) to (i).

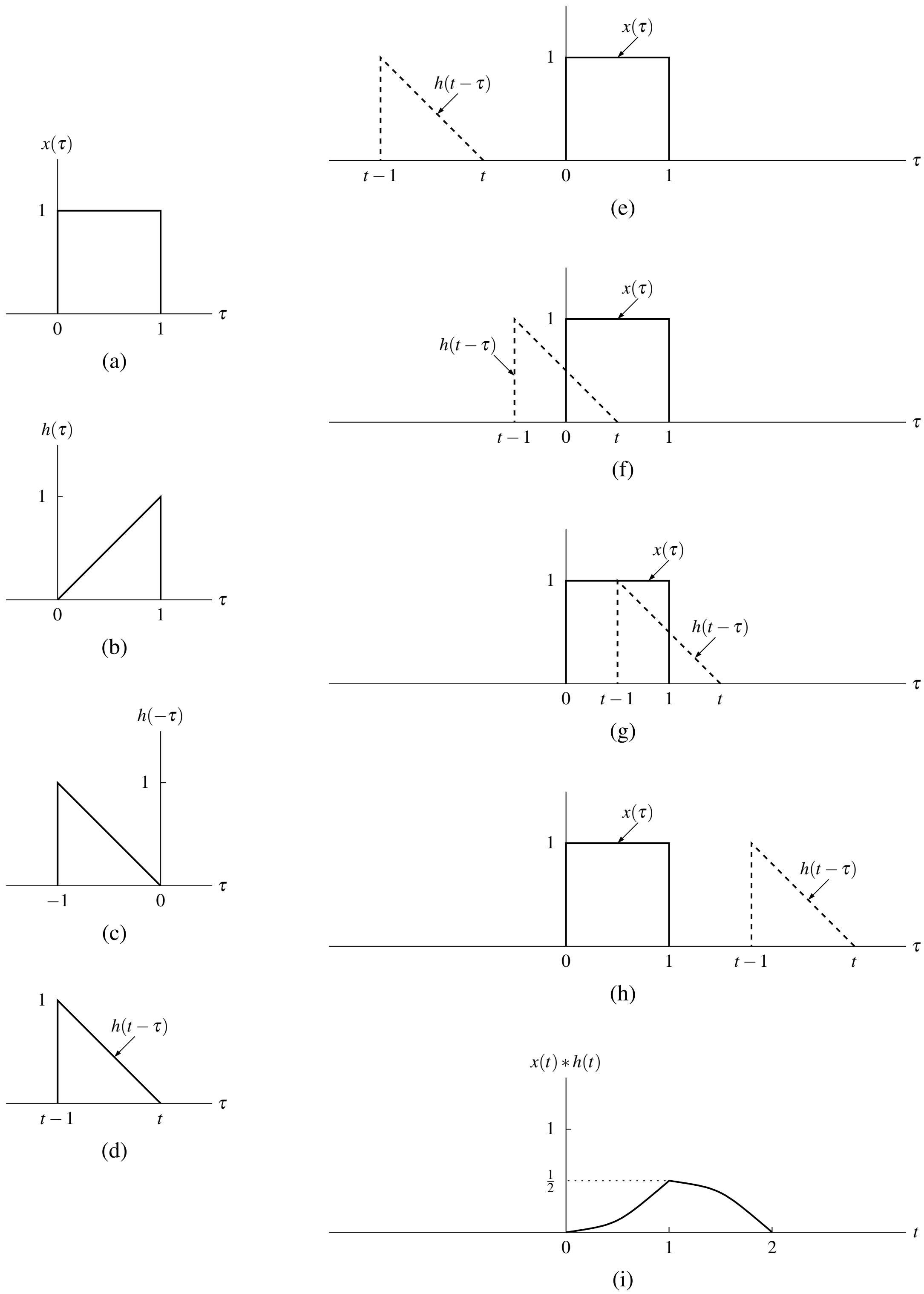


Figure 3.2: Evaluation of the convolution integral. The (a) input signal $x(\tau)$, (b) impulse response $h(\tau)$, (c) time-reversed impulse response $h(-\tau)$, and (d) impulse response after time-reversal and time-shifting $h(t-\tau)$. The functions associated with the product in the convolution integral for (e) $t<0$, (f) $0 \le t < 1$, (g) $1 \le t < 2$, and (h) $t \ge 2$, (i) The convolution result $x(t) * h(t)$.

First, we consider the case of $t < 0$. From Figure 3.3(e), we can see that

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{3.10}$$

Second, we consider the case of $0 \le t < 1$. From Figure 3.3(f), we can see that

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_0^t \tau d\tau \\ &= [\tfrac{1}{2}\tau^2]_0^t \\ &= \tfrac{1}{2}t^2.\end{aligned} \tag{3.11}$$

Third, we consider the case of $1 \le t < 2$. From Figure 3.3(g), we can see that

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{1} \tau d\tau + \int_1^t (-\tau+2)d\tau \\ &= [\tfrac{1}{2}\tau^2]_{t-1}^1 + [-\tfrac{1}{2}\tau^2+2\tau]_1^t \\ &= \tfrac{1}{2} - [\tfrac{1}{2}(t-1)^2] - \tfrac{1}{2}t^2 + 2t - [-\tfrac{1}{2}+2] \\ &= -t^2+3t-\tfrac{3}{2}.\end{aligned} \tag{3.12}$$

Fourth, we consider the case of $2 \le t < 3$. From Figure 3.3(h), we can see that

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{2} (-\tau+2)d\tau \\ &= [-\tfrac{1}{2}\tau^2+2\tau]_{t-1}^2 \\ &= 2 - [-\tfrac{1}{2}t^2+3t-\tfrac{5}{2}] \\ &= \tfrac{1}{2}t^2-3t+\tfrac{9}{2}.\end{aligned} \tag{3.13}$$

Lastly, we consider the case of $t \ge 3$. From Figure 3.3(i), we can see that

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau) = 0. \tag{3.14}$$

Combining the results of (3.10), (3.11), (3.12), (3.13), and (3.14) together, we have that

$$x(t)*h(t) = \begin{cases} 0 & \text{for } t < 0 \\ \tfrac{1}{2}t^2 & \text{for } 0 \le t < 1 \\ -t^2+3t-\tfrac{3}{2} & \text{for } 1 \le t < 2 \\ \tfrac{1}{2}t^2-3t+\tfrac{9}{2} & \text{for } 2 \le t < 3 \\ 0 & \text{for } t \ge 3. \end{cases}$$

The convolution result (i.e., $x(t)*h(t)$) is plotted in Figure 3.3(j). □

Example 3.4. Compute the convolution $y(t) = x(t)*h(t)$ where

$$\begin{aligned} x(t) &= e^{-at}u(t), \quad \text{and} \\ h(t) &= u(t), \end{aligned}$$

and a is a positive real constant.

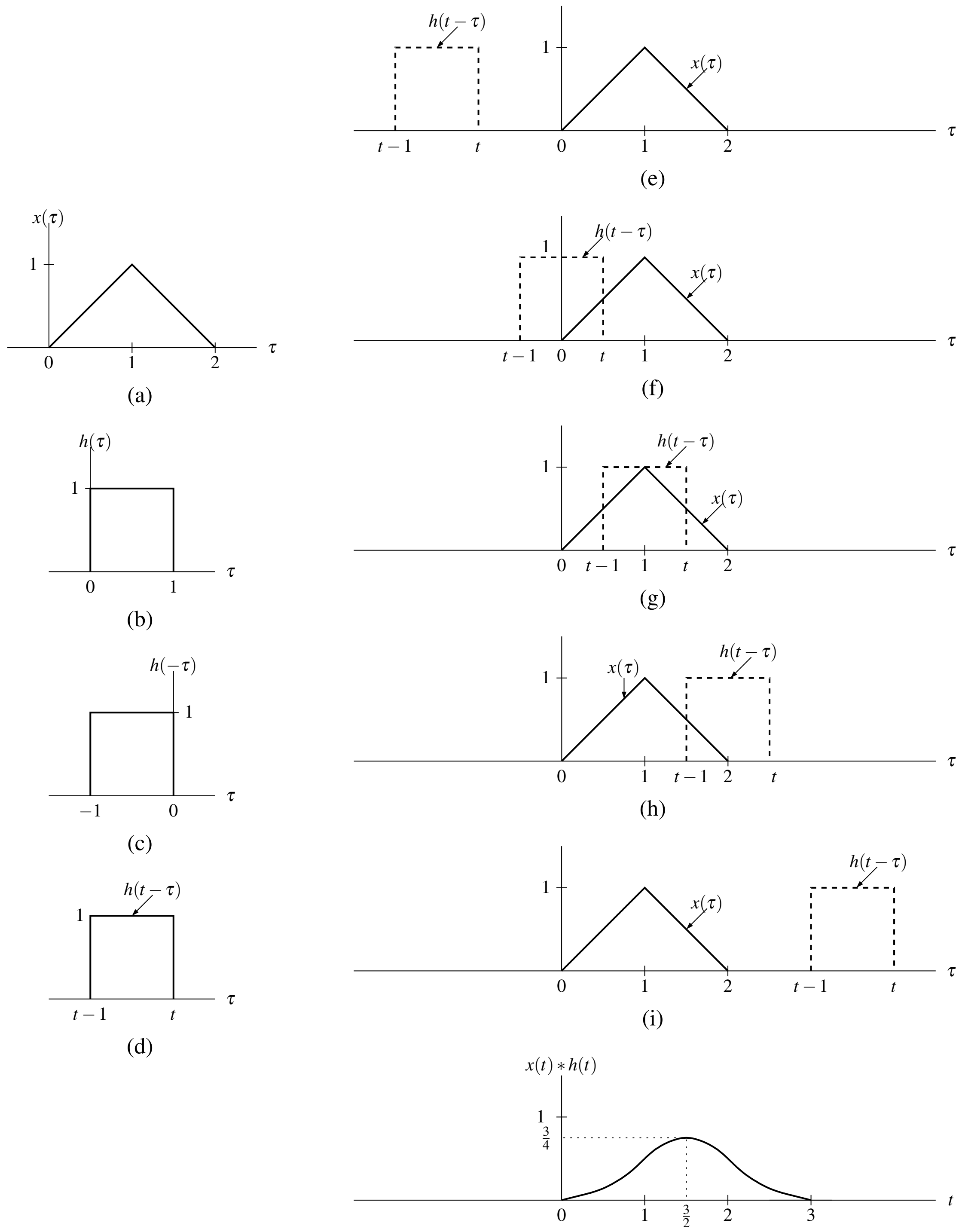


Figure 3.3: Evaluation of the convolution integral. The (a) input signal $x(\tau)$, (b) impulse response $h(\tau)$, (c) time-reversed impulse response $h(-\tau)$, and (d) impulse response after time-reversal and time-shifting $h(t-\tau)$. The functions associated with the product in the convolution integral for (e) $t<0$, (f) $0 \le t < 1$, (g) $1 \le t < 2$, (h) $2 \le t < 3$, and (i) $t \ge 3$. (j) The convolution result $x(t) * h(t)$.

Solution. Since $x(t)$ and $h(t)$ are relatively simple functions, we will solve this problem without the aid of graphs. Using the definition of the convolution operation, we can write:

$$\begin{aligned} y(t) = x(t) * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau. \end{aligned} \tag{3.15}$$

The integrand is zero for $\tau < 0$ or $\tau > t$. Conversely, the integrand can only be nonzero for $\tau \geq 0$ and $\tau \leq t$. So, if $t < 0$, the integrand will be zero, and $y(t) = 0$. Now, let us consider the case of $t > 0$. From (3.15), we can write:

$$\begin{aligned} y(t) &= \int_0^t e^{-a\tau}d\tau \\ &= [-\tfrac{1}{a}e^{-a\tau}]_0^t \\ &= \tfrac{1}{a}(1-e^{-at}). \end{aligned}$$

Thus, we have

$$\begin{aligned} y(t) &= \begin{cases} \frac{1}{a}(1-e^{-at}) & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \tfrac{1}{a}(1-e^{-at})u(t). \end{aligned}$$

(If some steps in the above solution are unclear, it would probably be helpful to sketch the corresponding graphs. This will yield the graphs shown in Figure 3.4.) □

3.3 Properties of Convolution

Since convolution is frequently employed in the study of LTI systems, it is important for us to know some of its basic properties. In what follows, we examine some of these properties. We will later use these properties in the context of LTI system.

3.3.1 Commutative Property

The convolution operation is commutative. That is, for any two signals $x(t)$ and $h(t)$, we have

$$x(t) * h(t) = h(t) * x(t). \tag{3.16}$$

Thus, the result of the convolution operation is not affected by the order of the operands.

We now provide a proof of the commutative property stated above. To begin, we expand the left-hand side of (3.16) as follows:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Next, we perform a change of variable. Let $v = t - \tau$ which implies that $\tau = t - v$ and $d\tau = -dv$. Using this change of variable, we can rewrite the previous equation as

$$\begin{aligned} x(t) * h(t) &= \int_{\infty}^{-\infty} x(t-v)h(v)(-dv) \\ &= \int_{-\infty}^{\infty} x(t-v)h(v)dv \\ &= \int_{-\infty}^{\infty} h(v)x(t-v)dv \\ &= h(t) * x(t). \end{aligned}$$

Thus, we have proven that convolution is commutative.

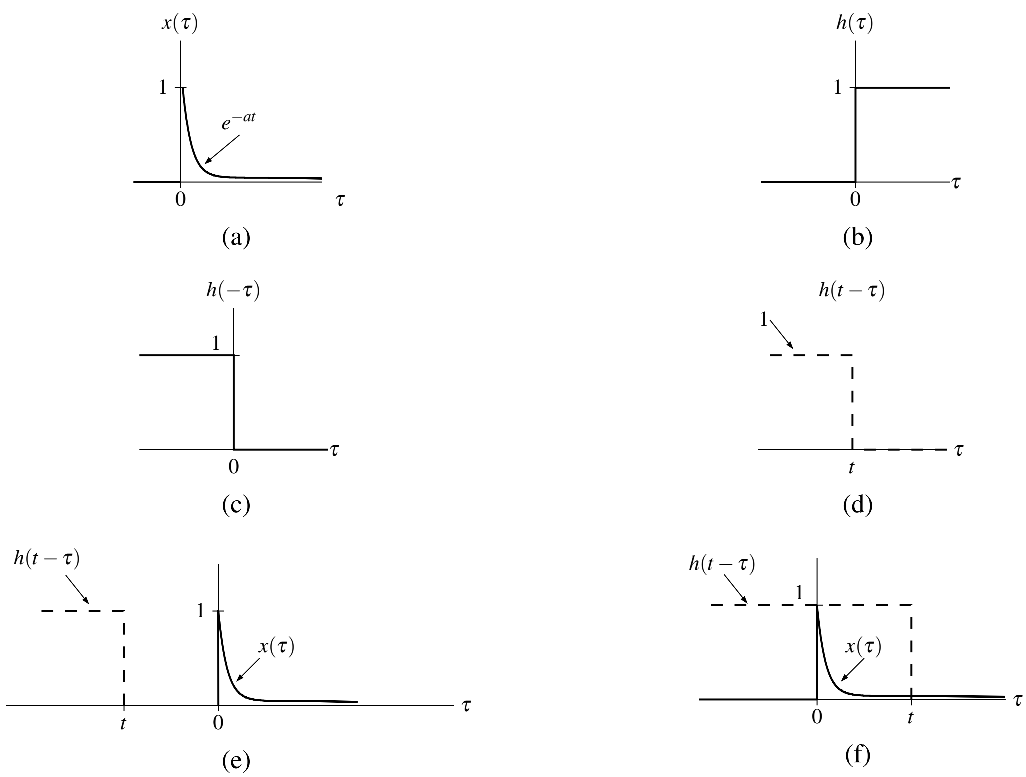


Figure 3.4: Evaluation of the convolution integral. The (a) input signal $x(\tau)$, (b) impulse response $h(\tau)$, (c) time-reversed impulse response $h(-\tau)$, and (d) impulse response after time-reversal and time-shifting $h(t-\tau)$. The functions associated with the product in the convolution integral for (e) $t < 0$ and (f) $t > 0$.

3.3.2 Associative Property

The convolution operation is associative. That is, for any signals $x(t)$, $h_1(t)$, and $h_2(t)$, we have

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]. \tag{3.17}$$

In other words, the final convolution result does not depend on how the intermediate operations are grouped.

The proof of this property is relatively straightforward, although somewhat confusing notationally. To begin, we use the definition of the convolution operation to expand the left-hand side of (3.17) as follows:

$$\begin{aligned}
[x(t) * h_1(t)] * h_2(t) &= \int_{-\infty}^{\infty} [x(v) * h_1(v)] h_2(t-v) dv \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) h_1(v-\tau) d\tau \right) h_2(t-v) dv.
\end{aligned}$$

Now, we change the order of integration to obtain

$$[x(t) * h_1(t)] * h_2(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h_1(v-\tau) h_2(t-v) dv d\tau.$$

Pulling the factor of $x(\tau)$ out of the inner integral yields

$$[x(t) * h_1(t)] * h_2(t) = \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h_1(v-\tau) h_2(t-v) dv d\tau.$$

Next, we perform a change of variable. Let $\lambda = v - \tau$ which implies that $v = \lambda + \tau$ and $d\lambda = dv$. Using this change of variable, we can write

$$\begin{aligned}
[x(t) * h_1(t)] * h_2(t) &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h_1(\lambda) h_2(t-\lambda-\tau) d\lambda d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h_1(\lambda) h_2([t-\tau]-\lambda) d\lambda \right) d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \, [h_1(\rho) * h_2(\rho)]|_{\rho=t-\tau} d\tau \\
&= x(t) * [h_1(t) * h_2(t)].
\end{aligned}$$

Thus, we have proven that convolution is associative.

3.3.3 Distributive Property

The convolution operation is distributive. That is, for any signals $x(t)$, $h_1(t)$, and $h_2(t)$, we have

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t). \tag{3.18}$$

The proof of this property is relatively simple. Expanding the left-hand side of (3.18), we have:

$$\begin{aligned}
x(t) * [h_1(t) + h_2(t)] &= \int_{-\infty}^{\infty} x(\tau) [h_1(t-\tau) + h_2(t-\tau)] d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) h_1(t-\tau) d\tau + \int_{-\infty}^{\infty} x(\tau) h_2(t-\tau) d\tau \\
&= x(t) * h_1(t) + x(t) * h_2(t).
\end{aligned}$$

Thus, we have shown that convolution is distributive.

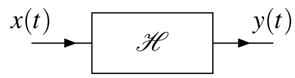


Figure 3.5: System.

3.4 Representation of Continuous-Time Signals Using Impulses

The unit-impulse function is of fundamental importance in many aspects of system theory. For this reason, we sometimes have the need to express an arbitrary signal in terms of impulse functions. In what follows, we develop a means for accomplishing just this.

Suppose that we have an arbitrary signal $x(t)$. From the equivalence property of the impulse function given in (2.30), we can write

$$x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau).$$

Now, let us integrate both sides of the preceding equation with respect to τ to obtain

$$\int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau. \tag{3.19}$$

The left-hand side of this equation can be further simplified as follows:

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau &= x(t)\int_{-\infty}^{\infty}\delta(t-\tau)d\tau \\ &= x(t)\int_{\infty}^{-\infty}\delta(\lambda)(-d\lambda) \\ &= x(t)\int_{-\infty}^{\infty}\delta(\lambda)d\lambda \\ &= x(t).\end{aligned} \tag{3.20}$$

(In the above simplification, we used the change of variable $\lambda = t-\tau$, $d\lambda = -d\tau$.) Combining (3.19) and (3.20), we obtain

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) * \delta(t). \tag{3.21}$$

Thus, we can represent any signal $x(t)$ using an expression containing the impulse function. Furthermore, we have also just shown that the unit impulse function is the convolutional identity. That is, we have that, for any $x(t)$,

$$x(t) * \delta(t) = x(t)$$

(i.e., convolving a function $x(t)$ with the unit-impulse function $\delta(t)$ simply yields $x(t)$).

3.5 Continuous-Time Unit-Impulse Response and Convolution Integral Representation of LTI Systems

Suppose that we have an arbitrary system with input $x(t)$ and output $y(t)$. Let us represent the processing performed by the system with the operator $\mathscr{H}$, so that $y(t) = \mathscr{H}\{x(t)\}$. In other words, we have the system shown in Figure 3.5. As a matter of terminology, the system response to the unit-impulse function input is referred to as the **impulse response**. Let us denote the impulse response of the system as $h(t)$. Mathematically, we can state the definition of the impulse response as

$$h(t) = \mathscr{H}\{\delta(t)\}. \tag{3.22}$$

Now, let us assume that the system is LTI (i.e., the operator $\mathscr{H}$ is both linear and time invariant). As we shall demonstrate below, the behavior of a LTI system is completely characterized by its impulse response. That is, if the impulse response of a system is known, we can determine the response of the system to *any* input.

From the earlier result, we know that we can represent any signal in the form of (3.21). So, let us express the input to the system in this form as follows:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau. \tag{3.23}$$

Now, let us consider the form of the output $y(t)$. To begin, we know that

$$y(t) = \mathscr{H}\{x(t)\}.$$

From (3.23), we can rewrite this equation as

$$y(t) = \mathscr{H}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}.$$

Since $\mathscr{H}$ is a linear operator, we can move $\mathscr{H}$ inside the integral, and simplify the result to obtain

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} \mathscr{H}\{x(\tau)\delta(t-\tau)\}d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\mathscr{H}\{\delta(t-\tau)\}d\tau. \end{aligned} \tag{3.24}$$

Since the system is time invariant (by assumption), we know that

$$h(t-\tau) = \mathscr{H}\{\delta(t-\tau)\}. \tag{3.25}$$

Thus, we can rewrite (3.24) as

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= x(t)*h(t). \end{aligned}$$

In other words, the output $y(t)$ is simply the convolution of the input $x(t)$ and the impulse response $h(t)$.

Clearly, the impulse response provides a very powerful tool for the study of LTI systems. If we know the impulse response of a system, we can determine the response of the system to *any* input.

Example 3.5. Suppose that we are given a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$ where

$$h(t) = u(t). \tag{3.26}$$

Show that this system is the integrator characterized by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau. \tag{3.27}$$

Solution. Since the system is LTI, we have that

$$y(t) = x(t)*h(t).$$

Substituting (3.26) into the preceding equation, and simplifying we obtain

$$\begin{aligned} y(t) &= x(t)*h(t) \\ &= x(t)*u(t) \\ &= \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau \\ &= \int_{-\infty}^{t} x(\tau)u(t-\tau)d\tau + \int_{t}^{\infty} x(\tau)u(t-\tau)d\tau \\ &= \int_{-\infty}^{t} x(\tau)d\tau. \end{aligned}$$

Therefore, the system with the impulse response $h(t)$ given by (3.26) is, in fact, the integrator given by (3.27). □

Example 3.6. Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$ where

$$h(t) = \begin{cases} 1 & \text{for } 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find and plot the response of the system to the particular input $x(t)$ given by

$$x(t) = \begin{cases} 1 & \text{for } 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Plots of $x(t)$ and $h(t)$ are given in Figures 3.6(a) and (b), respectively.

Solution. Since the system is LTI, we know that

$$y(t) = x(t) * h(t).$$

Thus, in order to find the response of the system to the input $x(t)$, we simply need to compute the convolution $x(t) * h(t)$.

We begin by plotting the signals $x(\tau)$ and $h(\tau)$ as shown in Figures 3.6(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 3.6(c). Second, we time-shift the resulting signal by t to obtain $h(t-\tau)$ as shown in Figure 3.6(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of $x(\tau)$ and $h(\tau)$, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 3.6(e) to (h).

First, we consider the case of $t < 0$. From Figure 3.6(e), we can see that

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{3.28}$$

Second, we consider the case of $0 \le t < 1$. From Figure 3.6(f), we can see that

$$\begin{aligned} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_0^t d\tau \\ &= [\tau]|_0^t \\ &= t. \end{aligned} \tag{3.29}$$

Third, we consider the case of $1 \le t < 2$. From Figure 3.6(g), we can see that

$$\begin{aligned} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{1} d\tau \\ &= [\tau]|_{t-1}^{1} \\ &= 1-(t-1) \\ &= 2-t. \end{aligned} \tag{3.30}$$

Fourth, we consider the case of $t \ge 2$. From Figure 3.6(h), we can see that

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{3.31}$$

Combining the results of (3.28), (3.29), (3.30), and (3.31), we have that

$$x(t) * h(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } 0 \le t < 1 \\ 2-t & \text{for } 1 \le t < 2 \\ 0 & \text{for } t \ge 2. \end{cases}$$

The convolution result $x(t) * h(t)$ is plotted in Figure 3.6(i). The response of the system to the specified input is simply the result of the convolution (i.e., $x(t) * h(t)$). $\square$

3.6 Unit-Step Response of LTI Systems

Suppose that we have a system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Consider the response $s(t)$ of the system to the unit-step function input $u(t)$. We call $s(t)$ the **step response** of the system. This response $s(t)$ is given by

$$\begin{aligned} s(t) &= u(t) * h(t) \\ &= h(t) * u(t) \\ &= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau \\ &= \int_{-\infty}^{t} h(\tau)d\tau. \end{aligned}$$

Taking the derivative of $s(t)$ with respect to t, we obtain

$$\begin{aligned} \frac{ds(t)}{dt} &= \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{-\infty}^{t+\Delta t} h(\tau)d\tau - \int_{-\infty}^{t} h(\tau)d\tau \right] \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{t}^{t+\Delta t} h(\tau)d\tau \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(h(t)\Delta t \right) \\ &= h(t). \end{aligned}$$

Thus, we have shown that

$$\frac{ds(t)}{dt} = h(t).$$

That is, the impulse response $h(t)$ of a system is equal to the derivative of its step response $s(t)$. Therefore, the impulse response of a system can be determined from its step response simply through differentiation.

The step response is often of great practical interest, since it can be used to determine the impulse response of a LTI system. From a practical point of view, the step response is more useful for characterizing a system based on experimental measurements. Obviously, we cannot directly measure the impulse response of a system because we cannot (in the real world) produce a unit-impulse signal. We can, however, produce a reasonably good approximation of the unit-step function in the real world. Thus, we can measure the step response and from it determine the impulse response.

3.7 Block Diagram Representation of Continuous-Time LTI Systems

Frequently, it is convenient to represent continuous-time LTI systems in block diagram form. Since such systems are completely characterized by their impulse response, we often label the system with its impulse response. That is, we represent a system with input $x(t)$, output $y(t)$, and impulse response $h(t)$, as shown in Figure 3.7.

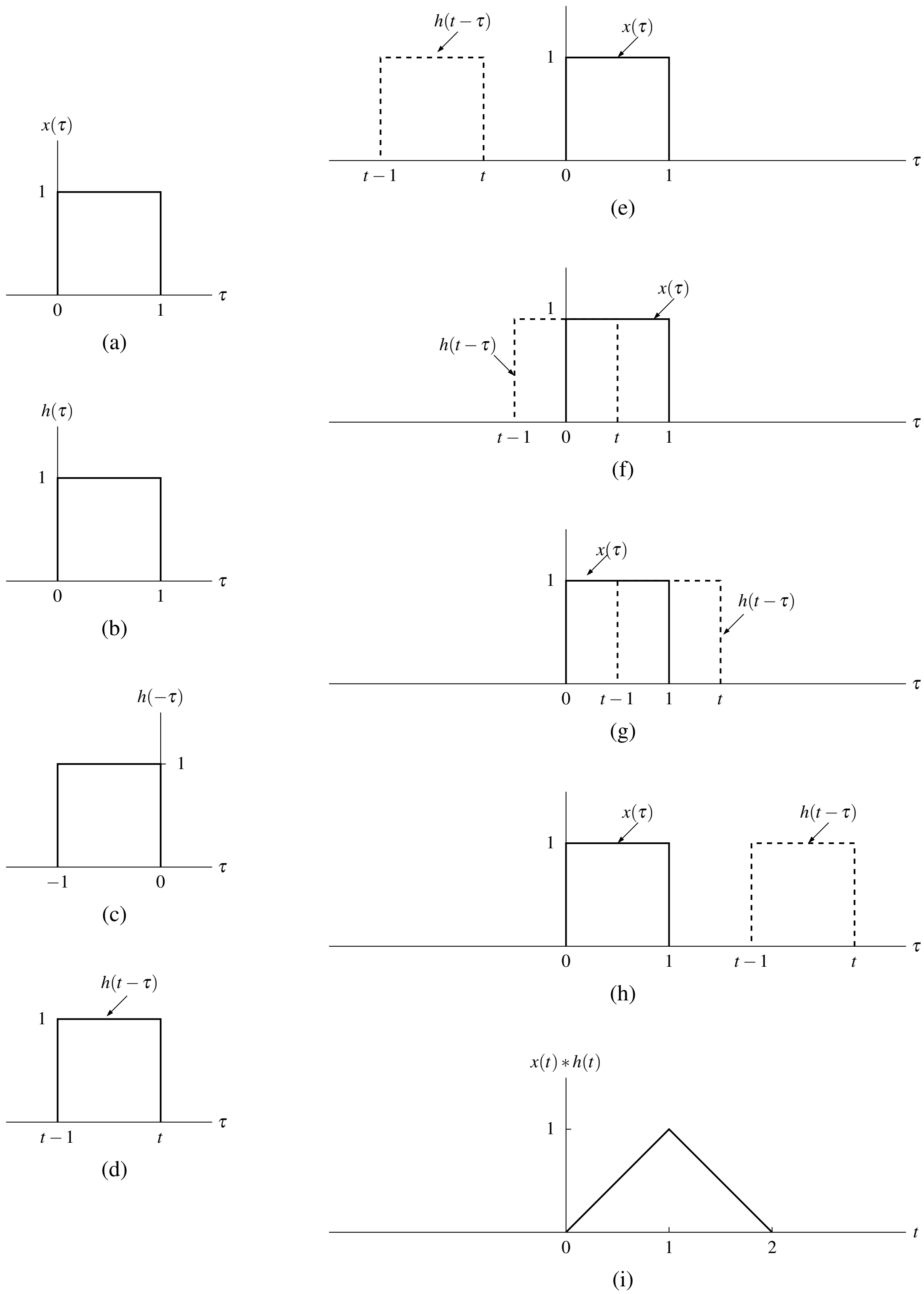


Figure 3.6: Evaluation of the convolution integral. The (a) input signal $x(\tau)$, (b) impulse response $h(\tau)$, (c) time-reversed impulse response $h(-\tau)$, and (d) impulse response after time-reversal and time-shifting $h(t-\tau)$. The functions associated with the product in the convolution integral for (e) $t < 0$, (f) $0 \leq t < 1$, (g) $1 \leq t < 2$, and (h) $t \geq 2$, (i) The convolution result $x(t) * h(t)$.

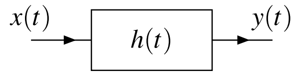


Figure 3.7: Block diagram representation of continuous-time LTI system.

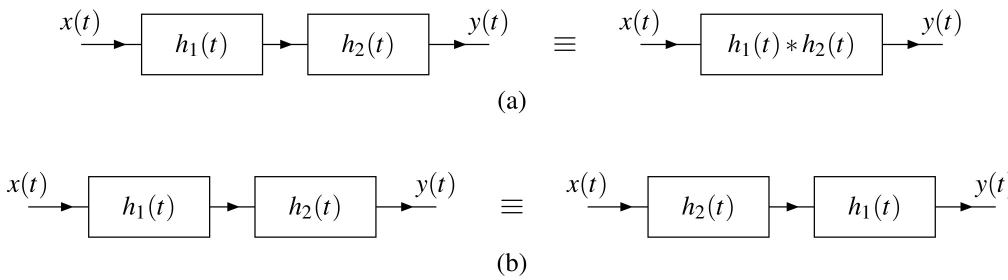


Figure 3.8: Series interconnection of continuous-time LTI systems.

3.8 Interconnection of Continuous-Time LTI Systems

Suppose that we have an LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. We know that $x(t)$ and $y(t)$ are related as $y(t) = x(t) * h(t)$. In other words, the system can be viewed as performing a convolution operation. From the properties of convolution introduced earlier, we can derive a number of equivalences involving the impulse responses of series- and parallel-interconnected systems.

Suppose that we have two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ that are connected in a series configuration, as shown on the left-side of Figure 3.8(a). For convenience, let us define $v(t) = x(t) * h_1(t)$. Using the associative property of convolution, we can simplify the expression for the output $y(t)$ as follows:

$$\begin{aligned} y(t) &= v(t) * h_2(t) \\ &= [x(t) * h_1(t)] * h_2(t) \\ &= x(t) * [h_1(t) * h_2(t)]. \end{aligned}$$

In other words, we have the equivalence shown in Figure 3.8(a).

Suppose that we have two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ that are connected in a series configuration, as shown on the left-side of Figure 3.8(b). Using the commutative property of convolution, we can simplify the expression for the output $y(t)$ as follows:

$$\begin{aligned} y(t) &= x(t) * h_1(t) * h_2(t) \\ &= x(t) * h_2(t) * h_1(t). \end{aligned}$$

In other words, we have the equivalence shown in Figure 3.8(b).

Suppose that we have two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ that are connected in a parallel configuration, as shown on the left-side of Figure 3.9. Using the distributive property of convolution, we can simplify the expression for the output $y(t)$ as follows:

$$\begin{aligned} y(t) &= x(t) * h_1(t) + x(t) * h_2(t) \\ &= x(t) * [h_1(t) + h_2(t)]. \end{aligned}$$

Thus, we have the equivalence shown in Figure 3.9.

Example 3.7. Consider the system shown in Figure 3.10 with input $x(t)$, output $y(t)$. Find the impulse response $h(t)$ of the system. Use the properties of convolution in order to accomplish this.

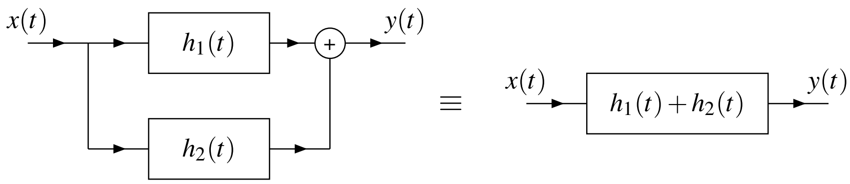


Figure 3.9: Parallel interconnection of continuous-time LTI systems.

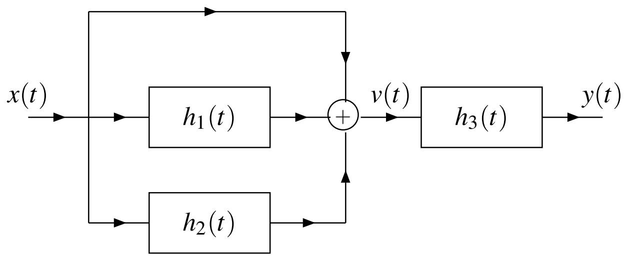


Figure 3.10: System interconnection example.

Solution. From the diagram, we can write:

$$\begin{aligned} v(t) &= x(t) + x(t) * h_1(t) + x(t) * h_2(t) \\ &= x(t) * \delta(t) + x(t) * h_1(t) + x(t) * h_2(t) \\ &= x(t) * [\delta(t) + h_1(t) + h_2(t)]. \end{aligned}$$

Similarly, we can write:

$$y(t) = v(t) * h_3(t).$$

Substituting the expression for $v(t)$ into the preceding equation we obtain:

$$\begin{aligned} y(t) &= v(t) * h_3(t) \\ &= x(t) * [\delta(t) + h_1(t) + h_2(t)] * h_3(t) \\ &= x(t) * [h_3(t) + h_1(t) * h_3(t) + h_2(t) * h_3(t)]. \end{aligned}$$

Therefore, the impulse response $h(t)$ of the overall system is

$$h(t) = h_3(t) + h_1(t) * h_3(t) + h_2(t) * h_3(t).$$

□

3.9 Properties of Continuous-Time LTI Systems

In the previous chapter, we introduced a number of properties that might be possessed by a system (e.g., memory, causality, stability, invertibility). Since a LTI system is completely characterized by its impulse response, one might wonder if there is a relationship between some of the properties introduced previously and the impulse response. In what follows, we explore some of these relationships.

3.9.1 Memory

Recall that a system is memoryless if its output $y(t)$ at any arbitrary time t_0 depends only on the value of its input $x(t)$ at that same time. Suppose now that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. The output $y(t)$ at some arbitrary time $t = t_0$ is given by

$$\begin{aligned} y(t_0) &= x(t) * h(t)|_{t=t_0} \\ &= h(t) * x(t)|_{t=t_0} \\ &= \left. \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right|_{t=t_0} \\ &= \int_{-\infty}^{\infty} h(\tau)x(t_0-\tau)d\tau. \end{aligned}$$

Consider the integral in the above equation. In order for the system to be memoryless, the result of the integration must depend only on $x(t)$ at $t = t_0$. This, however, is only possible if

$$h(t) = 0 \quad \text{for all } t \neq 0. \tag{3.32}$$

Consequently, $h(t)$ must be of the form

$$h(t) = K\delta(t) \tag{3.33}$$

where K is a complex constant. Thus, we have that a LTI system is memoryless if and only if its impulse response satisfies (3.32). This, in turn, implies that the impulse response is of the form of (3.33). As a consequence of this fact, we also have that all memoryless LTI systems must have an input-output relation of the form

$$\begin{aligned} y(t) &= x(t) * K\delta(t) \\ &= Kx(t). \end{aligned}$$

Example 3.8. Suppose that we have the LTI system with the impulse response $h(t)$ given by

$$h(t) = e^{-at}u(t)$$

where a is a real constant. Determine whether this system has memory.

Solution. The system has memory since $h(t) \neq 0$ for some $t \neq 0$ (e.g., $h(1) = e^{-a} \neq 0$). $\square$

Example 3.9. Suppose that we have the LTI system with the impulse response $h(t)$ given by

$$h(t) = \delta(t).$$

Determine whether this system has memory.

Solution. Clearly, $h(t)$ is only nonzero at $t = 0$. This follows immediately from the definition of the unit-impulse function $\delta(t)$. Therefore, the system is memoryless (i.e., does not have memory). $\square$

3.9.2 Causality

Recall that a system is causal if its output $y(t)$ at any arbitrary time t_0 depends only on its input $x(t)$ for $t \leq t_0$. Suppose that we have the LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. The value of the output $y(t)$ for $t = t_0$ is given by

$$\begin{aligned} y(t_0) &= [x(t) * h(t)]|_{t=t_0} \\ &= \left. \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \right|_{t=t_0} \\ &= \int_{-\infty}^{\infty} x(\tau)h(t_0-\tau)d\tau \\ &= \int_{-\infty}^{t_0} x(\tau)h(t_0-\tau)d\tau + \int_{t_0^+}^{\infty} x(\tau)h(t_0-\tau)d\tau. \end{aligned} \tag{3.34}$$

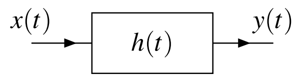


Figure 3.11: System.

In order for the expression for $y(t_0)$ in (3.34) not to depend on $x(t)$ for $t > t_0$, we must have that

$$h(t) = 0 \quad \text{for } t < 0 \tag{3.35}$$

(i.e., $h(t)$ is a causal signal). In this case, (3.34) simplifies to

$$y(t_0) = \int_{-\infty}^{t_0} x(\tau)h(t_0 - \tau)d\tau.$$

Clearly, the result of this integration does not depend on $x(t)$ for $t > t_0$ (since τ varies from $-\infty$ to t_0). Therefore, a LTI system is causal if its impulse response $h(t)$ satisfies (3.35).

Example 3.10. Suppose that we have the LTI system with impulse response $h(t)$ given by

$$h(t) = e^{-at}u(t),$$

where a is a real constant. Determine whether this system is causal.

Solution. Clearly, $h(t) = 0$ for $t < 0$ (due to the $u(t)$ factor in the expression for $h(t)$). Therefore, the system is causal. □

Example 3.11. Suppose that we have the LTI system with impulse response $h(t)$ given by

$$h(t) = \delta(t + t_0),$$

where t_0 is a strictly positive real constant. Determine whether this system is causal.

Solution. From the definition of $\delta(t)$, we can easily deduce that $h(t) = 0$ except at $t = -t_0$. Since $-t_0 < 0$, the system is not causal. □

3.9.3 Invertibility

Recall that a system is invertible if we can always uniquely determine its input $x(t)$ from its output $y(t)$. An equivalent way of stating this is that an inverse system must exist.

Suppose now that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Such a system is illustrated in Figure 3.11.

One can readily show that the inverse of a LTI system, if it exists, must also be LTI. This follows from the fact that the inverse of a linear system must be linear, and the inverse of a time-invariant system must be time invariant. Since the inverse system, if it exists, must be LTI, the system can be completely characterized by its impulse response. Let us denote the impulse response of the inverse system as $h^{\text{inv}}(t)$. Now, we want to find the relationship between $h(t)$ and $h^{inv}(t)$.

If the inverse system exists, then by definition, it must be such that

$$x(t) * h(t) * h^{\text{inv}}(t) = x(t). \tag{3.36}$$

This relationship is expressed diagrammatically in Figure 3.12. Since the unit-impulse function is the convolutional identity, we can equivalently rewrite (3.36) as

$$x(t) * h(t) * h^{\text{inv}}(t) = x(t) * \delta(t).$$

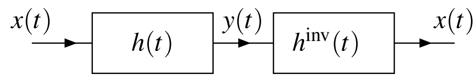


Figure 3.12: System in cascade with its inverse.

By comparing the left- and right-hand sides of the preceding equation, we have

$$h(t) * h^{\text{inv}}(t) = \delta(t). \tag{3.37}$$

Therefore, a system will have an inverse if and only if a solution for $h^{inv}(t)$ exists in (3.37). Thus, a system is invertible if and only if a solution to this equation exists.

Example 3.12. Suppose that we have the LTI system with impulse response $h(t)$ given by

$$h(t) = A\delta(t - t_0) \tag{3.38}$$

where A is a nonzero real constant and t_0 is a real constant. Determine whether this system is invertible. If it is invertible, find the impulse response $h^{\text{inv}}(t)$ of the inverse system.

Solution. The inverse system, if it exists, is given by the solution to the equation

$$h(t) * h^{\text{inv}}(t) = \delta(t).$$

So, let us attempt to solve this equation for $h^{\text{inv}}(t)$. Substituting (3.38) into (3.37) and using straightforward algebraic manipulation, we can write:

$$\begin{aligned}
& h(t) * h^{\text{inv}}(t) = \delta(t) \\
\Rightarrow \quad & \int_{-\infty}^{\infty} h(\tau) h^{\text{inv}}(t-\tau) d\tau = \delta(t) \\
\Rightarrow \quad & \int_{-\infty}^{\infty} A\delta(\tau - t_0) h^{\text{inv}}(t-\tau) d\tau = \delta(t) \\
\Rightarrow \quad & \int_{-\infty}^{\infty} \delta(\tau - t_0) h^{\text{inv}}(t-\tau) d\tau = \tfrac{1}{A}\delta(t).
\end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the integral expression in the preceding equation to obtain

$$h^{\text{inv}}(t - t_0) = \tfrac{1}{A}\delta(t). \tag{3.39}$$

Now, we perform a change of variable. Let $\tau = t - t_0$ so that $t = \tau + t_0$. Using this change of variable, we can rewrite (3.39) as

$$h^{\text{inv}}(\tau) = \tfrac{1}{A}\delta(\tau + t_0).$$

Since $A \neq 0$, the function $h^{\text{inv}}(t)$ is always well defined. Therefore, the inverse system exists and has the impulse response $h^{\text{inv}}(t)$ given by

$$h^{\text{inv}}(t) = \tfrac{1}{A}\delta(t + t_0).$$

□

Example 3.13. Consider the system shown in Figure 3.13 with the input $x(t)$ and output $y(t)$. Use the notion of an inverse system in order to express $y(t)$ in terms of $x(t)$.

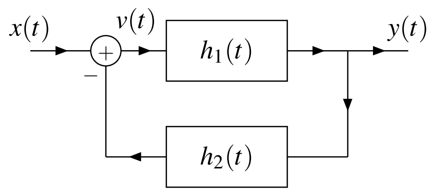


Figure 3.13: Feedback system.

Solution. From Figure 3.13, we can write:

$$v(t) = x(t) - y(t) * h_2(t) \quad \text{and} \tag{3.40}$$
$$y(t) = v(t) * h_1(t). \tag{3.41}$$

Substituting (3.40) into (3.41), and simplifying we obtain:

$$\begin{aligned}
& y(t) = [x(t) - y(t) * h_2(t)] * h_1(t) \\
\Rightarrow \quad & y(t) = x(t) * h_1(t) - y(t) * h_2(t) * h_1(t) \\
\Rightarrow \quad & y(t) + y(t) * h_2(t) * h_1(t) = x(t) * h_1(t) \\
\Rightarrow \quad & y(t) * \delta(t) + y(t) * h_2(t) * h_1(t) = x(t) * h_1(t) \\
\Rightarrow \quad & y(t) * [\delta(t) + h_2(t) * h_1(t)] = x(t) * h_1(t).
\end{aligned} \tag{3.42}$$

For convenience, we now define the function $g(t)$ as

$$g(t) = \delta(t) + h_2(t) * h_1(t). \tag{3.43}$$

So, we can rewrite (3.42) as

$$y(t) * g(t) = x(t) * h_1(t). \tag{3.44}$$

Thus, we have almost solved for $y(t)$ in terms of $x(t)$. To complete the solution, we need to eliminate $g(t)$ from the left-hand side of the equation. To do this, we use the notion of an inverse system. Consider the inverse of the system with impulse response $g(t)$. This inverse system has an impulse response $g^{\text{inv}}(t)$ given by

$$g(t) * g^{\text{inv}}(t) = \delta(t). \tag{3.45}$$

This relationship follows from the definition of an inverse system (associated with (3.37)). Now, we use $g^{\text{inv}}(t)$ in order to simplify (3.44) as follows:

$$\begin{aligned}
& y(t) * g(t) = x(t) * h_1(t) \\
\Rightarrow \quad & y(t) * g(t) * g^{\text{inv}}(t) = x(t) * h_1(t) * g^{\text{inv}}(t) \\
\Rightarrow \quad & y(t) * \delta(t) = x(t) * h_1(t) * g^{\text{inv}}(t) \\
\Rightarrow \quad & y(t) = x(t) * h_1(t) * g^{\text{inv}}(t).
\end{aligned}$$

Thus, we can express the output $y(t)$ in terms of the input $x(t)$ as

$$y(t) = x(t) * h_1(t) * g^{\text{inv}}(t),$$

where $g^{\text{inv}}(t)$ is given by (3.45) and $g(t)$ is given by (3.43). □

3.9.4 Stability

Recall that a system is BIBO stable if any arbitrary bounded input produces a bounded output. Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Further, suppose that $|x(t)| \leq A < \infty$ for all t (i.e., $x(t)$ is bounded). We can write

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= h(t) * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau. \end{aligned}$$

So, we have (by taking the magnitude of both sides of the preceding equation)

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right|. \tag{3.46}$$

One can show, for any two functions $f_1(t)$ and $f_2(t)$, that

$$\left| \int_{-\infty}^{\infty} f_1(t)f_2(t)dt \right| \leq \int_{-\infty}^{\infty} |f_1(t)f_2(t)|\,dt.$$

Using this inequality, we can rewrite (3.46) as

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|\,d\tau = \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\,d\tau.$$

We know (by assumption) that $|x(t)| \leq A$, so we can replace $|x(t)|$ by its bound A in the above inequality to obtain

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\,d\tau \leq \int_{-\infty}^{\infty} A\,|h(\tau)|\,d\tau = A\int_{-\infty}^{\infty} |h(\tau)|\,d\tau. \tag{3.47}$$

Thus, we have

$$|y(t)| \leq A\int_{-\infty}^{\infty} |h(\tau)|\,d\tau. \tag{3.48}$$

Since A is finite, we can deduce from (3.48) that $y(t)$ is bounded if

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty. \tag{3.49}$$

(i.e., $h(t)$ is absolutely integrable). Therefore, a LTI system is BIBO stable if its impulse response $h(t)$ is absolutely integrable (as in (3.49)). In other words, the absolute integrability of the impulse response $h(t)$ is a sufficient condition for BIBO stability.

As it turns out, condition (3.49) is also necessary for BIBO stability. Suppose that $h(t)$ is not absolutely integrable. That is, suppose that

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \infty.$$

If such is the case, we can show that the system is not BIBO stable. To begin, consider the particular input $x(t)$ given by the following:

$$x(t) = \operatorname{sgn}(h(-t)),$$

where

$$\operatorname{sgn}\alpha = \begin{cases} -1 & \text{for } \alpha < 0 \\ 0 & \text{for } \alpha = 0 \\ 1 & \text{for } \alpha > 0. \end{cases}$$

The signal $x(t)$ can only assume the values -1, 0, and 1. So, $x(t)$ is obviously bounded (i.e., $|x(t)| \leq 1$). The output $y(t)$ is given by

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} (\operatorname{sgn} h(-\tau))h(t-\tau)d\tau. \end{aligned} \tag{3.50}$$

Now, let us consider the output $y(t)$ at $t = 0$. From (3.50), we have

$$y(0) = \int_{-\infty}^{\infty} (\operatorname{sgn} h(-\tau))h(-\tau)d\tau. \tag{3.51}$$

Then, we observe that

$$\alpha \operatorname{sgn} \alpha = |\alpha| \quad \text{for any real } \alpha.$$

So $(\operatorname{sgn} h(-\tau))h(-\tau) = |h(-\tau)|$, and we can simplify (3.51) to obtain

$$\begin{aligned} y(0) &= \int_{-\infty}^{\infty} |h(-\tau)|\, d\tau \\ &= \int_{-\infty}^{\infty} |h(\tau)|\, d\tau \\ &= \infty. \end{aligned}$$

Thus, we have shown that the bounded input $x(t)$ will result in an unbounded output $y(t)$ (where $y(t)$ is unbounded at $t = 0$). Consequently, if the impulse response $h(t)$ is not absolutely integrable (i.e., does not satisfy (3.49)), the system is not BIBO stable. Consequently, the condition (3.49) is not only sufficient but also necessary for a system to be BIBO stable. In other words, we have that a LTI system is BIBO stable if and only if its impulse response $h(t)$ satisfies

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

(i.e., $h(t)$ is absolutely integrable).

Example 3.14. Suppose that we have the LTI system with impulse response $h(t)$ given by

$$h(t) = e^{at}u(t),$$

where a is a real constant. Determine for what values of the constant a the system is BIBO stable.

Solution. We need to determine for what values of a the impulse response $h(t)$ is absolutely integrable. Suppose that $a \neq 0$. We can write

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)|\, dt &= \int_{-\infty}^{\infty} \left|e^{at}u(t)\right| dt \\ &= \int_{0}^{\infty} \left|e^{at}\right| dt \\ &= \int_{0}^{\infty} e^{at}\, dt \\ &= [\tfrac{1}{a}e^{at}]\big|_{0}^{\infty} \\ &= \tfrac{1}{a}\left(e^{a\infty} - 1\right). \end{aligned}$$

We can see that the result of the above integration is finite if $a < 0$ and infinite if $a > 0$. In particular, if $a < 0$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)|\,dt &= 0 - \tfrac{1}{a} \\ &= -\tfrac{1}{a}.\end{aligned}$$

Let us now consider the case of $a = 0$. In this case, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} |u(t)|\,dt \\ &= \int_{0}^{\infty} dt \\ &= [t]|_{0}^{\infty} \\ &= \infty.\end{aligned}$$

Thus, we have shown that

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \begin{cases} -\frac{1}{a} & \text{for } a < 0 \\ \infty & \text{for } a \geq 0. \end{cases}$$

In other words, the impulse response $h(t)$ is absolutely integrable if $a < 0$. Consequently, the system is BIBO stable if $a < 0$. □

Example 3.15. Suppose that we have the LTI system with input $x(t)$ and output $y(t)$ defined by

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

(i.e., an ideal integrator). Determine whether this system is BIBO stable.

Solution. First, we find the impulse response $h(t)$ of the system.

$$\begin{aligned}h(t) &= \int_{-\infty}^{t} \delta(\tau)d\tau \\ &= \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} \\ &= u(t).\end{aligned}$$

Using this expression for $h(t)$, we now check to see if $h(t)$ is absolutely integrable. We have

$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} |u(t)|\,dt \\ &= \int_{0}^{\infty} dt \\ &= \infty.\end{aligned}$$

So, we have shown that $h(t)$ is not absolutely integrable. Therefore, the system is not BIBO stable. □

3.10 Response of Continuous-Time LTI Systems to Complex Exponential Signals

Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Consider now the response of the system to the complex exponential $x(t) = e^{st}$. We can write:

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= h(t) * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \\ &= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \\ &= H(s)e^{st} \end{aligned}$$

where $H(s)$ is the complex function given by

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau. \tag{3.52}$$

(The function $H(s)$ depends only on the complex quantity s.) From above, we can see that the response of a LTI system to a complex exponential is the same complex exponential multiplied by the complex factor $H(s)$. Thus, we have that e^{st} and $H(s)$ satisfy an eigenvalue problem of the form

$$\mathscr{H}\{\Phi(t)\} = \lambda\Phi(t)$$

where $\Phi(t) = e^{st}$ and $\lambda = H(s)$. As a matter of terminology, $\Phi(t)$ is said to be an **eigenfunction** and λ its corresponding **eigenvalue**. Therefore, e^{st} is an eigenfunction of a LTI system and $H(s)$ is the corresponding eigenvalue.

Suppose now that we can express some arbitrary input signal $x(t)$ as a sum of complex exponentials as follows:

$$x(t) = \sum_k a_k e^{s_k t}.$$

From the eigenvalue property, the response to the input $a_k e^{s_k t}$ is $a_k H(s_k)e^{s_k t}$. By using this knowledge and the superposition property, we can write

$$\begin{aligned} y(t) &= \mathscr{H}\{x(t)\} \\ &= \mathscr{H}\left\{\sum_k a_k e^{s_k t}\right\} \\ &= \sum_k a_k \mathscr{H}\{e^{s_k t}\} \\ &= \sum_k a_k H(s_k)e^{s_k t}. \end{aligned}$$

Thus, if an input to a LTI system can be represented as a linear combination of complex exponentials, the output can also be represented as linear combination of the same complex exponentials. As a matter of terminology, we refer to $H(s)$ as the **system function**.

Example 3.16. Suppose that we have the LTI system with the impulse response $h(t)$ given by

$$h(t) = \delta(t-1). \tag{3.53}$$

Find the system function $H(s)$.

Solution. Substituting (3.53) into (3.52), we obtain

$$\begin{aligned} H(s) &= \int_{-\infty}^{\infty} h(t)e^{-st}dt \\ &= \int_{-\infty}^{\infty} \delta(t-1)e^{-st}dt \\ &= e^{-s}. \end{aligned}$$

Therefore, the system function $H(s)$ is given by $H(s) = e^{-s}$. □

Example 3.17. Suppose that we have the LTI system from Example 3.16. Use the system function to determine the response $y(t)$ of the system to the particular input $x(t)$ given by

$$x(t) = e^t \cos(\pi t) = \tfrac{1}{2}e^{(1+j\pi)t} + \tfrac{1}{2}e^{(1-j\pi)t}.$$

Solution. The input $x(t)$ has already been expressed in the form

$$x(t) = \sum_{k=0}^{1} a_k e^{s_k t}$$

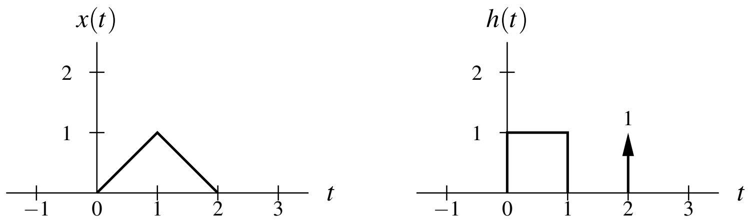
where $a_0 = a_1 = \frac{1}{2}$, $s_0 = 1 + j\pi$, and $s_1 = s_0^* = 1 - j\pi$. In Example 3.16, we found the system function $H(s)$ to be $H(s) = e^{-s}$. So we can calculate $y(t)$ by using the system function as follows:

$$\begin{aligned} y(t) &= \sum_k a_k H(s_k) e^{s_k t} \\ &= a_0 H(s_0) e^{s_0 t} + a_1 H(s_1) e^{s_1 t} \\ &= \tfrac{1}{2}H(1+j\pi)e^{(1+j\pi)t} + \tfrac{1}{2}H(1-j\pi)e^{(1-j\pi)t} \\ &= \tfrac{1}{2}e^{-(1+j\pi)}e^{(1+j\pi)t} + \tfrac{1}{2}e^{-(1-j\pi)}e^{(1-j\pi)t} \\ &= \tfrac{1}{2}e^{t-1+j\pi t-j\pi} + \tfrac{1}{2}e^{t-1-j\pi t+j\pi} \\ &= \tfrac{1}{2}e^{t-1}e^{j\pi(t-1)} + \tfrac{1}{2}e^{t-1}e^{-j\pi(t-1)} \\ &= e^{t-1}\left[\tfrac{1}{2}\left(e^{j\pi(t-1)} + e^{-j\pi(t-1)}\right)\right] \\ &= e^{t-1}\cos\pi(t-1). \end{aligned}$$

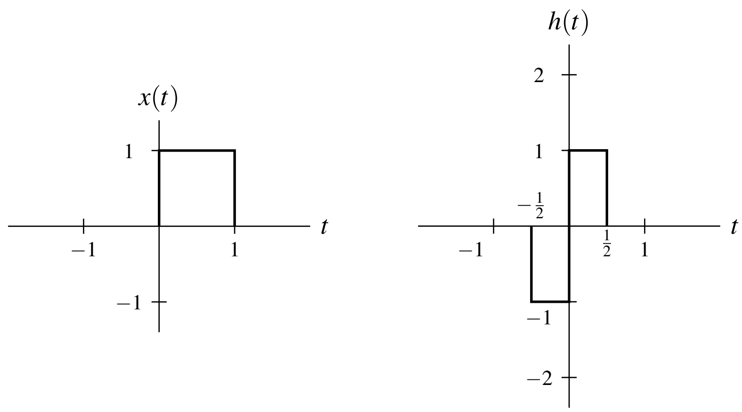
In passing, we note that we should expect the above result for $y(t)$, since the output of the system is nothing more than the input time-shifted by one. □

3.11 Problems

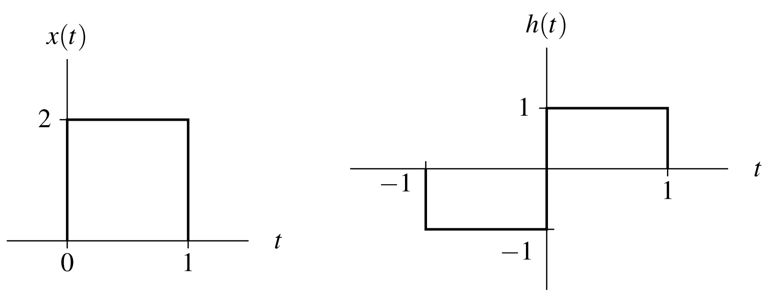
3.1 Using graphical methods, for each pair of signals $x(t)$ and $h(t)$ given in the figures below, compute the convolution $y(t) = x(t) * h(t)$.



(a)



(b)



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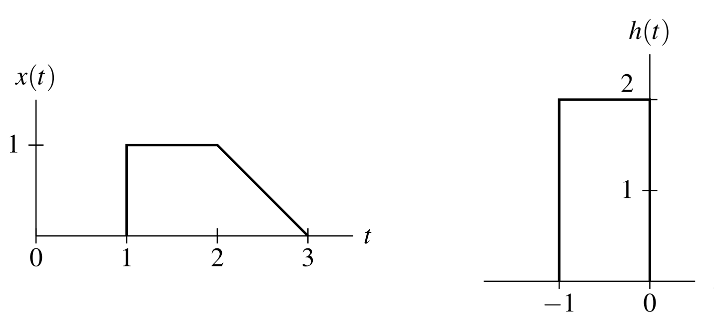

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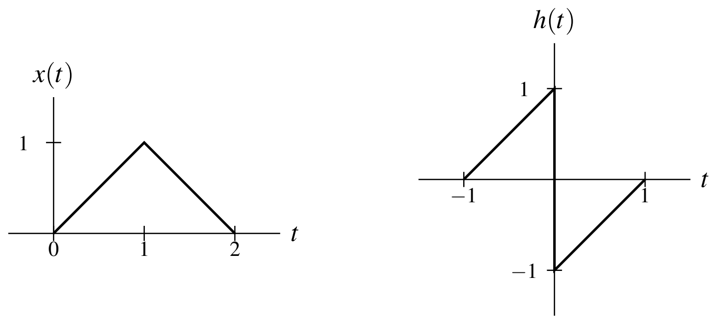
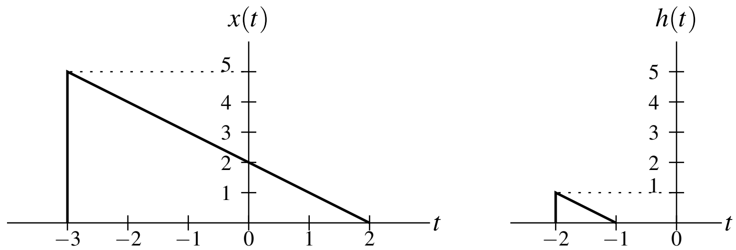


(e)



(f)

3.2 For each pair of signals $x(t)$ and $h(t)$ given below, compute the convolution $y(t) = x(t) * h(t)$.
(a) $x(t) = e^{at}u(t)$ and $h(t) = e^{-at}u(t)$ where a is a nonzero real constant;
(b) $x(t) = e^{-j\omega_0 t}u(t)$ and $h(t) = e^{j\omega_0 t}u(t)$ where ω_0 is a strictly positive real constant;
(c) $x(t) = u(t-2)$ and $h(t) = u(t+3)$;
(d) $x(t) = u(t)$ and $h(t) = e^{-2t}u(t-1)$;
(e) $x(t) = u(t-1) - u(t-2)$ and $h(t) = e^{t}u(-t)$.

3.3 Let $y(t) = x(t) * h(t)$. Given that

$$v(t) = \int_{-\infty}^{\infty} x(-\tau - b)h(\tau + at)d\tau,$$

where a and b are constants, express $v(t)$ in terms of $y(t)$.

3.4 Consider the convolution $y(t) = x(t) * h(t)$. Assuming that the convolution $y(t)$ exists, prove that each of the following assertions is true:
(a) If $x(t)$ is periodic then $y(t)$ is periodic.
(b) If $x(t)$ is even and $h(t)$ is odd, then $y(t)$ is odd.

3.5 From the definition of the convolution operation, show that if $y(t) = x(t) * h(t)$, then $\frac{d}{dt}y(t) = x(t) * [\frac{d}{dt}h(t)]$.

3.6 Let $x(t)$ and $h(t)$ be signals satisfying

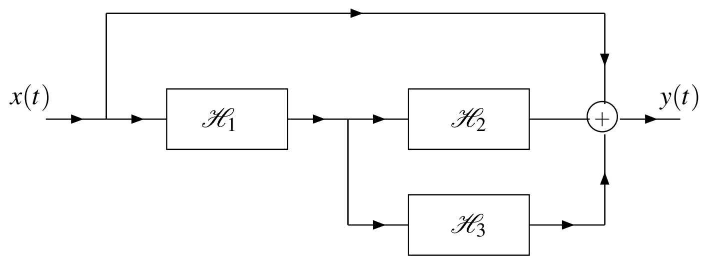
$$x(t) = 0 \quad \text{for } t < A_1 \text{ or } t > A_2, \quad \text{and}$$
$$h(t) = 0 \quad \text{for } t < B_1 \text{ or } t > B_2$$

(i.e., $x(t)$ and $h(t)$ are finite duration). Determine for which values of t the convolution $y(t) = x(t) * h(t)$ must be zero.

3.7 Find the impulse response of the LTI system characterized by each of the equations below. In each case, the input and output of the system are denoted as $x(t)$ and $y(t)$, respectively.
(a) $y(t) = \int_{-\infty}^{t+1} x(\tau)d\tau$;

(b) $y(t) = \int_{-\infty}^{\infty} x(\tau+5)e^{\tau-t+1}u(t-\tau-2)d\tau$;
(c) $y(t) = \int_{-\infty}^{t} x(\tau)v(t-\tau)d\tau$ and
(d) $y(t) = \int_{t-1}^{t} x(\tau)d\tau$.

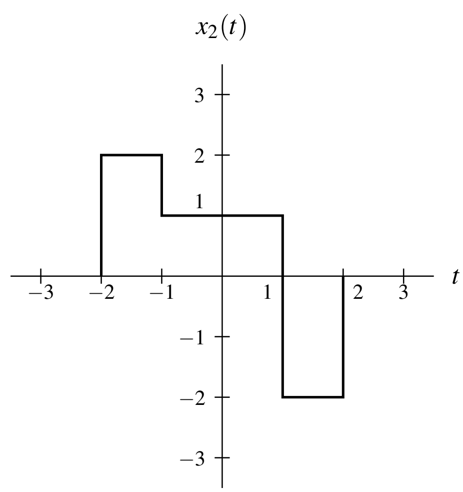
3.8 Consider the system with input $x(t)$ and output $y(t)$ as shown in the figure below. Suppose that the systems $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_3$ are LTI systems with impulse responses $h_1(t)$, $h_2(t)$, and $h_3(t)$, respectively.



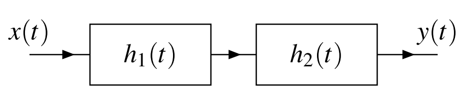
(a) Find the impulse response $h(t)$ of the overall system in terms of $h_1(t)$, $h_2(t)$, and $h_3(t)$.
(b) Determine the impulse response $h(t)$ in the specific case that

$$h_1(t) = \delta(t+1), \quad h_2(t) = \delta(t), \quad \text{and} \quad h_3(t) = \delta(t).$$

3.9 Consider a LTI system whose response to the signal $x_1(t) = u(t) - u(t-1)$ is the signal $y_1(t)$. Determine the response $y_2(t)$ of the system to the input $x_2(t)$ shown in the figure below in terms of $y_1(t)$.



3.10 Suppose that we have the system shown in the figure below with input $x(t)$ and output $y(t)$. This system is formed by the interconnection of two LTI systems with the impulse responses $h_1(t)$ and $h_2(t)$.



For each pair of $h_1(t)$ and $h_2(t)$ given below, find the output $y(t)$ if the input $x(t) = u(t)$.
(a) $h_1(t) = \delta(t)$ and $h_2(t) = \delta(t)$;
(b) $h_1(t) = \delta(t+1)$ and $h_2(t) = \delta(t+1)$;
(c) $h_1(t) = e^{-3t}u(t)$ and $h_2(t) = \delta(t)$.

3.11 Show that a linear system is invertible if and only if the only input $x(t)$ that produces the output $y(t) = 0$ for all t is $x(t) = 0$ for all t.

3.12 Consider the LTI systems with the impulse responses given below. Determine whether each of these systems is causal and/or memoryless.
(a) $h(t) = (t+1)u(t-1)$;
(b) $h(t) = 2\delta(t+1)$;
(c) $h(t) = \frac{\omega_c}{\pi} \operatorname{sinc} \omega_c t$;
(d) $h(t) = e^{-4t}u(t-1)$;
(e) $h(t) = e^t u(-1-t)$;
(f) $h(t) = e^{-3|t|}$; and
(g) $h(t) = 3\delta(t)$.

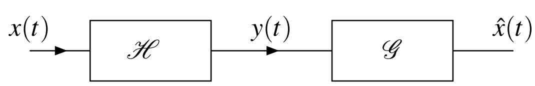
3.13 Consider the LTI systems with the impulse responses given below. Determine whether each of these systems is BIBO stable.
(a) $h(t) = e^{at}u(-t)$ where a is a strictly positive real constant;
(b) $h(t) = (1/t)u(t-1)$;
(c) $h(t) = e^t u(t)$;
(d) $h(t) = \delta(t-10)$;
(e) $h(t) = \operatorname{rect} t$; and
(f) $h(t) = e^{-|t|}$.

3.14 Suppose that we have two LTI systems with impulse responses

$$h_1(t) = \tfrac{1}{2}\delta(t-1) \quad \text{and} \quad h_2(t) = 2\delta(t+1).$$

Determine whether these systems are inverses of one another.

3.15 Consider the system shown in the figure below, where $\mathscr{H}$ is a LTI system and $\mathscr{G}$ is known to be the inverse system of $\mathscr{H}$. Let $y_1(t)$ and $y_2(t)$ denote the responses of the system $\mathscr{H}$ to the inputs $x_1(t)$ and $x_2(t)$, respectively.



(a) Determine the response of the system $\mathscr{G}$ to the input $a_1y_1(t) + a_2y_2(t)$ where a_1 and a_2 are complex constants.
(b) Determine the response of the system $\mathscr{G}$ to the input $y_1(t-t_0)$ where t_0 is a real constant.
(c) Using the results of the previous parts of this question, determine whether the system $\mathscr{G}$ is linear and/or time invariant.

3.16 Suppose that we have the systems $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}_3$, and $\mathscr{H}_4$, whose responses to a complex exponential input e^{j2t} are given by

$$e^{j2t} \xrightarrow{\mathscr{H}_1} 2e^{j2t},$$
$$e^{j2t} \xrightarrow{\mathscr{H}_2} te^{j2t},$$
$$e^{j2t} \xrightarrow{\mathscr{H}_3} e^{j2t+\pi/3}, \quad \text{and}$$
$$e^{j2t} \xrightarrow{\mathscr{H}_4} \cos 2t.$$

Indicate which of these systems cannot be LTI.

Chapter 4

Continuous-Time Fourier Series

4.1 Introduction

One very important tool in the study of signals and systems is the Fourier series. A very large class of signals can be represented using Fourier series, namely most practically useful periodic signals. The Fourier series represents a signal as a linear combination of complex sinusoids. This is often desirable since complex sinusoids are easy functions with which to work. For example, complex sinusoids are easy to integrate and differentiate. Also, complex sinusoids have important properties in relation to LTI systems. In particular, complex sinusoids are eigenfunctions of LTI systems. Therefore, the response of a LTI system to a complex sinusoid is the same complex sinusoid multiplied by a complex scaling factor.

4.2 Definition of Continuous-Time Fourier Series

Suppose that we have a set of **harmonically-related** complex sinusoids of the form

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t} \quad k = 0, \pm 1, \pm 2, \ldots.$$

The fundamental frequency of the kth complex sinusoid $\phi_k(t)$ is $k\omega_0$, an integer multiple of ω_0. Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of ω_0, a linear combination of these complex sinusoids must be periodic. More specifically, a linear combination of these complex sinusoids is periodic with period $T = 2\pi/\omega_0$.

Suppose that we can represent a periodic complex signal $x(t)$ as a linear combination of harmonically-related complex sinusoids:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \tag{4.1}$$

Such a representation is known as a **Fourier series**. More specifically, this is the **complex exponential form** of the Fourier series. The terms in the summation for $k = 1$ and $k = -1$ are known as the fundamental frequency components or **first harmonic components**, and have the fundamental frequency ω_0. More generally, the terms in the summation for $k = K$ and $k = -K$ are called the Kth **harmonic components**, and have the fundamental frequency $K\omega_0$. Since the complex sinusoids are harmonically related, the signal $x(t)$ is periodic with period $T = 2\pi/\omega_0$ (and frequency ω_0).

Since we often work with Fourier series, it is sometimes convenient to have an abbreviated notation to indicate that a signal is associated with particular Fourier series coefficients. If a signal $x(t)$ has the Fourier series coefficient sequence c_k, we sometimes indicate this using the notation

$$x(t) \overset{\mathscr{FS}}{\longleftrightarrow} c_k.$$

Consider the Fourier series representation of the periodic signal $x(t)$ given by (4.1). In the most general case, $x(t)$ is a complex signal, but let us now suppose that $x(t)$ is a real signal. In the case of real signals, an important relationship exists between the Fourier series coefficients c_k and c_{-k}. To show this, we proceed as follows. Suppose that we can represent $x(t)$ in the form of (4.1). So, we have

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \tag{4.2}$$

Taking the complex conjugate of both sides of the preceding equation, we obtain

$$\begin{aligned} x^*(t) &= \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} \left(c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}. \end{aligned} \tag{4.3}$$

Since $x(t)$ is real, we know that $x(t) = x^*(t)$, and we can rewrite (4.3) as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}.$$

Replacing k by $-k$ in the summation, we obtain

$$x(t) = \sum_{k=-\infty}^{\infty} c_{-k}^* e^{jk\omega_0 t}. \tag{4.4}$$

Comparing (4.2) and (4.4), we can see that

$$c_k = c_{-k}^*. \tag{4.5}$$

Consequently, if $x(t)$ is a real signal, we have that c_k and c_{-k} are complex conjugates of each other.

Using the relationship in (4.5), we can derive two alternative forms of the Fourier series for the case of real signals. We begin by rewriting (4.1) in a slightly different form. In particular, we rearrange the summation to obtain

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right].$$

Substituting $c_k = c_{-k}^*$ from (4.5), we obtain

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} \right].$$

Now, we observe that the two terms inside the summation are complex conjugates of each other. So, we can rewrite the equation as

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\{c_k e^{jk\omega_0 t}\}. \tag{4.6}$$

Let us now rewrite c_k in polar form as

$$c_k = |c_k|\, e^{j\theta_k},$$

where θ_k is real (i.e., $\theta_k = \arg c_k$). Substituting this expression for c_k into (4.6) yields

$$\begin{aligned}
x(t) &= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left\{|c_k|\, e^{j(k\omega_0 t+\theta_k)}\right\} \\
&= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left\{|c_k|\left[\cos(k\omega_0 t+\theta_k) + j\sin(k\omega_0 t+\theta_k)\right]\right\} \\
&= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left\{|c_k|\cos(k\omega_0 t+\theta_k) + j\,|c_k|\sin(k\omega_0 t+\theta_k)\right\}.
\end{aligned}$$

Finally, further simplification yields

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k|\cos(k\omega_0 t+\theta_k)$$

(where $\theta_k = \arg c_k$). This is known as the **combined trigonometric form** of a Fourier series.

A second alternative form of the Fourier series can be obtained by expressing c_k in Cartesian form as

$$c_k = \tfrac{1}{2}(a_k - jb_k).$$

where a_k and b_k are real. Substituting this expression for c_k into (4.6) from earlier yields

$$\begin{aligned}
x(t) &= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\{(\tfrac{1}{2}(a_k - jb_k)e^{jk\omega_0 t}\} \\
&= c_0 + \sum_{k=1}^{\infty} \operatorname{Re}\{(a_k - jb_k)(\cos k\omega_0 t + j\sin k\omega_0 t)\} \\
&= c_0 + \sum_{k=1}^{\infty} \operatorname{Re}\{a_k\cos k\omega_0 t + ja_k\sin k\omega_0 t - jb_k\cos k\omega_0 t + b_k\sin k\omega_0 t\}.
\end{aligned}$$

Further simplification yields

$$x(t) = c_0 + \sum_{k=1}^{\infty}\left[a_k\cos k\omega_0 t + b_k\sin k\omega_0 t\right]$$

(where $a_k = \operatorname{Re} 2c_k$ and $b_k = -\operatorname{Im} 2c_k$). This is known as the **trigonometric form** of a Fourier series.

By comparing the various forms of the Fourier series introduced above, we can see that the quantities c_k, a_k, b_k, and θ_k are related as follows:

$$2c_k = a_k - jb_k \quad \text{and} \quad c_k = |c_k|\, e^{j\theta_k}.$$

(Recall that a_k, b_k, and θ_k are real and c_k is complex.)

4.3 Determining the Fourier Series Representation of a Continuous-Time Periodic Signal

Given an arbitrary periodic signal $x(t)$, we need some means for finding its corresponding Fourier series representation. In other words, we need a method for calculating the Fourier series coefficients c_k. In what follows, we derive a formula for the calculation of the c_k. We begin with the definition of the Fourier series in (4.1). Multiplying both sides of this equation by $e^{-jn\omega_0 t}$ yields

$$\begin{aligned}
x(t)e^{-jn\omega_0 t} &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-jn\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} c_k e^{j(k-n)\omega_0 t}.
\end{aligned}$$

As a matter of notation, we use $\int_T$ to denote the integral over an arbitrary interval of length T (i.e., the interval (t_0, t_0+T) for arbitrary t_0). Integrating both sides of this equation over one period T of $x(t)$, we obtain

$$\int_T x(t)e^{-jn\omega_0 t}dt = \int_T \sum_{k=-\infty}^{\infty} c_k e^{j(k-n)\omega_0 t}dt.$$

Reversing the order of integration and summation yields

$$\int_T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} c_k \left(\int_T e^{j(k-n)\omega_0 t}dt \right). \tag{4.7}$$

Consider now the integral on the right-hand side of this equation. In order to evaluate this integral, we employ Euler's formula to write

$$\begin{aligned} \int_T e^{j(k-n)\omega_0 t}dt &= \int_T \left[\cos\left((k-n)\omega_0 t\right) + j\sin\left((k-n)\omega_0 t\right)\right] dt \\ &= \int_T \cos\left((k-n)\omega_0 t\right) dt + j\int_T \sin\left((k-n)\omega_0 t\right) dt. \end{aligned} \tag{4.8}$$

For $k \neq n$, $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ are both sinusoids of frequency $(k-n)\omega_0$ and period $T/|k-n|$. Since we are integrating over an interval of length T, both sinusoids will be integrated over an integral number of periods, resulting in a value of zero. For $k = n$, the integrand on the left-hand side of (4.8) is simply $e^{j0} = 1$, and the result of integration is T. To summarize, we have the following:

$$\int_T e^{j(k-n)\omega_0 t}dt = \begin{cases} T & \text{for } k = n \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

Substituting (4.9) into (4.7), we obtain

$$\int_T x(t)e^{-jn\omega_0 t}dt = c_n T. \tag{4.10}$$

Rearranging, we obtain the following expression for c_n:

$$c_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt.$$

Thus, we can calculate the Fourier series coefficient sequence c_k for an arbitrary periodic signal $x(t)$ using the formula:

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt. \tag{4.11}$$

As a matter of terminology, we refer to (4.11) as the **Fourier series analysis equation** and (4.1) as the **Fourier series synthesis equation**.

Suppose that we have a complex periodic function $x(t)$ with period T and Fourier series coefficient sequence c_k. One can easily show that the coefficient c_0 is the average value of $x(t)$ over a single period T. The proof is trivial. Consider the Fourier series analysis equation given by (4.11). Substituting $k = 0$ into this equation, we obtain

$$\begin{aligned} c_0 &= \left[\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \right] \Bigg|_{k=0} \\ &= \frac{1}{T}\int_T x(t)e^0 dt \\ &= \frac{1}{T}\int_T x(t)dt. \end{aligned}$$

Thus, c_0 is simply the average value of $x(t)$ over a single period.

Example 4.1 (Fourier series of a periodic square wave)**.** Find the Fourier series representation of the periodic square wave $x(t)$ shown in Figure 4.1.

Solution. Let us consider the single period of $x(t)$ for $0 \leq t < T$. For this range of t, we have

$$x(t) = \begin{cases} A & \text{for } 0 \leq t < \frac{T}{2} \\ -A & \text{for } \frac{T}{2} \leq t < T. \end{cases}$$

We use the Fourier series analysis equation (4.11) to write (and subsequently assume that $k \neq 0$):

$$\begin{aligned}
c_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\
&= \frac{1}{T} \left(\int_0^{T/2} A e^{-jk\omega_0 t} dt + \int_{T/2}^{T} (-A) e^{-jk\omega_0 t} dt \right) \\
&= \frac{1}{T} \left(\left[\frac{-A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_0^{T/2} + \left[\frac{A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_{T/2}^{T} \right) \\
&= \frac{-A}{j2\pi k} \left(\left[e^{-jk\omega_0 t} \right] \Big|_0^{T/2} - \left[e^{-jk\omega_0 t} \right] \Big|_{T/2}^{T} \right) \\
&= \frac{jA}{2\pi k} \left(\left[e^{-j\pi k} - 1 \right] - \left[e^{-j2\pi k} - e^{-j\pi k} \right] \right) \\
&= \frac{jA}{2\pi k} \left[2e^{-j\pi k} - e^{-j2\pi k} - 1 \right] \\
&= \frac{jA}{2\pi k} \left[2(e^{-j\pi})^k - (e^{-j2\pi})^k - 1 \right].
\end{aligned}$$

Now, we observe that $e^{-j\pi} = -1$ and $e^{-j2\pi} = 1$. So, we have

$$\begin{aligned}
c_k &= \frac{jA}{2\pi k} [2(-1)^k - 1^k - 1] \\
&= \frac{jA}{2\pi k} [2(-1)^k - 2] \\
&= \frac{jA}{\pi k} [(-1)^k - 1] \\
&= \begin{cases} \frac{-j2A}{\pi k} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even, } k \neq 0. \end{cases}
\end{aligned}$$

Now, we consider the case of c_0. We have

$$\begin{aligned}
c_0 &= \tfrac{1}{T} \int_T x(t) dt \\
&= \tfrac{1}{T} \left[\int_0^{T/2} A dt - \int_{T/2}^{T} A dt \right] \\
&= \tfrac{1}{T} \left[\tfrac{AT}{2} - \tfrac{AT}{2} \right] \\
&= 0.
\end{aligned}$$

Thus, the Fourier series of $x(t)$ is given by (4.1) where the coefficient sequence c_k is

$$c_k = \begin{cases} \frac{-j2A}{\pi k} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even.} \end{cases}$$

□

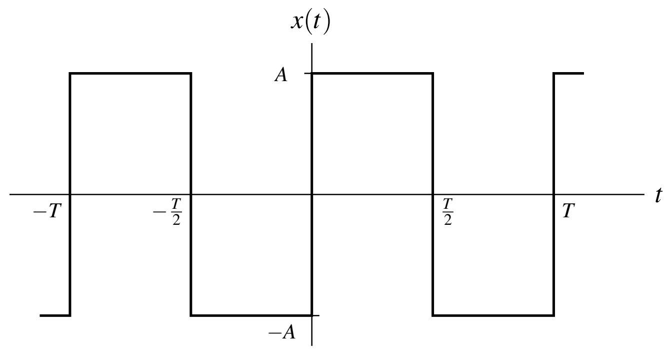


Figure 4.1: Periodic square wave.

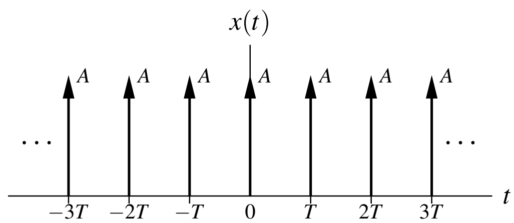


Figure 4.2: Periodic impulse train.

Example 4.2 (Fourier series of a periodic impulse train)**.** Suppose that we have the periodic impulse train $x(t)$ shown in Figure 4.2. Find the Fourier series representation of $x(t)$.

Solution. Let us consider the single period of $x(t)$ for $-\frac{T}{2} \leq t < \frac{T}{2}$. We use the Fourier series analysis equation (4.11) to write:

$$\begin{aligned} c_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} A\delta(t) e^{-jk\omega_0 t} dt \\ &= \frac{A}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt. \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

$$c_k = \frac{A}{T}.$$

Thus, the Fourier series for $x(t)$ is given by (4.1) where the coefficient sequence c_k is $c_k = \frac{A}{T}$. □

Example 4.3. Consider the periodic function $x(t)$ with fundamental period $T = 3$ as shown in Figure 4.3. Find the Fourier series representation of $x(t)$.

Solution. The signal $x(t)$ has the fundamental frequency $\omega_0 = 2\pi/T = \frac{2\pi}{3}$. Let us consider the single period of $x(t)$

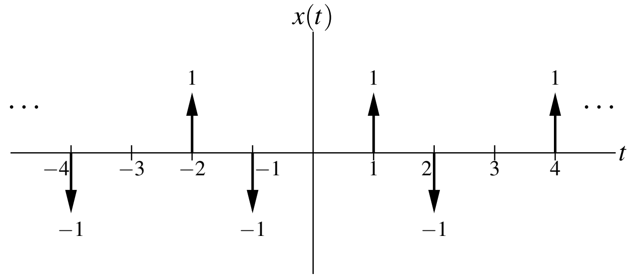


Figure 4.3: Periodic impulse train.

for $-\frac{T}{2} \leq t < \frac{T}{2}$ (i.e., $-\frac{3}{2} \leq t < \frac{3}{2}$). Computing the Fourier series coefficients c_k, we have

$$\begin{aligned}
c_k &= \tfrac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\
&= \tfrac{1}{T} \int_{-3/2}^{3/2} x(t) e^{-jk\omega_0 t} dt \\
&= \tfrac{1}{T} \left[\int_{-3/2}^{0} -\delta(t+1) e^{-jk\omega_0 t} dt + \int_0^{3/2} \delta(t-1) e^{-jk\omega_0 t} dt \right] \\
&= \tfrac{1}{T} \left[-e^{-jk\omega_0(-1)} + e^{-jk\omega_0(1)} \right] \\
&= \tfrac{1}{T} \left[e^{-jk\omega_0} - e^{jk\omega_0} \right] \\
&= \tfrac{1}{T} \left[2j \sin(-k\omega_0) \right] \\
&= \tfrac{2j}{T} \sin(-k\omega_0) \\
&= -\tfrac{2j}{T} \sin k\omega_0 \\
&= -\tfrac{2j}{3} \sin \tfrac{2\pi k}{3}.
\end{aligned}$$

Thus, $x(t)$ has the Fourier series representation

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} -\tfrac{2j}{3} (\sin \tfrac{2\pi k}{3}) e^{j2\pi kt/3}.
\end{aligned}$$

□

Example 4.4 (Fourier series of an even real function)**.** Let $x(t)$ be an arbitrary periodic real function that is even. Let c_k denote the Fourier series coefficient sequence for $x(t)$. Show that $\text{Im}\{c_k\} = 0$, $c_k = c_{-k}$ and $c_0 = \frac{1}{T}\int_0^T x(t)dt$.

Proof. From the Fourier series analysis equation (4.11) and using Euler's relation, we can write

$$\begin{aligned}
c_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\
&= \frac{1}{T} \int_T \left(x(t) \left[\cos(-k\omega_0 t) + j \sin(-k\omega_0 t) \right] \right) dt.
\end{aligned}$$

Since cos and sin are even and odd functions, respectively, we can rewrite the above equation as

$$\begin{aligned}
c_k &= \frac{1}{T} \int_T \left[x(t) \left(\cos k\omega_0 t - j \sin k\omega_0 t \right) \right] dt \\
&= \frac{1}{T} \left[\int_T x(t) \cos k\omega_0 t \, dt - j \int_T x(t) \sin k\omega_0 t \, dt \right].
\end{aligned}$$

Consider the first integral above (i.e., the one involving the cos function). Since $x(t)$ is even and $\cos k\omega_0 t$ is even, we have that $x(t)\cos k\omega_0 t$ is even. Thus, $\int_T x(t)\cos k\omega_0 t\,dt = 2\int_0^{T/2} x(t)\cos k\omega_0 t\,dt$. Consider the second integral above (i.e., the one involving the sin function). Since $x(t)$ is even and $\sin k\omega_0 t$ is odd, we have that $x(t)\sin k\omega_0 t$ is odd. If we integrate an odd periodic function over one period (or an integer multiple thereof), the result is zero. Therefore, the second integral is zero. Combining these results, we can write

$$\begin{aligned} c_k &= \frac{1}{T}\left[2\int_0^{T/2} x(t)\cos k\omega_0 t\,dt\right] \\ &= \frac{2}{T}\int_0^{T/2} x(t)\cos k\omega_0 t\,dt. \end{aligned} \tag{4.12}$$

Since $x(t)$ is real, the quantity c_k must also be real. Thus, we have that $\operatorname{Im}\{c_k\} = 0$.

Consider now the expression for c_{-k}. We substitute $-k$ for k in (4.12) to obtain

$$c_{-k} = \frac{2}{T}\int_0^{T/2} x(t)\cos(-k\omega_0 t)\,dt.$$

Since cos is an even function, we can simplify this expression to obtain

$$\begin{aligned} c_{-k} &= \frac{2}{T}\int_0^{T/2} x(t)\cos(k\omega_0 t)\,dt \\ &= c_k. \end{aligned}$$

Thus, $c_k = c_{-k}$.

Consider now the quantity c_0. Substituting $k = 0$ into (4.12), we can write

$$\begin{aligned} c_0 &= \frac{2}{T}\int_0^{T/2} x(t)\cos(0)\,dt \\ &= \frac{2}{T}\int_0^{T/2} x(t)\,dt \\ &= \frac{2}{T}\left[\frac{1}{2}\int_0^{T} x(t)\,dt\right] \\ &= \frac{1}{T}\int_0^{T} x(t)\,dt. \end{aligned}$$

Thus, we have shown that $c_0 = \frac{1}{T}\int_0^T x(t)\,dt$. Therefore, in summary, we have shown that $\operatorname{Im}\{c_k\} = 0$, $c_k = c_{-k}$, and $c_0 = \frac{1}{T}\int_0^T x(t)\,dt$. □

Example 4.5 (Fourier series of an odd real function)**.** Let $x(t)$ be a periodic real function that is odd. Let c_k denote the Fourier series coefficient sequence for $x(t)$. Show that $\operatorname{Re}\{c_k\} = 0$, $c_k = -c_{-k}$, and $c_0 = 0$.

Proof. From the Fourier series analysis equation (4.11) and Euler's formula, we can write

$$\begin{aligned} c_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt \\ &= \frac{1}{T}\left[\int_T x(t)\left[\cos(-k\omega_0 t) + j\sin(-k\omega_0 t)\right]dt\right]. \end{aligned}$$

Since cos and sin are even and odd functions, respectively, we can rewrite the above equation as

$$\begin{aligned} c_k &= \frac{1}{T}\left[\int_T x(t)\left[\cos k\omega_0 t - j\sin k\omega_0 t\right]dt\right] \\ &= \frac{1}{T}\left[\int_T x(t)\cos k\omega_0 t\,dt - j\int_T x(t)\sin k\omega_0 t\,dt\right]. \end{aligned}$$

Consider the first integral above (i.e., the one involving the cos function). Since $x(t)$ is odd and $\cos k\omega_0 t$ is even, we have that $x(t)\cos k\omega_0 t$ is odd. If we integrate an odd periodic function over a single period (or an integer multiple thereof), the result is zero. Therefore, the first integral is zero. Consider the second integral above (i.e., the one involving the sin function). Since $x(t)$ is odd and $\sin k\omega_0 t$ is odd, we have that $x(t)\sin k\omega_0 t$ is even. Thus, $\int_T x(t)\sin k\omega_0 t dt = 2\int_0^{T/2} x(t)\sin k\omega_0 t dt$. Combining these results, we can write

$$\begin{aligned} c_k &= \frac{-j}{T}\int_T x(t)\sin k\omega_0 t dt \\ &= \frac{-j2}{T}\int_0^{T/2} x(t)\sin k\omega_0 t dt. \end{aligned} \tag{4.13}$$

Since $x(t)$ is real, the result of the integration is real, and consequently c_k is purely imaginary. Thus, $\operatorname{Re}\{c_k\} = 0$.

Consider the quantity c_{-k}. Substituting $-k$ for k in (4.13), we obtain

$$\begin{aligned} c_{-k} &= \frac{-j2}{T}\int_0^{T/2} x(t)\sin(-k\omega_0 t)dt \\ &= \frac{-j2}{T}\int_0^{T/2} x(t)[-\sin(k\omega_0 t)]dt \\ &= \frac{j2}{T}\int_0^{T/2} x(t)\sin k\omega_0 t dt \\ &= -c_k. \end{aligned}$$

Thus, $c_k = -c_{-k}$.

Consider now the quantity c_0. Substituting $k = 0$ in the expression (4.13), we have

$$\begin{aligned} c_0 &= \frac{-j2}{T}\int_0^{T/2} x(t)\sin(0)dt \\ &= 0. \end{aligned}$$

Thus, $c_0 = 0$. Therefore, in summary, we have shown that $\operatorname{Re}\{c_k\} = 0$, $c_k = -c_{-k}$, and $c_0 = 0$. □

4.4 Convergence of Continuous-Time Fourier Series

So far we have assumed that a given periodic signal $x(t)$ can be represented by a Fourier series. Since a Fourier series consists of an infinite number of terms, we need to more carefully consider the issue of convergence. That is, we want to know under what circumstances the Fourier series of $x(t)$ converges (in some sense) to $x(t)$.

Suppose that we have an arbitrary periodic signal $x(t)$. This signal has the Fourier series representation given by (4.1) and (4.11). Let $x_N(t)$ denote the finite series

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

(i.e., $x_N(t)$ is a Fourier series truncated after the Nth harmonic components). The approximation error is given by

$$e_N(t) = x(t) - x_N(t).$$

Let us also define the mean-squared error (MSE) as

$$E_N = \frac{1}{T}\int_T |e_N(t)|^2 dt.$$

Before we can proceed further, we need to more precisely specify what we mean by convergence. This is necessary because convergence can be defined in more than one way. For example, two common types of convergence are:

pointwise and MSE. In the case of pointwise convergence, the error goes to zero at every point. If convergence is pointwise and the rate of convergence is the same everywhere, we call this uniform convergence. In the case of MSE convergence, the MSE goes to zero, which does not necessarily imply that the error goes to zero at every point.

Now, we introduce a few important results regarding the convergence of Fourier series for various types of periodic signals. The first result that we consider is for the case of continuous signals as given below.

Theorem 4.1 (Convergence of Fourier series (continuous case))**.** *If the periodic signal $x(t)$ is a continuous function of t, then its Fourier series converges uniformly (i.e., converges pointwise and at the same rate everywhere).*

In other words, in the above theorem, we have that if $x(t)$ is continuous, then as $N \to \infty$, $e_N(t) \to 0$ for all t. Often, however, we must work with signals that are not continuous. For example, many useful periodic signals are not continuous (e.g., the square wave). Consequently, we must consider the matter of convergence for signals with discontinuities.

Another important result regarding convergence applies to signals that have finite energy over a single period. Mathematically, a signal $x(t)$ has finite energy over a single period if it satisfies

$$\int_T |x(t)|^2 \, dt < \infty.$$

In the case of such a signal, we have the following important result.

Theorem 4.2 (Convergence of Fourier series (finite-energy case))**.** *If the periodic signal $x(t)$ has finite energy in a single period (i.e., $\int_T |x(t)|^2 \, dt < \infty$), the Fourier series converges in the MSE sense.*

In other words, in the above theorem, we have that if $x(t)$ is of finite energy, then as $N \to \infty$, $E_N \to 0$.

The last important result regarding convergence that we shall consider relates to what are known as the Dirichlet conditions. The Dirichlet[1] conditions for the periodic signal $x(t)$ are as follows:

1. Over a single period, $x(t)$ is absolutely integrable (i.e., $\int_T |x(t)| \, dt < \infty$).
2. In any finite interval of time, $x(t)$ is of bounded variation. In other words, there must be a finite number of maxima and minima in a single period of $x(t)$.
3. In any finite interval of time, $x(t)$ has a finite number of discontinuities, each of which is finite.

Theorem 4.3 (Convergence of Fourier series (Dirichlet case))**.** *If $x(t)$ is a periodic signal satisfying the Dirichlet conditions, then:*

1. *The Fourier series converges pointwise everywhere to $x(t)$, except at the points of discontinuity of $x(t)$.*
2. *At each point $t = t_a$ of discontinuity of $x(t)$, the Fourier series converges to $\frac{1}{2}(x(t_a^-) + x(t_a^+))$ where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the signal on the left- and right-hand sides of the discontinuity, respectively.*

In other words, if the Dirichlet conditions are satisfied, then as $N \to \infty$, $e_N(t) \to 0$ for all t except at discontinuities. Furthermore, at each discontinuity, the Fourier series converges to the average of the signal values on the left- and right-hand side of the discontinuity.

Example 4.6. Consider the periodic function $x(t)$ with period $T = 2$ as shown in Figure 4.4. Let $\hat{x}(t)$ denote the Fourier series representation of $x(t)$ (i.e., $\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, where $\omega_0 = \pi$). Determine the values $\hat{x}(0)$ and $\hat{x}(1)$.

Solution. We begin by observing that $x(t)$ satisfies the Dirichlet conditions. Consequently, Theorem 4.3 applies. Thus, we have that

$$\begin{aligned} \hat{x}(0) &= \tfrac{1}{2}\left[x(0^-) + x(0^+)\right] = \tfrac{1}{2}(0+1) = \tfrac{1}{2} \quad \text{and} \\ \hat{x}(1) &= \tfrac{1}{2}\left[x(1^-) + x(1^+)\right] = \tfrac{1}{2}(1+0) = \tfrac{1}{2}. \end{aligned}$$

□

[1] Pronounced Dee-ree-klay.

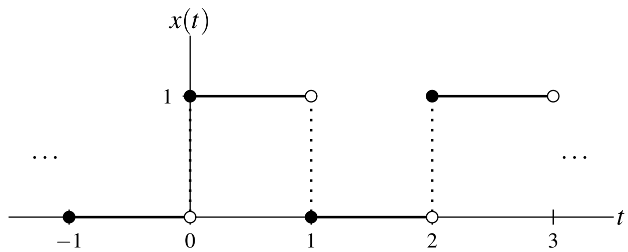


Figure 4.4: Periodic signal $x(t)$.

Although many signals of practical interest satisfy the Dirichlet conditions, not all signals satisfy these conditions. For example, consider the periodic signal $x(t)$ defined by

$$x(t) = 1/t \quad \text{for } 0 < t \leq 1 \quad \text{and} \quad x(t) = x(t+1).$$

This signal is plotted in Figure 4.5(a). This signal violates the first condition, since the signal is not absolutely integrable over a single period.

Consider the periodic signal $x(t)$ defined by

$$x(t) = \sin\left(\frac{2\pi}{t}\right) \quad \text{for } 0 < t \leq 1 \quad \text{and} \quad x(t) = x(t+1).$$

This signal is plotted in Figure 4.5(b). Since this signal has an infinite number of minima and maxima in a single period, the second condition is violated.

The third condition is violated by the periodic signal shown in Figure 4.5(c), which has an infinite number of discontinuities in a single period.

One might wonder how the Fourier series converges for periodic signals with discontinuities. Let us consider the periodic square wave from Example 4.1. In Figure 4.6, we have plotted the truncated Fourier series $x_N(t)$ for the square wave (with period $T = 1$ and amplitude $A = 1$) for several values of N. At the discontinuities of $x(t)$, we can see that the series appears to converge to the average of the signal values on either side of the discontinuity. In the vicinity of a discontinuity, however, the truncated series $x_N(t)$ exhibits ripples and the peak amplitude of the ripples does not seem to decrease with increasing N. As it turns out, as N increases, the ripples get compressed towards discontinuity, but, for any finite N, the peak amplitude of the ripples remains constant. This behavior is known as **Gibbs phenomenon**.

4.5 Properties of Continuous-Time Fourier Series

Fourier series representations possess a number of important properties. In the sections that follow, we introduce a few of these properties. For convenience, these properties are also summarized later in Table 4.1 (on page 91).

4.5.1 Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T and frequency $\omega_0 = 2\pi/T$. If

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \quad \text{and}$$
$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k,$$

then

$$Ax(t) + By(t) \overset{\mathcal{FS}}{\longleftrightarrow} Aa_k + Bb_k,$$

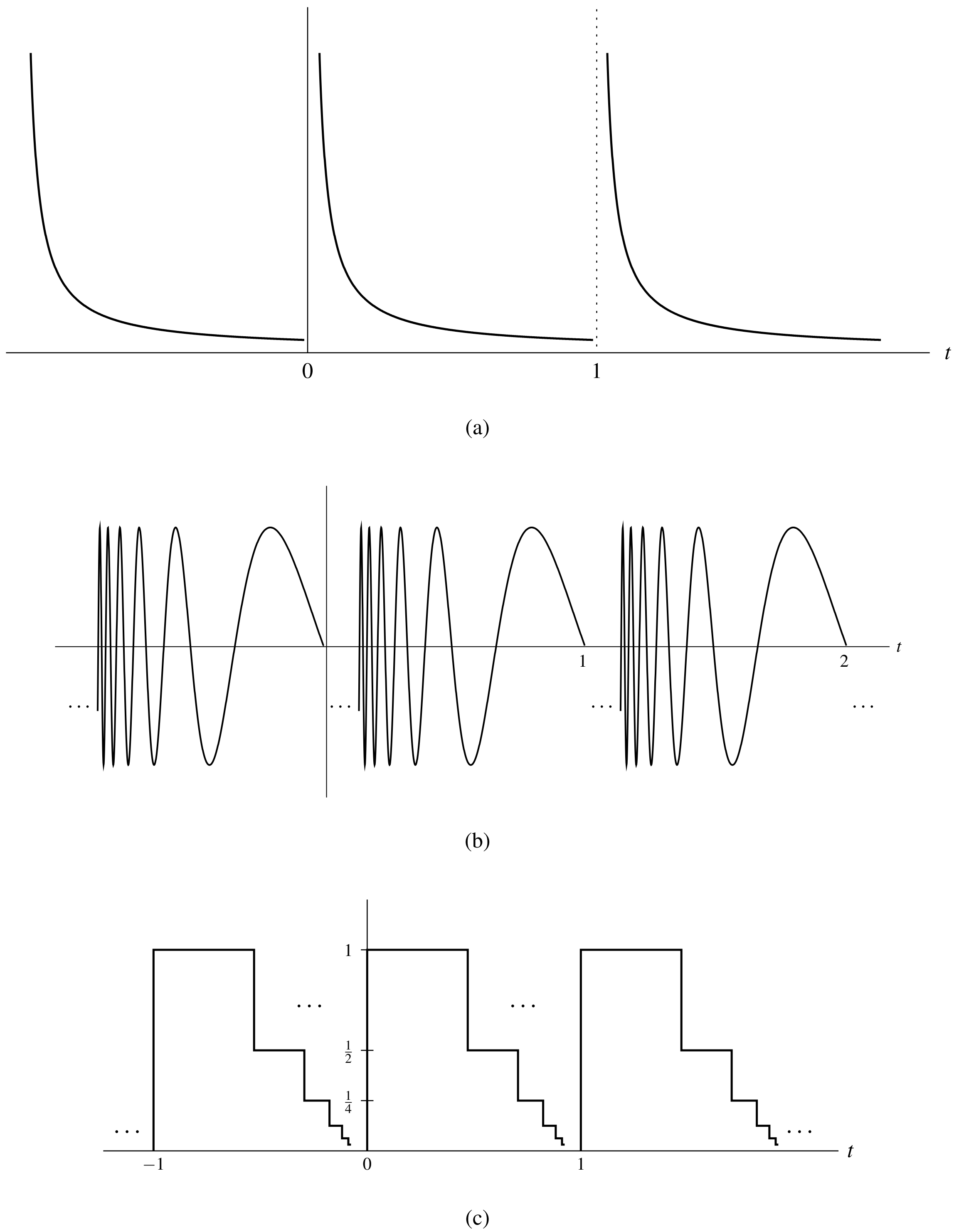


Figure 4.5: Examples of signals that violate the Dirichlet conditions. (a) A signal that is not absolutely integrable over a single period. (b) A signal that has an infinite number of maxima and minima over a single period. (c) A signal that has an infinite number of discontinuities over a single period.

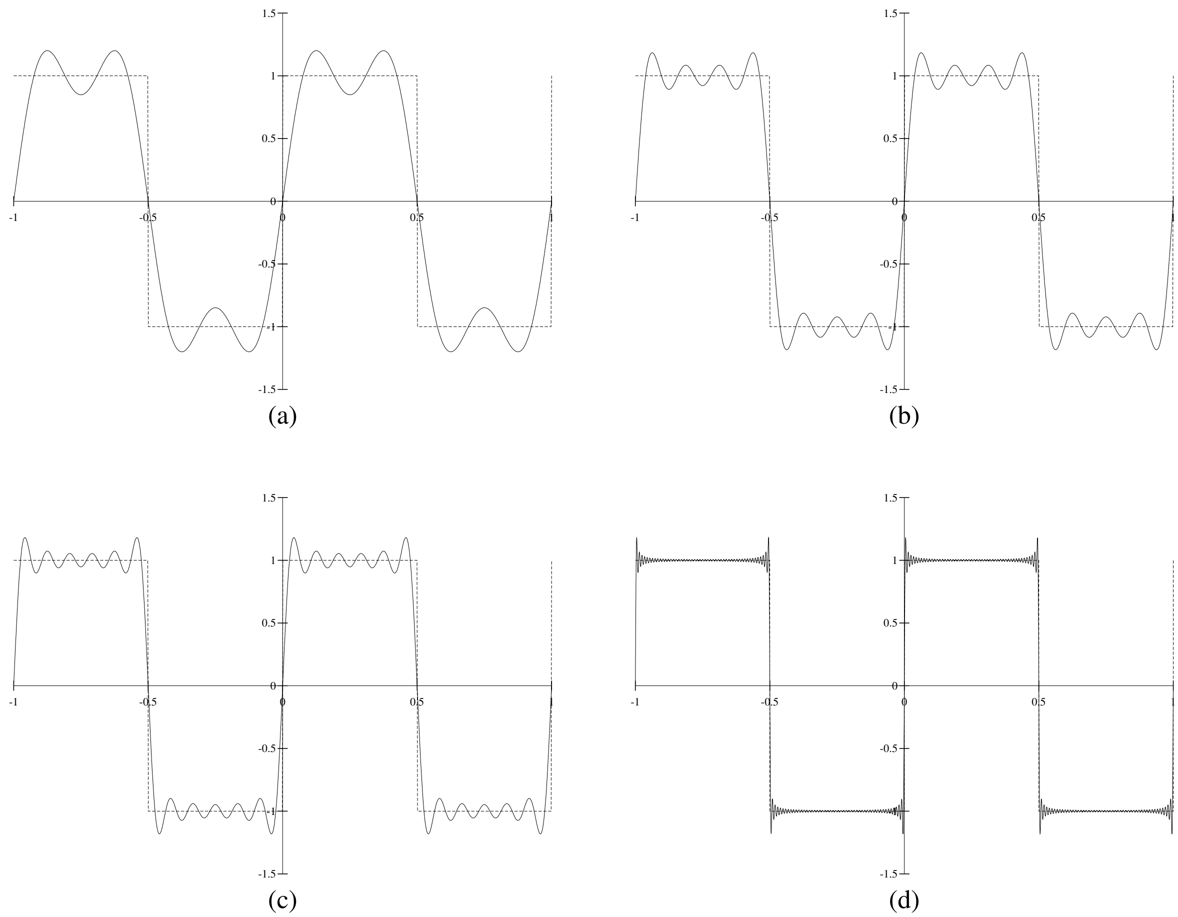


Figure 4.6: Gibbs phenomenon. The Fourier series for the periodic square wave truncated after the Nth harmonic components for (a) $N = 3$, (b) $N = 7$, (c) $N = 11$, and (d) $N = 101$.

where A and B are complex constants. In other words, a linear combination of signals produces the same linear combination of their Fourier series coefficients.

To prove the above property, we proceed as follows. First, we express $x(t)$ and $y(t)$ in terms of their corresponding Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{and}$$
$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}.$$

Now, we determine the Fourier series of $Ax(t) + By(t)$. We have

$$\begin{aligned} Ax(t) + By(t) &= A \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + B \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} A a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} B b_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} (A a_k + B b_k) e^{jk\omega_0 t}. \end{aligned}$$

Therefore, we have that $Ax(t) + By(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} Aa_k + Bb_k$.

4.5.2 Time Shifting

Let $x(t)$ denote a periodic signal with period T and frequency $\omega_0 = 2\pi/T$. If

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$

then

$$x(t - t_0) \stackrel{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k,$$

where t_0 is a real constant. From this, we can see that time shifting a periodic signal does not change the magnitude of its Fourier series coefficients (since $\left|e^{j\theta}\right| = 1$ for any real θ).

To prove the time-shifting property, we proceed as follows. The Fourier series of $x(t)$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{4.14}$$

We express $x(t - t_0)$ in terms of its Fourier series, and then use algebraic manipulation to obtain:

$$\begin{aligned} x(t - t_0) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t - t_0)} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jk\omega_0 t_0} \\ &= \sum_{k=-\infty}^{\infty} \left(a_k e^{-jk\omega_0 t_0}\right) e^{jk\omega_0 t}. \end{aligned} \tag{4.15}$$

Comparing (4.14) and (4.15), we have that $x(t - t_0) \stackrel{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k$.

Table 4.1: Fourier Series Properties

Property	Time Domain	Fourier Domain
Linearity	$Ax(t)+By(t)$	Aa_k+Bb_k
Time-Domain Shifting	$x(t-t_0)$	$e^{-jk\omega_0 t_0}a_k$
Time Reversal	$x(-t)$	a_{-k}

4.5.3 Time Reversal

Let $x(t)$ denote a periodic signal with period T and frequency $\omega_0 = 2\pi/T$. If

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$

then

$$x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}.$$

In other words, the time reversal of a signal results in the time reversal of the corresponding sequence of Fourier series coefficients. Furthermore, if $x(t)$ is a real signal, we have from (4.5) that $a_{-k} = a_k^*$ and obtain $x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k^*$.

To prove the time-reversal property, we proceed in the following manner. The Fourier series of $x(t)$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{4.16}$$

Now, we consider the Fourier series expansion of $x(-t)$. The Fourier series in this case is given by

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(-t)}. \tag{4.17}$$

Now, we employ a change of variable. Let $l = -k$ so that $k = -l$. Performing this change of variable, we can rewrite (4.17) to obtain

$$\begin{aligned} x(-t) &= \sum_{l=-\infty}^{\infty} a_{-l} e^{j(-l)\omega_0(-t)} \\ &= \sum_{l=-\infty}^{\infty} a_{-l} e^{jl\omega_0 t}. \end{aligned} \tag{4.18}$$

Comparing (4.16) and (4.18), we have that $x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}$.

4.6 Fourier Series and Frequency Spectra

The Fourier series represents a signal in terms of harmonically-related complex sinusoids. In this sense, the Fourier series captures information about the frequency content of a signal. Each complex sinusoid is associated with a particular frequency (which is some integer multiple of the fundamental frequency). So, these coefficients indicate at which frequencies the information/energy in a signal is concentrated. For example, if only the Fourier series coefficients for the low order harmonics have large magnitudes, then the signal is mostly associated with low frequencies. On the other hand, if a signal has many large magnitude coefficients for high order harmonics, then the signal has a considerable amount of information/energy associated with high frequencies. In this way, the Fourier series representation provides a means for measuring the frequency content of a signal. The distribution of the energy/information in a signal over different frequencies is referred to as the **frequency spectrum** of the signal.

To gain further insight into the role played by the Fourier series coefficients c_k in the context of the frequency spectrum of the signal $x(t)$, it is helpful to write the Fourier series with the c_k expressed in polar form as follows:

$$\begin{aligned}x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} |c_k| e^{j\arg c_k} e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} |c_k| e^{j(k\omega_0 t + \arg c_k)}.\end{aligned}$$

Clearly (from the last line of the above equation), the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $k\omega_0$ that has had its amplitude scaled by a factor of $|c_k|$ and has been time-shifted by an amount that depends on $\arg c_k$. For a given k, the larger $|c_k|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{jk\omega_0 t}$, and therefore the larger the contribution the kth term (which is associated with frequency $k\omega_0$) will make to the overall summation. In this way, we can use $|c_k|$ as a measure of how much information a signal $x(t)$ has at the frequency $k\omega_0$.

Various ways exist to illustrate the frequency spectrum of a signal. Typically, we plot the Fourier series coefficients as a function of frequency. Since, in general, the Fourier series coefficients are complex valued, we usually display this information using two plots. One plot shows the magnitude of the coefficients as a function of frequency. This is called the **magnitude spectrum**. The other plot shows the arguments of the coefficients as a function of frequency. In this context, the argument is referred to as the phase, and the plot is called the **phase spectrum** of the signal.

Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, we only have values to plot for these particular frequencies. In other words, the frequency spectrum is discrete in the independent variable (i.e., frequency). For this reason, we use a stem graph to plot such functions. Due to the general appearance of the graph (i.e., a number of vertical lines at various frequencies) we refer to such spectra as **line spectra**.

Recall that, for a real signal $x(t)$, the Fourier series coefficient sequence c_k satisfies $c_k = c_{-k}^*$. This, however, implies that $|c_k| = |c_{-k}|$ and $\arg c_k = -\arg c_{-k}$. Since $|c_k| = |c_{-k}|$, the magnitude spectrum of a real signal is always even. Similarly, since $\arg c_k = -\arg c_{-k}$, the phase spectrum of a real signal is always odd.

Example 4.7. The periodic square wave $x(t)$ in Example 4.1 has fundamental period T, fundamental frequency ω_0, and the Fourier series coefficient sequence given by

$$c_k = \begin{cases} \frac{-j2A}{\pi k} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even,} \end{cases}$$

where A is a positive constant. Find and plot the magnitude and phase spectra of this signal. Determine at what frequency (or frequencies) this signal has the most information.

Solution. First, we compute the magnitude spectrum of $x(t)$, which is given by $|c_k|$. We have

$$\begin{aligned}|c_k| &= \begin{cases} \left|\frac{-j2A}{\pi k}\right| & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases} \\ &= \begin{cases} \frac{2A}{\pi|k|} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even.} \end{cases}\end{aligned}$$

Next, we compute the phase spectrum of $x(t)$, which is given by $\arg c_k$. Using the fact that $\arg 0 = 0$ and $\arg \frac{-j2A}{\pi k} =$

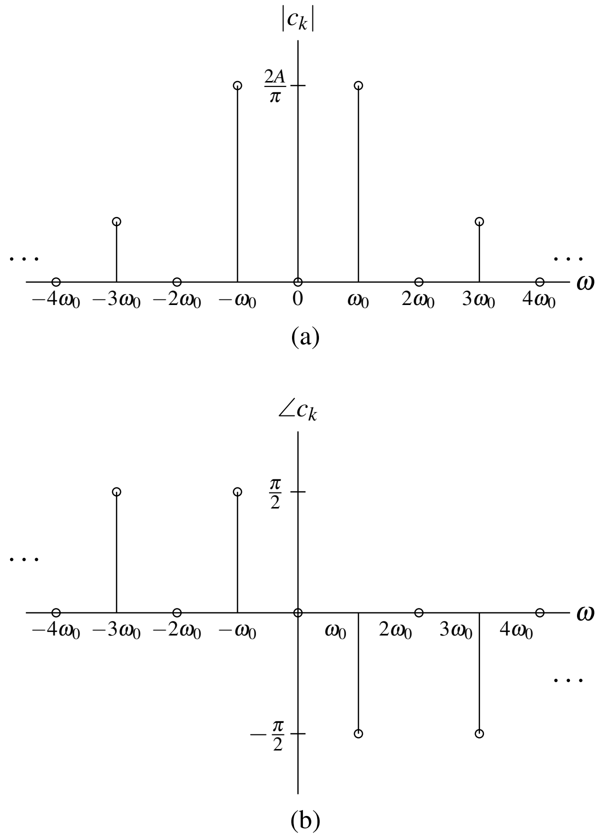


Figure 4.7: Frequency spectrum of the periodic square wave. (a) Magnitude spectrum and (b) phase spectrum.

$-\frac{\pi}{2}\operatorname{sgn} k$, we have

$$\begin{aligned}
\arg c_k &= \begin{cases} \arg \frac{-j2A}{\pi k} & \text{for } k \text{ odd} \\ \arg 0 & \text{for } k \text{ even} \end{cases} \\
&= \begin{cases} \frac{\pi}{2} & \text{for } k \text{ odd, } k < 0 \\ -\frac{\pi}{2} & \text{for } k \text{ odd, } k > 0 \\ 0 & \text{for } k \text{ even} \end{cases} \\
&= \begin{cases} -\frac{\pi}{2}\operatorname{sgn} k & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even.} \end{cases}
\end{aligned}$$

The magnitude and phase spectra of $x(t)$ are plotted in Figures 4.7(a) and (b), respectively. Note that the magnitude spectrum is an even function, while the phase spectrum is an odd function. This is what we should expect, since $x(t)$ is real. Since c_k is largest in magnitude for $k = -1$ and $k = 1$, the signal $x(t)$ has the most information at frequencies $-\omega_0$ and ω_0. □

Example 4.8. The periodic impulse train $x(t)$ in Example 4.2 has fundamental period T, fundamental frequency ω_0, and the Fourier series coefficient sequence given by

$$c_k = \frac{A}{T},$$

where A is a positive constant. Find and plot the magnitude and phase spectra of this signal.

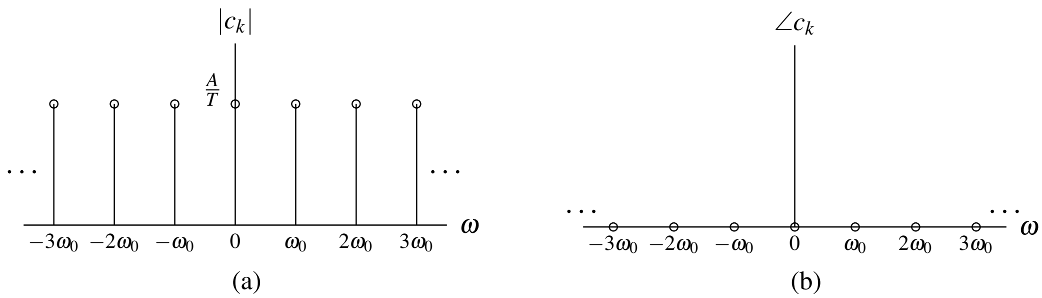


Figure 4.8: Frequency spectrum for the periodic impulse train. (a) Magnitude spectrum and (b) phase spectrum.

Solution. We have $|c_k| = \frac{A}{T}$ and $\arg c_k = 0$. The magnitude and phase spectra of $x(t)$ are plotted in Figures 4.8(a) and (b), respectively. □

4.7 Fourier Series and LTI Systems

In Section 3.10, we showed that complex exponentials are eigenfunctions of LTI systems. More specifically, for a LTI system defined by the operator $\mathscr{H}$ and having impulse response $h(t)$, we showed that

$$e^{st} \xrightarrow{\mathscr{H}} H(s)e^{st},$$

where

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \tag{4.19}$$

(i.e., $H(s)$ is the system function).

Often, we are interested in the case of $H(s)$ when $\operatorname{Re}\{s\} = 0$ (i.e., s is purely imaginary). Let $s = j\omega$ where ω is real. Substituting $s = j\omega$ for s in (4.19), we obtain

$$H(j\omega) = H(s)|_{s=j\omega} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt.$$

We call $H(j\omega)$ the **frequency response** of the system. From above, it follows that an LTI system must be such that

$$e^{j\omega t} \to H(j\omega)e^{j\omega t}. \tag{4.20}$$

Suppose now that we have a periodic signal $x(t)$ represented in terms of a Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Using (4.20) and the superposition property, we can determine the system response $y(t)$ to the input $x(t)$ as follows:

$$\begin{aligned}
y(t) &= \mathscr{H}\{x(t)\} \\
&= \mathscr{H}\left\{\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right\} \\
&= \sum_{k=-\infty}^{\infty} \mathscr{H}\left\{c_k e^{jk\omega_0 t}\right\} \\
&= \sum_{k=-\infty}^{\infty} c_k \mathscr{H}\left\{e^{jk\omega_0 t}\right\} \\
&= \sum_{k=-\infty}^{\infty} c_k H(jk\omega_0)e^{jk\omega_0 t}.
\end{aligned}$$

Therefore, we can view a LTI system as an entity that operates on the individual coefficients of a Fourier series. In particular, the system forms its output by multiplying each Fourier series coefficient by the value of the frequency response function at the frequency to which the Fourier series coefficient corresponds. In other words, if

$$x(t) \overset{\mathscr{FS}}{\longleftrightarrow} c_k$$

then

$$y(t) \overset{\mathscr{FS}}{\longleftrightarrow} H(jk\omega_0)c_k.$$

Example 4.9. Suppose that we have a LTI system with the frequency response

$$H(j\omega) = e^{-j\omega/4}.$$

Find the response $y(t)$ of the system to the input $x(t)$ where

$$x(t) = \tfrac{1}{2}\cos(2\pi t) = \tfrac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}).$$

Solution. The Fourier series for $x(t)$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$, $c_{-1} = \frac{1}{4}$, $c_1 = \frac{1}{4}$, and $c_k = 0$ for $k \notin \{-1, 1\}$. Thus, we can write

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} c_k H(jk\omega_0) e^{jk\omega_0 t} \\
&= c_{-1}H(-j\omega_0)e^{-j\omega_0 t} + c_1 H(j\omega_0)e^{j\omega_0 t} \\
&= \tfrac{1}{4}H(-j2\pi)e^{-j2\pi t} + \tfrac{1}{4}H(j2\pi)e^{j2\pi t} \\
&= \tfrac{1}{4}e^{j\pi/2}e^{-j2\pi t} + \tfrac{1}{4}e^{-j\pi/2}e^{j2\pi t} \\
&= \tfrac{1}{4}[e^{-j(2\pi t-\pi/2)} + e^{j(2\pi t-\pi/2)}] \\
&= \tfrac{1}{4}\left(2\cos(2\pi t - \tfrac{\pi}{2})\right) \\
&= \tfrac{1}{2}\cos(2\pi t - \tfrac{\pi}{2}) \\
&= \tfrac{1}{2}\cos\left(2\pi\left[t - \tfrac{1}{4}\right]\right).
\end{aligned}$$

In other words, the output $y(t)$ is just a shifted version of the input $x(t)$, namely $x(t - \frac{1}{4})$. As it turns out, the frequency response $H(j\omega)$ corresponds to an ideal delay of $\frac{1}{4}$. □

4.8 Filtering

In some applications, we want to change the relative amplitude of the frequency components of a signal or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a signal is referred to as **filtering**. Various types of filters exist. Frequency selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An ideal lowpass filter eliminates all frequency components with a frequency greater than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \le \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

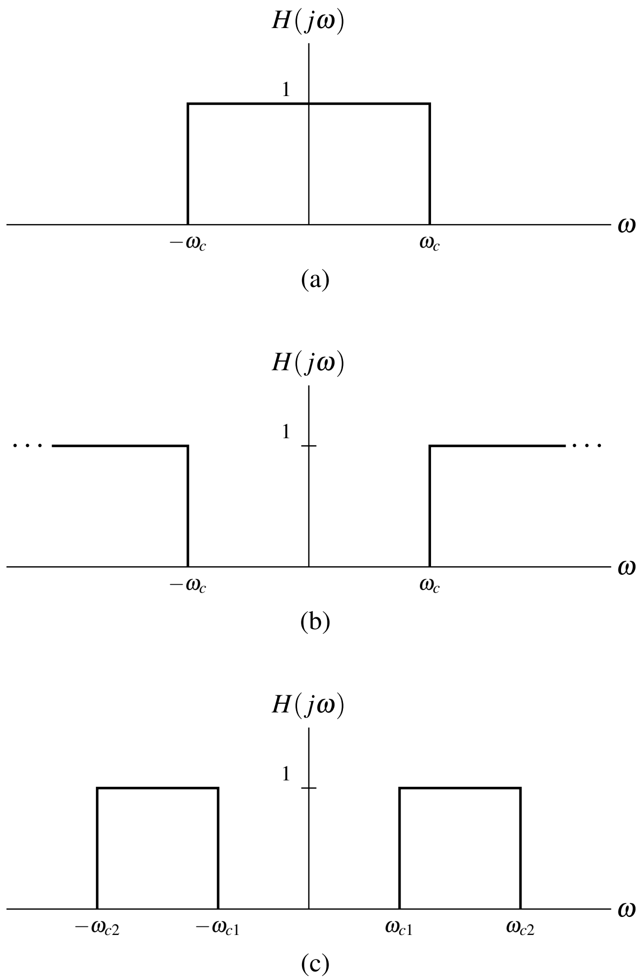


Figure 4.9: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

where ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 4.9(a).

The ideal highpass filter eliminates all frequency components with a frequency less than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 4.9(b).

An ideal bandpass filter eliminates all frequency components that do not lie in its passband, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

where the limits of the passband are ω_{c1} and ω_{c2}. A plot of this frequency response is given in Figure 4.9(c).

Example 4.10 (Lowpass filtering)**.** Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and frequency response $H(j\omega)$, where

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \leq 3\pi \\ 0 & \text{otherwise.} \end{cases}$$

Further, suppose that the input $x(t)$ is the periodic signal

$$x(t) = 1 + 2\cos 2\pi t + \cos 4\pi t + \tfrac{1}{2}\cos 6\pi t.$$

Find the Fourier series representation of $x(t)$. Use this representation in order to find the response $y(t)$ of the system to the input $x(t)$. Plot the frequency spectra of $x(t)$ and $y(t)$.

Solution. We begin by finding the Fourier series representation of $x(t)$. Using Euler's formula, we can re-express $x(t)$ as

$$\begin{aligned}
x(t) &= 1 + 2\cos 2\pi t + \cos 4\pi t + \tfrac{1}{2}\cos 6\pi t \\
&= 1 + 2\left[\tfrac{1}{2}(e^{j2\pi t} + e^{-j2\pi t})\right] + \left[\tfrac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})\right] + \tfrac{1}{2}\left[\tfrac{1}{2}(e^{j6\pi t} + e^{-j6\pi t})\right] \\
&= 1 + e^{j2\pi t} + e^{-j2\pi t} + \tfrac{1}{2}[e^{j4\pi t} + e^{-j4\pi t}] + \tfrac{1}{4}[e^{j6\pi t} + e^{-j6\pi t}] \\
&= \tfrac{1}{4}e^{-j6\pi t} + \tfrac{1}{2}e^{-j4\pi t} + e^{-j2\pi t} + 1 + e^{j2\pi t} + \tfrac{1}{2}e^{j4\pi t} + \tfrac{1}{4}e^{j6\pi t} \\
&= \tfrac{1}{4}e^{j(-3)(2\pi)t} + \tfrac{1}{2}e^{j(-2)(2\pi)t} + e^{j(-1)(2\pi)t} + e^{j(0)(2\pi)t} + e^{j(1)(2\pi)t} + \tfrac{1}{2}e^{j(2)(2\pi)t} + \tfrac{1}{4}e^{j(3)(2\pi)t}.
\end{aligned}$$

From the last line of the preceding equation, we deduce that $\omega_0 = 2\pi$, since a larger value for ω_0 would imply that some Fourier series coefficient indices are noninteger, which clearly makes no sense. Thus, we have that the Fourier series of $x(t)$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$ and

$$a_k = \begin{cases} 1 & \text{for } k = 0 \\ 1 & \text{for } k = \pm 1 \\ \tfrac{1}{2} & \text{for } k = \pm 2 \\ \tfrac{1}{4} & \text{for } k = \pm 3 \\ 0 & \text{otherwise.} \end{cases}$$

Since the system is LTI, we know that the output $y(t)$ has the form

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where

$$b_k = a_k H(jk\omega_0).$$

Using the results from above, we can calculate the b_k as follows:

$$\begin{aligned}
b_0 &= a_0 H(j[0][2\pi]) = 1(1) = 1, \\
b_1 &= a_1 H(j[1][2\pi]) = 1(1) = 1, \\
b_{-1} &= a_{-1} H(j[-1][2\pi]) = 1(1) = 1, \\
b_2 &= a_2 H(j[2][2\pi]) = \tfrac{1}{2}(0) = 0, \\
b_{-2} &= a_{-2} H(j[-2][2\pi]) = \tfrac{1}{2}(0) = 0, \\
b_3 &= a_3 H(j[3][2\pi]) = \tfrac{1}{4}(0) = 0, \quad \text{and} \\
b_{-3} &= a_{-3} H(j[-3][2\pi]) = \tfrac{1}{4}(0) = 0.
\end{aligned}$$

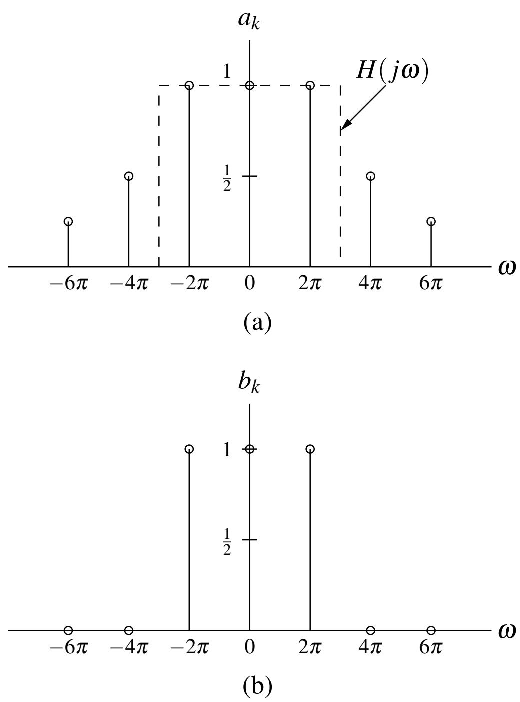


Figure 4.10: Frequency spectra of the input and output signals.

Thus, we have

$$b_k = \begin{cases} 1 & \text{for } k = 0 \\ 1 & \text{for } k = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

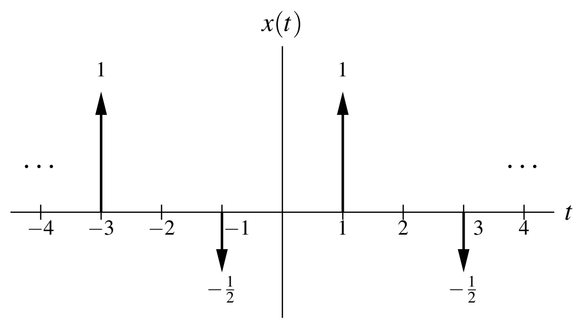
Lastly, we plot the frequency spectra of $x(t)$ and $y(t)$ in Figures 4.10(a) and (b), respectively. The frequency response $H(j\omega)$ is superimposed on the plot of the frequency spectrum of $x(t)$ for illustrative purposes.

□

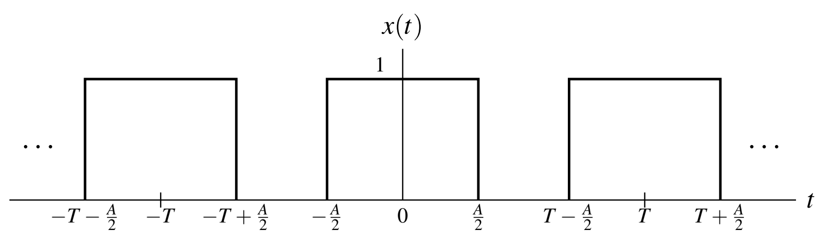
4.9 Problems

4.1 Find the Fourier series representation (in complex exponential form) of each of the signals given below. In each case, explicitly identify the fundamental period and Fourier series coefficient sequence c_k.
(a) $x(t) = 1 + \cos \pi t + \sin^2 \pi t$;
(b) $x(t) = [\cos 4t][\sin t]$; and
(c) $x(t) = |\sin 2\pi t|$. [Hint: $\int e^{ax} \sin bx dx = \frac{e^{ax}[a \sin bx - b \cos bx]}{a^2+b^2} + C$, where a and b are arbitrary complex and nonzero real constants, respectively.]

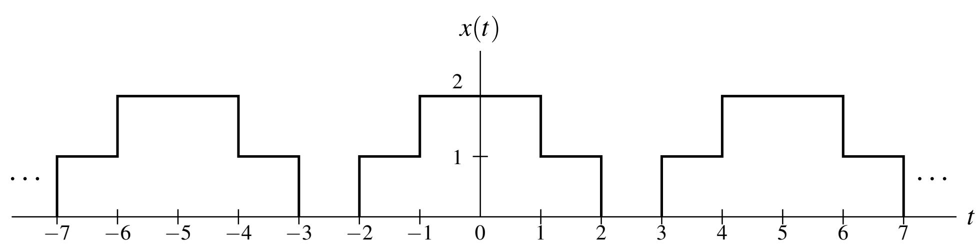
4.2 For each of the signals shown in the figure below, find the corresponding Fourier series coefficient sequence.



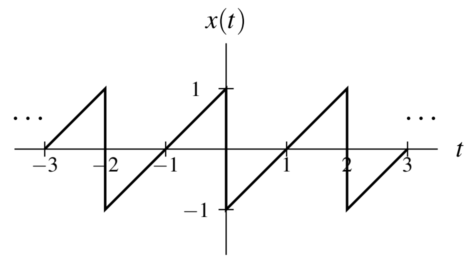
(a)



(b)



(c)



(d)

4.3 Let $x(t)$ be a periodic signal with the Fourier series coefficient sequence c_k given by

$$c_k = \begin{cases} 1 & \text{for } k = 0 \\ j\left(\frac{1}{2}\right)^{|k|} & \text{otherwise.} \end{cases}$$

Use the properties of the Fourier series to answer the following questions:
(a) Is $x(t)$ real?
(b) Is $x(t)$ even?
(c) Is $\frac{d}{dt}x(t)$ even? [Hint: Try Problems 4.4 and 4.5 first.]

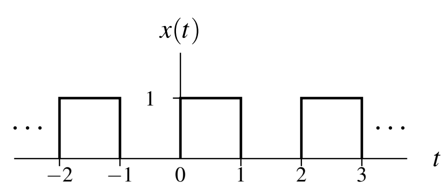
4.4 Show that, if a complex periodic signal $x(t)$ is even, then its Fourier series coefficient sequence c_k satisfies $c_k = c_{-k}$.

4.5 Suppose that the periodic signal $x(t)$ has the Fourier series coefficient sequence c_k. Determine the Fourier series coefficient sequence d_k of the signal $\frac{d}{dt}x(t)$.

4.6 A periodic signal $x(t)$ with period T and Fourier series coefficient sequence c_k is said to be odd harmonic if $c_k = 0$ for all even k.
(a) Show that if $x(t)$ is odd harmonic, then $x(t) = -x(t - \frac{T}{2})$ for all t.
(b) Show that if $x(t) = -x(t - \frac{T}{2})$ for all t, then $x(t)$ is odd harmonic.

4.7 Let $x(t)$ be a periodic signal with fundamental period T and Fourier series coefficient sequence c_k. Find the Fourier series coefficient sequence of each of the following signals in terms of c_k:
(a) $\text{Even}\{x(t)\}$
(b) $\text{Re}\{x(t)\}$.

4.8 Find the Fourier series coefficient sequence c_k of the periodic signal $x(t)$ shown in the figure below. Plot the frequency spectrum of this signal including the first five harmonics.



4.9 Suppose that we have a LTI system with frequency response

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \geq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Using frequency-domain methods, find the output $y(t)$ of the system if the input $x(t)$ is given by

$$x(t) = 1 + 2\cos 2t + 2\cos 4t + \tfrac{1}{2}\cos 6t.$$

4.10 MATLAB Problems

4.101 Consider the periodic signal $x(t)$ shown in Figure B of Problem 4.2 where $T = 1$ and $A = \frac{1}{2}$. We can show that this signal $x(t)$ has the Fourier series representation

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

where $c_k = \frac{1}{2} \operatorname{sinc} \frac{\pi k}{2}$ and $\omega_0 = 2\pi$. Let $\hat{x}_N(t)$ denote the above infinite series truncated after the Nth harmonic component. That is,

$$\hat{x}_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

(a) Use MATLAB to plot $\hat{x}_N(t)$ for $N = 1, 5, 10, 50, 100$. You should see that as N increases, $\hat{x}_N(t)$ converges to $x(t)$. [**HINT:** You may find the `sym`, `symsum`, `subs`, and `ezplot` functions useful for this problem. Please note that the MATLAB `sinc` function is NOT defined in the same way as in the lecture notes. The MATLAB `sinc` function is defined as $\operatorname{sinc} x = (\sin(\pi x))/(\pi x)$. So, it might be wise to avoid the use of this MATLAB function altogether.]
(b) By examining the graphs obtained in part (a), answer the following: As $N \to \infty$, does $\hat{x}_N(t)$ converge to $x(t)$ uniformly (i.e., equally fast) everywhere? If not, where is the rate of convergence slower?
(c) The signal $x(t)$ is not continuous everywhere. For example, the signal has a discontinuity at $t = \frac{1}{4}$. As $N \to \infty$, to what value does $\hat{x}_N(t)$ appear to converge at this point? Again, deduce your answer from the graphs obtained in part (a).

Chapter 5

Continuous-Time Fourier Transform

5.1 Introduction

The Fourier series provides an extremely useful representation for periodic signals. Often, however, we need to deal with signals that are not periodic. A more general tool than the Fourier series is needed in this case. In this chapter, we will introduce a tool for representing arbitrary (i.e., possibly aperiodic) signals, known as the Fourier transform.

5.2 Development of the Continuous-Time Fourier Transform

As demonstrated earlier, the Fourier series is an extremely useful signal representation. Unfortunately, this signal representation can only be used for periodic signals, since a Fourier series is inherently periodic. Many signals, however, are not periodic. Therefore, one might wonder if we can somehow use the Fourier series to develop a representation for aperiodic signals. As it turns out, this is possible. In order to understand why, we must make the following key observation. An aperiodic signal can be viewed as a periodic signal with a period of infinity. By viewing an aperiodic signal as this limiting case of a periodic signal where the period is infinite, we can use the Fourier series to develop a more general signal representation that can be used in the aperiodic case. (In what follows, our development of the Fourier transform is not completely rigorous, as we assume that various integrals, summations, and limits converge. Such assumptions are not valid in all cases. Our development is mathematically sound, however, provided that the Fourier transform of the signal being considered exists.)

Suppose that we have an arbitrary signal $x(t)$ that is not necessarily periodic. Let us define the signal $x_T(t)$ as

$$x_T(t) = \begin{cases} x(t) & \text{for } -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

In other words, $x_T(t)$ is identical to $x(t)$ over the interval $-\frac{T}{2} \leq t < \frac{T}{2}$ and zero otherwise. Let us now repeat the portion of $x_T(t)$ for $-\frac{T}{2} \leq t < \frac{T}{2}$ to form a periodic signal $\tilde{x}(t)$ with period T. That is, we define $\tilde{x}(t)$ as

$$\tilde{x}(t) = x_T(t) \;\; \text{for } -\tfrac{T}{2} \leq t < \tfrac{T}{2} \quad \text{and} \quad \tilde{x}(t) = \tilde{x}(t+T).$$

In Figures 5.1 and 5.2, we provide illustrative examples of the signals $x(t)$, $x_T(t)$, and $\tilde{x}(t)$.

Before proceeding further, we make two important observations that we will use later. First, from the definition of $x_T(t)$, we have

$$\lim_{T\to\infty} x_T(t) = x(t). \tag{5.2}$$

Second, from the definition of $x_T(t)$ and $\tilde{x}(t)$, we have

$$\lim_{T\to\infty} \tilde{x}(t) = x(t). \tag{5.3}$$

Now, let us consider the signal $\tilde{x}(t)$. Since $\tilde{x}(t)$ is periodic, we can represent it using a Fourier series as

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{5.4}$$

where

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt \tag{5.5}$$

and $\omega_0 = 2\pi/T$. Since $x_T(t) = \tilde{x}(t)$ for $-\frac{T}{2} \leq t < \frac{T}{2}$, we can rewrite (5.5) as

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t) e^{-jk\omega_0 t} dt.$$

Furthermore, since $x_T(t) = 0$ for $t < -\frac{T}{2}$ and $t \geq \frac{T}{2}$, we can rewrite the preceding expression for a_k as

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x_T(t) e^{-jk\omega_0 t} dt.$$

Substituting this expression for a_k into (5.4) and rearranging, we obtain the following Fourier series representation for $\tilde{x}(t)$:

$$\begin{aligned}
\tilde{x}(t) &= \sum_{k=-\infty}^{\infty} \left[\frac{1}{T}\int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau} d\tau\right] e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} \left[\frac{\omega_0}{2\pi}\int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau} d\tau\right] e^{jk\omega_0 t} \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau} d\tau\right] e^{jk\omega_0 t}\omega_0.
\end{aligned}$$

Substituting the above expression for $\tilde{x}(t)$ into (5.3), we obtain

$$x(t) = \lim_{T\to\infty} \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau} d\tau\right] e^{jk\omega_0 t}\omega_0. \tag{5.6}$$

Now, we must evaluate the above limit. As $T \to \infty$, we have that $\omega_0 \to 0$. Thus, in the limit above, $k\omega_0$ becomes a continuous variable which we denote as ω, ω_0 becomes the infinitesimal $d\omega$, and the summation becomes an integral. This is illustrated in Figure 5.3. Also, as $T \to \infty$, we have that $x_T(t) \to x(t)$. Combining these results, we can rewrite (5.6) to obtain

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega,$$

where

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

Thus, we have found a representation of the arbitrary signal $x(t)$ in terms of complex sinusoids at all frequencies. We call this the Fourier transform representation of the signal $x(t)$.

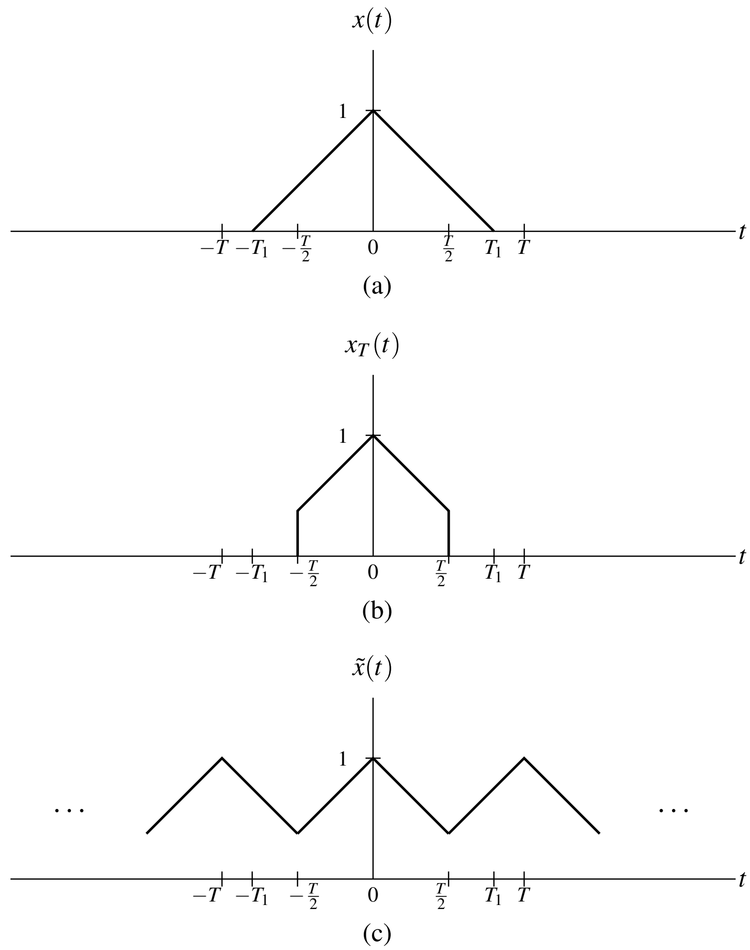


Figure 5.1: Example of signals used in derivation of Fourier transform representation (where $T_1 > \frac{T}{2}$).

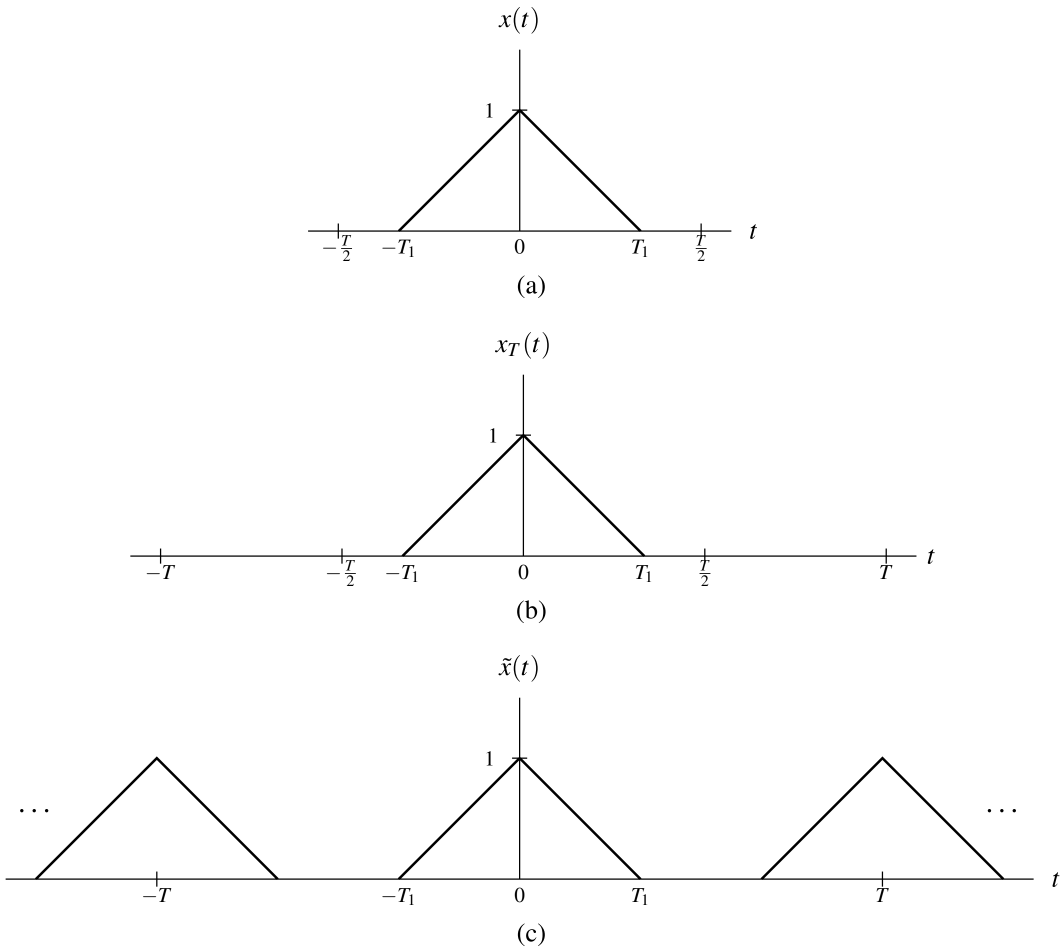


Figure 5.2: Example of signals used in derivation of Fourier transform representation (where $T_1 < \frac{T}{2}$).

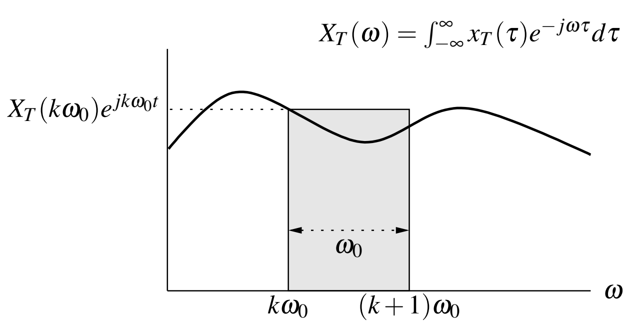


Figure 5.3: Integral obtained in the derivation of the Fourier transform representation.

5.3 Definition of the Continuous-Time Fourier Transform

In the previous section, we derived the Fourier transform representation of an arbitrary signal. This representation expresses a signal in terms of complex sinusoids at all frequencies. Given a signal $x(t)$, its Fourier transform representation is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega, \tag{5.7a}$$

where

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \tag{5.7b}$$

We refer to (5.7b) as the **Fourier transform analysis equation** and (5.7a) as the **Fourier transform synthesis equation**.

The quantity $X(\omega)$ is called the Fourier transform of $x(t)$. That is, the Fourier transform of the signal $x(t)$, denoted as $\mathscr{F}\{x(t)\}$ or $X(\omega)$, is defined as

$$\mathscr{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \tag{5.8}$$

Similarly, the inverse Fourier transform of $X(\omega)$ is given by

$$\mathscr{F}^{-1}\{X(\omega)\} = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega. \tag{5.9}$$

If a signal $x(t)$ has the Fourier transform $X(\omega)$ we often denote this as

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega).$$

As a matter of terminology, $x(t)$ and $X(\omega)$ are said to be a **Fourier transform pair**.

Example 5.1 (Fourier transform of the unit-impulse function)**.** Find the Fourier transform $X(\omega)$ of the signal $x(t) = A\delta(t-t_0)$. Then, from this result, write the Fourier transform representation of $x(t)$.

Solution. From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \mathscr{F}\{x(t)\} \\ &= \mathscr{F}\{A\delta(t-t_0)\} \\ &= \int_{-\infty}^{\infty} A\delta(t-t_0)e^{-j\omega t}dt \\ &= A\int_{-\infty}^{\infty} \delta(t-t_0)e^{-j\omega t}dt. \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

$$X(\omega) = Ae^{-j\omega t_0}.$$

Thus, we have shown that

$$A\delta(t-t_0) \overset{\mathscr{F}}{\longleftrightarrow} Ae^{-j\omega t_0}.$$

From the Fourier transform analysis and synthesis equations, we have that the Fourier transform representation of $x(t)$ is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

where

$$X(\omega) = Ae^{-j\omega t_0}.$$

□

Example 5.2 (Inverse Fourier transform of the unit-impulse function)**.** Find the inverse Fourier transform of $X(\omega) = 2\pi A\delta(\omega - \omega_0)$.

Solution. From the definition of the inverse Fourier transform, we can write

$$\begin{aligned}\mathscr{F}^{-1}\{2\pi A\delta(\omega-\omega_0)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi A\delta(\omega-\omega_0)e^{j\omega t}d\omega \\ &= A\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}d\omega.\end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the preceding equation to obtain

$$\mathscr{F}^{-1}\{2\pi A\delta(\omega-\omega_0)\} = Ae^{j\omega_0 t}.$$

Thus, we have that

$$Ae^{j\omega_0 t} \stackrel{\mathscr{F}}{\longleftrightarrow} 2\pi A\delta(\omega-\omega_0).$$

□

Example 5.3 (Fourier transform of the rectangular pulse)**.** Find the Fourier transform $X(\omega)$ of the signal $x(t) = \operatorname{rect} t$.

Solution. From the definition of the Fourier transform, we can write

$$X(\omega) = \mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty}[\operatorname{rect} t]e^{-j\omega t}dt.$$

From the definition of the rectangular pulse function, we can simplify this equation as follows:

$$\begin{aligned}X(\omega) &= \int_{-1/2}^{1/2}[\operatorname{rect} t]e^{-j\omega t}dt \\ &= \int_{-1/2}^{1/2} e^{-j\omega t}dt.\end{aligned}$$

Evaluating the integral and simplifying, we obtain

$$\begin{aligned}X(\omega) &= \left[-\frac{1}{j\omega}e^{-j\omega t}\right]\Big|_{-1/2}^{1/2} \\ &= \frac{1}{j\omega}[e^{j\omega/2} - e^{-j\omega/2}] \\ &= \frac{1}{j\omega}[2j\sin\frac{\omega}{2}] \\ &= \frac{2}{\omega}\sin\omega/2 \\ &= \operatorname{sinc}\omega/2.\end{aligned}$$

Thus, we have shown that

$$\operatorname{rect} t \stackrel{\mathscr{F}}{\longleftrightarrow} \operatorname{sinc}\omega/2.$$

□

5.4 Convergence of the Continuous-Time Fourier Transform

When deriving the Fourier transform representation earlier, we implicitly made some assumptions about the convergence of the integrals and other expressions involved. These assumptions are not always valid. For this reason, a more careful examination of the convergence properties of the Fourier transform is in order.

Suppose that we have an arbitrary signal $x(t)$. This signal has the Fourier transform representation $\hat{x}(t)$ given by

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega,$$

where

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

Now, we need to concern ourselves with the convergence properties of this representation. In other words, we want to know when $\hat{x}(t)$ is a valid representation of $x(t)$. In our earlier derivation of the Fourier transform, we relied heavily on the Fourier series. Therefore, one might expect that the convergence of the Fourier transform representation is closely related to the convergence properties of Fourier series. This is, in fact, the case. The convergence properties of the Fourier transform are very similar to the convergence properties of the Fourier series (as studied in Section 4.4).

The first important result concerning convergence relates to finite-energy signals as stated by the theorem below.

Theorem 5.1 (Convergence of Fourier transform (finite-energy case))**.** *If a signal $x(t)$ is of finite energy (i.e., $\int_{-\infty}^{\infty}|x(t)|^2 dt < \infty$), then its Fourier transform representation converges in the MSE sense.*

In other words, if $x(t)$ is of finite energy, then

$$E = \int_{-\infty}^{\infty} |\hat{x}(t) - x(t)|^2 dt = 0.$$

Although $x(t)$ and $\hat{x}(t)$ may differ at individual points, the energy E in the difference is zero.

The other important result concerning convergence that we shall consider relates to what are known as the Dirichlet conditions. The Dirichlet conditions for the signal $x(t)$ are as follows:

1. The signal $x(t)$ is absolutely integrable (i.e., $\int_{-\infty}^{\infty}|x(t)|\,dt < \infty$).
2. The signal $x(t)$ has a finite number of maxima and minima on any finite interval.
3. The signal $x(t)$ has a finite number of discontinuities on any finite interval, and each discontinuity is itself finite.

For a signal satisfying the Dirichlet conditions, we have the important convergence result stated below.

Theorem 5.2 (Convergence of Fourier transform (Dirichlet case))**.** *If a signal $x(t)$ satisfies the Dirichlet conditions, then its Fourier transform representation $\hat{x}(t)$ converges pointwise for all t, except at points of discontinuity. Furthermore, at each discontinuity point $t = t_a$, we have that*

$$\hat{x}(t_a) = \tfrac{1}{2}[x(t_a^+) + x(t_a^-)],$$

where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the signal $x(t)$ on the left- and right-hand sides of the discontinuity, respectively.

In other words, if a signal $x(t)$ satisfies the Dirichlet conditions, then the Fourier transform representation $\hat{x}(t)$ of $x(t)$ converges to $x(t)$ for all t, except at points of discontinuity where $\hat{x}(t)$ instead converges to the average of $x(t)$ on the two sides of the discontinuity.

The finite-energy and Dirichlet conditions mentioned above are only sufficient conditions for the convergence of the Fourier transform representation. They are not necessary. In other words, a signal may violate these conditions and still have a valid Fourier transform representation.

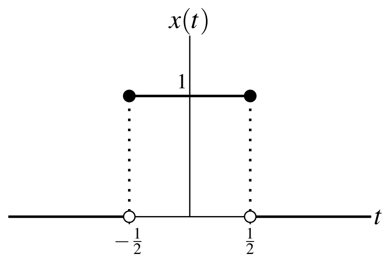


Figure 5.4: Signal $x(t)$.

Example 5.4. Consider the function $x(t)$ shown in Figure 5.4. Let $\hat{x}(t)$ denote the Fourier transform representation of $x(t)$ (i.e., $\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$, where $X(\omega)$ denotes the Fourier transform of $x(t)$). Determine the values $\hat{x}(-\frac{1}{2})$ and $\hat{x}(\frac{1}{2})$.

Solution. We begin by observing that $x(t)$ satisfies the Dirichlet conditions. Consequently, Theorem 5.2 applies. Thus, we have that

$$\hat{x}(-\tfrac{1}{2}) = \tfrac{1}{2}\left[x(-\tfrac{1}{2}^-) + x(-\tfrac{1}{2}^+)\right] = \tfrac{1}{2}(0+1) = \tfrac{1}{2} \quad \text{and}$$
$$\hat{x}(\tfrac{1}{2}) = \tfrac{1}{2}\left[x(\tfrac{1}{2}^-) + x(\tfrac{1}{2}^+)\right] = \tfrac{1}{2}(1+0) = \tfrac{1}{2}.$$

□

5.5 Properties of the Continuous-Time Fourier Transform

The Fourier transform has a number of important properties. In the sections that follow, we introduce several of these properties. For convenience, these properties are also later summarized in Table 5.1 (on page 123).

5.5.1 Linearity

If $x_1(t) \overset{\mathscr{F}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\mathscr{F}}{\longleftrightarrow} X_2(\omega)$, then

$$a_1x_1(t) + a_2x_2(t) \overset{\mathscr{F}}{\longleftrightarrow} a_1X_1(\omega) + a_2X_2(\omega),$$

where a_1 and a_2 are arbitrary complex constants. This is known as the linearity property of the Fourier transform.

To prove the above property, we proceed as follows:

$$\begin{aligned}
\mathscr{F}\{a_1x_1(t) + a_2x_2(t)\} &= \int_{-\infty}^{\infty}[a_1x_1(t) + a_2x_2(t)]e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} a_1x_1(t)e^{-j\omega t}dt + \int_{-\infty}^{\infty} a_2x_2(t)e^{-j\omega t}dt \\
&= a_1\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt + a_2\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt \\
&= a_1\mathscr{F}\{x_1(t)\} + a_2\mathscr{F}\{x_2(t)\} \\
&= a_1X_1(\omega) + a_2X_2(\omega).
\end{aligned}$$

Thus, we have shown that the linearity property holds.

Example 5.5 (Linearity property of the Fourier transform)**.** Find the Fourier transform $X(\omega)$ of the signal $x(t) = A\cos\omega_0 t$.

Solution. We recall that $\cos\alpha = \frac{1}{2}[e^{j\alpha} + e^{-j\alpha}]$ for any real α. Thus, we can write

$$\begin{aligned} X(\omega) &= \mathscr{F}\{x(t)\} \\ &= \mathscr{F}\{A\cos\omega_0 t\} \\ &= \mathscr{F}\{\tfrac{A}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]\}. \end{aligned}$$

Then, we use the linearity property of the Fourier transform to obtain

$$X(\omega) = \tfrac{A}{2}\mathscr{F}\{e^{j\omega_0 t}\} + \tfrac{A}{2}\mathscr{F}\{e^{-j\omega_0 t}\}.$$

From Example 5.2, we know that $e^{j\omega_0 t} \stackrel{\mathscr{F}}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0)$ and $e^{-j\omega_0 t} \stackrel{\mathscr{F}}{\longleftrightarrow} 2\pi\delta(\omega+\omega_0)$. Thus, we can further simplify the above expression for $X(\omega)$ as follows:

$$\begin{aligned} X(\omega) &= \tfrac{A}{2}[2\pi\delta(\omega+\omega_0)] + \tfrac{A}{2}[2\pi\delta(\omega-\omega_0)] \\ &= A\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]. \end{aligned}$$

Thus, we have shown that

$$A\cos\omega_0 t \stackrel{\mathscr{F}}{\longleftrightarrow} A\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)].$$

□

Example 5.6 (Fourier transform of the unit-step function)**.** Suppose that $\operatorname{sgn} t \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{2}{j\omega}$. Find the Fourier transform $X(\omega)$ of the signal $x(t) = u(t)$.

Solution. First, we observe that $x(t)$ can be expressed in terms of the signum function as

$$x(t) = u(t) = \tfrac{1}{2} + \tfrac{1}{2}\operatorname{sgn} t.$$

Taking the Fourier transform of both sides of this equation yields

$$X(\omega) = \mathscr{F}\{x(t)\} = \mathscr{F}\{\tfrac{1}{2} + \tfrac{1}{2}\operatorname{sgn} t\}.$$

Using the linearity property of the Fourier transform, we can rewrite this as

$$X(\omega) = \tfrac{1}{2}\mathscr{F}\{1\} + \tfrac{1}{2}\mathscr{F}\{\operatorname{sgn} t\}.$$

From Example 5.5 (with $\omega_0 = 0$), we know that $1 \stackrel{\mathscr{F}}{\longleftrightarrow} 2\pi\delta(\omega)$. Also, we are given that $\operatorname{sgn} t \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{2}{j\omega}$. Using these facts, we can rewrite the expression for $X(\omega)$ as

$$\begin{aligned} X(\omega) &= \tfrac{1}{2}[2\pi\delta(\omega)] + \tfrac{1}{2}(\tfrac{2}{j\omega}) \\ &= \pi\delta(\omega) + \tfrac{1}{j\omega}. \end{aligned}$$

Thus, we have shown that

$$u(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \pi\delta(\omega) + \tfrac{1}{j\omega}.$$

□

5.5.2 Time-Domain Shifting

If $x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(\omega)$, then

$$x(t-t_0) \stackrel{\mathscr{F}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega),$$

where t_0 is an arbitrary real constant. This is known as the time-shifting property of the Fourier transform.

To prove the above property, we proceed as follows. To begin, we have

$$\mathscr{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt.$$

Now, we use a change of variable. Let $\lambda = t - t_0$ so that $t = \lambda + t_0$ and $dt = d\lambda$. Performing the change of variable and simplifying, we obtain

$$\begin{aligned}
\mathscr{F}\{x(t-t_0)\} &= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega(\lambda+t_0)} d\lambda \\
&= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} e^{-j\omega t_0} d\lambda \\
&= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} d\lambda \\
&= e^{-j\omega t_0} \mathscr{F}\{x(t)\} \\
&= e^{-j\omega t_0} X(\omega).
\end{aligned}$$

Thus, we have proven that the time-shifting property holds.

Example 5.7 (Time-domain shifting property of the Fourier transform)**.** Find the Fourier transform $X(\omega)$ of the signal $x(t) = A\cos(\omega_0 t + \theta)$.

Solution. Let $v(t) = A\cos\omega_0 t$ so that $x(t) = v(t + \frac{\theta}{\omega_0})$. From Example 5.5, we have that

$$\begin{aligned}
\mathscr{F}\{v(t)\} = V(\omega) &= \mathscr{F}\{A\cos\omega_0 t\} \\
&= A\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)].
\end{aligned}$$

From the definition of $v(t)$ and the time-shifting property of the Fourier transform, we have

$$\begin{aligned}
X(\omega) &= \mathscr{F}\{x(t)\} \\
&= e^{j\omega\theta/\omega_0} V(\omega) \\
&= e^{j\omega\theta/\omega_0} A\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)].
\end{aligned}$$

Thus, we have shown that

$$A\cos(\omega_0 t + \theta) \stackrel{\mathscr{F}}{\longleftrightarrow} A\pi e^{j\omega\theta/\omega_0}[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)].$$

□

5.5.3 Frequency-Domain Shifting

If $x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(\omega)$, then

$$e^{j\omega_0 t} x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(\omega - \omega_0),$$

where ω_0 is an arbitrary real constant. This is known as the frequency-domain shifting property of the Fourier transform.

To prove the above property, we proceed as follows. From the definition of the Fourier transform and straightforward algebraic manipulation, we can write

$$\begin{aligned}\mathcal{F}\{e^{j\omega_0 t}x(t)\} &= \int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt \\ &= X(\omega-\omega_0).\end{aligned}$$

Thus, we have shown that the frequency-domain shifting property holds.

Example 5.8 (Frequency-domain shifting property of the Fourier transform)**.** Find the Fourier transform $X(\omega)$ of the signal $x(t) = (\cos\omega_0 t)(\cos 20\pi t)$.

Solution. Recall that $\cos\alpha = \frac{1}{2}[e^{j\alpha}+e^{-j\alpha}]$ for any real α. Using this relationship and the linearity property of the Fourier transform, we can write

$$\begin{aligned}X(\omega) &= \mathcal{F}\{x(t)\} \\ &= \mathcal{F}\{(\cos\omega_0 t)(\tfrac{1}{2})[e^{j20\pi t}+e^{-j20\pi t}]\} \\ &= \mathcal{F}\{\tfrac{1}{2}e^{j20\pi t}\cos\omega_0 t + \tfrac{1}{2}e^{-j20\pi t}\cos\omega_0 t\} \\ &= \tfrac{1}{2}\mathcal{F}\{e^{j20\pi t}\cos\omega_0 t\} + \tfrac{1}{2}\mathcal{F}\{e^{-j20\pi t}\cos\omega_0 t\}.\end{aligned}$$

From Example 5.5 (where we showed $\cos\omega_0 t \stackrel{\mathcal{F}}{\longleftrightarrow} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$) and the frequency-domain shifting property of the Fourier transform, we have

$$\begin{aligned}X(\omega) &= \tfrac{1}{2}\left[\pi\left[\delta(v-\omega_0)+\delta(v+\omega_0)\right]\right]\big|_{v=\omega-20\pi} + \tfrac{1}{2}\left[\pi\left[\delta(v-\omega_0)+\delta(v+\omega_0)\right]\right]\big|_{v=\omega+20\pi} \\ &= \tfrac{1}{2}\left(\pi[\delta(\omega+\omega_0-20\pi)+\delta(\omega-\omega_0-20\pi)]\right) + \tfrac{1}{2}\left(\pi[\delta(\omega+\omega_0+20\pi)+\delta(\omega-\omega_0+20\pi)]\right) \\ &= \tfrac{\pi}{2}\left[\delta(\omega+\omega_0-20\pi)+\delta(\omega-\omega_0-20\pi)+\delta(\omega+\omega_0+20\pi)+\delta(\omega-\omega_0+20\pi)\right].\end{aligned}$$

□

5.5.4 Time- and Frequency-Domain Scaling

If $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)$, then

$$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{\omega}{a}\right),$$

where a is an arbitrary nonzero real constant. This is known as the time/frequency-scaling property of the Fourier transform.

To prove the above property, we proceed as follows. From the definition of the Fourier transform, we can write

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt.$$

Now, we use a change of variable. Let $\lambda = at$ so that $t = \lambda/a$ and $dt = d\lambda/a$. Performing the change of variable (and being mindful of the change in the limits of integration), we obtain

$$\begin{aligned}\mathcal{F}\{x(at)\} &= \begin{cases}\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}(\frac{1}{a})d\lambda & \text{for } a>0 \\ \int_{\infty}^{-\infty} x(\lambda)e^{-j(\omega/a)\lambda}(\frac{1}{a})d\lambda & \text{for } a<0\end{cases} \\ &= \begin{cases}\frac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}d\lambda & \text{for } a>0 \\ -\frac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}d\lambda & \text{for } a<0.\end{cases}\end{aligned}$$

Combining the two cases (i.e., for $a < 0$ and $a > 0$), we obtain

$$\begin{aligned}\mathscr{F}\{x(at)\} &= \frac{1}{|a|}\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}d\lambda \\ &= \frac{1}{|a|}X\left(\frac{\omega}{a}\right).\end{aligned}$$

Thus, we have shown that the time/frequency-scaling property holds.

Example 5.9 (Time scaling property of the Fourier transform)**.** Find the Fourier transform $X(\omega)$ of the signal $x(t) = \operatorname{rect}(at)$.

Solution. Let $v(t) = \operatorname{rect}(t)$ so that $x(t) = v(at)$. From Example 5.3, we know that

$$\mathscr{F}\{v(t)\} = V(\omega) = \mathscr{F}\{\operatorname{rect} t\} = \operatorname{sinc}\omega/2. \tag{5.10}$$

From the definition of $v(t)$ and the time-scaling property of the Fourier transform, we have

$$\begin{aligned}X(\omega) &= \mathscr{F}\{x(t)\} \\ &= \frac{1}{|a|}V\left(\frac{\omega}{a}\right).\end{aligned}$$

Substituting the expression for $V(\omega)$ in (5.10) into the preceding equation, we have

$$X(\omega) = \frac{1}{|a|}\operatorname{sinc}\frac{\omega}{2a}.$$

Thus, we have shown that

$$\operatorname{rect}(at) \xleftrightarrow{\mathscr{F}} \tfrac{1}{|a|}\operatorname{sinc}\tfrac{\omega}{2a}.$$

□

5.5.5 Conjugation

If $x(t) \xleftrightarrow{\mathscr{F}} X(\omega)$, then

$$x^*(t) \xleftrightarrow{\mathscr{F}} X^*(-\omega).$$

This is known as the conjugation property of the Fourier transform.

A proof of the above property is quite simple. From the definition of the Fourier transform, we have

$$\mathscr{F}\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt.$$

From the properties of conjugation, we can rewrite this equation as

$$\begin{aligned}\mathscr{F}\{x^*(t)\} &= \left[\left(\int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt\right)^*\right]^* \\ &= \left[\int_{-\infty}^{\infty} [x(t)^*]^*\left[e^{-j\omega t}\right]^* dt\right]^* \\ &= \left[\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt\right]^* \\ &= X^*(-\omega).\end{aligned}$$

Thus, we have shown that the conjugation property holds.

Example 5.10 (Fourier transform of a real signal)**.** Show that the Fourier transform $X(\omega)$ of any real signal $x(t)$ must satisfy $X(\omega) = X^*(-\omega)$, and this condition implies that $|X(\omega)| = |X(-\omega)|$ and $\arg X(\omega) = -\arg X(-\omega)$ (i.e., $|X(\omega)|$ and $\arg X(\omega)$ are even and odd functions, respectively).

Solution. From the conjugation property of the Fourier transform, we have

$$\mathscr{F}\{x^*(t)\} = X^*(-\omega).$$

Since $x(t)$ is real, $x(t) = x^*(t)$ which implies

$$\mathscr{F}\{x(t)\} = X^*(-\omega),$$

or equivalently

$$X(\omega) = X^*(-\omega). \tag{5.11}$$

Taking the magnitude of both sides of (5.11) and observing that $|z| = |z^*|$ for any complex z, we have

$$\begin{aligned} |X(\omega)| &= |X^*(-\omega)| \\ &= |X(-\omega)|. \end{aligned}$$

Taking the argument of both sides of (5.11) and observing that $\arg z^* = -\arg z$ for any complex z, we have

$$\begin{aligned} \arg X(\omega) &= \arg X^*(-\omega) \\ &= -\arg X(-\omega). \end{aligned}$$

□

5.5.6 Duality

If $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega)$, then

$$X(t) \overset{\mathscr{F}}{\longleftrightarrow} 2\pi x(-\omega)$$

This is known as the duality property of the Fourier transform. This property follows from the similarity/symmetry in the definition of the forward and inverse Fourier transforms. That is, the forward Fourier transform equation given by (5.8) and the inverse Fourier transform equation given by (5.9) are identical except for a factor of 2π and different sign in the parameter for the exponential function.

To prove the above property, we proceed as follows. From the Fourier transform synthesis equation, we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$$

Substituting $-t$ for t, we obtain

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{-j\omega t}d\omega.$$

Now, we multiply both sides of the equation by 2π to yield

$$\begin{aligned} 2\pi x(-t) &= \int_{-\infty}^{\infty} X(\omega)e^{-j\omega t}d\omega \\ &= \mathscr{F}\{X(t)\}. \end{aligned}$$

Thus, we have shown that the duality property holds.

Example 5.11 (Fourier transform of the sinc function)**.** Given that $\operatorname{rect}(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \operatorname{sinc}\omega/2$, find the Fourier transform $X(\omega)$ of the signal $x(t) = \operatorname{sinc}(t/2)$.

Solution. From the given Fourier transform pair and the duality property, we have that

$$X(\omega) = \mathscr{F}\{\operatorname{sinc} t/2\} = 2\pi \operatorname{rect}(-\omega).$$

Since $\operatorname{rect}(-\omega) = \operatorname{rect}(\omega)$, we can simplify this to obtain

$$X(\omega) = 2\pi \operatorname{rect}(\omega).$$

Thus, we have shown that

$$\operatorname{sinc} t/2 \stackrel{\mathscr{F}}{\longleftrightarrow} 2\pi \operatorname{rect}\omega.$$

□

5.5.7 Time-Domain Convolution

If $x_1(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1(t) * x_2(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X_1(\omega)X_2(\omega).$$

This is known as the time-domain convolution property of the Fourier transform.

The proof of this property is as follows. From the definition of the Fourier transform and convolution, we have

$$\begin{aligned}
\mathscr{F}\{x_1(t) * x_2(t)\} &= \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\, e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right] e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-j\omega t} d\tau dt.
\end{aligned}$$

Changing the order of integration, we obtain

$$\mathscr{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-j\omega t} dt d\tau.$$

Now, we use a change of variable. Let $\lambda = t - \tau$ so that $t = \lambda + \tau$ and $d\lambda = dt$. Applying the change of variable and simplifying, we obtain

$$\begin{aligned}
\mathscr{F}\{x_1(t) * x_2(t)\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(\lambda)e^{-j\omega(\lambda+\tau)} d\lambda d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(\lambda)e^{-j\omega\lambda}e^{-j\omega\tau} d\lambda d\tau \\
&= \int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}\left[\int_{-\infty}^{\infty} x_2(\lambda)e^{-j\omega\lambda} d\lambda\right] d\tau \\
&= \left[\int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau} d\tau\right]\left[\int_{-\infty}^{\infty} x_2(\lambda)e^{-j\omega\lambda} d\lambda\right] \\
&= \mathscr{F}\{x_1(t)\}\mathscr{F}\{x_2(t)\} \\
&= X_1(\omega)X_2(\omega).
\end{aligned}$$

Thus, we have shown that the time-domain convolution property holds.

Example 5.12 (Time-domain convolution property of the Fourier transform)**.** With the aid of Table 5.2, find the Fourier transform $X(\omega)$ of the signal $x(t) = x_1(t) * x_2(t)$ where

$$\begin{aligned} x_1(t) &= e^{-2t}u(t) \quad \text{and} \\ x_2(t) &= u(t). \end{aligned}$$

Solution. Let $X_1(\omega)$ and $X_2(\omega)$ denote the Fourier transforms of $x_1(t)$ and $x_2(t)$, respectively. From the time-domain convolution property of the Fourier transform, we know that

$$X(\omega) = \mathscr{F}\{x_1(t) * x_2(t)\} = X_1(\omega)X_2(\omega). \tag{5.12}$$

From Table 5.2, we know that

$$\begin{aligned} X_1(\omega) &= \mathscr{F}\{x_1(t)\} = \mathscr{F}\{e^{-2t}u(t)\} = \tfrac{1}{2+j\omega} \quad \text{and} \\ X_2(\omega) &= \mathscr{F}\{x_2(t)\} = \mathscr{F}\{u(t)\} = \pi\delta(\omega) + \tfrac{1}{j\omega}. \end{aligned}$$

Substituting these expressions for $X_1(\omega)$ and $X_2(\omega)$ into (5.12), we obtain

$$\begin{aligned} X(\omega) &= [\tfrac{1}{2+j\omega}](\pi\delta(\omega) + \tfrac{1}{j\omega}) \\ &= \tfrac{\pi}{2+j\omega}\delta(\omega) + \tfrac{1}{j\omega}(\tfrac{1}{2+j\omega}) \\ &= \tfrac{\pi}{2+j\omega}\delta(\omega) + \tfrac{1}{j2\omega-\omega^2} \\ &= \tfrac{\pi}{2}\delta(\omega) + \tfrac{1}{j2\omega-\omega^2}. \end{aligned}$$

□

5.5.8 Frequency-Domain Convolution

If $x_1(t) \overset{\mathscr{F}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\mathscr{F}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1(t)x_2(t) \overset{\mathscr{F}}{\longleftrightarrow} \tfrac{1}{2\pi}X_1(\omega) * X_2(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta)X_2(\omega-\theta)d\theta.$$

This is known as the frequency-domain convolution (or time-domain multiplication) property of the Fourier transform.

To prove the above property, we proceed as follows. From the definition of the inverse Fourier transform, we have

$$\begin{aligned} \mathscr{F}^{-1}\left\{\frac{1}{2\pi}X_1(\omega) * X_2(\omega)\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}X_1(\omega) * X_2(\omega)\right]e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{1}{2\pi}X_1(\lambda)X_2(\omega-\lambda)d\lambda\right]e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}X_1(\lambda)X_2(\omega-\lambda)e^{j\omega t}d\lambda d\omega. \end{aligned}$$

Reversing the order of integration, we obtain

$$\mathscr{F}^{-1}\left\{\frac{1}{2\pi}X_1(\omega) * X_2(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}X_1(\lambda)X_2(\omega-\lambda)e^{j\omega t}d\omega d\lambda.$$

Now, we employ a change of variable. Let $v = \omega - \lambda$ so that $\omega = v + \lambda$ and $dv = d\lambda$. Applying the change of variable and simplifying yields

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\frac{1}{2\pi}X_1(\omega)*X_2(\omega)\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}X_1(\lambda)X_2(v)e^{j(v+\lambda)t}dvd\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}X_1(\lambda)X_2(v)e^{jvt}e^{j\lambda t}dvd\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(\lambda)e^{j\lambda t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X_2(v)e^{jvt}dv\right]d\lambda \\
&= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(\lambda)e^{j\lambda t}d\lambda\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X_2(v)e^{jvt}dv\right] \\
&= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(\omega)e^{j\omega t}d\omega\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X_2(\omega)e^{j\omega t}d\omega\right] \\
&= \mathcal{F}^{-1}\{X_1(\omega)\}\mathcal{F}^{-1}\{X_2(\omega)\} \\
&= x_1(t)x_2(t).
\end{aligned}$$

Thus, we have shown that the frequency-domain convolution property holds.

Example 5.13 (Frequency-domain convolution property)**.** Suppose that we have the signal

$$y(t) = x(t)\cos\omega_c t$$

where ω_c is a nonzero real constant. Find the Fourier transform $Y(\omega)$ of the signal $y(t)$ in terms of $X(\omega) = \mathcal{F}\{x(t)\}$.

Solution. Taking the Fourier transform of both sides of the above equation for $y(t)$, we have

$$Y(\omega) = \mathcal{F}\{x(t)\cos\omega_c t\}.$$

Using the frequency-domain convolution property of the Fourier transform, we can write

$$\begin{aligned}
Y(\omega) &= \mathcal{F}\{x(t)\cos\omega_c t\} \\
&= \tfrac{1}{2\pi}\mathcal{F}\{x(t)\}*\mathcal{F}\{\cos\omega_c t\} \\
&= \tfrac{1}{2\pi}X(\omega)*[\pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]] \\
&= \tfrac{1}{2}X(\omega)*[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)] \\
&= \tfrac{1}{2}[X(\omega)*\delta(\omega-\omega_c)+X(\omega)*\delta(\omega+\omega_c)] \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty}X(\lambda)\delta(\omega-\lambda-\omega_c)d\lambda+\int_{-\infty}^{\infty}X(\lambda)\delta(\omega-\lambda+\omega_c)d\lambda\right] \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty}X(\lambda)\delta(\lambda-\omega+\omega_c)d\lambda+\int_{-\infty}^{\infty}X(\lambda)\delta(\lambda-\omega-\omega_c)d\lambda\right] \\
&= \tfrac{1}{2}[X(\omega-\omega_c)+X(\omega+\omega_c)] \\
&= \tfrac{1}{2}X(\omega-\omega_c)+\tfrac{1}{2}X(\omega+\omega_c).
\end{aligned}$$

□

5.5.9 Time-Domain Differentiation

If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, then

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(\omega).$$

This is known as the time-domain differentiation property of the Fourier transform.

To prove the above property, we begin by using the definition of the inverse Fourier transform to write

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$$

Now, we differentiate both sides of the preceding equation with respect to t and simplify to obtain

$$\begin{aligned}\frac{dx(t)}{dt} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)(j\omega)e^{j\omega t}d\omega \\ &= \mathscr{F}^{-1}\{j\omega X(\omega)\}.\end{aligned}$$

Thus, we have shown that the time-differentiation property holds. In passing, we also note that by repeating the above argument, we have

$$\frac{d^n}{dt^n}x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} (j\omega)^n X(\omega).$$

Example 5.14 (Time-domain differentiation property)**.** Find the Fourier transform $X(\omega)$ of the signal $x(t) = \frac{d}{dt}\delta(t)$.

Solution. Taking the Fourier transform of both sides of the above equation for $x(t)$ yields

$$X(\omega) = \mathscr{F}\{\tfrac{d}{dt}\delta(t)\}.$$

Using the time-domain differentiation property of the Fourier transform, we can write

$$\begin{aligned}X(\omega) &= \mathscr{F}\{\tfrac{d}{dt}\delta(t)\} \\ &= j\omega\mathscr{F}\{\delta(t)\} \\ &= j\omega(1) \\ &= j\omega.\end{aligned}$$

□

5.5.10 Frequency-Domain Differentiation

If $x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(\omega)$, then

$$tx(t) \stackrel{\mathscr{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega).$$

This is known as the frequency-domain differentiation property of the Fourier transform.

To prove the above property, we proceed as follows. From the definition of the Fourier transform, we can write

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

Now, we differentiate both sides of this equation with respect to ω and simplify to obtain

$$\begin{aligned}\frac{d}{d\omega}X(\omega) &= \int_{-\infty}^{\infty} x(t)(-jt)e^{-j\omega t}dt \\ &= -j\int_{-\infty}^{\infty} tx(t)e^{-j\omega t}dt \\ &= -j\mathscr{F}\{tx(t)\}.\end{aligned}$$

Multiplying both sides of the preceding equation by j yields

$$j\frac{d}{d\omega}X(\omega) = \mathscr{F}\{tx(t)\}.$$

Thus, we have shown that the frequency-domain differentiation property holds.

Example 5.15 (Frequency-domain differentiation property)**.** Find the Fourier transform $X(\omega)$ of the signal $x(t) = t\cos\omega_0 t$ where ω_0 is a nonzero real constant.

Solution. Taking the Fourier transform of both sides of the equation for $x(t)$ yields

$$X(\omega) = \mathscr{F}\{t\cos\omega_0 t\}.$$

From the frequency-domain differentiation property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \mathscr{F}\{t\cos\omega_0 t\} \\ &= j\tfrac{d}{d\omega}\left[\mathscr{F}\{\cos\omega_0 t\}\right] \\ &= j\tfrac{d}{d\omega}\left[\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]\right] \\ &= j\pi\tfrac{d}{d\omega}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] \\ &= j\pi\tfrac{d}{d\omega}\delta(\omega-\omega_0)+j\pi\tfrac{d}{d\omega}\delta(\omega+\omega_0). \end{aligned}$$

□

5.5.11 Time-Domain Integration

If $x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega).$$

This is known as the time-domain integration property of the Fourier transform.

The above property can be proven as follows. First, we observe that

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t)*u(t).$$

Taking the Fourier transform of both sides of the preceding equation and using the time-domain convolution property of the Fourier transform, we have

$$\begin{aligned} \mathscr{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} &= \mathscr{F}\{x(t)*u(t)\} \\ &= X(\omega)\mathscr{F}\{u(t)\}. \end{aligned} \tag{5.13}$$

From Example 5.6, we know that $u(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \pi\delta(\omega)+\frac{1}{j\omega}$. Using this fact, we can rewrite (5.13) as

$$\begin{aligned} \mathscr{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} &= X(\omega)[\pi\delta(\omega)+\tfrac{1}{j\omega}] \\ &= \tfrac{1}{j\omega}X(\omega)+\pi X(\omega)\delta(\omega). \end{aligned}$$

From the equivalence property of the unit-impulse function, we have

$$\mathscr{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \tfrac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega).$$

Thus, we have shown that the time-domain integration property holds.

Example 5.16 (Time-domain integration property of the Fourier transform)**.** Use the time-domain integration property of the Fourier transform in order to find the Fourier transform $X(\omega)$ of the signal $x(t) = u(t)$.

Solution. We begin by observing that $x(t)$ can be expressed in terms of an integral as follows:

$$x(t) = u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau.$$

Now, we consider the Fourier transform of $x(t)$. We have

$$\begin{aligned} X(\omega) &= \mathscr{F}\{u(t)\} \\ &= \mathscr{F}\left\{\int_{-\infty}^{t} \delta(\tau) d\tau\right\}. \end{aligned}$$

From the time-domain integration property, we can write

$$\begin{aligned} X(\omega) &= \frac{1}{j\omega}\mathscr{F}\{\delta(t)\} + \pi \left[\mathscr{F}\{\delta(t)\}\right]\big|_{\omega=0} \delta(\omega) \\ &= \frac{1}{j\omega}(1) + \pi(1)\delta(\omega) \\ &= \frac{1}{j\omega} + \pi\delta(\omega). \end{aligned}$$

Thus, we have shown that $u(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)$. □

5.5.12 Parseval's Relation

Recall that the energy of a signal $f(t)$ is given by $\int_{-\infty}^{\infty} |f(t)|^2 dt$. If $x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega. \tag{5.14}$$

That is, the energy of $x(t)$ and energy of $X(\omega)$ are equal within a scaling factor of 2π. This relationship is known as Parseval's relation.

To prove the above relationship, we proceed as follows. Consider the left-hand side of (5.14) which we can write as

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)x^*(t) dt \\ &= \int_{-\infty}^{\infty} x(t)\mathscr{F}^{-1}\{\mathscr{F}\{x^*(t)\}\} dt. \end{aligned}$$

From the conjugation property of the Fourier transform, we have that $x^*(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X^*(-\omega)$. So, we can rewrite the above equation as

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)\mathscr{F}^{-1}\{X^*(-\omega)\} dt \\ &= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(-\omega)e^{j\omega t} d\omega\right] dt. \end{aligned}$$

Now, we employ a change of variable (i.e., replace ω by $-\omega$) to obtain

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t} d\omega\right] dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)X^*(\omega)e^{-j\omega t} d\omega dt. \end{aligned}$$

Reversing the order of integration and simplifying, we have

$$\begin{aligned}\int_{-\infty}^{\infty}|x(t)|^2\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(t)X^*(\omega)e^{-j\omega t}\,dt\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(\omega)\left[\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\right]d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(\omega)X(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega.\end{aligned}$$

Thus, Parseval's relation holds.

Example 5.17 (Energy of the sinc signal)**.** Suppose that we have the signal $x(t) = \operatorname{sinc} t/2$ and $\mathscr{F}\{x(t)\} = 2\pi \operatorname{rect}\omega$. Compute the energy of the signal $x(t)$.

Solution. We could directly compute the energy of the signal $x(t)$ as

$$\begin{aligned}E &= \int_{-\infty}^{\infty}|x(t)|^2\,dt \\ &= \int_{-\infty}^{\infty}|\operatorname{sinc} t/2|^2\,dt.\end{aligned}$$

This integral is not so easy to compute, however. Instead, we use Parseval's relation to write

$$\begin{aligned}E &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|2\pi\operatorname{rect}\omega|^2\,d\omega \\ &= \frac{1}{2\pi}\int_{-1/2}^{1/2}(2\pi)^2 d\omega \\ &= 2\pi\int_{-1/2}^{1/2}d\omega \\ &= 2\pi[\omega]|_{-1/2}^{1/2} \\ &= 2\pi[\tfrac{1}{2}+\tfrac{1}{2}] \\ &= 2\pi.\end{aligned}$$

Thus, we have

$$E = \int_{-\infty}^{\infty}\left|\operatorname{sinc}\tfrac{t}{2}\right|^2 dt = 2\pi.$$

□

5.6 Continuous-Time Fourier Transform of Periodic Signals

The Fourier transform can be applied to both aperiodic and periodic signals. In what follows, we derive a method for computing the Fourier transform of a periodic signal.

Suppose that we have a periodic signal $x(t)$ with period T. Since $x(t)$ is periodic, we can express it using a Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{5.15}$$

Table 5.1: Fourier Transform Properties

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time-Domain Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time-Domain Convolution	$x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
Frequency-Domain Convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Frequency-Domain Differentiation	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
Time-Domain Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$

Property	
Parseval's Relation	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2d\omega$

where

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt. \tag{5.16}$$

Let us define the signal $x_T(t)$ to contain a single period of $x(t)$ as follows:

$$x_T(t)=\begin{cases}x(t) & \text{for } -\frac{T}{2}\le t<\frac{T}{2}\\ 0 & \text{otherwise.}\end{cases} \tag{5.17}$$

Let $X_T(\omega)$ denote the Fourier transform of $x_T(t)$. Consider the expression for a_k in (5.16). Since $x_T(t)=x(t)$ for a single period of $x(t)$ and is zero otherwise, we can rewrite (5.16) as

$$\begin{aligned}a_k&=\frac{1}{T}\int_{-\infty}^{\infty}x_T(t)e^{-jk\omega_0 t}dt\\&=\frac{1}{T}X_T(k\omega_0).\end{aligned} \tag{5.18}$$

Now, let us consider the Fourier transform of $x(t)$. By taking the Fourier transform of both sides of (5.15), we obtain

$$\begin{aligned}\mathscr{F}\{x(t)\}&=\mathscr{F}\left\{\sum_{k=-\infty}^{\infty}a_ke^{jk\omega_0 t}\right\}\\&=\int_{-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty}a_ke^{jk\omega_0 t}\right]e^{-j\omega t}dt.\end{aligned}$$

Reversing the order of summation and integration, we have

$$\begin{aligned}\mathscr{F}\{x(t)\}&=\sum_{k=-\infty}^{\infty}a_k\int_{-\infty}^{\infty}e^{jk\omega_0 t}e^{-j\omega t}dt\\&=\sum_{k=-\infty}^{\infty}a_k\mathscr{F}\{e^{jk\omega_0 t}\}.\end{aligned} \tag{5.19}$$

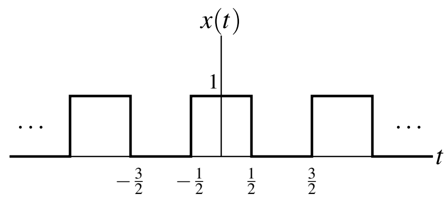


Figure 5.5: Periodic function $x(t)$.

From Example 5.2, we know that $\mathscr{F}\{e^{j\lambda t}\} = 2\pi\delta(\omega-\lambda)$. So, we can simplify (5.19) to obtain

$$\begin{aligned}\mathscr{F}\{x(t)\} &= \sum_{k=-\infty}^{\infty} a_k[2\pi\delta(\omega-k\omega_0)] \\ &= \sum_{k=-\infty}^{\infty} 2\pi a_k\delta(\omega-k\omega_0). \qquad (5.20)\end{aligned}$$

Thus, the Fourier transform of a periodic function is a series of impulse functions located at integer multiples of the fundamental frequency ω_0. The weight of each impulse is 2π times the corresponding Fourier series coefficient. Furthermore, by substituting (5.18) into (5.20), we have

$$\begin{aligned}\mathscr{F}\{x(t)\} &= \sum_{k=-\infty}^{\infty} 2\pi[\tfrac{1}{T}X_T(k\omega_0)]\delta(\omega-k\omega_0) \\ &= \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega-k\omega_0). \qquad (5.21)\end{aligned}$$

This provides an alternative expression for the Fourier transform of $x(t)$ in terms of the Fourier transform $X_T(\omega)$ of a single period of $x(t)$.

In summary, we have shown that the periodic signal $x(t)$ with period T and frequency $\omega_0 = \frac{2\pi}{T}$ has the Fourier transform $X(\omega)$ given by

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\delta(\omega-k\omega_0). \qquad (5.22)$$

(as in (5.20)), or equivalently,

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega-k\omega_0) \qquad (5.23)$$

(as in (5.21), where a_k is the Fourier series coefficient sequence of $x(t)$, $X_T(\omega)$ is the Fourier transform of $x_T(t)$, and $x_T(t)$ is a function equal to $x(t)$ over a single period and zero elsewhere (e.g., as in (5.17)). Furthermore, we have also shown that the Fourier series coefficients a_k of $x(t)$ are related to $X_T(\omega)$ by the equation

$$a_k = \frac{1}{T}X_T(k\omega_0).$$

Thus, we have that the Fourier series coefficient sequence a_k of the periodic signal $x(t)$ is produced by sampling the Fourier transform of $x_T(t)$ at integer multiples of the fundamental frequency ω_0 and scaling the resulting sequence by $\frac{1}{T}$.

Example 5.18. Consider the periodic function $x(t)$ with period $T = 2$ as shown in Figure 5.5. Using the Fourier transform, find the Fourier series representation of $x(t)$.

Table 5.2: Fourier Transform Pairs

Pair	$x(t)$	$X(\omega)$
1	$\delta(t)$	1
2	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
3	1	$2\pi\delta(\omega)$
4	$\operatorname{sgn}(t)$	$\frac{2}{j\omega}$
5	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
6	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
7	$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
8	$\operatorname{rect}(t/T)$	$T\operatorname{sinc}(\omega T/2)$
9	$\frac{B}{\pi}\operatorname{sinc}Bt$	$\operatorname{rect}\frac{\omega}{2B}$
10	$e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
11	$t^{n-1}e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{(n-1)!}{(a+j\omega)^n}$
12	$\operatorname{tri}(t/T)$	$\frac{T}{2}\operatorname{sinc}^2(T\omega/4)$

Solution. We have that $\omega_0=\frac{2\pi}{T}=\pi$. Let $y(t)=\operatorname{rect}t$ (i.e., $y(t)$ corresponds to a single period of the periodic function $x(t)$). Thus, we have that

$$x(t)=\sum_{k=-\infty}^{\infty}y(t-2k).$$

Taking the Fourier transform of $y(t)$, we obtain

$$Y(\omega)=\mathscr{F}\{\operatorname{rect}t\}=\operatorname{sinc}\omega/2.$$

Now, we seek to find the Fourier series representation of $x(t)$, which has the form

$$x(t)=\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 t}.$$

Using the Fourier transform, we compute c_k as follows:

$$\begin{aligned}c_k&=\tfrac{1}{T}Y(k\omega_0)\\&=\tfrac{1}{2}\operatorname{sinc}(\tfrac{k\omega_0}{2})\\&=\tfrac{1}{2}\operatorname{sinc}(k\pi/2).\end{aligned}$$

□

5.7 Fourier Transforms

Throughout this chapter, we have derived a number of Fourier transform pairs. Some of these and other important transform pairs are listed in Table 5.2. Using the various Fourier transform properties listed in Table 5.1 and the Fourier transform pairs listed in Table 5.2, we can determine (more easily) the Fourier transform of more complicated signals.

Example 5.19. Suppose that $x(t)\overset{\mathscr{F}}{\longleftrightarrow}X(\omega)$, $x_1(t)\overset{\mathscr{F}}{\longleftrightarrow}X_1(\omega)$, and

$$x_1(t)=\frac{d^2}{dt^2}x(t-2).$$

Express $X_1(\omega)$ in terms of $X(\omega)$.

Solution. Let $v_1(t) = x(t-2)$. From the definition of $v_1(t)$ and the time-shifting property of the Fourier transform, we have

$$V_1(\omega) = e^{-j2\omega} X(\omega). \tag{5.24}$$

From the definition of $v_1(t)$, we have

$$x_1(t) = \frac{d^2}{dt^2} v_1(t).$$

Thus, from the time-differentiation property of the Fourier transform, we can write

$$\begin{aligned} X_1(\omega) &= (j\omega)^2 V_1(\omega) \\ &= -\omega^2 V_1(\omega). \end{aligned} \tag{5.25}$$

Combining (5.24) and (5.25), we obtain

$$X_1(\omega) = -\omega^2 e^{-j2\omega} X(\omega).$$

□

Example 5.20. Suppose that $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, $x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)$, and

$$x_1(t) = x(at-b),$$

where a is a nonzero real constant and b is a real constant. Express $X_1(\omega)$ in terms of $X(\omega)$.

Solution. We rewrite $x_1(t)$ as

$$x_1(t) = v_1(at)$$

where

$$v_1(t) = x(t-b).$$

We now take the Fourier transform of both sides of each of the preceding equations. Using the time-shifting property of the Fourier transform, we can write

$$V_1(\omega) = e^{-jb\omega} X(\omega). \tag{5.26}$$

Using the time-scaling property of the Fourier transform, we can write

$$X_1(\omega) = \frac{1}{|a|} V_1\left(\frac{\omega}{a}\right). \tag{5.27}$$

Substituting the expression for $V_1(\omega)$ in (5.26) into (5.27), we obtain

$$X_1(\omega) = \frac{1}{|a|} e^{-j(b/a)\omega} X\left(\frac{\omega}{a}\right).$$

□

Example 5.21. Suppose that we have the periodic signal $x(t)$ given by

$$x(t) = \sum_{k=-\infty}^{\infty} x_0(t-kT)$$

where a single period of $x(t)$ is given by

$$x_0(t) = A\,\mathrm{rect}(2t/T).$$

Find the Fourier transform $X(\omega)$ of the signal $x(t)$.

Solution. From (5.23), we know that

$$\begin{aligned}\mathscr{F}\{x(t)\} &= \mathscr{F}\left\{\sum_{k=-\infty}^{\infty} x_0(t-kT)\right\} \\ &= \sum_{k=-\infty}^{\infty} \omega_0 X_0(k\omega_0)\delta(\omega-k\omega_0).\end{aligned}$$

So, we need to find $X_0(\omega)$. This quantity is computed (by using the linearity property of the Fourier transform and Table 5.2) as follows:

$$\begin{aligned}X_0(\omega) &= \mathscr{F}\{x_0(t)\} \\ &= \mathscr{F}\{A\operatorname{rect}(2t/T)\} \\ &= A\mathscr{F}\{\operatorname{rect}(2t/T)\} \\ &= \tfrac{AT}{2}\operatorname{sinc}(\omega T/4).\end{aligned}$$

Thus, we have that

$$\begin{aligned}X(\omega) &= \sum_{k=-\infty}^{\infty} \omega_0 \tfrac{AT}{2}\operatorname{sinc}(k\omega_0 T/4)\delta(\omega-k\omega_0) \\ &= \sum_{k=-\infty}^{\infty} \pi A\operatorname{sinc}(\tfrac{\pi k}{2})\delta(\omega-k\omega_0).\end{aligned}$$

□

Example 5.22. Suppose that we have the signal $x(t)$ given by

$$x(t) = \int_{-\infty}^{t} e^{-(3+j2)\tau}u(\tau)d\tau.$$

Find the Fourier transform $X(\omega)$ of the signal $x(t)$.

Solution. We can rewrite $x(t)$ as

$$x(t) = \int_{-\infty}^{t} v_1(\tau)d\tau, \tag{5.28}$$

where

$$v_1(t) = e^{-j2t}v_2(t), \quad \text{and} \tag{5.29}$$

$$v_2(t) = e^{-3t}u(t). \tag{5.30}$$

From (5.28) and the time-domain integration property of the Fourier transform, we have

$$X(\omega) = \tfrac{1}{j\omega}V_1(\omega) + \pi V_1(0)\delta(\omega). \tag{5.31}$$

From (5.29) and the frequency-domain shifting property of the Fourier transform, we have

$$V_1(\omega) = V_2(\omega+2). \tag{5.32}$$

From (5.30) and Table 5.2 (i.e., the entry for $\mathscr{F}\{e^{-at}u(t)\}$), we have

$$V_2(\omega) = \frac{1}{3+j\omega}. \tag{5.33}$$

Combining (5.31), (5.32), and (5.33), we obtain

$$\begin{aligned} X(\omega) &= \frac{1}{j\omega}V_1(\omega) + \pi V_1(0)\delta(\omega) \\ &= \frac{1}{j\omega}V_2(\omega+2) + \pi V_2(2)\delta(\omega) \\ &= \frac{1}{j\omega}\left(\frac{1}{3+j(\omega+2)}\right) + \pi\left(\frac{1}{3+j2}\right)\delta(\omega). \end{aligned}$$

□

Example 5.23. Let $X(\omega)$ and $Y(\omega)$ denote the Fourier transforms of $x(t)$ and $y(t)$, respectively. Suppose that $y(t) = x(t)\cos at$, where a is a nonzero real constant. Find an expression for $Y(\omega)$ in terms of $X(\omega)$.

Solution. Essentially, we need to take the Fourier transform of both sides of the given equation. There are two obvious ways in which to do this. One is to use the time-domain multiplication property of the Fourier transform, and another is to use the frequency-domain shifting property. We will solve this problem using each method in turn in order to show that the two approaches do not involve an equal amount of effort.

FIRST SOLUTION (USING AN UNENLIGHTENED APPROACH). We use the time-domain multiplication property. Taking the Fourier transform of both sides of the given equation, we obtain

$$\begin{aligned} Y(\omega) &= \mathscr{F}\{x(t)\cos at\} \\ &= \tfrac{1}{2\pi}\left[X(\omega) * \mathscr{F}\{\cos at\}\right] \\ &= \tfrac{1}{2\pi}\left[X(\omega) * [\pi(\delta(\omega-a)+\delta(\omega+a))]\right] \\ &= \tfrac{1}{2}\left[X(\omega) * [\delta(\omega-a)+\delta(\omega+a)]\right] \\ &= \tfrac{1}{2}\left[X(\omega)*\delta(\omega-a) + X(\omega)*\delta(\omega+a)\right] \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\tau)\delta(\omega-\tau-a)d\tau + \int_{-\infty}^{\infty} X(\tau)\delta(\omega-\tau+a)d\tau\right] \\ &= \tfrac{1}{2}\left[X(\omega-a)+X(\omega+a)\right] \\ &= \tfrac{1}{2}X(\omega-a)+\tfrac{1}{2}X(\omega+a). \end{aligned}$$

Note that the above solution is identical to the one appearing earlier in Example 5.13 on page 118.

SECOND SOLUTION (USING AN ENLIGHTENED APPROACH). We use the frequency-domain shifting property. Taking the Fourier transform of both sides of the given equation, we obtain

$$\begin{aligned} Y(\omega) &= \mathscr{F}\{x(t)\cos at\} \\ &= \mathscr{F}\{\tfrac{1}{2}(e^{jat}+e^{-jat})x(t)\} \\ &= \tfrac{1}{2}\mathscr{F}\{e^{jat}x(t)\} + \tfrac{1}{2}\mathscr{F}\{e^{-jat}x(t)\} \\ &= \tfrac{1}{2}X(\omega-a)+\tfrac{1}{2}X(\omega+a). \end{aligned}$$

COMMENTARY. Clearly, of the above two solution methods, the second approach is simpler and much less error prone. Generally, the use of the time-domain multiplication property tends to lead to less clean solutions, as this forces a convolution to be performed in the frequency domain and convolution is often best avoided if possible. □

Example 5.24 (Fourier transform of an even signal)**.** Show that if a signal $x(t)$ is even, then its Fourier transform $X(\omega)$ is even.

Solution. From the definition of the Fourier transform, we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

Since $x(t)$ is even, we can rewrite this as

$$X(\omega)=\int_{-\infty}^{\infty}x(-t)e^{-j\omega t}dt.$$

Now, we employ a change of variable. Let $\lambda=-t$ so that $d\lambda=-dt$. Applying the change of variable, we obtain

$$\begin{aligned}X(\omega)&=\int_{\infty}^{-\infty}x(\lambda)e^{j\omega\lambda}(-1)d\lambda\\&=-\int_{\infty}^{-\infty}x(\lambda)e^{j\omega\lambda}d\lambda\\&=\int_{-\infty}^{\infty}x(\lambda)e^{j\omega\lambda}d\lambda\\&=\int_{-\infty}^{\infty}x(\lambda)e^{-j(-\omega)\lambda}d\lambda\\&=X(-\omega).\end{aligned}$$

Therefore, $X(\omega)$ is even. □

Example 5.25 (Fourier transform of an odd signal)**.** Show that if a signal $x(t)$ is odd, then its Fourier transform $X(\omega)$ is odd.

Solution. From the definition of the Fourier transform, we have

$$X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt.$$

Since $x(t)$ is odd, we can rewrite this as

$$X(\omega)=\int_{-\infty}^{\infty}-x(-t)e^{-j\omega t}dt.$$

Now, we employ a change of variable. Let $\lambda=-t$ so that $d\lambda=-dt$. Applying this change of variable, we obtain

$$\begin{aligned}X(\omega)&=\int_{\infty}^{-\infty}-x(\lambda)e^{-j\omega(-\lambda)}(-1)d\lambda\\&=\int_{\infty}^{-\infty}x(\lambda)e^{j\omega\lambda}d\lambda\\&=-\int_{-\infty}^{\infty}x(\lambda)e^{-j(-\omega)\lambda}d\lambda\\&=-X(-\omega).\end{aligned}$$

Therefore, $X(\omega)$ is odd. □

5.8 Frequency Spectra of Signals

The Fourier transform representation expresses a signal as a function of complex sinusoids at all frequencies. In this sense, the Fourier transform representation captures information about the frequency content of a signal. For example, suppose that we have a signal $x(t)$ with Fourier transform $X(\omega)$. If $X(\omega)$ is nonzero for some frequency ω_0, then the signal $x(t)$ contains some information at the frequency ω_0. On the other hand, if $X(\omega)$ is zero at the frequency ω_0, then the signal has no information at that frequency. In this way, the Fourier transform representation provides a means for measuring the frequency content of a signal. This distribution of information in a signal over different frequencies is referred to as the **frequency spectrum** of the signal. That is, $X(\omega)$ is the frequency spectrum of $x(t)$.

To gain further insight into the role played by the Fourier transform $X(\omega)$ in the context of the frequency spectrum of $x(t)$, it is helpful to write the Fourier transform representation of $x(t)$ with $X(\omega)$ expressed in polar form as follows:

$$\begin{aligned} x(t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\, e^{j\arg X(\omega)}e^{j\omega t}d\omega \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\, e^{j[\omega t+\arg X(\omega)]}d\omega. \end{aligned}$$

In effect, the quantity $|X(\omega)|$ is a weight that determines how much the complex sinusoid at frequency ω contributes to the integration result $x(t)$. Perhaps, this can be more easily seen if we express the above integral as the limit of a sum, derived from an approximation of the integral using the area of rectangles. Expressing $x(t)$ in this way, we obtain

$$\begin{aligned} x(t) &= \lim_{\Delta\omega\to 0}\tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty}\Delta\omega\,|X(k\Delta\omega)|\,e^{j[k\Delta\omega t+\arg X(k\Delta\omega)]} \\ &= \lim_{\Delta\omega\to 0}\tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty}\Delta\omega\,\left|X(\omega')\right|\,e^{j[\omega' t+\arg X(\omega')]}, \end{aligned}$$

where $\omega' = k\Delta\omega$. From the last line of the above equation, the kth term in the summation (associated with the frequency $\omega' = k\Delta\omega$) corresponds to a complex sinusoid with fundamental frequency ω' that has had its amplitude scaled by a factor of $|X(\omega')|$ and has been time-shifted by an amount that depends on $\arg X(\omega')$. For a given $\omega' = k\Delta\omega$ (which is associated with the kth term in the summation), the larger $|X(\omega')|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\omega' t}$ will be, and therefore the larger the contribution the kth term will make to the overall summation. In this way, we can use $|X(\omega')|$ as a measure of how much information a signal $x(t)$ has at the frequency ω'.

Note that, since the Fourier transform $X(\omega)$ is a function of a real variable (namely, ω), a signal can, in the most general case, have information at *any arbitrary* real frequency. This is different from the case of frequency spectra in the Fourier series context (which deals only with periodic signals), where a signal can only have information at certain specific frequencies (namely, at integer multiples of the fundamental frequency). There is no inconsistency here, however. As we saw in Section 5.6, in the case of periodic signals the Fourier transform will also be zero, except possibly at integer multiples of the fundamental frequency.

Since the frequency spectrum is complex (in the general case), it is usually represented using two plots, one showing the magnitude and one showing the argument of $X(\omega)$. We refer to $|X(\omega)|$ as the **magnitude spectrum** of the signal $x(t)$. Similarly, we refer to $\arg X(\omega)$ as the **phase spectrum** of the signal $x(t)$. In the special case that $X(\omega)$ is a real-valued (or purely imaginary) function, we usually plot the frequency spectrum directly on a single graph.

Consider the signal $x(t) = \operatorname{sgn}(t-1)$. We can show that the Fourier transform $X(\omega)$ of this signal is $X(\omega) = \frac{2}{j\omega}e^{-j\omega}$. In this case, $X(\omega)$ is neither purely real nor purely imaginary, so we use two separate graphs to represent the frequency spectrum of the signal. We plot the magnitude spectrum and phase spectrum as shown in Figures 5.6(a) and (b), respectively.

Consider the signal $x(t) = \delta(t)$. This signal has the Fourier transform $X(\omega) = 1$. Since, in this case, $X(\omega)$ is a real-valued function, we can plot the frequency spectrum $X(\omega)$ on a single graph, as shown in Figure 5.7.

Consider the signal $x(t) = \operatorname{sinc} t/2$. This signal has the Fourier transform $X(\omega) = 2\pi \operatorname{rect}\omega$. Since, in this case, $X(\omega)$ is a real-valued function, we can plot the frequency spectrum $X(\omega)$ on a single graph, as shown in Figure 5.8.

Suppose that we have a signal $x(t)$ with Fourier transform $X(\omega)$. If $x(t)$ is real, then

$$\begin{aligned} |X(\omega)| &= |X(-\omega)| \quad \text{and} \\ \arg X(\omega) &= -\arg X(-\omega) \end{aligned}$$

(i.e., $|X(\omega)|$ is an even function and $\arg X(\omega)$ is an odd function). (Earlier, these identities were shown to hold. See Example 5.10 for a detailed proof.)

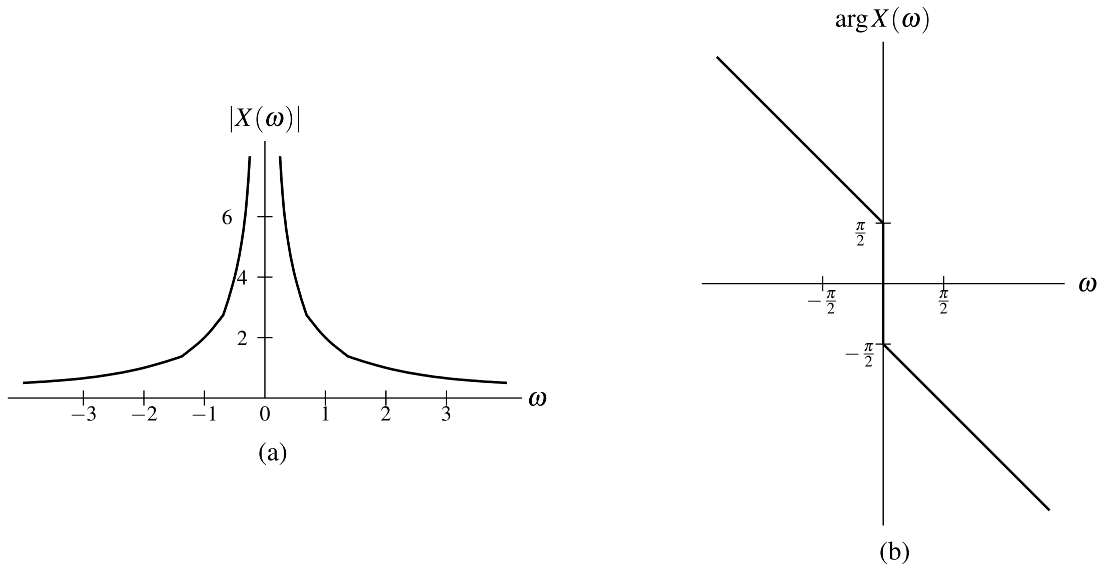


Figure 5.6: Frequency spectrum of the time-shifted signum signal. (a) Magnitude spectrum and (b) Phase spectrum.

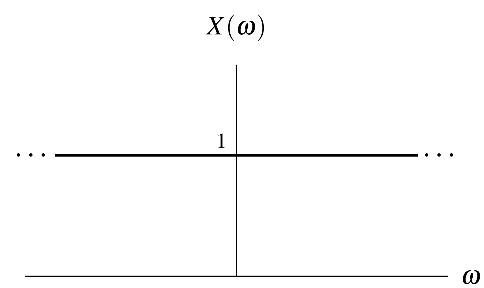


Figure 5.7: Frequency spectrum of the unit-impulse function.

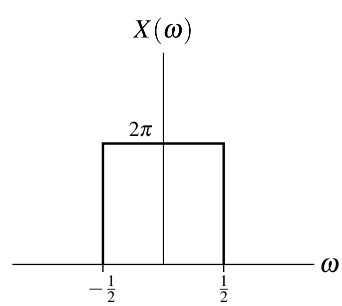


Figure 5.8: Frequency spectrum of the sinc function.

Example 5.26 (Frequency spectrum of the shifted signum signal)**.** Suppose that we have the signal $x(t) = \operatorname{sgn}(t-1)$ with the Fourier transform $X(\omega) = \left(\frac{2}{j\omega}\right)e^{-j\omega}$. Find and plot the magnitude and phase spectra of this signal. Determine at what frequency (or frequencies) the signal $x(t)$ has the most information.

Solution. First, we must find the magnitude spectrum $|X(\omega)|$. From the expression for $X(\omega)$, we can write

$$\begin{aligned}
|X(\omega)| &= \left|\tfrac{2}{j\omega}e^{-j\omega}\right| \\
&= \left|\tfrac{2}{j\omega}\right|\left|e^{-j\omega}\right| \\
&= \left|\tfrac{2}{j\omega}\right| \\
&= \tfrac{2}{|\omega|}.
\end{aligned}$$

Next, we find the phase spectrum $\arg X(\omega)$. From the expression for $X(\omega)$, we can write

$$\begin{aligned}
\arg X(\omega) &= \arg\left\{\tfrac{2}{j\omega}e^{-j\omega}\right\} \\
&= \arg e^{-j\omega} + \arg\tfrac{2}{j\omega} \\
&= -\omega + \arg\tfrac{2}{j\omega} \\
&= -\omega + \arg(-\tfrac{j2}{\omega}) \\
&= \begin{cases} -\frac{\pi}{2} - \omega & \text{for } \omega > 0 \\ \frac{\pi}{2} - \omega & \text{for } \omega < 0 \end{cases} \\
&= -\tfrac{\pi}{2}\operatorname{sgn}\omega - \omega.
\end{aligned}$$

Note that

$$\arg\tfrac{2}{j\omega} = \arg(-\tfrac{j2}{\omega}) = \begin{cases} -\frac{\pi}{2} & \text{for } \omega > 0 \\ \frac{\pi}{2} & \text{for } \omega < 0. \end{cases}$$

Finally, using numerical calculation, we can plot the graphs of the functions $|X(\omega)|$ and $\arg X(\omega)$ to obtain the results shown previously in Figures 5.6(a) and (b). Since $|X(\omega)|$ is largest for $\omega = 0$, the signal $x(t)$ has the most information at the frequency 0. □

Example 5.27 (Frequency spectra of images)**.** The human visual system is more sensitive to the phase spectrum of an image than its magnitude spectrum. This can be aptly demonstrated by separately modifying the magnitude and phase spectra of an image, and observing the effect. Below, we consider two variations on this theme.

Consider the `potatohead` and `hongkong` images shown in Figures 5.9(a) and (b), respectively. Replacing the magnitude spectrum of the `potatohead` image with the magnitude spectrum of the `hongkong` image (and leaving the phase spectrum unmodified), we obtain the new image shown in Figure 5.9(c). Although changing the magnitude spectrum has led to distortion, the basic essence of the original image has not been lost. On the other hand, replacing the phase spectrum of the `potatohead` image with the phase spectrum of the `hongkong` image (and leaving the magnitude spectrum unmodified), we obtain the image shown in Figure 5.9(d). Clearly, by changing the phase spectrum of the image, the fundamental nature of the image has been altered, with the new image more closely resembling the `hongkong` image than the original `potatohead` image.

A more extreme scenario is considered in Figure 5.10. In this case, we replace each of the magnitude and phase spectra of the `potatohead` image with random data, with this data being taken from the image consisting of random noise shown in Figure 5.10(b). When we completely replace the magnitude spectrum of the `potatohead` image with random values, we can still recognize the resulting image in Figure 5.10(c) as a very grainy version of the original `potatohead` image. On the other hand, when the phase spectrum of the `potatohead` image is replaced with random values, all visible traces of the original `potatohead` image are lost in the resulting image in Figure 5.10(d).

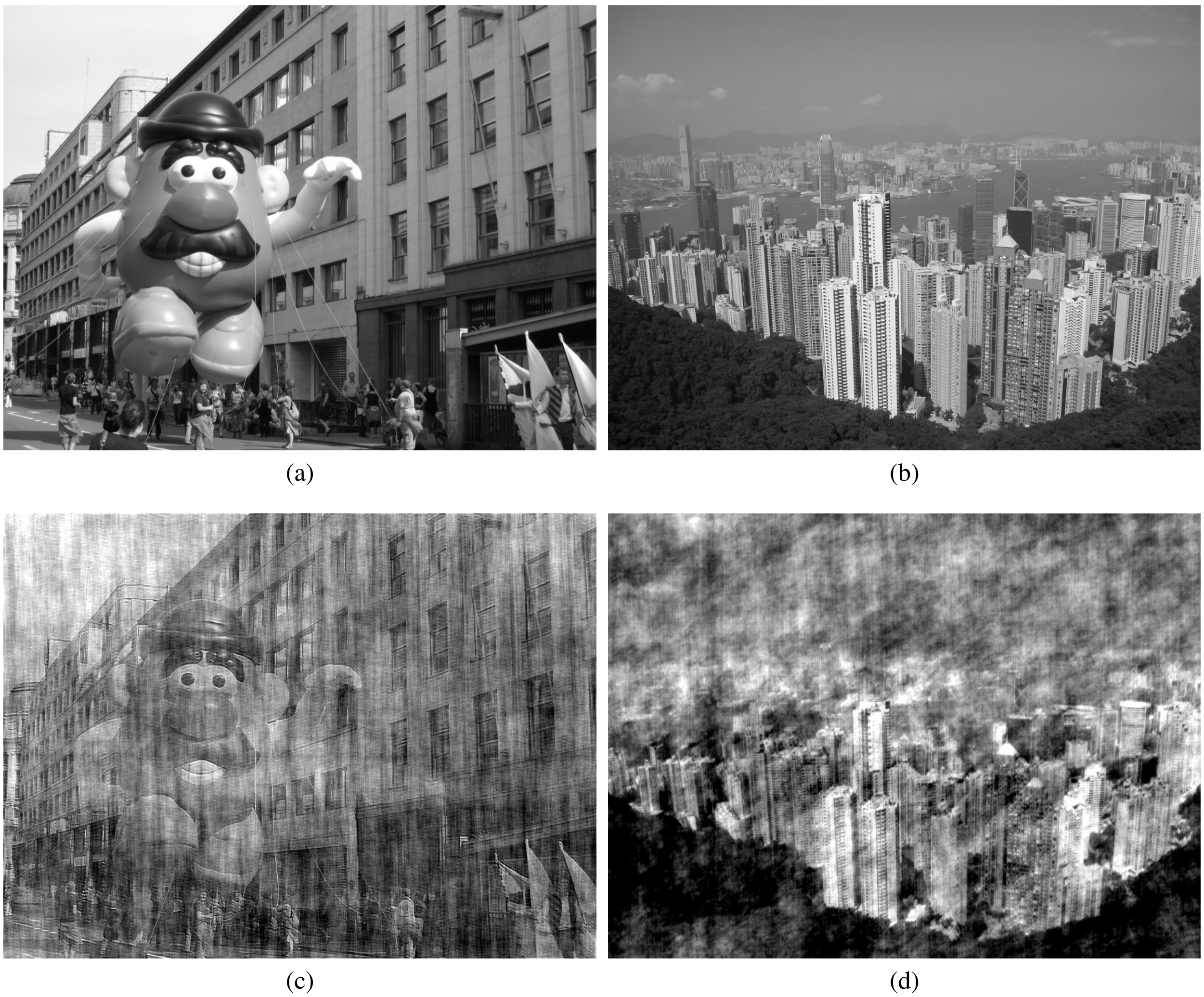
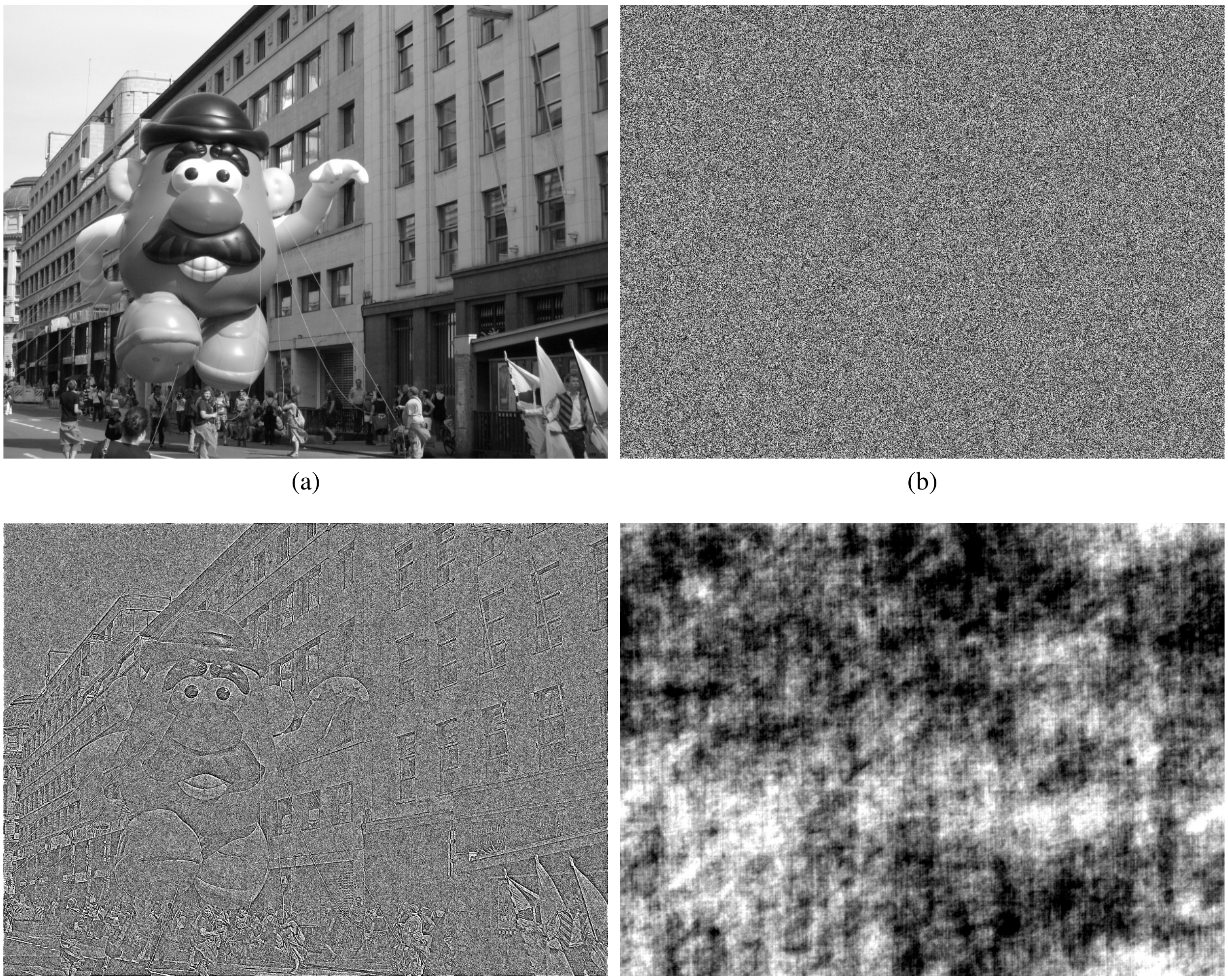


Figure 5.9: Importance of phase information in images. The (a) `potatohead` and (b) `hongkong` images. (c) The `potatohead` image after having its magnitude spectrum replaced with the magnitude spectrum of the `hongkong` image. (d) The `potatohead` image after having its phase spectrum replaced with the phase spectrum of the `hongkong` image.



(a) (b)

(c) (d)

Figure 5.10: Importance of phase information in images. The (a) `potatohead` and (b) `random` images. (c) The `potatohead` image after having its magnitude spectrum replaced with the magnitude spectrum of the `random` image. (d) The `potatohead` image after having its phase spectrum replaced with the phase spectrum of the `random` image.

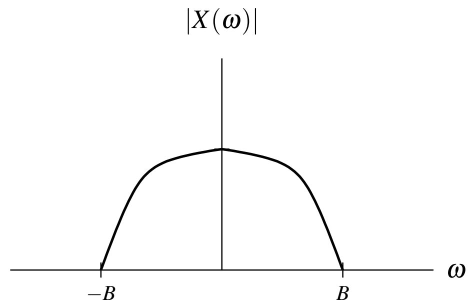


Figure 5.11: Bandwidth of a signal.

5.9 Bandwidth of Signals

A signal $x(t)$ with Fourier transform $X(\omega)$ is said to be **bandlimited** if, for some nonnegative real constant B, $X(\omega) = 0$ for all ω satisfying $|\omega| > B$. We sometimes refer to B as the **bandwidth** of the signal $x(t)$. An illustrative example is provided in Figure 5.11.

One can show that a signal cannot be both time limited and bandlimited. To help understand why this is so we recall the time/frequency scaling property of the Fourier transform. From this property, we know that as we compress a signal $x(t)$ along the t-axis, its Fourier transform $X(\omega)$ will expand along the ω-axis. Similarly, a compression of the Fourier transform $X(\omega)$ along the ω-axis corresponds to an expansion of $x(t)$ along the t-axis. Clearly, there is an inverse relationship between the time-extent and bandwidth of a signal.

5.10 Frequency Response of LTI Systems

Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Such a system is depicted in Figure 5.12. The behavior of such a system is governed by the equation

$$y(t) = x(t) * h(t). \tag{5.34}$$

Let $X(\omega)$, $Y(\omega)$, and $H(\omega)$ denote the Fourier transforms of $x(t)$, $y(t)$, and $h(t)$, respectively. Taking the Fourier transform of both sides of (5.34) yields

$$Y(\omega) = \mathscr{F}\{x(t) * h(t)\}.$$

From the time-domain convolution property of the Fourier transform, however, we can rewrite this as

$$Y(\omega) = X(\omega)H(\omega). \tag{5.35}$$

This result provides an alternative way of viewing the behavior of an LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output signals. In other words, we have a system resembling that in Figure 5.13. In this case, however, the convolution operation from the time domain is replaced by multiplication in the frequency domain. The frequency spectrum (i.e., Fourier transform) of the output is the product of the frequency spectrum (i.e., Fourier transform) of the input and the frequency spectrum (i.e., Fourier transform) of the impulse response. As a matter of terminology, we refer to $H(\omega)$ as the **frequency response** of the system. The system behavior is completely characterized by the frequency response $H(\omega)$. If we know the input, we can compute its Fourier transform $X(\omega)$, and then determine the Fourier transform $Y(\omega)$ of the output. Using the inverse Fourier transform, we can then determine the output $y(t)$.

In the general case, $H(\omega)$ is a complex-valued function. Thus, we can represent $H(\omega)$ in terms of its magnitude $|H(\omega)|$ and argument $\arg H(\omega)$. We refer to $|H(\omega)|$ as the **magnitude response** of the system. Similarly, we call $\arg H(\omega)$ the **phase response** of the system.

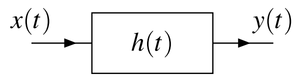


Figure 5.12: Time-domain view of LTI system.

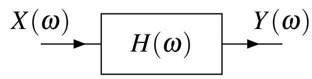


Figure 5.13: Frequency-domain view of LTI system.

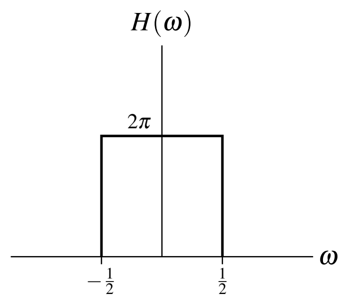


Figure 5.14: Frequency response of example system.

From (5.35), we can write

$$\begin{aligned}|Y(\omega)| &= |X(\omega)H(\omega)| \\ &= |X(\omega)|\,|H(\omega)| \quad \text{and}\end{aligned} \tag{5.36}$$

$$\begin{aligned}\arg Y(\omega) &= \arg(X(\omega)H(\omega)) \\ &= \arg X(\omega) + \arg H(\omega).\end{aligned} \tag{5.37}$$

From (5.36), we can see that the magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude spectrum of the impulse response. From (5.37), we have that the phase spectrum of the output equals the phase spectrum of the input plus the phase spectrum of the impulse response.

Since the frequency response $H(\omega)$ is simply the frequency spectrum of the impulse response $h(t)$, for the reasons explained in Section 5.8, if $h(t)$ is real, then

$$\begin{aligned}|H(\omega)| = |H(-\omega)| \quad \text{and} \\ \arg H(\omega) = -\arg H(-\omega)\end{aligned}$$

(i.e., the magnitude response $|H(\omega)|$ is an even function and the phase response $\arg H(\omega)$ is an odd function).

Example 5.28. Suppose that we have a LTI system with impulse response $h(t) = \operatorname{sinc} t/2$. The frequency response of the system is $H(\omega) = 2\pi \operatorname{rect} \omega$. In this particular case, $H(\omega)$ is real. So, we can plot the frequency response $H(\omega)$ on a single graph as shown in Figure 5.14.

5.11 Frequency Response and Differential Equation Representations of LTI Systems

Many LTI systems of practical interest can be represented using an Nth-order linear differential equation with constant coefficients. Suppose that we have such a system with input $x(t)$ and output $y(t)$. Then, the input-output behavior of the system is given by an equation of the form

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t)$$

(where $M \leq N$). Let $X(\omega)$ and $Y(\omega)$ denote the Fourier transforms of $x(t)$ and $y(t)$, respectively. Taking the Fourier transform of both sides of this equation yields

$$\mathscr{F}\left\{\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t)\right\} = \mathscr{F}\left\{\sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t)\right\}.$$

Using the linearity property of the Fourier transform, we can rewrite this as

$$\sum_{k=0}^{N} b_k \mathscr{F}\left\{\frac{d^k}{dt^k} y(t)\right\} = \sum_{k=0}^{M} a_k \mathscr{F}\left\{\frac{d^k}{dt^k} x(t)\right\}.$$

Using the time-differentiation property of the Fourier transform, we can re-express this as

$$\sum_{k=0}^{N} b_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} a_k (j\omega)^k X(\omega).$$

Then, factoring we have

$$Y(\omega) \sum_{k=0}^{N} b_k (j\omega)^k = X(\omega) \sum_{k=0}^{M} a_k (j\omega)^k.$$

Rearranging this equation, we find the frequency response $H(\omega)$ of the system to be

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} a_k (j\omega)^k}{\sum_{k=0}^{N} b_k (j\omega)^k} = \frac{\sum_{k=0}^{M} a_k j^k \omega^k}{\sum_{k=0}^{N} b_k j^k \omega^k}.$$

Observe that, for a system of the form considered above, the frequency response is a rational function—hence, our interest in rational functions.

Example 5.29 (Resistors, inductors, and capacitors)**.** The basic building blocks of many electrical networks are resistors, inductors, and capacitors. The resistor, shown in Figure 5.15(a), is governed by the relationship

$$v(t) = Ri(t) \quad \stackrel{\mathscr{F}}{\longleftrightarrow} \quad V(\omega) = RI(\omega)$$

where R, $v(t)$ and $i(t)$ denote the resistance of, voltage across, and current through the resistor, respectively. The inductor, shown in Figure 5.15(b), is governed by the relationship

$$v(t) = L\tfrac{d}{dt} i(t) \quad \stackrel{\mathscr{F}}{\longleftrightarrow} \quad V(\omega) = j\omega L I(\omega)$$

or equivalently

$$i(t) = \tfrac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau \quad \stackrel{\mathscr{F}}{\longleftrightarrow} \quad I(\omega) = \tfrac{1}{j\omega L} V(\omega)$$

where L, $v(t)$, and $i(t)$ denote the inductance of, voltage across, and current through the inductor, respectively. The capacitor, shown in Figure 5.15(c), is governed by the relationship

$$v(t) = \tfrac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \stackrel{\mathscr{F}}{\longleftrightarrow} \quad V(\omega) = \tfrac{1}{j\omega C} I(\omega)$$

or equivalently

$$i(t) = C\tfrac{d}{dt} v(t) \quad \stackrel{\mathscr{F}}{\longleftrightarrow} \quad I(\omega) = j\omega C V(\omega)$$

where C, $v(t)$, and $i(t)$ denote the capacitance of, voltage across, and current through the capacitor, respectively.

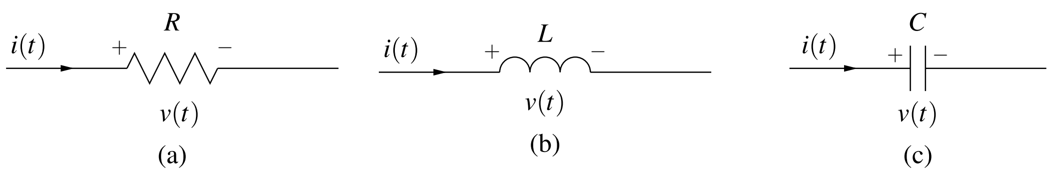


Figure 5.15: Basic electrical components. (a) Resistor, (b) inductor, and (c) capacitor.

Example 5.30 (Simple RL network)**.** Suppose that we have the RL network shown in Figure 5.16 with input $v_1(t)$ and output $v_2(t)$. This system is LTI, since it can be characterized by a linear differential equation with constant coefficients. (a) Find the frequency response $H(\omega)$ of the system. (b) Find the response $v_2(t)$ of the system to the input $v_1(t) = \operatorname{sgn} t$.

Solution. (a) From basic circuit analysis, we can write

$$v_1(t) = Ri(t) + L\tfrac{d}{dt}i(t) \quad \text{and} \tag{5.38}$$

$$v_2(t) = L\tfrac{d}{dt}i(t). \tag{5.39}$$

(Recall that the voltage $v(t)$ across an inductor L is related to the current $i(t)$ through the inductor as $v(t) = L\frac{d}{dt}i(t)$.) Taking the Fourier transform of (5.38) and (5.39) yields

$$\begin{aligned} V_1(\omega) &= RI(\omega) + j\omega LI(\omega) \\ &= (R + j\omega L)I(\omega) \quad \text{and} \end{aligned} \tag{5.40}$$

$$V_2(\omega) = j\omega LI(\omega). \tag{5.41}$$

From (5.40) and (5.41), we have

$$\begin{aligned} H(\omega) &= \frac{V_2(\omega)}{V_1(\omega)} \\ &= \frac{j\omega LI(\omega)}{(R + j\omega L)I(\omega)} \\ &= \frac{j\omega L}{R + j\omega L}. \end{aligned} \tag{5.42}$$

Thus, we have found the frequency response of the system.

(b) Now, suppose that $v_1(t) = \operatorname{sgn} t$ (as given). Taking the Fourier transform of the input $v_1(t)$ (with the aid of Table 5.2), we have

$$V_1(\omega) = \frac{2}{j\omega}. \tag{5.43}$$

From the definition of the system, we know

$$V_2(\omega) = H(\omega)V_1(\omega). \tag{5.44}$$

Substituting (5.43) and (5.42) into (5.44), we obtain

$$\begin{aligned} V_2(\omega) &= \left(\frac{j\omega L}{R + j\omega L}\right)\left(\frac{2}{j\omega}\right) \\ &= \frac{2L}{R + j\omega L}. \end{aligned}$$

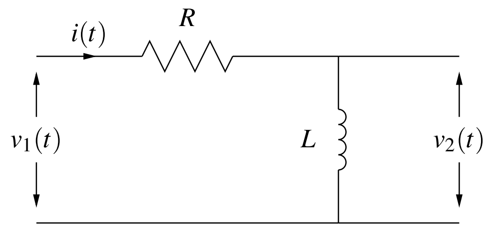


Figure 5.16: Simple RL network.

Taking the inverse Fourier transform of both sides of this equation, we obtain

$$\begin{aligned} v_2(t) &= \mathscr{F}^{-1}\left\{\frac{2L}{R+j\omega L}\right\} \\ &= \mathscr{F}^{-1}\left\{\frac{2}{R/L+j\omega}\right\} \\ &= 2\mathscr{F}^{-1}\left\{\frac{1}{R/L+j\omega}\right\}. \end{aligned}$$

Using Table 5.2, we can simplify to obtain

$$v_2(t) = 2e^{-(R/L)t}u(t).$$

Thus, we have found the response $v_2(t)$ to the input $v_1(t) = \operatorname{sgn} t$. □

5.12 Energy Spectral Density

Suppose that we have a signal $x(t)$ with finite energy E and Fourier transform $X(\omega)$. By definition, the energy contained in $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt.$$

We can use Parseval's relation (5.14) to express E in terms of $X(\omega)$ as

$$\begin{aligned} E &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega \\ &= \int_{-\infty}^{\infty} \tfrac{1}{2\pi}\, |X(\omega)|^2\, d\omega. \end{aligned}$$

Thus, the energy E is given by

$$E = \int_{-\infty}^{\infty} E_x(\omega) d\omega$$

where

$$E_x(\omega) = \tfrac{1}{2\pi}\, |X(\omega)|^2\, .$$

We refer to $E_x(\omega)$ as the **energy spectral density** of the signal $x(t)$. The quantity $E_x(\omega)$ indicates how the energy in the signal $x(t)$ is distributed as a function of frequency (in units of energy per rad/unit-time). For example, the energy contributed by frequency components in range $\omega_1 \le \omega \le \omega_2$ is simply given by

$$\int_{\omega_1}^{\omega_2} E_x(\omega) d\omega.$$

Example 5.31. Compute the energy spectral density $E_x(\omega)$ of the signal $x(t) = \operatorname{sinc} t/2$. Determine the amount of energy contained in the frequency components in the range $|\omega| \le \frac{1}{4}$. Also, determine the total amount of energy in the signal.

Solution. First, we compute the Fourier transform $X(\omega)$ of $x(t)$. We obtain $X(\omega) = 2\pi \operatorname{rect} \omega$. Next, we find the energy spectral density function $E_x(\omega)$ as follows:

$$\begin{aligned} E_x(\omega) &= \tfrac{1}{2\pi} |X(\omega)|^2 \\ &= \tfrac{1}{2\pi} |2\pi \operatorname{rect} \omega|^2 \\ &= 2\pi \operatorname{rect}^2 \omega \\ &= 2\pi \operatorname{rect} \omega. \end{aligned}$$

Let E_1 denote the energy contained in the signal for frequencies $|\omega| \le \frac{1}{4}$. Then, we have

$$\begin{aligned} E_1 &= \int_{-1/4}^{1/4} E_x(\omega) d\omega \\ &= \int_{-1/4}^{1/4} 2\pi \operatorname{rect} \omega d\omega \\ &= \int_{-1/4}^{1/4} 2\pi d\omega \\ &= \pi. \end{aligned}$$

Let E denote the total amount of energy in the signal. We can compute E as follows:

$$\begin{aligned} E &= \int_{-\infty}^{\infty} E_x(\omega) d\omega \\ &= \int_{-\infty}^{\infty} 2\pi \operatorname{rect} \omega d\omega \\ &= \int_{-1/2}^{1/2} 2\pi d\omega \\ &= 2\pi. \end{aligned}$$

□

5.13 Power Spectral Density

Suppose that we have a signal $x(t)$ with finite power P and Fourier transform $X(\omega)$. Let us define $x_T(t)$ as the following truncated (or "windowed") version of $x(t)$:

$$x_T(t) = x(t) \operatorname{rect}(t/T).$$

Also, let $X_T(\omega)$ denote the Fourier transform of $x_T(t)$. From this definition of $x_T(t)$, we have that

$$\lim_{T \to \infty} x_T(t) = x(t).$$

The average power of the signal $x_T(t)$ is given by

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} |x_T(t)|^2 dt.$$

Using Parseval's relation (5.14), we can rewrite this expression for P in terms of $X(\omega)$ to obtain

$$\begin{aligned}P &= \lim_{T\to\infty} \frac{1}{T}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}|X_T(\omega)|^2\,d\omega\right]\\ &= \lim_{T\to\infty}\left[\frac{1}{2\pi T}\int_{-\infty}^{\infty}|X_T(\omega)|^2\,d\omega\right].\end{aligned}$$

Interchanging the order of the limit operation and integration, we obtain

$$P = \int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{2\pi T}\,|X_T(\omega)|^2\,d\omega].$$

Thus, we have that

$$P = \int_{-\infty}^{\infty} S_x(\omega)d\omega$$

where

$$S_x(\omega) = \lim_{T\to\infty}\frac{1}{2\pi T}\,|X_T(\omega)|^2.$$

We refer to $S_x(\omega)$ as the **power spectral density** of the signal $x(t)$. This quantity indicates how the power in the signal $x(t)$ is distributed as a function of frequency (in units of power per rad/unit-time). We can, therefore, determine the amount of power contained in frequency components over the range $\omega_1 \leq \omega \leq \omega_2$ as

$$\int_{\omega_1}^{\omega_2} S_x(\omega)d\omega.$$

5.14 Filtering

In some applications, we want to change the relative amplitude of the frequency components of a signal or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a signal is referred to as **filtering**. Various types of filters exist. One type is frequency-selective filters. Frequency selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An ideal lowpass filter eliminates all frequency components with a frequency greater than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases}1 & \text{for } |\omega| \leq \omega_c\\ 0 & \text{otherwise,}\end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 5.17(a).

The ideal highpass filter eliminates all frequency components with a frequency less than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases}1 & \text{for } |\omega| \geq \omega_c\\ 0 & \text{otherwise,}\end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 5.17(b).

An ideal bandpass filter eliminates all frequency components that do not lie in its passband, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases}1 & \text{for } \omega_{c1} \leq |\omega| \leq \omega_{c2}\\ 0 & \text{otherwise,}\end{cases}$$

where the limits of the passband are ω_{c1} and ω_{c2}. A plot of this frequency response is given in Figure 5.17(c).

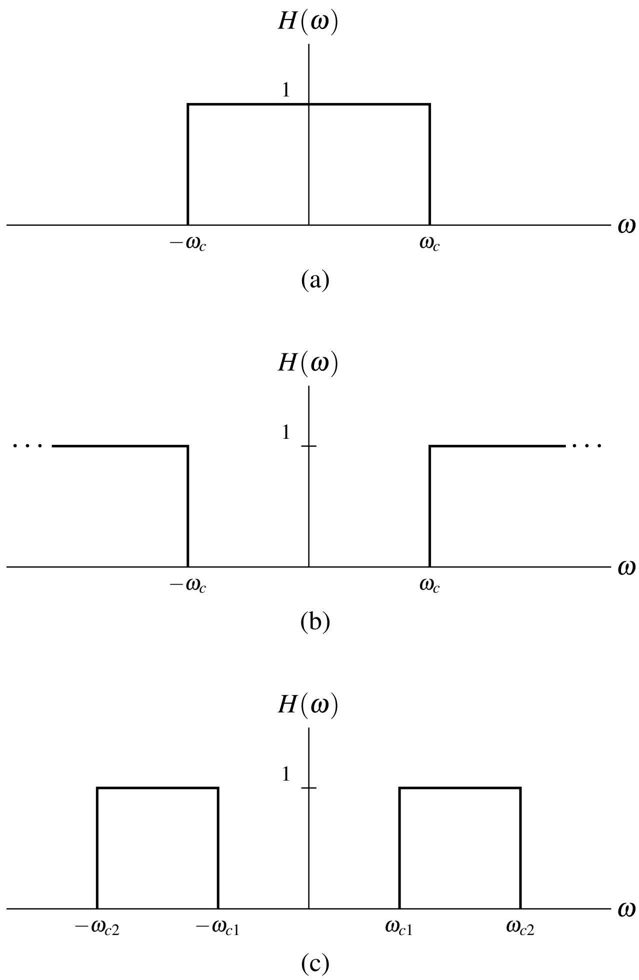


Figure 5.17: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

Example 5.32 (Ideal filters)**.** For each of the following impulse responses, find and plot the frequency response of the corresponding system:

$$\begin{aligned} h_{\text{LP}}(t) &= \tfrac{\omega_c}{\pi}\operatorname{sinc}\omega_c t, \\ h_{\text{HP}}(t) &= \delta(t) - \tfrac{\omega_c}{\pi}\operatorname{sinc}\omega_c t, \quad \text{and} \\ h_{\text{BP}}(t) &= \tfrac{2\omega_b}{\pi}[\operatorname{sinc}\omega_b t]\cos\omega_a t, \end{aligned}$$

where ω_c, ω_a, and ω_b are positive real constants. In each case, identify the type of frequency-selective filter to which the system corresponds.

Solution. In what follows, let us denote the input and output of the system as $x(t)$ and $y(t)$, respectively. Also, let $X(\omega)$ and $Y(\omega)$ denote the Fourier transforms of $x(t)$ and $y(t)$, respectively.

First, let us consider the system with impulse response $h_{\text{LP}}(t)$. The frequency response $H_{\text{LP}}(\omega)$ of the system is simply the Fourier transform of the impulse response $h_{\text{LP}}(t)$. Thus, we have

$$\begin{aligned} H_{\text{LP}}(\omega) &= \mathscr{F}\{h_{\text{LP}}(t)\} \\ &= \mathscr{F}\{\tfrac{\omega_c}{\pi}\operatorname{sinc}\omega_c t\} \\ &= \tfrac{\omega_c}{\pi}\mathscr{F}\{\operatorname{sinc}\omega_c t\} \\ &= \tfrac{\omega_c}{\pi}\left[\tfrac{\pi}{\omega_c}\operatorname{rect}\tfrac{\omega}{2\omega_c}\right] \\ &= \operatorname{rect}\tfrac{\omega}{2\omega_c} \\ &= \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The frequency response $H_{\text{LP}}(\omega)$ is plotted in Figure 5.18(a). Since $Y(\omega) = H_{\text{LP}}(\omega)X(\omega)$ and $H_{\text{LP}}(\omega) = 0$ for $|\omega| > \omega_c$, $Y(\omega)$ will contain only those frequency components in $X(\omega)$ that lie in the frequency range $|\omega| \leq \omega_c$. In other words, only the lower frequency components from $X(\omega)$ are kept. Thus, the system represents a lowpass filter.

Second, let us consider the system with impulse response $h_{\text{HP}}(t)$. The frequency response $H_{\text{HP}}(\omega)$ of the system is simply the Fourier transform of the impulse response $h_{\text{HP}}(t)$. Thus, we have

$$\begin{aligned} H_{\text{HP}}(\omega) &= \mathscr{F}\{h_{\text{HP}}(t)\} \\ &= \mathscr{F}\{\delta(t) - \tfrac{\omega_c}{\pi}\operatorname{sinc}\omega_c t\} \\ &= \mathscr{F}\{\delta(t)\} - \tfrac{\omega_c}{\pi}\mathscr{F}\{\operatorname{sinc}\omega_c t\} \\ &= 1 - \tfrac{\omega_c}{\pi}[\tfrac{\pi}{\omega_c}\operatorname{rect}\tfrac{\omega}{2\omega_c}] \\ &= 1 - \operatorname{rect}\tfrac{\omega}{2\omega_c} \\ &= \begin{cases} 1 & \text{for } |\omega| \geq \omega_c \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The frequency response $H_{\text{HP}}(\omega)$ is plotted in Figure 5.18(b). Since $Y(\omega) = H_{\text{HP}}(\omega)X(\omega)$ and $H_{\text{HP}}(\omega) = 0$ for $|\omega| < \omega_c$, $Y(\omega)$ will contain only those frequency components in $X(\omega)$ that lie in the frequency range $|\omega| \geq \omega_c$. In other words, only the higher frequency components from $X(\omega)$ are kept. Thus, the system represents a highpass filter.

Third, let us consider the system with impulse response $h_{\text{BP}}(t)$. The frequency response $H_{\text{BP}}(\omega)$ of the system is simply the Fourier transform of the impulse response $h_{\mathrm{BP}}(t)$. Thus, we have

$$\begin{aligned}
H_{\mathrm{BP}}(\omega) &= \mathscr{F}\{h_{\mathrm{BP}}(t)\} \\
&= \mathscr{F}\{\tfrac{2\omega_b}{\pi}[\operatorname{sinc}\omega_b t]\cos\omega_a t\} \\
&= \tfrac{\omega_b}{\pi}\mathscr{F}\{[\operatorname{sinc}\omega_b t](2\cos\omega_a t)\} \\
&= \tfrac{\omega_b}{\pi}\mathscr{F}\{[\operatorname{sinc}\omega_b t][e^{j\omega_a t}+e^{-j\omega_a t}]\} \\
&= \tfrac{\omega_b}{\pi}\left[\mathscr{F}\{e^{j\omega_a t}\operatorname{sinc}\omega_b t\}+\mathscr{F}\{e^{-j\omega_a t}\operatorname{sinc}\omega_b t\}\right] \\
&= \tfrac{\omega_b}{\pi}\left[\tfrac{\pi}{\omega_b}\operatorname{rect}\tfrac{\omega-\omega_a}{2\omega_b}+\tfrac{\pi}{\omega_b}\operatorname{rect}\tfrac{\omega+\omega_a}{2\omega_b}\right] \\
&= \operatorname{rect}\tfrac{\omega-\omega_a}{2\omega_b}+\operatorname{rect}\tfrac{\omega+\omega_a}{2\omega_b} \\
&= \begin{cases} 1 & \text{for } \omega_a-\omega_b \le |\omega| \le \omega_a+\omega_b \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The frequency response $H_{\mathrm{BP}}(\omega)$ is plotted in Figure 5.18(c). Since $Y(\omega) = H_{\mathrm{BP}}(\omega)X(\omega)$ and $H_{\mathrm{BP}}(\omega) = 0$ for $|\omega| < \omega_a-\omega_b$ or $|\omega| > \omega_a+\omega_b$, $Y(\omega)$ will contain only those frequency components in $X(\omega)$ that lie in the frequency range $\omega_a-\omega_b \le |\omega| \le \omega_a+\omega_b$. In other words, only the middle frequency components of $X(\omega)$ are kept. Thus, the system represents a bandpass filter. □

Example 5.33 (Lowpass filtering)**.** Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$, where

$$h(t) = 300\operatorname{sinc}300\pi t.$$

Using frequency-domain methods, find the response $y(t)$ of the system to the input $x(t) = x_1(t)$, where

$$x_1(t) = \tfrac{1}{2}+\tfrac{3}{4}\cos 200\pi t+\tfrac{1}{2}\cos 400\pi t-\tfrac{1}{4}\cos 600\pi t.$$

Solution. To begin, we must find the frequency spectrum $X_1(\omega)$ of the signal $x_1(t)$. Computing $X_1(\omega)$, we have

$$\begin{aligned}
X_1(\omega) &= \mathscr{F}\left\{\tfrac{1}{2}+\tfrac{3}{4}\cos 200\pi t+\tfrac{1}{2}\cos 400\pi t-\tfrac{1}{4}\cos 600\pi t\right\} \\
&= \tfrac{1}{2}\mathscr{F}\{1\}+\tfrac{3}{4}\mathscr{F}\{\cos 200\pi t\}+\tfrac{1}{2}\mathscr{F}\{\cos 400\pi t\}-\tfrac{1}{4}\mathscr{F}\{\cos 600\pi t\} \\
&= \tfrac{1}{2}[2\pi\delta(\omega)]+\tfrac{3\pi}{4}[\delta(\omega+200\pi)+\delta(\omega-200\pi)]+\tfrac{\pi}{2}[\delta(\omega+400\pi)+\delta(\omega-400\pi)] \\
&\qquad -\tfrac{\pi}{4}[\delta(\omega+600\pi)+\delta(\omega-600\pi)] \\
&= -\tfrac{\pi}{4}\delta(\omega+600\pi)+\tfrac{\pi}{2}\delta(\omega+400\pi)+\tfrac{3\pi}{4}\delta(\omega+200\pi)+\pi\delta(\omega)+\tfrac{3\pi}{4}\delta(\omega-200\pi) \\
&\qquad +\tfrac{\pi}{2}\delta(\omega-400\pi)-\tfrac{\pi}{4}\delta(\omega-600\pi).
\end{aligned}$$

A plot of the frequency spectrum $X_1(\omega)$ is shown in Figure 5.19(a). Using the results of Example 5.32, we can determine the frequency response $H(\omega)$ of the system to be

$$\begin{aligned}
H(\omega) &= \mathscr{F}\{300\operatorname{sinc}300\pi t\} \\
&= \operatorname{rect}\tfrac{\omega}{2(300\pi)} \\
&= \begin{cases} 1 & \text{for } |\omega| \le 300\pi \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The frequency response $H(\omega)$ is shown in Figure 5.19(b). The frequency spectrum $Y(\omega)$ of the output can be computed as

$$\begin{aligned}
Y(\omega) &= H(\omega)X(\omega) \\
&= \tfrac{3\pi}{4}\delta(\omega+200\pi)+\pi\delta(\omega)+\tfrac{3\pi}{4}\delta(\omega-200\pi).
\end{aligned}$$

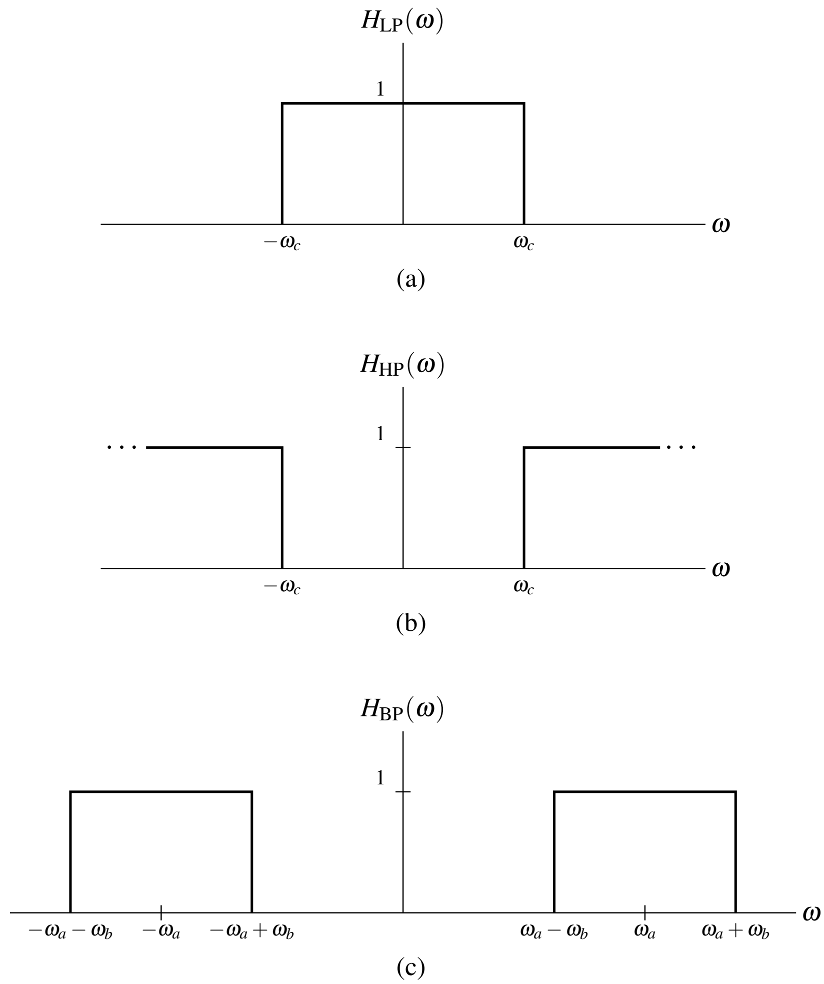


Figure 5.18: Frequency responses of systems from example.

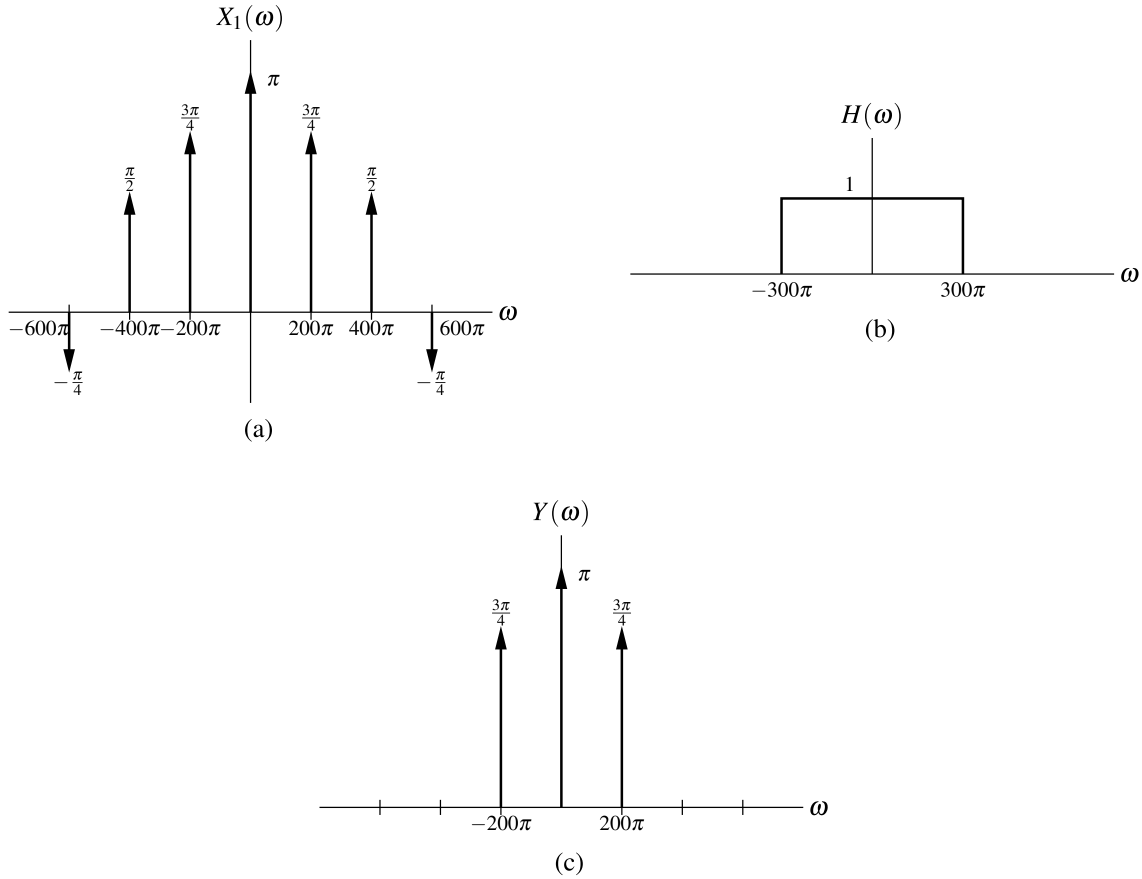


Figure 5.19: Frequency spectra for lowpass filtering example.

The frequency spectrum $Y(\omega)$ is shown in Figure 5.19(c). Taking the inverse Fourier transform of $Y(\omega)$ yields

$$\begin{aligned}
y(t) &= \mathscr{F}^{-1}\left\{\tfrac{3\pi}{4}\delta(\omega+200\pi)+\pi\delta(\omega)+\tfrac{3\pi}{4}\delta(\omega-200\pi)\right\}\\
&= \pi\mathscr{F}^{-1}\{\delta(\omega)\}+\tfrac{3}{4}\mathscr{F}^{-1}\{\pi[\delta(\omega+200\pi)+\delta(\omega-200\pi)]\}\\
&= \pi(\tfrac{1}{2\pi})+\tfrac{3}{4}\cos 200\pi t\\
&= \tfrac{1}{2}+\tfrac{3}{4}\cos 200\pi t.
\end{aligned}$$

□

Example 5.34 (Bandpass filtering)**.** Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$, where

$$h(t) = (200\operatorname{sinc}100\pi t)\cos 400\pi t.$$

Using frequency-domain methods, find the response $y(t)$ of the system to the input $x(t) = x_1(t)$ where $x_1(t)$ is as defined in Example 5.33.

Solution. From Example 5.33, we already know the frequency spectrum $X_1(\omega)$. In particular, we previously found that

$$\begin{aligned}
X_1(\omega) = &-\tfrac{\pi}{4}\delta(\omega+600\pi)+\tfrac{\pi}{2}\delta(\omega+400\pi)+\tfrac{3\pi}{4}\delta(\omega+200\pi)+\pi\delta(\omega)+\tfrac{3\pi}{4}\delta(\omega-200\pi)\\
&+\tfrac{\pi}{2}\delta(\omega-400\pi)-\tfrac{\pi}{4}\delta(\omega-600\pi).
\end{aligned}$$

The frequency spectrum $X_1(\omega)$ is shown in Figure 5.20(a). Now, we compute the frequency response $H(\omega)$ of the system. Using the results of Example 5.32, we can determine $H(\omega)$ to be

$$\begin{aligned}
H(\omega) &= \mathscr{F}\{(200\operatorname{sinc}100\pi t)\cos 400\pi t\} \\
&= \operatorname{rect}\tfrac{\omega-400\pi}{2(100\pi)} + \operatorname{rect}\tfrac{\omega+400\pi}{2(100\pi)} \\
&= \begin{cases} 1 & \text{for } 300\pi \leq |\omega| \leq 500\pi \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The frequency response $H(\omega)$ is shown in Figure 5.20(b). The frequency spectrum $Y(\omega)$ of the output is given by

$$\begin{aligned}
Y(\omega) &= H(\omega)X(\omega) \\
&= \tfrac{\pi}{2}\delta(\omega+400\pi) + \tfrac{\pi}{2}\delta(\omega-400\pi).
\end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned}
y(t) &= \mathscr{F}^{-1}\left\{\tfrac{\pi}{2}\delta(\omega+400\pi) + \tfrac{\pi}{2}\delta(\omega-400\pi)\right\} \\
&= \tfrac{1}{2}\mathscr{F}^{-1}\left\{\pi[\delta(\omega+400\pi) + \delta(\omega-400\pi)]\right\} \\
&= \tfrac{1}{2}\cos 400\pi t.
\end{aligned}$$

□

5.15 Sampling and Interpolation

Often, we encounter situations in which we would like to process a continuous-time signal in the discrete-time domain or vice versa. For example, we might have a continuous-time audio signal that we would like to process using a digital computer (which is a discrete-time system), or we might have a discrete-time audio signal that we wish to play on a loudspeaker (which is a continuous-time system). Clearly, some means is needed to link the continuous- and discrete-time domains. This connection is established through processes known as sampling and interpolation. In what follows, we will formally introduce these processes and study them in some detail.

Sampling allows us to create a discrete-time signal from a continuous-time signal. Although sampling can be performed in many different ways, the most commonly used scheme is **periodic sampling**. With this scheme, a sequence $y[n]$ of samples is obtained from a continuous-time signal $x(t)$ according to the relation

$$y[n] = x(nT) \quad \text{for all integer } n, \tag{5.45}$$

where T is a positive real constant. As a matter of terminology, we refer to T as the **sampling period**, and $\omega_s = 2\pi/T$ as the (angular) **sampling frequency**. A system such as that described by (5.45) is known as an **ideal continuous-to-discrete-time (C/D) converter**, and is shown diagrammatically in Figure 5.21. An example of periodic sampling is shown in Figure 5.22. In Figure 5.22(a), we have the original continuous-time signal $x(t)$. This signal is then sampled with sampling period $T = 10$, yielding the sequence $y[n]$ in Figure 5.22(b).

Interpolation allows us to construct a continuous-time signal from a discrete-time signal. In effect, this process is one of assigning values to a signal between its sample points. Although there are many different ways in which to perform interpolation, we will focus our attention in subsequent sections on one particular scheme known as bandlimited interpolation. Interpolation produces a continuous-time signal $\hat{x}(t)$ from a sequence $y[n]$ according to the relation

$$\hat{x}(t) = f(y[n]),$$

where f is some function of the sample values $y[n]$. The precise form of the function f depends on the particular interpolation scheme employed. The interpolation process is performed by a system known as an **ideal discrete-to-continuous-time (D/C) converter**, as shown in Figure 5.23.

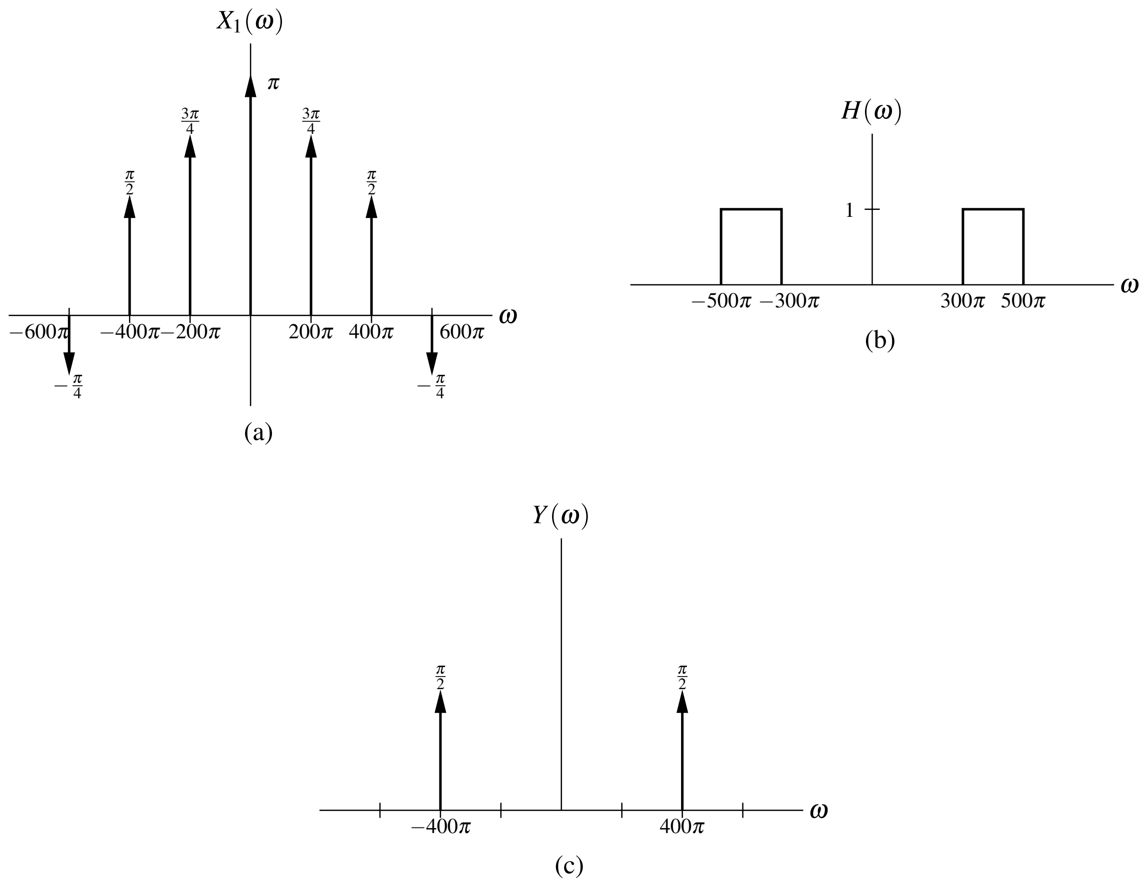


Figure 5.20: Frequency spectra for bandpass filtering example.

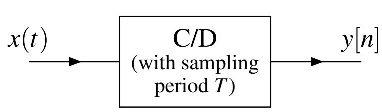


Figure 5.21: Ideal C/D converter.

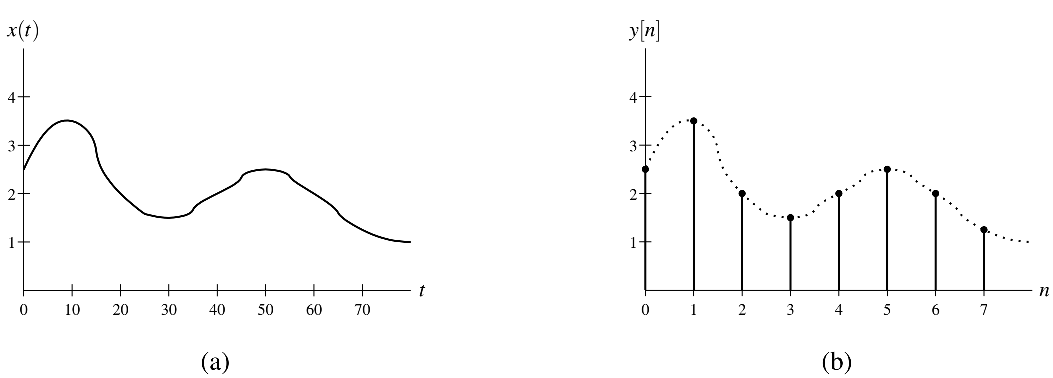


Figure 5.22: Periodic sampling of signal.

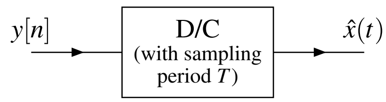


Figure 5.23: Ideal D/C converter.

In the absence of any constraints, a continuous-time signal cannot usually be uniquely determined from a sequence of its equally-spaced samples. In other words, the sampling process is not generally invertible. Consider, for example, the continuous-time signals $x_1(t)$ and $x_2(t)$ given by

$$x_1(t) = 0 \quad \text{and}$$
$$x_2(t) = \sin(2\pi t).$$

If we sample each of these signals with the sampling period $T = 1$, we obtain the respective sequences

$$y_1[n] = x_1(nT) = x_1(n) = 0 \quad \text{and}$$
$$y_2[n] = x_2(nT) = \sin(2\pi n) = 0.$$

Thus, $y_1[n] = y_2[n]$ for all n, although $x_1(t) \neq x_2(t)$ for all noninteger t. This example trivially shows that if no constraints are placed upon a continuous-time signal, then the signal cannot be uniquely determined from its samples.

Fortunately, under certain circumstances, a continuous-time signal can be recovered exactly from its samples. In particular, in the case that the signal being sampled is bandlimited, we can show that a sequence of its equally-spaced samples uniquely determines the signal if the sampling period is sufficiently small. This result, known as the sampling theorem, is of paramount importance in the study of signals and systems.

5.15.1 Sampling

In order to gain some insight into sampling, we need a way in which to mathematically model this process. To this end, we employ the simple model for the ideal C/D converter shown in Figure 5.24. In short, we may view the process of sampling as impulse train modulation followed by conversion of an impulse train to a sequence of sample values. More specifically, to sample a signal $x(t)$ with sampling period T, we first multiply the signal $x(t)$ by the periodic impulse train $p(t)$ to obtain

$$s(t) = x(t)p(t) \tag{5.46}$$

where

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

Then, we take the weights of successive impulses in $s(t)$ to form a sequence $y[n]$ of samples. The sampling frequency is given by $\omega_s = 2\pi/T$. As a matter of terminology, $p(t)$ is referred to as a sampling function. From the diagram, we can see that the signals $s(t)$ and $y[n]$, although very closely related, have some key differences. The impulse train $s(t)$ is a *continuous-time* signal that is zero everywhere except at integer multiples of T (i.e., at sample points), while $y[n]$ is a *discrete-time* signal, defined only on integers with its values corresponding to the weights of successive impulses in $s(t)$. The various signals involved in sampling are illustrated in Figure 5.25.

In passing, we note that the above model of sampling is only a mathematical convenience. That is, the model provides us with a relatively simple way in which to study the mathematical behavior of sampling. The above model, however, is not directly useful as a means for actually realizing sampling in a real world system. Obviously, the impulse train employed in the above model poses some insurmountable problems as far as implementation is concerned.

Now, let us consider the above model of sampling in more detail. In particular, we would like to find the relationship between the frequency spectra of the original signal $x(t)$ and its impulse-train sampled version $s(t)$. Since $p(t)$ is T-periodic, it can be represented in terms of a Fourier series as

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}. \tag{5.47}$$

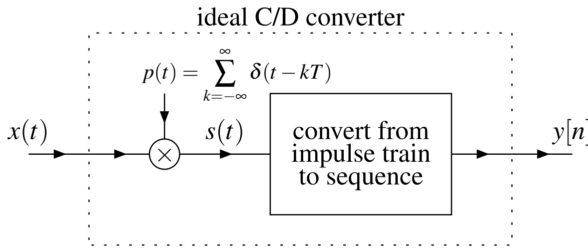


Figure 5.24: Model of ideal C/D converter.

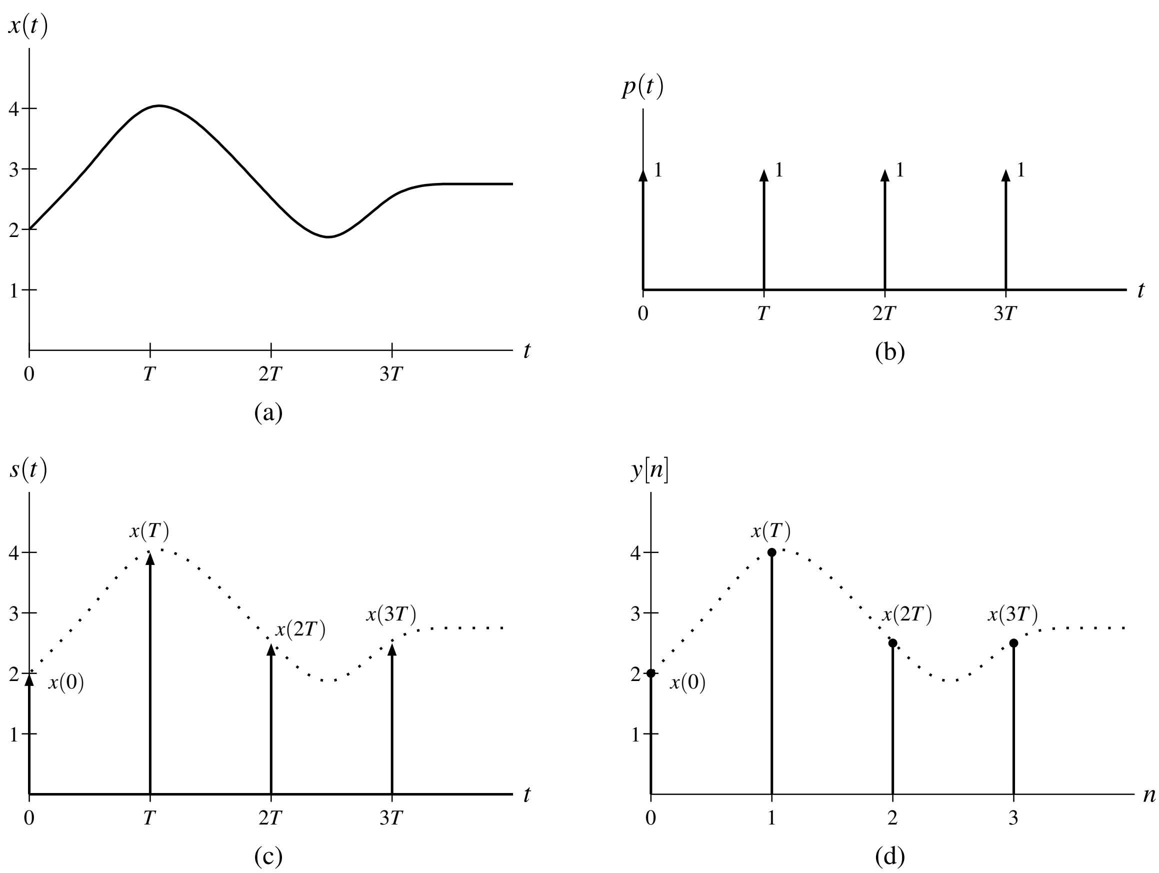


Figure 5.25: The various signals involved in the sampling process. (a) The original continuous-time signal $x(t)$. (b) The sampling function $p(t)$. (c) The impulse-modulated signal $s(t)$. (d) The discrete-time signal $y[n]$.

Using the Fourier series analysis equation, we calculate the coefficients c_k to be

$$\begin{aligned} c_k &= \tfrac{1}{T}\int_{-T/2}^{T/2} p(t)e^{-jk\omega_s t}dt \\ &= \tfrac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_s t}dt \\ &= \tfrac{1}{T} \\ &= \tfrac{\omega_s}{2\pi}. \end{aligned} \tag{5.48}$$

Substituting (5.47) and (5.48) into (5.46), we obtain

$$\begin{aligned} s(t) &= x(t)\sum_{k=-\infty}^{\infty} \tfrac{\omega_s}{2\pi} e^{jk\omega_s t} \\ &= \tfrac{\omega_s}{2\pi}\sum_{k=-\infty}^{\infty} x(t)e^{jk\omega_s t}. \end{aligned}$$

Taking the Fourier transform of $s(t)$ yields

$$S(\omega) = \tfrac{\omega_s}{2\pi}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s). \tag{5.49}$$

Thus, the spectrum of the impulse-train sampled signal $s(t)$ is a scaled sum of an infinite number of shifted copies of the spectrum of the original signal $x(t)$.

Now, we consider a simple example to further illustrate the behavior of the sampling process in the frequency domain. Suppose that we have a signal $x(t)$ with the Fourier transform $X(\omega)$ where $|X(\omega)| = 0$ for $|\omega| > \omega_m$ (i.e., $x(t)$ is bandlimited). To simplify the visualization process, we will assume $X(\omega)$ has the particular form shown in Figure 5.26(a). In what follows, however, we only actually rely on the bandlimited nature of $x(t)$ and not the particular shape of $X(\omega)$. So, the results that we derive in what follows generally apply to any bandlimited signal. Let $S(\omega)$ denote the Fourier transform of $s(t)$. From (5.49), we know that $S(\omega)$ is formed by the superposition of an infinite number of shifted copies of $X(\omega)$. Upon more careful consideration, we can see that two distinct situations can arise. That is, the shifted copies of $X(\omega)$ used to form $S(\omega)$ can either: 1) overlap or, 2) not overlap. These two cases are illustrated in Figures 5.26(b) and 5.26(c), respectively. From these graphs, we can see that the shifted copies of $X(\omega)$ will not overlap if

$$\omega_m < \omega_s - \omega_m \quad \text{and} \quad -\omega_m > -\omega_s + \omega_m$$

or equivalently

$$\omega_s > 2\omega_m.$$

Consider the case in which the copies of the original spectrum $X(\omega)$ in $S(\omega)$ do not overlap, as depicted in Figure 5.26(b). In this situation, the spectrum $X(\omega)$ of the original signal is clearly discernable in the spectrum $S(\omega)$. In fact, one can see that the original spectrum $X(\omega)$ can be obtained directly from $S(\omega)$ through a lowpass filtering operation. Thus, the original signal $x(t)$ can be exactly recovered from $s(t)$.

Now, consider the case in which copies of the original spectrum $X(\omega)$ in $S(\omega)$ do overlap. In this situation, multiple frequencies in the spectrum $X(\omega)$ of the original signal are mapped to the same frequency in $S(\omega)$. This phenomenon is referred to as **aliasing**. Clearly, aliasing leads to individual periods of $S(\omega)$ having a different shape than the original spectrum $X(\omega)$. When aliasing occurs, the shape of the original spectrum $X(\omega)$ is no longer discernable from $S(\omega)$. Consequently, we are unable to recover the original signal $x(t)$ from $s(t)$ in this case.

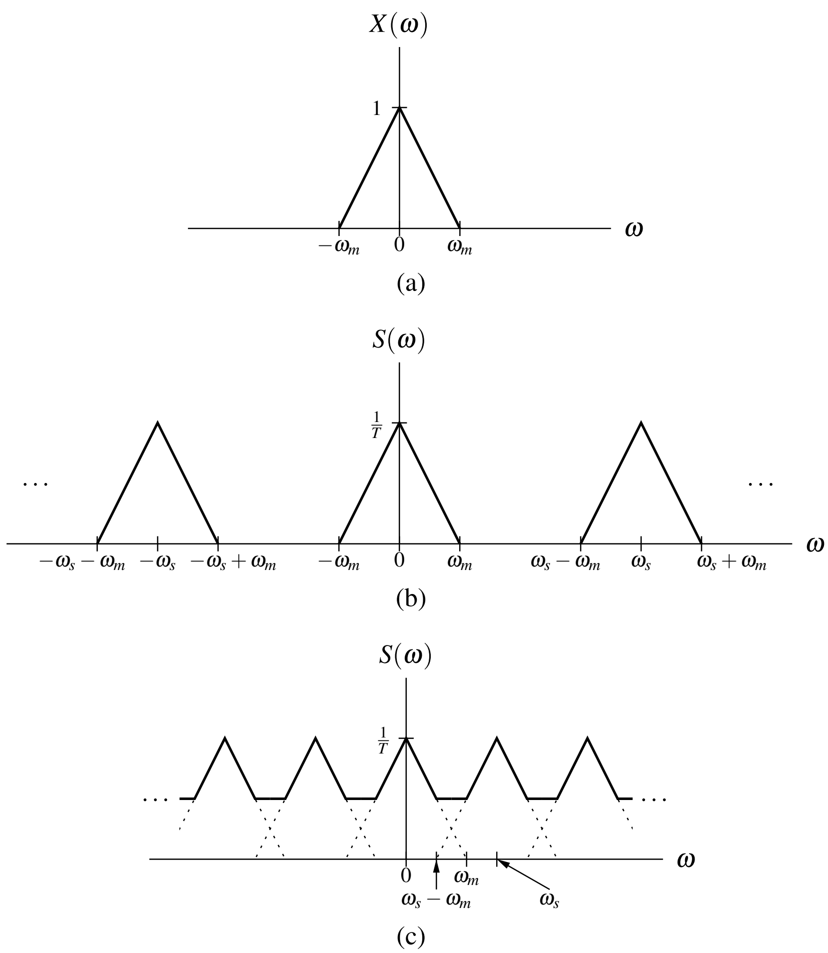


Figure 5.26: Effect of impulse-train sampling on frequency spectrum. (a) Spectrum of original signal $x(t)$. (b) Spectrum of $s(t)$ in the absence of aliasing. (c) Spectrum of $s(t)$ in the presence of aliasing.

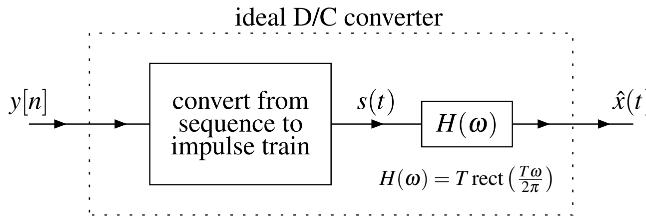


Figure 5.27: Model of ideal D/C converter.

5.15.2 Interpolation and Reconstruction of a Signal From Its Samples

Interpolation allows us to construct a continuous-time signal from a discrete-time one. This process is essentially responsible for determining the value of a continuous-time signal between sample points. Except in very special circumstances, it is not generally possible to exactly reproduce a continuous-time signal from its samples. Although many interpolation schemes exist, we shall focus our attention shortly on one particular scheme. The interpolation process can be modeled with the simple ideal D/C converter system, shown in Figure 5.27.

Recall the ideal C/D converter of Figure 5.24. Since the process of converting an impulse train to a sequence is invertible, we can reconstruct the original signal $x(t)$ from its sampled version $y[n]$ if we can somehow recover $x(t)$ from $s(t)$. Let us suppose now that $x(t)$ is bandlimited. As we saw in the previous section, we can recover $x(t)$ from $s(t)$ provided that $x(t)$ is bandlimited and sampled at a sufficiently high rate so as to avoid aliasing. In the case that aliasing does not occur, we can reconstruct the original continuous-time signal $x(t)$ from $y[n]$ using the ideal D/C converter shown in Figure 5.27. In what follows, we will derive a formula for computing the original continuous-time signal $\hat{x}(t)$ from its samples $y[n]$. Consider the model of the D/C converter. We have a lowpass filter with frequency response

$$H(\omega) = T \operatorname{rect}\left(\frac{T\omega}{2\pi}\right) = \frac{2\pi}{\omega_s} \operatorname{rect}\left(\frac{\omega}{\omega_s}\right) = \begin{cases} T & |\omega| < \frac{\omega_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

and impulse response

$$h(t) = \operatorname{sinc}\left(\frac{\pi t}{T}\right) = \operatorname{sinc}\left(\frac{\omega_s t}{2}\right).$$

First, we convert the sequence $y[n]$ to the impulse train $s(t)$ to obtain

$$s(t) = \sum_{n=-\infty}^{\infty} y[n]\delta(t - nT).$$

Then, we filter the resulting signal $s(t)$ with the lowpass filter having impulse response $h(t)$, yielding

$$\begin{aligned}
\hat{x}(t) &= s(t) * h(t) \\
&= \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(t-\tau) \sum_{n=-\infty}^{\infty} y[n]\delta(\tau - nT)d\tau \\
&= \sum_{n=-\infty}^{\infty} y[n] \int_{-\infty}^{\infty} h(t-\tau)\delta(\tau - nT)d\tau \\
&= \sum_{n=-\infty}^{\infty} y[n]h(t-nT) \\
&= \sum_{n=-\infty}^{\infty} y[n] \operatorname{sinc}\left(\tfrac{\pi}{T}(t-nT)\right).
\end{aligned}$$

If $x(t)$ is bandlimited and aliasing is avoided, $\hat{x}(t) = x(t)$ and we have a formula for exactly reproducing $x(t)$ from its samples $y[n]$.

5.15.3 Sampling Theorem

In the preceding sections, we have established the important result given by the theorem below.

Theorem 5.3 (Sampling Theorem)**.** *Let $x(t)$ be a signal with Fourier transform $X(\omega)$, and suppose that $|X(\omega)| = 0$ for all ω satisfying $|\omega| > \omega_M$ (i.e., $x(t)$ is bandlimited to the interval $[-\omega_M, \omega_M]$). Then, $x(t)$ is uniquely determined by its samples $y[n] = x(nT)$ for $n = 0, \pm 1, \pm 2, \ldots$, if*

$$\omega_s > 2\omega_M, \tag{5.50}$$

where $\omega_s = 2\pi/T$. In particular, if (5.50) is satisfied, we have that

$$x(t) = \sum_{n=-\infty}^{\infty} y[n] \operatorname{sinc}(\tfrac{\pi}{T}(t - nT)),$$

or equivalently (i.e., rewritten in terms of ω_s instead of T),

$$x(t) = \sum_{n=-\infty}^{\infty} y[n] \operatorname{sinc}(\tfrac{\omega_s}{2}t - \pi n).$$

As a matter of terminology, we refer to (5.50) as the **Nyquist condition** (or Nyquist criterion). Also, we call $\omega_s/2$ the **Nyquist frequency** and $2\omega_M$ the **Nyquist rate**. It is important to note that the Nyquist condition is a strict inequality. Therefore, to ensure aliasing does not occur in the most general case, one must choose the sampling rate larger than the Nyquist rate. One can show, however, that if the frequency spectrum does not have impulses at the Nyquist frequency, it is sufficient to sample at exactly the Nyquist rate.

Example 5.35. Let $x(t)$ denote a continuous-time audio signal with Fourier transform $X(\omega)$. Suppose that $|X(\omega)| = 0$ for all $|\omega| \geq 44100\pi$. Determine the largest period T with which $x(t)$ can be sampled that will allow $x(t)$ to be exactly recovered from its samples.

Solution. The signal $x(t)$ is bandlimited to frequencies in the range $(-\omega_m, \omega_m)$, where $\omega_m = 44100\pi$. From the sampling theorem, we know that the minimum sampling rate required is given by

$$\begin{aligned} \omega_s &= 2\omega_m \\ &= 2(44100\pi) \\ &= 88200\pi. \end{aligned}$$

Thus, the largest permissible sampling period is given by

$$\begin{aligned} T &= \tfrac{2\pi}{\omega_s} \\ &= \tfrac{2\pi}{88200\pi} \\ &= \tfrac{1}{44100}. \end{aligned}$$

□

Although the sampling theorem provides an upper bound on the sampling rate that holds in the case of arbitrary bandlimited signals, in some special cases it may be possible to employ an even smaller sampling rate. This point is further illustrated by way of the example below.

Example 5.36. Suppose that we have a signal $x(t)$ with the Fourier transform $X(\omega)$ shown in Figure 5.28 (where $\omega_c \gg \omega_a$). (a) Using the sampling theorem directly, determine the largest permissible sampling period T that will allow $x(t)$ to be exactly reconstructed from its samples. (b) Explain how one can exploit the fact that $X(\omega) = 0$ for a large portion of the interval $[-\omega_c - \omega_a, \omega_c + \omega_a]$ in order to reduce the rate at which $x(t)$ must be sampled.

Solution. (a) The signal $x(t)$ is bandlimited to $(-\omega_m, \omega_m)$, where $\omega_m = \omega_c + \omega_a$. Thus, the minimum sampling rate required is given by

$$\begin{aligned}\omega_s &= 2\omega_m \\ &= 2(\omega_c + \omega_a) \\ &= 2\omega_c + 2\omega_a.\end{aligned}$$

and the maximum sampling period is calculated as

$$\begin{aligned}T &= \tfrac{2\pi}{\omega_s} \\ &= \tfrac{2\pi}{2\omega_c + 2\omega_a} \\ &= \tfrac{\pi}{\omega_c + \omega_a}.\end{aligned}$$

(b) We can modulate and lowpass filter $x(t)$ in order to compress all of its spectral information into the frequency range $[-2\omega_a, 2\omega_a]$, yielding the signal $x_1(t)$. That is, we have

$$x_1(t) = [x(t)\cos([\omega_c - \omega_a]t)] * h(t)$$

where

$$h(t) = \tfrac{4\omega_a}{\pi}\operatorname{sinc}(2\omega_a t) \quad \stackrel{\mathscr{F}}{\longleftrightarrow} \quad H(\omega) = 2\operatorname{rect}\left(\tfrac{\omega}{4\omega_a}\right).$$

This process can be inverted (by modulation and filtering) to obtain $x(t)$ from $x_1(t)$. In particular, we have that

$$x(t) = (x_1(t)\cos([\omega_c - \omega_a]t)) * h_2(t)$$

where

$$h_2(t) = \delta(t) - \tfrac{2(\omega_c - \omega_a)}{\pi}\operatorname{sinc}([\omega_c - \omega_a]t) \quad \stackrel{\mathscr{F}}{\longleftrightarrow} \quad H_2(\omega) = 2 - 2\operatorname{rect}\left(\tfrac{\omega}{4(\omega_c - \omega_a)}\right).$$

Let $X_1(\omega)$ denote the Fourier transform of $x_1(t)$. The Fourier transform $X_1(\omega)$ is as shown in Figure 5.29. Applying the sampling theorem to $x_1(t)$ we find that the minimum sampling rate is given by

$$\begin{aligned}\omega_s &= 2(2\omega_a) \\ &= 4\omega_a\end{aligned}$$

and the largest sampling period is given by

$$\begin{aligned}T &= \tfrac{2\pi}{\omega_s} \\ &= \tfrac{2\pi}{4\omega_a} \\ &= \tfrac{\pi}{2\omega_a}.\end{aligned}$$

Since $\omega_c \gg \omega_a$ (by assumption), this new sampling period is larger than the one computed in the first part of this problem. □

5.16 Amplitude Modulation

In communication systems, we often need to transmit a signal using a frequency range that is different from that of the original signal. For example, voice/audio signals typically have information in the range of 0 to 20 kHz. Often, it is not practical to transmit such a signal using its original frequency range. Two potential problems with such an approach

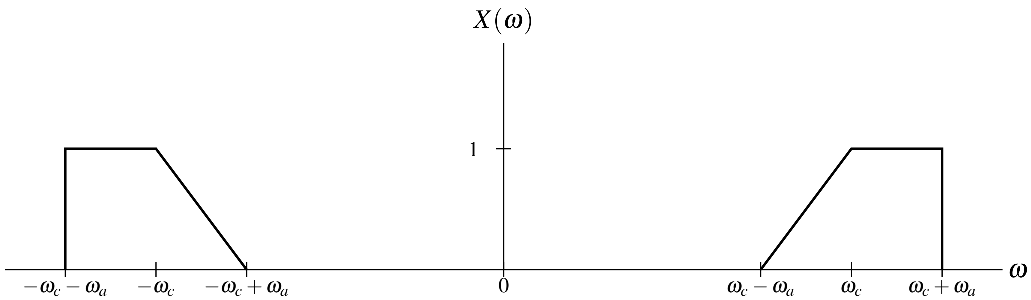


Figure 5.28: Frequency spectrum of signal $x(t)$.

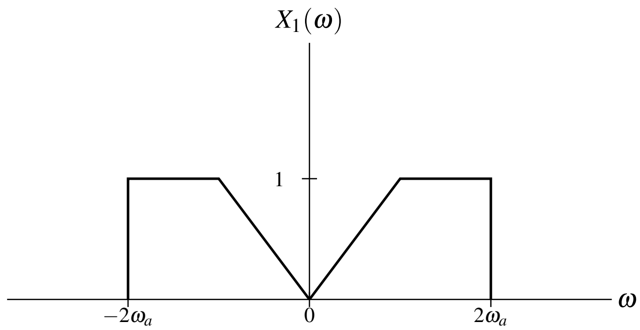


Figure 5.29: Frequency spectrum of signal $x_1(t)$.

are: 1) interference and 2) constraints on antenna length. Since many signals are broadcast over the airwaves, we need to ensure that no two transmitters use the same frequency bands in order to avoid interference. Also, in the case of transmission via electromagnetic waves (e.g., radio waves), the length of antenna required becomes impractically large for the transmission of relatively low frequency signals. For the preceding reasons, we often need to change the frequency range associated with a signal before transmission. In what follows, we consider one possible scheme for accomplishing this. This scheme is known as amplitude modulation.

Amplitude modulation (AM) is used in many communication systems. Numerous variations on amplitude modulation are possible. Here, we consider two of the simplest variations: double-side-band/suppressed-carrier (DSB/SC) and single-side-band/suppressed-carrier (SSB/SC).

5.16.1 Modulation With a Complex Sinusoid

Suppose that we have the communication system shown in Figure 5.30. First, let us consider the transmitter in Figure 5.30(a). The transmitter is a system with input $x(t)$ and output $y(t)$. Mathematically, the behavior of this system is given by

$$y(t) = x(t)c_1(t) \tag{5.51}$$

where

$$c_1(t) = e^{j\omega_c t}.$$

Let $X(\omega)$, $Y(\omega)$, and $C_1(\omega)$ denote the Fourier transforms of $x(t)$, $y(t)$, and $c_1(t)$, respectively. Taking the Fourier transform of both sides of (5.51), we obtain

$$\begin{aligned} Y(\omega) &= \mathscr{F}\{c_1(t)x(t)\} \\ &= \mathscr{F}\left\{e^{j\omega_c t}x(t)\right\} \\ &= X(\omega-\omega_c). \end{aligned} \tag{5.52}$$

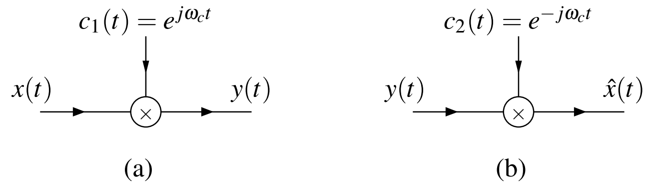


Figure 5.30: Simple communication system. (a) Transmitter and (b) receiver.

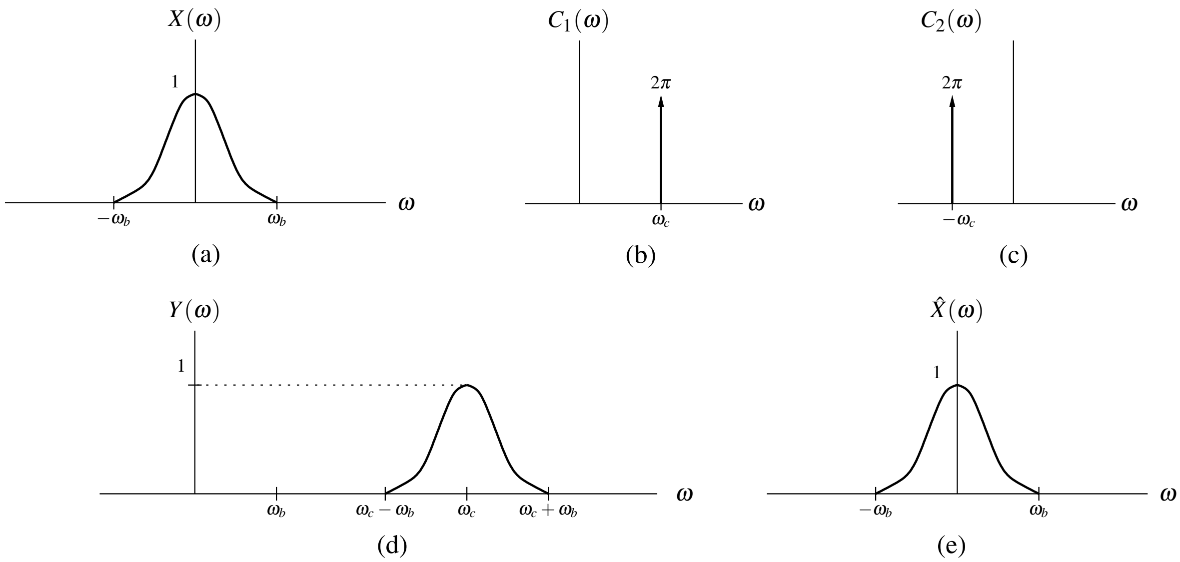


Figure 5.31: Frequency spectra for modulation with a complex sinusoid.

Thus, the frequency spectrum of the output is simply the frequency spectrum of the input shifted by ω_c. The relationship between the frequency spectra of the input and output is illustrated in Figure 5.31. Clearly, the output signal has been shifted to a different frequency range as desired. Now, we need to determine whether the receiver can recover the original signal $x(t)$ from the transmitted signal $y(t)$.

Now, let us consider the receiver shown in Figure 5.30(b). The receiver is a system with input $y(t)$ and output $\hat{x}(t)$. Mathematically, this system is given by

$$\hat{x}(t) = y(t)c_2(t) \tag{5.53}$$

where

$$c_2(t) = e^{-j\omega_c t}$$

(i.e., $c_2(t) = c_1^*(t)$). In order for the communication system to be useful, we need for the received signal $\hat{x}(t)$ to be equal to the original signal $x(t)$ from the transmitter. Let $Y(\omega)$, $\hat{X}(\omega)$, and $C_2(\omega)$ denote the Fourier transform of $y(t)$, $\hat{x}(t)$, and $c_2(t)$, respectively. Taking the Fourier transform of both sides of (5.53), we obtain

$$\begin{aligned}\hat{X}(\omega) &= \mathscr{F}\{c_2(t)y(t)\}\\ &= \mathscr{F}\{e^{-j\omega_c t}y(t)\}\\ &= Y(\omega+\omega_c).\end{aligned}$$

Substituting the expression for $Y(\omega)$ in (5.52) into this equation, we obtain

$$\begin{aligned}\hat{X}(\omega) &= X([\omega+\omega_c]-\omega_c) \\ &= X(\omega).\end{aligned}$$

Since $\hat{X}(\omega) = X(\omega)$, we have that the received signal $\hat{x}(t)$ is equal to the original signal $x(t)$ from the transmitter. Thus, the communication system has the desired behavior. The relationship between the frequency spectra of the various signals in the AM system is illustrated in Figure 5.31.

Although the above result is quite interesting mathematically, it does not have direct practical application. The difficulty here is that $c_1(t)$, $c_2(t)$, and $y(t)$ are complex signals, and we cannot realize complex signals in the physical world. This communication system is not completely without value, however, as it leads to the development of the practically useful system that we consider next.

5.16.2 DSB/SC Amplitude Modulation

Now, let us consider the communication system shown in Figure 5.32. This system is known as a double-side-band/suppressed-carrier (DSB/SC) amplitude modulation (AM) system. This system is very similar to the one in Figure 5.30. In the new system, however, we have replaced the complex sinusoid $c_1(t)$ with a real sinusoid $c(t)$. The new system also requires that the input signal $x(t)$ be bandlimited to frequencies in the interval $[-\omega_b, \omega_b]$ and that $\omega_b < \omega_{c0} < 2\omega_c - \omega_b$. The reasons for this restriction will become clear after having studied this system in more detail.

Consider the transmitter shown in Figure 5.32(a). The transmitter is a system with input $x(t)$ and output $y(t)$. Mathematically, we can describe the behavior of the system as

$$y(t) = x(t)c(t) \tag{5.54}$$

where $c(t) = \cos\omega_c t$. (Note that we can rewrite $c(t)$ as $c(t) = \frac{1}{2}[e^{j\omega_c t} + e^{-j\omega_c t}]$.) Taking the Fourier transform of both sides of (5.54), we obtain

$$\begin{aligned}Y(\omega) &= \mathscr{F}\{x(t)c(t)\} \\ &= \mathscr{F}\left\{\tfrac{1}{2}[e^{j\omega_c t} + e^{-j\omega_c t}]x(t)\right\} \\ &= \tfrac{1}{2}\left[\mathscr{F}\{e^{j\omega_c t}x(t)\} + \mathscr{F}\{e^{-j\omega_c t}x(t)\}\right] \\ &= \tfrac{1}{2}\left[X(\omega-\omega_c) + X(\omega+\omega_c)\right].\end{aligned} \tag{5.55}$$

Thus, the frequency spectrum of the output is the average of two shifted versions of the frequency spectrum of the input. The relationship between the frequency spectra of the input and output is illustrated in Figure 5.33(d). Observe that we have managed to shift the frequency spectrum of the input signal into a different range of frequencies for transmission as desired. Now, we must determine whether the receiver can recover the original signal $x(t)$.

Consider the receiver shown in Figure 5.32(b). The receiver is a system with input $y(t)$ and output $\hat{x}(t)$. Let $Y(\omega)$, $V(\omega)$ and $\hat{X}(\omega)$ denote the Fourier transforms of $y(t)$, $v(t)$ and $\hat{x}(t)$, respectively. Then, the input-output behavior of the system is characterized by the equations

$$v(t) = c(t)y(t) \quad \text{and} \tag{5.56}$$

$$\hat{X}(\omega) = H(\omega)V(\omega) \tag{5.57}$$

where

$$H(\omega) = \begin{cases} 2 & \text{for } |\omega| \leq \omega_{c0} \\ 0 & \text{otherwise} \end{cases}$$

and $\omega_b < \omega_{c0} < 2\omega_c - \omega_b$. Taking the Fourier transform of both sides of (5.56) yields

$$\begin{aligned}V(\omega) &= \mathscr{F}\{c(t)y(t)\} \\ &= \mathscr{F}\left\{\tfrac{1}{2}\left[e^{j\omega_c t} + e^{-j\omega_c t}\right]y(t)\right\} \\ &= \tfrac{1}{2}\left[\mathscr{F}\left\{e^{j\omega_c t}y(t)\right\} + \mathscr{F}\left\{e^{-j\omega_c t}y(t)\right\}\right] \\ &= \tfrac{1}{2}[Y(\omega-\omega_c) + Y(\omega+\omega_c)].\end{aligned}$$

Substituting the expression for $Y(\omega)$ in (5.55) into this equation, we obtain

$$\begin{aligned}V(\omega) &= \tfrac{1}{2}\left[\tfrac{1}{2}\left[X([\omega-\omega_c]-\omega_c) + X([\omega-\omega_c]+\omega_c)\right] + \tfrac{1}{2}\left[X([\omega+\omega_c]-\omega_c) + X([\omega+\omega_c]+\omega_c)\right]\right] \\ &= \tfrac{1}{2}X(\omega) + \tfrac{1}{4}X(\omega-2\omega_c) + \tfrac{1}{4}X(\omega+2\omega_c). \qquad (5.58)\end{aligned}$$

The relationship between $V(\omega)$ and $X(\omega)$ is depicted graphically in Figure 5.33(e). Substituting the above expression for $V(\omega)$ into (5.57) and simplifying, we obtain

$$\begin{aligned}\hat{X}(\omega) &= H(\omega)V(\omega) \\ &= H(\omega)\left[\tfrac{1}{2}X(\omega) + \tfrac{1}{4}X(\omega-2\omega_c) + \tfrac{1}{4}X(\omega+2\omega_c)\right] \\ &= \tfrac{1}{2}H(\omega)X(\omega) + \tfrac{1}{4}H(\omega)X(\omega-2\omega_c) + \tfrac{1}{4}H(\omega)X(\omega+2\omega_c) \\ &= \tfrac{1}{2}[2X(\omega)] + \tfrac{1}{4}(0) + \tfrac{1}{4}(0) \\ &= X(\omega).\end{aligned}$$

(In the above simplification, since $H(\omega) = 2\operatorname{rect}(\frac{\omega}{2\omega_{c0}})$ and $\omega_b < \omega_{c0} < 2\omega_c - \omega_b$, we were able to deduce that $H(\omega)X(\omega) = 2X(\omega)$, $H(\omega)X(\omega-2\omega_c) = 0$, and $H(\omega)X(\omega+2\omega_c) = 0$.) The relationship between $\hat{X}(\omega)$ and $X(\omega)$ is depicted in Figure 5.33(f). Thus, we have that $\hat{X}(\omega) = X(\omega)$ which implies $\hat{x}(t) = x(t)$. So, we have recovered the original signal $x(t)$ at the receiver. This system has managed to shift $x(t)$ into a different frequency range before transmission and then recover $x(t)$ at the receiver. This is exactly what we wanted to accomplish.

5.16.3 SSB/SC Amplitude Modulation

By making a minor modification to the DSB/SC amplitude modulation system, we can reduce the bandwidth requirements of the system by half. The resulting system is referred to as a SSB/SC amplitude modulation system. This modified system is illustrated in Figure 5.34. In this new system, $G(\omega)$ and $H(\omega)$ are given by

$$G(\omega) = \begin{cases}1 & \text{for } |\omega| \geq \omega_c \\ 0 & \text{otherwise,}\end{cases} \quad \text{and}$$

$$H(\omega) = \begin{cases}4 & \text{for } |\omega| \leq \omega_{c0} \\ 0 & \text{otherwise.}\end{cases}$$

Let $X(\omega)$, $Y(\omega)$, $Q(\omega)$, $V(\omega)$, $\hat{X}(\omega)$, and $C(\omega)$ denote the Fourier transforms of $x(t)$, $y(t)$, $q(t)$, $v(t)$, $\hat{x}(t)$, and $c(t)$, respectively. Figure 5.35 depicts the transformations the signal undergoes as it passes through the system. Again, the output from the receiver is equal to the input to the transmitter.

5.17 Equalization

Often, we find ourselves faced with a situation where we have a system with a particular frequency response that is undesirable for the application at hand. As a result, we would like to change the frequency response of the system to be something more desirable. This process of modifying the frequency response in this way is referred to as

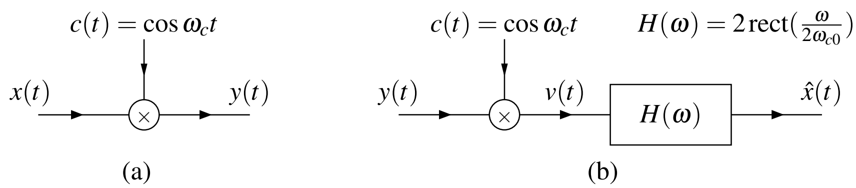


Figure 5.32: DSB/SC amplitude modulation system. (a) Transmitter and (b) receiver.

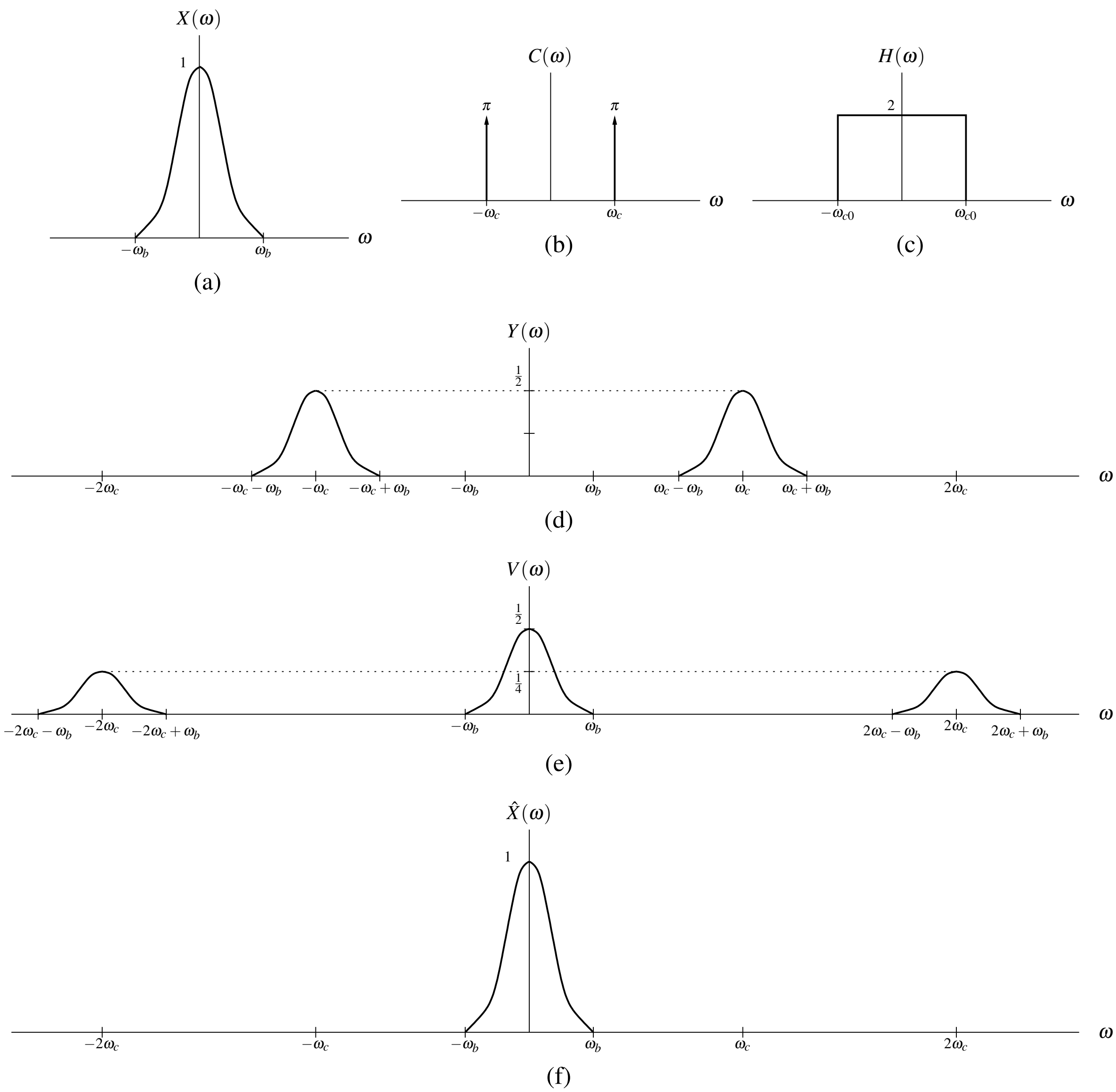


Figure 5.33: Signal spectra for DSB/SC amplitude modulation.

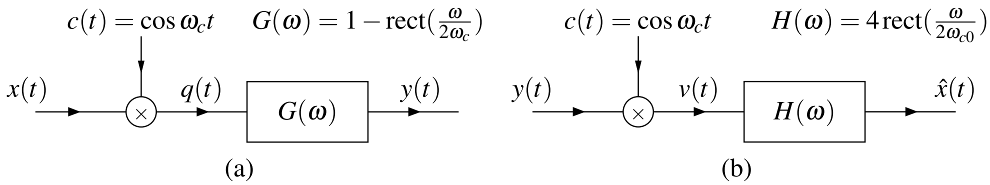


Figure 5.34: SSB/SC amplitude modulation system. (a) Transmitter and (b) receiver.

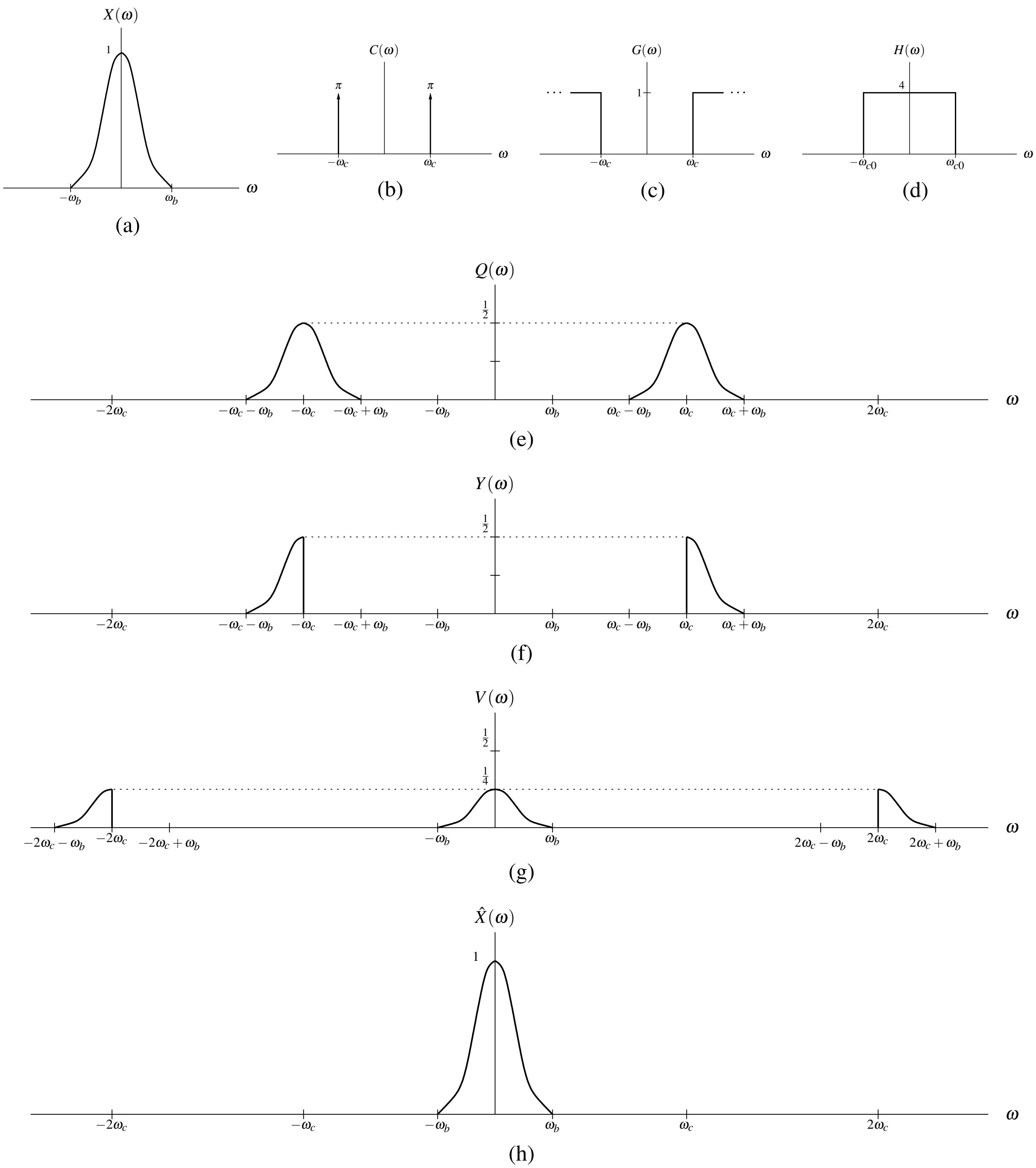


Figure 5.35: Signal spectra for SSB/SC amplitude modulation.

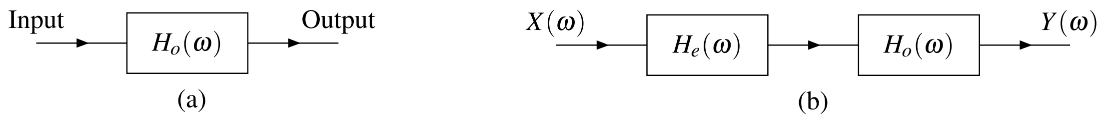


Figure 5.36: Equalization example. (a) Original system. (b) New system with equalization.

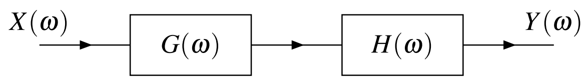


Figure 5.37: Equalization system.

equalization. Essentially, equalization is just a filtering operation, where the filtering is applied with the specific goal of obtaining a more desirable frequency response.

Let us now examine the mathematics behind equalization. Consider the LTI system with frequency response $H_o(\omega)$ as shown in Figure 5.36(a). Suppose that the frequency response $H_o(\omega)$ is undesirable for some reason (i.e., the system does not behave in a way that is good for the application at hand). Consequently, we would instead like to have a system with frequency response $H_d(\omega)$. In effect, we would like to somehow change the frequency response $H_o(\omega)$ of the original system to $H_d(\omega)$. This can be accomplished by using another system called an **equalizer**. More specifically, consider the new system shown in Figure 5.36(b) which consists of a LTI equalizer with frequency response $H_e(\omega)$ connected in series with the original system having frequency response $H_o(\omega)$. From the block diagram, we have

$$Y(\omega) = H(\omega)X(\omega),$$

where $H(\omega) = H_o(\omega)H_e(\omega)$. In effect, we want to force $H(\omega)$ to be equal to $H_d(\omega)$ so that the overall (i.e., series-interconnected) system has the frequency response desired. So, we choose the equalizer to be such that $H_e(\omega) = \frac{H_d(\omega)}{H_o(\omega)}$. Then, we have

$$\begin{aligned} H(\omega) &= H_o(\omega)H_e(\omega) \\ &= H_o(\omega)\left[\frac{H_d(\omega)}{H_o(\omega)}\right] \\ &= H_d(\omega). \end{aligned}$$

Thus, the system in Figure 5.36(b) has the frequency response $H_d(\omega)$ as desired.

Equalization is used in many applications. In real-world communication systems, equalization is used to eliminate or minimize the distortion introduced when a signal is sent over a (nonideal) communication channel. In audio applications, equalization can be employed to emphasize or de-emphasize certain ranges of frequencies. For example, often we like to boost the bass (i.e., emphasize the low frequencies) in the audio output of a stereo.

Example 5.37 (Communication channel equalization)**.** Suppose that we have a LTI communication channel with frequency response $H(\omega) = \frac{1}{3+j\omega}$. Unfortunately, this channel has the undesirable effect of attenuating higher frequencies. Find the frequency response $G(\omega)$ of an equalizer that when connected in series with the communication channel yields an ideal (i.e., distortionless) channel. The new system with equalization is shown in Figure 5.37.

Solution. An ideal communication channel has a frequency response equal to one for all frequencies. Consequently, we want $H(\omega)G(\omega) = 1$ or equivalently $G(\omega) = 1/H(\omega)$. Thus, we conclude that

$$G(\omega) = \frac{1}{H(\omega)} = \frac{1}{\frac{1}{3+j\omega}} = 3 + j\omega.$$

□

5.18 Problems

5.1 Using the Fourier transform analysis equation, find the Fourier transform of each of the following signals:
(a) $x(t) = A\delta(t-t_0)$ where t_0 and A are real and complex constants, respectively;
(b) $x(t) = \operatorname{rect}(t-t_0)$ where t_0 is a constant;
(c) $x(t) = e^{-4t}u(t-1)$;
(d) $x(t) = 3[u(t)-u(t-2)]$; and
(e) $x(t) = e^{-|t|}$.

5.2 Use a Fourier transform table and properties of the Fourier transform to find the Fourier transform of each of the signals below.
(a) $x(t) = \cos(t-5)$;
(b) $x(t) = e^{-j5t}u(t+2)$;
(c) $x(t) = [\cos t]u(t)$;
(d) $x(t) = 6[u(t)-u(t-3)]$;
(e) $x(t) = 1/t$;
(f) $x(t) = t\operatorname{rect}(2t)$;
(g) $x(t) = e^{-j3t}\sin(5t-2)$;
(h) $x(t) = \cos(5t-2)$;
(i) $x(t) = e^{-j2t}\frac{1}{3t+1}$;
(j) $x(t) = \int_{-\infty}^{5t} e^{-\tau-1}u(\tau-1)d\tau$;
(k) $x(t) = (t+1)\sin(5t-3)$;
(l) $x(t) = (\sin 2\pi t)\delta(t-\frac{\pi}{2})$;
(m) $x(t) = e^{-jt}\frac{1}{3t-2}$;
(n) $x(t) = e^{j5t}(\cos 2t)u(t)$; and
(o) $x(t) = e^{-j2t}\operatorname{sgn}(-t-1)$.

5.3 Compute the Fourier transform $X(\omega)$ of the signal $x(t)$ given by

$$x(t) = \sum_{k=0}^{\infty} a^k \delta(t-kT),$$

where a is a constant satisfying $|a| < 1$. (Hint: Recall the formula for the sum of an infinite geometric series. That is, $b + br + br^2 + \ldots = \frac{b}{1-r}$ if $|r| < 1$.)

5.4 The ideal Hilbert transformer is a LTI system with the frequency response

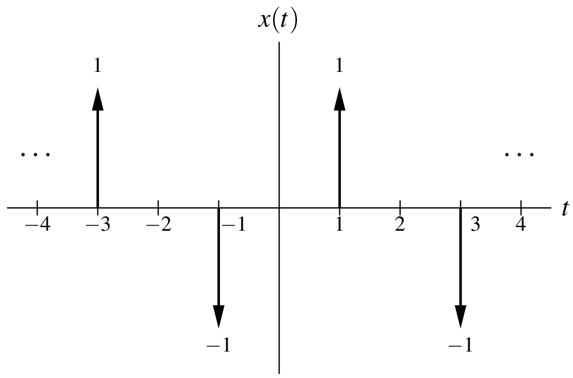
$$H(\omega) = \begin{cases} -j & \text{for } \omega > 0 \\ 0 & \text{for } \omega = 0 \\ j & \text{for } \omega < 0. \end{cases}$$

This type of system is useful in a variety of signal processing applications (e.g., SSB/SC amplitude modulation). By using the duality property of the Fourier transform, find the impulse response $h(t)$ of this system.

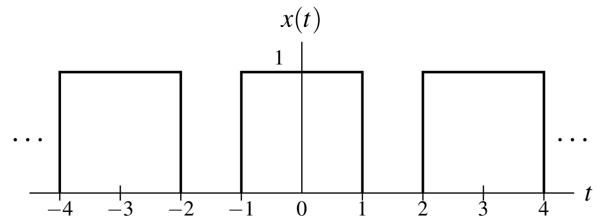
5.5 Given that $x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(\omega)$ and $y(t) \stackrel{\mathscr{F}}{\longleftrightarrow} Y(\omega)$, express $Y(\omega)$ in terms of $X(\omega)$ for each of the following:
(a) $y(t) = x(at-b)$ where a and b are constants and $a \neq 0$;
(b) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(c) $y(t) = \int_{-\infty}^{t} x^2(\tau)d\tau$;
(d) $y(t) = \frac{d}{dt}[x(t) * x(t)]$;
(e) $y(t) = tx(2t-1)$;
(f) $y(t) = e^{j2t}x(t-1)$;
(g) $y(t) = \left(te^{-j5t}x(t)\right)^*$;

(h) $y(t) = \left[\frac{d}{dt}x(t)\right] * \left[e^{-jt}x(t)\right]$;
(i) $y(t) = \int_{-\infty}^{3t} x^*(\tau - 1)d\tau$;
(j) $y(t) = [\cos(3t-1)]\,x(t)$;
(k) $y(t) = \left[\frac{d}{dt}x(t)\right]\sin(t-2)$;
(l) $y(t) = tx(t)\sin 3t$; and
(m) $y(t) = e^{j7t}\,[x(\lambda) * x(\lambda)]|_{\lambda=t-1}$.

5.6 Find the Fourier transform of each of the periodic signals shown below.



(a)



(b)

5.7 Using the time-domain convolution property of the Fourier transform, compute the convolution $h(t) = h_1(t) * h_2(t)$ where

$$h_1(t) = 2000\,\mathrm{sinc}(2000\pi t) \quad \text{and} \quad h_2(t) = \delta(t) - 1000\,\mathrm{sinc}(1000\pi t).$$

5.8 Compute the energy contained in the signal $x(t) = 200\,\mathrm{sinc}(200\pi t)$.

5.9 Compute the frequency spectrum of each of the signals specified below. In each case, also find and plot the corresponding magnitude and phase spectra.
(a) $x(t) = e^{-at}u(t)$, where a is a positive real constant; and
(b) $x(t) = \mathrm{sinc}\,\frac{t-1}{200}$.

5.10 Suppose that we have the LTI systems defined by the differential/integral equations given below, where $x(t)$ and $y(t)$ denote the system input and output, respectively. Find the frequency response of each of these systems.
(a) $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + y(t) + 3\frac{d}{dt}x(t) - x(t) = 0$; and
(b) $\frac{d}{dt}y(t) + 2y(t) + \int_{-\infty}^{t} 3y(\tau)d\tau + 5\frac{d}{dt}x(t) - x(t) = 0$.

5.11 Suppose that we have the LTI systems with the frequency responses given below. Find the differential equation that characterizes each of these systems.
(a) $H(\omega) = \dfrac{j\omega}{1 + j\omega}$; and
(b) $H(\omega) = \dfrac{j\omega + \frac{1}{2}}{-j\omega^3 - 6\omega^2 + 11j\omega + 6}$.

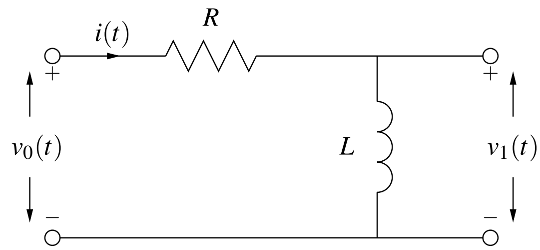
5.12 Suppose that we have a LTI system with input $x(t)$ and output $y(t)$, and impulse response $h(t)$, where

$$h(t) = \delta(t) - 300\,\mathrm{sinc}\,300\pi t.$$

Using frequency-domain methods, find the response $y(t)$ of the system to the input $x(t) = x_1(t)$, where

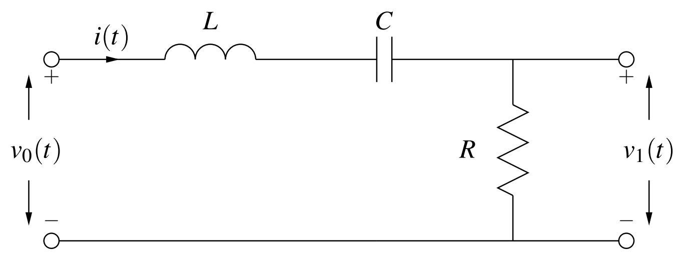
$$x_1(t) = \tfrac{1}{2} + \tfrac{3}{4}\cos 200\pi t + \tfrac{1}{2}\cos 400\pi t - \tfrac{1}{4}\cos 600\pi t.$$

5.13 Consider the LTI system with input $v_0(t)$ and output $v_1(t)$ as shown in the figure below, where $R = 1$ and $L = 1$.



(a) Find the frequency response $H(\omega)$ of the system.
(b) Determine the magnitude and phase responses of the system.
(c) Find the impulse response $h(t)$ of the system.

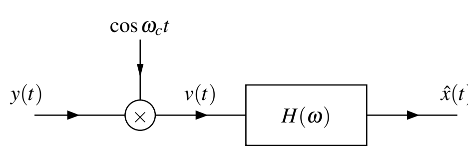
5.14 Consider the LTI system with input $v_0(t)$ and output $v_1(t)$ as shown in the figure below, where $R = 1$, $C = \frac{1}{1000}$, and $L = \frac{1}{1000}$.



(a) Find the frequency response $H(\omega)$ of the system.
(b) Use a computer to plot the magnitude and phase responses of the system.
(c) From the plots in part (b), identify the type of ideal filter that this system approximates.

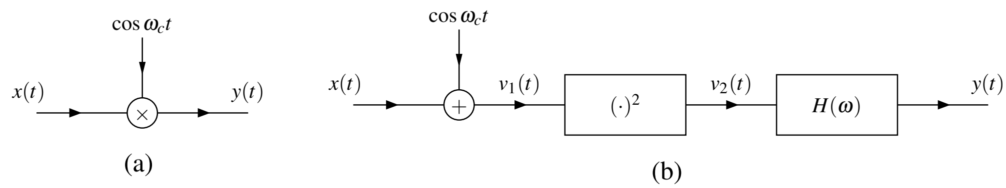
5.15 Let $x(t)$ be a real signal with Fourier transform $X(\omega)$ satisfying $X(\omega) = 0$ for $|\omega| > \omega_b$. We use amplitude modulation to produce the signal $y(t) = x(t)\sin\omega_c t$. Note that $\omega_c \gg \omega_b$. In order to recover the original signal $x(t)$, it is proposed that the system shown in the figure below be used. The frequency response $H(\omega)$ is given by

$$H(\omega) = \begin{cases} 2 & \text{for } |\omega| < \omega_b \\ 0 & \text{otherwise.} \end{cases}$$



Let $Y(\omega)$, $V(\omega)$, and $\hat{X}(\omega)$ denote the Fourier transforms of $y(t)$, $v(t)$, and $\hat{x}(t)$, respectively.
(a) Find an expression for $Y(\omega)$ in terms of $X(\omega)$. Find an expression for $\hat{X}(\omega)$ in terms of $V(\omega)$. Find a simplified expression for $\hat{X}(\omega)$.
(b) Compare $\hat{x}(t)$ and $x(t)$. Comment on the utility of the proposed system.

5.16 When discussing DSB/SC amplitude modulation, we saw that a system of the form shown below in Figure A is often useful. In practice, however, the multiplier unit needed by this system is not always easy to implement. For this reason, we sometimes employ a system like that shown below in Figure B. In this second system, we sum the sinusoidal carrier and modulating signal $x(t)$ and then pass the result through a nonlinear squaring device (i.e., $v_2(t) = [v_1(t)]^2$).

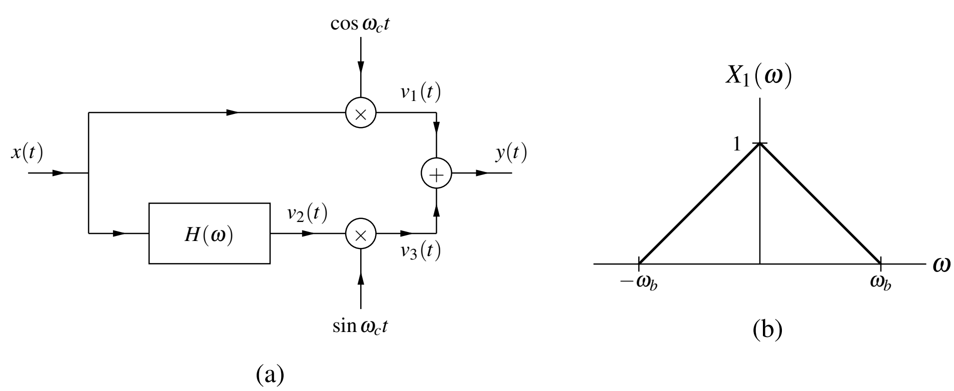


Let $X(\omega)$, $V_1(\omega)$, and $V_2(\omega)$ denote the Fourier transforms of $x(t)$, $v_1(t)$, and $v_2(t)$, respectively. Suppose that $X(\omega) = 0$ for $|\omega| > \omega_b$ (i.e., $x(t)$ is bandlimited).
(a) Find an expression for $v_1(t)$, $v_2(t)$ and $V_2(\omega)$. (Hint: If $X(\omega) = 0$ for $|\omega| > \omega_b$, then using the time-domain convolution property of the Fourier transform, we can deduce that the Fourier transform of $x^2(t)$ is zero for $|\omega| > 2\omega_b$.)
(b) Determine the frequency response $H(\omega)$ required for the system shown in Figure B to be equivalent to the system in Figure A. State any assumptions made with regard to the relationship between ω_c and ω_b. (Hint: It might be helpful to sketch $X(\omega)$ and $V_2(\omega)$ for the case of some simple $X(\omega)$. Then, compare $V_2(\omega)$ to $X(\omega)$ in order to deduce your answer.)

5.17 Consider the system with input $x(t)$ and output $y(t)$ as shown in Figure A below. The frequency response $H(\omega)$ is that of an ideal Hilbert transformer, which is given by

$$H(\omega) = -j \operatorname{sgn} \omega.$$

Let $X(\omega)$, $Y(\omega)$, $V_1(\omega)$, $V_2(\omega)$, and $V_3(\omega)$ denote the Fourier transforms of $x(t)$, $y(t)$, $v_1(t)$, $v_2(t)$, and $v_3(t)$, respectively.

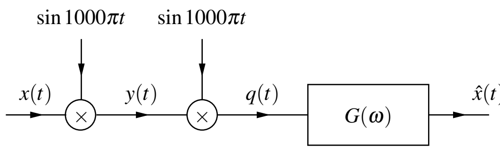


(a) Suppose that $X(\omega) = 0$ for $|\omega| > \omega_b$, where $\omega_b \ll \omega_c$. Find expressions for $V_1(\omega)$, $V_2(\omega)$, $V_3(\omega)$, and $Y(\omega)$ in terms of $X(\omega)$.
(b) Suppose that $X(\omega) = X_1(\omega)$ where $X_1(\omega)$ is as shown in Figure B. Sketch $V_1(\omega)$, $V_2(\omega)$, $V_3(\omega)$, and $Y(\omega)$ in this case.
(c) Draw the block diagram of a system that could be used to recover $x(t)$ from $y(t)$.

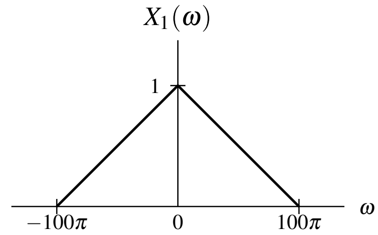
5.18 Consider the system shown below in Figure A with input $x(t)$ and output $\hat{x}(t)$, where

$$G(\omega) = \begin{cases} 2 & \text{for } |\omega| \leq 100\pi \\ 0 & \text{otherwise.} \end{cases}$$

Let $X(\omega)$, $\hat{X}(\omega)$, $Y(\omega)$, and $Q(\omega)$ denote the Fourier transforms of $x(t)$, $\hat{x}(t)$, $y(t)$, and $q(t)$, respectively.



(a)



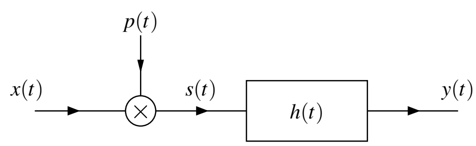
(b)

(a) Suppose that $X(\omega) = 0$ for $|\omega| > 100\pi$. Find expressions for $Y(\omega)$, $Q(\omega)$, and $\hat{X}(\omega)$ in terms of $X(\omega)$.
(b) If $X(\omega) = X_1(\omega)$ where $X_1(\omega)$ is as shown in Figure B, sketch $Y(\omega)$, $Q(\omega)$, and $\hat{X}(\omega)$.

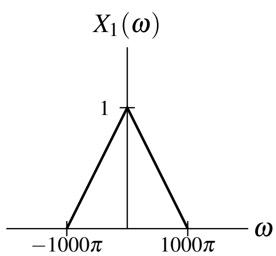
5.19 Consider the system shown below in Figure A with input $x(t)$ and output $y(t)$. Let $X(\omega)$, $P(\omega)$, $S(\omega)$, $H(\omega)$, and $Y(\omega)$ denote the Fourier transforms of $x(t)$, $p(t)$, $s(t)$, $h(t)$, and $y(t)$, respectively. Suppose that

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - \tfrac{n}{1000}) \quad \text{and} \quad H(\omega) = \tfrac{1}{1000}\operatorname{rect}(\tfrac{\omega}{2000\pi}).$$

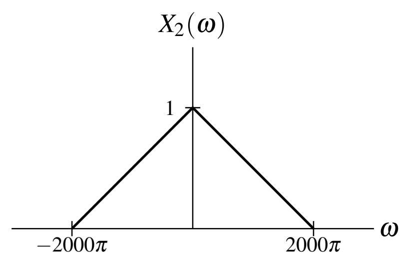
(a) Derive an expression for $S(\omega)$ in terms of $X(\omega)$. Derive an expression for $Y(\omega)$ in terms of $S(\omega)$ and $H(\omega)$.
(b) Suppose that $X(\omega) = X_1(\omega)$, where $X_1(\omega)$ is as shown in Figure B. Using the results of part (a), plot $S(\omega)$ and $Y(\omega)$. Indicate the relationship (if any) between the input $x(t)$ and output $y(t)$ of the system.
(c) Suppose that $X(\omega) = X_2(\omega)$, where $X_2(\omega)$ is as shown in Figure C. Using the results of part (a), plot $S(\omega)$ and $Y(\omega)$. Indicate the relationship (if any) between the input $x(t)$ and output $y(t)$ of the system.



(a)



(b)



(c)

5.19 MATLAB Problems

5.101 (a) Consider a frequency response $H(\omega)$ of the form

$$H(\omega) = \frac{\sum_{k=0}^{M-1} a_k \omega^k}{\sum_{k=0}^{N-1} b_k \omega^k},$$

where a_k and b_k are complex constants. Write a MATLAB function called `freqw` that evaluates a function of the above form at an arbitrary number of specified points. The function should take three input arguments: 1) a vector containing the a_k coefficients, 2) a vector containing the b_k coefficients, 3) a vector containing the values of ω at which to evaluate $H(\omega)$. The function should generate two return values: 1) a vector of function values, and 2) a vector of points at which the function was evaluated. If the function is called with no output arguments

