Handin6

April 2, 2022

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A The sin-collection

We start out with the space of differentiable functions $f : [0, \pi] \to \mathbb{R}$, with the L^2-inner product:

$$\langle f, g\rangle = \int_0^\pi f(x)g(x)dx$$

We're given the following set of functions, where $x = [0, \pi]$:

$$sin(x), sin(2x), ..., sin((k-1)x)$$

Here we're asked to show that the collection above is orthogonal on the inner product. Taking inspiration from example 13.10 in the notes, we have:

$$\langle sin(mx), sin(nx)\rangle = \int_0^\pi sin(mx)sin(nx)dx,$$

where $m \neq n$ and $m, n \in \mathbb{N}$. Using the identities in 9.11, we get:

$$= \frac{1}{2}\int_0^\pi -cos((m+n)x) + cos((m-n)x)dx$$

$$= \frac{1}{2}\left[\frac{-1}{m+n}sin((m+n)x) + \frac{1}{m-n}sin((m-n)x)\right]_0^\pi$$

$$= \frac{1}{2}\Bigg(\Big(\frac{-1}{m+n}sin((m+n)\pi) + \frac{1}{m-n}sin((m-n)\pi)\Big) - \Big(\frac{-1}{m+n}sin((m+n)0) + \frac{1}{m-n}sin((m-n)0)\Big)\Bigg)$$

$$= \frac{1}{2}\Bigg(\Big(\frac{0}{m+n} + \frac{0}{m-n}\Big) - \Big(\frac{0}{m+n} + \frac{0}{m-n}\Big)\Bigg)$$

$$= 0 \Rightarrow sin(\alpha x) \text{ is an orthogonal collection, where } \{\ \alpha \in \mathbb{N} \mid \alpha > 0\ \}.$$

For cases where $m = n$ the result will be $\frac{\pi}{2}$. This is due to $cos((m-n)x) = cos(0) = 1$, which remains when we integrate with $x = \pi$, then multiply it by $\frac{1}{2}$. This result will be used later in this assignment.

B Even and uneven

We're informed that $f(x) \equiv 1$ is perpendicular (Yes, I had to look that up) on $sin(2x)$ and $sin(4x)$, but not on $sin(x)$ or $sin(3x)$, and asked to clarify, why this is. Firstly, using four cases is not enough for us, so we extend this to generalized terms:

$$\begin{aligned}
\langle f, sin(\alpha x)\rangle &= \int_0^{\pi} 1 \cdot sin(\alpha x), \{\ \alpha \in \mathbb{N} \mid \alpha > 0\ \} \\
&= \left[-\frac{1}{\alpha}cos(\alpha x)\right]_0^{\pi} \\
&= \left(-\frac{1}{\alpha}cos(\alpha\pi)\right) - \left(-\frac{1}{\alpha}cos(0)\right) \\
&= \left(-\frac{1}{\alpha}cos(\alpha\pi)\right) + \frac{1}{\alpha}
\end{aligned}$$

Using our knowledge of cosines, given a multiplum of π, we get ± 1:

$$cos(t\pi) = (-1)^t$$

Substituting with this in our calculations, we get:

$$= \frac{-1}{\alpha}(-1)^{\alpha} + \frac{1}{\alpha}$$

$$= \frac{(-1)^{\alpha+1}}{\alpha} + \frac{1}{\alpha}$$

We see here, that should α be even, the equation results in 0, and $\frac{2}{\alpha}$, should the number be uneven.

C The overkill

To stay in the mindset of generalized functions, we take the projection function that we know, and substitute with the results above, to make it useable with $sin(kx)$-functions, where $k > 0 \wedge k \in \mathbb{N}$:

$$pr_{sin(kx)}(1) = \frac{\left(\frac{(-1)^{k+1}}{k} + \frac{1}{k}\right)}{\pi/2} sin(kx)$$

For the sake of more fun, we can refactor this formula to:

$$pr_{sin(kx)}(1) = \frac{4 \cdot (k\ mod\ 2)}{k\pi} sin(kx)$$

**Quick note: This is purely for the sake of 'fun with mathematics', and completely overkill for the use on only $sin(x)$ and $sin(3x)$.

Our actual use of this is takes the formula 13.3 in the notes (due to our collection being orthogonal), and we setup the following:

$$pr_{sin(x),sin(3x)}(1) = \frac{\langle 1, sin(x)\rangle}{||sin(x)||_2^2} sin(x) + \frac{\langle 1, sin(3x)\rangle}{||sin(3x)||_2^2} sin(3x)$$

Here we use our knowledge that the multiplier k in $sin(kx)$, is uneven, with 1 and 3, so we get:

$$= \frac{2/1}{\pi/2} sin(x) + \frac{2/3}{\pi/2} sin(3x)$$

$$= \frac{4}{\pi} sin(x) + \frac{4}{3\pi} sin(3x)$$

From this point, we want to end up with a function, that is perpendicular on $sin(x)$ and $sin(3x)$, in the form of:

$$1 + c_1 sin(x) + c_3 sin(3x)$$

If we take a look in the notes, we notice just below figure 8.1, that $u - pr_v(u)$, is perpendicular on v. If we apply this logic to our formula, we get that

$$1 - \left(\frac{4}{\pi} sin(x) + \frac{4}{3\pi} sin(3x)\right) \text{ is perpendicular on } sin(x) \text{ and } sin(3x)$$

Now we want to do this for $sin(2x)$ and $sin(4x)$ (The remaining functions in the $sin(kx)$ with k = 5), which means that we have to prove the following:

$$\langle 1 - \frac{4}{\pi} \sin(x) + \frac{4}{3\pi} \sin(3x), \sin(kx) \rangle = 0, \text{ for } k \in \{2, 4\}$$

If we apply the definitions from 13.1 in the notes, we get:

$$\langle 1, \sin(kx) \rangle - \frac{4}{\pi} \langle \sin(x), \sin(kx) \rangle - \frac{4}{3\pi} \langle \sin(3x), \sin(kx) \rangle = 0$$

From here, we simply reuse our knowledge, gained in this assignment. For the second and third part, of the equation, we know that this is 0 and 0, based on assignment a:

$$\langle 1, \sin(kx) \rangle - \frac{4}{\pi} 0 - \frac{4}{3\pi} 0 = 0$$

For the left side, we already know that if n is even, which it is, the inner-product then results in 0:

$$0 - \frac{4}{\pi} 0 - \frac{4}{3\pi} 0 = 0$$

$$0 = 0$$

D Actual python?

In this part of the assigment, we turn our nose to python, to help us with an approximation of:

$$f(x) = 1 - e^{-x} \quad \text{for } 0 \le x \le \pi.$$

Our answer includes elements from chapter 13.2, and 13.3, from the notes. The formula 13.4 is used as a base for this code:

[87]:
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import numpy as np
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x, h = np.linspace(0, np.pi, 20, retstep=True)

# Creating the vector of sin(x)
base = sinGen(np.ones_like(x))

# Shorthand syntax for the function f
f = lambda x: 1-(np.e**-x)

# Returns the vector sin(m*x)
def sinGen(lin, m = 1):
    return np.vectorize(lambda k: (np.sin(m*k)))(f_lin)

# Calculating the inner product of the given f-function, over given space, with␣
↪the given vector g
def inner_product(f, g, x):
    return np.dot(np.vectorize(f)(x), g)

# Calculating the projection of f, onto the span of {sin(x), sin(2x), ... ,␣
↪sin((k-1)x)}
def proj(f, k, x):
    out = (inner_product(f, base, x)/ np.dot(base, base) * base)
    for m in range(2, k):
        out += (inner_product(f, sinGen(x, m), x)/ np.dot(np.sin(m*x), np.
↪sin(m*x)) * np.sin(m*x))
    return out

# Plotting the approximation, with the origional function f
fig, ax = plt.subplots(figsize=(gridWidth,gridHeight))
ax.set_aspect('equal')
plt.grid()
ax.plot(x, f(x))
ax.plot(x, proj(f, 5, x))
```

[87]: [<matplotlib.lines.Line2D at 0x11620ba60>]