DM545

Linear and Integer Programming

Lecture 2

The Simplex Method

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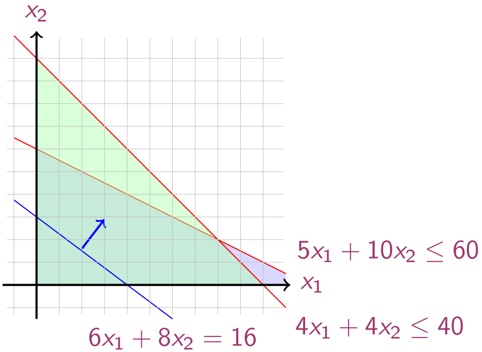
Outline

Mathematical Model

Machines/Materials A and B
Products 1 and 2

$$
\begin{array}{rrcrcr}
\max & 6x_1 & + & 8x_2 & & \\
 & 5x_1 & + & 10x_2 & \leq & 60 \\
 & 4x_1 & + & 4x_2 & \leq & 40 \\
 & & & x_1 & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

Graphical Representation:



Outline

Linear Programming

Abstract mathematical model:

Decision Variables

Criterion

Constraints

$$
\begin{array}{rrcll}
\text{objective func.} & \max / \min c^T \cdot x & & & c \in \mathbb{R}^n \\
\text{constraints} & A \cdot x & \substack{\geq \\ = \\ \leq} & b & A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \\
 & x & \geq & 0 & x \in \mathbb{R}^n, 0 \in \mathbb{R}^n
\end{array}
$$

- Any vector $x \in \mathbb{R}^n$ satisfying all constraints is a feasible solution.
- Each $x^* \in \mathbb{R}^n$ that gives the best possible value for $c^T x$ among all feasible x is an optimal solution or optimum
- The value $c^T x^*$ is the optimum value

In Matrix Form

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \dots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & \leq & b_2 \\
 & \dots & & & & & & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \dots, x_n & \geq & 0
\end{array}
$$

$$
c^T = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}
$$

$$
\begin{array}{rrcl}
\max & z & = & c^T x \\
 & Ax & = & b \\
 & x & \geq & 0
\end{array}
$$

$$
A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{31} & a_{32} & \dots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

Definitions

- $\mathbb{N}$ natural numbers, $\mathbb{Z}$ integer numbers, $\mathbb{Q}$ rational numbers, $\mathbb{R}$ real numbers

- column vector and matrices
  scalar product: $y^T x = \sum_{i=1}^n y_i x_i$

- linear combination

$$\begin{aligned} x &\in \mathbb{R}^k \\ x_1, \dots, x_k &\in \mathbb{R} \\ \lambda = (\lambda_1, \dots, \lambda_k)^T &\in \mathbb{R}^k \end{aligned} \qquad x = \sum_{i=1}^k \lambda_i x_i$$

  moreover:

$$\begin{aligned} \lambda \geq 0 & \qquad \text{conic combination} \\ \lambda^T 1 = 1 \quad (\textstyle\sum_{i=1}^k \lambda_i = 1) & \qquad \text{affine combination} \\ \lambda \geq 0 \text{ and } \lambda^T 1 = 1 & \qquad \text{convex combination} \end{aligned}$$

Definitions

- set S is linear independent if no element of it can be expressed as combination of the others
  Eg: $S \subseteq \mathbb{R} \implies$ max n lin. indep.

- rank of a matrix for columns (= for rows)
  if (m, n)-matrix has rank $= \min\{m, n\}$ then the matrix is full rank
  if (n, n)-matrix is full rank then it is regular and admits an inverse

- $G \subseteq \mathbb{R}^n$ is an hyperplane if $\exists a \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$:

  $$G = \{x \in \mathbb{R}^n \mid a^T x = \alpha\}$$

- $H \subseteq \mathbb{R}^n$ is an halfspace if $\exists a \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$:

  $$H = \{x \in \mathbb{R}^n \mid a^T x \leq \alpha\}$$

  ($a^T x = \alpha$ is a supporting hyperplane of H)

Definitions

- a set $S \subset \mathbb{R}$ is a polyhedron if $\exists m \in \mathbb{Z}^+, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$:

  $$P = \{x \in \mathbb{R} \mid Ax \leq b\} = \cap_{i=1}^{m} \{x \in \mathbb{R}^n \mid A_{i\cdot}x \leq b_i\}$$

- a polyhedron P is a polytope if it is bounded: $\exists B \in \mathbb{R}, B > 0$:

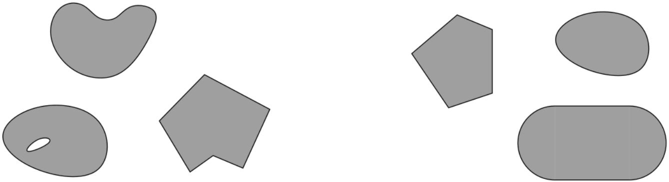
  $$p \subseteq \{x \in \mathbb{R}^n \mid\| x \|\leq B\}$$

- Theorem: every polyhedron $P \neq \mathbb{R}^n$ is determined by finitely many halfspaces

Definitions

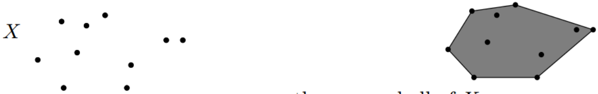
- General optimization problem:
  $\max\{\varphi(x) \mid x \in F\}$, $\qquad F$ is feasible region for x
- If A and b are rational numbers, $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is a rational polyhedron
- convex set: if $x, y \in P$ and $0 \leq \lambda \leq 1$ then $\lambda x + (1-\lambda)y \in P$
- convex function if its epigraph $\{(x, y) \in \mathbb{R}^2 : y \geq f(x)\}$ is a convex set or $f : X \to \mathbb{R}$, if $\forall x, y \in X, \lambda \in [0, 1]$ it holds that
  $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$

Definitions



nonconvex convex

- Given a set of points $X \subseteq \mathbb{R}^n$ the convex hull **conv**(X) is the convex linear combination of the points



the convex hull of X

$$\textbf{conv}(X) = \{\lambda_1\vec{x}_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n | \vec{x}_i \in X; \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0 \text{ and } \sum_i \lambda_i = 1\}$$

Definitions

- A face of P is $F = \{x \in P | ax = \alpha\}$. Hence F is either P itself or the intersection of P with a supporting hyperplane. It is said to be proper if $F \neq \emptyset$ and $F \neq P$.
- A point x for which $\{x\}$ is a face is called a vertex of P and also a basic solution of $Ax \leq b$
- A facet is a maximal face distinct from P
  $cx \leq d$ is facet defining if $cx = d$ is a supporting hyperplane of P

Linear Programming Problem

Input: a matrix $A \in \mathbb{R}^{m \times n}$ and column vectors $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

Task: optimization $c^T x$ subject to $Ax \leq b$

1. decide that $\{x \in \mathbb{R}^n; Ax \leq b\}$ is empty (prob. infeasible), or
2. find a column vector $x \in \mathbb{R}^n$ such that $Ax \leq b$ and $c^T x$ is max, or
3. decide that for all $\alpha \in \mathbb{R}$ there is an $x \in \mathbb{R}^n$ with $Ax \leq b$ and $c^T x > \alpha$ (prob. unbounded)

1. $F = \emptyset$
2. $F \neq \emptyset$ and $\exists$ solution
    1. one solution
    2. infinite solution
3. $F \neq \emptyset$ and $\nexists$ solution

Linear Programming and Linear Algebra

- Linear algebra: linear equations (Gaussian elimination)
- Integer linear algebra: linear diophantine equations
- Linear programming: linear inequalities (simplex method)
- Integer linear programming: linear diophantine inequalities

Outline

1. Definitions and Basics
2. Fundamental Theorem of LP
3. Gaussian Elimination
4. Simplex Method
   - Standard Form
   - Basic Feasible Solutions
   - Algorithm

Fundamental Theorem of LP

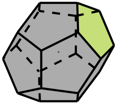
Theorem (Fundamental Theorem of Linear Programming)

Given:

$$\min\{c^T x \mid x \in P\} \;\; \textit{where } P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

If P is a bounded polyhedron and not empty and x^* is an optimal solution to the problem, then:

- *x^* is an extreme point (vertex) of P, or*
- *x^* lies on a face $F \subset P$ of optimal solution*



Proof:

- assume x^* not a vertex of P then $\exists$ a ball around it still in P. Show that a point in the ball has better cost

- if x^* is not a vertex then it is a convex combination of vertices. Show that all points are also optimal.

Implications:

- the optimal solution is at the intersection of hyperplanes supporting halfspaces.
- hence finitely many possibilities
- Solution method: write all inequalities as equalities and solve all $\binom{n}{m}$ systems of linear equalities (n # variables, m # constraints)
- for each point we need then to check if feasible and if best in cost.
- each system is solved by Gaussian elimination

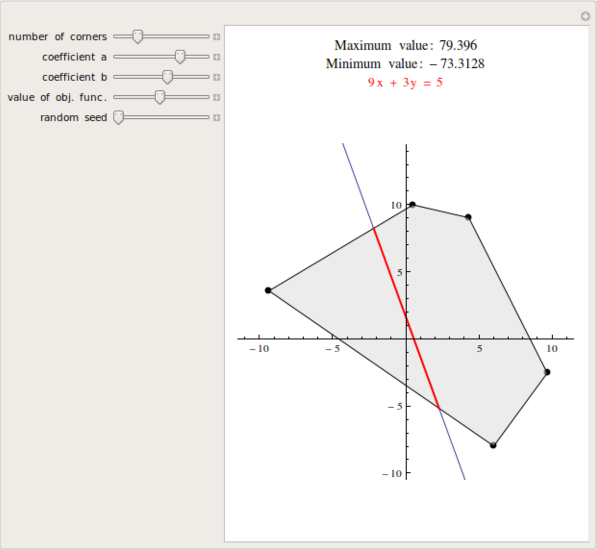
Simplex Method

1. find a solution that is at the intersection of some n hyperplanes

2. try systematically to produce the other points by exchanging one hyperplane with another

3. check optimality, proof provided by duality theory

Demo


number of corners
coefficient a
coefficient b
value of obj. func.
random seed
Maximum value: 79.396
Minimum value: – 73.3128
9 x + 3y = 5
10
5
– 5
– 10
– 10
– 5
5
10

Outline

Gaussian Elimination

1. Forward elimination
   reduces the system to triangular (row echelon) form (or degenerate)
   elementary row operations (or LU decomposition)
2. back substitution

Example:

$$
\begin{array}{rcrcrcrr}
2x & + & y & - & z & = & 8 & (I) \\
-3x & - & y & + & 2z & = & -11 & (II) \\
-2x & + & y & + & 2z & = & -3 & (III)
\end{array}
$$

```
|-----------+---+-----+-----+---|
|           | 2 |   1 |  -1 | 8 |
| 3/2 I+II  | 0 | 1/2 | 1/2 | 1 |
| I+III     | 0 |   2 |   1 | 5 |
|-----------+---+-----+-----+---|
```

$$\begin{array}{rcrcrcrr} 2x & + & y & - & z & = & 8 & (I) \\ & + & \frac{1}{2}y & + & \frac{1}{2}z & = & 1 & (II) \\ & + & 2y & + & 1z & = & 5 & (III) \end{array}$$

```
|-----------+---+-----+-----+---|
|           | 2 |   1 |  -1 | 8 |
|           | 0 | 1/2 | 1/2 | 1 |
| -4 II+III | 0 |   0 |  -1 | 1 |
|-----------+---+-----+-----+---|
```

$$\begin{array}{rcrcrcrr} 2x & + & y & - & z & = & 8 & (I) \\ & + & \frac{1}{2}y & + & \frac{1}{2}z & = & 1 & (II) \\ & & & - & z & = & 1 & (III) \end{array}$$

```
|---+-----+-----+---|
| 2 |   1 |  -1 | 8 |
| 0 | 1/2 | 1/2 | 1 |
| 0 |   0 |  -1 | 1 |
|---+-----+-----+---|
```

$$\begin{array}{rcrcrcrr} 2x & + & y & - & z & = & 8 & (I) \\ & + & \frac{1}{2}y & + & \frac{1}{2}z & = & 1 & (II) \\ & & & - & z & = & 1 & (III) \end{array}$$

```
|---+---+---+----|
| 1 | 0 | 0 |  2 | => x=2
| 0 | 1 | 0 |  3 | => y=3
| 0 | 0 | 1 | -1 | => z=-1
|---+---+---+----|
```

$$\begin{array}{cccrrr} x & & & = & 2 & (I) \\ & y & & = & 3 & (II) \\ & & z & = & -1 & (III) \end{array}$$

Polynomial time $O(n^2 m)$ but needs to guarantee that all the numbers during the run can be represented by polynomially bounded bits

Outline

A Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\ & x_j \geq 0, \quad j = 1, \ldots, n \end{aligned}$$

$$\begin{array}{rrcrcr} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{aligned} \max \quad & c^T x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

$$\max \quad \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

Outline

Standard Form

Each linear program can be converted in the form:

$$\begin{array}{lrcl} \max & c^T x & & \\ & Ax & \leq & b \\ & x & \in & \mathbb{R}^n \end{array}$$

$$c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

- if equations, then put two constraints, $ax \leq b$ and $ax \geq b$
- if $ax \geq b$ then $-ax \leq -b$
- if $\min c^T x$ then $\max(-c^T x)$

and then be put in standard (or equational) form

$$\boxed{\begin{array}{lrcl} \max & c^T x & & \\ & Ax & = & b \\ & x & \geq & 0 \\ \\ \multicolumn{4}{l}{x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m} \end{array}}$$

1. "$=$" constraints
2. $x \geq 0$ nonnegativity constraints
3. ($b \geq 0$)
4. max

Transformation into Std Form

Every LP can be transformed in std. form

1. introduce slack variables (or surplus)

$$\begin{array}{ccccccc} 5x_1 & + & 10x_2 & + & x_3 & = & 60 \\ 4x_1 & + & 4x_2 & + & x_4 & = & 40 \end{array}$$

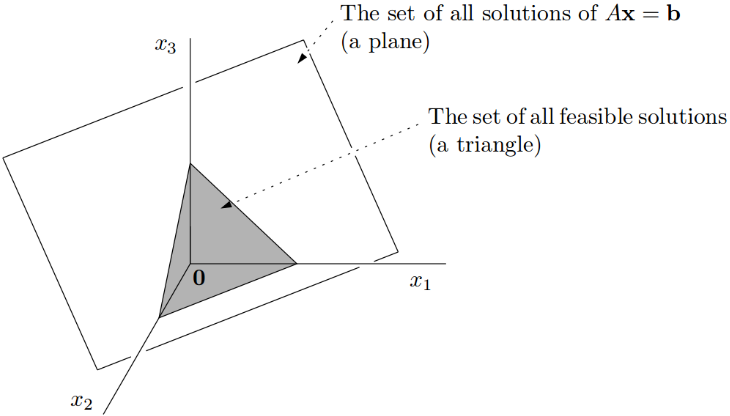
2. if $x_1 \gtreqless 0$ then $\begin{array}{l} x_1 = x_1' - x_1'' \\ x_1' \geq 0 \\ x_1'' \geq 0 \end{array}$

3. $(b \geq 0)$

4. $\min c^T x \equiv \max(-c^T x)$

LP in $n \times m$ converted into LP with at most $(m + 2n)$ variables and m equations (n # original variables, m # constraints)

Geometry

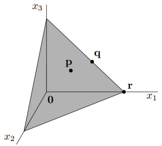

The set of all solutions of $A\mathbf{x} = \mathbf{b}$
(a plane)
The set of all feasible solutions
(a triangle)
x_3
$\mathbf{0}$
x_1
x_2

- $Ax = b$ is a system of equations that we can solve by Gaussian elimination
- Elementary row operations of $\begin{bmatrix} A & \mid & b \end{bmatrix}$ do not affect set of feasible solutions
  - multiplying all entries in some row of $\begin{bmatrix} A & \mid & b \end{bmatrix}$ by a nonzero real number λ
  - replacing the ith row of $\begin{bmatrix} A & \mid & b \end{bmatrix}$ by the sum of the ith row and jth row for some $i \neq j$
- We assume $\mathrm{rank}(\begin{bmatrix} A & \mid & b \end{bmatrix}) = \mathrm{rank}(A) = m$, ie, rows of A are linearly independent
  otherwise, remove linear dependent rows

Outline

Basic Feasible Solutions

Basic feasible solutions are the vertices of the feasible region:



More formally:
Let $B = \{1 \ldots m\}$, $N = \{m+1 \ldots n+m\}$ be subsets of columns
A_B is made of columns of A indexed by B:

Definition

$x \in \mathbb{R}^n$ is a basic feasible solution of the linear program $\max\{c^T x \mid Ax = b, x \geq 0\}$ for an index set B if:

- $x_j = 0 \, \forall j \notin B$
- the square matrix A_B is nonsingular, ie, all columns indexed by B are lin. indep.
- $x_B = A_B^{-1} b$ is nonnegative, ie, $x_B \geq 0$ (feasibility)

We call $x_j, j \in B$ basic variables and remaining variables nonbasic variables.

Theorem

A basic feasible solution is uniquely determined by the set B.

Proof:

$$
\begin{aligned}
Ax = & A_B x_B + A_N x_N = b \\
& x_B + A_B^{-1} A_N x_N = A_B^{-1} b \\
& x_B = A_B^{-1} b
\end{aligned}
$$

A_B is singular hence one solution

Note: we call B a (feasible) basis

Extreme points and basic feasible solutions are geometric and algebraic manifestations of the same concept:

Theorem

Let P be a (convex) polyhedron from LP in std. form. For a point $v \in P$ the following are equivalent:

(i) *v is an extreme point (vertex) of P*

(ii) *v is a basic feasible solution of LP*

Proof: by recognizing that vertices of P are linear independent and such are the columns in A_B

Theorem

Let $LP = \max\{c^T x \mid Ax = b, x \geq 0\}$ be feasible and bounded, then the optimal solution is a basic feasible solution.

Proof. consequence of previous theorem and fundamental theorem of linear programming

Idea for solution method:
examine all basic solutions.
There are finitely many: $\binom{m+n}{m}$.
However, if $n = m$ then $\binom{2m}{m} \approx 4^m$.

Outline

Simplex Method

$$\max \quad z = \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Canonical std. form: one decision variable is isolated in each constraint and does not appear in the other constraints or in the obj. func. and b terms are positive

It gives immediately a feasible solution:

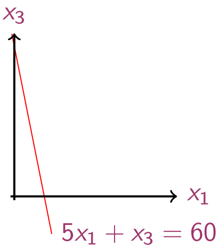
$$x_1 = 0,\ x_2 = 0,\ x_3 = 60,\ x_4 = 40$$

Is it optimal? Look at signs in $z \rightsquigarrow$ if positive then an increase would improve.

Let's try to increase a promising variable, ie, x_1, one with positive coefficient in z (is the best choice?)

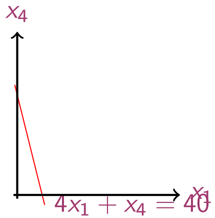
$$5x_1 + x_3 = 60$$
$$x_1 = \frac{60}{5} - \frac{x_3}{5}$$
$$x_3 = 60 - 5x_1 \geq 0$$

If $x_1 > 12$ then $x_3 < 0$



$$4x_1 + x_4 = 40$$
$$x_1 = \frac{40}{4} - \frac{x_4}{4}$$
$$x_4 = 40 - 4x_1 \geq 0$$

If $x_1 > 10$ then $x_4 < 0$



we can take the minimum of the two $\rightsquigarrow$ x_1 increased to 10
x_4 exits the basis and x_1 enters

Simplex Tableau

First simplex tableau:

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

we want to reach this new tableau

	x_1	x_2	x_3	x_4	$-z$	b
x_3	0	?	1	?	0	?
x_1	1	?	0	?	0	?
	0	?	0	?	1	?

Pivot operation:

1. Choose pivot:

   column: one s with positive coefficient in obj. func. (to discuss later)

   row: ratio between coefficient b and pivot column: choose the one with smallest ratio:

$$\theta = \min_i \left\{ \frac{b_i}{a_{is}} : a_{is} > 0 \right\}, \qquad \theta \text{ increase value of entering var.}$$

2. elementary row operations to update the tableau

- x_4 leaves the basis, x_1 enters the basis
  - Divide row pivot by pivot
  - Send to zero the coefficient in the pivot column of the first row
  - Send to zero the coefficient of the pivot column in the third (cost) row

	x1	x2	x3	x4	-z	b
I'=I-5II'	0	5	1	-5/4	0	10
II'=II/4	1	1	0	1/4	0	10
III'=III-6II'	0	2	0	-6/4	1	-60

From the last row we read: $2x_2 - 3/2x_4 - z = -60$, that is:
$z = 60 + 2x_2 - 3/2x_4$.
Since x_2 and x_4 are nonbasic we have $z = 60$ and
$x_1 = 10, x_2 = 0, x_3 = 10, x_4 = 0$.

- Done? No! Let x_2 enter the basis

	x1	x2	x3	x4	-z	b
I'=I/5	0	1	1/5	-1/4	0	2
II'=II-I'	1	0	-1/5	1/2	0	8
III'=III-2I'	0	0	-2/5	-1	1	-64

Reduced costs: the coefficients in the objective function of the nonbasic variables, $\bar{c}_N$

Optimality:
The basic solution is optimal when the reduced costs in the corresponding simplex tableau are nonpositive, ie, such that:

$$\bar{c}_N \leq 0$$

Graphical Representation