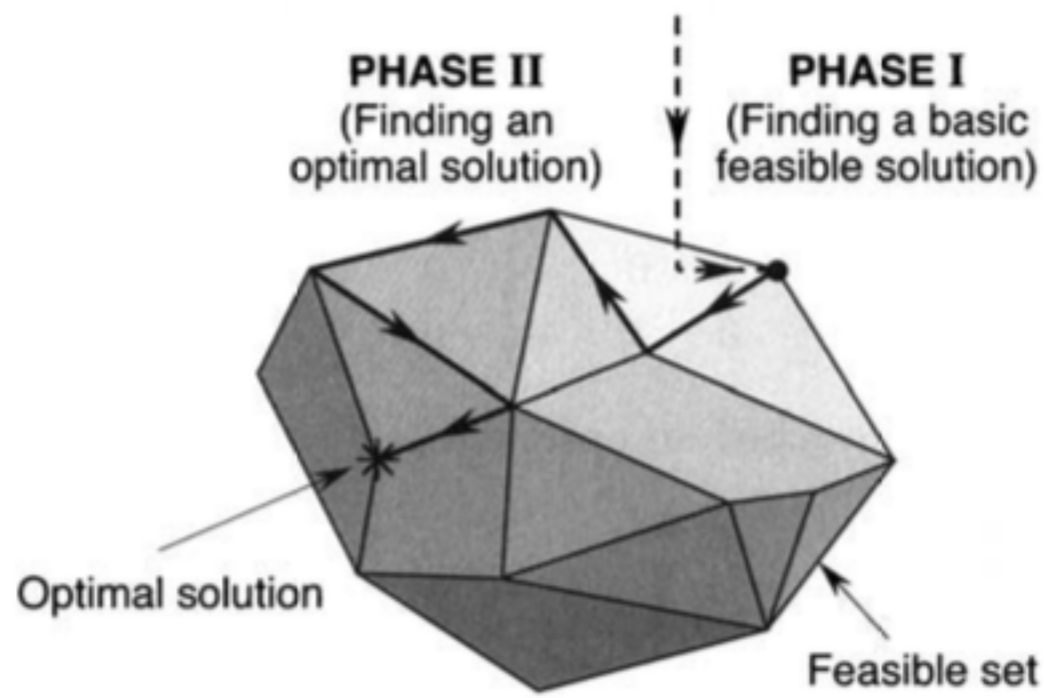
Phase 1: How to find a first basic feasible solution?



Ievgen Redko
(based on slides by Prof. James B. Orlin, Marco Cuturi)

Quote of the Day

“Everyone designs who devises courses of action aimed at changing existing situations into preferred ones.”

-- Herbert Simon

Today's Lecture

- Very quick review of the simplex algorithm.
- Phase 1: How to obtain the initial bfs
- Finiteness (assuming bases do not repeat)
  - Degeneracy
  - Anti-cycling rule(s)
- Alternative optima

A very quick review

-z	x_1	x_2	x_3	s_1	s_2	s_3		RHS
1	5	4.5	6	0	0	0		0
0	6	5	8	1	0	0		60
0	10	20	10	0	1	0		150
0	1	0	0	0	0	1		8

The ***basic variables*** here are -z, s_1, s_2, s_3.

It is optional whether to call -z basic.

The **basic feasible solution** (**bfs**) is

$z = 0$; $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $s_1 = 60$; $s_2 = 150$; $s_3 = 8$

A very quick review

-z	x_1	x_2	x_3	s_1	s_2	s_3		RHS	Ratio
1	5	4.5	6	0	0	0		0	
0	6	5	8	1	0	0		60	60/6
0	10	20	10	0	1	0		150	150/10
0	1	0	0	0	0	1		8	8/1

If all reduced costs are ≤ 0, then you are optimal. Otherwise, choose a reduced cost that is positive.

We could have chosen the 5 or the 4.5 or the 6.

Use the min ratio rule to determine the pivot element (and the exiting variable).

Ending conditions: Optimality

If all coefficients in the z-row are nonpositive ($\bar{c}_i \leq 0$ for all i), then the current basic solution is optimal.

Basic Variable	-z	x_1	x_2	x_3	x_4	x_5		RHS
-z	1	0	-5	0	0	-1	=	-1
x_3	0	0	2	1	0	-2	=	1
x_4	0	0	-1	0	1	-2	=	7
x_1	0	1	6	0	0	0	=	3

Ending conditions: Unboundedness

If the z-row coefficient of x_s is positive for some s, and if all (other) coefficients in the column for x_s are nonpositive, then the optimal objective value is unbounded from above.

BV	-z	x_1	x_2	x_3	x_4	x_5		RHS
-z	1	0	-2	0	0	+1	=	-6
x_3	0	0	2	1	0	-2	=	4
x_4	0	0	-1	0	1	-2	=	2
x_1	0	1	6	0	0	0	=	3

$z = 6 + \otimes$

$x_1 = 3$

$x_2 = 0$

$x_3 = 4 + 2\otimes$

$x_4 = 2 + 2\otimes$

$x_5 = \otimes$

The pivot rule (min ratio version)

Basic Var	-z	x_1	x_2	x_3	x_4	x_5		RHS	
-z	1	0	-2	0	0	6	=	-11	$= - z_0$
x_3	0	0	2	1	0	2	=	4	
x_4	0	0	-1	0	1	-2	=	1	
x_1	0	1	6	0	0	3	=	9	

Choose a variable x_s (column) for which the z-row coefficient is positive.

Determine the constraint for which the following ratio is minimum. {RHS coeff / Col coeff : Col coeff > 0}

Constraint	(1)	(2)	(3)
Ratio	4/2	-2 < 0	9/3

The pivot

Basic Var	-z	x_1	x_2	x_3	x_4	x_5		RHS
-z	1	0	-2	0	0	6	=	-11
x_3	0	0	2	1	0	2	=	4
x_4	0	0	-1	0	1	-2	=	1
x_1	0	1	6	0	0	3	=	9

Basic Var	-z	x_1	x_2	x_3	x_4	x_5		RHS
-z	1	0	-8	-3	0	0	=	-23
x_5	0	0	1	0.5	0	1	=	2
x_4	0	0	1	1	1	0	=	5
x_1	0	1	3	-1.5	0	0	=	3

Examples 5, 7, 6

How do we find the first bfs?

- Fact 1: If start with a basic feasible solution, we can use the simplex algorithm to find an optimal basic feasible solution.
- Fact 2: If we start with an LP with "<=" constraints and non-negative RHS, it is easy to find an initial bfs.
- How can we use these facts to find the first bfs for problem P?

$$\begin{array}{ll} \max & z = -3x_1 + x_2 + x_3 \\ \text{s.t.} & x_1 + x_2 + x_3 = 4 \\ & -2x_1 + x_2 - x_3 = 1 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{array}$$

Example of Problem P

How do we find the first bfs?

We will create a new problem P* such that

1. It is easy to find a bfs for P*
2. An optimal solution for P* is feasible for P.

Choose a solution *x*

s.t.
$$x_1 + x_2 + x_3 \le 4$$
$$-2x_1 + x_2 - x_3 \le 1$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$$

Problem P*

and so that $x_1 + x_2 + x_3$ is as close to 4 as possible
and $-2 x_1 + x_2 - x_3$ is as close to 1 as possible.

The Phase 1 Problem

minimize $\quad y_1 + y_2$ $\qquad$ **maximize** $\quad v = -y_1 - y_2$

s.t.

$$x_1 + x_2 + x_3 + y_1 = 4$$

$$-2x_1 + x_2 - x_3 + y_2 = 1$$

Problem P*

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ y_1 \geq 0,\ y_2 \geq 0$$

-v	x_1	x_2	x_3	y_1	y_2	RHS
1	0	0	0	-1	-1	0
0	1	1	1	1	0	4
0	-2	1	-1	0	1	1

Rules for creating Problem P*

Assume we start with equality constraints and RHS >= 0.

Change the equality constraints to "≤ constraints".

Add "artificial variables" y as slack variables.

Minimize $y_1 + y_2 + \ldots$

$$\begin{aligned}
&\textbf{minimize} \quad y_1 + y_2 \\
&\textbf{s.t.} \quad x_1 + x_2 + x_3 + y_1 = 4 \\
&\qquad -2x_1 + x_2 - x_3 + y_2 = 1 \\
&\qquad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ y_1 \geq 0,\ y_2 \geq 0
\end{aligned}$$

Problem P*

The Phase 1 Problem in canonical form

-v	x_1	x_2	x_3	y_1	y_2	RHS
1	0	0	0	-1	-1	0
0	1	1	1	1	0	4
0	-2	1	-1	0	1	1

Add constraints 1 and 2 to the objective in order to get into canonical form.

-v	x_1	x_2	x_3	y_1	y_2	
1	-1	2	0	0	0	5
0	1	1	1	1	0	4
0	-2	1	-1	0	1	1

Time for a mental break

Even smart people get it wrong occasionally.

"Even considering the improvements possible... the gas turbine could hardly be considered a feasible application to airplanes because of the difficulties of complying with the stringent weight requirements."

-- US National Academy Of Science, 1940

"People have been talking about a 3,000 mile high-angle rocket shot from one continent to another, carrying an atomic bomb and so directed as to be a precise weapon... I think we can leave that out of our thinking."

-- Dr. Vannevar Bush, 1945

Fooling around with alternating current is a waste of time. Nobody will use it, ever.

-- Thomas Edison

There is not the slightest indication that nuclear energy will be obtainable.

-- Albert Einstein 1932

Rail travel at high speed is not possible because passengers, unable to breathe, would die of asphyxia.

-- Dr. Dionysus Lardner, 1793-1859

Inventions have long since reached their limit, and I see no hope for future improvements.

-- Julius Frontenus, 10 AD

The Phase 1 Problem

$-v$	x_1	x_2	x_3	y_1	y_2	
1	-1	2	0	0	0	5
0	1	1	1	1	0	4
0	-2	1	-1	0	1	1

The variables y_1, y_2, y_3 are called ***artificial variables***.

Theorem. ***There is a feasible solution for P if and only if the optimal objective value for P* is 0.***

The next pivot

$-v$	x_1	x_2	x_3	y_1	y_2	
1	-1	2	0	0	0	5
0	1	1	1	1	0	4
0	-2	1	-1	0	1	1

$-v$	x_1	x_2	x_3	y_1	y_2	
1	3	0	2	0	-2	3
0	3	0	2	1	-1	3
0	-2	1	-1	0	1	1

One more pivot till the optimum for Phase 1

-v	x_1	x_2	x_3	y_1	y_2	
1	3	0	2	0	-2	3
0	3	0	2	1	-1	3
0	-2	1	-1	0	1	1

-v	x_1	x_2	x_3	y_1	y_2	
1	0	0	0	-1	-1	0
0	3/2	0	1	1/2	-1/2	3/2
0	-1/2	1	0	1/2	1/2	5/2

Let P be the original linear program. Let P* be the LP after adding artificial variables. Suppose $y_j > 0$ in the optimal solution for P*, where y_j is artificial. Then

1. The problem P has no feasible solution.
2. The problem P is unbounded from above.
3. If we ignore y_j, the solution is feasible for P.
4. Either (1) or (2) is true.

Phase 1, Phase 2

- If there is a feasible solution for P, then Phase 1 ends with a feasible basis.

- To start Phase 2, put back the original objective function. Then put the tableau in canonical form. (The basis is almost in canonical form. But the z-row is not yet right.)

- Then pivot until optimal (or until there is proof of unboundedness.)

End of Phase 1.

$-v$	x_1	x_2	x_3	y_1	y_2	
1	0	0	0	-1	-1	0
0	3/2	0	1	1/2	-1/2	3/2
0	-1/2	1	0	1/2	1/2	5/2

$-z$	x_1	x_2	x_3
1	-3	1	1

Example

Problem:

- Objective Function:
  - $\max Z = X1 + 5\ X2$
- Constraints:
  - $X1 + X2 - E3 = 1$
  - $2\ X1 + X2 + E4 = 4$
  - $X1,\ X2,\ E3,\ E4 \geq 0$

Example

Problem:

- Objective Function:
  - max $Z = X1 + 5\ X2$
- Constraints:
  - $X1 + X2 - E3 = 1$
  - $2\ X1 + X2 + E4 = 4$
  - $X1, X2, E3, E4 \geq 0$

Initial dictionary (not feasible):

$$\boxed{\begin{array}{l} E3 = -1 + X1 + X2 \\ E4 = 4 - 2\ X1 - X2 \\ \hline Z = 0 + X1 + 5\ X2 \end{array}}$$

Example

Problem:

- Objective Function:
  - max Z = X1 + 5 X2
- Constraints:
  - X1 + X2 - E3 = 1
  - 2 X1 + X2 + E4 = 4
  - X1, X2, E3, E4 $\geq$ 0

Phase 1 problem:

- Phase 1 Objective Function:
  - max Z' = - R5
- Phase 1 Constraints:
  - X1 + X2 - E3 + R5 = 1
  - 2 X1 + X2 + E4 = 4
  - X1, X2, E3, E4, R5 $\geq$ 0

Initial dictionary (not feasible):

E3	=	-1	+	X1	+	X2
E4	=	4	-	2 X1	-	X2
Z	=	0	+	X1	+	5 X2

Example

Problem:

- Objective Function:
  - max $Z = X1 + 5\ X2$
- Constraints:
  - $X1 + X2 - E3 = 1$
  - $2\ X1 + X2 + E4 = 4$
  - $X1, X2, E3, E4 \geq 0$

Initial dictionary (not feasible):

$$\begin{array}{|l|}\hline E3 = -1 + X1 + X2 \\ E4 = 4 - 2\ X1 - X2 \\ \hline Z = 0 + X1 + 5\ X2 \\ \hline\end{array}$$

Phase 1 problem:

- Phase 1 Objective Function:
  - max $Z' = -\ R5$
- Phase 1 Constraints:
  - $X1 + X2 - E3 + R5 = 1$
  - $2\ X1 + X2 + E4 = 4$
  - $X1, X2, E3, E4, R5 \geq 0$

Phase 1 Dictionary:

$$\begin{array}{|l|}\hline R5 = 1 - X1 - X2 + E3 \\ E4 = 4 - 2\ X1 - X2 \\ \hline Z' = -1 + X1 + X2 - E3 \\ \hline\end{array}$$

Example

Problem:

- Objective Function:
  - max $Z = X1 + 5\ X2$
- Constraints:
  - $X1 + X2 - E3 = 1$
  - $2\ X1 + X2 + E4 = 4$
  - $X1, X2, E3, E4 \geq 0$

Initial dictionary (not feasible):

$$\begin{array}{|l|}\hline E3 = -1 + X1 + X2 \\ E4 = 4 - 2\ X1 - X2 \\ \hline Z = 0 + X1 + 5\ X2 \\ \hline \end{array}$$

Phase 1 problem:

- Phase 1 Objective Function:
  - max $Z' = -\ R5$
- Phase 1 Constraints:
  - $X1 + X2 - E3 + R5 = 1$
  - $2\ X1 + X2 + E4 = 4$
  - $X1, X2, E3, E4, R5 \geq 0$

Phase 1 Dictionary:

$$\begin{array}{|l|}\hline R5 = 1 - X1 - X2 + E3 \\ E4 = 4 - 2\ X1 - X2 \\ \hline Z' = -1 + X1 + X2 - E3 \\ \hline \end{array}$$

- This new dictionary is feasible, we can solve the phase 1 problem.

Example: Phase 1

Solving Phase 1 problem:

E4	=	4	-	2 X1	-	1 X2					
R5	=	1	-	1 X1	-	1 X2	+	1 E3			
Z'	=	-1	+	1 X1	+	1 X2	-	1 E3			

B			X1	X2	E3	E4	R5
E4	4	=	2	1	0	1	0
R5	1	=	1	1	-1	0	1
	Z'+1	=	1	1	-1	0	0

Example: Phase 1

Solving Phase 1 problem:

E4	=	4	-	2	X1	-	1	X2			
R5	=	1	-	1	X1	-	1	X2	+	1	E3
Z'	=	-1	+	1	X1	+	1	X2	-	1	E3

B			X1	X2	E3	E4	R5
E4	4	=	2	1	0	1	0
R5	1	=	1	1	-1	0	1
	Z'+1	=	1	1	-1	0	0

E4	=	4	-	2	X1	-	1	X2			
R5	=	1	-	1	X1	-	1	X2	+	1	E3
Z'	=	-1	+	1	X1	+	1	X2	-	1	E3

B			X1	X2	E3	E4	R5
E4	4	=	2	1	0	1	0
R5	1	=	1	1	-1	0	1
	Z'+1	=	1	1	-1	0	0

Example: Phase 1

Solving Phase 1 problem:

E4	=	4	-	2	X1	-	1	X2			
R5	=	1	-	1	X1	-	1	X2	+	1	E3
Z'	=	-1	+	1	X1	+	1	X2	-	1	E3

B			X1	X2	E3	E4	R5
E4	4	=	2	1	0	1	0
R5	1	=	1	1	-1	0	1
	Z'+1	=	1	1	-1	0	0

E4	=	4	-	2	X1	-	1	X2			
R5	=	1	-	1	X1	-	1	X2	+	1	E3
Z'	=	-1	+	1	X1	+	1	X2	-	1	E3

B			X1	X2	E3	E4	R5
E4	4	=	2	1	0	1	0
R5	1	=	1	1	-1	0	1
	Z'+1	=	1	1	-1	0	0

E4	=	2	+	1	X2	-	2	E3	+	2	R5
X1	=	1	-	1	X2	+	1	E3	-	1	R5
Z'	=	0							-	1	R5

B			X1	X2	E3	E4	R5
E4	2	=	0	-1	2	1	-2
X1	1	=	1	1	-1	0	1
	Z'-0	=	0	0	0	0	-1

Example: Phase 1

- Solving Phase 1 problem:

E4	$= 2 + 1\ X2 - 2\ E3 + 2\ R5$
X1	$= 1 - 1\ X2 + 1\ E3 - 1\ R5$
Z'	$= 0 \qquad\qquad\qquad\qquad - 1\ R5$

B			X1	X2	E3	E4	R5
E4	2	=	0	-1	2	1	-2
X1	1	=	1	1	-1	0	1
	Z'-0	=	0	0	0	0	-1

Example: Phase 1

- Solving Phase 1 problem:

E4 = 2 + 1 X2 - 2 E3 + 2 R5	
X1 = 1 - 1 X2 + 1 E3 - 1 R5	
Z' = 0	- 1 R5

B			X1	X2	E3	E4	R5
E4	2	=	0	-1	2	1	-2
X1	1	=	1	1	-1	0	1
	Z'-0	=	0	0	0	0	-1

- The solution of the phase 1 problem is:
  X2 = E3 = R5 = 0, E4 = 2, X1 = 1 and Z' = 0

Example: Phase 1

- Solving Phase 1 problem:

E4 = 2 + 1 X2 - 2 E3 + 2 R5	
X1 = 1 - 1 X2 + 1 E3 - 1 R5	
Z' = 0	- 1 R5

B			X1	X2	E3	E4	R5
E4	2	=	0	-1	2	1	-2
X1	1	=	1	1	-1	0	1
	Z'-0	=	0	0	0	0	-1

- The solution of the phase 1 problem is:
  X2 = E3 = R5 = 0, E4 = 2, X1 = 1 and Z' = 0
- Z' = 0, So a feasible solution for the initial problem is:
  X2 = E3 = 0, E4 = 2 and X1 = 1 and Z = 1

Example: Phase 1

- Solving Phase 1 problem:

E4	=	2	+ 1 X2	- 2 E3	+ 2 R5
X1	=	1	- 1 X2	+ 1 E3	- 1 R5
Z'	=	0			- 1 R5

B			X1	X2	E3	E4	R5
E4	2	=	0	-1	2	1	-2
X1	1	=	1	1	-1	0	1
	Z'-0	=	0	0	0	0	-1

- The solution of the phase 1 problem is:
  X2 = E3 = R5 = 0, E4 = 2, X1 = 1 and Z' = 0
- Z' = 0, So a feasible solution for the initial problem is:
  X2 = E3 = 0, E4 = 2 and X1 = 1 and Z = 1
- We can start phase 2 with the final dictionary / tableau of phase 1 (remove R5 and replace Z' by Z = X1 + 5 X2):

X1	=	1	- 1 X2	+ 1 E3
E4	=	2	+ 1 X2	- 2 E3
Z	=	1	+ 4 X2	+ 1 E3

B			X1	X2	E3	E4
X1	1	=	1	1	-1	0
E4	2	=	0	-1	2	1
	Z-1	=	0	4	1	0

Example: Phase 2

X1	=	1	-	1	X2	+	1	E3
E4	=	2	+	1	X2	-	2	E3
Z	=	1	+	4	X2	+	1	E3

B			X1	X2	E3	E4
X1	1	=	1	1	-1	0
E4	2	=	0	-1	2	1
	Z-1	=	0	4	1	0

Example: Phase 2

$$
\begin{array}{|l|}
\hline
X1 = 1 - 1\ X2 + 1\ E3 \\
E4 = 2 + 1\ X2 - 2\ E3 \\
\hline
Z\ = 1 + 4\ X2 + 1\ E3 \\
\hline
\end{array}
$$

B			X1	X2	E3	E4
X1	1	=	1	1	-1	0
E4	2	=	0	-1	2	1
	Z-1	=	0	4	1	0

$$
\begin{array}{|l|}
\hline
X1 = 1 - 1\ X2 + 1\ E3 \\
E4 = 2 + 1\ X2 - 2\ E3 \\
\hline
Z\ = 1 + 4\ X2 + 1\ E3 \\
\hline
\end{array}
$$

B			X1	X2	E3	E4
X1	1	=	1	1	-1	0
E4	2	=	0	-1	2	1
	Z-1	=	0	4	1	0

Example: Phase 2

X1	=	1	-	1 X2	+	1	E3
E4	=	2	+	1 X2	-	2	E3
Z	=	1	+	4 X2	+	1	E3

X1	=	1	-	1 X2	+	1	E3
E4	=	2	+	1 X2	-	2	E3
Z	=	1	+	4 X2	+	1	E3

E4	=	3	-	1 X1	-	1	E3
X2	=	1	-	1 X1	+	1	E3
Z	=	5	-	4 X1	+	5	E3

B			X1	X2	E3	E4
X1	1	=	1	1	-1	0
E4	2	=	0	-1	2	1
	Z-1	=	0	4	1	0

B			X1	X2	E3	E4
X1	1	=	1	1	-1	0
E4	2	=	0	-1	2	1
	Z-1	=	0	4	1	0

B			X1	X2	E3	E4
E4	3	=	1	0	1	1
X2	1	=	1	1	-1	0
	Z-5	=	-4	0	5	0

Example: Phase 2

$$\begin{array}{|l|}\hline E4 = 3 - 1\ X1\ -\ 1\ E3 \\ X2 = 1 - 1\ X1 + 1\ E3 \\ \hline Z\ = 5 - 4\ X1 + 5\ E3 \\ \hline \end{array}$$

B			X1	X2	E3	E4
E4	3	=	1	0	1	1
X2	1	=	1	1	-1	0
	Z-5	=	-4	0	5	0

Example: Phase 2

E4	=	3	-	1 X1	-	1 E3
X2	=	1	-	1 X1	+	1 E3
Z	=	5	-	4 X1	+	5 E3

B			X1	X2	E3	E4
E4	3	=	1	0	1	1
X2	1	=	1	1	-1	0
	Z-5	=	-4	0	5	0

E4	=	3	-	1 X1	-	1 E3
X2	=	1	-	1 X1	+	1 E3
Z	=	5	-	4 X1	+	5 E3

B			X1	X2	E3	E4
E4	3	=	1	0	1	1
X2	1	=	1	1	-1	0
	Z-5	=	-4	0	5	0

If the RHS is greater than 0, then the next bfs has greater objective value.

-z	x_1	x_2	x_3	s_1	s_2	s_3		RHS
1	5	4.5	6	0	0	0		0
0	6	5	8	1	0	0		60
0	10	20	10	0	1	0		150
0	1	0	0	0	0	1		8

1	0.5	0.75	0	-0.75	0	0		-45
	0.75	0.625	1	0.125				7.5
	2.5	13.75		-1.25	1			75
	1					1		8

Is the Simplex Method Finite?

Theorem. ***If the objective value improves at every iteration, then every basic feasible solution is different, and the simplex method is finite.***

Proof. Each canonical tableau is uniquely determined by choosing n basic variables out of n variables. The number of bases is at most:

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

If the RHS is 0, it is possible that the solution stays the same after a pivot.

-z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
1	5	4.5	6	0	0	0	0
0	6	5	8	1	0	0	0
0	10	20	10	0	1	0	150
0	1	0	0	0	0	1	8

							RHS
1	0.5	0.75	0	-0.75	0	0	0
	0.75	0.625	1	0.125			0
	2.5	13.75		-1.25	1		150
	1					1	8

If one of the basic variables is 0 (RHS is 0), we say that the tableau is *degenerate*.

If the RHS is 0, it is possible that the objective increases.

-z	x_1	x_2	x_3	s_1	s_2	s_3		RHS
1	5	4.5	6	0	0	0		0
0	6	5	8	1	0	0		60
0	10	20	10	0	1	0		150
0	1	0	0	0	0	1		0

1	0.5	0.75	0	-0.75	0	0		-45
	0.75	0.625	1	0.125				7.5
	2.5	13.75		-1.25	1			75
	1					1		0

If many bases are degenerate, it is possible for the simplex algorithm to *cycle*, that is, repeat a sequence of basic feasible solutions.

-z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	0.75	-20	0.5	-6	0	0	0	-3
0	0.25	-8	-1	9	1	0	0	0
0	0.5	-12	-0.5	3	0	1	0	0
0	0	0	1	0	0	0	1	1

1	0	4	3.5	-33	-3	0	0	-3
	1	-32	-4	36	4			0
		4	1.5	-15	-2	1		0
			1				1	1

The Klee and Minty example, which can cycle.

Microsoft® Excel

Bland’s Rule

- There are several ways of guaranteeing that no set of basic variables repeats.
- The simplest way of avoiding “cycling” is Bland’s rule.

Bland’s Rule:

1. Among variables that have a positive coefficient in the z-row, choose the one with least index.
2. Among rows that satisfy the min ratio rule, choose the one with least index.

Theorem. The simplex method with Bland’s rule is finite.

Example of Bland's rule

Simplex Cycle

Without the Bland Rule:
Entering variable: the one with the largest coefficient in Z
Leaving variable: the one with the smallest index (among those that can leave)

B			X1	X2	X3	X4	X5	X6	X7
X5	0	=	1/4	-60	$-1/25$	9	1	0	0
X6	0	=	1/2	-90	$-1/50$	3	0	1	0
X7	1	=	0	0	1	0	0	0	1
	Z-0	=	3/4	-150	1/50	-6	0	0	0

B			X1	X2	X3	X4	X5	X6	X7
X5	0	=	1/4	-60	$-1/25$	9	1	0	0
X6	0	=	1/2	-90	$-1/50$	3	0	1	0
X7	1	=	0	0	1	0	0	0	1
	Z-0	=	3/4	-150	1/50	-6	0	0	0

B			X1	X2	X3	X4	X5	X6	X7
X6	0	=	0	30	3/50	-15	-2	1	0
X7	1	=	0	0	1	0	0	0	1
X1	0	=	1	-240	$-4/25$	36	4	0	0
	Z-0	=	0	30	7/50	-33	-3	0	0

B			X1	X2	X3	X4	X5	X6	X7
X6	0	=	0	30	3/50	-15	-2	1	0
X7	1	=	0	0	1	0	0	0	1
X1	0	=	1	-240	$-4/25$	36	4	0	0
	Z-0	=	0	30	7/50	-33	-3	0	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	0	0	1	0	0	0	1
X1	0	=	1	0	8/25	-84	-12	8	0
X2	0	=	0	1	1/500	$-1/2$	$-1/15$	1/30	0
	Z-0	=	0	0	2/25	-18	-1	-1	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	0	0	1	0	0	0	1
X1	0	=	1	0	8/25	-84	-12	8	0
X2	0	=	0	1	1/500	$-1/2$	$-1/15$	1/30	0
	Z-0	=	0	0	2/25	-18	-1	-1	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	$-25/8$	0	0	525/2	75/2	-25	1
X2	0	=	$-1/160$	1	0	1/40	1/120	$-1/60$	0
X3	0	=	25/8	0	1	$-525/2$	$-75/2$	25	0
	Z-0	=	$-1/4$	0	0	3	2	-3	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	$-25/8$	0	0	525/2	75/2	-25	1
X2	0	=	$-1/160$	1	0	1/40	1/120	$-1/60$	0
X3	0	=	25/8	0	1	$-525/2$	$-75/2$	25	0
	Z-0	=	$-1/4$	0	0	3	2	-3	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	125/2	-10500	0	0	-50	150	1
X3	0	=	$-125/2$	10500	1	0	50	-150	0
X4	0	=	$-1/4$	40	0	1	1/3	$-2/3$	0
	Z-0	=	1/2	-120	0	0	1	-1	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	125/2	-10500	0	0	-50	150	1
X3	0	=	$-125/2$	10500	1	0	50	-150	0
X4	0	=	$-1/4$	40	0	1	1/3	$-2/3$	0
	Z-0	=	1/2	-120	0	0	1	-1	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	0	0	1	0	0	0	1
X4	0	=	1/6	-30	$-1/150$	1	0	1/3	0
X5	0	=	$-5/4$	210	1/50	0	1	-3	0
	Z-0	=	7/4	-330	$-1/50$	0	0	2	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	0	0	1	0	0	0	1
X4	0	=	1/6	-30	$-1/150$	1	0	1/3	0
X5	0	=	$-5/4$	210	1/50	0	1	-3	0
	Z-0	=	7/4	-330	$-1/50$	0	0	2	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	0	0	1	0	0	0	1
X5	0	=	1/4	-60	$-1/25$	9	1	0	0
X6	0	=	1/2	-90	$-1/50$	3	0	1	0
	Z-0	=	3/4	-150	1/50	-6	0	0	0

This is the same pb as the first one.

With the Bland Rule:
Entering variable: the one with the smallest index (among those with a positive coefficient in Z)
Leaving variable: the one with the smallest index (among those that can leave)

B			X1	X2	X3	X4	X5	X6	X7
X5	0	=	1/4	-60	$-1/25$	9	1	0	0
X6	0	=	1/2	-90	$-1/50$	3	0	1	0
X7	1	=	0	0	1	0	0	0	1
	Z-0	=	3/4	-150	1/50	-6	0	0	0

B			X1	X2	X3	X4	X5	X6	X7
X5	0	=	1/4	-60	$-1/25$	9	1	0	0
X6	0	=	1/2	-90	$-1/50$	3	0	1	0
X7	1	=	0	0	1	0	0	0	1
	Z-0	=	3/4	-150	1/50	-6	0	0	0

B			X1	X2	X3	X4	X5	X6	X7
X6	0	=	0	30	3/50	-15	-2	1	0
X7	1	=	0	0	1	0	0	0	1
X1	0	=	1	-240	$-4/25$	36	4	0	0
	Z-0	=	0	30	7/50	-33	-3	0	0

B			X1	X2	X3	X4	X5	X6	X7
X6	0	=	0	30	3/50	-15	-2	1	0
X7	1	=	0	0	1	0	0	0	1
X1	0	=	1	-240	$-4/25$	36	4	0	0
	Z-0	=	0	30	7/50	-33	-3	0	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	0	0	1	0	0	0	1
X1	0	=	1	0	8/25	-84	-12	8	0
X2	0	=	0	1	1/500	$-1/2$	$-1/15$	1/30	0
	Z-0	=	0	0	2/25	-18	-1	-1	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	0	0	1	0	0	0	1
X1	0	=	1	0	8/25	-84	-12	8	0
X2	0	=	0	1	1/500	$-1/2$	$-1/15$	1/30	0
	Z-0	=	0	0	2/25	-18	-1	-1	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	$-25/8$	0	0	525/2	75/2	-25	1
X2	0	=	$-1/160$	1	0	1/40	1/120	$-1/60$	0
X3	0	=	25/8	0	1	$-525/2$	$-75/2$	25	0
	Z-0	=	$-1/4$	0	0	3	2	-3	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	$-25/8$	0	0	525/2	75/2	-25	1
X2	0	=	$-1/160$	1	0	1/40	1/120	$-1/60$	0
X3	0	=	25/8	0	1	$-525/2$	$-75/2$	25	0
	Z-0	=	$-1/4$	0	0	3	2	-3	0

Up to this step, this is the same as without the Bland Rule. But now, whereas in the cycle version the entering variable is X5, it is now X1.

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	125/2	-10500	0	0	-50	150	1
X3	0	=	$-125/2$	10500	1	0	50	-150	0
X4	0	=	$-1/4$	40	0	1	1/3	$-2/3$	0
	Z-0	=	1/2	-120	0	0	1	-1	0

B			X1	X2	X3	X4	X5	X6	X7
X7	1	=	125/2	-10500	0	0	-50	150	1
X3	0	=	$-125/2$	10500	1	0	50	-150	0
X4	0	=	$-1/4$	40	0	1	1/3	$-2/3$	0
	Z-0	=	1/2	-120	0	0	1	-1	0

B			X1	X2	X3	X4	X5	X6	X7
X3	1	=	0	0	1	0	0	0	1
X4	1/250	=	0	-2	0	1	2/15	$-1/15$	1/250
X1	2/125	=	1	-168	0	0	$-4/5$	12/5	2/125
	Z-1/125	=	0	-36	0	0	7/5	$-11/5$	$-1/125$

B			X1	X2	X3	X4	X5	X6	X7
X3	1	=	0	0	1	0	0	0	1
X4	1/250	=	0	-2	0	1	2/15	$-1/15$	1/250
X1	2/125	=	1	-168	0	0	$-4/5$	12/5	2/125
	Z-1/125	=	0	-36	0	0	7/5	$-11/5$	$-1/125$

B			X1	X2	X3	X4	X5	X6	X7
X3	1	=	0	0	1	0	0	0	1
X1	1/25	=	1	-180	0	6	0	2	1/25
X5	3/100	=	0	-15	0	15/2	1	$-1/2$	3/100
	Z-1/20	=	0	-15	0	$-21/2$	0	$-3/2$	$-1/20$

Non-degeneracy and finiteness.

Lemma. ***If the RHS of a tableau is positive, then the next pivot will lead to an improved objective function value.***

If a coefficient of the RHS of a tableau is 0, the tableau is *degenerate* (and the bfs is *degenerate*). If a bfs is degenerate, it is possible that the next pivot will lead to a different basis, but the same solution.

Theorem. ***If no basis is degenerate, then the simplex method is finite.***

Alternative Optima

	$-z$	x_1	x_2	x_3	x_4	x_5		RHS
A_0	1	0	0	0	0	-1	=	-2
A_1	0	0	2	1	0	-1	=	4
A_2	0	0	-1	0	1	2	=	1
A_3	0	1	6	0	0	3	=	3

Let $x_2 = \Delta$;

$x_1 = 3 - 6\Delta$
$x_2 = \Delta$
$x_3 = 4 - 2\Delta$
$x_4 = 1 + \Delta$
$x_5 = 0$
$z = 2$

This tableau satisfies the optimality conditions.

If a tableau satisfies the optimality conditions, and if $\bar{c}_j = 0$ for a nonbasic variable, then there may be multiple alternative optima solutions.

Non-degeneracy guarantees that we can choose $\Delta > 0$.

Alternative Optima and Pivoting

	-z	x_1	x_2	x_3	x_4	x_5		RHS
A_0	1	0	0	0	0	-1	=	-2
A_1	0	0	2	1	0	-1	=	4
A_2	0	0	-1	0	1	2	=	1
A_3	0	1	6	0	0	3	=	3

If a tableau satisfies the optimality conditions, and if $\bar{c}_j = 0$ for a nonbasic variable, we can pivot to get an alternative optimal bfs. (or prove that there is a ray along which the objective stays the same).

	-z	x_1	x_2	x_3	x_4	x_5		RHS
$B_0 = A_0$	1	0	0	0	0	-1	=	-2
$B_1 = A_1 - 2\,B_3$	0	-1/3	0	1	0	-2	=	3
$B_2 = A_2 + B_3$	0	1/6	0	0	1	2.5	=	1.5
$B_3 = A_3/6$	0	1/6	1	0	0	.5	=	.5

Overview

- The simplex method has been a huge success in optimization.
  - It solves linear programs efficiently
  - We can solve problems with millions of variables
  - It can be a starting point for problems that are not linear
- The simplex method requires some simple techniques to get started
  - Transformation into standard form
  - Phase 1 of the simplex algorithm
  - In practice, it requires lots of implementation care
- Degeneracy and techniques to avoid "cycling".
- Alternative optima

Measures of Efficiency

Simplex: efficiency

- We have examined the simplex algorithm completely.
- Works in every situation: unbounded, infeasible, degenerate, etc
- The question is now: how **fast**?

Simplex: efficiency

- We have examined the simplex algorithm completely.
- Works in every situation: unbounded, infeasible, degenerate, etc
- The question is now: how **fast**?

- Quite important one: the simplex is a **discrete** and **combinatorial** algorithm.
- The **combinatorial** makes it a suspect for being quite time consuming.

Simplex: efficiency

- Efficiency measures:
  - Should be a function of the problem size, characteristics
  - Should be easy to compute
  - Should work for entire classes of problems

Simplex: efficiency

- Efficiency measures:
  - Should be a function of the problem size, characteristics
  - Should be easy to compute
  - Should work for entire classes of problems

- For linear programming, two classic answers:
  - **Worst case**: Time it takes the simplex to find a solution to the hardest problem in a class
  - **Average case**: Time it takes the simplex to finish, averaged over random problems in a class

Simplex: efficiency

- **Worst case** analysis:
  - Most common measure
  - More tractable
  - Does not reflect **practical** performance
- **Average case** analysis:
  - Hard to evaluate explicitly
  - Equally difficult to define the priors for the problem
  - Mostly empirical results
- Why: it's easier to measure the complexity of only **one bad** program, it also produces an **upper bound** on the computing time.

Simplex: efficiency

- Measures of problem **size**:
  - Number of constraints m, number of variables d
  - We assume canonical forms usually, with obvious "standardizations" if necessary.
  - Number of parameters $(d+1)(m+1)$
- Measures of **computing time**:
  - Total number of iterations
  - CPU time of each iteration (in flops, or floating point operations)
- Up to a multiplicative constant: written $O(n^2)$ for example if require time is proportional to n^2.. . .

Simplex: efficiency

- Problems are usually classified according to their worst-case complexity:
  - **Polynomial** problems: the worst-case total CPU time is a polynomial function of the problem size
  - **Non polynomial** problems: the worst-case total CPU time grows faster than **all** polynomial functions of the problem size (very often: exponential)
- Examples:
  - Computing a matrix times vector product is $O(d^2)$ in $\mathbf{R}^d$
  - Combinatorial problems are usually exponential (sparse linear programs, integer programs, etc)

Simplex: efficiency

- Verdict on the **simplex method** :
  - For **all** known deterministic pivot rules, there are problems for which the **simplex** method takes an exponential (λ^m) number of pivots.
  - However, **good** performance in practice.
  - In applications, the convergence only takes **a few times** m steps.

What about Linear Programming?

- However, **linear programming** is (relatively) **easy**:
  - Can prove **theoretically** that linear programming is polynomial
  - This is also true in **practice**: interior point algorithms produce a solution in $O(d^{3.5})$.
  - Interesting contrast: bounds for the **simplex** are usually in the **number of constraints**, **IPM** to the **number of variables**.

  - Finding a pivot rule that makes the simplex polynomial in the worst case is still an open problem: we do not know whether such a rule exists.

- any intuition why?

Polyhedra, number of vertices

Simplex: Intuitions on the Number of Vertices

- Feasible region is $\{A\mathbf{x} \leq \mathbf{b}\} \cap \{\mathbf{x} \geq \mathbf{0}\}$.
- We assume nonnegativity constraints are in the m inequalities $\Rightarrow m > d$.
- Any vertex is at the intersections of at least d hyperplanes of those described in m.
- A very loose upper-bound would be $\binom{m}{d}$ vertices.
- For instance, using Stirling's approximations and assuming $m \gg d$,

$$\binom{m}{d} \approx \frac{m^d}{d!}$$

- Order of m^d...

Klee Minty counterexample

No. Real issues for some pathological cases.

Klee Minty counterexample

First such example by Klee and Minty in 1972:

$$
\begin{array}{lrll}
\text{maximize} & \displaystyle\sum_{j=1}^{d} 10^{d-j} x_j & & \\
\text{subject to} & \displaystyle 2\sum_{j=1}^{i-1} 10^{i-j} x_j + x_i & \leq 100^{i-1} & i = 1, 2, \ldots, m \\
& x_j & \geq 0 & j = 1, 2, \ldots, d.
\end{array}
$$

In practice, this looks like:

$$
\begin{array}{rcrcrcr}
x_1 & & & & & \leq & 1 \\
20x_1 & + & x_2 & & & \leq & 100 \\
200x_1 & + & 20x_2 & + & x_3 & \leq & 10000.
\end{array}
$$

Simplex: efficiency

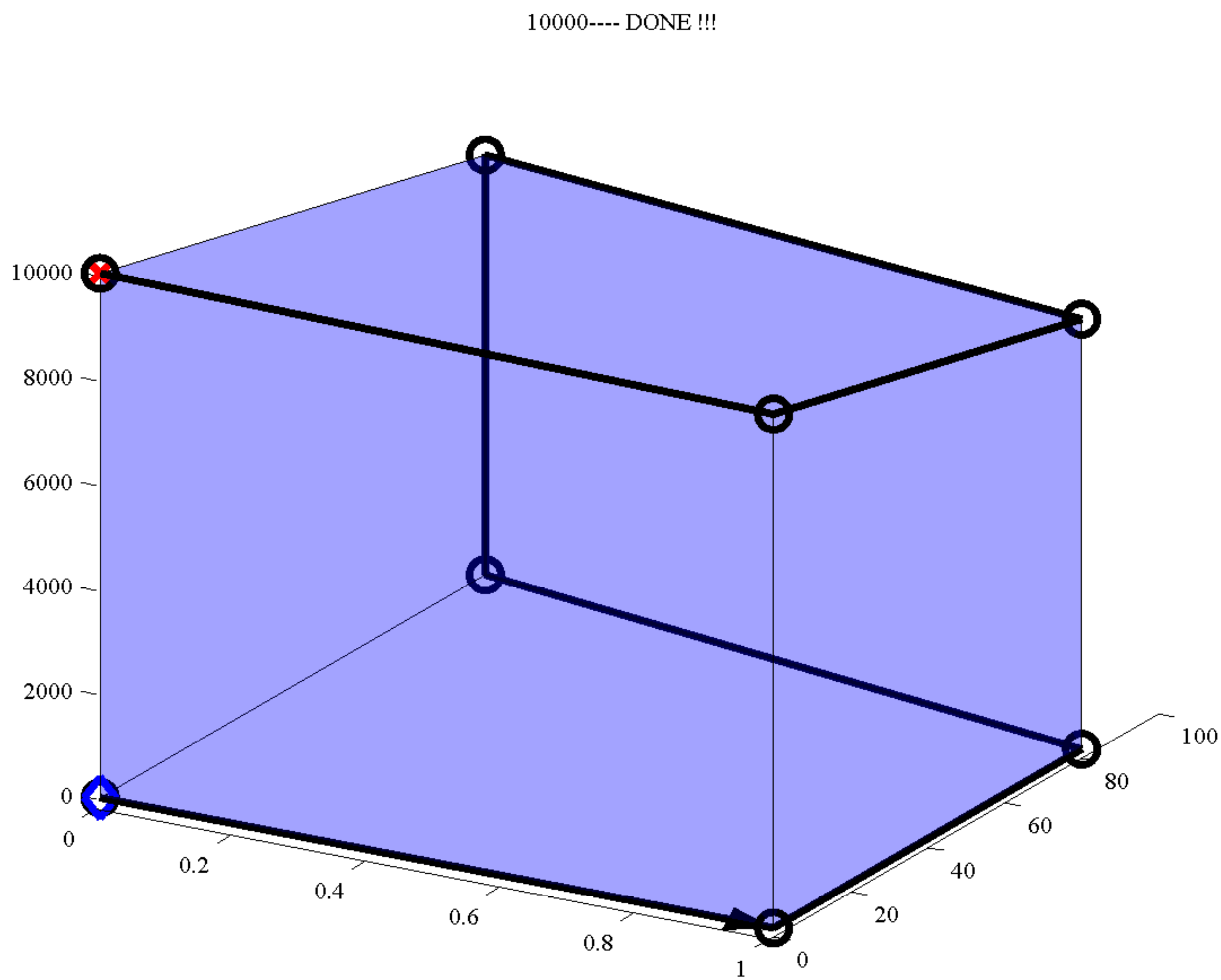
Intuition behind this problem:

- A hypercube in dimension m has 2^m vertices
- The constraints in the K&M problem are *roughly* equivalent to:

$$
\begin{aligned}
0 \leq \quad & x_1 & \leq 1 \\
0 \leq \quad & x_2 & \leq 100 \\
& \vdots & \\
0 \leq \quad & x_d & \leq 100^{d-1}.
\end{aligned}
$$

- The pivot rule choosing the largest **reduced cost coefficient** will visit every vertex of that box before reaching the solution

Klee-Minty: Matlab demo



Tableaux for Klee Minty

- Let us check the corresponding tableaux.

1	0	0	1	0	0	1
20	1	0	0	1	0	100
200	20	1	0	0	1	10000
100	10	1	0	0	0	0

1	0	0	1	0	0	1
0	**1**	0	-20	1	0	80
0	20	1	-200	0	1	9800
0	**10**	1	-100	0	0	-100

1	0	0	**1**	0	0	1
0	1	0	-20	1	0	80
0	0	1	200	-20	1	8200
0	0	1	**100**	-10	0	-900

1	0	0	1	0	0	1
20	1	0	0	1	0	100
-200	0	**1**	0	-20	1	8000
-100	0	**1**	0	-10	0	-1000

Tableaux for Klee Minty

1	0	0	1	0	0	1
20	1	0	0	1	0	100
-200	0	1	0	-20	1	8000
100	0	0	0	10	-1	-9000

1	0	0	1	0	0	1
0	1	0	-20	**1**	0	80
0	0	1	200	-20	1	8200
0	0	0	-100	**10**	-1	-9100

1	0	0	**1**	0	0	1
0	1	0	-20	1	0	80
0	20	1	-200	0	1	9800
0	-10	0	**100**	0	-1	-9900

1	0	0	1	0	0	1
20	1	0	0	1	0	100
200	20	1	0	0	1	10000
-100	**-10**	0	0	0	**-1**	-10000

Simplex: efficiency

- Suppose now that we do a simple change of variables:

$$u_j = 100^{1-j} x_j$$

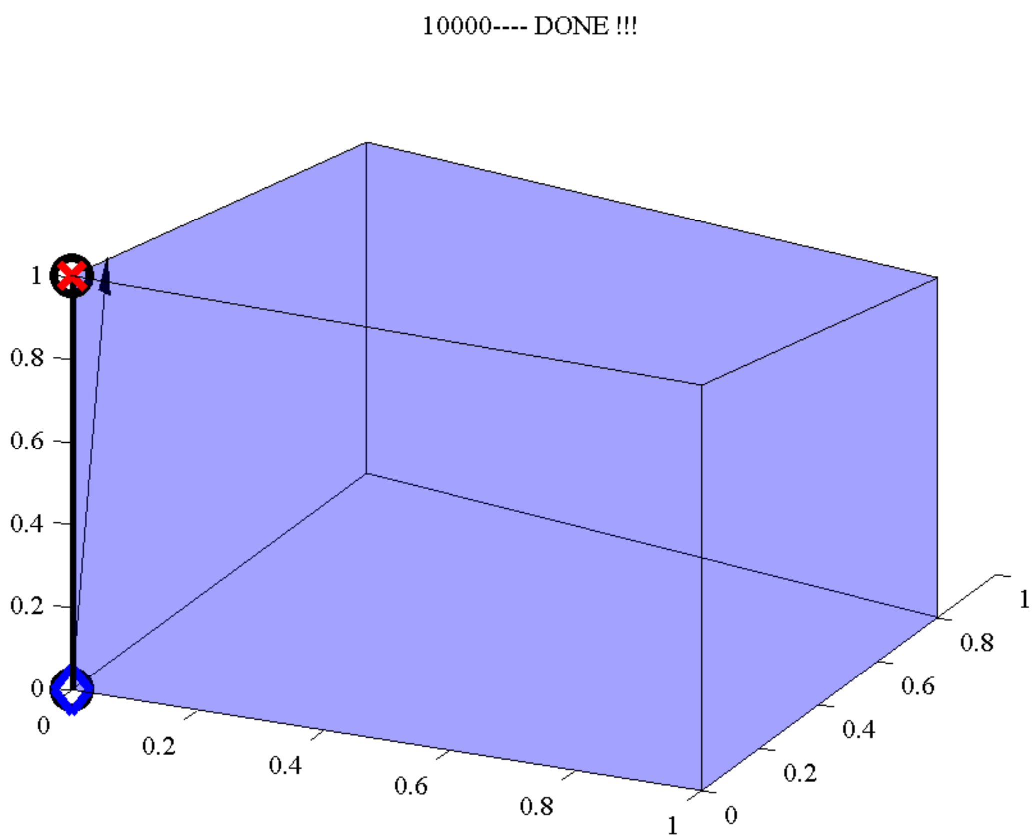
- that is $u_1 = x_1, 100u_2 = x_2$ and $10000u_3 = x_3$
- This is just a scaling of the variables and should not (ideally) affect the complexity of the problem
- The constraint become:

$$\begin{array}{rcrcrcr} u_1 & & & & & \leq & 1 \\ 20u_1 & + & 100u_2 & & & \leq & 100 \\ 200u_1 & + & 2000u_2 & + & 10000u_3 & \leq & 10000. \end{array}$$

- The objective: maximize $100u_1 + 1000u_2 + 10000u_3$.

Simplex: efficiency

- everything should be the same yet...



Tableaux for Klee Minty

- Only one pivot

1	0	0	1	0	0	1
20	100	0	0	1	0	100
200	2000	**10000**	0	0	1	10000
100	1000	**10000**	0	0	0	0

1	0	0	1	0	0	1
20	100	0	0	1	0	100
0	0	1	0	0	0	1
−100	**−1000**	0	0	0	**−1**	−10000

- Embarassing!

Simplex: efficiency

- After the change of variable, the simplex method performs much better...
- This means that the **largest reduced cost coefficient** rule is probably not the most reasonable choice.
- There exist pivot rules for the simplex that are **scale invariant**
- However: **K&M examples have also been found** for most of these rules

Simplex: efficiency

- Klee and Minty show that the largest coefficient rule takes $2^m - 1$ pivots to solve a given problem with m variables and constraints.
- For $m = 70$, this means
$$2^m = 1.2\ 10^{21} \text{ pivots}$$
- At 1000 iterations per second, it will take 40 billion years to solve the problem. (The age of the universe is estimated at 15 billion years)
- On the other hand, very large problems are solved routinely with $m = 10,000$.
- Conclusion here: simplex can take an exponential amount of time on pathological problems.

Simplex: efficiency

Complexity: a few examples for comparison. . .

- Sorting: fast algorithm $O(n \log n)$, simple one $O(n^2)$
- Matrix - matrix product: $O(n^3)$
- Matrix inverse: $O(n^3)$
- Linear Programming with Simplex, worst case: $O(n^2 2^n)$
- Linear Programming with Simplex, average case: $O(n^3)$
- Linear Programming with interior point methods: $O(n^{3.5})$

Simplex: efficiency

n	n^2	n^3	2^n
1	1	1	2
2	4	8	4
3	9	27	8
4	16	64	16
5	25	125	32
6	36	216	64
7	49	343	128
8	64	512	256
9	81	729	512
10	100	1000	1024
12	144	1728	4096
14	196	2744	16384
16	256	4096	65536
18	324	5832	262144
20	400	8000	1048576
22	484	10648	4194304
24	576	13824	16777216
26	676	17576	67108864
28	784	21952	268435456

Simplex: empirical efficiency

Simplex: Complexity History

- Monte-Carlo simulations were pioneered in the 63 (Kuhn& Quandt)
- Objective $\mathbf{c} = \mathbf{1}$, $\mathbf{b} = 10000 \cdot \mathbf{1}$ and and each entry of A selected uniformly between 1 and 1000.
- Limited computational powers: dimensions 5 to 25.
- Computations made on a super-computer in the E-quad (Von Neumann Hall).
- 9 different pivot rules.
- Very successful: again, convergence below $3 \cdot m$ pivots.