NP-Completeness of Dominating Set

Let $G = (V, E)$ be a simple undirected graph. A dominating set in a graph G is a subset of vertices $S \subseteq V$ such that each vertex in V is either in S or is adjacent to some vertex in S. That is, for every vertex $u \in V - S$, there exists a vertex $v \in S$ such that $uv \in E$. A dominating set is minimal if S cannot be contracted further; that is, there exists no vertex $w \in S$ such that $S - \{w\}$ is also a dominating set in G. The problem of finding a minimal dominating set of minimum cardinality is a hard problem.

The DOMINATING SET (DS) decision problem is the following.

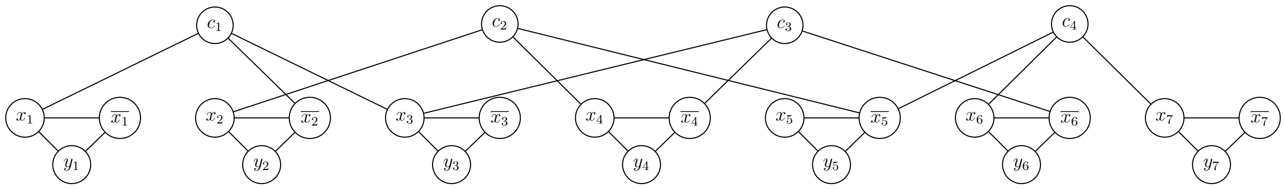
INSTANCE : Given a graph G and an integer k.
QUESTION : Does G have a dominating set of size at most k ?

Theorem : DS is NP-complete.

Proof **:** For each vertex in the graph G, check whether that vertex is in the dominating set S or adjacent to a vertex in S. That is, for each vertex we check every edge incident to it in G to see if that edge connects the vertex to at least one vertex in S. If we ever find a vertex which is not in S and is not adjacent to S, we reject. Otherwise we accept S as the dominating set of the graph G. The overall algorithm runs in polynomial time, so DS is in NP.

The following reduction from 3SAT to DS will establish the NP-completeness of DS. Let E be an instance of the 3SAT problem with n variables $X_1, X_2, ..., X_n$ and m clauses $C_1, C_2, ..., C_m$. Given this instance, we construct a graph G with $3n + m$ vertices. Three vertices correspond to each variable; these are labelled $x_i, \overline{x_i}, y_i$. One vertex corresponds to each clause; these are labelled c_i. The graph G has $3n + 3m$ edges. The three vertices corresponding to each variable are connected by an edge (that is, $x_i, \overline{x_i}, y_i$ forms a triangle). Each clause vertex is connected to its component terms (that is, the clause vertex c_p corresponding to the clause $X_i + \overline{X_j} + X_k$ is connected by edges to the vertices $x_i, \overline{x_j}, x_k$).

For example, consider the 3SAT instance
$E = (X_1 + \overline{X_2} + X_3)(X_2 + X_4 + \overline{X_5})(X_3 + \overline{X_4} + \overline{X_6})(\overline{X_5} + X_6 + X_7)$.
Then the graph G is as given below.



It should be straightforward that this construction can be done in polynomial time in the length of the number of variables and number of clauses in E. We must show that this correctly reduces 3SAT to DS. That is :

E is satisfiable if and only if G has a dominating set of size at most n.

Assume E is satisfiable. Then there exists an assignment of $\top$ or $\perp$ values to the variables such that all clauses evaluate to true. We will form a set S which includes x_i for all X_i which are true in this assignment, and $\overline{x_i}$ for all X_i which are false. This set contains exactly n vertices. All the vertices which came from variables will be covered by this set, since one of $x_i, \overline{x_i}$ was selected for every i. Since the truth assignment makes every clause true, one of the component terms of each clause is true and therefore is in S. It follows that all the vertices are adjacent to vertices in S, and the set is a dominating set. Thus if E is satisfiable then G has a dominating set of size at most n.

Assume there exists a dominating set S of size at most n. Since y_i is either in S or adjacent to a vertex of S for all i, and y_i is connected by edges only to x_i and $\overline{x_i}$, it follows that for every i, either $x_i, \overline{x_i}$, or y_i is in S. This already specifies n vertices, so exactly one vertex for each variable is included in S. To dominate the c-vertices note that none of the y-vertices can be selected to be in S. We create a truth assignment as follows. X_i will be assigned true if x_i is in S. Otherwise X_i will be assigned false. Consider clause C_j. The vertex c_j was not in the dominating set (which only uses variable correlated vertices). So c_j is adjacent to some x_h or $\overline{x_i}$ in the dominating set. If c_j is adjacent to x_h in S, then since X_h is set to true it follows that c_j will be true. If c_j is adjacent to $\overline{x_i}$ in S, then x_i is not in S and X_i will be false, so C_j will be true. It follows that this assignment is a solution for E and E is satisfiable. Thus if G has a dominating set of size at most n then E is satisfiable.

Note that the set of c-vertices will yield a dominating set had we not included the y-vertices as in the construction. That would have given a dominating set in G but not a natural truth assignment for E.

Thus there is a polynomial time reduction from 3SAT to DS. Since 3SAT is NP-complete, it follows that DS is also NP-complete. □

Reference

[1] J. E. Hopcroft, R. Motwani, and J. D. Ullman. *Introduction to Automata Theory, Languages, and Computation*, third edition. Pearson Addison-Wesley, 2007.