ELL205 Project : **Power Spectral Density**

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1 What is the Power Spectral Density $S_{xx}(\omega)$ of a power signal $x(t)$?

1.1 Power Signals

Power Signal has a finite average power

$$0 < \overline{x^2(t)} < \infty$$

where

$$P_{avg} = \overline{x^2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \left|x(t)\right|^2 \, dt$$

and that a periodic signal is a power signal having an average power evaluated over one period

$$P = \overline{x^2(t)} = \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} \left|x(t)\right|^2 \, dt \; = \sum_{n=-\infty}^{+\infty} \left|a_n\right|^2$$

where a_n is the Fourier coefficient of the periodic signal x(t)

1.2 Cross and Auto-correlation of Power signals

Let $x_1(t)$ and $x_2(t)$ be two real power signals. The cross-correlation $R_{x_1x_2}(t)$ is given by

$$R_{x_1x_2}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x_1(t+\tau)x_2(\tau) \, d\tau$$

$$R_{x_1x_2}(-t) = R_{x_1x_2}(t)$$

if $x_1(t)$ and $x_2(t)$ are complex then

$$R_{x_1x_2}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x_1(t+\tau)x_2^*(\tau) \, d\tau$$

$$R_{x_1x_2}(-t) = R^*_{x_1x_2}(t)$$

So, Auto-correlation of a real signal $x(t)$ is given by

$$R_{xx}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x(t+\tau)x(\tau) \, d\tau$$

$$R_{xx}(-t) = R_{xx}(t)$$

$$R_{xx}(0) = \overline{x^2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \left|x(t)\right|^2 \, dt \ = P_{avg}$$

For a real periodic signal $x(t)$ with period T , its auto-correlation $R_{xx}(t)$ is periodic defined by

$$R_{xx}(t) = \frac{1}{T} \int_0^T x(t+\tau)x(\tau)\, d\tau = \frac{1}{T} \int_0^T x(\tau) \sum_{n=-\infty}^{+\infty} a_n e^{\iota n\omega_o(t+\tau)} \, d\tau = \sum_{n=-\infty}^{\infty} |a_n|^2 \, e^{\iota n\omega_o t}, \omega_o = \frac{2\pi}{T}$$

1.3 Power Spectral Density

For a real power signal $x(t)$ the power spectral density denoted by $S_{xx}(\omega)$ is by definition the Fourier transform of the auto-correlation function.

$$S_{xx}(\omega) = \mathcal{F}[R_{xx}(t)]$$

Since $R_{xx}(t)$ is real and even its transform $S_{xx}(\omega)$ is also real and even . We have

$$S_{xx}(\omega) = 2 \int_0^\infty R_{xx}(t) cos(\omega t)\, dt$$

$$R_{xx}(t) = \frac{1}{\pi} \int_0^\infty S_{xx}(\omega) cos(\omega t)\, d\omega$$

Power Spectral density of a periodic Signal x(t) is

$$S_{xx}(\omega) = \mathcal{F}[R_{xx}(t)] = \mathcal{F}[\sum_{n=-\infty}^{\infty} |a_n|^2 \, e^{\iota n\omega_o t}] = 2\pi \sum_{n=-\infty}^{+\infty} |a_n|^2 \, \delta(\omega - n\omega_o)$$

2 Power Spectral Density of different power signals $x(t)$

2.1 x(t)=K, where K is a constant

$$R_{xx}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} K^2 \, d\tau = K^2$$

$$S_{xx}(\omega) = \mathcal{F}[R_{xx}(t)] = 2\pi K^2 \delta(\omega)$$

The total average power by direct evaluation is $P_{avg} = K^2$ and alternatively can be calculated

$$P_{avg} = \overline{x^2(t)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{xx}(\omega)\, d\omega \ = K^2 = R_{xx}(t=0)$$

2.2 x(t)=Ku(t), u(t) is the unit step function

$$R_{xx}(t) = \lim_{T\to\infty} \frac{K^2}{2T} \int_{-T}^{+T} u(t+\tau)u(\tau)\, d\tau$$

for $t \geq 0$ we have

$$R_{xx}(t) = \lim_{T\to\infty} \frac{K^2}{2T} \int_{0}^{T} d\tau = \frac{K^2}{2}$$

as $R_{xx}(t)$ is even , So for $t < 0$ also $R_{xx}(t) = \frac{K^2}{2}$

$$S_{xx}(\omega) = \mathcal{F}[R_{xx}(t)] = 2\pi \frac{K^2}{2} \delta(\omega) = \pi K^2 \delta(\omega)$$

2.3 x(t)=Kcos(ω_o t)

$$x(t) = \frac{K}{2}(e^{\iota\omega_o t} + e^{-\iota\omega_o t})$$

$$a_n = \begin{cases} \frac{K}{2}, & \text{if } n = \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

$$S_{xx}(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} |a_n|^2 \, \delta(\omega - n\omega_o) = \pi \frac{K^2}{2} (\delta(\omega - \omega_o) + \delta(\omega + \omega_o))$$

2.4 $x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n\mathrm{T})$

$$a_n = \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} \delta(t) e^{-\iota\omega_o n t}\, dt \; = \frac{1}{T}$$

$$S_{xx}(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} |a_n|^2 \, \delta(\omega - n\omega_o) = \frac{2\pi}{T^2} \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_o)$$

2.5 $x(t)$ is periodic ramp signal with period T=1

$$a_n = \frac{1}{T} \int_{0}^{T} x(t) e^{-\iota\omega_o n t}\, dt \; = \int_{0}^{1} t e^{-\iota 2\pi n t}\, dt \; = \begin{cases} \int_0^1 t\, dt \text{ if } n = 0 \\ [\frac{t e^{-2\pi\iota n t}}{-2\pi\iota n}]_{t=0}^{t=1} + \frac{1}{2\pi\iota n} \int_0^1 e^{-2\pi\iota n t}\, dt \text{ , if } n \neq 0 \end{cases} = \begin{cases} \frac{1}{2}, & \text{if } n = 0 \\ \frac{\iota}{2\pi n}, & \text{if } n \neq 0 \end{cases}$$

$$S_{xx}(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} |a_n|^2 \, \delta(\omega - n\omega_o) = \frac{\pi}{2}\delta(\omega) + \sum_{n=-\infty, n\neq 0}^{+\infty} \frac{1}{2\pi n^2} \delta(\omega - n\omega_o)$$

2.6 x(t) is symmetric square wave of width a with period T, $a \leq \frac{T}{2}$

$$a_n = \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t) e^{-\iota\omega_o n t}\, dt \; = \frac{1}{T} \int_{-a}^{+a} e^{-\iota\omega_o n t}\, dt \; = \begin{cases} \frac{2a}{T}, & \text{if } n = 0 \\ \frac{\sin(\omega_o a n)}{n\pi}, & \text{if } n \neq 0 \end{cases}$$

$$S_{xx}(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} |a_n|^2 \, \delta(\omega - n\omega_o) = \frac{2a^2\omega_o^2}{\pi} \delta(\omega) + \sum_{n=-\infty, n\neq 0}^{n=+\infty} 2\frac{\sin(\omega_o a n)^2}{n^2\pi} \delta(\omega - n\omega_o)$$

Objective :

- Understand the theory related to power spectral density including autocorrelation and cross-correlation functions
- Find out power spectral density of some selected power signals
- Learn about the PSD of white noise
- Apply the theory learned to come up with some useful application

What we have done so far :

- We studied the theory related to power spectral density and the way to find out PSD for signals from the links provided in the google sheet and extra references.
- We also learned about white noise and its PSD.
- Using the concept of autocorrelation functions and Fourier transform, we have managed to find the PSD of the following power signals so far :

    1. x(t)=K, where K is a constant
    2. x(t)=Ku(t), u(t) is the unit step function
    3. x(t)=Kcos(ωo t)
    4. x(t) = periodic impulse train with period T
    5. x(t) is periodic ramp signal with period T=1
    6. x(t) is symmetric square wave of width a with period T, a ≤ T /2

- Currently we are working on writing Matlab
- codes to plot the PSD graphs

References and relevant links used :

1. https://www.researchgate.net/publication/300166409_SIGNAL_PROCESSING_FOR_POWER_SPECTRAL_DENSITY_PSD
2. https://www.youtube.com/watch?v=-Nt0FaofWL4
3. Signals and systems - Alan V. Oppenheim, Alan S. Willsky, with S. Hamid Nawab. – 2nd ed
4. Discrete time Signal Processing – Alan V. Oppenheim , Ronald W. Schafer – 3rd ed

Power Spectral Density Code and Underlying algorithm:

Power spectral density (PSD) shows how the power of a signal is distributed over frequencies.

Let's consider a discrete signal $\mathbf{x} = (x_1, \ldots, x_N)$, where N is the length of the signal. This can be either the complete signal or a N-length window of a bigger signal.

Furthermore, let's assume that the signal is sampled at frequency $F = \frac{1}{\Delta t}$, where Δt is the sample interval in seconds.

The total signal or window duration is given by $T = (N-1)\Delta t$.

This leads us to definition of discrete-time power spectral density at frequency f

$$\bar{S}_{xx}(f) = \lim_{N\to\infty} \frac{(\Delta t)^2}{T} \left| \sum_{n=1}^{N} x_n e^{-i2fn\Delta t} \right|^2$$

Note that the theoretical PSD assumes that the signal length approaches infinity. This is of course not the case in real life. Thus, in practice, we consider finite N.

This is not PSD but rather a periodogram, which converges to the actual PSD as $N \to \infty$.

Taking into account the finite signal length, in practice, we consider PSD via periodogram as

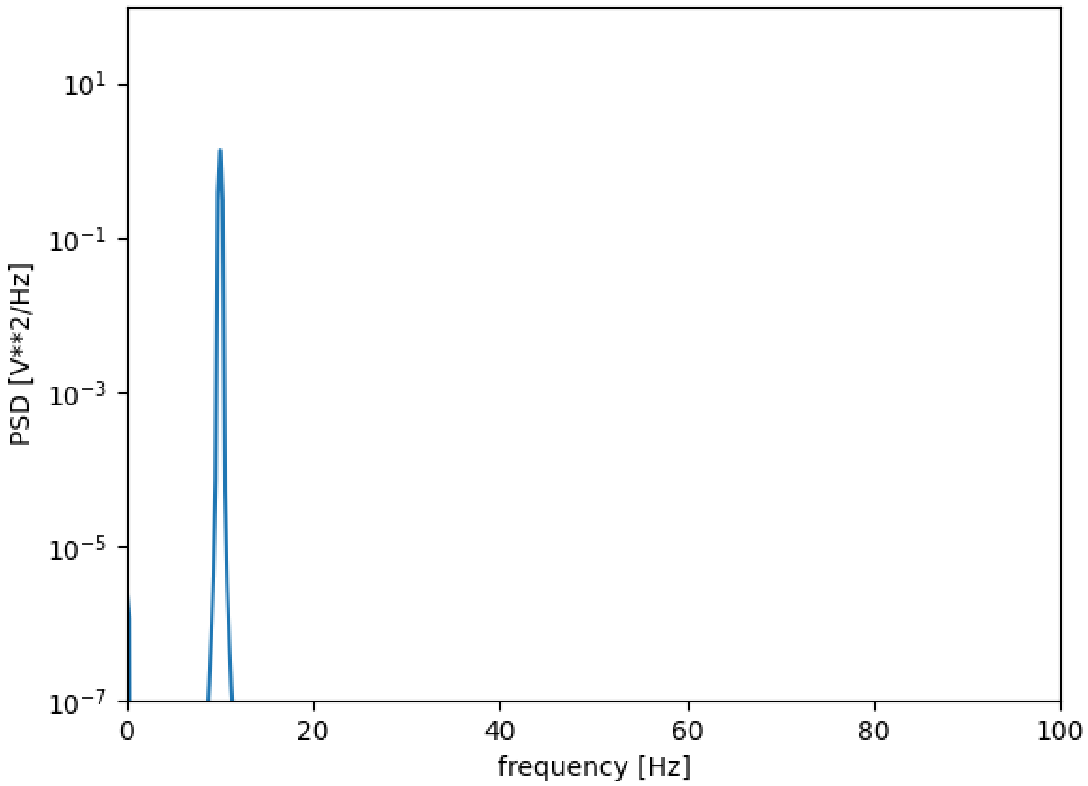
$$S_{xx}(f) = \frac{(\Delta t)^2}{T} \left| \sum_{n=1}^{N} x_n e^{-i2fn\Delta t} \right|^2$$

Estimating PSD using Scipy & Welch's method (Python Code) :

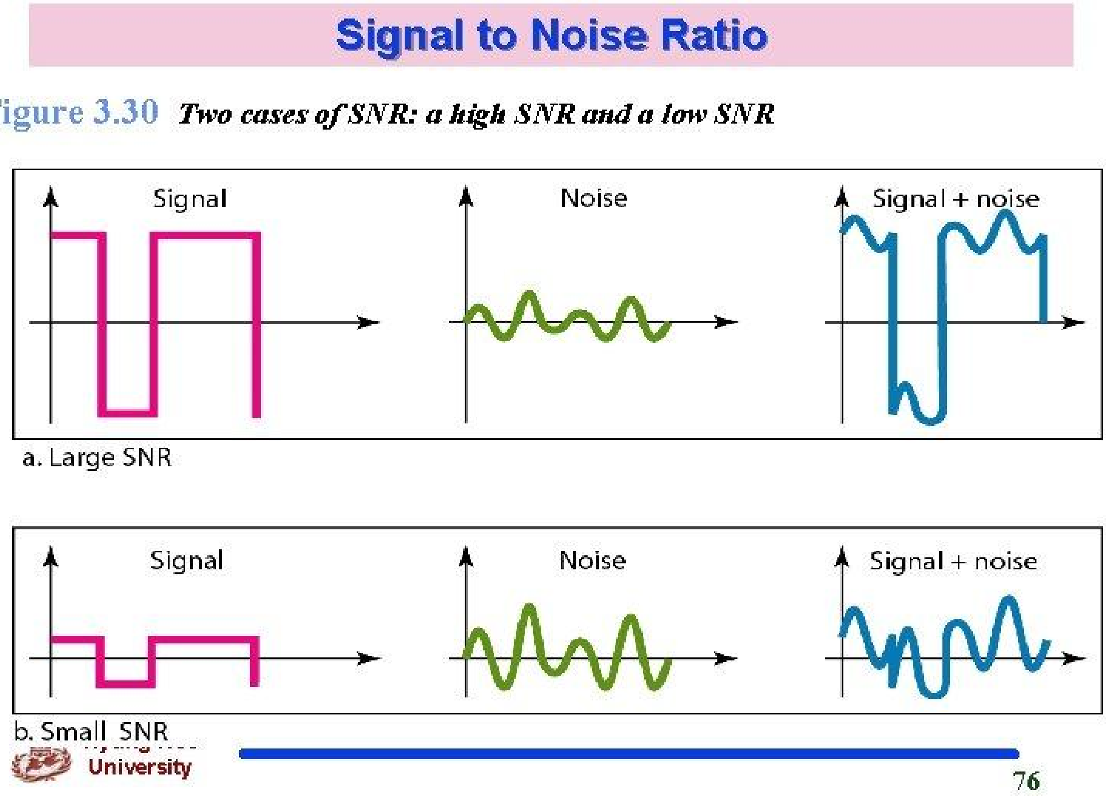
PSD of a cosine wave (Python Code):

```
fs = 1000
F1=10
T=10
t = np.r_[0:T:(1/fs)] # Sample time
signal = np.cos(2 * F1 * np.pi * t)
# (S, f) = plt.psd(signal, Fs)
(f, S) = scipy.signal.welch(signal, fs, nperseg=1024*4)
plt.semilogy(f, S)
plt.ylim([1e-7, 1e2])
plt.xlim([0, 100])
plt.xlabel('frequency [Hz]')
plt.ylabel('PSD [V**2/Hz]')
plt.show()
```

Output of the Code as a PSD plot of the signal :



Noise Filtering using spectral analysis :



The PSD of a discrete-time noise signal is given by the FFT of its autocorrelation R(k).

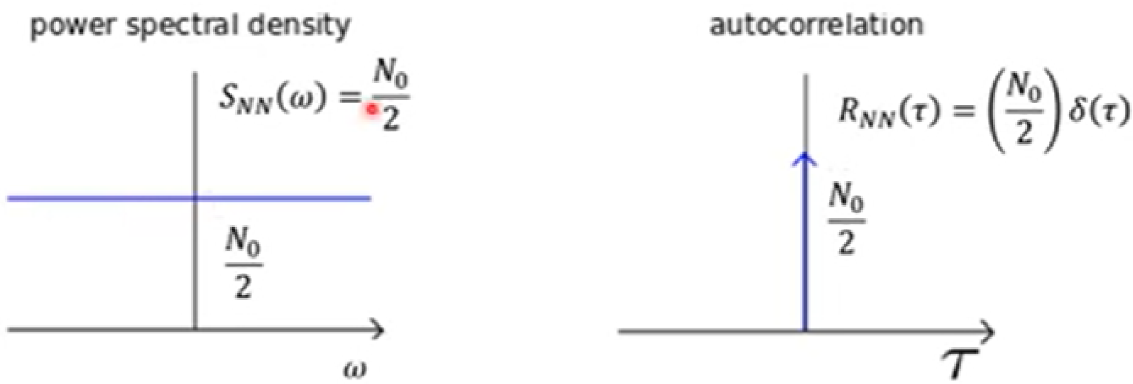
$$S(f) = \sum_{k=1}^{N} R(k) e^{-i2\pi f k}$$

$$R(k) = \frac{1}{N} \sum_{n=1}^{N} x(n) x(n-k)$$

For discrete-time signals, FFT is the most convenient tool for calculating spectral noise power distribution. However, we go for the FFT of the autocorrelation function instead of computing the FFT of the direct discrete-time noise signal. Why!?

Because, in random stochastic signals such as noise, the past and present values are statistically analysed to predict future occurrences. The correlation function is a statistical function that investigates the similarity between two random functions. When we measure the similarity of random discrete-time function x[n] with itself vs x[n-k], the correlation function becomes an autocorrelation function. The autocorrelation function is the best statistical quantity to represent the causality of the stochastic noise signal and hence, find its way into PSD computation. The autocorrelation function R(k) reduces to total average noise power when we measure the similarity of x[n] with itself, i.e, by putting k=0.

White Noise (Constant PSD):



It is called white because it has all different colours (all frequencies) .