Analysis 1 - Exercise class 4

Amr Umeri

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Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.
In class you defined what a **converging sequence** is and what a **Cauchy sequence** is.

Remark (not important, difficult)
In class you defined the real numbers using axioms. However It is possible to construct the real numbers from the rational numbers without using axioms. The process of this construction is called completion (Vervollständigung) and it is done by defining the real numbers to be the set of certain equivalence classes of Cauchy sequences with rational entries. Then we say that $\mathbb{R}$ is the completion of $\mathbb{Q}$.

In class you proved the following important result:

Theorem

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.
The sequence is a Cauchy sequence if and only if it is a converging sequence.

Remark (maybe not important)

It is possible that we have a cauchy sequence which consists *only* of rational numbers, but which does *not* converge to a rational number, for example you can construct a sequence of rational numbers which converges to $\sqrt{2}$. This is actually a subtle point, because the same cannot happen for cauchy sequences of real numbers: the limit of the sequence is always a real number and not an element of some bigger space.

Exercise 3.3

Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that:

$$|q_n - q_{n+1}| \to 0 \quad n \to \infty$$

Does this imply that $(q_n)_{n\in\mathbb{N}}$ is a Cauchy sequence? No.

Counterexample:

$$q_n = \sum_{k=1}^{n} \frac{1}{k}$$

Why is this a counterexample? Why is the above sequence not a Cauchy sequence?

Exercise 4.2

Show: The recursively defined sequence, $(a_n)_{n\in\mathbb{N}}$, with:

$$a_1 := 1, \quad a_{n+1} := \sqrt{1+a_n}, \quad n \leq 1$$

is convergent, this means: $\exists a \in \mathbb{R},\ a = \lim_{n\to\infty} a_n$.

Proof: The idea of the proof is the following: We show that the function is **bounded** from above and increasing then we apply the **Satz von Weierstrass**, using the recursive definition of the sequence, we can then determine the limit.

Lemma (Satz von Weierstrass)

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers which is increasing and bounded. Then the limit exists, this means:

$$\exists a \in \mathbb{R},\ a = \lim_{n\to\infty} a_n.$$

Proof (continuation): First we show that the sequence is bounded by 2 using induction:

$$a_1 \leq 2, \quad a_{n+1} = \sqrt{1 + a_n} \leq \sqrt{1+2} \leq 2$$

Above we applied the induction hypothesis which states that $a_n \leq 2$.

Proof (continuation): Next we show that the function is increasing, again by induction. Clearly we have $a_2 \geq a_1$. Furthermore:

$$a_{n+2} = \sqrt{1 + a_{n+1}} \geq \sqrt{1 + a_n} = a_{n+1}$$

Again we applied the induction hypothesis. Hence we showed $a_{n+2} \geq a_{n+1}$.

Proof (continuation): Hence Weierstrass tells us that the limit, $a \in \mathbb{R}$, exists. Now we do the following steps:

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{1 + a_n} = \sqrt{1 + a}$$

Then you just solve for a and you must notice that the solution must be in in the inverval $[1, 2]$ because the sequence only takes values between 1 and 2.

In class you defined **series** and what it means for a series to be converging or to be a Cauchy sequence.

Exercise 4.4

Let $\sum_{k=1}^{\infty} a_k$ an absolutely convergent series, $\sum_{k=1}^{\infty} b_k$ an converging series.
Show: The series $\sum_{k=1}^{\infty} b_k \sin(a_k)$ converges

Proof: From the assumption it follows that $(b_k)_{k\in\mathbb{N}}$ is a zero-sequence, especially it is bounded, since converging sequences are always bounded. This means:

$$\exists C \geq 0 \,\forall k \in \mathbb{N}, |\, b_k \,|\leq C.$$

Next we have that:

$$|\, b_k \sin(a_k) \,|\leq C \cdot \,|\, a_k \,|$$

This implies (think about it) that $\sum_{k=1}^{\infty} b_k \sin(a_k)$ converges absolutely, hence it converges. (Why?)

Lemma

Let $\sum_{k=1}^{\infty} a_k$ an absolutely convergent series, then it converges. (the contrary is not true in general)

Proof: Absolute convergence of the series implies that the sequence $t_n := \sum_{k=1}^{n} |a_k|$ is a Cauchy-sequence. This means the following:

$$\forall \epsilon > 0 \, \exists n_0 \in \mathbb{N} \, \forall m, n \geq n_0, \sum_{k=n}^{m} |a_k| < \epsilon$$

But then

$$| \sum_{k=n}^{m} a_k | \leq \sum_{k=n}^{m} |a_k| < \epsilon$$

This implies that the sequence $s_n := \sum_{k=1}^{n} a_k$ is a Cauchy-sequence aswell. This proves that the series converges.

Analysis 1 - Exercise class 5

Amr Umeri

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Let $\sum_{k=1}^{\infty} a_k$ be a series. In class you defined what it means for a series to be **converging** or to be **absolutely converging**.

Definition

The series **converges** if the **sequence of partial sums** $s_n := \sum_{k=1}^{n} a_k$ converges. Then we denote the limit as $\sum_{k=1}^{\infty} a_k$.

Definition

The series **converges absolutely** if the sequence $t_n := \sum_{k=1}^{n} |a_k|$ converges.

Remark

Last time we have seen that absolute convergence implies convergence of a series. The other implication is not true in general. (Do you know a counter-example?)

Here is an important Satz to test if a series converges or not:

Lemma

Let $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ be series with the following property: $0 \leq a_k \leq b_k \ \forall k \in \mathbb{N}$. Then if $\sum_{k=1}^{\infty} b_k$ converges, then also $\sum_{k=1}^{\infty} a_k$.

Proof: (different from the notes)
Convergence of $\sum_{k=1}^{\infty} b_k$ implies that the sequence $\sum_{k=1}^{n} b_k$ is a Cauchy-sequence. This means the following:

$$\forall \epsilon > 0 \, \exists n_0 \in \mathbb{N} \, \forall m, n \geq n_0, \sum_{k=n}^{m} b_k < \epsilon$$

But then

$$\sum_{k=n}^{m} a_k \leq \sum_{k=n}^{m} b_k < \epsilon$$

This implies that the sequence $\sum_{k=1}^{n} a_k$ is a Cauchy-sequence aswell. This proves that the series $\sum_{k=1}^{\infty} a_k$ converges.

More generally we can prove the following:

Lemma

Let $\sum_{k=1}^{\infty} a_k$ *be a series with the following property:* $0 \leq a_k \ \forall k \in \mathbb{N}$. *If the sequence of partial sums is bounded, this means:*

$$\exists C \geq 0 \, \forall n \in \mathbb{N}, \ \sum_{k=1}^{n} a_k \ \leq C,$$

then the series converges.

Proof:

This can be proven by Satz von Weierstrass. The previous lemma can also be proven with the Satz von Weierstrass.

Exercise 4.3, (b)

Consider the series $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt[3]{k^7+k^4+1}}$. Here we only sum non-negative numbers, hence absolute convergence is the same as convergence for this series. Does the series converge?

We have the following estimates:

$$\sum_{k=1}^{n} \frac{\sqrt{k}}{\sqrt[3]{k^7+k^4+1}} \leq \sum_{k=1}^{n} \frac{\sqrt{k}}{\sqrt[3]{k^7}} = \sum_{k=1}^{n} \frac{1}{k^{\frac{11}{6}}}$$

Hence our series converges if the series on the right converges. Why does the series on the right converge?

Riemann-Zeta-function

Consider the series $\sum_{k=1}^{\infty} \frac{1}{k^a}$ for $a \in \mathbb{R}$. The series converges for $a > 1$. For $a = 1$ we get the harmonic series.

We have the following estimates:

$$\frac{1}{k^a} \leq \int_{k-1}^{k} \frac{1}{t^a} dt$$

This implies:

$$\sum_{k=1}^{\infty} \frac{1}{k^a} \leq 1 + \int_{1}^{\infty} \frac{1}{t^a} dt = \frac{a}{a-1}$$

Hence the series converges for $a > 1$. It is possible to show convergence for $a \in \mathbb{C}$, for complex numbers we require that $Re(a) > 1$. Notice that we are using tools, that have not yet been introduced in class.

An important class of series are the **geometric series**.

Consider the series $\sum_{k=0}^{\infty} q^k$ for $q \in \mathbb{C}$. The sequence of partial sums looks like this:

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}.$$

Hence the series converges if and only if $|q| < 1$. Then the limit is:

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}.$$

Example: Geometric series

Consider the series $\sum_{k=0}^{\infty}(-1)^k z^{2k}$. Here $q = -z^2$. Hence the series converges for $|z| < 1$, this is the **radius of convergence** for the series. The limit is:

$$\sum_{k=0}^{\infty}(-1)^k z^{2k} = \frac{1}{1+z^2}.$$

Remark

The series only converges for $|z| < 1$ and for these values it coincides with the fraction. However the fraction also makes sense for values $|z| > 1$ (!). Hence the equality above of both expressions holds only in the radius of convergence. For $z = i$ the fraction is not defined, also the series does not converge for this value.

Here is another important Satz to test if a series converges or not:

Lemma (Leibniz)

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with the following properties:

1. $(a_n)_{n\in\mathbb{N}}$ *is decreasing.*
2. $\forall n \in \mathbb{N} \ a_n \ \geq 0$
3. $\lim_{n\to\infty} a_n = 0$

Then the alternating series $\sum_{k=1}^{\infty}(-1)^k a_k$ converges.

Remark

This lemma shows that the **alternating harmonic series** $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k}$ converges (but it does not converge absolutely!).

This is an important tool to determine whether a series converges:

Lemma (Quotientenkriterium)

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with $a_n \neq 0\, \forall n \in \mathbb{N}$. If $\limsup\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Remark

If $\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, then the Quotientenkriterium does not give us any information, the series can converge or diverge. Consider for example the following series:

$\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Both of them satisfy $\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, but one of them converges and the other diverges.

Exercise 5.2 (a)

$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$. Does the sequence converge/converge absolutely?

We will apply the Quotientenkriterium. Since the sequence only takes non-negative values, absolute convergence is the same as convergence.
$a_n = \frac{(n!)^2}{(2n)!}$.
Then we have the following:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{(n+1)}{(4n+2)} \to \frac{1}{4} \text{ as } n \to \infty.$$

Hence the series converges.
(If the limit exists, then it is equal to the limes superior)

This is another important tool to determine whether a series converges:

Lemma (Wurzelkriterium)

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. If $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Remark

Often you have to decide whether criterion you should use for a particular exercise. Often one criterion is better than the other (for example if you have to do less calulations).

Consider the following series. Decide whether it is better to use the Wurzelkriterium or the Quotientenkriterium and determine whether the series converges or not.

Exercise

1. $\sum_{n=1}^{\infty} \frac{1}{n^n}$
2. $\sum_{n=1}^{\infty} \frac{1}{n!}$

Exercise 5.2 (c)

$\sum_{n=1}^{\infty} \frac{5^n}{n^{n+1}}$. Does the sequence converge/converge absolutely?

We will apply the Wurzelkriterium. Since the sequence only takes non-negative values, absolute convergence is the same as convergence.
$a_n = \frac{5^n}{n^{n+1}}$.
Then we have the following:

$$\sqrt[n]{\frac{5^n}{n^{n+1}}} = \frac{5}{n\sqrt[n]{n}} \to 0 \text{ as } n \to \infty.$$

Hence the series converges.

In class you defined **power series**. It is a series which depends on a complex (or real) parameter $z \in \mathbb{C}$.

Definition

Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of complex (or real) numbers. A **power series** is given formally for any $z \in \mathbb{C}$ by the series:

$$\sum_{n=0}^{\infty} c_n z^n.$$

We take powers of $z \in \mathbb{C}$, hence the name.

Remark

We want to know for which values $z \in \mathbb{C}$ the series converges. The power series then defines a function on the set of points on which it converges.

Remark

To determine the set of points on which the series converges, one can use the Wurzelkriterium or the Quotientenkriterium. In class you have seen that the set of points on which the series converges is given by a ball around 0, without the boundary. The radius of the ball $\rho > 0$ is then called the **radius of convergence**. It can be calculated by one of the criteria from above. If you consider only real parameters, $z \in \mathbb{R}$, then the ball is actually given by some open interval around 0. It is possible that for some points on the boundary the series converges and for other points the series diverges. (See exercise 5.3 (b))

Remark

If you have an open interval, $(a, b) \subseteq \mathbb{R}$ for $a, b \in \mathbb{R}$, then the boundary points are given by: $a, b \in \mathbb{R}$.

Exercise 5.3 (b)

$\sum_{n=0}^{\infty} \frac{(-3)^n}{n+2} x^n$. For which $x \in \mathbb{R}$ does the power series converge?

We will apply the Wurzelkriterium. For a given $x \in \mathbb{R}$, we have $a_n = \frac{(-3)^n}{n+2} x^n$. Then the series converges if $\limsup_{n\to\infty} \mid \frac{a_{n+1}}{a_n} \mid < 1$.

We have the following:

$$\mid \frac{a_{n+1}}{a_n} \mid = \frac{3(n+3)}{n+2} \mid x \mid .$$

Hence the series converges if $\mid x \mid < \frac{1}{3}$. (Why?). However we don't know yet what happens at the boundary points!

Here the boundary points are $x = \frac{1}{3}$, $x = -\frac{1}{3}$. If you insert $x = \frac{1}{3}$ into the power series, you get:

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{n+2} (\frac{1}{3})^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$$

This converges by Leibniz. If you insert $x = -\frac{1}{3}$ into the power series, you get:

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{n+2} (-\frac{1}{3})^n = \sum_{n=0}^{\infty} \frac{1}{n+2}$$

This does not converge.
To summarize: we found that the series converges for the values in the non-open interval $(-\frac{1}{3}, \frac{1}{3}]$.

Analysis 1 - Exercise class 6

Amr Umeri

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We want to review the real **absolute value function**.

Definition

Define the function $|\cdot|\colon \mathbb{R} \mapsto \mathbb{R}$ by setting

$$|x| := \max\{x, -x\} \text{ for } x \in \mathbb{R}.$$

This is the real **absolute value function**.
Can you draw a picture of the graph of this function?

We want to prove an useful property of the absolute value, which you have seen already in class. It is called **triangle inequality**. This inequality is used in a lot of proofs.

Lemma (Triangle inequality)

$\forall x, y \in \mathbb{R}$ we have:
$|x + y| \leq |x| + |y|$.

Proof:
Let $x, y \in \mathbb{R}$ arbitrary.
We have $x \leq |x|$, $y \leq |y|$ and $-x \leq |x|$, $-y \leq |y|$.
Hence we have:
$x + y \leq |x| + |y|$ and $-x - y \leq |x| + |y|$.
This just means that:
$\max\{x+y, -x-y\} \leq |x| + |y|$.
But we have $\max\{x+y, -x-y\} = |x+y|$ by definition.
Hence the lemma is proven.

We want to prove another lemma, we will see an application of it later. Namely we will use it to show that the absolute value function is continuous.

Lemma (Reverse triangle inequality)

$\forall x, y \in \mathbb{R}$ *we have:*
$||x| - |y|| \leq |x - y|$.

Can you already see how one could prove continuity of the absolute value function using this inequality?

Proof:

Let $x, y \in \mathbb{R}$ arbitrary.

We trivially have $x = x - y + y$. But then $|x| = |x - y + y|$.

Using the triangle inequality we get:

$|x| \leq |x - y| + |y|$

But this just means after substraction:

$|x| - |y| \leq |x - y|$

Hence the lemma is almost proven. Can you finish the proof?

Now we start our discussion of continuous functions.
Let $D \subseteq \mathbb{R}$ be a domain, let $f : D \to \mathbb{R}$ be a function, let $x_0 \in D$ be an arbitrary point in the domain.

Definition

We say that the function $f : D \to \mathbb{R}$ is **continuous at the point** $x_0 \in D$ if the following is true:

$$\forall \epsilon > 0 \, \exists \delta > 0, \left(|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon \right)$$

Remark

The above statement means that for arbitrary $\epsilon > 0$ we can construct a $\delta > 0$ such that if some point $x \in D$ lies in the interval $(x_0 - \delta, x_0 + \delta)$, then the image of this point $f(x) \in \mathbb{R}$ lies in the interval $(f(x_0) - \epsilon, f(x_0) + \epsilon)$.

Above we defined what it means for a function to be continuous at some point. We now want to define what it means if the function is **continuous**.

Definition

We say that the function $f : D \to \mathbb{R}$ is **continuous** if it is continuous at every point in the domain. This means if the following is true:

$$\forall x_0 \in D \, \forall \epsilon > 0 \, \exists \delta > 0, \Big(|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon \Big)$$

Remark

The above definition is known as the "$\epsilon - \delta$"-definition of continuity.

Remark

We will see in the examples that we can give an explicit formula for the $\delta > 0$ to prove continuity. The formula will obviously depend on the function itself, it will also depend on the point $x_0 \in D$ and on $\epsilon > 0$. But it should not depend on any other point of the domain D.

Remark

Often proving continuity using the "$\epsilon - \delta$"-definition is quite difficult and the proofs are not obvious. We will see some examples later. In the lecture you will see other equivalent definitions of continuity. Often these definitions are more practical.

Example: Linear functions

Consider a linear function $f:\mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$, where $a, b \in \mathbb{R}$, $a \neq 0$. This function is continuous.

Let $x_0 \in \mathbb{R}$ arbitrary point.
What we must prove is that given any $\epsilon > 0$ there is some $\delta > 0$ such that if $|x - x_0| < \delta$, then $|ax + b - (ax_0 + b)| < \epsilon$. We have the following:

$$|ax + b - (ax_0 + b)| = |a||x - x_0|.$$

Hence we put $\delta := \frac{\epsilon}{|a|} > 0$.
Then continuity is proven. (Why?)

Example: Quadratic function

Consider the quadratic function $f:\mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. This function is continuous.

Let $x_0 \in \mathbb{R}$ arbitrary point. There are 2 possible cases. Either $x_0 = 0$ or $x_0 \neq 0$. In the first case we can choose $\delta := \sqrt{\epsilon} > 0$.

Then this proves continuity of the function at $x_0 = 0$. (Why?)
Because then we found $\delta > 0$ such that
$|x - 0| < \delta$ implies $|x^2 - 0^2| < \epsilon$.

Now let us consider the second case, namely if $x_0 \neq 0$. We have the following estimates, where we used the triangle inequality twice:

$$\begin{aligned} |x^2 - x_0^2| &= |x + x_0| \cdot |x - x_0| \\ &\leq |x| \cdot |x - x_0| + |x_0| \cdot |x - x_0| \\ &= |x - x_0 + x_0| \cdot |x - x_0| + |x_0| \cdot |x - x_0| \\ &\leq |x - x_0|^2 + 2\,|x_0| \cdot |x - x_0| \end{aligned}$$

Remember that we must prove the following: given any $\epsilon > 0$ there is some $\delta > 0$ such that if $|x - x_0| < \delta$, then $|x^2 - x_0^2| < \epsilon$.

I claim that we should choose $\delta := \min\{\sqrt{\frac{\epsilon}{2}}, \frac{\epsilon}{4|x_0|}\}$.
This choice may seem arbitrary, but notice that we then can show the following:
$|x - x_0|^2 < \delta^2 \leq \frac{\epsilon}{2}$
$2 \cdot |x_0| \cdot |x - x_0| < 2 \cdot |x_0| \cdot \delta \leq \frac{\epsilon}{2}$

Remember that we have proven:

$$|x^2 - x_0^2| \leq |x - x_0|^2 + 2|x_0| \cdot |x - x_0|$$

But then:

$$|x^2 - x_0^2| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence we have proven continuity.

Example: Absolute value function

Consider the absolute value function $|\cdot|\colon \mathbb{R} \mapsto \mathbb{R}$. This function is continuous.

Let $x_0 \in \mathbb{R}$ arbitrary point.
What we must prove is that given any $\epsilon > 0$ there is some $\delta > 0$ such that if $|x - x_0| < \delta$, then $||x| - |x_0|| < \epsilon$.
But we just choose $\delta := \epsilon$ (!) and we apply the reverse triangle inequality which says precisely that:
$||x| - |x_0|| \leq |x - x_0|$.
Hence the claim follows. (Why?)

Example: Square-root function

Consider the square-root function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $f(x) = \sqrt{x}$. This function is continuous.

Example: Reciprocal function

Consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Here the domain $D = \mathbb{R} \setminus \{0\}$. This function is continuous. That means the function is continuous on every point $x_0 \in \mathbb{R}$ but not for $x_0 = 0$.

The proofs are equally complicated. However if you want to see more "$\epsilon - \delta$"-arguments, I can include the proofs in an upgraded version of the slides.

Remark

In the lecture you have seen also a different (but equivalent) definition of continuity, so-called "sequential continuity". Often it is better to work with this definition, since you know how to work with sequences already. We will work with this definition in the next exercise class.

What does it mean if a function is **not continuous at some point**? We just take the negation of the definition of continuity at some point.
Let $D \subseteq \mathbb{R}$ be a domain, let $f : D \to \mathbb{R}$ be a function, let $x_0 \in D$ be an arbitrary point in the domain.

Definition

We say that the function $f : D \to \mathbb{R}$ is **not continuous at the point** $x_0 \in D$ if the following is true:

$$\exists \epsilon > 0 \, \forall \delta > 0, \left(\, |x - x_0| < \delta \wedge |f(x) - f(x_0)| \geq \epsilon \right)$$

Can you write down a function which satisfies the above property and which is hence not continuous at some point? See for example exercise 6.2 (a).

Definition

Let $f:[0,1]\to[0,1]$ be a function. A **fix point** is a point $x_0\in[0,1]$ such that $f(x_0)=x_0$.

Theorem (Zwischenwertsatz)

Let $f:[0,1]\to[0,1]$ be a **continuous** *function. Then there exists a fix point $x_0\in[0,1]$.*
In other words this just means that the graph of your function intersects the graph of the identity function.

Proof:

Analysis 1 - Exercise class 7

Amr Umeri

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Let $D \subseteq \mathbb{R}$ be a domain, let $f : D \to \mathbb{R}$ be a function, let $x_0 \in D$ be an arbitrary point in the domain. Last timewe reviewed the "$\epsilon - \delta$"-definition of continuity. In class you have seen another definition of continuity, so-called "sequential continuity". One can prove that these definitions are equivalent:

Theorem

$$f : D \to \mathbb{R} \textit{ is continuous at the point } x_0 \in D$$

$$\iff$$

$$\forall\, (x_n)_{n\in\mathbb{N}} \textit{ sequence in } D, \Big(\lim_{n\to\infty} x_n = x_0 \implies \lim_{n\to\infty} f(x_n) = f(x_0) \Big)$$

We also have a statement about **continuous** functions:

Theorem

$$f : D \to \mathbb{R} \textit{ is continuous}$$

$$\iff$$

$$\forall x_0 \in D \forall (x_n)_{n\in\mathbb{N}} \textit{ sequence in } D, \left(\lim_{n\to\infty} x_n = x_0 \implies \lim_{n\to\infty} f(x_n) = f(x_0)\right)$$

Remark

What it means is that if you have any point $x_0 \in D$, if you have any sequence $(x_n)_{n\in\mathbb{N}}$, and if this sequence happens to converge to x_0, then the sequence $(f(x_n))_{n\in\mathbb{N}}$ must converge to $f(x_0)$. In less rigorous words, this means that we can take "the limit inside of the function": $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x_0)$.

We want to review some statements about **converging sequences**. These are the "calculus rules" for converging sequences:

Lemma

Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ *be converging sequences in* $\mathbb{R}$ *with* $\lim_{n\to\infty} a_n = a, \lim_{n\to\infty} b_n = b$ *for some numbers* $a, b \in \mathbb{R}$.
Then $(a_n + b_n)_{n\in\mathbb{N}}, (a_n b_n)_{n\in\mathbb{N}}$ *converge and:*

$$\lim_{n\to\infty} a_n + b_n = a + b, \quad \lim_{n\to\infty} a_n b_n = ab.$$

Last time we have proven the continuity of the quadratic function using the "$\epsilon - \delta$"-definition. Now we want to prove it using the other definition.

Example: Quadratic function

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. This function is continuous.

Let $x_0 \in \mathbb{R}$ arbitrary point. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence with $\lim_{n\to\infty} x_n = x_0$.
What we must prove is that $\lim_{n\to\infty} x_n^2 = x_0^2$.
This is just a statement about sequences and we know that it is true! (Why?)
This proves continuity of the quadratic function.

We also want to discuss how to prove continuity of the absolute value function using the "new" definition.

Example: Absolute value function

$|\cdot|\colon \mathbb{R} \mapsto \mathbb{R}$. This function is continuous.

Let $x_0 \in \mathbb{R}$ arbitrary point. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $\lim_{n\to\infty} x_n = x_0$.
What we must prove is that $\lim_{n\to\infty} |x_n| = |x_0|$.
The assumption is that $\lim_{n\to\infty} x_n = x_0$. This means:
$\forall \epsilon > 0 \, \exists n_0 \in \mathbb{N} \, \forall n \geq n_0, |x_n - x_0| < \epsilon$.

Remember we can apply the reverse triangle inequality which implies:
$\big|\,|x_n| - |x_0|\,\big| \leq |x_n - x_0|$.
This means we can show:
$\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0, \big|\,|x_n| - |x_0|\,\big| < \epsilon$.
But this is **precisely** the meaning of $\lim_{n\to\infty} |x_n| = |x_0|$.
This proves continuity of the absolute value function.

Let $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ be functions, let $x_0 \in D$ be an arbitrary point in the domain. Let $\lambda \in \mathbb{R}$. In class you have seen how to define new functions pointwise:
$(f+g) : D \to \mathbb{R}$, $(f \cdot g) : D \to \mathbb{R}$, $(\lambda \cdot f) : D \to \mathbb{R}$.
The set of functions $\{f : D \to \mathbb{R}\}$ is a commutative ring and a vector space over $\mathbb{R}$.
You also proved the following:

Lemma

If f, g are continuous functions, then also $f+g, f \cdot g$, $\lambda \cdot f$.

Remark

What it means in algebraic terms is that the set of continuous functions $\{f : D \to \mathbb{R} \mid \text{continuous}\}$ is a subring of $\{f : D \to \mathbb{R}\}$ and vector space itself.

We want to review injective, surjective and bijective maps. Recall the following definitions:
Let A, B be any sets. Let $f : A \to B$ be a map of sets.

Definition

$f : A \to B$ is **injective** $\iff \forall a_1, a_2 \in A\Big(f(a_1) = f(a_2) \implies a_1 = a_2\Big)$.

Remark

Equivalently we can define injectivity like this:
$\forall a_1, a_2 \in A\Big(a_1 \neq a_2 \implies f(a_1) \neq f(a_2)\Big)$
Why is this definition equivalent to the above one?
Can you write down a function which is injective/not injective?

Let A, B be any sets. Let $f : A \to B$ be a map of sets.

Definition

$f : A \to B$ is **surjective** $\iff \forall b \in B \exists a \in A, f(a) = b$.

Remark

Surjectivity depends on the range (Wertebereich), for example set: $f(A) := \{b \in B \mid \exists a \in A, f(a) = b\}$. Then the map $f : A \to f(A)$ is surjective, but the map $f : A \to B$ does not have to be surjective. Indeed the map is surjective if and only if $B = f(A)$. Can you write down a map which is not surjective?

Let A, B be any sets. Let $f : A \to B$ be a map of sets.

Definition

$f : A \to B$ is **bijective** $\iff$ $f : A \to B$ is surjective and bijective.

Can you write down a map which is bijective?
Consider for example the following trivial examples, the identity maps on the sets A, B:
$Id_A : A \to A, Id_A(a) = a$
$Id_B : B \to B, Id_B(b) = b.$
These are always bijective.

Let A, B be any sets. Let $f : A \to B$ be a map of sets.
If $g : B \to A$ is another map of sets, but this time from B to A, then we can take the composition and we can get two different maps:
$g \circ f : A \to A, \quad f \circ g : B \to B.$
We can prove the following result:

Lemma

$f : A \to B$ is **bijective** $\iff$ *$\exists g : B \to A$ such that $g \circ f = Id_A$ and $f \circ g = Id_B$.*

Remark

Hence bijectivity is equivalent to the existence of an inverse map.

Proof:

We will only prove one direction, the other direction is a good exercise.

Assume $f : A \to B$ is bijective. We want to construct a function $g : B \to A$.

Let $b \in B$, we want to map it to some element $a \in A$.

Lets study the following set:

$f^{-1}(\{b\}) := \{a \in A \mid f(a) = b\}$.

This is the set of all elements $a \in A$ which map to b

Can we just define $g(b) := f^{-1}(\{b\})$? Yes we can, but we have to make sure that this defines a function (!).
This means $f^{-1}(\{b\})$ should be a single element in A since a function does not take more than one value for the same input.
Also a function does not map to the empty set. So we have to show that $f^{-1}(\{b\})$ is not more than one element and it is not the empty set.

First we show that there is at least one element in $f^{-1}(\{b\})$. For that we will need surjectivity. Directly from the definition of surjectivity, we have: $\exists a \in A, f(a) = b$, this means $a \in f^{-1}(\{b\})$ and the set is non-empty.
Now we want to show that there is not more than one element in $f^{-1}(\{b\})$. For that we will need injectivity. Assume we have $a_1, a_2 \in f^{-1}(\{b\})$. By construction of the set this means $f(a_1) = b = f(a_2)$. Then injectivity implies $a_1 = a_2$.
Hence there is just one element in $f^{-1}(\{b\})$.
Finally we just define:
$g\colon B \to A,\ \ g(b) \coloneqq f^{-1}(\{b\})$.
This is indeed a function and satisfies the properties of the lemma.

Exercise 7.2

Define the function $f\colon(-1,1)\to\mathbb{R}$, $f(x)=\frac{x}{\sqrt{1-x^2}}$.
Show: The function is bijective.

Proof: To prove that we will construct an inverse function. Define: $g\colon\mathbb{R}\to(-1,1)$, $g(x)=\frac{x}{\sqrt{1+x^2}}$.
Then consider the following:

$$g\circ f(x)=g(f(x))=\frac{\frac{x}{\sqrt{1-x^2}}}{\sqrt{1+\frac{x^2}{1-x^2}}}=\frac{\frac{x}{\sqrt{1-x^2}}}{\sqrt{\frac{1-x^2}{1-x^2}+\frac{x^2}{1-x^2}}}=\frac{\frac{x}{\sqrt{1-x^2}}}{\sqrt{\frac{1}{1-x^2}}}=x.$$

Similarly we have $f\circ g(x)=x$. This proves bijectivity.

We want to discuss now **sequence of functions**. Let $D \subseteq \mathbb{R}$ be a domain, we can consider a sequence of functions, this means for each $n \in \mathbb{N}$ we have a function:
$f_n : D \to \mathbb{R}$, we then write $(f_n)_{n \in \mathbb{N}}$. We want to understand what **convergence** means for sequences of functions.
Today we will only discuss **pointwise convergence**.

Definition

Remark
Notice that if $x_0 \in D$, then $(f_n(x_0))_{n \in \mathbb{N}}$ is just an ordinary sequence.

Analysis 1 - Exercise class 8

Amr Umeri

April 21, 2020

We want to discuss **sequence of functions**. Let $D \subseteq \mathbb{R}$ be a domain, we can consider a sequence of functions, this means for each $n \in \mathbb{N}$ we have a function:
$f_n : D \to \mathbb{R}$, we then write $(f_n)_{n\in\mathbb{N}}$. This resembles the notation for a sequence of real numbers, $(a_n)_{n\in\mathbb{N}}$.

Similarly as we did for sequences of real numbers, we want to understand what **convergence** means for a **sequence of functions**. There are many ways to define convergence for sequences of functions. In class you have seen 2 definitions:

pointwise convergence (punktweise Konvergenz) and **uniform convergence** (gleichmässige Konvergenz). The key point is that these definitions are **not equivalent**. The goal of this exercise class is to understand what this means.

Remark

Let $f_n : D \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions.
Notice that if $x_0 \in D$, then $(f_n(x_0))_{n\in\mathbb{N}}$ is just an ordinary sequence of real numbers.

Example

Consider the sequence of functions
$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) = \frac{x}{n}, \quad n \in \mathbb{N}.$

Then we have for example:
$f_{10}(0.5) = \frac{1}{20},\ \ f_n(1) = \frac{1}{n}.$

We start with **pointwise convergence**.

Definition

Let $f_n : D \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions. Let $f : D \to \mathbb{R}$ be a function.
We say that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ **converges pointwise** to f if the following is true:

$$\forall x \in D, \lim_{n\to\infty} f_n(x) = f(x).$$

We then say that f, if it exists, is the **pointwise limit**.

Remark

One can prove that the pointwise limit of a sequence of functions, if it exists, is unique.

Remark

More precisely, the above definition means:

$$\forall x \in D \, \forall \epsilon > 0 \, \exists n_0 \in \mathbb{N} \, \forall n \geq n_0, |\, f_n(x) - f(x) | \leq \epsilon.$$

Notice that this definition means that the number $n_0 \in \mathbb{N}$ can also depend on the point $x \in D$ and not just on $\epsilon > 0$.
This will be important when we discuss the difference between this definition of convergence and the definition of uniform convergence.

Example

Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) = \frac{x}{n}, \quad n \in \mathbb{N}$.
Consider the function which only takes 0 as a value
$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 0$.

Can you draw the functions for $n = 1, 3, 5, 10, \ldots$ and see how it "evolves"?
We claim that the sequence of functions $(f_n)_{n \in \mathbb{N}}$ converges pointwise to the zero-function f.

Can you prove the above claim rigorously using only the definition of pointwise convergence and your knowledge about sequences of real numbers?

Here is an example from the lectures. Notice that here the domain is $[0,1]$.

Example

Consider the sequence of functions
$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) = x^n, \quad n \in \mathbb{N}.$

Consider the function $f : [0,1] \to \mathbb{R}, \quad f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}.$

Notice that this function is not continuous.

Can you draw the functions for $n = 1, 3, 5, 10, \ldots$ and see how it "evolves"?
We claim that the sequence of functions $(f_n)_{n \in \mathbb{N}}$ converges pointwise to the function f.

Proof:

We consider 2 cases. First let us consider the case where $x = 1$.
Then obviously:
$\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1^n = 1 = f(1)$.
Now consider the other case, namely if $x \in [0, 1)$. Then we have:
$\lim_{n\to\infty} f_n(x) = 0 = f(x)$.
Hence we have proven pointwise convergence.

Here is some **important observation** which we can deduce from the last example:

Remark

In the last example we considered a sequence of functions, $(f_n)_{n\in\mathbb{N}}$, which consisted only of **continuous functions**. We showed that the sequence of functions converges pointwise to some function. But the pointwise limit of this sequence is **not continuous**.

Hence **pointwise limits of sequences of continuous functions are not necessarily continuous.**

We will now review the definition of **uniform convergence**.

Definition

Let $f_n : D \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions. Let $f : D \to \mathbb{R}$ be a function.

We say that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ **converges uniformly** to f if the following is true:

$$\forall \epsilon > 0 \, \exists n_0 \in \mathbb{N} \, \forall n \geq n_0 \, \forall x \in D, |\, f_n(x) - f(x) | \leq \epsilon.$$

We then say that f, if it exists, is the **uniform limit**.

Remark

This definition means that the number $n_0 \in \mathbb{N}$ does not depend on the point $x \in D$ and depends only $\epsilon > 0$.

Remark (important)
One can prove using the above definition that uniform convergence implies pointwise convergence. The reverse is not true: If a sequence convergs pointwise, it does not have to converge uniformly. Can you explain why the reverse conclusion is not true given the remark some slide before?

In class you have proven the following important theorem:

Theorem

Let $f_n : D \to \mathbb{R}$, $n \in \mathbb{N}$, *be a sequence of* **continuous functions**. *Let* $f : D \to \mathbb{R}$ *be a function. If* $(f_n)_{n\in\mathbb{N}}$ *converges uniformly to* f, *then the function* f *is also* **continuous**.

Remark

This means **uniform limits of sequences of continuous functions are always continuous.**

This is the key difference between uniform convergence and pointwise convergence.

Remember this example:

Example

Consider the sequence of functions
$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) = x^n, \quad n \in \mathbb{N}.$

Consider the function $f : [0,1] \to \mathbb{R}, \quad f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$

The sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the function f. But the function f is not continuous. This actually implies that the sequnce of functions $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly to f. Hence it does not converge uniformly at all.

Example

Consider the sequence of functions
$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) = x^n \cdot (1-x)^n, \quad n \in \mathbb{N}.$
Consider the function which only takes 0 as a value
$f : [0,1] \to \mathbb{R}, \quad f(x) = 0.$

We claim that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly to the zero-function f.

Proof:

First notice the following triviallity:

$(x - \frac{1}{2})^2 \geq 0.$

This implies that $x \cdot (1 - x) \leq \frac{1}{4}$.

Hence we have:

$f_n(x) = x^n \cdot (1 - x)^n \leq \frac{1}{4^n}$.

The sequence on the right converges to zero.

Can you prove the above claim using these facts?

This is essentially exercise 8.2 (b).

Example

Consider the sequence of functions
$f_n : [0, 1] \to \mathbb{R}, \quad f_n(x) = (x^{\frac{1}{2}} + \frac{1}{n})^2, \quad n \in \mathbb{N}.$
Consider the identity function $f : [0, 1] \to \mathbb{R}, \quad f(x) = x.$
Notice that here the domain is a compact interval: $[0, 1]$.
(compact means that it is a bounded and closed interval).

We claim that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly to the identity function f.

Proof:
Notice the following:
$x \in [0,1]$.
$| f_n(x) - f(x) | = | (x^{\frac{1}{2}} + \frac{1}{n})^2 - x | = | x^{\frac{1}{2}} \cdot \frac{2}{n} + \frac{1}{n^2} | \leq \frac{3}{n}$.
Notice that the sequence on the right converges to zero.
We used that $x \leq 1$ and $\frac{1}{n^2} \leq \frac{1}{n}$ for $n \in \mathbb{N}$.
Can you prove uniform convergence?

Analysis 1 - Exercise class 9, 10

Amr Umeri

May 5, 2020

We want to discuss differentiable functions. We want to review the definition of **differentiability** given in the lecture. We will see how to use the definition to calculate the derivative of a function at some point. Then we will discuss extreme points, minima and maxima.

For motivation: A function is differentiable if and only if we can approximate it locally by a linear function.

Let $D \subseteq \mathbb{R}$ be a domain, let $f : D \to \mathbb{R}$ be a function, let $x_0 \in D$ be a limit point (Häufungspunkt) of D.

Definition

Let $x_0 \in D$ be a limit point of D. We say that the function $f : D \to \mathbb{R}$ is **differentiable at the point** $x_0 \in D$ if the following is true:
The limit:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists.
If this is the case, we put:

$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

and call it the derivative (Ableitung).

Remark

If you want to know why we require that $x_0 \in D$ is a limit point (Häufungspunkt) of D in the definition of differentiability, then have a look at chapter *3.10* of M. Burger's notes, especially *3.10.3* and *3.10.4*. This is a technical requirement to make sense of the limit. For example if $D = \mathbb{R}$, then we don't care about this technical requirement because every point is a limit point.

Above we defined what it means for a function to be differentiable at some point. We now want to define what it means if the function is **differentiable**.

Definition

We say that the function $f : D \to \mathbb{R}$ is **differentiable** if it is differentiable at every limit point $x_0 \in D$.

Remark

There is an equivalent definition of differentiability given in the following way, see also remark *3.10.4* in Marc Burger's notes:

Definition

Let $x_0 \in D$ be a limit point of D. The function $f : D \to \mathbb{R}$ is **differentiable at the point** $x_0 \in D$ if:
$\forall\, (h_n)_{n\in\mathbb{N}}$ sequence in $\mathbb{R} \setminus \{0\}$,
$\left(\lim_{n\to\infty} h_n = 0 \implies \lim_{n\to\infty} \frac{f(x_0+h_n)-f(x_0)}{h_n} = f'(x_0) \right)$.

Remark

The above definition implies that the limit $f'(x_0)$ should not depend on how we choose the sequence $(h_n)_{n\in\mathbb{N}}$. Also it is not enough if we show "differentiability" for a particular sequence, for example: $h_n = \frac{1}{n}$. We must show that the limit $f'(x_0)$ exists for *any* sequence $(h_n)_{n\in\mathbb{N}}$ which converges to 0.

Differentiability is best understood using examples.
I assume I don't need to review the differentiation rules (Ableitungsregeln). You should know about the addition rule, product rule and chain rule and their implication: Sums, products and compositions of differentiable functions are differentiable. And you should know about the quotient rule, which is a corrolary of the other rules.

Example: Linear functions

Consider a linear function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$, where $a, b \in \mathbb{R}$. This function is differentiable for all $x_0 \in D$.
The derivative is given by: $f'(x_0) = a$.

Let $x_0 \in \mathbb{R}$ arbitrary point.
Then we have the following:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{a \cdot (x_0 + h) - a \cdot x_0}{h} = \lim_{h \to 0} \frac{a \cdot h}{h} = a.$$

Example: Quadratic function

Consider the quadratic function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. This function is differentiable for all $x_0 \in D$.
The derivative is given by: $f'(x_0) = 2x_0$.

Let $x_0 \in \mathbb{R}$ arbitrary point.
Then we have the following:

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+h)^2 - x_0^2}{h} =$$

$$\lim_{h\to 0} \frac{2hx_0 + h^2}{h} = \lim_{h\to 0} 2x_0 + h = 2x_0.$$

Here is an example of a function which is not differentiable.

Example: Absolute value function

Consider the absolute value function $|\cdot|\colon \mathbb{R} \mapsto \mathbb{R}$. This function is not differentiable at the point $0 \in \mathbb{R}$.

Here $x_0 = 0$. We want to show that it is not differentiable at this point.
First let $h_n = \frac{1}{n}$ and consider the following:

$$\lim_{n\to\infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n\to\infty} = \lim_{n\to\infty} \frac{\mid \frac{1}{n} \mid -0}{\frac{1}{n}} = 1.$$

Now let $h_n = -\frac{1}{n}$.

$$\lim_{n\to\infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n\to\infty} = \lim_{n\to\infty} \frac{\mid -\frac{1}{n} \mid -0}{-\frac{1}{n}} = -1.$$

Hence the limits are different and the function can't be differentiable at 0.

Exercise 9.1 (c)

Consider the function $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} xsin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

This function is **continuous but not differentiable** at 0.

It is a good exercise to prove continuity (Hint: To prove continuity at 0, use the sequential definition of continuity and use sandwich-lemma, this will be done in the exercise class.)

Let $x = 0$. We want to show that this function is not differentiable at this point, for that consider the following:

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{hsin(\frac{1}{h})}{h} = \lim_{h \to 0} sin(\frac{1}{h}).$$

This limit does not even exist (!). Hence the claim follows.

Let us consider a slight variation of the last example.

Example

Consider the function $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x^2 sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$

This function is **continuous and differentiable** at 0.

Let $x = 0$. We want to show that this function is differentiable at this point, for that consider the following:

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 sin(\frac{1}{h})}{h} = \lim_{h \to 0} h \cdot sin(\frac{1}{h}) = 0.$$

The last assertion is proven by sandwich-lemma and by the fact that $| sin(x) | \leq 1$ for any $x \in \mathbb{R}$.

Remark

We can even show that if $f: \mathbb{R} \to \mathbb{R}, \quad$ is any bounded (not necessarily differentiable) function, then the function $g(x) = x^2 \cdot f(x)$ is differentiable at 0. Question: Is this function differentiable everywhere?

Exercise 9.4

Consider the function $f : (-1, 1) \to \mathbb{R}$,
$f(t) = \begin{cases} t + 2t^2 sin(\frac{1}{t}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. This function is differentiable at 0.

We have the following:

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h + 2h^2 sin(\frac{1}{h})}{h} = \lim_{h \to 0} 1 + 2h sin(\frac{1}{h}) = 1.$$

Hence $f'(0) = 1$.

The next example will be needed for exercise 10.3.

Consider the function $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$. This function is differentiable for $x \neq 0$ by chain rule. But this function is also differentiable at 0. The derivative at this point is given by $f'(0) = 0$. In the exercises you will have to show that this function is infinitely often differentiable!

We want to show that this function is differentiable at 0. First observe the following estimate for the exponential function for $x > 0$.

$$e^{\frac{1}{x}} \geq \frac{1}{2} \cdot \frac{1}{x^2}.$$

(Can you prove this inequality?).
Then we have the following:
The rest of the proof will be given in the exercise class.

One can prove the following:

$f'(x) = \begin{cases} \frac{1}{x^2} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$. Then one shows that this function is also differentiable using similar ideas from before, then one shows inductively that this function is infinitely often differentiable.

Now we want to discuss **local minima, local maxima, extreme points and critical points**.

Definition

Let $f : D \to \mathbb{R}$ be a function, let $x_0 \in D$.
x_0 is called **local maxima** if the following is true:

$$\exists \delta > 0 \forall x \in D, |x - x_0| < \delta \implies f(x) \leq f(x_0).$$

x_0 is called **local minima** if the following is true:

$$\exists \delta > 0 \forall x \in D, |x - x_0| < \delta \implies f(x) \geq f(x_0).$$

x_0 is called **extreme point** if it is either a local maxima or a local minima.

In the class you have proven the following important result. Lets give an alternative proof.

Lemma

Let $f : D \to \mathbb{R}$ be a function, let f be differentiable at $x_0 \in D$ and let it be an extreme point. Then $f'(x_0) = 0$. (These points are also called critical points.)

Remark

This lemma gives you the following important information: A polynomial of degree n can have at most $n-1$ extreme points, because the extreme points must be zeros of the derivative of the polynomial (which is a polynomial of degree $n-1$).

Proof

Since f is differentiable at $x_0 \in D$, we have that the limit $\frac{f(x_0+h_n)-f(x_0)}{h_n} = f'(x_0)$ exists for any sequence $(h_n)_{n\in\mathbb{N}}$ which converges to 0. Since x_0 is an extreme point, it is either a maxima or a minima. Lets assume it is a maxima (the proof for the other case is similar).

First take $h_n = \frac{1}{n}$.

Then consider the following:

$\frac{f(x_0+h_n)-f(x_0)}{h_n} \leq 0$.

This implies: $f'(x_0) \leq 0$. Now take $h_n = \frac{-1}{n}$.

Then consider the following:

$\frac{f(x_0+h_n)-f(x_0)}{h_n} \geq 0$.

This implies: $f'(x_0) \geq 0$.

Hence we have $f'(x_0) = 0$. This proves the lemma.

Lets see how to work with the definitions of local minima and maxima.

Example

Consider the function $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3 - x^2$.
It has a local maxima at $x_0 = 0$ and a local minima at $x_0 = \frac{2}{3}$.
You can guess these points by plotting the function.

First of all, notice that the derivative is given by $f'(x) = 3x^2 - 2x$.
This is a polynomial of degree 2, hence f can have at most 2 extreme points.

First let us show that $x_0 = 0$ is a local maxima. For that choose $\delta = \frac{1}{2}$. Then take $x \in \mathbb{R}$ with $\mid x \mid < \frac{1}{2}$.
Then we see that
$f(x) = x^2 \cdot (x - 1) \leq \frac{1}{4} \cdot \frac{-1}{2} \leq 0 = f(0)$.

Now let us show that $x_0 = \frac{2}{3}$ is a local minima. For that choose $\delta = \frac{1}{3}$. Then take $x \in \mathbb{R}$ with $\mid x - x_0 \mid < \frac{1}{3}$.
Then we see that
$f(x) = x^2 \cdot (x-1) \geq 0 \geq f(\frac{2}{3})$.

Analysis 1 - Exercise class 11, 12, 13

Amr Umeri

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We want to discuss integration. We will review the definition of **integrability** given in the lecture. The idea of the **integral** is, as you know already, to assign a value to the area which is enclosed by the graph of some function and the x-axis.

Let $f : [0,1] \to \mathbb{R}$ be a bounded function.
There are many ways to approximate the integral of f. For example you can partition your interval $[0,1]$ into 4 sub-intervals, for simplicity with equal length:
$[0,0.25],[0.25,0.5],[0.5,0.75],[0.75,1]$. Then you approximate the integral by the area given by some rectangles:
$\sum_{k=1}^{4} f(\frac{k}{4}) \cdot \frac{1}{4}$.

This means we approximated the integral by the area of 4 rectangles all with length 0.25 given by the sub-intervals and with height given by evaluating the function on the *right* on each sub-interval. This is an example of a Riemann-Sum approximation.

Can you draw a sketch describing this situation?

Remark

It is important to notice that we can generalize the partition by allowing arbitrary number of division of the intervals $[0, 1]$ and the sub-intervals don't need to have equal lenght also we don't need to evaluate on the right and we can choose any point on each sub-interval.

In class you have defined 2 special types of **Riemann-Sums**, let's review the definition.
Let $f : [0, 1] \to \mathbb{R}$ be a bounded function. Let P be a finite partition of the interval $[0, 1]$, this means we have n+1 points $x_0 = 0 < x_1 \cdots < x_n = 1$.
Then we define the **lower Riemann-Sum** and the **upper Riemann-Sum** the following way:

Definition

$s(f, P) \coloneqq \sum_{k=1}^{n} \inf_{x \in [x_{k-1}, x_k]} f(x) \cdot (x_k - x_{k-1}) \quad \text{(Lower Sum)}$
$S(f, P) \coloneqq \sum_{k=1}^{n} \sup_{x \in [x_{k-1}, x_k]} f(x) \cdot (x_k - x_{k-1}) \quad \text{(Upper sum)}$

Remark

If f is continious, then we can replace $\inf, \sup$ with $\min, \max$. (Satz 3.4.5. in M. Burger's notes).

Let $\mathcal{P}$ be the set of all finite partitions of the interval $[0, 1]$. Let $f : [0, 1] \to \mathbb{R}$ be a bounded function.
Let's define now what it means if the function is **Riemann-integrable**.

Definition

f is **Riemann-integrable** if the following is true:

$$\sup_{P \in \mathcal{P}} s(f, P) = \inf_{P \in \mathcal{P}} S(f, P).$$

Then we set this value equal to this expression:

$$\int_0^1 f(x)dx.$$

Remark

The above means that a function is integrable if the supremum over all possible lower sums is equal to the infimum over all possible upper sums. Then we set the integral of this function to be equal to this value.

We want to review an example from the lecture.

Example

Consider the function $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ not rational} \end{cases}.$$

This is the rational indicator function. It is **not integrable**.

Given an arbitrary partition P of $[0, 1]$, let's say $x_0 = 0 < x_1 \cdots < x_n = 1$. Then each of the intervals $[x_0, x_1] \ldots [x_{n-1}, x_n]$ contains a rational number and an irrational number. Then we have:
$\inf_{x\in[x_0,x_1]} f(x) = \cdots = \inf_{x\in[x_{n-1},x_n]} f(x) = 0$. This implies $s(f, P) = 0$, and since the partition P was arbitrary, we have $\sup_{P\in\mathcal{P}} s(f, P) = 0$.
Similarly we have the following:
$\sup_{x\in[x_0,x_1]} f(x) = \cdots = \sup_{x\in[x_{n-1},x_n]} f(x) = 1$. This implies $S(f, P) = 1$, and since the partition P was arbitrary, we have $\inf_{P\in\mathcal{P}} S(f, P) = 1$.
This means $\sup_{P\in\mathcal{P}} s(f, P) = 0 \neq 1 = \inf_{P\in\mathcal{P}} S(f, P)$. Hence the function is not integrable because these values are different.

In the lecture you proved the following important result:

Lemma (Riemann criterion for integrability)

Let $f : [0,1] \to \mathbb{R}$ be a bounded function.
f is Riemann-integrable if and only if the following is true:

$$\forall \epsilon > 0 \exists P \in \mathcal{P}, S(f,P) - s(f,P) \leq \epsilon.$$

In the lecture you have seen an application of the previous lemma, let's see another example.

Example

Consider the function $f : [0,1] \to \mathbb{R}, \quad f(x) = x^2$.
This function is integrable with integral $\int_0^1 x^2 dx = \frac{1}{3}$.

Consider for a given $n \in \mathbb{N}$ the equi-distant partition of $[0,1]$ and let's call it P_n:
$x_0 = 0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \cdots < x_n = 1.$
Then let us consider the lower sum and the upper sum corresponding to the partition:

$s(f, P_n) = \sum_{k=1}^{n} \inf_{x \in [\frac{k-1}{n}, \frac{k}{n}]} f(x) \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} (k-1)^2.$
$S(f, P_n) = \sum_{k=1}^{n} \sup_{x \in [\frac{k-1}{n}, \frac{k}{n}]} f(x) \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2.$

Then consider the distance (compare with previous lemma):
$S(f, P) - s(f, P) = \frac{1}{n^3} \sum_{k=1}^{n} k^2 - (k-1)^2 = \frac{1}{n}.$

Using the previous lemma and the fact that for all $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ (remember the Archimedean principle from the first exercise class), we conclude that the function is integrable.

The integral is given by:

$\lim_{n\to\infty} S(f, P_n) = \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 =$
$\lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}.$

(Here one should argue why the limit exists and why it converges to $\inf_{P\in\mathcal{P}} S(f, P)$, for that see the lecture.)

We used the following equality, which you can prove by induction: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

This is Satz 5.2.7 from M. Burger's notes:

Lemma

Let $f : [a, b] \to \mathbb{R}$ be a function. If it is continuous, then it is integrable.

Remark

For the proof it is important that the interval $[a, b]$ is compact (for us this just means that the interval is closed and bounded). The proof uses the fact that continuous functions on compact intervals are uniformly continuous, which is just a stronger version of continuity (don't confuse this with uniform convergence).
See exercise 11.1.

This is the **fundamental theorem of calculus**, see Satz 5.4.1 in M. Burger's notes:

Theorem (Fundamental theorem of Calculus)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Consider the function: $F : [a, b] \to \mathbb{R}$, $F(x) = \int_a^x f(t)dt$. This function is differentiable and we have: $F' = f$. (This means the derivative of F is f.)
Moreover we have:

$$\int_a^b f(t)dt = F(b) - F(a).$$

Remark
This theorem is fundamental, because it gives a relation between integration and differentiation. We can use the second part of the theorem to calculate integrals of continuous functions without using partitions.

Lets see how to use the previous theorem in order to calculate integrals of a continuous functions.

Example

Consider the function $f : [0,1] \to \mathbb{R}, \;\; f(x) = x^2$.
This function is integrable with integral $\int_0^1 x^2 dx = \frac{1}{3}$.
To prove that, notice that we can choose $F(x) = \frac{x^3}{3}$, since $F' = f$.
Then we have:

$$\int_0^1 x^2 dx = F(1) - F(0) = \frac{1}{3}.$$

Exercise 11.3 (c)

Consider the function $f : \mathbb{R} \to \mathbb{R}, \;\; f(x) = e^{\cos(x)} sin(x)$.
We want to find a differentiable function $F : \mathbb{R} \to \mathbb{R}$ such that $F' = f$.

Define $F(x) = -e^{\cos(x)}$, then the result follows by chain rule.
See also exercise 12.3 (c).

Exercise 11.3 (d)

Consider the function $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{x}{\sqrt{1+5x^2}}$.
We want to find a differentiable function $F : \mathbb{R} \to \mathbb{R}$ such that $F' = f$.

Define $F(x) = \frac{\sqrt{1+5x^2}}{5}$.
See also exercise 12.3 (h).

Lets discuss now integration methods, especially **partial integration**.

Lemma

Let $f : [a, b] \to \mathbb{R}, g : [a, b] \to \mathbb{R}$ be differentiable functions. Then we have the following:

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx.$$

Remark

The proof used the product rule in differentiation and the fundamental theorem, you can memorize the statement like this:
$(fg)' = fg' + f'g$ by product rule. Then by "integrating" we have:
$fg = \int fg' + \int f'g$.
(This is not a proof).

Lets see examples of partial integration:

Example

We want to calculate:
$\int_0^{2\pi} x \sin(x) dx$
using partial integration.

Set $f(x) = x$ and $g'(x) = \sin(x)$. Hence we have $f'(x) = 1$ and $g(x) = -\cos(x)$. Then using partial integration we have:

$$\int_0^{2\pi} x \sin(x) dx = -2\pi + \int_0^{2\pi} \cos(x) dx = -2\pi.$$

Example

We want to calculate:
$\int_0^{2\pi} x^2 \cos(x) dx$
using partial integration.

Set $f(x) = x^2$ and $g'(x) = \cos(x)$. Hence we have $f'(x) = 2x$ and $g(x) = \sin(x)$. Then using partial integration we have:

$$\int_0^{2\pi} x^2 \cos(x) dx = -\int_0^{2\pi} 2x \sin(x) = -2\int_0^{2\pi} x \sin(x) = 4\pi.$$

(We used the previous example for the last step).

Lets discuss the so-called Wallis product. This is similair to exercise 12.4.
The formula gives a sequence of rational numbers which converges to π. See also section 8.7.1 of the book on Analysis by Einsiedler/Wieser (link on the webpage).

Example: Wallis product

For each $n \geq 0$ we want to define the following integrals:

$$I(n) := \int_0^\pi \sin^n(x)dx$$

We want to calculate the integrals using partial integration and induction on $n \geq 0$.

We have the base cases:
$I(0) = \int_0^\pi 1dx = \pi$,
$I(1) = \int_0^\pi \sin(x)dx = -\cos(\pi) + \cos(0) = 2$.

Then lets do partial integration for $n \geq 2$:
We set $f(x) = \sin^{n-1}(x)$ and $g'(x) = \sin(x)$. Hence we have $f'(x) = (n-1)\sin^{n-2}(x)\cos(x)$ and $g(x) = -cos(x)$.

$$I(n) = \int_0^\pi \sin^{n-1}(x)\sin(x)dx = \int_0^\pi (n-1)\sin^{n-2}(x)\cos^2(x)dx.$$

For the left integral we apply the equality: $\cos^2(x) = 1 - \sin^2(x)$.

We have the following:

$$I(n) = (n-1)\int_0^{\pi} \sin^{n-2}(x)\cdot(1-\sin^2(x))dx.$$

This means:

$$\begin{aligned} I(n) &= (n-1)\int_0^{\pi} \sin^{n-2}(x) - (n-1)\int_0^{\pi} \sin^{n-2}(x)\sin^2(x)dx \\ &= (n-1)\cdot I(n-2) - (n-1)\cdot I(n). \end{aligned}$$

Hence we have:

$$n\cdot I(n) = (n-1)\cdot I(n-2)$$

Hence we have found the recursive formula:

$$I(n) = \frac{n-1}{n}\cdot I(n-2).$$

Using the recursive formula and the base cases we can get an explicit formula for $I(n)$.

Lets discuss now **integration by substitution**.

Lemma

Let $I \subseteq \mathbb{R}$ some interval. Let $f : I \to \mathbb{R}$ be a continuous function. Let $\phi : [a, b] \to \mathbb{R}$ be a differentiable function such that $\phi([a, b] \subseteq I$. Then we have the following:

$$\int_{\phi(a)}^{\phi(b)} f(x)dx = \int_a^b f(\phi(t))\phi'(t)dt.$$

Proof:
The proof is an application of the fundamental theorem, the best way to remember the above formula is by remembering the proof.

Since f is continuous we have by the fundamental theorem: $\int_{\phi(a)}^{\phi(b)} f(x)dx = F(\phi(b)) - F(\phi(a))$, where F is an indefinite integral (Stammfunktion) of f. But by chain rule we have that $F \circ \phi$ is the indefinite integral of $t \mapsto f(\phi(t))\phi'(t)$.
Hence we have:
$\int_a^b f(\phi(t))\phi'(t)dt = F(\phi(b)) - F(\phi(a))$. This proves the lemma.

Lets see examples of substitution:

Exercise 12.3 (b)

We want to calculate:
$\int_1^9 (\sqrt{x} - 1)(x+1)dx$
using substitution.

Let $\phi : [1,3] \to \mathbb{R}, \phi(t) = t^2$. This satisfies the conditions of the lemma. By substitution we get:
$\int_1^9 (\sqrt{x} - 1)(x+1)dx = \int_{\phi(1)}^{\phi(3)} (\sqrt{x} - 1)(x+1)dx =$
$\int_1^3 (t-1)(t^2+1)2tdt.$

Lets review a lemma from the lecture. This is Satz 5.5.1 in M.Burger's lecture notes.

Lemma

Let $f_n : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of integrable functions, which converge uniformly to a function $f : [a, b] \to \mathbb{R}$. Then this function is integrable aswell and we have the following:

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx.$$

Heuristically this means that we can take "the limit inside the function".

Remark

Uniform convergence is not necessary, but sufficient.

Lets see an example where we do not have uniform convergence but where the above lemma still holds.

Example

Let $f_n : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$. $f_n(x) = x^n$. We have seen in Exercise class 8, that this function does not converge uniformly. It only converges pointwise. Remember that the pointwise limit was given by the following function:

$$f : [0,1] \to \mathbb{R}, \quad f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases} .$$

The function f is actually integrable with integral $\int_0^1 f(x)dx = 0$.. This can be seen by taking equi-distant partition, then the lower sum is obviously 0 and the upper sum can be bounded by $\frac{1}{n}$ for any $n \in \mathbb{N}$, hence is also equal to 0.

Now consider the integrals:
$\int_0^1 f_n(x)dx = \int_0^1 x^n dx = \frac{1}{n+1}$.
This clearly converges to 0.

Lets review a lemma from the lecture. This is the so-called Majoranten-Kriterium.

Lemma

Let $f : [a, \infty) \to \mathbb{R}$ *be a bounded function which is integrable on every* $[a, b]$ *with* $b > a$. *Let* $g : [a, \infty) \to \mathbb{R}$ *be an integrable function. Assume we have:* $\mid f(x) \mid \leq g(x), x \geq a$. *Then* f *is integrable at* $[a, \infty)$ *aswell and we have:*

$$\int_0^\infty f(x)dx \leq \int_0^\infty g(x)dx.$$

Lets see an example of the previous lemma.

Exercise 13.2 (a)

Consider the integral:

$$\int_1^{+\infty} \frac{1}{x^3}\sqrt{\frac{x}{x+1}}dx.$$

We want to know if this integral converges.

We have an estimate $\sqrt{\frac{x}{x+1}} \leq 1$. Hence we have:

$$\int_1^{+\infty} \frac{1}{x^3}\sqrt{\frac{x}{x+1}}dx \leq \int_1^{+\infty} \frac{1}{x^3} = \frac{1}{2}.$$

Hence the integral exists and it wil be less than $\frac{1}{2}$. To calculate the

integral, you can apply substitution.