Strong Schemes for the Numerical Solution of Stochastic Differential Equations (SDE's)

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Overview

1. Introduction
2. Stochastic Integration
3. Stochastic Differential Equations
4. Numerical Analysis

Introduction

SDE = Differential equation + Noise term (Randomness)
Simbolycal:

$$\mathrm{d}X_t = a(X_t)\mathrm{d}t + b(X_t)\mathrm{d}W_t.$$

integral form:

$$X_t = x_0 + \int_0^t a(X_s)\,\mathrm{d}s + \underbrace{\int_0^t b(X_s)\,\mathrm{d}W_s}_{\text{Stochastic integral}}, \quad t \in [0, T]$$

$$X_0 = x_0 \text{ initial value}$$

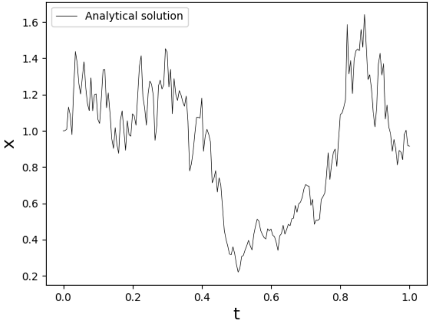
Introduction

Questions:

1. How to give a mathematical interpretation?
2. What are solutions of such equations?
   Existence, uniqueness? Probability distribution?
3. How to solve such equations analytically?

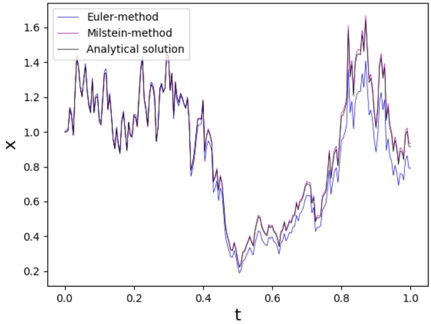
Introduction

Solutions of stochastic differential equations are **stochastic processes** (if it exists).
Figure: one **realization** of a stochastic process.



Main goal: Construction of numerical methods for SDE's

We want to construct numerical methods for the strong approximation of SDE's.
Strong: **pathwise approximation**.



Terminology

Given a probability space $(\Omega, \mathcal{F}, P)$ and T$\in \mathbb{R}$ finite.

Definition (Stochastic process)

Collection of random variables indexed by time:

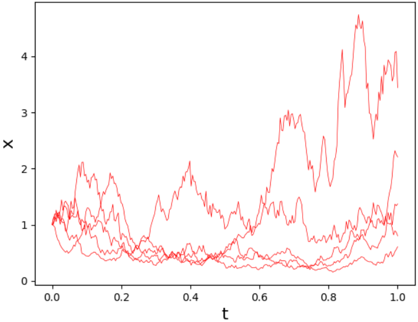
$$X(t, \omega) \colon [0, T] \times \Omega \to \mathbb{R}$$

We will use the notation: $\{X_t\}_{t \in [0,T]}$.
X_t is a random variable for t$\in [0, T]$ fixed.

For each $\omega_0 \in \Omega$ fixed we can define the mapping:

$$t \mapsto X(t, \omega_0)$$

We will call this mapping a realization/**sample path**/trajectory of the stochastic process $\{X_t\}_{t \in [0,T]}$.



Terminology

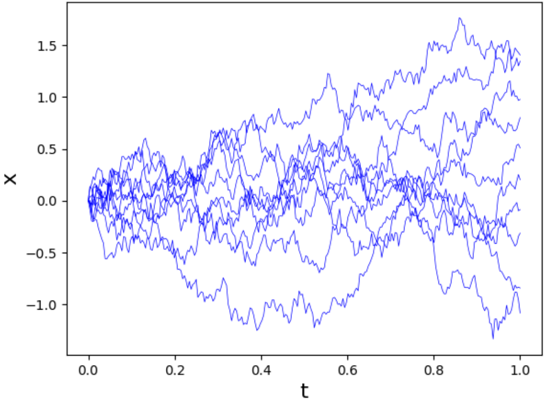
Definition (Wiener process)

The Wiener process $\{W_t\}_{t\in[0,T]}$ is a stochastic process with the following properties:

1. $W_0 = 0$
2. $W_t - W_s \sim \mathcal{N}(0, t-s)$
3. $W_{t_1} - W_{s_1}, W_{t_2} - W_{s2}$ are independent for disjoint intervals $[s_1, t_1],\ [s_2, t_2]$.

Terminology

Possible sample paths of the **discretized** Wiener process.



The first stochastic integral

We want to define:

$$\int_0^T 2W_t \,\mathrm{d}W_t.$$

reasonable approximations:
Given P_n partition of [0,T] with n equidistant time steps.

$$R_1(P_n) := \sum_{k=0}^{n-1} 2W_{t_k} \cdot (W_{t_{k+1}} - W_{t_k})$$

$$R_2(P_n) := \sum_{k=0}^{n-1} 2W_{t_{k+1}} \cdot (W_{t_{k+1}} - W_{t_k})$$

The first stochastic integral

We want to define:

$$\int_0^T 2W_t \, \mathrm{d}W_t.$$

It is possible to show:

Lemma

$$R_1(P_n) \to W_T^2 - T \quad (\textit{in } L^2[\Omega]) \textit{ as } n \to \infty$$

$$R_2(P_n) \to W_T^2 + T \quad (\textit{in } L^2[\Omega]) \textit{ as } n \to \infty$$

Questions:

1. Which definition to use? (Itô)
2. For which stochastic processes $\{X_t\}_{t\in[0,T]}$ is it possible to define the Itô stochastic integral:

$$\int_0^T X_t \,\mathrm{d}W_t \,?$$

Terminology

Definition (Filtration induced by the Wiener process)

Filtration $\mathcal{F}^{\mathcal{W}}{}_t := \sigma(\{W_s \mid 0 \leq s \leq t\})$ for t $\in [0, T]$.
$\{\mathcal{F}^{\mathcal{W}}{}_t\}_{t\in[0,T]}$ is an increasing sequence of σ-algebras.

Definition (Adapted process)

We will call $\{X_t\}_{t\in[0,T]}$ to be $\mathcal{F}^{\mathcal{W}}{}_t$-adapted if it is $\mathcal{F}^{\mathcal{W}}{}_t$-measurable $\forall\ t \in [0, T]$.

Terminology

Definition (L^2-norm for stochastic processes)

$\|\{X_t\}_{t\in[0,T]}\|_{L^2[0,T]} := \mathbb{E}[\int_0^T X_t^2 \mathrm{d}t]^{\frac{1}{2}}$

Definition (Class of adapted, square-integrable processes)

$$C := \{X : [0,\, T] \times \Omega \to \mathbb{R} \mid Properties\}.$$

Properties

1. $\{X_t\}_{t\in[0,T]}$ is measurable
2. $\|\{X_t\}_{t\in[0,T]}\|_{L^2[0,T]} < \infty$
3. $\{X_t\}_{t\in[0,T]}$ is $\mathcal{F}^{\mathcal{W}}{}_t$-adapted

Stochastic integration

For each $\{X_t\}_{t\in[0,T]} \in \mathcal{C}$ the Itô stochastic integral

$$\int_0^T X_t \,\mathrm{d}W_t$$

exists as an element in $L^2[\Omega]$.

Construction:

Via "random step functions"

Stochastic integration

Theorem (Itô isometry)

Let $X_t \in \mathcal{C}$. Then

$$\mathbb{E}[(\int_0^T X_t \,\mathrm{d}W_t)^2] = \mathbb{E}[\int_0^T (X_t)^2 \,\mathrm{d}t]$$

$$\| \int_0^T X_t \,\mathrm{d}W_t \|_{L^2[\Omega]} = \| \{X_t\}_{t\in[0,T]} \|_{L^2[0,T]}$$

Holds only for stochastic integrals in the Itô-sense.

Lemma of Itô

Given a stochastic differential equation (Also called **Itô-process**):

$$X_t = x_0 + \int_0^t a(X_s)\,\mathrm{d}s + \int_0^t b(X_s)\,\mathrm{d}W_s, \quad t \in [0, T]$$

$$X_0 = x_0 \text{ initial value}$$

Given **a smooth map u**:$\mathbb{R} \to \mathbb{R}$.

Question:

1. What is the **representation** of $u(X_t)$?

Lemma of Itô

Given a stochastic differential equation as before. Given a smooth map u:$\mathbb{R} \to \mathbb{R}$.
Then:

$$u(X_t) = u(x_0) + \int_0^t u_x(X_s)a(X_s) + \frac{1}{2}u_{xx}(X_s)b(X_s)^2 \,\mathrm{d}s +$$

$$\int_0^t u_x(X_s)b(X_s)\,\mathrm{d}W_s$$

If we are lucky: the lemma gives us the solution of the SDE.

Stochastic Differential Equations

Consider again a stochastic differential equation:

$$X_t = x_0 + \int_0^t a(X_s)\,\mathrm{d}s + \int_0^t b(X_s)\,\mathrm{d}W_s, \quad t \in [0, T],$$

$$X_0 = x_0 \text{ initial value}$$

Conditions on the coefficients such that the solution $\{X_t\}_{t\in[0,T]}$ exists and is unique.
We are interested in the stochastic process which solves the above equation.
Examples?

Geometric Brownian motion

Set $a(x) = \mu x$ and $b(x) = \sigma x$. Then the SDE reads:

$$X_t = x_0 + \int_0^t \mu X_s \, \mathrm{d}s + \int_0^t \sigma X_s \, \mathrm{d}W_s$$

Using the Itô-lemma we can get the solution for all $t \in [0, T]$:

$$X_t = x_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma W_t).$$

This is the **geometric Brownian motion.**

Ornstein-Uhlenbeck process

Set $a(x) = -\beta x$ and $b(x) = \sigma$. Then the SDE reads:

$$X_t = x_0 + \int_0^t -\beta X_s \, \mathrm{d}s + \int_0^t \sigma \, \mathrm{d}W_s$$

Using the Itô-lemma we can get the solution for all $t \in [0, T]$:

$$X_t = x_0 \exp(-\beta t) + \sigma \int_0^t \exp(-\beta(t-s)) \, \mathrm{d}W_s.$$

This is the **Ornstein-Uhlenbeck process.**

Stochastic Taylor expansions

Given a stochastic differential equation as before:

$$X_t = x_0 + \int_0^t a(X_s)\,\mathrm{d}s + \int_0^t b(X_s)\,\mathrm{d}W_s, \quad t \in [0, T]$$

$$X_0 = x_0 \text{ initial value}$$

Question:

1. What happens if we apply the Itô-lemma on $a(x)$ and $b(x)$?

Stochastic Taylor expansions

$$a(X_s) = a(X_0) + \int_0^s a_x(X_r)a(X_r) + \frac{1}{2}a_{xx}(X_r)b(X_r)^2 \,\mathrm{d}r + \int_0^s a_x(X_r)b(X_r)\,\mathrm{d}W_r$$

$$b(X_s) = b(X_0) + \int_0^s b_x(X_r)a(X_r) + \frac{1}{2}b_{xx}(X_r)b(X_r)^2 \,\mathrm{d}r + \int_0^s b_x(X_r)b(X_r)\,\mathrm{d}W_r$$

Stochastic Taylor expansions

Plugging into the original equation yields:

$$X_t = x_0 + \int_0^t a(X_0)\,\mathrm{d}s + \int_0^t b(X_0)\,\mathrm{d}W_s + R$$

$$X_t = x_0 + a(X_0) \cdot t + b(X_0) \cdot W_t + R$$

$$t \in [0, T]$$

R is the remainder term.
This is the **1. Stochastic Taylor expansion**.

Stochastic Taylor expansions

A third application of the Itô-lemma yields:

$$X_t = x_0 + a(X_0)t + b(X_0)W_t + \underbrace{\frac{1}{2}b_x(X_0)b(X_0)(W_t^2 - t)}_{\text{Additional term}} + R$$

$$t \in [0, T]$$

R is the remainder term.
This is the **2. Stochastic Taylor expansion**.

Stochastic Taylor expansions

Numerical methods for SDE's:
Expand the equation iteratively using the Itô-lemma and remove terms of higher order (in $L^2[\Omega]$-norm).
It is possible to show that these truncated stochastic Taylor expansions converge to X_t in $L^2[\Omega]$.

Euler-Method for SDE's

Euler-Method for SDE's: 1. Stochastic Taylor expansion for n points $t_1, \cdots, t_n \in [0, T]$ and remove the remainder term. Defined recursively:

1. Initialize $\overline{X}_{t_0}^n = x_0$
2. For k = 0 to n-1
    1. $\overline{X}_{t_{k+1}}^n = \overline{X}_{t_k}^n + a(\overline{X}_{t_k}^n)(t_{k+1} - t_k) + b(\overline{X}_{t_k}^n)(W_{t_{k+1}} - W_{t_k})$
3. Linear interpolation.

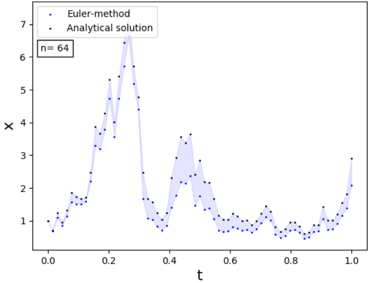
Milstein-Method for SDE's

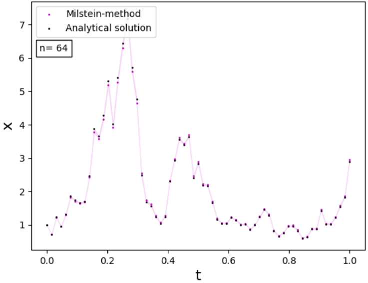
Milstein-Method for SDE's: 2. Stochastic Taylor expansion for n points $t_1, \cdots, t_n \in [0, T]$ and remove the remainder term.
Defined recursively:

1. Initialize $\overline{X}^n_{t_0} = x_0$
2. For k = 0 to n-1
    1. $\overline{X}^n_{t_{k+1}} = \overline{X}^n_{t_k} + a(\overline{X}^n_{t_k})(t_{k+1} - t_k) + b(\overline{X}^n_{t_k})(W_{t_{k+1}} - W_{t_k}) + \frac{1}{2} b(\overline{X}^n_{t_k}) b_x(\overline{X}^n_{t_k})((W_{t_{k+1}} - W_{t_k})^2 - (t_{k+1} - t_k))$
3. Linear interpolation.

Illustration

Approximation of the sample paths of the geometric Brownian motion.
Using the **Euler** and the **Milstein** method.





Numerical analysis

How to measure the **approximation error**?
$\{\overline{X}_t^n\}_{t\in[0,T]}$ is the approximation for n points, linearly interpolated.
$\{X_t\}_{t\in[0,T]}$ is the true, analytical solution.

Approximation error

possible criterion for the approximation error:
$e_n := \mathbb{E}[(X_T - \overline{X}_T^n)^2]^{\frac{1}{2}}$

Numerical analysis

How to measure the **convergence order** of the methods?

Convergence order - Error bound

$$e_n \leq K \cdot n^{-\gamma}$$

For some constants.
γ is called the *strong order of convergence* of the method.

How to estimate γ?
- Analytically
- Simulation

Numerical analysis

It is possible to show:
- Euler-Method has order of convergence 0.5
- Milstein-Method has order of convergence 1.0

The End