Advanced Topics in Machine Learning, GI13, 2010/11

Answer any THREE questions. Each question is worth 20 marks. Use separate answer books for PART A and PART B. **Gatsby PhD students only**: answer *either* TWO questions from PART A and ONE question from PART B; *or* ONE question from PART A and TWO questions from PART B.

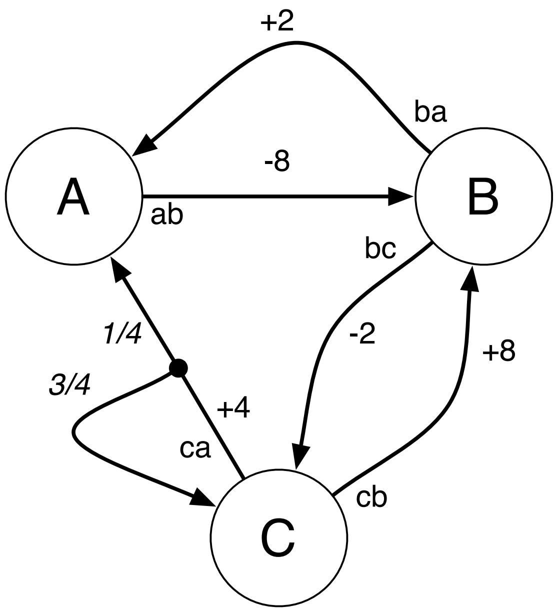
Marks for each part of each question are indicated in square brackets

Calculators are NOT permitted

Part A: Kernel Methods

Part B: Reinforcement Learning

1. Consider the following Markov Decision Process (MDP) with discount factor $\gamma = 0.5$. Upper case letters A, B, C represent states; arcs represent state transitions; lower case letters ab, ba, bc, ca, cb represent actions; signed integers represent rewards; and fractions represent transition probabilities.



- Define the *state-value function* $V^{\pi}(s)$ for a discounted MDP

[1 marks]

Answer: $V^{\pi}(s) = \mathbb{E}_{\pi}\left[r_{t+1} + \gamma r_{t+2} + ... | s_t = s\right]$

- Write down the *Bellman expectation equation* for state-value functions

[2 marks]

Answer:

$$V^{\pi}(s) = \mathbb{E}_{\pi}\left[r_{t+1} + \gamma V^{\pi}(s_{t+1}) | s_t = s\right] \textbf{ or}$$

$$V^{\pi}(s) = \sum_{a \in \mathcal{A}} \pi(s,a) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a V^{\pi}(s') \right) \textbf{ or}$$

$$V^{\pi} = \mathcal{R}^{\pi} + \gamma \mathcal{P}^{\pi} V^{\pi}$$

- Consider the uniform random policy π_1(s,a) that takes all actions from state s with equal probability. Starting with an initial value function of $V_1(A) = V_1(B) = V_1(C) = 2$, apply one synchronous iteration of iterative policy evaluation (i.e. one backup for each state) to compute a new value function $V_2(s)$

[3 marks]

Answer:

- $V_2(A) = -8 + 0.5V_1(B) = -7$
  $V_2(B) = 0.5(2 + 0.5V_1(A)) + 0.5(-2 + 0.5V_1(C)) = 1$
  $V_2(C) = 0.5(8 + 0.5V_1(B)) + 0.5(4 + 0.5(1/4V_1(A) + 3/4V_1(C))) = 7$

- Apply one iteration of greedy policy improvement to compute a new, deterministic policy $\pi_2(s)$

[2 marks]

Answer: $\pi_2(A) = ab$
$Q_2(B, ba) = 2 + 0.5V_2(A) = -1.5,$
$Q_2(B, bc) = -2 + 0.5V_2(C) = 1.5 \implies \pi_2(B) = bc$
$Q_2(C, ca) = 4 + 0.5(1/4V_2(A) + 3/4V_2(B)) = 5.75,$
$Q_2(C, cb) = 8 + 0.5V_2(B) = 8.5 \implies \pi_2(C) = cb$

- Consider a deterministic policy π(s). Prove that if a new policy π' is greedy with respect to V^{π} then it must be better than or equal to π, i.e. $V^{\pi'}(s) \geq V^{\pi}(s)$ for all s; and that if $V^{\pi'}(s) = V^{\pi}(s)$ for all s then π' must be an optimal policy.

[5 marks]

Answer: Greedy policy improvement is given by $\pi'(s) = \operatorname*{argmax}_{a \in \mathcal{A}} Q^{\pi}(s,a)$. This is an improvement over one step because for any state s, $Q^{\pi}(s, \pi'(s)) = \max_{a \in \mathcal{A}} Q^{\pi}(s,a) \geq$

$Q^{\pi}(s, \pi(s)) = V^{\pi}(s)$. It therefore improves the value function, $V^{\pi}(s) \leq Q^{\pi}(s, \pi'(s)) = \mathbb{E}_{\pi'}[r_{t+1} + \gamma V^{\pi}(s_{t+1}) | s_t = s] \leq \mathbb{E}_{\pi'}[r_{t+1} + \gamma Q^{\pi}(s_{t+1}, \pi'(s_{t+1})) | s_t = s] \leq \ldots \leq \mathbb{E}_{\pi'}[r_{t+1} + \gamma r_{t+2} + \ldots | s_t = s] = V^{\pi'}(s)$. If improvements stop, i.e. $Q^{\pi}(s, \pi'(s)) = \max_{a \in \mathcal{A}} Q^{\pi}(s, a) = Q^{\pi}(s, \pi(s)) = V^{\pi}(s)$ then V^{π} has satisfied the Bellman optimality equation, so π and must be an optimal policy.

- Define the *optimal state-value function* $V^*(s)$ for an MDP

[1 marks]

Answer: $V^*(s) = \max_{\pi} V^{\pi}(s)$

- Write down the *Bellman optimality equation* for state-value functions

[2 marks]

Answer:

$$V^*(s) = \max_a \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a V^*(s') \textbf{ or}$$

$$V^*(s) = \max_{\pi} \mathbb{E}_{\pi}[r_{t+1} + \gamma V^*(s_{t+1}) | s_t = s] \textbf{ or}$$

$$V^* = \max_a \mathcal{R}^a + \gamma \mathcal{P}^a V^*$$

- Starting with an initial value function of $V_1(A) = V_1(B) = V_1(C) = 2$, apply one synchronous iteration of value iteration (i.e. one backup for each state) to compute a new value function $V_2(s)$.

[3 marks]

Answer: $V_2(A) = -7, V_2(B) = 3, V_2(C) = 9$

- Is your new value function $V_2(s)$ optimal? Justify your answer.

[1 marks]

Answer: Applying one more iteration, $V_3(A) = -6.5 \neq V_2(A)$ hence V_2 is not a fixed point of the Bellman optimality equation.

[Total 20 marks]

2. Consider an undiscounted Markov Reward Process with two states A and B. The transition matrix and reward function are unknown, but you have observed two sample episodes:

$$A+3 \to A+2 \to B-4 \to A+4 \to B-3 \to \text{terminate}$$
$$B-2 \to A+3 \to B-3 \to \text{terminate}$$

In the above episodes, sample state transitions and sample rewards are shown at each step, e.g. $A+3 \to A$ indicates a transition from state A to state A, with a reward of $+3$.

- Using first-visit Monte-Carlo evaluation, estimate the state-value function $V(A), V(B)$

[2 marks]

Answer:

$$V(A) = 1/2(2+0) = 1$$
$$V(B) = 1/2(-3+-2) = -5/2$$

- Using every-visit Monte-Carlo evaluation, estimate the state-value function $V(A), V(B)$

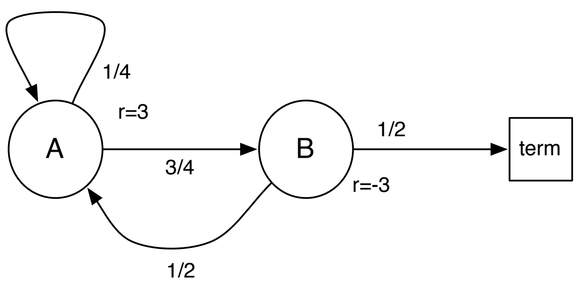
[2 marks]

Answer:

$$V(A) = 1/4(2+-1+1+0) = 1/2$$
$$V(B) = 1/4(-3+-3+-2+-3) = -11/4$$

- Draw a diagram of the Markov Reward Process that best explains these two episodes (i.e. the model that maximises the likelihood of the data - although it is not necessary to prove this fact). Show rewards and transition probabilities on your diagram.

[4 marks]



Answer:

- Define the Bellman equation for a Markov reward process

[2 marks]

Answer:

$$V(s) = \mathbb{E}[r_{t+1} + \gamma V(s_{t+1}) | s_t = s] \textbf{ or}$$
$$V(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} V^{\pi}(s') \textbf{ or}$$
$$V = \mathcal{R} + \gamma \mathcal{P} V$$

- Solve the Bellman equation to give the true state-value function $V(A), V(B)$. Hint: solve the Bellman equations directly, rather than iteratively. **Answer:**

$$V(A) = 3 + 1/4V(A) + 3/4V(B)$$
$$V(B) = -3 + 1/2V(A)$$
$$V(A) = 2$$
$$V(B) = -2$$

[4 marks]

- What value function would batch TD(0) find, i.e. if TD(0) was applied repeatedly to these two episodes?

[2 marks]

Answer: The solution to the above MDP,

$$V(A) = 2$$
$$V(B) = -2$$

- What value function would batch TD(1) find, using accumulating eligibility traces?

[2 marks]

Answer: The same as every-visit Monte-Carlo

$$V(A) = 1/2$$
$$V(B) = -11/4$$

- What value function would LSTD(0) find?

[2 marks]

Answer: The same as batch TD(0)

$$V(A) = 2$$

$$V(B) = -2$$

[Total 20 marks]

3. A rat is involved in an experiment. It experiences one episode. At the first step it hears a bell. At the second step it sees a light. At the third step it both hears a bell and sees a light. It then receives some food, worth +1 reward, and the episode terminates on the fourth step. All other rewards were zero. The experiment is undiscounted.

- Represent the rat's state s by a vector of two binary features, $bell(s) \in \{0,1\}$ and $light(s) \in \{0,1\}$. Write down the sequence of feature vectors corresponding to this episode.

  [3 marks]

  Answer: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- Approximate the state-value function by a linear combination of these features with two parameters: $b \cdot bell(s) + l \cdot light(s)$. If $b = 2$ and $l = -2$ then write down the sequence of approximate values corresponding to this episode.

  [3 marks]

  Answer: $2, -2, 0$ and also 0 for the terminal state

- Define the λ-return v_t^λ

  [1 marks]

  Answer:

$$v_t^{(n)} = r_{t+1} + \gamma r_{t+2} + ... + \gamma^{n-1} r_{t+n} + \gamma^n V(s_{t+n})$$
$$v_t^\lambda = (1-\lambda) \sum_{n=1}^{\infty} \lambda^{n-1} v_t^{(n)}$$

- Write down the sequence of λ-returns v_t^λ corresponding to this episode, for $\lambda = 0.5$ and $b = 2, l = -2$

  [3 marks]

  Answer:

$$v_1^\lambda = 0.5(-2 + 0.5 \times 0 + 0.5 \times 1) = -3/4$$
$$v_2^\lambda = 0.5(0 + 1 \times 1) = 1/2$$
$$v_3^\lambda = 0.5(2 \times 1) = 1$$

- Using the forward-view TD(λ) algorithm and your linear function approximator, what are the sequence of updates to weight b? What is the total update to weight b? Use $\lambda = 0.5, \gamma = 1, \alpha = 0.5$ and start with $b = 2, l = -2$

  [3 marks]

  Answer:

  $$\begin{aligned}
  \Delta b_1 &= \alpha(v_1^\lambda - V(s_1))bell(s_1) = 0.5(-3/4 - 2)1 = -11/8 \\
  \Delta b_2 &= \alpha(v_2^\lambda - V(s_2))bell(s_2) = 0.5(1/2 - -2)0 = 0 \\
  \Delta b_3 &= \alpha(v_3^\lambda - V(s_3))bell(s_3) = 0.5(1 - 0)1 = 1/2 \\
  \sum \Delta b &= (-11/8 + -1/2) = -7/8
  \end{aligned}$$

- Define the TD(λ) *accumulating eligibility trace* $\mathbf{e_t}$ when using linear value function approximation

  [1 marks]

  Answer: $e_t = \gamma\lambda e_{t-1} + \phi(s)$

- Write down the sequence of eligibility traces $\mathbf{e_t}$ corresponding to the bell, using $\lambda = 0.5, \gamma = 1$

  [3 marks]

  Answer: $1, 1/2, 5/4$

- Using the backward-view TD(λ) algorithm and your linear function approximator, what are the sequence of updates to weight b? (Use offline updates, i.e. do not actually change your weights, just accumulate your updates). What is the total update to weight b? Use $\lambda = 0.5, \gamma = 1, \alpha = 0.5$ and start with $b = 2, l = -2$

  [3 marks]

  $$\begin{aligned}
  \Delta b_1 &= \alpha\delta_1 e_1 = 0.5(0 + -2 - 2)1 = -2 \\
  \Delta b_2 &= \alpha\delta_2 e_2 = 0.5(0 + 0 - -2)1/2 = 1/2 \\
  \Delta b_3 &= \alpha\delta_3 e_3 = 0.5(1 + 0 - 0)5/4 = 5/8 \\
  \sum \Delta b &= (-2 + 1/2 + 5/8) = -7/8
  \end{aligned}$$

[Total 20 marks]

END OF PAPER