THE

THEORY

OF

INTEREST

Second Edition

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Contents

1 The Measurement of Interest

1.1 Introduction

Interest

- compensation a borrower of capital pays to a lender of capital
- lender has to be compensated since they have temporarily lost use of their capital
- interest and capital are almost always expressed in terms of money

1.2 The Accumulation Function and the Amount Function

The Financial Transaction

- an amount of money or capital (Principal) is invested for a period of time
- at the end of the investment period, a total amount (Accumulated Value) is returned
- difference between the Accumulated Value and the Principal is the Interest Earned

Accumulation Function: $a(t)$

- let t be the number of investment years ($t \geq 0$), where $a(0) = 1$
- assume that $a(t)$ is continuously increasing
- $a(t)$ defines the pattern of accumulation for an investment of amount 1

Amount Function: $A(t) = k \cdot a(t)$

- let k be the initial principal invested ($k > 0$) where $A(0) = k$
- $A(t)$ is continuously increasing
- $A(t)$ defines the Accumulated Value that amount k grows to in t years

Interest Earned during the nth period: $I_n = A(n) - A(n-1)$

- interest earned is the difference between the Accumulated Value at the end of a period and the Accumulated Value at the beginning of the period

1.3 The Effective Rate of Interest: i

Definition

- i is the amount of interest earned over a one-year period when 1 is invested
- let i_n be the effective rate of interest earned during the nth period of the investment where interest is paid at the end of the period
- i is also defined as the ratio of the amount of Interest Earned during the period to the Accumulated Value at the beginning of the period

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{I_n}{A(n-1)}, \text{for integral } n \geq 1$$

1.4 Simple Interest

- assume that interest accrues for t years and is then reinvested for another s years where $s < 1$
- let interest earned each year on an investment of 1 be constant at i and

$$a(0) = 1,\ a(1) = 1 + i$$

- simple interest is a linear accumulation function, $a(t) = 1 + it$, for integral $t \geq 0$
- simple interest has the property that interest is **NOT** reinvested to earn additional interest
- a constant rate of simple interest implies a decreasing effective rate of interest:

$$\begin{aligned} i_n &= \frac{A(n) - A(n-1)}{A(n-1)} = \frac{k \cdot a(n) - k \cdot a(n-1)}{k \cdot a(n-1)} \\ &= \frac{a(n) - a(n-1)}{a(n-1)} = \frac{1 + i \cdot n - [1 + i \cdot (n-1)]}{1 + i \cdot (n-1)} \\ &= \frac{i}{1 + i \cdot (n-1)} \end{aligned}$$

Nonintegral Value of t

- assume that interest accrues for t years and is then reinvested for another s years where $s < 1$
- if no interest is credited for fractional periods, then $a(t)$ becomes a step function with discontinuities
- assume that interest accrues proportionately over fractional periods

$$a(t+s) = a(t) + a(s) - 1 = 1 + it + 1 + is - 1 = 1 + i(t+s)$$

- amount of Interest Earned to time t is

$$I = A(0) \cdot it$$

1.5 Compound Interest

- let interest earned each year on an investment of 1 be constant at i and

$$a(0) = 1,\ a(1) = 1 + i$$

- compound interest is an exponential accumulation function $a(t) = (1+i)^t$, for integral $t \geq 0$
- compound interest has the property that interest is reinvested to earn additional interest
- compound interest produces larger accumulations than simple interest when $t > 1$

– a constant rate of compound interest implies a constant effective rate of interest

$$\begin{aligned} i_n &= \frac{A(n) - A(n-1)}{A(n-1)} = \frac{k \cdot a(n) - k \cdot a(n-1)}{k \cdot a(n-1)} \\ &= \frac{a(n) - a(n-1)}{a(n-1)} = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} \\ &= i \end{aligned}$$

Nonintegral Value of t

– assume that interest accrues for t years and is then reinvested for another s years where $s < 1$

– $a(t+s) = a(t) \cdot a(s) = (1+i)^t \cdot (1+i)^s = (1+i)^{t+s}$

1.6 Present Value

Discounting

– Accumulated Value is a future value pertaining to payment(s) made in the past

– Discounted Value is a present value pertaining to payment(s) to be made in the future

– discounting determines how much must be invested initially (X) so that 1 will be accumulated after t years

$$X \cdot (1+i)^t = 1 \rightarrow \; X = \frac{1}{(1+i)^t}$$

– X represents the present value of 1 to be paid in t years

– let $v = \dfrac{1}{1+i}$, v is called a discount factor or present value factor

$$X = 1 \cdot v^t$$

Discount Function: $a^{-1}(t)$

– let $a^{-1}(t) = \dfrac{1}{a(t)}$

– simple interest: $a^{-1}(t) = \dfrac{1}{1+it}$

– compound interest: $a^{-1}(t) = \dfrac{1}{(1+i)^t} = v^t$

– compound interest produces smaller Discount Values than simple interest when $t > 1$

1.7 The Effective Rate of Discount: d

Definition

- an effective rate of interest is taken as a percentage of the balance at the beginning of the year, while an effective rate of discount is at the end of the year.

  eg. if 1 is invested and 6% interest is paid at the end of the year, then the Accumulated Value is 1.06

  eg. if 0.94 is invested after a 6% discount is paid at the beginning of the year, then the Accumulated Value at the end of the year is 1.00

- let d_n be the effective rate of discount earned during the nth period of the investment where discount is paid at the beginning of the period
- d is also defined as the ratio of the amount of interest (amount of discount) earned during the period to the amount invested at the end of the period

$$d_n = \frac{A(n) - A(n-1)}{A(n)} = \frac{I_n}{A(n)}, \text{for integral } n \geq 1$$

- if interest is constant, $i_m = i$, then discount is constant, $d_m = d$

Relationship Between i and d

- if 1 is borrowed and interest is paid at the beginning of the year then $1 - d$ remains
- the accumulated value of $1 - d$ at the end of the year is 1:

$$(1-d)(1+i) = 1$$

- interest rate is the ratio of the discount paid to the amount at the beginning of the period:

$$i = \frac{I_1}{A(0)} = \frac{d}{1-d}$$

- discount rate is the ratio of the interest paid to the amount at the end of the period:

$$d = \frac{I_1}{A(1)} = \frac{i}{1+i}$$

- the present value of interest paid at the end of the year is the discount paid at the beginning of the year

$$iv = d$$

- the present value of 1 to be paid at the end of the year is the same as borrowing $1 - d$ and repaying 1 at the end of the year (if both have the same value at the end of the year, then they have to have the same value at the beginning of the year)

$$1 \cdot v = 1 - d$$

- the difference between interest paid at the end and at the beginning of the year depends on the difference that is borrowed at the beginning of the year and the interest earned on that difference

$$i - d = i[1 - (1 - d)] = i \cdot d \geq 0$$

Discount Function: $a^{-1}(t)$

- let $d_n = d$
- under the simple discount model, the discount function is

$$a^{-1}(t) = 1 - dt \quad \text{for } \ 0 \leq t < 1/d$$

- under the compound discount model, the discount function is

$$a^{-1}(t) = (1 - d)^t = v^t \quad \text{for } \ t \geq 0$$

- a constant rate of simple discount implies an increasing effective rate of discount

$$\begin{aligned}
d_n &= \frac{A(n) - A(n-1)}{A(n)} = \frac{k \cdot a(n) - k \cdot a(n-1)}{k \cdot a(n)} \\
&= 1 - \frac{a(n-1)}{a(n)} = 1 - \frac{a^{-1}(n)}{a^{-1}(n-1)} \\
&= 1 - \frac{(1 - d \cdot n)}{1 - d(n-1)} = \frac{1 - d \cdot n + d - 1 + d \cdot n}{1 - d(n-1)} \\
&= \frac{d}{1 - d(n-1)}
\end{aligned}$$

- a constant rate of compound discount implies a constant effective rate of discount

$$\begin{aligned}
d_n &= \frac{A(n) - A(n-1)}{A(n)} = \frac{k \cdot a(n) - k \cdot a(n-1)}{k \cdot a(n)} \\
&= 1 - \frac{a(n-1)}{a(n)} = 1 - \frac{a^{-1}(n)}{a^{-1}(n-1)} \\
&= 1 - \frac{(1-d)^n}{(1-d)^{n-1}} \\
&= 1 - (1 - d) \\
&= d
\end{aligned}$$

1.8 Nominal Rate of Interest and Discount Convertible m^{th}ly: $i^{(m)}, d^{(m)}$

Definition

- an effective rate of interest (discount) is paid once per year at the end(beginning) of the year
- a nominal rate of interest (discount) is paid more frequently during the year (m times) and at the end (beginning) of the sub-period (nominal rates are also quoted as annual rates)
- nominal rates are adjusted to reflect the rate to be paid during the sub–period

$$i^{(2)} = 10\% \rightarrow \frac{i^{(2)}}{2} = \frac{10\%}{2} = 5\% \text{ paid every 6 months}$$

Equivalency to Effective Rates of Interest: $i, i^{(m)}$

- with effective interest, you have interest, i, paid at the end of the year
- with nominal interest, you have interest $\frac{i^{(m)}}{m}$, paid at the end of each sub-period and this is done m times over the year (m sub-periods per year)

$$(1+i) = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

- if given an effective rate of interest, a nominal rate of interest can be determined

$$i^{(m)} = m[(1+i)^{1/m} - 1]$$

- the interest rate per sub-period can be determined, if given the effective interest rate

$$\frac{i^{(m)}}{m} = (1+i)^{1/m} - 1$$

Equivalency to Effective Rates of Discount: $d, d^{(m)}$

- with effective discount, you have discount, d, paid at the beginning of the year
- with nominal discount, you have discount $\frac{d^{(m)}}{m}$, paid at the beginning of each sub-period and this is done m times over the year (m sub-periods per year)

$$(1-d) = \left(1 - \frac{d^{(m)}}{m}\right)^m$$

- if given an effective rate of discount, a nominal rate of discount can be determined

$$d^{(m)} = m[1 - (1-d)^{1/m}]$$

- the discount rate per sub-period can be determined, if given the effective discount rate

$$\frac{d^{(m)}}{m} = 1 - (1-d)^{1/m}$$

Relationship Between $\dfrac{i^{(m)}}{m}$ and $\dfrac{d^{(m)}}{m}$

– when using effective rates, you must have $(1+i)$ or $(1-d)^{-1}$ by the end of the year

$$(1+i) = \frac{1}{v} = \frac{1}{(1-d)} = (1-d)^{-1}$$

– when replacing the effective rate formulas with their nominal rate counterparts, you have

$$\left(1+\frac{i^{(m)}}{m}\right)^{m} = \left(1-\frac{d^{(p)}}{p}\right)^{-p}$$

– when $p = m$

$$\begin{aligned}
\left(1+\frac{i^{(m)}}{m}\right)^{m} &= \left(1-\frac{d^{(m)}}{m}\right)^{-m} \\
1+\frac{i^{(m)}}{m} &= \left(1-\frac{d^{(m)}}{m}\right)^{-1} \\
1+\frac{i^{(m)}}{m} &= \frac{m}{m \cdot d^{(m)}} \\
\frac{i^{(m)}}{m} &= \frac{m}{m \cdot d^{(m)}} - 1 = \frac{m-m+d^{(m)}}{m-d^{(m)}} \\
\frac{i^{(m)}}{m} &= \frac{d^{(m)}}{m-d^{(m)}} \\
\frac{i^{(m)}}{m} &= \frac{\dfrac{d^{(m)}}{m}}{1-\dfrac{d^{(m)}}{m}}
\end{aligned}$$

– the interest rate over the sub-period is the ratio of the discount paid to the amount at the beginning of the sub-period (principle of the interest rate still holds)

$$\frac{d^{(m)}}{m} = \frac{\dfrac{i^{(m)}}{m}}{1+\dfrac{i^{(m)}}{m}}$$

– the discount rate over the sub–period is the ratio of interest paid to the amount at the end of the sub-period (principle of the discount rate still holds)

– the difference between interest paid at the end and at the beginning of the sub-period depends on the difference that is borrowed at the beginning of the sub-period and on the interest earned on that difference (principle of the interest and discount rates still holds)

$$\begin{aligned}
\frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} &= \frac{i^{(m)}}{m}\left[1-\left(1-\frac{d^{(m)}}{m}\right)\right] \\
&= \frac{i^{(m)}}{m} \cdot \frac{d^{(m)}}{m} \geq 0
\end{aligned}$$

1.9 Forces of Interest and Discount: δ_n^i, δ_n^d

Definitions

- annual effective rate of interest and discount are applied over a one-year period
- annual nominal rate of interest and discount are applied over a sub-period once the rates have been converted
- annual force of interest and discount are applied over the smallest sub-period imaginable (at a moment in time) i.e. $m \to \infty$

Annual Force of Interest At Time $n : \delta_n^i$

- recall that the interest rate over a sub-period is the ratio of the Interest Earned during that period to the Accumulated Value at the beginning of the period

$$\frac{i^{(m)}}{m} = \frac{A\left(n + \frac{1}{m}\right) - A(n)}{A(n)}$$

- if $m = 12$, $\dfrac{i^{(12)}}{12} = \dfrac{A\left(n + \frac{1}{12}\right) - A(n)}{A(n)}$ = monthly rate; monthly rate x 12 = annual rate
- if $m = 365$, $\dfrac{i^{(365)}}{365} = \dfrac{A\left(n + \frac{1}{365}\right) - A(n)}{A(n)}$ = daily rate; daily rate x 365 = annual rate
- if $m = 8760$, $\dfrac{i^{(8760)}}{8760} = \dfrac{A\left(n + \frac{1}{8760}\right) - A(n)}{A(n)}$ = hourly rate; hourly rate x 8760 = annual rate
- if $m \to \infty$, $\displaystyle\lim_{m\to\infty} \frac{i^{(m)}}{m} = \lim_{m\to\infty} \frac{A\left(n + \frac{1}{m}\right) - A(n)}{A(n)}$ = instantaneous rate
- let $\displaystyle\delta_n^i = \lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} \frac{\left[\dfrac{A\left(n + \frac{1}{m}\right) - A(n)}{\frac{1}{m}}\right]}{A(n)}$ = Force of Interest At Time n

$$\delta_n^i = \frac{\frac{d}{dn}A(n)}{A(n)} = \frac{\frac{d}{dn}k \cdot a(n)}{k \cdot a(n)} = \frac{\frac{d}{dn}a(n)}{a(n)}$$

$$\delta_n^i = \frac{d}{dn}ln[A(n)] = \frac{d}{dn}ln[a(n)]$$

Accumulation Function Using the Force of Interest

– recall that the Force of Interest is defined as

$$\delta_n^i = \frac{d}{dn} ln[a(n)] \rightarrow \delta_n^i \cdot dn = d(ln[a(n)])$$

– integrating both sides from time 0 to t results in

$$\begin{aligned}\int_0^t \delta_n^i \cdot dn &= \int_0^t d(ln[a(n)]) \\ &= ln[a(t)] - ln[a(0)] \\ &= ln\left[\frac{a(t)}{a(0)}\right] \\ \int_0^t \delta_n^i \cdot dn &= ln[a(t)])\end{aligned}$$

– taking the exponential function of both sides results in

$$e^{\int_0^t \delta_n^i \cdot dn} = a(t)$$

– the Accumulation Function can therefore be defined as an exponential function where the annual force of interest is converted into an infinitesimally small rate [$\delta_n^i \cdot dn$]; this small rate is then applied over every existing moment from time 0 to time t

Interest Earned Over t Years Using the Force of Interest

– recall that the Force of Interest is also defined as

$$\delta_n^i = \frac{\frac{d}{dn} A(n)}{A(n)} \rightarrow A(n) \cdot \delta_n^i \cdot dn = d(A(n))$$

– integrating both sides from time 0 to t results in

$$\begin{aligned}\int_0^t A(n) \cdot \delta_n^i \cdot dn &= \int_0^t d(A(n)) \\ \int_0^t A(n) \cdot \delta_n^i \cdot dn &= A(t) - A(0)\end{aligned}$$

– the Interest Earned over a t year period can be found by applying the interest rate that exists at a certain moment, $\delta_n^i \cdot dn$, to the balance at that moment, A(n), and evaluating it for every moment from time 0 to t

Annual Force of Discount At Time n = Annual Force of Interest At Time n : $\delta_n^d = \delta_n^i$

– recall that the interest rate over a sub-period is the ratio of the Interest Earned during that period to the Accumulated Value at the end of the period

$$\frac{d^{(m)}}{m} = \frac{A\left(n+\frac{1}{m}\right) - A(n)}{A(n+\frac{1}{m})}$$

– if $m = 12$, $\dfrac{d^{(12)}}{12} = \dfrac{A\left(n+\frac{1}{12}\right) - A(n)}{A(n+\frac{1}{12})}$ = monthly rate; monthly rate x 12 = annual rate

– if $m = 365$, $\dfrac{d^{(365)}}{365} = \dfrac{A\left(n+\frac{1}{365}\right) - A(n)}{A(n+\frac{1}{365})}$ = daily rate; daily rate x 365 = annual rate

– if $m = 8760$, $\dfrac{d^{(8760)}}{8760} = \dfrac{A\left(n+\frac{1}{8760}\right) - A(n)}{A(n+\frac{1}{8760})}$ = hourly rate; hourly rate x 8760 = annual rate

– if $m \to \infty$, $\displaystyle\lim_{m\to\infty} \frac{d^{(m)}}{m} = \lim_{m\to\infty} \frac{A\left(n+\frac{1}{m}\right) - A(n)}{A(n+\frac{1}{m})}$ = instantaneous rate

– let $\displaystyle\delta_n^d = \lim_{m\to\infty} d^{(m)} = \lim_{m\to\infty} \frac{\left[\dfrac{A\left(n+\frac{1}{m}\right) - A(n)}{\frac{1}{m}}\right]}{A(n+\frac{1}{m})}$ = Force of Discount At Time n

$$\delta_n^d = \lim_{m\to\infty} \frac{\frac{d}{dn}A(n)}{A(n+\frac{1}{m})} \cdot \frac{A(n)}{A(n)}$$

$$\delta_n^d = \frac{\frac{d}{dn}A(n)}{A(n)} \cdot \lim_{m\to\infty} \frac{A(n)}{A(n+\frac{1}{m}))} = \frac{\frac{d}{dn}A(n)}{A(n)} \cdot 1$$

$$\delta_n^d = \delta_n^i$$

– an alternative approach to determine the Force of Discount is to take the derivative of the discount functions (remember that δ_n^i took the derivative of the accumulation functions)

$$\delta_n^d = \frac{-\frac{d}{dn}a^{-1}(n)}{a^{-1}(n)} = \frac{-\frac{d}{dn}\frac{1}{a(n)}}{\frac{1}{a(n)}} = \frac{-(-1)\frac{1}{a(n)^2}\frac{d}{dn}a(n)}{\frac{1}{a(n)}} = \frac{\frac{d}{dn}a(n)}{a(n)}$$

$$\delta_n^d = \delta_n^i$$

– from now on, we will use δ_n instead of δ_n^i or δ_n^d

Force of Interest When Interest Rate Is Constant

– δ_n can vary at each instantaneous moment

– let the Force of Interest be constant each year: $\delta_n = \delta \rightarrow i_n = i$, then

$$
\begin{aligned}
a(t) &= e^{\int_0^t \delta_n^i \cdot dn} = e^{\int_0^t \delta \cdot dn} \\
&= e^{\delta \cdot t} = (1+i)^t \rightarrow \underbrace{e^{\delta} = 1+i}_{\delta = ln[1+i]} \rightarrow e^{-\delta} = v
\end{aligned}
$$

– note that nominal rates can now be introduced

$$
1+i = \left(1 + \frac{i^{(m)}}{m}\right)^m = \left(1 - \frac{d^{(p)}}{p}\right)^{-p} = e^{\delta}
$$

Force of Interest Under Simple Interest

– a constant rate of simple interest implies a decreasing force of interest

$$
\begin{aligned}
\delta_n &= \frac{\frac{d}{dn}a(n)}{a(n)} = \frac{\frac{d}{dn}(1 + i \cdot n)}{1 + i \cdot n} \\
&= \frac{i}{1 + i \cdot n}
\end{aligned}
$$

Force of Interest Under Simple Discount

– a constant rate of simple discount implies an increasing force of interest

$$
\begin{aligned}
\delta_n &= \frac{-\frac{d}{dn}a^{-1}(n)}{a^{-1}(n)} = \frac{-\frac{d}{dn}(1 - d \cdot n)}{1 - d \cdot n} \\
&= \frac{d}{1 - d \cdot n} \quad \text{for } \ 0 \leq t < 1/d
\end{aligned}
$$

1.10 Varying Interest

Varying Force of Interest

– recall the basic formula

$$a(t) = e^{\int_0^t \delta_n dn}$$

– if δ_n is readily integrable, then $a(t)$ can be derived easily

– if δ_n is not readily integrable, then approximate methods of integration are required

Varying Effective Rate of Interest

– the more common application

$$a(t) = \prod_{k=1}^{t} (1 + i_k)$$

and

$$a^{-1}(t) = \prod_{k=1}^{t} \frac{1}{(1 + i_k)}$$

1.11 Summary of Results

Rate of interest or discount	$a(t)$	$a^{-1}(t)$
Compound interest		
i	$(1+i)^t$	$v^t = (1+i)^{-t}$
$i^{(m)}$	$\left[1 + \frac{i^{(m)}}{m}\right]^{mt}$	$\left[1 + \frac{i^{(m)}}{m}\right]^{-mt}$
d	$(1-d)^t$	$(1-d)^t$
$d^{(m)}$	$\left[1 - \frac{d^{(m)}}{m}\right]^{-mt}$	$\left[1 - \frac{d^{(m)}}{m}\right]^{mt}$
δ	$e^{\delta t}$	$e^{-\delta t}$
Simple interest		
i	$1 + it$	$(1+it)^{-1}$
Simple discount		
d	$(1-dt)^{-1}$	$1 - dt$

2 Solution of Problems in Interest

2.1 Introduction

How to Solve an Interest Problem

– use basic principles

– develop a systematic approach

2.2 Obtaining Numerical Results

– using a calculator with exponential functions is the obvious first choice

– in absence of such a calculator, using the Table of Compound Interest Functions: Appendix I (page 376 – 392) would be the next option

– series expansions could be used as a last resort

$$\text{e.g.}\quad (1+i)^k = 1 + ki + \frac{k(k-1)}{2!}i^2 + \frac{k(k-1)(k-2)}{3!}i^3 + \cdots$$
$$\text{e.g.}\quad e^{k\delta} = 1 + k\delta + \frac{(k\delta)^2}{2!} + \frac{(k\delta)^3}{3!} + \cdots$$

A Common Problem

– using compound interest for integral periods of time and using simple interest for fractional periods is an exercises in linear interpolation

$$\begin{aligned}\text{e.g.}\quad (1+i)^{n+k} &\approx (1-k)(1+i)^n + k(1+i)^{n+1}\\ &= (1+i)^n[(1-k)+k(1+i)]\\ &= (1+i)^n(1+ki)\\ \text{e.g.}\quad (1-d)^{n+k} &\approx (1-k)(1-d)^n + k(1-d)^{n+1}\\ &= (1-d)^n(1-kd)\end{aligned}$$

2.3 Determining Time Periods

– when using simple interest, there are 3 different methods for counting the days in an investment period

(i) exact simple interest approach: count actual number of days where one year equals 365 days

(ii) ordinary simple interest approach: one month equals 30 days; total number of days between D_2, M_2, Y_2 and D_1, M_1, Y_1 is

$$360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1)$$

(iii) Banker's Rule: count actual number of days where one year equals 360 days

2.4 The Basic Problem

– there are 4 variables required in order to solve an interest problem

(a) original amount(s) invested
(b) length of investment period(s)
(c) interest rate
(d) accumulated value(s) at the end of the investment period

– if you have 3 of the above variables, then you can solve for the unknown 4th variable

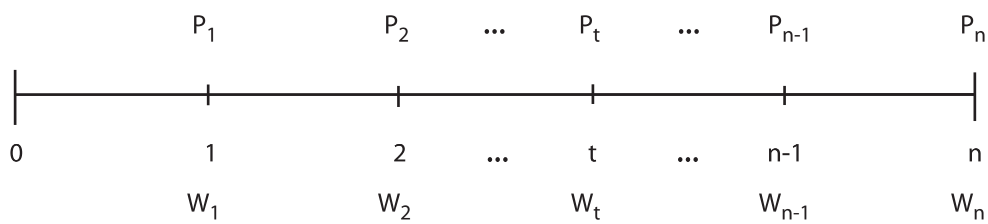
2.5 Equations of Value

– the value at any given point in time, t, will be either a present value or a future value (sometimes referred to as the time value of money)

– the time value of money depends on the calculation date from which payment(s) are either accumulated or discounted to

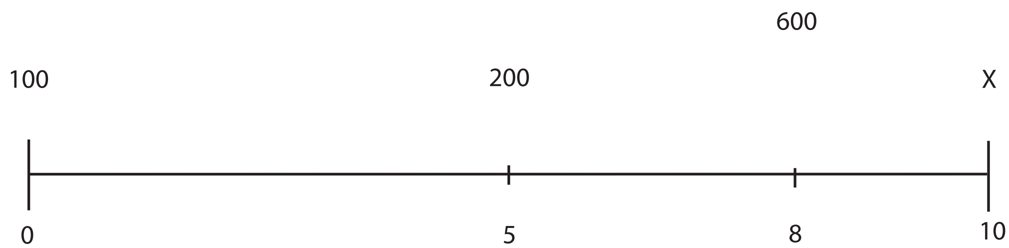
Time Line Diagrams

– it helps to draw out a time line and plot the payments and withdrawals accordingly

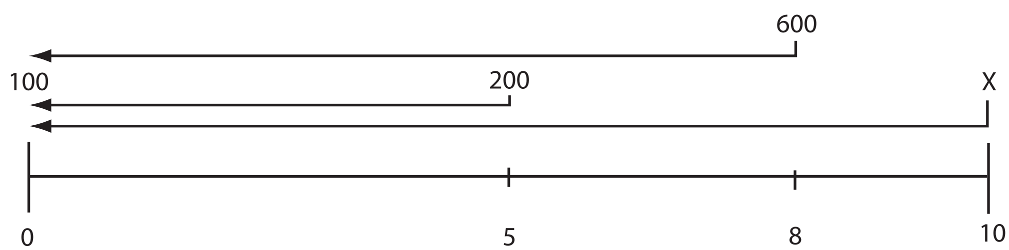


Example

– a \$600 payment is due in 8 years; the alternative is to receive \$100 now, \$200 in 5 years and \$$X$ in 10 years. If $i = 8\%$, find \$$X$, such that the value of both options is equal.

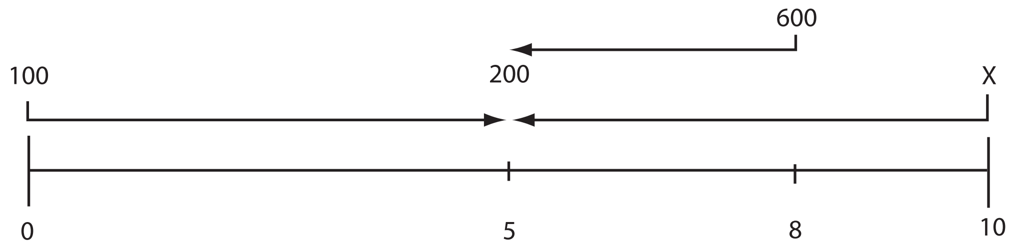


– compare the values at $t = 0$



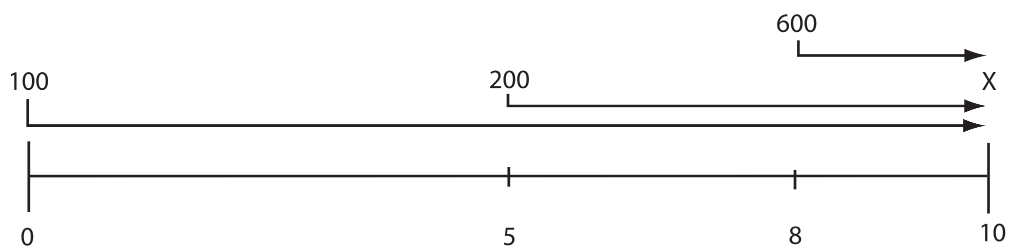
$$600v_{8\%}^{8} = 100 + 200v_{8\%}^{5} + Xv_{8\%}^{10}$$
$$X = \frac{600v_{8\%}^{8} - 100 - 200v_{8\%}^{5}}{v_{8\%}^{10}} = 190.08$$

– compare the values at $t = 5$



$$600v^3 = 100(1+i)^5 + 200 + Xv^5$$
$$X = \frac{600v^3 - 100(1+i)^5 - 200}{v^5} = 190.08$$

– compare the values at $t = 10$



$$600(1+i)^2 = 100(1+i)^{10} + 200(1+i)^5 + X$$
$$X = 600(1+i)^2 - 100(1+i)^{10} - 200(1+i)^5 = 190.08$$

– all 3 equations gave the same answer because all 3 equations treated the value of the payments consistently at a given point of time.

2.6 Unknown Time

Single Payment

– the easiest approach is to use logarithms

<u>Example</u>

– How long does it take money to double at $i = 6\%$?

$$\begin{aligned}(1.06)^n &= 2 \\ n\ln[1.06] &= \ln[2] \\ n &= \frac{\ln[2]}{\ln[1.06]} = 11.89566 \text{ years}\end{aligned}$$

– if logarithms are not available, then use an interest table from Appendix I (page 376 – 392) and perform a linear interpolation

$$(1.06)^n = 2$$

go to page 378, and find:

$$\begin{aligned}&(1.06)^{11} = 1.89830 \quad \text{and} \\ &(1.06)^{12} = 2.01220 \\ &\therefore \quad n = 11 + \frac{2 - 1.89830}{2.01220 - 1.89830} = 11.89 \text{ years}\end{aligned}$$

– **Rule of 72** for doubling a single payment

$$\begin{aligned}n &= \frac{\ln[2]}{\ln[1+i]} = \frac{0.6931}{i} \cdot \frac{i}{\ln[1+i]} \\ &= \frac{0.6931}{i}(1.0395), \quad \text{when } i = 8\% \\ n &\approx \frac{0.72}{i}\end{aligned}$$

– **Rule of 114** for tripling a single payment

$$\begin{aligned}n &= \frac{\ln[3]}{\ln[1+i]} = \frac{1.0986}{i} \cdot \frac{i}{\ln[1+i]} \\ &= \frac{1.0986}{i}(1.0395), \quad \text{when} i = 8\% \\ n &\approx \frac{1.14}{i}\end{aligned}$$

An Approximate Approach For Multiple Payments

– let S_t represent a payment made at time t such that

– we wish to replace the multiple payments with a single payment equal to $\displaystyle\sum_{k=1}^{n} S_k$ such that the present value of this single payment at a single moment in time (call it t) is equal to the present value of the multiple payments.

– to find the true value of t:

$$
\begin{aligned}
(S_1 + S_2 + \cdots + S_n) \cdot v^t &= S_1 v^{t_1} + S_2 v^{t_2} + \cdots + S_n v^{t_n} \\
\left(\sum_{k=1}^{n} S_k \right) \cdot v^t &= \sum_{k=1}^{n} S_k v^{t_k} \\
v^t &= \frac{\displaystyle\sum_{k=1}^{n} S_k v^{t_k}}{\displaystyle\sum_{k=1}^{n} S_k} \\
t \ln[v] &= \ln \left[\frac{\displaystyle\sum_{k=1}^{n} S_k v^{t_k}}{\displaystyle\sum_{k=1}^{n} S_k} \right] \\
t &= \frac{\ln \left[\frac{\displaystyle\sum_{k=1}^{n} S_k v^{t_k}}{\displaystyle\sum_{k=1}^{n} S_k} \right]}{\ln[v]} \\
t &= \frac{\ln \left[\displaystyle\sum_{k=1}^{n} S_k v^{t_k} \right] - \ln \left[\displaystyle\sum_{k=1}^{n} S_k \right]}{\ln[v]}
\end{aligned}
$$

– to find an approximate value of t:

– let $\bar{t}$ equal the weighted average of time (weighted by the payments)

$$\bar{t} = \frac{S_1 \cdot t_1 + S_2 \cdot t_2 + \cdots + S_{n-1} \cdot t_{n-1} + S_n \cdot t_n}{S_1 + S_2 + \cdots + S_{n-1} + S_n}$$

$$\bar{t} = \frac{\sum_{k=1}^{n} S_k t_k}{\sum_{k=1}^{n} S_k} \qquad \text{method of equated time}$$

– if $\bar{t} > t$, then the present value using the method of equated time will be less than the present value using exact t

Algebraic Proof: $\bar{t} > t$

– let v^{t_k} be the present value of a future payment of 1 at timet_k and let S_k be the number of payments made at time k

(a) arithmetic weighted mean of present values

$$\frac{S_1 v^{t_1} + S_2 v^{t_2} + \cdots + S_n v^{t_n}}{S_1 + S_2 + \cdots + S_n} = \frac{\sum_{k=1}^{n} S_k v^{t_k}}{\sum_{k=1}^{n} S_k}$$

(b) geometric weighted mean of present value

$$\begin{aligned}
&\left[\left(v^{t_1}\right)^{S_1} \cdot \left(v^{t_2}\right)^{S_2} \cdots \left(v^{t_n}\right)^{S_n}\right]^{\frac{1}{S_1 + S_2 + \cdots + S_n}} \\
&= \left[v^{S_1 \cdot t_1 + S_2 \cdot t_2 + \cdots + S_n \cdot t_n}\right]^{\frac{1}{S_1 + S_2 + \cdots + S_n}} \\
&= v^{\frac{S_1 \cdot t_1 + S_2 \cdot t_2 + \cdots + S_n \cdot t_n}{S_1 + S_2 + \cdots + S_n}} \\
&= v^{\bar{t}}
\end{aligned}$$

Since geometric means are less than arithmetic means,

$$v^{\bar{t}} < \frac{\sum_{k=1}^{n} S_k v^{t_k}}{\sum_{k=1}^{n} S_k}$$

$$\left[\sum_{k=1}^{n} S_k\right] \cdot v^{\bar{t}} < \sum_{k=1}^{n} S_k v^{t_k} = \left[\sum_{k=1}^{n} S_k\right] \cdot v^{t}$$

Present Value: Method of Equated Time < Present Value: Exact t

2.7 Unknown Rate of Interest

– it is quite common to have a financial transaction where the rate of return needs to be determined

Single Payment

– interest rate is easy to determine if a calculator with exponential and logarithmic functions is available
Example

  – \$100 investment triples in 10 years at nominal rate of interest convertible quarterly. Find $i^{(4)}$.

$$1,000\left(1+\frac{i^{(4)}}{4}\right)^{4\times 10} = 3,000$$

$$i^{(4)} = 4\left(3^{\frac{1}{40}} - 1\right) = 0.1114$$

Multiple Payments

– interest rate is easy to determine if there are only a small number of payments and the equation of value can be reduced to a polynomial that is not too difficult to solve
Example

  – At what effective interest rate will the present value of \$200 at the end of 5 years and \$300 at the end of 10 years be equal to \$500?

$$\begin{aligned}
&200v^5 + 300v^{10} = 500 \\
&\underbrace{3}_{a}\left(v^5\right)^2 \underbrace{+2}_{b} v^5 \underbrace{-5}_{c} = 0 \qquad \rightarrow \text{ quadratic formula} \\
&v^5 = \frac{-2+\sqrt{2^2-4(3)(-5)}}{2(3)} = \frac{-2+\sqrt{64}}{6} = \frac{-2+8}{6} = 1 \\
&v^5 = 1 \qquad \rightarrow (1+i)^5 = 1 \quad \rightarrow i = 0\%
\end{aligned}$$

- when a quadratic formula cannot be found, then linear interpolation may be used
  Example
  - At what effective interest rate will an investment of \$100 immediately and \$500 3 years from now accumulate to \$1000 10 years from now?

$$100(1+i)^{10} + 500(1+i)^{7} = 1000$$
$$(1+i)^{10} + 5(1+i)^{7} = 10 = f(i)$$

  Use trial and error and find where $f(i^{-}) < 10$ and $f(i^{+}) > 10$ and then linearly interpolate. The closer to 10 you can get, the more accurate will be the answer:

$$f(9\%) = 9.68$$
$$f(i) = 10$$
$$f(10\%) = 10.39$$
$$i = 9 + \frac{10 - 9.68}{10.39 - 9.68} = 9.45\%$$

  The actual answer is 9.46%.
- a higher level of accuracy can be achieved if the linear interpolation is repeated until the desired numbers of decimal accuracy is achieved

2.8 Practical Examples

In the real world, interest rates are expressed in a number of ways:

- e.g. A bank advertising deposit rates as "5.87%/6%" yield is saying $i^{(4)} = 5.87\%$ and $i = 6\%$. (they often neglect to mention the conversion rate).
- e.g. United States Treasury bills (T-bills) are $13, 26$ or 52 week deposits where the interest rates quoted are actually discount rates. Longer–term Treasury securities will quote interest rates.
- e.g. Short–term commercial transactions often are based using discount rates on a simple discount basis
- e.g. Credit cards charge interest on the ending balance of the prior month. In other words, a card holder who charges in October will not be charged with interest until November. The card holder is getting an interest-free loan from the time of their purchase to the end of the month if they pay off the whole balance. This is why interest rates on credit cards are high; companies need to make up for the lack of interest that is not charged during the month of purchase.

3 Basic Annuities

3.1 Introduction

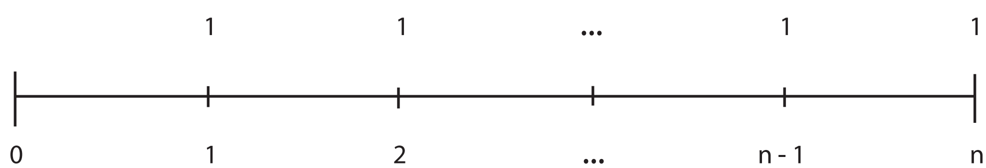
Definition of An Annuity

- a series of payments made at equal intervals of time (annually or otherwise)
- payments made for certain for a fixed period of time are called an annuity-certain
- the payment frequency and the interest conversion period are equal (this will change in Chapter 4)
- the payments are level (this will also change in Chapter 4)

3.2 Annuity-Immediate

Definition

- payments of 1 are made at the end of every year for n years



- the present value (at $t = 0$) of an annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $a_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}
a_{\overline{n}|i} &= (1)v + (1)v^2 + \cdots + (1)v^{n-1} + (1)v^n \\
&= v(1 + v + v^2 + \cdots + v^{n-2} + v^{n-1}) \\
&= \left(\frac{1}{1+i}\right)\left(\frac{1-v^n}{1-v}\right) \\
&= \left(\frac{1}{1+i}\right)\left(\frac{1-v^n}{d}\right) \\
&= \left(\frac{1}{1+i}\right)\left(\frac{1-v^n}{\frac{i}{1+i}}\right) \\
&= \frac{1-v^n}{i}
\end{aligned}$$

– the accumulated value (at $t = n$) of an annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $s_{\overline{n|}i}$ and is calculated as follows:

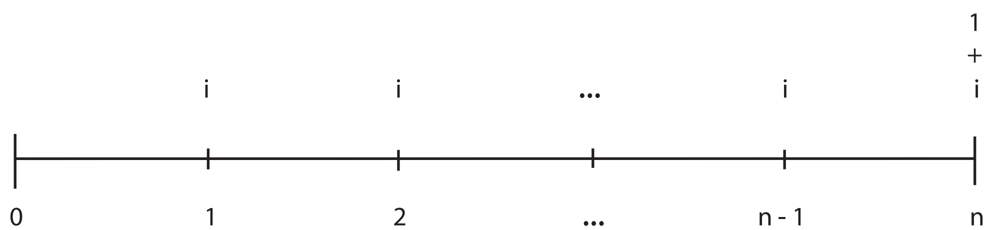
$$
\begin{aligned}
s_{\overline{n|}i} &= 1 + (1)(1+i) + \cdots + (1)(1+i)^{n-2} + (1)(1+i)^{n-1} \\
&= \frac{1-(1+i)^n}{1-(1+i)} \\
&= \frac{1-(1+i)^n}{-i} \\
&= \frac{(1+i)^n - 1}{i}
\end{aligned}
$$

Basic Relationship $1 : 1 = i \cdot a_{\overline{n|}} + v^n$

Consider an n–year investment where 1 is invested at time 0.

The present value of this single payment income stream at $t = 0$ is 1.

Alternatively, consider a n–year investment where 1 is invested at time 0 and produces annual interest payments of $(1) \cdot i$ at the end of each year and then the 1 is refunded at $t = n$.



The present value of this multiple payment income stream at $t = 0$ is $i \cdot a_{\overline{n|}} + (1)v^n$.

Note that $a_{\overline{n|}} = \dfrac{1-v^n}{i} \rightarrow 1 = i \cdot a_{\overline{n|}} + v^n$.

Therefore, the present value of both investment opportunities are equal.

Basic Relationship $2: PV(1+i)^n = FV$ **and** $PV = FV \cdot v^n$

– if the future value at time n, $s_{\overline{n}|}$, is discounted back to time 0, then you will have its present value, $a_{\overline{n}|}$

$$\begin{aligned} s_{\overline{n}|} \cdot v^n &= \left[\frac{(1+i)^n - 1}{i}\right] \cdot v^n \\ &= \frac{(1+i)^n \cdot v^n - v^n}{i} \\ &= \frac{1 - v^n}{i} \\ &= a_{\overline{n}|} \end{aligned}$$

– if the present value at time 0, $a_{\overline{n}|}$, is accumulated forward to time n, then you will have its future value, $s_{\overline{n}|}$

$$\begin{aligned} a_{\overline{n}|} \cdot (1+i)^n &= \left[\frac{1 - v^n}{i}\right](1+i)^n \\ &= \frac{(1+i)^n - v^n(1+i)^n}{i} \\ &= \frac{(1+i)^n - 1}{i} \\ &= s_{\overline{n}|} \end{aligned}$$

Basic Relationship $3: \dfrac{1}{a_{\overline{n}|}} = \dfrac{1}{s_{\overline{n}|}} + i$

Consider a loan of 1, to be paid back over n years with equal annual payments of P made at the end of each year. An annual effective rate of interest, i, is used. The present value of this single payment loan must be equal to the present value of the multiple payment income stream.

$$P \cdot a_{\overline{n}|i} = 1$$

$$P = \frac{1}{a_{\overline{n}|i}}$$

Alternatively, consider a loan of 1, where the annual interest due on the loan, $(1)i$, is paid at the end of each year for n years and the loan amount is paid back at time n.

In order to produce the loan amount at time n, annual payments at the end of each year, for n years, will be made into an account that credits interest at an annual effective rate of interest i.

The future value of the multiple deposit income stream must equal the future value of the single payment, which is the loan of 1.

$$D \cdot s_{\overline{n}|i} = 1$$

$$D = \frac{1}{s_{\overline{n}|i}}$$

The total annual payment will be the interest payment and account payment:

$$i + \frac{1}{s_{\overline{n}|i}}$$

Note that

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-v^n} \times \frac{(1+i)^n}{(1+i)^n} = \frac{i(1+i)^n}{(1+i)^n - 1}$$

$$= \frac{i(1+i)^n + i - i}{(1+i)^n - 1} = \frac{i[(1+i)^n - 1] + i}{(1+i)^n - 1}$$

$$= i + \frac{i}{(1+i)^n - 1} = i + \frac{1}{s_{\overline{n}|}}$$

Therefore, a level annual annuity payment on a loan is the same as making an annual interest payment each year plus making annual deposits in order to save for the loan repayment.

Interest Repayment Options

Given a loan of 1, there are 3 options in repaying back the loan over the next n years:

Option 1: Pay back the loan and all interest due at time n.

$$\begin{aligned}\text{Total Interest Paid} &= A(n) - A(0)\\ &= \text{Loan} \times (1+i)^n - \text{Loan}\\ &= \text{Loan} \times [(1+i)^n - 1]\end{aligned}$$

Option 2: Pay at the end of each year, the interest that comes due on the loan and then pay back the loan at time n.

$$\begin{aligned}\text{Annual Interest Payment} &= i \cdot \text{Loan}\\ \text{Total Interest Paid} &= i \cdot \text{Loan} \times n\\ &= \text{Loan} \times (i \cdot n)\end{aligned}$$

Option 3: Pay a level annual amount at the end of each year for the next n years.

$$\begin{aligned}\text{Annual Payment} &= \frac{\text{Loan}}{a_{\overline{n}|}}\\ \text{Total Payments} &= \left(\frac{\text{Loan}}{a_{\overline{n}|}}\right) \times n\\ \text{Total Interest Paid} &= \text{Total Payments} - \text{Loan}\\ &= \left(\frac{\text{Loan}}{a_{\overline{n}|}}\right) \times n - \text{Loan}\\ &= \text{Loan}\left(\frac{n}{a_{\overline{n}|}} - 1\right)\\ &= \text{Loan}\left(\frac{i \cdot n}{1-v^n} - 1\right)\end{aligned}$$

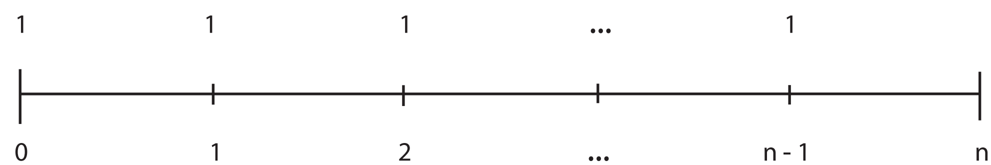
Option 1 and 2 is a comparison between compound v.s. simple interest. Therefore, less interest is paid under Option 2. This would make sense because if you pay off interest as it comes due, the loan can not grow, as it does under Option 1.

Option 2 and 3 is a mathematical comparison that shows less interest being paid under Option 3.

3.3 Annuity–Due

Definition

– payments of 1 are made at the beginning of every year for n years



– the present value (at $t = 0$) of an annuity–due, where the annual effective rate of interest is i, shall be denoted as $\ddot{a}_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}\ddot{a}_{\overline{n}|i} &= 1 + (1)v + (1)v^2 + \cdots + (1)v^{n-2} + (1)v^{n-1} \\ &= \frac{1 - v^n}{1 - v} \\ &= \frac{1 - v^n}{d}\end{aligned}$$

– the accumulated value (at $t = n$) of an annuity–due, where the annual effective rate of interest is i, shall be denoted as $\ddot{s}_{\overline{n}|i}$ and is calculated as follows:

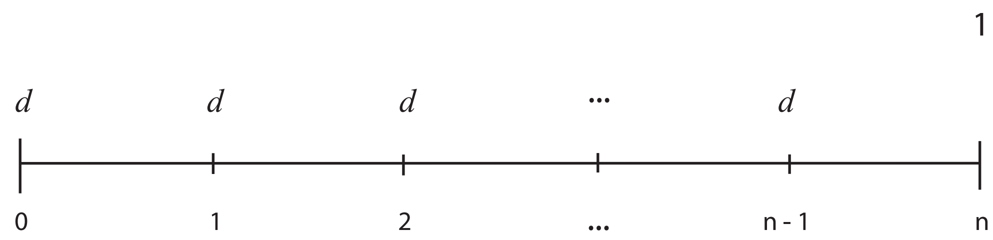
$$\begin{aligned}\ddot{s}_{\overline{n}|i} &= (1)(1+i) + (1)(1+i)^2 + \cdots + (1)(1+i)^{n-1} + (1)(1+i)^n \\ &= (1+i)[1 + (1+i) + \cdots + (1+i)^{n-2} + (1+i)^{n-1}] \\ &= (1+i)\left[\frac{1-(1+i)^n}{1-(1+i)}\right] \\ &= (1+i)\left[\frac{1-(1+i)^n}{-i}\right] \\ &= (1+i)\left[\frac{(1+i)^n - 1}{i}\right] \\ &= \frac{(1+i)^n - 1}{d}\end{aligned}$$

Basic Relationship 1 : $1 = d \cdot \ddot{a}_{\overline{n}|} + v^n$

Consider an n–year investment where 1 is invested at time 0.

The present value of this single payment income stream at $t = 0$ is 1.

Alternatively, consider a n–year investment where 1 is invested at time 0 and produces annual interest payments of $(1) \cdot d$ at the beginning of each year and then have the 1 refunded at $t = n$.



The present value of this multiple payment income stream at $t = 0$ is $d \cdot \ddot{a}_{\overline{n|}} + (1)v^n$.

Note that $\ddot{a}_{\overline{n|}} = \dfrac{1 - v^n}{d} \rightarrow 1 = d \cdot \ddot{a}_{\overline{n|}} + v^n$.

Therefore, the present value of both investment opportunities are equal.

Basic Relationship $2 : PV(1 + i)^n = FV$ **and** $PV = FV \cdot v^n$

– if the future value at time n, $\ddot{s}_{\overline{n|}}$, is discounted back to time 0, then you will have its present value, $\ddot{a}_{\overline{n|}}$

$$\begin{aligned}
\ddot{s}_{\overline{n|}} \cdot v^n &= \left[\frac{(1+i)^n - 1}{d}\right] \cdot v^n \\
&= \frac{(1+i)^n \cdot v^n - v^n}{d} \\
&= \frac{1 - v^n}{d} \\
&= \ddot{a}_{\overline{n|}}
\end{aligned}$$

– if the present value at time 0, $\ddot{a}_{\overline{n}|}$, is accumulated forward to time n, then you will have its future value, $\ddot{s}_{\overline{n}|}$

$$\begin{aligned}\ddot{a}_{\overline{n}|}\cdot(1+i)^n &= \left[\frac{1-v^n}{d}\right](1+i)^n\\ &= \frac{(1+i)^n - v^n(1+i)^n}{d}\\ &= \frac{(1+i)^n-1}{d}\\ &= \ddot{s}_{\overline{n}|}\end{aligned}$$

Basic Relationship 3 : $\dfrac{1}{\ddot{a}_{\overline{n}|}} = \dfrac{1}{\ddot{s}_{\overline{n}|}} + d$

Consider a loan of 1, to be paid back over n years with equal annual payments of P made at the beginning of each year. An annual effective rate of interest, i, is used. The present value of the single payment loan must be equal to the present value of the multiple payment stream.

$$P\cdot \ddot{a}_{\overline{n}|i} = 1$$

$$P = \frac{1}{\ddot{a}_{\overline{n}|i}}$$

Alternatively, consider a loan of 1, where the annual interest due on the loan, $(1)\cdot d$, is paid at the beginning of each year for n years and the loan amount is paid back at time n.

In order to produce the loan amount at time n, annual payments at the beginning of each year, for n years, will be made into an account that credits interest at an annual effective rate of interest i.

The future value of the multiple deposit income stream must equal the future value of the single payment, which is the loan of 1.

$$D\cdot \ddot{s}_{\overline{n}|i} = 1$$

$$D = \frac{1}{\ddot{s}_{\overline{n}|i}}$$

The total annual payment will be the interest payment and account payment:

$$d + \frac{1}{\ddot{s}_{\overline{n}|i}}$$

Note that

$$\begin{aligned}\frac{1}{\ddot{a}_{\overline{n}|i}} &= \frac{d}{1-v^n}\times\frac{(1+i)^n}{(1+i)^n} = \frac{d(1+i)^n}{(1+i)^n-1}\\ &= \frac{d(1+i)^n + d - d}{(1+i)^n-1} = \frac{d[(1+i)^n-1]+d}{(1+i)^n-1}\\ &= d + \frac{d}{(1+i)^n-1} = d + \frac{1}{\ddot{s}_{\overline{n}|}}\end{aligned}$$

Therefore, a level annual annuity payment is the same as making an annual interest payment each year and making annual deposits in order to save for the loan repayment.

Basic Relationship 4**: Due= Immediate** $\times(1+i)$

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d} = \frac{1-v^n}{i} \cdot (1+i) = a_{\overline{n}|} \cdot (1+i)$$

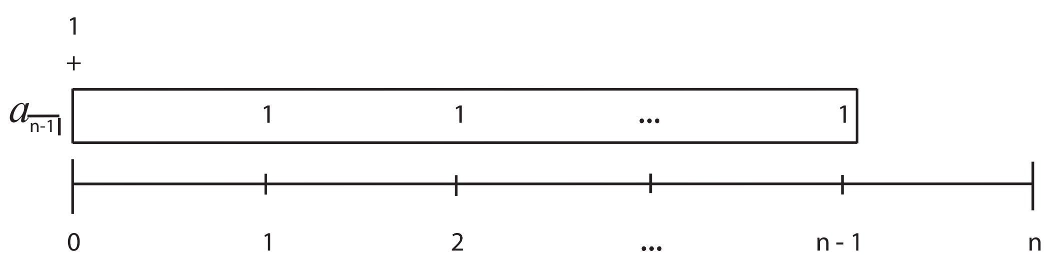
$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d} = \left[\frac{(1+i)^n - 1}{i}\right] \cdot (1+i) = s_{\overline{n}|} \cdot (1+i)$$

An annuity–due starts one period earlier than an annuity-immediate and as a result, earns one more period of interest, hence it will be larger.

Basic Relationship $5: \ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$

$$\begin{aligned}
\ddot{a}_{\overline{n}|} &= 1 + [v + v^2 + \cdots + v^{n-2} + v^{n-1}] \\
&= 1 + v[1 + v + \cdots + v^{n-3} + v^{n-2}] \\
&= 1 + v\left(\frac{1-v^{n-1}}{1-v}\right) \\
&= 1 + \left(\frac{1}{1+i}\right)\left(\frac{1-v^{n-1}}{d}\right) \\
&= 1 + \left(\frac{1}{1+i}\right)\left(\frac{1-v^{n-1}}{i/1+i}\right) \\
&= 1 + \frac{1-v^{n-1}}{i} \\
&= 1 + a_{\overline{n-1}|}
\end{aligned}$$

This relationship can be visualized with a time line diagram.

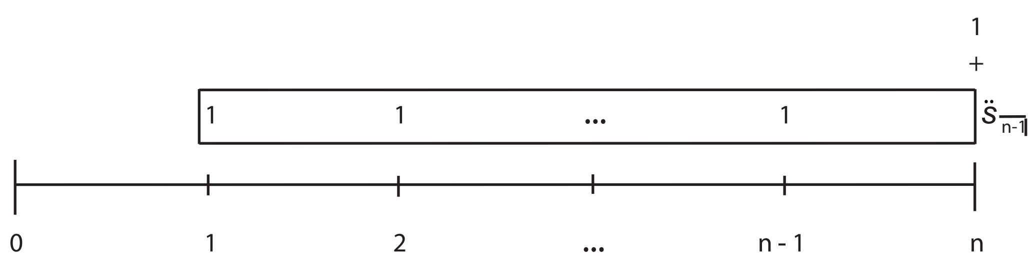


An additional payment of 1 at time 0 results in $a_{\overline{n-1}|}$ becoming n payments that now commence at the beginning of each year which is $\ddot{a}_{\overline{n}|}$.

Basic Relationship 6 : $s_{\overline{n}|} = 1 + \ddot{s}_{\overline{n-1}|}$

$$\begin{aligned} s_{\overline{n}|} &= 1 + [(1+i) + (1+i)^2 + \cdots + (1+i)^{n-2} + (1+i)^{n-1}] \\ &= 1 + (1+i)[1 + (1+i) + \cdots + (1+i)^{n-3} + (1+i)^{n-2}] \\ &= 1 + (1+i)\left[\frac{1-(1+i)^{n-1}}{1-(1+i)}\right] \\ &= 1 + (1+i)\left[\frac{1-(1+i)^{n-1}}{-i}\right] \\ &= 1 + (1+i)\left[\frac{(1+i)^{n-1}-1}{i}\right] \\ &= 1 + \frac{(1+i)^{n-1}-1}{d} \\ &= 1 + \ddot{s}_{\overline{n-1}|} \end{aligned}$$

This relationship can also the visualized with a time line diagram.



An additional payment of 1 at time n results in $\ddot{s}_{\overline{n-1}|}$ becoming n payments that now commerce at the end of each year which is $s_{\overline{n}|}$

3.4 Annuity Values On Any Date

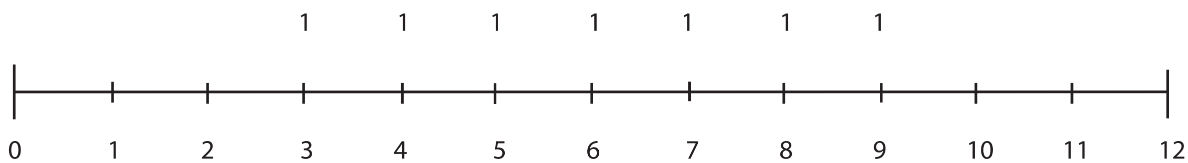
- There are three alternative dates to valuing annuities rather than at the beginning of the term ($t = 0$) or at the end of the term ($t = n$)

  (i) present values more than one period before the first payment date

  (ii) accumulated values more than one period after the last payment date

  (iii) current value between the first and last payment dates

- The following example will be used to illustrate the above cases. Consider a series of payments of 1 that are made at time $t = 3$ to $t = 9$, inclusive.



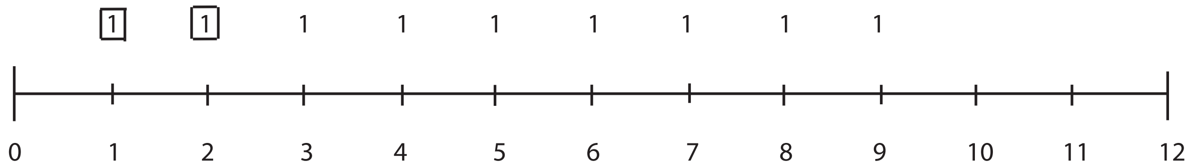
Present Values More than One Period Before The First Payment Date
At $t = 2$, there exists 7 future end-of-year payments whose present value is represented by $a_{\overline{7|}}$. If this value is discounted back to time $t = 0$, then the value of this series of payments (2 periods before the first end-of-year payment) is

$$v^2 \cdot a_{\overline{7|}}.$$

Alternatively, at $t = 3$, there exists 7 future beginning-of-year payments whose present value is represented by $\ddot{a}_{\overline{7|}}$. If this value is discounted back to time $t = 0$, then the value of this series of payments (3 periods before the first beginning-of-year payment) is

$$v^3 \cdot \ddot{a}_{\overline{7|}}.$$

Another way to examine this situation is to pretend that there are 9 end-of-year payments. This can be done by adding 2 more payments to the existing 7. In this case, let the 2 additional payments be made at $t = 1$ and 2 and be denoted as $\boxed{1}$.



At t = 0, there now exists 9 end-of-year payments whose present value is $a_{\overline{9|}}$. This present value of 9 payments would then be reduced by the present value of the two imaginary payments, represented by $a_{\overline{2|}}$. Therefore, the present value at $t = 0$ is

$$a_{\overline{9|}} - a_{\overline{2|}},$$

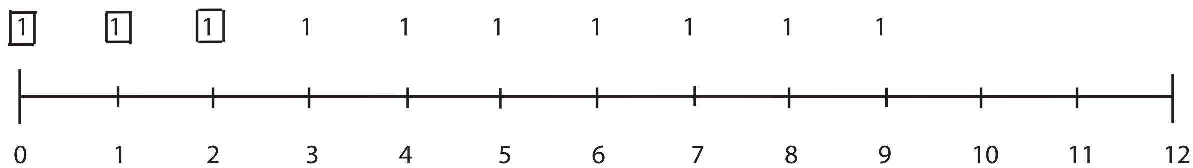
and this results in

$$v^2 \cdot a_{\overline{7|}} = a_{\overline{9|}} - a_{\overline{2|}}.$$

The general form is

$$v^m \cdot a_{\overline{n|}} = a_{\overline{m+n|}} - a_{\overline{m|}}.$$

With the annuity–due version, one can pretend that there are 10 payments being made. This can be done by adding 3 payments to the existing 7 payments. In this case, let the 3 additional payments be made at $t = 0$, 1 and 2 and be denoted as $\boxed{1}$.



At t = 0, there now exists 10 beginning-of-year payments whose present value is $\ddot{a}_{\overline{10}|}$. This present value of 10 payments would then be reduced by the present value of the three imaginary payments, represented by $\ddot{a}_{\overline{3}|}$. Therefore, the present value at $t = 0$ is

$$\ddot{a}_{\overline{10}|} - \ddot{a}_{\overline{3}|},$$

and this results in

$$v^3 \cdot \ddot{a}_{\overline{7}|} = \ddot{a}_{\overline{10}|} - \ddot{a}_{\overline{3}|}.$$

The general form is

$$v^m \cdot \ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|}.$$

Accumulated Values More Than One Period After The Last Payment Date

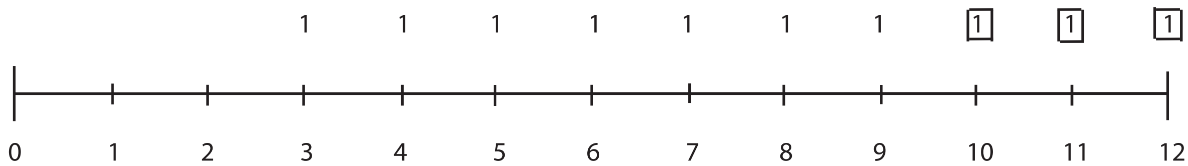
At $t = 9$, there exists 7 past end-of-year payments whose accumulated value is represented by $s_{\overline{7}|}$. If this value is accumulated forward to time $t = 12$, then the value of this series of payments (3 periods after the last end-of-year payment) is

$$s_{\overline{7}|} \cdot (1 + i)^3.$$

Alternatively, at $t = 10$, there exists 7 past beginning-of-year payments whose accumulated value is represented by $\ddot{s}_{\overline{7}|}$. If this value is accumulated forward to time $t = 12$, then the value of this series of payments (2 periods after the last beginning-of-year payment) is

$$\ddot{s}_{\overline{7}|} \cdot (1 + i)^2.$$

Another way to examine this situation is to pretend that there are 10 end-of-year payments. This can be done by adding 3 more payments to the existing 7. In this case, let the 3 additional payments be made at $t = 10$, 11 and 12 and be denoted as $\boxed{1}$.



At t = 12, there now exists 10 end-of-year payments whose present value is $s_{\overline{10}|}$. This future value of 10 payments would then be reduced by the future value of the three imaginary payments, represented by $s_{\overline{3}|}$. Therefore, the accumulated value at $t = 12$ is

$$s_{\overline{10}|} - s_{\overline{3}|},$$

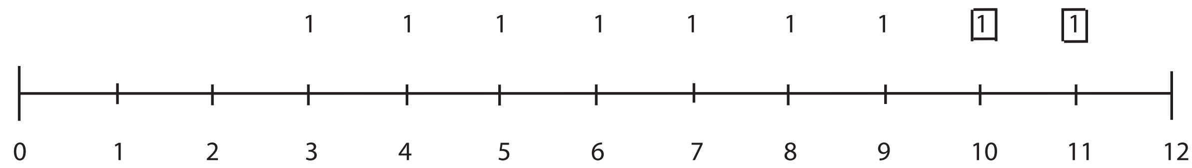
and this results in

$$s_{\overline{7}|} \cdot (1+i)^3 = s_{\overline{10}|} - s_{\overline{3}|}.$$

The general form is

$$s_{\overline{n}|} \cdot (1+i)^m = s_{\overline{m+n}|} - s_{\overline{m}|}.$$

With the annuity–due version, one can pretend that there are 9 payments being made. This can be done by adding 2 payments to the existing 7 payments. In this case, let the 2 additional payments be made at $t = 10$ and 11 and be denoted as $\boxed{1}$.



At t = 12, there now exists 9 beginning-of-year payments whose accumulated value is $\ddot{s}_{\overline{9|}}$. This future value of 9 payments would then be reduced by the future value of the two imaginary payments, represented by $\ddot{s}_{\overline{2|}}$. Therefore, the accumulated value at $t = 12$ is

$$\ddot{s}_{\overline{9|}} - \ddot{s}_{\overline{2|}},$$

and this results in

$$\ddot{s}_{\overline{7|}} \cdot (1+i)^2 = \ddot{s}_{\overline{9|}} - \ddot{s}_{\overline{2|}}.$$

The general form is

$$\ddot{s}_{\overline{n|}} \cdot (1+i)^m = \ddot{s}_{\overline{m+n|}} - \ddot{s}_{\overline{m|}}.$$

Current Values Between The First And Last Payment Dates

The 7 payments can be represented by an annuity-immediate or by an annuity-due depending on the time that they are evaluated at.

For example, at $t=2$, the present value of the 7 end-of-year payments is $a_{\overline{7}|}$. At $t=9$, the future value of those same payments is $s_{\overline{7}|}$. There is a point between time 2 and 9 where the present value and the future value can be accumulated to and discounted back, respectively. At $t=6$, for example, the present value would need to be accumulated forward 4 years, while the accumulated value would need to be discounted back 3 years.

$$a_{\overline{7}|} \cdot (1+i)^4 = v^3 \cdot s_{\overline{7}|}$$

The general form is

$$a_{\overline{n}|} \cdot (1+i)^m = v^{(n-m)} \cdot s_{\overline{n}|}$$

Alternatively, at $t=3$, one can view the 7 payments as being paid at the beginning of the year where the present value of the payments is $\ddot{a}_{\overline{7}|}$. The future value at $t=10$ would then be $\ddot{s}_{\overline{7}|}$. At $t=6$, for example, the present value would need to be accumulated forward 3 years, while the accumulated value would need to be discounted back 4 years.

$$\ddot{a}_{\overline{7}|} \cdot (1+i)^3 = v^4 \cdot \ddot{s}_{\overline{7}|}.$$

The general form is

$$\ddot{a}_{\overline{n}|} \cdot (1+i)^m = v^(n-m) \cdot \ddot{s}_{\overline{n}|}.$$

At any time during the payments, there will exists a series of past payments and a series of future payments.

For example, at $t=6$, one can define the past payments as 4 end-of-year payments whose accumulated value is $s_{\overline{4}|}$. The 3 end-of-year future payments at $t=6$ would then have a present value (at $t=6$) equal to $a_{\overline{3}|}$. Therefore, the current value as at $t=6$ of the 7 payments is

$$s_{\overline{4}|} + a_{\overline{3}|}.$$

Alternatively, if the payments are viewed as beginning-of-year payments at $t=6$, then there are 3 past payments and 4 future payments whose accumulated value and present value are respectively, $\ddot{s}_{\overline{3}|}$ and $\ddot{a}_{\overline{4}|}$. Therefore, the current value as at $t=6$ of the 7 payments can also be calculated as

$$\ddot{s}_{\overline{3}|} + \ddot{a}_{\overline{4}|}.$$

This results in

$$s_{\overline{4}|} + a_{\overline{3}|} = \ddot{s}_{\overline{3}|} + \ddot{a}_{\overline{4}|}.$$

The general form is

$$s_{\overline{m}|} + a_{\overline{n}|} = \ddot{s}_{\overline{n}|} + \ddot{a}_{\overline{m}|}.$$

3.5 Perpetuities

Definition Of A Perpetuity-Immediate

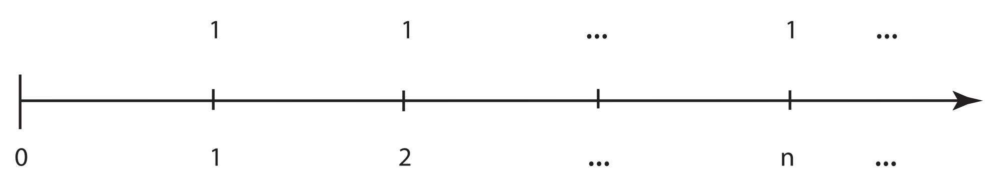
- payments of 1 are made at the end of every year forever i.e. $n = \infty$
- the present value (at $t = 0$) of a perpetuity–immediate, where the annual effective rate of interest is i, shall be denoted as $a_{\overline{\infty}|i}$ and is calculated as follows:

$$\begin{aligned}
a_{\overline{\infty}|i} &= (1)v + (1)v^2 + (1)v^3 + \cdots \\
&= v(1 + v + v^2 + \cdots) \\
&= \left(\frac{1}{1+i}\right)\left(\frac{1-v^{\infty}}{1-v}\right) \\
&= \left(\frac{1}{1+i}\right)\left(\frac{1-0}{d}\right) \\
&= \left(\frac{1}{1+i}\right)\left(\frac{1}{\frac{i}{1+i}}\right) \\
&= \frac{1}{i}
\end{aligned}$$

- one could also derive the above formula by simply substituting $n = \infty$ into the original present value formula:

$$a_{\overline{\infty}|i} = \frac{1-v^{\infty}}{i} = \frac{1-0}{i} = \frac{1}{i}$$

- Note that $\frac{1}{i}$ represents an initial amount that can be invested at $t = 0$. The annual interest payments, payable at the end of the year, produced by this investment is $\left(\frac{1}{i}\right) \cdot i = 1$.
- $s_{\overline{\infty}|}$ is not defined since it would equal ∞



Basic Relationship $1 : a_{\overline{n}|} = a_{\overline{\infty}|} - v_n \cdot a_{\overline{\infty}|}$

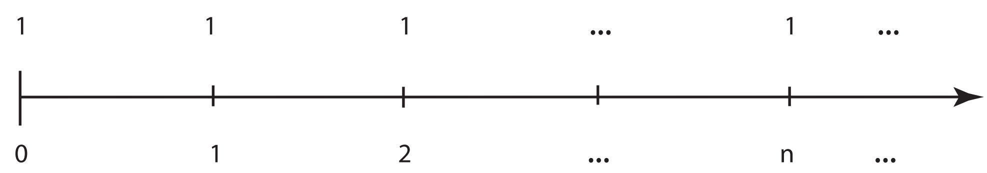
The present value formula for an annuity-immediate can be expressed as the difference between two perpetuity-immediates:

$$a_{\overline{n}|} = \frac{1 - v^n}{i} = \frac{1}{i} - \frac{v^n}{i} = \frac{1}{i} - v^n \cdot \frac{1}{i} = a_{\overline{\infty}|} - v^n \cdot a_{\overline{\infty}|}.$$

In this case, a perpetuity-immediate that is payable forever is reduced by perpetuity-immediate payments that start after n years. The present value of both of these income streams, at $t = 0$, results in end-of-year payments remaining only for the first n years.

Definition Of A Perpetuity-Due

– payments of 1 are made at the beginning of every year forever i.e. $n = \infty$



– the present value (at $t = 0$) of a perpetuity–due, where the annual effective rate of interest is i, shall be denoted as $\ddot{a}_{\overline{\infty}|i}$ and is calculated as follows:

$$\begin{aligned}\ddot{a}_{\overline{\infty}|i} &= (1) + (1)v^1 + (1)v^2 + \cdots \\ &= \left(\frac{1 - v^{\infty}}{1 - v}\right) \\ &= \left(\frac{1 - 0}{d}\right) \\ &= \frac{1}{d}\end{aligned}$$

– one could also derive the above formula by simply substituting $n = \infty$ into the original present value formula:

$$\ddot{a}_{\overline{\infty}|d} = \frac{1 - v^{\infty}}{d} = \frac{1 - 0}{d} = \frac{1}{d}$$

– Note that $\frac{1}{d}$ represents an initial amount that can be invested at $t = 0$. The annual interest payments, payable at the beginning of the year, produced by this investment is $\left(\frac{1}{d}\right) \cdot d = 1$.

– $\ddot{s}_{\overline{\infty}|}$ is not defined since it would equal ∞

Basic Relationship $1 : \ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{\infty}|} - v_n \cdot \ddot{a}_{\overline{\infty}|}$

The present value formula for an annuity-due can be expressed as the difference between two perpetuity-dues:

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d} = \frac{1}{d} - \frac{v^n}{d} = \frac{1}{d} - v^n \cdot \frac{1}{d} = \ddot{a}_{\overline{\infty}|} - v^n \cdot \ddot{a}_{\overline{\infty}|}.$$

In this case, a perpetuity-due that is payable forever is reduced by perpetuity-due payments that start after n years. The present value of both of these income streams, at $t = 0$, results in beginning-of-year payments remaining only for the first n years.

3.6 Nonstandard Terms and Interest Rates

– **material not tested in SoA Exam FM**

3.7 Unknown Time

– When solving for n, you often will not get an integer value

– An adjustment to the payments can be made so that n does become an integer

– Example
How long will it take to payoff a \$1000 loan if \$100 is paid at the end of every year and the annual effective rate of interest is 5%?

$$\begin{aligned} \$1,000 &= \$100\, a_{\overline{n}|5\%} \\ n &= 14.2067 \text{ (by financial calculator)} \end{aligned}$$

– This says that we need to pay \$100 at the end of every year, for 14 years and then make a \$100 payment at the end of 14.2067 year? No!

– The payment required at time 14.2067 years is:

$$\begin{aligned} \$1,000 &= \$100\, a_{\overline{14}|5\%} + X \cdot v_{5\%}^{14.2067} \\ X &= \$20.27 \end{aligned}$$

– Therefore, the last payment at 14.2067 will be \$20.27 and is the *exact payment.*

- Question

– What if we wanted to pay off the loan in exactly 14 years?

- Solution

$$\begin{aligned}\$1,000 &= \$100\, a_{\overline{14}|5\%} + Y \cdot v_{5\%}^{14} \\ Y &= \$20.07\end{aligned}$$

– Therefore, the last payment will be \$120.07 and is called a *balloon payment*.

– Note, that since we made the last payment 0.2067 years earlier than we had to, the extra \$20.07 is equal to the exact payment discounted back, $Y = X \cdot v_{5\%}^{.2067}$.

- Question

– What if we wanted to make one last payment at the end of 15 years?

- Solution

$$\begin{aligned}\$1,000 &= \$100\, a_{\overline{14}|5\%} + Z \cdot v_{5\%}^{15} \\ Z &= \$21.07\end{aligned}$$

– Therefore, the last payment will be \$21.07 and is called a *drop payment*.

– Note that since we delayed the payment one year, it is equal to the balloon payment with interest, $Z = Y \cdot (1 + i)$.

– Note that since delayed the payment 0.7933 years, it is equal to the exact payment with interest, $Z = X \cdot (1 + i)^{0.7933}$.

3.8 Unknown Rate of Interest

– Assuming that you do not have a financial calculator

– This section will look at 3 approaches to solving for an unknown rate of interest when $a_{\overline{n}|i} = k$.

1. Algebraic Techniques

   – Note that $a_{\overline{n}|i} = k = v_i + v_i^2 + \cdots + v_i^n$ is an n degree polynomial and can be easily solve if n is small

   – When n gets too big, use a series expansion of $a_{\overline{n}|}$, or even better, $1/a_{\overline{n}|i}$. Now you are solving a quadratic formula.

$$\begin{aligned}a_{\overline{n}|i} = k &= n - \frac{n(n+1)}{2!} i + \frac{n(n+1)(n+2)}{3!} i^2 - \cdots \\ \frac{1}{a_{\overline{n}|i}} = \frac{1}{k} &= \frac{1}{n} + \frac{(n+1)}{2n} i + \frac{(n^2-1)}{12n} i^2 + \cdots\end{aligned}$$

2. Linear Interpolation

   – need to find the value of $a_{\overline{n}|}$ at two different interest rates where $a_{\overline{n}|i_1} = k_1 < k$ and $a_{\overline{n}|i_2} = k_2 > k$.

   –
$$\left.\begin{aligned} a_{\overline{n}|i_1} &= k_1 \\ a_{\overline{n}|i} &= k \\ a_{\overline{n}|i_2} &= k_2 \end{aligned}\right\} \; i \approx i_1 + \frac{k_1 - k}{k_1 - k_2}(i_2 - i_1)$$

3. Successive Approximation (Iteration)

   – considered the best way to go if no calculator and precision is really important

   – there are two techniques that can be used

      i. Solve for i

$$\begin{aligned} a_{\overline{n}|i} &= k \\ \frac{1 - v_i^n}{i} &= k \\ i_{s+1} &= \frac{1 - (1 + i_s)^{-n}}{k} \end{aligned}$$

      – What is a good starting value for i_0 $(s = 0)$?

      – one could use linear interpolation to find i_0

      – one could use the first two terms of $\frac{1}{a_{\overline{n}|}}$ and solve for i

$$\begin{aligned} \frac{1}{k} &= \frac{1}{n} + \left(\frac{n+1}{2n}\right) i \\ i_0 &= \frac{2(n-k)}{k(n+1)} \end{aligned}$$

      – another approach to derive a starting value is to use

$$i_0 = \frac{1 - \left(\frac{k}{n}\right)^2}{k}$$

      ii. Newton-Raphson Method

      – This method will see convergence very rapidly

      –
$$i_{s+1} = i_s - \frac{f(i_s)}{f'(i_s)}$$

– let

$$\begin{aligned}
a_{\overline{n}|i} &= k \\
a_{\overline{n}|i} - k &= 0 \\
\frac{1-(1+i_s)^{-n}}{i} - k &= 0 \\
f(i_s) = 1-(1+i_s)^{-n} - i_s k &= 0 \\
f'(i_s) = -(-n)(1+i_s)^{-n-1} - k
\end{aligned}$$

$$\begin{aligned}
\therefore \quad i_{s+1} &= i_s - \frac{1-(1+i_s)^{-n} - i_s k}{n(1+i_s)^{-n-1} - k} \\
&= i_s \left[1 + \frac{1-(1+i_s)^{-n} - k \cdot i_s}{1-(1+i_s)^{-n-1}\{1+i_s(n+1)\}} \right]
\end{aligned}$$

What about $s_{\overline{n}|i} = k$?

1. Algebraic Techniques

$$\begin{aligned}
s_{\overline{n}|i} = k &= n + \frac{n(n+1)}{2!} i + \frac{n(n-1)(n-2)}{3!} i^2 - \cdots \\
\frac{1}{s_{\overline{n}|i}} = \frac{1}{k} &= \frac{1}{n} - \frac{(n-1)}{2n} i + \frac{(n^2-1)}{12n} i^2 + \cdots
\end{aligned}$$

2. Linear Interpolation

$$\left.\begin{aligned}
s_{\overline{n}|i_1} &= k_1 \\
s_{\overline{n}|i} &= k \\
s_{\overline{n}|i_2} &= k_2
\end{aligned}\right\} \; i \approx i_1 + \frac{k_1 - k}{k_1 - k_2}(i_2 - i_1)$$

3. Successive Approximation (Iteration)

   – there are two techniques that can be used

   i. Solve for i

$$\begin{aligned}
s_{\overline{n}|i} &= k \\
\frac{(1+i)^n - 1}{i} &= k \\
i_{s+1} &= \frac{(1+i_s)^n - 1}{k}
\end{aligned}$$

- What is a good starting value for i_0 ($s = 0$)?
- one could use linear interpolation to find i_0

- one could use the first two terms of $\frac{1}{s_{\overline{n}|}}$ and solve for i

$$\frac{1}{k} = \frac{1}{n} - \left(\frac{n-1}{2n}\right) i$$
$$i_0 = \frac{2(n-k)}{k(n-1)}$$

- another approach to derive a starting value is to use

$$i_0 = \frac{\left(\frac{k}{n}\right)^2 - 1}{k}$$

ii. Newton-Raphson Method

–
$$i_{s+1} = i_s - \frac{f(i_s)}{f'(i_s)}$$

– let

$$\begin{aligned}
&s_{\overline{n}|i} = k \\
&s_{\overline{n}|i} - k = 0 \\
&\frac{(1+i_s)^n - 1}{i} - k = 0 \\
&f(i_s) = (1+i_s)^n - 1 - i_s k = 0 \\
&f'(i_s) = n(1+i_s)^{n-1} - k
\end{aligned}$$

$$\begin{aligned}
\therefore \quad i_{s+1} &= i_s - \frac{(1+i_s)^n - 1 - i_s k}{n(1+i_s)^{n-1} - k} \\
&= i_s \left[1 + \frac{(1+i_s)^n - 1 - k \cdot i_s}{(1+i_s)^{n-1}\{1 - i_s(n-1)\}} - 1\right] \quad = i_s \left[1 + \frac{(1+i_s)^n - 1 - k \cdot i_s}{k \cdot i_s - n \cdot i_s (1+i_s)^{-n-1}}\right]
\end{aligned}$$

3.9 Varying Interest

- if the annual effective rate of interest varies from one year to the next, $i \neq i_k$, then the present value and accumulated value of annuity payments needs to be calculated directly

- the varying rate of interest can be defined in one of two ways:

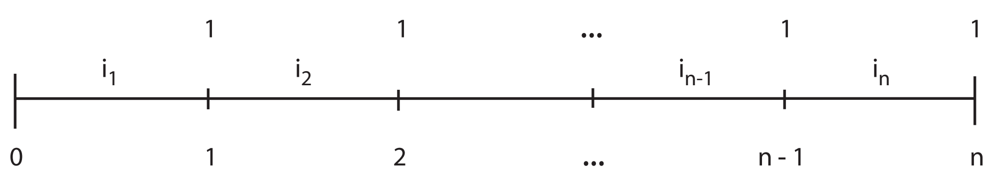
  (i) effective rate of interest for period k, i_k, is only used for period k

  (ii) effective rate of interest for period k, i_k, is used for all periods

  If i_k Is Only Used For Period k

  A payment made during year k will need to be discounted or accumulated over each past or future period at the interest rate that was in effect during that period.

  The time line diagram below illustrates the varying interest rates for an annuity-immediate.



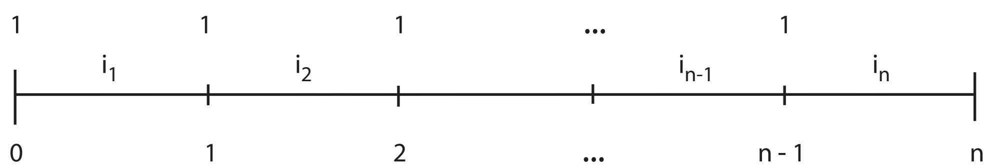
The present value of an annuity–immediate is determined as follows:

$$
\begin{aligned}
a_{\overline{n}|} &= \frac{1}{1+i_1} + \frac{1}{1+i_1}\cdot\frac{1}{1+i_2} + \cdots + \frac{1}{1+i_1}\cdot\frac{1}{1+i_2}\cdots\frac{1}{1+i_n} \\
&= \sum_{k=1}^{n}\prod_{j=1}^{k}\frac{1}{1+i_j}
\end{aligned}
$$

The accumulated value of an annuity–immediate is determined as follows:

$$
\begin{aligned}
s_{\overline{n}|} &= 1 + (1+i_n) + (1+i_n)\cdot(1+i_{n-1}) + \cdots + (1+i_n)\cdot(1+i_{n-1})\cdots(1+i_2) \\
&= 1 + \sum_{k=1}^{n-1}\prod_{j=1}^{k}(1+i_{n-j+1})
\end{aligned}
$$

The time line diagram below illustrates the varying interest rates for an annuity-due.



The present value of an annuity–due is determined as follows:

$$\begin{aligned}\ddot{a}_{\overline{n}|} &= 1 + \frac{1}{1+i_1} + \frac{1}{1+i_1}\cdot\frac{1}{1+i_2} + \cdots + \frac{1}{1+i_1}\cdot\frac{1}{1+i_2}\cdots\frac{1}{1+i_{n-1}} \\ &= 1 + \sum_{k=1}^{n-1}\prod_{j=1}^{k}\frac{1}{1+i_j}\end{aligned}$$

The accumulated value of an annuity–due is determined as follows:

$$\begin{aligned}\ddot{s}_{\overline{n}|} &= (1+i_n) + (1+i_n)\cdot(1+i_{n-1}) + \cdots + (1+i_n)\cdot(1+i_{n-1})\cdots(1+i_1) \\ &= \sum_{k=1}^{n}\prod_{j=1}^{k}(1+i_{n-j+1})\end{aligned}$$

– Note that the accumulated value of an annuity–immediate can also be solved by using the above annuity–due formula and applying Basic Relationship 6 from Section 3.3: $s_{\overline{n}|} = 1 + \ddot{s}_{\overline{n-1}|}$.

– Note that the present value of an annuity–due can also be solved by using the above annuity–immediate formula and applying Basic Relationship 5 from Section 3.3: $\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$.

If i_k Is Used For All Periods

A payment made during year k will be discounted or accumulated at the interest rate that was in effect at the time of the payment. For example, if the interest rate during year 10 was 6%, then the payment made during year 10 will discounted back or accumulated forward at 6% for each year.

The present value of an annuity–immediate is $a_{\overline{n}|} = \sum_{k=1}^{n}\frac{1}{(1+i_k)^k}$.

The accumulated value of an annuity–immediate is $s_{\overline{n}|} = 1 + \sum_{k=1}^{n-1}(1+i_{n-k})^k$.

The present value of an annuity–due is $\ddot{a}_{\overline{n}|} = 1 + \sum_{k=1}^{n-1}\frac{1}{(1+i_{k+1})^k}$.

The accumulated value of an annuity–due is $\ddot{s}_{\overline{n}|} = \sum_{k=1}^{n}(1+i_{n-k+1})^k$.

– Note that the accumulated value of an annuity–immediate can also be solved by using the above annuity–due formula and applying Basic Relationship 6 from Section 3.3: $s_{\overline{n}|} = 1 + \ddot{s}_{\overline{n-1}|}$.

– Note that the present value of an annuity–due can also be solved by using the above annuity–immediate formula and applying Basic Relationship 5 from Section 3.3: $\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$.

3.10 Annuities Not Involving Compound Interest

– **material not tested in SoA Exam FM**

4 More General Annuities

4.1 Introduction

- in Chapter 3, annuities were described as having level payments payable at the same frequency as what the interest rate was being converted at

- in this chapter, non-level payments are examined as well as the case where the interest conversion period and the payment frequency no longer coincide

4.2 Annuities Payable At A Different Frequency Than Interest Is Convertible

- let the payments remain level for the time being

- when the interest conversion period does not coincide with the payment frequency, one can take the given rate of interest and convert to an interest rate that does coincide

  Example
  Find the accumulated value in 10 years if semi-annual due payments of 100 are being made into a fund that credits a nominal rate of interest at 10%, convertible semiannually.

  $$FV_{10} = 100\ddot{s}_{\overline{10 \times 12}|j}$$

  Interest rate j will need to be a monthly rate and is calculated based on the semiannual rate that was given:

  $$j = \frac{i^{(12)}}{12} = \left(1 + \underbrace{\frac{i^{(2)}}{2}}_{6\text{ month rate}}\right)^{1/6} - 1 = \left(1 + \frac{10\%}{2}\right)^{1/6} - 1 = 0.8165\%$$

  Therefore, the accumulated value at time $t = 10$ is

  $$FV_{10} = 100\ddot{s}_{\overline{120}|0.8165\%} = 20,414.52$$

4.3 Further Analysis of Annuities Payable $\boxed{\text{Less}}$ Frequency Than Interest Is Convertible

– **material not tested in SoA Exam FM**

Annuity–Immediate

– let $i^{(k)}$ be a nominal rate of interest convertible k times a year and let there be level end-of-year payments of 1

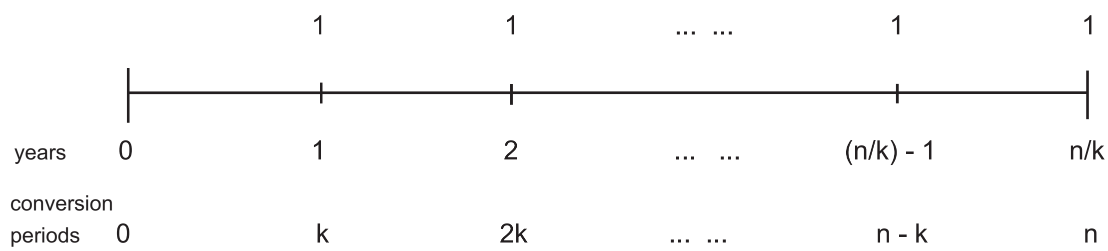
– after the first payment has been made, interest has been converted k times

– after the second payment has been made, interest has been converted $2k$ times

– after the last payment has been made, interest has been converted n times

– therefore, the term of the annuity (and obviously, the number of payments) will be $\frac{n}{k}$ years (i.e. if $n = 144$ and $i^{(12)}$ is used, then the term of the annuity is $\frac{144}{12} = 12$ years).

– The time line diagram will detail the above scenario:



– the present value (at $t = 0$) of an annual annuity–immediate where payments are made every

k conversion periods and where the rate of interest is $j = \dfrac{i^{(k)}}{k}$ shall be calculated as follows:

$$\begin{aligned}
PV_0 &= (1)v_i^1 + (1)v_i^2 + \cdots + (1)v_i^{n/k-1} + (1)v_i^{n/k} \\
&= (1)v_j^k + (1)v_j^{2k} + \cdots + (1)v_j^{n-k} + (1)v_j^n \\
&= v_j^k \left(1 + v_j^k + \cdots + v_j^{n-2k} + v_j^{n-k}\right) \\
&= v_j^k \left(\frac{1-(v_j^k)^{n/k}}{1-v_j^k}\right) \\
&= v_j^k \left(\frac{1-v^n}{1-v_j^k}\right) \\
&= \frac{1}{(1+j)^k}\left(\frac{1-v^n}{1-v_j^k}\right) \\
&= \left(\frac{1-v^n}{(1+j)^k-1}\right)\cdot\frac{j}{j} \\
&= \frac{a_{\overline{n}|j}}{s_{\overline{k}|j}}
\end{aligned}$$

– There is an alternative approach in determining the present value of this annuity–immediate:

  The payment of 1 made at the end of each year can represent the accumulated value of smaller level end-of-conversion-period payments that are made k times during the year.

$$P \cdot s_{\overline{k}|j} = 1$$

  These smaller level payments are therefore equal to $P = \dfrac{1}{s_{\overline{k}|j}}$. If these smaller payments were to be made at the end of every conversion period, during the term of the annuity, and there are n conversion periods in total, then the present value (at $t = 0$) of these n smaller payments is determined to be

$$PV_0 = \frac{1}{s_{\overline{k}|j}} \cdot a_{\overline{n}|j}.$$

– the accumulated value (at $t = n/k$ years or $t = n$ conversion periods) of an annual annuity–immediate where payments are made every k conversion periods and where the rate of interest is $j = \dfrac{i^{(k)}}{k}$ shall be calculated as follows:

$$\begin{aligned}
FV_{\frac{n}{k}} &= 1 + (1)(1+i)^1 + \cdots + (1)(1+i)^{\frac{n}{k}-2} + (1)(1+i)^{\frac{n}{k}-1} \\
&= 1 + (1)(1+j)^k + \cdots + (1)(1+j)^{n-2k} + (1)(1+j)^{n-k} \\
&= \frac{1 - [(1+j)^k]^{n/k}}{1-(1+j)^k} \\
&= \frac{1-(1+j)^n}{1-(1+j)^k} \\
&= \left[\frac{(1+j)^n - 1}{(1+j)^k - 1}\right] \times \frac{j}{j} \\
&= \frac{s_{\overline{n}|j}}{s_{\overline{k}|j}}
\end{aligned}$$

Another approach in determining the accumulated value would be to go back to a basic relationship where a future value is equal to its present value carried forward with interest:

$$\begin{aligned}
FV_{\frac{n}{k}} &= FV_n \\
&= PV_0 \cdot (1+j)^n \\
&= \frac{a_{\overline{n}|j}}{s_{\overline{k}|j}} \cdot (1+j)^n \\
&= \frac{s_{\overline{n}|j}}{s_{\overline{k}|j}}
\end{aligned}$$

The accumulated value can also be derived by again considering that each end-of-year payment of 1 represents the accumulated value of smaller level end-of-conversion-period payments that are made k times during the year:

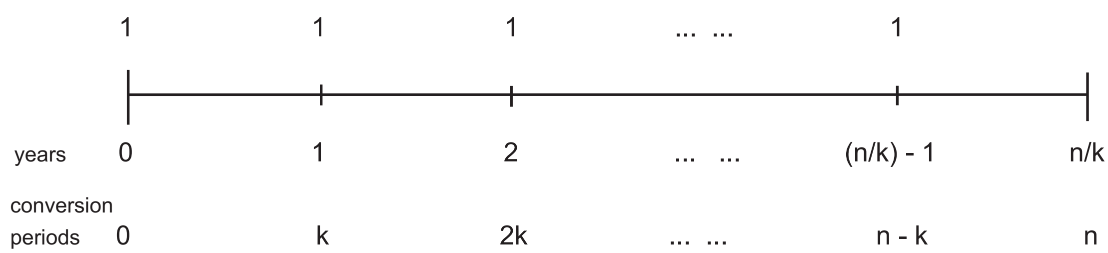
$$P \cdot s_{\overline{k|}j} = 1$$

These smaller level payments are therefore equal to $P = \dfrac{1}{s_{\overline{k|}j}}$. If these smaller payments are made at the end of every conversion period, over the term of the annuity, and there are n conversion periods in total, then the accumulated value (at $t = n/k$ years or $t = n$ conversion periods) of these smaller payments is determined to be

$$FV_{\frac{n}{k}} = \left(\frac{1}{s_{\overline{k|}j}}\right) \cdot s_{\overline{n|}j}.$$

Annuity–Due

- let $i^{(k)}$ be a nominal rate of interest convertible k times a year and let there be level beginning-of-year payments of 1
- after the second payment has been made, interest has been converted k times
- after the third payment has been made, interest has been converted $2k$ times
- after the last payment has been made, interest has been converted $n-k$ times
- therefore, the term of the annuity (and obviously, the number of payments) will be $1+\dfrac{n-k}{k} = \dfrac{n}{k}$ years (i.e. if $n = 144$ and $i^{(12)}$ is used, then the term of the annuity is $\dfrac{144}{12} = 12$ years).
- The time line diagram will detail the above scenario:



- the present value (at $t = 0$) of an annual annuity–due where payments are made every k conversion periods and where the rate of interest is $j = \dfrac{i^{(k)}}{k}$ shall be calculated as follows:

$$\begin{aligned}
PV_0 &= 1 + (1)v_i^1 + (1)v_i^2 + \cdots + (1)v_i^{n/k-1} \\
&= 1 + (1)v_j^k + (1)v_j^{2k} + \cdots + (1)v_j^{n-k} \\
&= 1 + v_j^k + \cdots + v_j^{n-2k} + v_j^{n-k} \\
&= \frac{1-(v_j^k)^{n/k}}{1-v_j^k} \\
&= \frac{1-v^n}{1-v_j^k} \\
&= \left(\frac{1-v^n}{1-v_j^k}\right)\cdot\frac{j}{j} \\
&= \frac{a_{\overline{n}|j}}{a_{\overline{k}|j}}
\end{aligned}$$

– There is an alternative approach in determining the present value of this annuity–due:

The payment of 1 made at the beginning of each year can represent the present value of smaller level end-of-conversion-period payments that are made k times during the year.

$$P \cdot a_{\overline{k}|j} = 1$$

These smaller level payments are therefore equal to $P = \dfrac{1}{a_{\overline{k}|j}}$. If these smaller payments were to be made at the end of every conversion period, during the term of the annuity, and there are n conversion periods in total, then the present value (at $t = 0$) of these n smaller payments is determined to be

$$PV_0 = \frac{1}{a_{\overline{k}|j}} \cdot a_{\overline{n}|j}.$$

– the accumulated value (at $t = n/k$ years or $t = n$ conversion periods) of an annual annuity–due where payments are made every k conversion periods and where the rate of interest is $j = \dfrac{i^{(k)}}{k}$ shall be calculated as follows:

$$\begin{aligned}
FV_{\frac{n}{k}} &= (1)(1+i)^1 + (1)(1+i)^2 + \cdots + (1)(1+i)^{\frac{n}{k}-1} + (1)(1+i)^{\frac{n}{k}} \\
&= (1)(1+j)^k + (1)(1+j)^{2k} + \cdots + (1)(1+j)^{n-k} + (1)(1+j)^n \\
&= (1+j)^k \left(\frac{1 - [(1+j)^k]^{n/k}}{1 - (1+j)^k} \right) \\
&= (1+j)^k \left(\frac{1 - (1+i)^n}{1 - (1+j)^k} \right) \\
&= \frac{1}{v_j^k} \left(\frac{(1+j)^n - 1}{(1+j)^k - 1} \right) \\
&= \left[\frac{(1+j)^n - 1}{1 - v_j^k} \right] \times \frac{j}{j} \\
&= \frac{s_{\overline{n}|j}}{a_{\overline{k}|j}}
\end{aligned}$$

Another approach in determining the accumulated value would be to go back to a basic relationship where a future value is equal to its present value carried forward with interest:

$$\begin{aligned}
FV_{\frac{n}{k}} &= FV_n \\
&= PV_0 \cdot (1+j)^n \\
&= \frac{a_{\overline{n}|j}}{a_{\overline{k}|j}} \cdot (1+j)^n \\
&= \frac{s_{\overline{n}|j}}{a_{\overline{k}|j}}
\end{aligned}$$

The accumulated value can also be derived by again considering that each beginning-of-year payment of 1 represents the present value of smaller level end-of-conversion-period payments that are made k times during the year:

$$P \cdot a_{\overline{k}|j} = 1$$

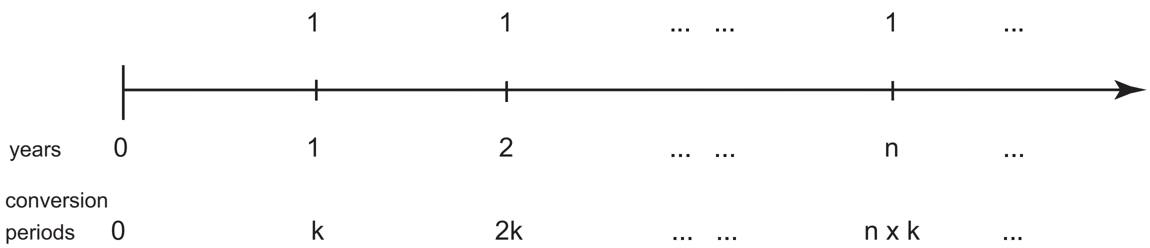
These smaller level payments are therefore equal to $P = \dfrac{1}{a_{\overline{k}|j}}$. If these smaller payments are made at the end of every conversion period, over the term of the annuity, and there are n conversion periods in total, then the accumulated value (at $t = n/k$ years or $t = n$ conversion periods) of these smaller payments is determined to be

$$FV_{\frac{n}{k}} = \left(\frac{1}{a_{\overline{k}|j}}\right) \cdot s_{\overline{n}|j}.$$

Other Considerations

<u>Perpetuity–Immediate</u>

– a perpetuity–immediate with annual end-of-year payments of 1 and where the nominal credited interest rate is convertible more frequently than annually, can be illustrated as:



– the present value (at $t = 0$) of an annual perpetuity–immediate where payments are made every k conversion periods and where the rate of interest is $j = \dfrac{i^{(k)}}{k}$ shall be calculated as follows:

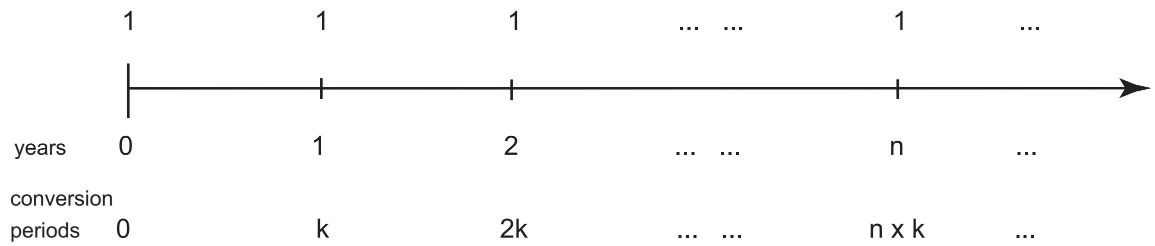
$$
\begin{aligned}
PV_0 &= (1)v_i^1 + (1)v_i^2 + (1)v_i^3 + \cdots \\
&= (1)v_j^k + (1)v_j^{2k} + (1)v_j^{3k} + \cdots \\
&= v_j^k(1 + v_j^k + v_j^{2k} + v_j^{3k} + \cdots) \\
&= v_j^k \left(\frac{1 - (v^k)^\infty}{1 - v_j^k} \right) \\
&= v_j^k \left(\frac{1 - 0}{1 - v_j^k} \right) \\
&= \frac{1}{(1+j)^k} \left(\frac{1}{1 - v_j^k} \right) \\
&= \left(\frac{1}{(1+j)^k - 1} \right) \cdot \frac{j}{j} \\
&= \frac{1}{j \cdot s_{\overline{k}|j}}
\end{aligned}
$$

– one could also derive the above formula by simply substituting $n = \infty$ into the original present value formula

$$PV_0 = \frac{a_{\overline{\infty}|j}}{s_{\overline{k}|j}} = \frac{1}{j} \times \frac{1}{s_{\overline{k}|j}}$$

Perpetuity–Due

– a perpetuity–due with annual beginning-of-year payments of 1 and where the nominal credited interest rate is convertible more frequently than annually, can be illustrated as:



– the present value (at $t = 0$) of an annual perpetuity–due where payments are made every k conversion periods and where the rate of interest is $j = \dfrac{i^{(k)}}{k}$ shall be calculated as follows:

$$
\begin{aligned}
PV_0 &= 1 + (1)v_i^1 + (1)v_i^2 + (1)v_i^3 + \cdots \\
&= 1 + (1)v_j^k + (1)v_j^{2k} + (1)v_j^{3k} + \cdots \\
&= 1 + v_j^k + v_j^{2k} + v_j^{3k} + \cdots \\
&= \left(\frac{1 - (v^k)^\infty}{1 - v_j^k} \right) \\
&= \left(\frac{1 - 0}{1 - v_j^k} \right) \\
&= \left(\frac{1}{1 - v_j^k} \right) \cdot \frac{j}{j} \\
&= \frac{1}{j \cdot a_{\overline{k}|j}}
\end{aligned}
$$

– one could also derive the above formula by simply substituting $n = \infty$ into the original present value formula

$$PV_0 = \frac{a_{\overline{\infty}|j}}{a_{\overline{k}|j}} = \frac{1}{j} \times \frac{1}{a_{\overline{k}|j}}$$

Interest Is Convertible Continuously: $i^{(\infty)} = \delta$

- the problem under this situation is that k is infinite (so is the total number of conversion periods over the term). Therefore, the prior formulas will not work.

- for example, the present value (at $t = 0$) of an annuity–immediate where payments of $\frac{1}{12}$ are made every month for n years (or $12n$ periods) and where the annual force of interest is δ can be calculated as follows:

$$
\begin{aligned}
PV_0 =& (\frac{1}{12})v_i^{\frac{1}{12}} + (\frac{1}{12})v_i^{\frac{2}{12}} + \cdots + (\frac{1}{12})v_i^{\frac{11}{12}} + (\frac{1}{12})v_i^{\frac{12}{12}} && (1st \text{ year})\\
&+(\frac{1}{12})v_i^{\frac{13}{12}} + (\frac{1}{12})v_i^{\frac{14}{12}} + \cdots + (\frac{1}{12})v_i^{\frac{23}{12}} + (\frac{1}{12})v_i^{\frac{24}{12}} && (2nd \text{ year})\\
&\vdots\\
&+(\frac{1}{12})v_i^{\frac{12(n-1)+1}{12}} + (\frac{1}{12})v_i^{\frac{12(n-1)+2}{12}} + \cdots + (\frac{1}{12})v_i^{\frac{12n-1}{12}} + (\frac{1}{12})v_i^{\frac{12n}{12}} && (\text{last year})\\
\\
=& (\frac{1}{12})v_i^{\frac{1}{12}}\left[1 + v_i^{\frac{1}{12}} + \cdots + v_i^{\frac{10}{12}} + v_i^{\frac{11}{12}}\right] && (1st \text{ year})\\
&+(\frac{1}{12})v_i^{\frac{13}{12}}\left[1 + v_i^{\frac{1}{12}} + \cdots + v_i^{\frac{10}{12}} + v_i^{\frac{11}{12}}\right] && (2nd \text{ year})\\
&\vdots\\
&+(\frac{1}{12})v_i^{\frac{12(n-1)+1}{12}}\left[1 + v_i^{\frac{1}{12}} + \cdots + v_i^{\frac{10}{12}} + v_i^{\frac{11}{12}}\right] && (\text{last year})\\
=& \left((\frac{1}{12})v_i^{\frac{1}{12}} + (\frac{1}{12})v_i^{\frac{13}{12}} + \cdots + (\frac{1}{12})v_i^{\frac{12(n-1)+1}{12})}\right)\left[1 + v_i^{\frac{1}{12}} + \cdots + v_i^{\frac{10}{12}} + v_i^{\frac{11}{12}}\right]\\
=& (\frac{1}{12})v_i^{\frac{1}{12}}\left(1 + v_i^{\frac{12}{12}} + \cdots + v_i^{(n-1)}\right)\left[\frac{1-(v_i^{\frac{1}{12}})^{12}}{1-v_i^{\frac{1}{12}}}\right]\\
=& (\frac{1}{12})\frac{1}{(1+i)^{\frac{1}{12}}}\left(\frac{1-v_i^n}{1-v_i^1}\right)\cdot\left[\frac{1-v_i^1}{1-v_i^{\frac{1}{12}}}\right]\\
=& (\frac{1}{12})\frac{1}{(1+i)^{\frac{1}{12}}}\left(\frac{1-v_i^n}{1-v_i^{\frac{1}{12}}}\right)\\
=& (\frac{1}{12})\left(\frac{1-v_i^n}{(1+i)^{\frac{1}{12}}-1}\right)\times\frac{i}{i}\\
=& (\frac{1}{12})\left(\frac{i}{(1+i)^{\frac{1}{12}}-1}\right)\times a_{\overline{n}|i} = (\frac{1}{12})\left(\frac{\left(1+\frac{i^{(12)}}{12}\right)^{12}-1}{\frac{i^{(12)}}{12}}\right)\times a_{\overline{n}|i}\\
=& \left[(\frac{1}{12})\cdot s_{\overline{12}|\frac{i^{(12)}}{12}}\right]\times a_{\overline{n}|i} = \left[(\frac{1}{12})\cdot s_{\overline{12}|\frac{i^{(12)}}{12}=e^{\frac{\delta}{12}}-1}\right]\times a_{\overline{n}|i=e^{\delta}-1}
\end{aligned}
$$

- in this case, the monthly payments for each year are converted to end-of-year lump sums that are discounted back to $t = 0$ at an annual effective rate of interest, i, which was converted from δ.

4.4 Further Analysis of Annuities Payable $\boxed{\text{More}}$ Frequency Than Interest Is Convertible

Annuity–Immediate

- payments of $\frac{1}{m}$ are made at the end of every $\frac{1}{m}$th of year for the next n years
- the present value (at $t=0$) of an m^{th}ly annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $a^{(m)}_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}
a^{(m)}_{\overline{n}|i} =&(\frac{1}{m})v_i^{\frac{1}{m}} + (\frac{1}{m})v_i^{\frac{2}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{m-1}{m}} + (\frac{1}{m})v_i^{\frac{m}{m}} && (1st\text{ year})\\
&+(\frac{1}{m})v_i^{\frac{m+1}{m}} + (\frac{1}{m})v_i^{\frac{m+2}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{2m-1}{m}} + (\frac{1}{m})v_i^{\frac{2m}{m}} && (2nd\text{ year})\\
&\vdots\\
&+(\frac{1}{m})v_i^{\frac{(n-1)m+1}{m}} + (\frac{1}{m})v_i^{\frac{(n-1)m+2}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{nm-1}{m}} + (\frac{1}{m})v_i^{\frac{nm}{m}} && (\text{last year})\\
\\
&=(\frac{1}{m})v_i^{\frac{1}{m}}\left[1+v_i^{\frac{1}{m}}+\cdots+v_i^{\frac{m-2}{m}}+v_i^{\frac{m-1}{m}}\right] && (1st\text{ year})\\
&+(\frac{1}{m})v_i^{\frac{m+1}{m}}\left[1+v_i^{\frac{1}{m}}+\cdots+v_i^{\frac{m-2}{m}}+v_i^{\frac{m-1}{m}}\right] && (2nd\text{ year})\\
&\vdots\\
&+(\frac{1}{m})v_i^{\frac{(n-1)m+1}{m}}\left[1+v_i^{\frac{1}{m}}+\cdots+v_i^{\frac{m-2}{m}}+v_i^{\frac{m-1}{m}}\right] && (\text{last year})\\
\\
&=\left((\frac{1}{m})v_i^{\frac{1}{m}} + (\frac{1}{m})v_i^{\frac{m+1}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{(n-1)m+1}{m}})\right)\left[1+v_i^{\frac{1}{m}}+\cdots+v_i^{\frac{m-2}{m}}+v_i^{\frac{m-1}{m}}\right]\\
&=(\frac{1}{m})v_i^{\frac{1}{m}}\left(1+v_i^{\frac{m}{m}}+\cdots+v_i^{(n-1)}\right)\left[\frac{1-(v_i^{\frac{1}{m}})^m}{1-v_i^{\frac{1}{m}}}\right]\\
&=(\frac{1}{m})\frac{1}{(1+i)^{\frac{1}{m}}}\left(\frac{1-v_i^n}{1-v_i^1}\right)\cdot\left[\frac{1-v_i^1}{1-v_i^{\frac{1}{m}}}\right]\\
&=(\frac{1}{m})\frac{1}{(1+i)^{\frac{1}{m}}}\left(\frac{1-v_i^n}{1-v_i^{\frac{1}{m}}}\right)\\
&=(\frac{1}{m})\left(\frac{1-v_i^n}{(1+i)^{\frac{1}{m}}-1}\right)\\
&=\frac{1-v_i^n}{m\left[(1+i)^{\frac{1}{m}}-1\right]}\\
&=\frac{1-v_i^n}{i^{(m)}} = \left(\frac{1}{m}\times m\right)\cdot\frac{1-v_i^n}{i^{(m)}}
\end{aligned}$$

– the accumulated value (at $t = n$) of an m^{th}ly annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $s^{(m)}_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}
s^{(m)}_{\overline{n}|i} =& (\frac{1}{m}) + (\frac{1}{m})(1+i)^{\frac{1}{m}} + \cdots + (\frac{1}{m})(1+i)^{\frac{m-2}{m}} + (\frac{1}{m})(1+i)^{\frac{m-1}{m}} && \text{(last year)}\\
&+(\frac{1}{m})(1+i)^{\frac{m}{m}} + (\frac{1}{m})(1+i)^{\frac{m+1}{m}} + \cdots + (\frac{1}{m})(1+i)^{\frac{2m-2}{m}} + (\frac{1}{m})(1+i)^{\frac{2m-1}{m}} && \text{(}2nd\text{ last year)}\\
&\vdots\\
&+(\frac{1}{m})(1+i)^{\frac{(n-1)m}{m}} + (\frac{1}{m})(1+i)^{\frac{(n-1)m+1}{m}} + \cdots + (\frac{1}{m})(1+i)^{\frac{nm-2}{m}} + (\frac{1}{m})(1+i)^{\frac{nm-1}{m}} && \text{(first year)}\\
\\
=& (\frac{1}{m}) \left[1 + (1+i)^{\frac{1}{m}} + \cdots + (1+i)^{\frac{m-2}{m}} + (1+i)^{\frac{m-1}{m}}\right] && \text{(last year)}\\
&+(\frac{1}{m})(1+i)^{\frac{m}{m}} \left[1 + (1+i)^{\frac{1}{m}} + \cdots + (1+i)^{\frac{m-2}{m}} + (1+i)^{\frac{m-1}{m}}\right] && \text{(}2nd\text{ last year)}\\
&\vdots\\
&+(\frac{1}{m})(1+i)^{\frac{(n-1)m}{m}} \left[1 + (1+i)^{\frac{1}{m}} + \cdots + (1+i)^{\frac{m-2}{m}} + (1+i)^{\frac{m-1}{m}}\right] && \text{(first year)}\\
\\
=& \left((\frac{1}{m}) + (\frac{1}{m})(1+i)^{\frac{m}{m}} + \cdots + (\frac{1}{m})(1+i)^{\frac{(n-1)m}{m})}\right) \left[1 + (1+i)^{\frac{1}{m}} + \cdots + (1+i)^{\frac{m-2}{m}} + (1+i)^{\frac{m-1}{m}}\right]\\
=& (\frac{1}{m}) \left(1 + (1+i)^{\frac{m}{m}} + \cdots + (1+i)^{(n-1)}\right) \left[\frac{1-((1+i)^{\frac{1}{m}})^m}{1-(1+i)^{\frac{1}{m}}}\right]\\
=& (\frac{1}{m}) \left(\frac{1-(1+i)^n}{1-(1+i)^1}\right) \cdot \left[\frac{1-(1+i)^1}{1-(1+i)^{\frac{1}{m}}}\right]\\
=& (\frac{1}{m}) \left(\frac{1-(1+i)^n}{1-(1+i)^{\frac{1}{m}}}\right)\\
=& \frac{(1+i)^n - 1}{m\left[(1+i)^{\frac{1}{m}} - 1\right]}\\
=& \frac{(1+i)^n - 1}{i^{(m)}} = \left(\frac{1}{m} \times m\right) \cdot \frac{(1+i)^n - 1}{i^{(m)}}
\end{aligned}$$

Basic Relationship $1 : 1 = i^{(m)} \cdot a_{\overline{n}|}^{(m)} + v^n$

Basic Relationship $2 : PV(1+i)^n = FV$ **and** $PV = FV \cdot v^n$

- if the future value at time n, $s_{\overline{n}|}^{(m)}$, is discounted back to time 0, then you will have its present value, $a_{\overline{n}|}^{(m)}$

$$
\begin{aligned}
s_{\overline{n}|}^{(m)} \cdot v^n &= \left[\frac{(1+i)^n - 1}{i^{(m)}}\right] \cdot v^n \\
&= \frac{(1+i)^n \cdot v^n - v^n}{i^{(m)}} \\
&= \frac{1 - v^n}{i^{(m)}} \\
&= a_{\overline{n}|}^{(m)}
\end{aligned}
$$

- if the present value at time 0, $a_{\overline{n}|}^{(m)}$, is accumulated forward to time n, then you will have its future value, $s_{\overline{n}|}^{(m)}$

$$
\begin{aligned}
a_{\overline{n}|}^{(m)} \cdot (1+i)^n &= \left[\frac{1 - v^n}{i^{(m)}}\right](1+i)^n \\
&= \frac{(1+i)^n - v^n(1+i)^n}{i^{(m)}} \\
&= \frac{(1+i)^n - 1}{i^{(m)}} \\
&= s_{\overline{n}|}^{(m)}
\end{aligned}
$$

Basic Relationship $3 : \dfrac{1}{m \times a_{\overline{n}|}^{(m)}} = \dfrac{1}{m \times s_{\overline{n}|}^{(m)}} + \dfrac{i^{(m)}}{m}$

- Consider a loan of 1, to be paid back over n years with equal m^{th}ly payments of P made at the end of each m^{th} of a year. An annual effective rate of interest, i, and nominal rate of interest, $i^{(m)}$, is used. The present value of this single payment loan must be equal to the present value of the multiple payment income stream.

$$(P \times m) \cdot a_{\overline{n}|i}^{(m)} = 1$$

$$P = \frac{1}{m \times a_{\overline{n}|i}^{(m)}}$$

- Alternatively, consider a loan of 1, where the m^{th}ly interest due on the loan, $(1) \times \frac{i^{(m)}}{m}$, is paid at the end of each m^{th} of a year for n years and the loan amount is paid back at time n.
- In order to produce the loan amount at time n, payments of D at the end of each m^{th} of a year, for n years, will be made into an account that credits interest at an m^{th}ly rate of interest $\frac{i^{(m)}}{m}$.

- The future value of the multiple deposit income stream must equal the future value of the single payment, which is the loan of 1.

$$(D \times m) \cdot s^{(m)}_{\overline{n}|i} = 1$$

$$D = \frac{1}{m \times s^{(m)}_{\overline{n}|i}}$$

- The total m^{th}ly payment will be the interest payment and account payment:

$$\frac{i^{(m)}}{m} + \frac{1}{m \times s^{(m)}_{\overline{n}|i}}$$

- Note that

$$\frac{1}{a^{(m)}_{\overline{n}|i}} = \frac{i^{(m)}}{1 - v^n} \times \frac{(1+i)^n}{(1+i)^n} = \frac{i^{(m)}(1+i)^n}{(1+i)^n - 1}$$

$$= \frac{i^{(m)}(1+i)^n + i^{(m)} - i^{(m)}}{(1+i)^n - 1} = \frac{i^{(m)}[(1+i)^n - 1] + i^{(m)}}{(1+i)^n - 1}$$

$$= i^{(m)} + \frac{i^{(m)}}{(1+i)^n - 1} = i^{(m)} + \frac{1}{s^{(m)}_{\overline{n}|}}$$

- Therefore, a level m^{th}ly annuity payment on a loan is the same as making an m^{th}ly interest payment each m^{th} of a year plus making m^{th}ly deposits in order to save for the loan repayment.

Basic Relationship $4 : a^{(m)}_{\overline{n}|} = \frac{i}{i^{(m)}} \cdot a_{\overline{n}|}$**,** $s^{(m)}_{\overline{n}|} = \frac{i}{i^{(m)}} \cdot s_{\overline{n}|}$

- Consider payments of $\frac{1}{m}$ made at the end of every $\frac{1}{m}$th of year for the next n years. Over a one-year period, payments of $\frac{1}{m}$ made at the end of each m^{th} period will accumulate at the end of the year to a lump sum of $\left(\frac{1}{m} \times m\right) \cdot s^{(m)}_{\overline{1}|}$. If this end-of-year lump sum exists for each year of the n-year annuity-immediate, then the present value (at $t = 0$) of these end-of-year lump sums is the same as $\left(\frac{1}{m} \times m\right) \cdot a^{(m)}_{\overline{n}|}$:

$$\left(\frac{1}{m} \times m\right) \cdot a^{(m)}_{\overline{n}|} = \left(\frac{1}{m} \times m\right) \cdot s^{(m)}_{\overline{1}|} \cdot a_{\overline{n}|}$$

$$a^{(m)}_{\overline{n}|} = \frac{i}{i^{(m)}} \cdot a_{\overline{n}|}$$

- Therefore, the accumulated value (at $t = n$) of these end-of-year lump sums is the same as $\left(\frac{1}{m} \times m\right) \cdot s^{(m)}_{\overline{n}|}$:

$$\left(\frac{1}{m} \times m\right) \cdot s^{(m)}_{\overline{n}|} = \left(\frac{1}{m} \times m\right) \cdot s^{(m)}_{\overline{1}|} \cdot s_{\overline{n}|}$$

$$s^{(m)}_{\overline{n}|} = \frac{i}{i^{(m)}} \cdot s_{\overline{n}|}$$

Annuity–Due

– payments of $\frac{1}{m}$ are made at the beginning of every $\frac{1}{m}$th of year for the next n years

– the present value (at $t = 0$) of an m^{th}ly annuity–due, where the annual effective rate of interest is i, shall be denoted as $\ddot{a}^{(m)}_{\overline{n|}i}$ and is calculated as follows:

$$
\begin{aligned}
\ddot{a}^{(m)}_{\overline{n|}i} =& (\frac{1}{m}) + (\frac{1}{m})v_i^{\frac{1}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{m-2}{m}} + (\frac{1}{m})v_i^{\frac{m-1}{m}} && \text{(1}st\text{ year)}\\
&+(\frac{1}{m})v_i^{\frac{m}{m}} + (\frac{1}{m})v_i^{\frac{m+1}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{2m-2}{m}} + (\frac{1}{m})v_i^{\frac{2m-1}{m}} && \text{(2}nd\text{ year)}\\
&\vdots\\
&+(\frac{1}{m})v_i^{\frac{(n-1)m}{m}} + (\frac{1}{m})v_i^{\frac{(n-1)m+1}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{nm-2}{m}} + (\frac{1}{m})v_i^{\frac{nm-1}{m}} && \text{(last year)}\\
\\
=&(\frac{1}{m})\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right] && \text{(1}st\text{ year)}\\
&+(\frac{1}{m})v_i^{\frac{m}{m}}\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right] && \text{(2}nd\text{ year)}\\
&\vdots\\
&+(\frac{1}{m})v_i^{\frac{(n-1)m}{m}}\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right] && \text{(last year)}\\
\\
=&\left((\frac{1}{m}) + (\frac{1}{m})v_i^{\frac{m}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{(n-1)m}{m}})\right)\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right]\\
=&(\frac{1}{m})\left(1 + v_i^{\frac{m}{m}} + \cdots + v_i^{(n-1)}\right)\left[\frac{1-(v_i^{\frac{1}{m}})^m}{1-v_i^{\frac{1}{m}}}\right]\\
=&(\frac{1}{m})\left(\frac{1-v_i^n}{1-v_i^1}\right)\cdot\left[\frac{1-v_i^1}{1-v_i^{\frac{1}{m}}}\right]\\
=&(\frac{1}{m})\left(\frac{1-v_i^n}{1-v_i^{\frac{1}{m}}}\right)\\
=&(\frac{1}{m})\left(\frac{1-v_i^n}{1-(1-d)^{\frac{1}{m}}}\right)\\
=&\frac{1-v_i^n}{m\left[1-(1-d)^{\frac{1}{m}}\right]}\\
=&\frac{1-v_i^n}{d^{(m)}} = \left(\frac{1}{m}\times m\right)\cdot\frac{1-v_i^n}{d^{(m)}}
\end{aligned}
$$

– the accumulated value (at $t = n$) of an m^{th}ly annuity–due, where the annual effective rate of interest is i, shall be denoted as $\ddot{s}^{(m)}_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}
\ddot{s}^{(m)}_{\overline{n}|i} =& (\frac{1}{m})(1+i)^{\frac{1}{m}} + (\frac{1}{m})(1+i)^{\frac{2}{m}} + \cdots + (\frac{1}{m})(1+i)^{\frac{m-1}{m}} + (\frac{1}{m})(1+i)^{\frac{m}{m}} && \text{(last year)} \\
&+ (\frac{1}{m})(1+i)^{\frac{m+1}{m}} + (\frac{1}{m})(1+i)^{\frac{m+2}{m}} + \cdots + (\frac{1}{m})(1+i)^{\frac{2m-1}{m}} + (\frac{1}{m})(1+i)^{\frac{2m}{m}} && \text{(}2nd\text{ last year)} \\
&\vdots \\
&+ (\frac{1}{m})(1+i)^{\frac{(n-1)m+1}{m}} + (\frac{1}{m})(1+i)^{\frac{(n-1)m+2}{m}} + \cdots + (\frac{1}{m})(1+i)^{\frac{nm-1}{m}} + (\frac{1}{m})(1+i)^{\frac{nm}{m}} && \text{(first year)} \\
\\
=& (\frac{1}{m})(1+i)^{\frac{1}{m}} \left[1 + (1+i)^{\frac{1}{m}} + \cdots + (1+i)^{\frac{m-2}{m}} + (1+i)^{\frac{m-1}{m}}\right] && \text{(last year)} \\
&+ (\frac{1}{m})(1+i)^{\frac{m+1}{m}} \left[1 + (1+i)^{\frac{1}{m}} + \cdots + (1+i)^{\frac{m-2}{m}} + (1+i)^{\frac{m-1}{m}}\right] && \text{(}2nd\text{ last year)} \\
&\vdots \\
&+ (\frac{1}{m})(1+i)^{\frac{(n-1)m+1}{m}} \left[1 + (1+i)^{\frac{1}{m}} + \cdots + (1+i)^{\frac{m-2}{m}} + (1+i)^{\frac{m-1}{m}}\right] && \text{(first year)} \\
\\
=& \left((\frac{1}{m})(1+i)^{\frac{1}{m}} + (\frac{1}{m})(1+i)^{\frac{m+1}{m}} + \cdots + (\frac{1}{m})(1+i)^{\frac{(n-1)m+1}{m}})\right) \left[\frac{1 - ((1+i)^{\frac{1}{m}})^m}{1 - (1+i)^{\frac{1}{m}}}\right] \\
=& (\frac{1}{m})(1+i)^{\frac{1}{m}} \left(1 + (1+i)^{\frac{m}{m}} + \cdots + (1+i)^{(n-1)}\right) \left[\frac{1 - ((1+i)^{\frac{1}{m}})^m}{1 - (1+i)^{\frac{1}{m}}}\right] \\
=& (\frac{1}{m})(1+i)^{\frac{1}{m}} \left(\frac{1 - (1+i)^n}{1 - (1+i)^1}\right) \cdot \left[\frac{1 - (1+i)^1}{1 - (1+i)^{\frac{1}{m}}}\right] \\
=& (\frac{1}{m})(1+i)^{\frac{1}{m}} \left(\frac{1 - (1+i)^n}{1 - (1+i)^{\frac{1}{m}}}\right) \\
=& \frac{(1+i)^n - 1}{m \cdot v_i^{\frac{1}{m}} \left[(1+i)^{\frac{1}{m}} - 1\right]} \\
=& \frac{(1+i)^n - 1}{m\left[1 - v_i^{\frac{1}{m}}\right]} \\
=& \frac{(1+i)^n - 1}{m\left[1 - (1-d)^{\frac{1}{m}}\right]} \\
=& \frac{(1+i)^n - 1}{d^{(m)}} = \left(\frac{1}{m} \times m\right) \cdot \frac{(1+i)^n - 1}{d^{(m)}}
\end{aligned}$$

Basic Relationship $1 : 1 = d^{(m)} \cdot \ddot{a}^{(m)}_{\overline{n}|} + v^n$

Basic Relationship $2 : PV(1+i)^n = FV$ **and** $PV = FV \cdot v^n$

– if the future value at time n, $\ddot{s}^{(m)}_{\overline{n}|}$, is discounted back to time 0, then you will have its present value, $\ddot{a}^{(m)}_{\overline{n}|}$

$$\begin{aligned}
\ddot{s}^{(m)}_{\overline{n}|} \cdot v^n &= \left[\frac{(1+i)^n - 1}{d^{(m)}}\right] \cdot v^n \\
&= \frac{(1+i)^n \cdot v^n - v^n}{d^{(m)}} \\
&= \frac{1 - v^n}{d^{(m)}} \\
&= \ddot{a}^{(m)}_{\overline{n}|}
\end{aligned}$$

– if the present value at time 0, $\ddot{a}^{(m)}_{\overline{n}|}$, is accumulated forward to time n, then you will have its future value, $\ddot{s}^{(m)}_{\overline{n}|}$

$$\begin{aligned}
\ddot{a}^{(m)}_{\overline{n}|} \cdot (1+i)^n &= \left[\frac{1 - v^n}{d^{(m)}}\right] (1+i)^n \\
&= \frac{(1+i)^n - v^n(1+i)^n}{d^{(m)}} \\
&= \frac{(1+i)^n - 1}{d^{(m)}} \\
&= \ddot{s}^{(m)}_{\overline{n}|}
\end{aligned}$$

Basic Relationship $3 : \dfrac{1}{m \times \ddot{a}^{(m)}_{\overline{n}|}} = \dfrac{1}{m \times \ddot{s}^{(m)}_{\overline{n}|}} + \dfrac{d^{(m)}}{m}$

– Consider a loan of 1, to be paid back over n years with equal m^{th}ly payments of P made at the beginning of each m^{th} of a year. An annual effective rate of interest, i, and nominal rate of discount, $d^{(m)}$, is used. The present value of this single payment loan must be equal to the present value of the multiple payment income stream.

$$(P \times m) \cdot \ddot{a}^{(m)}_{\overline{n}|i} = 1$$

$$P = \frac{1}{m \times \ddot{a}^{(m)}_{\overline{n}|i}}$$

– Alternatively, consider a loan of 1, where the m^{th}ly discount due on the loan, $(1) \times \frac{d^{(m)}}{m}$, is paid at the beginning of each m^{th} of a year for n years and the loan amount is paid back at time n.

– In order to produce the loan amount at time n, payments of D at the beginning of each m^{th} of a year, for n years, will be made into an account that credits interest at an m^{th}ly rate of discount $\frac{d^{(m)}}{m}$.

- The future value of the multiple deposit income stream must equal the future value of the single payment, which is the loan of 1.

$$(D \times m) \cdot \ddot{s}^{(m)}_{\overline{n}|i} = 1$$

$$D = \frac{1}{m \times \ddot{s}^{(m)}_{\overline{n}|i}}$$

- The total m^{th}ly payment will be the discount payment and account payment:

$$\frac{d^{(m)}}{m} + \frac{1}{m \times \ddot{s}^{(m)}_{\overline{n}|i}}$$

- Note that

$$\frac{1}{\ddot{a}^{(m)}_{\overline{n}|i}} = \frac{d^{(m)}}{1 - v^n} \times \frac{(1+i)^n}{(1+i)^n} = \frac{d^{(m)}(1+i)^n}{(1+i)^n - 1}$$

$$= \frac{d^{(m)}(1+i)^n + d^{(m)} - d^{(m)}}{(1+i)^n - 1} = \frac{d^{(m)}[(1+i)^n - 1] + d^{(m)}}{(1+i)^n - 1}$$

$$= i^{(m)} + \frac{d^{(m)}}{(1+i)^n - 1} = d^{(m)} + \frac{1}{\ddot{s}^{(m)}_{\overline{n}|}}$$

- Therefore, a level m^{th}ly annuity payment on a loan is the same as making an m^{th}ly discount payment each m^{th} of a year plus making m^{th}ly deposits in order to save for the loan repayment.

Basic Relationship $4 : \ddot{a}^{(m)}_{\overline{n}|} = \frac{d}{d^{(m)}} \cdot \ddot{a}_{\overline{n}|}$, $\ddot{s}^{(m)}_{\overline{n}|} = \frac{d}{d^{(m)}} \cdot \ddot{s}_{\overline{n}|}$

- Consider payments of $\frac{1}{m}$ made at the beginning of every $\frac{1}{m}$th of year for the next n years. Over a one-year period, payments of $\frac{1}{m}$ made at the beginning of each m^{th} period will accumulate at the end of the year to a lump sum of $\left(\frac{1}{m} \times m\right) \cdot \ddot{s}^{(m)}_{\overline{1}|}$. If this end-of-year lump sum exists for each year of the n-year annuity-immediate, then the present value (at $t = 0$) of these end-of-year lump sums is the same as $\left(\frac{1}{m} \times m\right) \cdot \ddot{a}^{(m)}_{\overline{n}|}$:

$$\left(\frac{1}{m} \times m\right) \cdot \ddot{a}^{(m)}_{\overline{n}|} = \left(\frac{1}{m} \times m\right) \cdot \ddot{s}^{(m)}_{\overline{1}|} \cdot a_{\overline{n}|}$$

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{i}{d^{(m)}} \cdot a_{\overline{n}|}$$

- Therefore, the accumulated value (at $t = n$) of these end-of-year lump sums is the same as $\left(\frac{1}{m} \times m\right) \cdot \ddot{s}^{(m)}_{\overline{n}|}$:

$$\left(\frac{1}{m} \times m\right) \cdot \ddot{s}^{(m)}_{\overline{n}|} = \left(\frac{1}{m} \times m\right) \cdot \ddot{s}^{(m)}_{\overline{1}|} \cdot s_{\overline{n}|}$$

$$\ddot{s}^{(m)}_{\overline{n}|} = \frac{i}{i^{(m)}} \cdot s_{\overline{n}|}$$

Basic Relationship 5: **Due= Immediate** $\times(1+i)^{\frac{1}{m}}$

$$\ddot{a}_{\overline{n}|}^{(m)}=\frac{1-v^n}{d^{(m)}}=\frac{1-v^n}{\left(\frac{i^{(m)}}{1+\frac{i^{(m)}}{m}}\right)}=a_{\overline{n}|}^{(m)}\cdot\left(1+\frac{i^{(m)}}{m}\right)=a_{\overline{n}|}^{(m)}\cdot(1+i)^{\frac{1}{m}}$$

$$\ddot{s}_{\overline{n}|}^{(m)}=\frac{(1+i)^n-1}{d^{(m)}}=\frac{(1+i)^n-1}{\left(\frac{i^{(m)}}{1+\frac{i^{(m)}}{m}}\right)}=s_{\overline{n}|}^{(m)}\cdot\left(1+\frac{i^{(m)}}{m}\right)=s_{\overline{n}|}^{(m)}\cdot(1+i)^{\frac{1}{m}}$$

An m^{th}ly annuity–due starts one m^{th} of a year earlier than an m^{th}ly annuity-immediate and as a result, earns one m^{th} of a year more interest, hence it will be larger.

Basic Relationship 6 : $\ddot{a}_{\overline{n}|}^{(m)}=\dfrac{1}{m}+a_{\overline{n-\frac{1}{m}}|}^{(m)}$

$$\begin{aligned}
\ddot{a}_{\overline{n}|}^{(m)} &= (\frac{1}{m})+\left[(\frac{1}{m})v^{\frac{1}{m}}+\cdots+(\frac{1}{m})v^{\frac{nm-2}{m}}+(\frac{1}{m})v^{\frac{nm-1}{m}}\right] \\
&= (\frac{1}{m})+(\frac{1}{m})v^{\frac{1}{m}}\left[1+v^{\frac{1}{m}}+\cdots+v^{\frac{nm-3}{m}}+v^{\frac{nm-2}{m}}\right] \\
&= (\frac{1}{m})+(\frac{1}{m})v^{\frac{1}{m}}\left[\frac{1-\left(v^{\frac{1}{m}}\right)^{nm-1}}{1-v^{\frac{1}{m}}}\right] \\
&= (\frac{1}{m})+\frac{1}{m(1+i)^{\frac{1}{m}}}\left[\frac{1-v^{n-\frac{1}{m}}}{1-v^{\frac{1}{m}}}\right] \\
&= (\frac{1}{m})+\left[\frac{1-v^{n-\frac{1}{m}}}{m\left[(1+i)^{\frac{1}{m}}-1\right]}\right] \\
&= (\frac{1}{m})+\left[\frac{1-v^{n-\frac{1}{m}}}{m\cdot\frac{i^{(m)}}{m}}\right] \\
&= \frac{1}{m}+\frac{1-v^{n-\frac{1}{m}}}{i^{(m)}} \\
&= \frac{1}{m}+a_{\overline{n-\frac{1}{m}}|}^{(m)}
\end{aligned}$$

An additional payment of $\frac{1}{m}$ at time 0 results in $a_{\overline{n-\frac{1}{m}}|}^{(m)}$ becoming nm $(=nm-1+1)$ payments that now commence at the beginning of each m^{th} of a year which is $\ddot{a}_{\overline{n}|}^{(m)}$.

Basic Relationship 7 : $s^{(m)}_{\overline{n}|} = \dfrac{1}{m} + \ddot{s}^{(m)}_{\overline{n-\frac{1}{m}}|}$

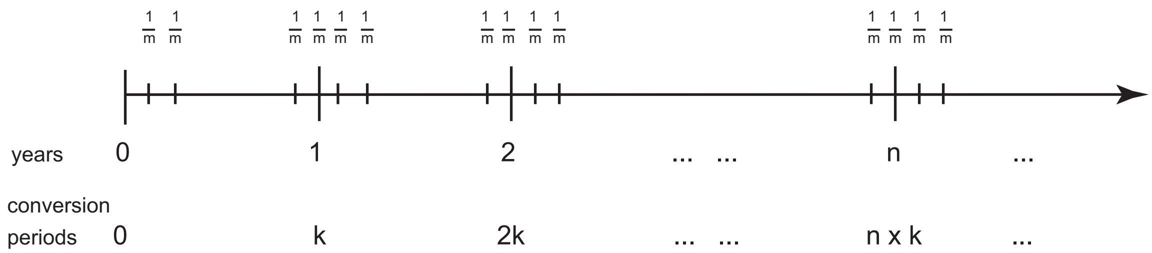
$$\begin{aligned}
s^{(m)}_{\overline{n}|} &= (\frac{1}{m}) + \left[(\frac{1}{m})(1+i)^{\frac{1}{m}} + \cdots + (\frac{1}{m})(1+i)^{\frac{nm-2}{m}} + (\frac{1}{m})(1+i)^{\frac{nm-1}{m}}\right] \\
&= (\frac{1}{m}) + (\frac{1}{m})(1+i)^{\frac{1}{m}}\left[1 + (1+i)^{\frac{1}{m}} + \cdots + (1+i)^{\frac{nm-3}{m}} + (1+i)^{\frac{nm-2}{m}}\right] \\
&= (\frac{1}{m}) + (\frac{1}{m})(1+i)^{\frac{1}{m}}\left[\frac{1-\left((1+i)^{\frac{1}{m}}\right)^{nm-1}}{1-(1+i)^{\frac{1}{m}}}\right] \\
&= (\frac{1}{m}) + \frac{1}{m \cdot v^{\frac{1}{m}}}\left[\frac{1-(1+i)^{n-\frac{1}{m}}}{1-(1+i)^{\frac{1}{m}}}\right] \\
&= (\frac{1}{m}) + \left[\frac{1-(1+i)^{n-\frac{1}{m}}}{m\left[v^{\frac{1}{m}}-1\right]}\right] \\
&= (\frac{1}{m}) + \left[\frac{(1+i)^{n-\frac{1}{m}}-1}{m\left[1-v^{\frac{1}{m}}\right]}\right] \\
&= (\frac{1}{m}) + \left[\frac{(1+i)^{n-\frac{1}{m}}-1}{m\left[1-(1-d)^{\frac{1}{m}}\right]}\right] \\
&= (\frac{1}{m}) + \left[\frac{(1+i)^{n-\frac{1}{m}}-1}{m \cdot \frac{d^{(m)}}{m}}\right] \\
&= \frac{1}{m} + \frac{(1+i)^{n-\frac{1}{m}}-1}{d^{(m)}} \\
&= \frac{1}{m} + \ddot{s}^{(m)}_{\overline{n-\frac{1}{m}}|}
\end{aligned}$$

An additional payment of $\frac{1}{m}$ at time n results in $\ddot{s}^{(m)}_{\overline{n-\frac{1}{m}}|}$ becoming nm $(= nm - 1 + 1)$ payments that now commerce at the end of each m^{th} of a year which is $s^{(m)}_{\overline{n}|}$.

Other Considerations

Perpetuity–Immediate

– payments of $\frac{1}{m}$ are made at the end of every $\frac{1}{m}$th of year forever.



– the present value (at $t = 0$) of an m^{th}ly perpetuity–immediate, where the annual effective rate of interest is i, shall be denoted as $a^{(m)}_{\overline{\infty}|i}$ and is calculated as follows:

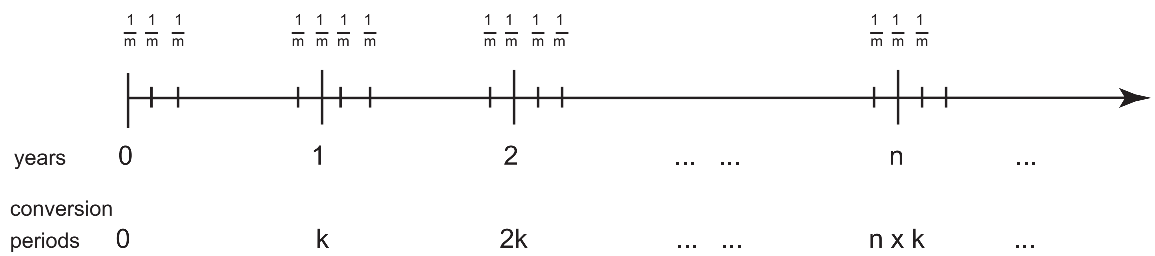
$$\begin{aligned}
a^{(m)}_{\overline{\infty}|i} =&(\frac{1}{m})v_i^{\frac{1}{m}} + (\frac{1}{m})v_i^{\frac{2}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{m-1}{m}} + (\frac{1}{m})v_i^{\frac{m}{m}} && (1st\ \text{year})\\
&+(\frac{1}{m})v_i^{\frac{m+1}{m}} + (\frac{1}{m})v_i^{\frac{m+2}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{2m-1}{m}} + (\frac{1}{m})v_i^{\frac{2m}{m}} && (2nd\ \text{year})\\
&\vdots\\
&+(\frac{1}{m})v_i^{\frac{(n-1)m+1}{m}} + (\frac{1}{m})v_i^{\frac{(n-1)m+2}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{nm-1}{m}} + (\frac{1}{m})v_i^{\frac{nm}{m}} && (nth\ \text{year})\\
&\vdots\\
=&(\frac{1}{m})v_i^{\frac{1}{m}}\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right] && (1st\ \text{year})\\
&+(\frac{1}{m})v_i^{\frac{m+1}{m}}\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right] && (2nd\ \text{year})\\
&\vdots\\
&+(\frac{1}{m})v_i^{\frac{(n-1)m+1}{m}}\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right] && (n\text{th year})\\
&\vdots\\
=&\left((\frac{1}{m})v_i^{\frac{1}{m}} + (\frac{1}{m})v_i^{\frac{m+1}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{(n-1)m+1}{m})} + \cdots\right)\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right]\\
=&(\frac{1}{m})v_i^{\frac{1}{m}}\left(1 + v_i^{\frac{m}{m}} + \cdots + v_i^{(n-1)} + \cdots\right)\left[\frac{1-(v_i^{\frac{1}{m}})^m}{1-v_i^{\frac{1}{m}}}\right]\\
=&(\frac{1}{m})\frac{1}{(1+i)^{\frac{1}{m}}}\left(\frac{1-v_i^{\infty}}{1-v_i^1}\right)\cdot\left[\frac{1-v_i^1}{1-v_i^{\frac{1}{m}}}\right]\\
=&(\frac{1}{m})\frac{1}{(1+i)^{\frac{1}{m}}}\left(\frac{1-0}{1-v_i^{\frac{1}{m}}}\right)\\
=&(\frac{1}{m})\left(\frac{1}{(1+i)^{\frac{1}{m}}-1}\right)\\
=&\frac{1}{m\left[(1+i)^{\frac{1}{m}}-1\right]}\\
=&\frac{1}{i^{(m)}} = \left(\frac{1}{m}\times m\right)\cdot\frac{1}{i^{(m)}}
\end{aligned}$$

– one could also derive the above formula by simply substituting $n = \infty$ into the original present value formula:

$$a^{(m)}_{\overline{\infty}|i} = \frac{1-v^{\infty}}{i^{(m)}} = \frac{1-0}{i^{(m)}} = \frac{1}{i^{(m)}}$$

Perpetuity–Due

– payments of $\frac{1}{m}$ are made at the beginning of every $\frac{1}{m}$th of year forever.



– the present value (at $t = 0$) of an m^{th}ly perpetuity–due, where the annual effective rate of interest is i, shall be denoted as $\ddot{a}^{(m)}_{\overline{\infty}|i}$ and is calculated as follows:

$$
\begin{aligned}
\ddot{a}^{(m)}_{\overline{\infty}|i} =& (\frac{1}{m}) + (\frac{1}{m})v_i^{\frac{1}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{m-2}{m}} + (\frac{1}{m})v_i^{\frac{m-1}{m}} && (1st \text{ year})\\
&+(\frac{1}{m})v_i^{\frac{m}{m}} + (\frac{1}{m})v_i^{\frac{m+1}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{2m-2}{m}} + (\frac{1}{m})v_i^{\frac{2m-1}{m}} && (2nd \text{ year})\\
&\vdots\\
&+(\frac{1}{m})v_i^{\frac{(n-1)m}{m}} + (\frac{1}{m})v_i^{\frac{(n-1)m+1}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{nm-2}{m}} + (\frac{1}{m})v_i^{\frac{nm-1}{m}} && (n\text{th year})\\
&\vdots\\
\\
=&(\frac{1}{m})\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right] && (1st \text{ year})\\
&+(\frac{1}{m})v_i^{\frac{m}{m}}\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right] && (2nd \text{ year})\\
&\vdots\\
&+(\frac{1}{m})v_i^{\frac{(n-1)m}{m}}\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right] && (n\text{th year})\\
&\vdots\\
=&\left((\frac{1}{m}) + (\frac{1}{m})v_i^{\frac{m}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{(n-1)m}{m})} + \cdots\right)\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right]\\
=&(\frac{1}{m})\left(1 + v_i^{\frac{m}{m}} + \cdots + v_i^{(n-1)} + \cdots\right)\left[\frac{1-(v_i^{\frac{1}{m}})^m}{1-v_i^{\frac{1}{m}}}\right]\\
=&(\frac{1}{m})\left(\frac{1-v_i^{\infty}}{1-v_i^1}\right)\cdot\left[\frac{1-v_i^1}{1-v_i^{\frac{1}{m}}}\right]\\
=&(\frac{1}{m})\left(\frac{1-0}{1-v_i^{\frac{1}{m}}}\right)\\
=&(\frac{1}{m})\left(\frac{1}{1-(1-d)^{\frac{1}{m}}}\right)\\
=&\frac{1}{m\left[1-(1-d)^{\frac{1}{m}}\right]}\\
=&\frac{1}{d^{(m)}} = \left(\frac{1}{m}\times m\right)\cdot\frac{1}{d^{(m)}}
\end{aligned}
$$

– one could also derive the above formula by simply substituting $n = \infty$ into the original present value formula:

$$\ddot{a}^{(m)}_{\overline{\infty}|i} = \frac{1-v^{\infty}}{d^{(m)}} = \frac{1-0}{d^{(m)}} = \frac{1}{d^{(m)}}$$

4.5 Continuous Annuities

- payments are made continuously every year for the next n years (i.e. $m = \infty$)
- the present value (at $t = 0$) of a continuous annuity, where the annual effective rate of interest is i, shall be denoted as $\bar{a}_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}\bar{a}_{\overline{n}|i} &= \int_0^n v^t dt \\ &= \int_0^n e^{-\delta t} dt \\ &= -\frac{1}{\delta} e^{-\delta t}\Big]_0^n \\ &= -\frac{1}{\delta}\left[e^{-\delta n} - e^{-\delta 0}\right] \\ &= \frac{1}{\delta}\left[1 - e^{-\delta n}\right] \\ &= \frac{1 - v_i^n}{\delta}\end{aligned}$$

- one could also derive the above formula by simply substituting $m = \infty$ into one of the original m^{th}ly present value formulas:

$$\begin{aligned}\bar{a}_{\overline{n}|i} &= a_{\overline{n}|i}^{(\infty)} = \frac{1 - v_i^n}{i^{(\infty)}} = \frac{1 - v_i^n}{\delta} \\ &= \ddot{a}_{\overline{n}|i}^{(\infty)} = \frac{1 - v_i^n}{d^{(\infty)}} = \frac{1 - v_i^n}{\delta}\end{aligned}$$

- the accumulated value (at $t = n$) of a continuous annuity, where the annual effective rate of interest is i, shall be denoted as $\bar{s}_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}\bar{s}_{\overline{n}|i} &= \int_0^n (1+i)^{n-t} dt \\ &= \int_0^n (1+i)^t dt \\ &= \int_0^n e^{\delta t} dt \\ &= \frac{1}{\delta} e^{\delta t} dt\Big]_0^n \\ &= \frac{1}{\delta}\left[e^{\delta n} - e^{\delta 0}\right] \\ &= \frac{(1+i)^n - 1}{\delta}\end{aligned}$$

Basic Relationship $1 : 1 = \delta \cdot \bar{a}_{\overline{n}|} + v^n$

Basic Relationship $2 : PV(1+i)^n = FV$ **and** $PV = FV \cdot v^n$

- if the future value at time n, $\bar{s}_{\overline{n}|}$, is discounted back to time 0, then you will have its present value, $\bar{a}_{\overline{n}|}$

$$\begin{aligned}\bar{s}_{\overline{n}|} \cdot v^n &= \left[\frac{(1+i)^n - 1}{\delta}\right] \cdot v^n \\ &= \frac{(1+i)^n \cdot v^n - v^n}{\delta} \\ &= \frac{1 - v^n}{\delta} \\ &= \bar{a}_{\overline{n}|}\end{aligned}$$

- if the present value at time 0, $\bar{a}_{\overline{n}|}$, is accumulated forward to time n, then you will have its future value, $\bar{s}_{\overline{n}|}$

$$\begin{aligned}\bar{a}_{\overline{n}|} \cdot (1+i)^n &= \left[\frac{1 - v^n}{\delta}\right] (1+i)^n \\ &= \frac{(1+i)^n - v^n(1+i)^n}{\delta} \\ &= \frac{(1+i)^n - 1}{\delta} \\ &= \bar{s}_{\overline{n}|}\end{aligned}$$

Basic Relationship $3 : \dfrac{1}{\bar{a}_{\overline{n}|}} = \dfrac{1}{\bar{s}_{\overline{n}|}} + \delta$

- Consider a loan of 1, to be paid back over n years with annual payments of P that are paid continuously each year, for the next n years. An annual effective rate of interest, i, and annual force of interest, δ, is used. The present value of this single payment loan must be equal to the present value of the multiple payment income stream.

$$P \cdot \bar{a}_{\overline{n}|i} = 1$$

$$P = \frac{1}{\bar{a}_{\overline{n}|i}}$$

- Alternatively, consider a loan of 1, where the annual interest due on the loan, $(1) \times \delta$, is paid continuously during the year for n years and the loan amount is paid back at time n.
- In order to produce the loan amount at time n, annual payments of D are paid continuously each year, for the next n years, into an account that credits interest at an annual force of interest, δ.

- The future value of the multiple deposit income stream must equal the future value of the single payment, which is the loan of 1.

$$D \cdot \bar{s}_{\overline{n}|i} = 1$$

$$D = \frac{1}{\bar{s}_{\overline{n}|i}}$$

- The total annual payment will be the interest payment and account payment:

$$\delta + \frac{1}{\bar{s}_{\overline{n}|i}}$$

- Note that

$$\begin{aligned}
\frac{1}{\bar{a}_{\overline{n}|i}} &= \frac{\delta}{1-v^n} \times \frac{(1+i)^n}{(1+i)^n} = \frac{\delta(1+i)^n}{(1+i)^n - 1} \\
&= \frac{\delta(1+i)^n + \delta - \delta}{(1+i)^n - 1} = \frac{\delta[(1+i)^n - 1] + \delta}{(1+i)^n - 1} \\
&= \delta + \frac{\delta}{(1+i)^n - 1} = \delta + \frac{1}{\bar{s}_{\overline{n}|}}
\end{aligned}$$

- Therefore, a level continuous annual annuity payment on a loan is the same as making an annual continuous interest payment each year plus making level annual continuous deposits in order to save for the loan repayment.

Basic Relationship 4 : $\bar{a}_{\overline{n}|} = \frac{i}{\delta} \cdot a_{\overline{n}|}$, $\bar{s}_{\overline{n}|} = \frac{i}{\delta} \cdot s_{\overline{n}|}$

- Consider annual payments of 1 made continuously each year for the next n years. Over a one-year period, the continuous payments will accumulate at the end of the year to a lump sum of $\bar{s}_{\overline{1}|}$. If this end-of-year lump sum exists for each year of the n-year annuity-immediate, then the present value (at $t = 0$) of these end-of-year lump sums is the same as $\bar{a}_{\overline{n}|}$:

$$\begin{aligned}
\bar{a}_{\overline{n}|} &= \bar{s}_{\overline{1}|} \cdot a_{\overline{n}|} \\
&= \frac{i}{\delta} \cdot a_{\overline{n}|}
\end{aligned}$$

- Therefore, the accumulated value (at $t = n$) of these end-of-year lump sums is the same as $\bar{s}_{\overline{n}|}$:

$$\begin{aligned}
\bar{s}_{\overline{n}|} &= \bar{s}_{\overline{1}|} \cdot s_{\overline{n}|} \\
&= \frac{i}{\delta} \cdot s_{\overline{n}|}
\end{aligned}$$

Basic Relationship 5 : $\frac{d}{dt}\bar{s}_{\overline{t|}} = 1 + \delta \cdot \bar{s}_{\overline{t|}}$, $\frac{d}{dt}\bar{a}_{\overline{t|}} = 1 - \delta \cdot \bar{a}_{\overline{t|}}$

- First off, consider how the accumulated value (as at time t) of an annuity-immediate changes from one payment period to the next:

$$\begin{aligned} s_{\overline{t+1|}} =& 1 + s_{\overline{t|}} \cdot (1+i) \\ =& 1 + s_{\overline{t|}} + i \cdot s_{\overline{t|}} \\ s_{\overline{t+1|}} - s_{\overline{t|}} =& 1 + i \cdot s_{\overline{t|}} \\ \Delta s_{\overline{t|}} =& 1 + i \cdot s_{\overline{t|}} \end{aligned}$$

- Therefore, the annual change in the accumulated value at time t will simply be the interest earned over the year, plus the end-of-year payment that was made.
- For an annuity-due, the annual change in the accumulated value will be:

$$\begin{aligned} \ddot{s}_{\overline{t+1|}} =& 1 \cdot (1+i) + \ddot{s}_{\overline{t|}} \cdot (1+i) \\ =& 1 \cdot (1+i) + \ddot{s}_{\overline{t|}} + i \cdot \ddot{s}_{\overline{t|}} \\ \ddot{s}_{\overline{t+1|}} - \ddot{s}_{\overline{t|}} =& 1 \cdot (1+i) + i \cdot \ddot{s}_{\overline{t|}} \\ \Delta \ddot{s}_{\overline{t|}} =& 1 \cdot (1+i) + i \cdot \ddot{s}_{\overline{t|}} \end{aligned}$$

- Note that, in general, a continuous annuity can be expressed as:

$$\bar{s}_{\overline{t+h|}} = 1 \cdot \int_0^h (1+i)^{h-t} dt + \bar{s}_{\overline{t|}} \cdot (1+i)^h$$

- The change in the accumulated value over a period of h is then:

$$\bar{s}_{\overline{t+h|}} - \bar{s}_{\overline{t|}} = 1 \cdot \int_0^h (1+i)^{h-t} dt + \bar{s}_{\overline{t|}} \cdot [(1+i)^h - 1]$$

- The derivative with respect to time of the accumulated value can be defined as:

$$\begin{aligned} \frac{d}{dt}\bar{s}_{\overline{t|}} &= \lim_{h \to 0} \frac{\bar{s}_{\overline{t+h|}} - \bar{s}_{\overline{t|}}}{h} \\ &= \lim_{h \to 0} \frac{1 \cdot \int_0^h (1+i)^{h-t} dt}{h} + \lim_{h \to 0} \frac{\bar{s}_{\overline{t|}} \cdot [(1+i)^h - 1]}{h} \\ &= \lim_{h \to 0} \frac{1 \cdot \frac{d}{dh} \int_0^h (1+i)^{h-t} dt}{\frac{d}{dh} \cdot h} + \lim_{h \to 0} \frac{[(1+i)^{t+h} - (1+i)^t]}{h \cdot (1+i)^t} \cdot \bar{s}_{\overline{t|}} \\ &= 1 + \delta \cdot \bar{s}_{\overline{t|}} \end{aligned}$$

- The derivative with respect to time of the present value can be defined as:

$$\begin{aligned} \frac{d}{dt}\bar{a}_{\overline{t|}} =& \frac{d}{dt}\left(\bar{s}_{\overline{t|}} \cdot v^t\right) \\ =& \left(\frac{d}{dt}\bar{s}_{\overline{t|}}\right) \cdot v^t + \bar{s}_{\overline{t|}} \cdot \left(\frac{d}{dt} v^t\right) \\ =& (1 + \delta \cdot \bar{s}_{\overline{t|}}) \cdot v^t + \bar{s}_{\overline{t|}} \cdot \left(v^t \cdot ln[v]\right) \\ =& \left(v^t + \delta \cdot \bar{a}_{\overline{t|}}\right) + \bar{a}_{\overline{t|}} \cdot (-\delta) \\ =& 1 - \delta \cdot \bar{a}_{\overline{t|}} \end{aligned}$$

4.6 Basic Varying Annuities

- in this section, payments will now vary; but the interest conversion period will continue to coincide with the payment frequency
- 3 types of varying annuities are detailed in this section:

  (i) payments varying in arithmetic progression

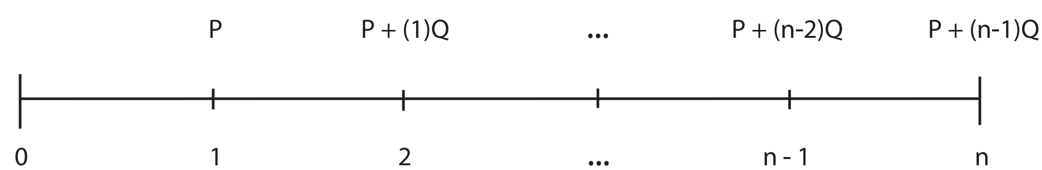
  (ii) payments varying in geometric progression

  (iii) other payment patterns

Payments Varying In Arithmetic Progression

Annuity-Immediate

An annuity-immediate is payable over n years with the first payment equal to P and each subsequent payment increasing by Q. The time line diagram below illustrates the above scenario:



The present value (at $t = 0$) of this annual annuity–immediate, where the annual effective rate of interest is i, shall be calculated as follows:

$$\begin{aligned}
PV_0 &= [P]v + [P+Q]v^2 + \cdots + [P+(n-2)Q]v^{n-1} + [P+(n-1)Q]v^n \\
&= P[v + v^2 + \cdots + v^{n-1} + v^n] + Q[v^2 + 2v^3 \cdots + (n-2)v^{n-1} + (n-1)v^n] \\
&= P[v + v^2 + \cdots + v^{n-1} + v^n] + Qv^2[1 + 2v \cdots + (n-2)v^{n-3} + (n-1)v^{n-2}] \\
&= P[v + v^2 + \cdots + v^{n-1} + v^n] + Qv^2\frac{d}{dv}[1 + v + v^2 + \cdots + v^{n-2} + v^{n-1}] \\
&= P \cdot a_{\overline{n}|i} + Qv^2\frac{d}{dv}\left[\ddot{a}_{\overline{n}|i}\right] \\
&= P \cdot a_{\overline{n}|i} + Qv^2\frac{d}{dv}\left[\frac{1-v^n}{1-v}\right] \\
&= P \cdot a_{\overline{n}|i} + Qv^2\left[\frac{(1-v)\cdot(-nv^{n-1}) - (1-v^n)\cdot(-1)}{(1-v)^2}\right] \\
&= P \cdot a_{\overline{n}|i} + \frac{Q}{(1+i)^2}\left[\frac{-nv^n(v^{-1}-1) + (1-v^n)}{(i/1+i)^2}\right] \\
&= P \cdot a_{\overline{n}|i} + Q\left[\frac{(1-v^n) - nv^{n-1} - nv^n}{i^2}\right] \\
&= P \cdot a_{\overline{n}|i} + Q\left[\frac{(1-v^n) - nv^n(v^{-1}-1)}{i^2}\right] \\
&= P \cdot a_{\overline{n}|i} + Q\left[\frac{(1-v^n) - nv^n(1+i-1)}{i^2}\right] \\
&= P \cdot a_{\overline{n}|i} + Q\left[\frac{\frac{(1-v^n)}{i} - \frac{nv^n(i)}{i}}{i}\right] \\
&= P \cdot a_{\overline{n}|i} + Q\left[\frac{a_{\overline{n}|i} - nv^n}{i}\right]
\end{aligned}$$

The accumulated value (at $t = n$) of an annuity–immediate, where the annual effective rate of interest is i, can be calculated using the same approach as above or calculated by using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$\begin{aligned}
FV_n &= PV_0 \cdot (1+i)^n \\
&= \left(P \cdot a_{\overline{n}|i} + Q\left[\frac{a_{\overline{n}|i} - nv^n}{i}\right]\right)(1+i)^n \\
&= P \cdot a_{\overline{n}|i} \cdot (1+i)^n + Q\left[\frac{a_{\overline{n}|i} \cdot (1+i)^n - nv^n \cdot (1+i)^n}{i}\right] \\
&= P \cdot s_{\overline{n}|i} + Q\left[\frac{s_{\overline{n}|i} - n}{i}\right]
\end{aligned}$$

Let $P = 1$ and $Q = 1$. In this case, the payments start at 1 and increase by 1 every year until the final payment of n is made at time n.

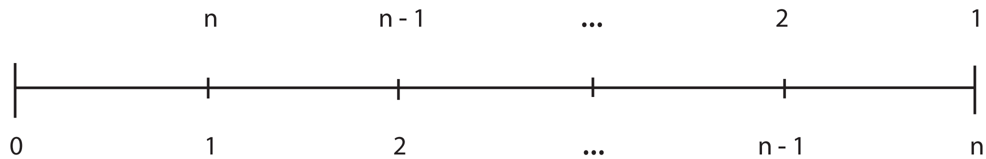
The present value (at $t = 0$) of this annual increasing annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $(Ia)_{\overline{n|}i}$ and is calculated as follows:

$$
\begin{aligned}
(Ia)_{\overline{n|}i} &= (1) \cdot a_{\overline{n|}i} + (1) \cdot \left[\frac{a_{\overline{n|}i} - nv^n}{i} \right] \\
&= \frac{1 - v^n}{i} + \frac{a_{\overline{n|}i} - nv^n}{i} \\
&= \frac{1 - v^n + a_{\overline{n|}i} - nv^n}{i} \\
&= \frac{\ddot{a}_{\overline{n|}i} - nv^n}{i}
\end{aligned}
$$

The accumulated value (at $t = n$) of this annual increasing annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $(Is)_{\overline{n|}i}$ and can be calculated using the same general approach as above, or alternatively, by simply using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$
\begin{aligned}
(Is)_{\overline{n|}i} &= (Ia)_{\overline{n|}i} \cdot (1+i)^n \\
&= \left(\frac{\ddot{a}_{\overline{n|}i} - nv^n}{i} \right) \cdot (1+i)^n \\
&= \frac{\ddot{a}_{\overline{n|}i} \cdot (1+i)^n - nv^n \cdot (1+i)^n}{i} \\
&= \frac{\ddot{s}_{\overline{n|}i} - n}{i}
\end{aligned}
$$

Let $P = n$ and $Q = -1$. In this case, the payments start at n and decrease by 1 every year until the final payment of 1 is made at time n.



The present value (at $t = 0$) of this annual decreasing annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $(Da)_{\overline{n}|i}$ and is calculated as follows:

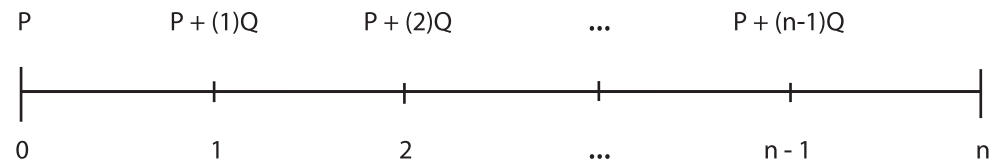
$$\begin{aligned}(Da)_{\overline{n}|i} &= (n) \cdot a_{\overline{n}|i} + (-1) \cdot \left[\frac{a_{\overline{n}|i} - nv^n}{i}\right] \\ &= n \cdot \frac{1 - v^n}{i} - \frac{a_{\overline{n}|i} - nv^n}{i} \\ &= \frac{n - nv^n - a_{\overline{n}|i} + nv^n}{i} \\ &= \frac{n - a_{\overline{n}|i}}{i}\end{aligned}$$

The accumulated value (at $t = n$) of this annual decreasing annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $(Ds)_{\overline{n}|i}$ and can be calculated by using the same general approach as above, or alternatively, by simply using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$\begin{aligned}(Ds)_{\overline{n}|i} &= (Da)_{\overline{n}|i} \cdot (1+i)^n \\ &= \left(\frac{n - a_{\overline{n}|i}}{i}\right) \cdot (1+i)^n \\ &= \frac{n \cdot (1+i)^n - s_{\overline{n}|i}}{i}\end{aligned}$$

Annuity-Due

An annuity-due is payable over n years with the first payment equal to P and each subsequent payment increasing by Q. The time line diagram below illustrates the above scenario:



The present value (at $t = 0$) of this annual annuity–due, where the annual effective rate of interest is i, shall be calculated as follows:

$$\begin{aligned}
PV_0 &= [P] + [P+Q]v + \cdots + [P+(n-2)Q]v^{n-2} + [P+(n-1)Q]v^{n-1} \\
&= P[1 + v + \cdots + v^{n-2} + v^{n-1}] + Q[v + 2v^2 \cdots + (n-2)v^{n-2} + (n-1)v^{n-1}] \\
&= P[1 + v + \cdots + v^{n-2} + v^{n-1}] + Qv[1 + 2v \cdots + (n-2)v^{n-3} + (n-1)v^{n-2}] \\
&= P[1 + v + \cdots + v^{n-2} + v^{n-1}] + Qv\frac{d}{dv}[1 + v + v^2 + \cdots + v^{n-2} + v^{n-1}] \\
&= P \cdot \ddot{a}_{\overline{n}|i} + Qv\frac{d}{dv}\left[\ddot{a}_{\overline{n}|i}\right] \\
&= P \cdot \ddot{a}_{\overline{n}|i} + Qv\frac{d}{dv}\left[\frac{1-v^n}{1-v}\right] \\
&= P \cdot \ddot{a}_{\overline{n}|i} + Qv\left[\frac{(1-v)\cdot(-nv^{n-1}) - (1-v^n)\cdot(-1)}{(1-v)^2}\right] \\
&= P \cdot \ddot{a}_{\overline{n}|i} + \frac{Q}{(1+i)}\left[\frac{-nv^n(v^{-1}-1) + (1-v^n)}{(i/1+i)^2}\right] \\
&= P \cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{(1-v^n) - nv^{n-1} - nv^n}{i^2}\right]\cdot(1+i) \\
&= P \cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{(1-v^n) - nv^n(v^{-1}-1)}{i^2}\right]\cdot(1+i) \\
&= P \cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{(1-v^n) - nv^n(1+i-1)}{i^2}\right]\cdot(1+i) \\
&= P \cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{\frac{(1-v^n)}{i} - \frac{nv^n(i)}{i}}{i}\right]\cdot(1+i) \\
&= P \cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{a_{\overline{n}|i} - nv^n}{i}\right]\cdot(1+i) \\
&= P \cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{a_{\overline{n}|i} - nv^n}{d}\right]
\end{aligned}$$

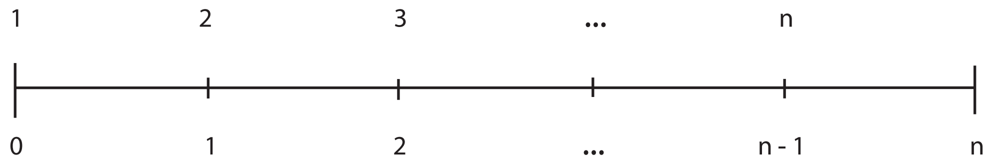
This present value of the annual annuity-due could also have been calculated using the basic principle that since payments under an annuity-due start one year earlier than under an annuity-immediate, the annuity–due will earn one more year of interest and thus, will be greater than an annuity-immediate by $(1 + i)$:

$$\begin{aligned}
PV_0^{due} &= PV_0^{immediate} \times (1+i) \\
&= \left(P \cdot a_{\overline{n}|i} + Q\left[\frac{a_{\overline{n}|i} - nv^n}{i}\right]\right) \times (1+i) \\
&= (P \cdot a_{\overline{n}|i}) \times (1+i) + Q\left[\frac{a_{\overline{n}|i} - nv^n}{i}\right] \times (1+i) \\
&= P \cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{a_{\overline{n}|i} - nv^n}{d}\right]
\end{aligned}$$

The accumulated value (at $t = n$) of an annuity–due, where the annual effective rate of interest is i, can be calculated using the same general approach as above, or alternatively, calculated by using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$\begin{aligned} FV_n &= PV_0 \cdot (1+i)^n \\ &= \left(P \cdot \ddot{a}_{\overline{n}|i} + Q \left[\frac{a_{\overline{n}|i} - nv^n}{d} \right] \right) (1+i)^n \\ &= P \cdot \ddot{a}_{\overline{n}|i} \cdot (1+i)^n + Q \left[\frac{a_{\overline{n}|i} \cdot (1+i)^n - nv^n \cdot (1+i)^n}{d} \right] \\ &= P \cdot \ddot{s}_{\overline{n}|i} + Q \left[\frac{s_{\overline{n}|i} - n}{d} \right] \end{aligned}$$

Let $P = 1$ and $Q = 1$. In this case, the payments start at 1 and increase by 1 every year until the final payment of n is made at time $n - 1$.



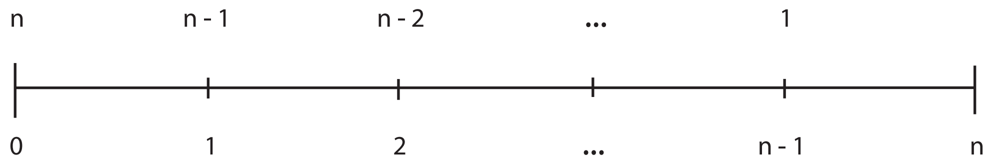
The present value (at $t = 0$) of this annual increasing annuity–due, where the annual effective rate of interest is i, shall be denoted as $(I\ddot{a})_{\overline{n|}i}$ and is calculated as follows:

$$\begin{aligned}
(I\ddot{a})_{\overline{n|}i} &= (1) \cdot \ddot{a}_{\overline{n|}i} + (1) \cdot \left[\frac{a_{\overline{n|}i} - nv^n}{d}\right] \\
&= \frac{1 - v^n}{d} + \frac{a_{\overline{n|}i} - nv^n}{d} \\
&= \frac{1 - v^n + a_{\overline{n|}i} - nv^n}{d} \\
&= \frac{\ddot{a}_{\overline{n|}i} - nv^n}{d}
\end{aligned}$$

The accumulated value (at $t = n$) of this annual increasing annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $(I\ddot{s})_{\overline{n|}i}$ and can be calculated using the same approach as above or by simply using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$\begin{aligned}
(I\ddot{s})_{\overline{n|}i} &= (I\ddot{a})_{\overline{n|}i} \cdot (1+i)^n \\
&= \left(\frac{\ddot{a}_{\overline{n|}i} - nv^n}{d}\right) \cdot (1+i)^n \\
&= \frac{\ddot{a}_{\overline{n|}i} \cdot (1+i)^n - nv^n \cdot (1+i)^n}{d} \\
&= \frac{\ddot{s}_{\overline{n|}i} - n}{d}
\end{aligned}$$

Let $P = n$ and $Q = -1$. In this case, the payments start at n and decrease by 1 every year until the final payment of 1 is made at time n.



The present value (at $t = 0$) of this annual decreasing annuity–due, where the annual effective rate of interest is i, shall be denoted as $(D\ddot{a})_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}
(D\ddot{a})_{\overline{n}|i} &= (n)\cdot \ddot{a}_{\overline{n}|i} + (-1)\cdot\left[\frac{a_{\overline{n}|i} - nv^n}{d}\right] \\
&= n\cdot\frac{1-v^n}{d} - \frac{a_{\overline{n}|i} - nv^n}{d} \\
&= \frac{n - nv^n - a_{\overline{n}|i} + nv^n}{d} \\
&= \frac{n - a_{\overline{n}|i}}{d}
\end{aligned}$$

The accumulated value (at $t = n$) of this annual decreasing annuity–due, where the annual effective rate of interest is i, shall be denoted as $(D\ddot{s})_{\overline{n}|i}$ and can be calculated using the same approach as above or by simply using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$\begin{aligned}
(D\ddot{s})_{\overline{n}|i} &= (\ddot{D}a)_{\overline{n}|i}\cdot(1+i)^n \\
&= \left(\frac{n - a_{\overline{n}|i}}{d}\right)\cdot(1+i)^n \\
&= \frac{n\cdot(1+i)^n - s_{\overline{n}|i}}{d}
\end{aligned}$$

Basic Relationship $1 : \ddot{a}_{\overline{n}|} = i\cdot(Ia)_{\overline{n}|} + nv^n$
Consider an n–year investment where 1 is invested at the beginning of each year. The present value of this multiple payment income stream at $t = 0$ is $\ddot{a}_{\overline{n}|}$.

Alternatively, consider a n–year investment where 1 is invested at the beginning of each year and produces increasing annual interest payments progressing to $n\cdot i$ by the end of the last year with the total payments $(n\times 1)$ refunded at $t = n$.

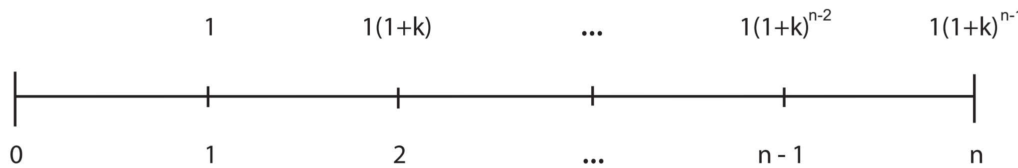
The present value of this multiple payment income stream at $t = 0$ is $i\cdot(Ia)_{\overline{n}|} + nv^n$.

Note that $(Ia)_{\overline{n}|} = \dfrac{\ddot{a}_{\overline{n}|} - n\cdot v^n}{i} \rightarrow \ddot{a}_{\overline{n}|} = i\cdot(Ia)_{\overline{n}|} + nv^n$. Therefore, the present value of both investment opportunities are equal.

Payments Varying In Geometric Progression

<u>Annuity-Immediate</u>

An annuity-immediate is payable over n years with the first payment equal to 1 and each subsequent payment increasing by $(1+k)$. The time line diagram below illustrates the above scenario:



The present value (at $t = 0$) of this annual geometrically increasing annuity–immediate, where the annual effective rate of interest is i, shall be calculated as follows:

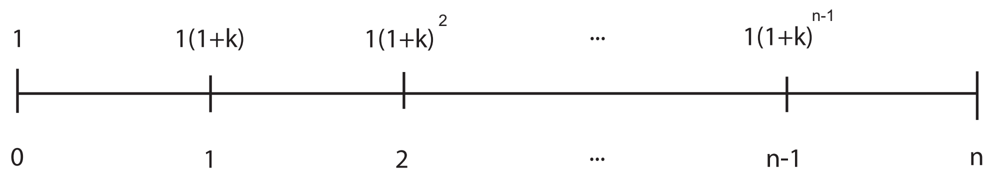
$$\begin{aligned}
PV_0 &= (1)v_i + (1+k)v_i^2 + \cdots + (1+k)^{n-2}v_i^{n-1} + (1+k)^{n-1}v_i^n \\
&= v_i[1 + (1+k)v_i + \cdots + (1+k)^{n-2}v_i^{n-2} + (1+k)^{n-1}v_i^{n-1}] \\
&= \left(\frac{1}{1+i}\right)\left[1 + \left(\frac{1+k}{1+i}\right) + \cdots + \left(\frac{1+k}{1+i}\right)^{n-2} + \left(\frac{1+k}{1+i}\right)^{n-1}\right] \\
&= \left(\frac{1}{1+i}\right)\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{1-\left(\frac{1+k}{1+i}\right)}\right] \\
&= \left(\frac{1}{1+i}\right)\left[\frac{1-v^n_{j=\frac{1+i}{1+k}-1}}{1-v_{j=\frac{1+i}{1+k}-1}}\right] \\
&= \left(\frac{1}{1+i}\right)\cdot \ddot{a}_{\overline{n}|j=\frac{1+i}{1+k}-1}
\end{aligned}$$

The accumulated value (at $t = n$) of an annual geometric increasing annuity-immediate, where the annual effective rate of interest is i, can be calculated using the same approach as above or calculated by using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$\begin{aligned}
FV_n &= PV_0 \cdot (1+i)^n \\
&= \left(\frac{1}{1+i}\right)\cdot \ddot{a}_{\overline{n}|j=\frac{1+i}{1+k}-1}(1+i)^n \\
&= \left(\frac{1}{1+i}\right)\cdot \ddot{s}_{\overline{n}|j=\frac{1+i}{1+k}-1}
\end{aligned}$$

Annuity-Due

An annuity-due is payable over n years with the first payment equal to 1 and each subsequent payment increasing by $(1+k)$. The time line diagram below illustrates the above scenario:



The present value (at $t=0$) of this annual geometrically increasing annuity–due, where the annual effective rate of interest is i, shall be calculated as follows:

$$\begin{aligned}
PV_0 &= (1) + (1+k)v_i + \cdots + (1+k)^{n-2}v_i^{n-2} + (1+k)^{n-1}v_i^{n-1} \\
&= 1 + \left(\frac{1+k}{1+i}\right) + \cdots + \left(\frac{1+k}{1+i}\right)^{n-2} + \left(\frac{1+k}{1+i}\right)^{n-1} \\
&= \frac{1-\left(\frac{1+k}{1+i}\right)^n}{1-\left(\frac{1+k}{1+i}\right)} \\
&= \frac{1-v^n_{j=\frac{1+i}{1+k}-1}}{1-v_{j=\frac{1+i}{1+k}-1}} \\
&= \ddot{a}_{\overline{n}|_{j=\frac{1+i}{1+k}-1}}
\end{aligned}$$

This present value could also have been achieved by simply multiplying the annuity-immediate version by $(1+i)$; $\left(\frac{1}{1+i}\right)\cdot \ddot{a}_{\overline{n}|_{j=\frac{1+i}{1+k}-1}}\cdot(1+i)$.

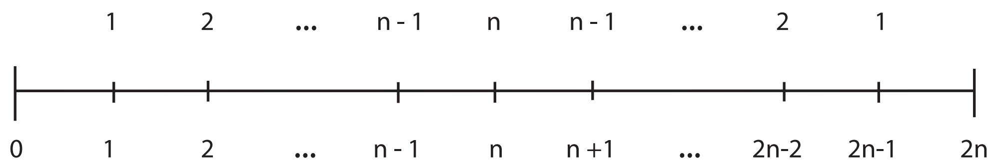
The accumulated value (at $t=n$) of an annual geometric increasing annuity-due, where the annual effective rate of interest is i, can be calculated using the same approach as above or calculated by using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$\begin{aligned}
FV_n &= PV_0\cdot(1+i)^n \\
&= \ddot{a}_{\overline{n}|_{j=\frac{1+i}{1+k}-1}}(1+i)^n \\
&= \ddot{s}_{\overline{n}|_{j=\frac{1+i}{1+k}-1}}
\end{aligned}$$

Other Payment Patterns

If the pattern is unrecognizable or cannot be manipulated into a combination of recognizable patterns, then each individual payment will to be discounted or accumulated accordingly.

<u>Example</u>
The present value of an annuity-immediate where the first payment is 1 and increases by 1 until it reaches n at time n and then decreases by 1 thereafter, is often referred to as a "pyramid" annuity-immediate since the payments increase by 1, peak at n and decrease by 1. This pattern of payments is as follows:

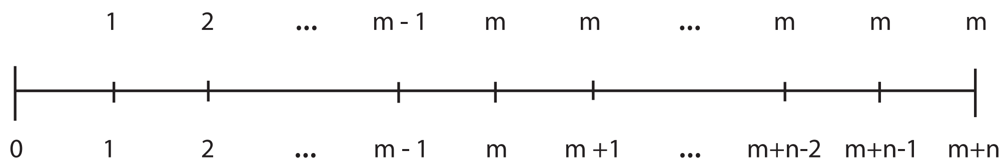


The pattern of payments can be broken down into 2 familiar patterns; namely, an n-year increasing annuity-immediate, followed by an $(n-1)$-year decreasing annuity immediate. The second pattern will need to be discounted back to time 0.
The present value of this pattern is:

$$
\begin{aligned}
PV_0 &= (Ia)_{\overline{n}|} + v^n (Da)_{\overline{n-1}|} \\
&= \frac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{i} + v^n \cdot \frac{(n-1) - a_{\overline{n-1}|}}{i} \\
&= \frac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{i} + \frac{v^n \cdot (n-1) - v^n \cdot (\ddot{a}_{\overline{n}|} - 1)}{i} \\
&= \frac{\ddot{a}_{\overline{n}|} - nv^n + nv^n - v^n - v^n \cdot \ddot{a}_{\overline{n}|} + v^n}{i} \\
&= \frac{\ddot{a}_{\overline{n}|}(1 - v^n)}{i} \\
&= \ddot{a}_{\overline{n}|} \cdot a_{\overline{n}|}
\end{aligned}
$$

Example

The present value of an annuity-immediate where the first payment is 1 and increases by 1 until it reaches m at time m and then remains at m for another n years where $0 < m < n$, is denoted as $(I_{\overline{m}|}a)_{\overline{n}|}$. This pattern of payments is as follows:



The pattern of payments can be broken down into 3 sets of 2 familiar patterns;

(i) an m-year increasing annuity-immediate, followed by an n-year level annuity-immediate. The present value is then:

$$(I_{\overline{m}|}a)_{\overline{n}|} = (Ia)_{\overline{m}|} + v^m \cdot m \times a_{\overline{n}|}.$$

(ii) an $n+m$-year increasing annuity-immediate; but this is overstating the payments after time m, so a reduction for the next n years is required. The present value is then:

$$(I_{\overline{m}|}a)_{\overline{n}|} = (Ia)_{\overline{m+n}|} - v^m \cdot (Ia)_{\overline{n}|}.$$

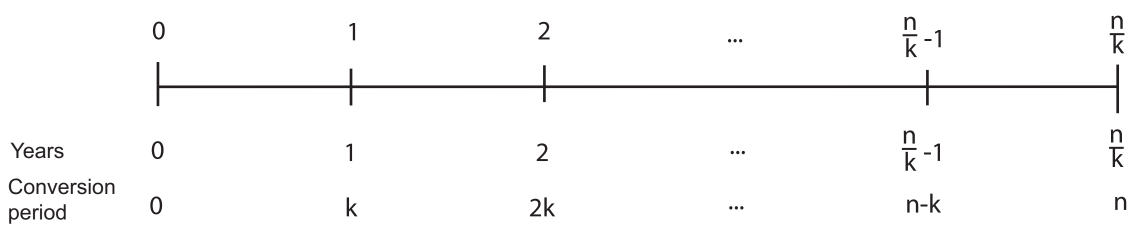
(iii) an $n+m$-year level annuity-immediate with payments of m; but this is overstating the payments for the first $m-1$ years, so a reduction for the next $m-1$ years is required. The present value is then:

$$(I_{\overline{m}|}a)_{\overline{n}|} = m \times a_{\overline{m+n}|} - (Da)_{\overline{m-1}|}.$$

4.7 More General Varying Annuities

Analysis of Annuities Payable $\boxed{\text{Less}}$ Frequency Than Interest Is Convertible

- let $i^{(k)}$ be a nominal rate of interest convertible k times a year and let there be increasing end-of-year payments starting with 1 and increasing by 1.
- after the first payment has been made, interest has been converted k times
- after the second payment has been made, interest has been converted $2k$ times
- after the last payment has been made, interest has been converted n times
- therefore, the term of the annuity (and obviously, the number of payments) will be $\frac{n}{k}$ years (i.e. if $n = 144$ and $i^{(12)}$ is used, then the term of the annuity is $\frac{144}{12} = 12$ years).
- The time line diagram will detail the above scenario:



– the present value (at $t = 0$) of an annual increasing annuity-immediate where increasing payments are being made every k conversion periods and where the rate of interest is $j = \dfrac{i^{(k)}}{k}$ shall be calculated as follows:

$$\begin{aligned}
PV_0 &= (1)v_i^1 + (2)v_i^2 + \cdots + \left(\frac{n}{k} - 1\right) v_i^{\frac{n}{k}-1} + \left(\frac{n}{k}\right) v_i^{\frac{n}{k}} \\
&= (1)v_j^k + (2)v_j^{2k} + \cdots + \left(\frac{n}{k} - 1\right) v_j^{n-k} + \left(\frac{n}{k}\right) v_j^n \\
(1+j)^k \cdot PV_0 &= (1+j)^k \cdot \left[(1)v_j^k + (2)v_j^{2k} + \cdots + \left(\frac{n}{k} - 1\right) v_j^{n-k} + \left(\frac{n}{k}\right) v_j^n\right] \\
&= (1) + (2)v_j^k + \cdots + \left(\frac{n}{k} - 1\right) v_j^{n-2k} + \left(\frac{n}{k}\right) v_j^{n-k} \\
(1+j)^k \cdot PV_0 - PV_0 &= 1 + (2-1)v_j^k + (3-2)v_j^{2k} + \cdots + \left(\frac{n}{k} - \left(\frac{n}{k} - 1\right)\right) v_j^{n-k} - \left(\frac{n}{k}\right) v_j^n \\
[(1+j)^k - 1] \cdot PV_0 &= 1 + (1)v_j^k + (1)v_j^{2k} + \cdots + (1)v_j^{n-k} - \left(\frac{n}{k}\right) v_j^n \\
PV_0 &= \frac{1 + (1)v_j^k + (1)v_j^{2k} + \cdots + (1)v_j^{n-k} - \left(\frac{n}{k}\right) v_j^n}{[(1+j)^k - 1]} \\
&= \frac{\frac{a_{\overline{n}|j}}{a_{\overline{k}|j}} - \left(\frac{n}{k}\right) v_j^n}{j \cdot s_{\overline{k}|j}}
\end{aligned}$$

– There is an alternative approach in determining the present value of this annuity–immediate:

The increasing payments made at the end of each year can represent the accumulated value of smaller level end-of-conversion-period payments that are made k times during the year.

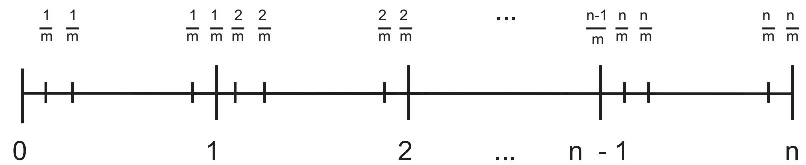
$$\begin{aligned}
P \cdot s_{\overline{k}|j} &= 1 \\
2P \cdot s_{\overline{k}|j} &= 2 \\
&\vdots \\
\left(\frac{n}{k}\right) P \cdot s_{\overline{k}|j} &= \frac{n}{k}
\end{aligned}$$

These smaller level payments are therefore equal to $\dfrac{1}{s_{\overline{k}|j}}$ for the first year, $\dfrac{2}{s_{\overline{k}|j}}$ for the second year, $\dfrac{2}{s_{\overline{k}|j}}$ for the third year and up to $\dfrac{\frac{n}{k}}{s_{\overline{k}|j}}$ for the last year. If these smaller payments were to be made at the end of every conversion period, during the term of the annuity, and there are n conversion periods in total, then the present value (at $t = 0$) of these n smaller payments is determined to be:

$$
\begin{aligned}
PV_0 = {} & \left(\frac{1}{s_{\overline{k}|j}}\right) v_i^{\frac{1}{k}} + \left(\frac{1}{s_{\overline{k}|j}}\right) v_i^{\frac{2}{k}} + \cdots + \left(\frac{1}{s_{\overline{k}|j}}\right) v_i^{\frac{k-1}{k}} + \left(\frac{1}{s_{\overline{k}|j}}\right) v_i^{\frac{k}{k}} && (1st \text{ year}) \\
& + \left(\frac{2}{s_{\overline{k}|j}}\right) v_i^{\frac{k+1}{k}} + \left(\frac{2}{s_{\overline{k}|j}}\right) v_i^{\frac{k+2}{k}} + \cdots + \left(\frac{2}{s_{\overline{k}|j}}\right) v_i^{\frac{2k-1}{k}} + \left(\frac{2}{s_{\overline{k}|j}}\right) v_i^{\frac{2k}{k}} && (2nd \text{ year}) \\
& \vdots \\
& + \left(\frac{n/k}{s_{\overline{k}|j}}\right) v_i^{\frac{(\frac{n}{k}-1)k+1}{k}} + \left(\frac{n/k}{s_{\overline{k}|j}}\right) v_i^{\frac{(\frac{n}{k}-1)k+2}{k}} + \cdots + \left(\frac{n/k}{s_{\overline{k}|j}}\right) v_i^{\frac{\frac{n}{k}k-1}{k}} + \left(\frac{n/k}{s_{\overline{k}|j}}\right) v_i^{\frac{\frac{n}{k}k}{k}} && (\frac{n}{k}th \text{ year}) \\
\\
= {} & \left(\frac{1}{s_{\overline{k}|j}}\right) v_j + \left(\frac{1}{s_{\overline{k}|j}}\right) v_j^2 + \cdots + \left(\frac{1}{s_{\overline{k}|j}}\right) v_j^{k-1} + \left(\frac{1}{s_{\overline{k}|j}}\right) v_j^k && (1st \text{ year}) \\
& + \left(\frac{2}{s_{\overline{k}|j}}\right) v_j^{k+1} + \left(\frac{2}{s_{\overline{k}|j}}\right) v_j^{k+2} + \cdots + \left(\frac{2}{s_{\overline{k}|j}}\right) v_j^{2k-1} + \left(\frac{2}{s_{\overline{k}|j}}\right) v_j^{2k} && (2nd \text{ year}) \\
& \vdots \\
& + \left(\frac{n/k}{s_{\overline{k}|j}}\right) v_j^{n-k+1} + \left(\frac{n/k}{s_{\overline{k}|j}}\right) v_j^{n-k+2} + \cdots + \left(\frac{n/k}{s_{\overline{k}|j}}\right) v_j^{n-1} + \left(\frac{n/k}{s_{\overline{k}|j}}\right) v_j^{n} && (\frac{n}{k}th \text{ year}) \\
\\
= {} & \left(\frac{1}{s_{\overline{k}|j}}\right) v_j + \left(\frac{2}{s_{\overline{k}|j}}\right) v_j^{k+1} + \cdots + \left(\frac{n/k}{s_{\overline{k}|j}}\right) v_j^{n-k+1} \\
& + \left(\frac{1}{s_{\overline{k}|j}}\right) v_j^2 + \left(\frac{2}{s_{\overline{k}|j}}\right) v_j^{k+2} + \cdots + \left(\frac{n/k}{s_{\overline{k}|j}}\right) v_j^{n-k+2} \\
& \vdots \\
& + \left(\frac{1}{s_{\overline{k}|j}}\right) v_j^{k-1} + \left(\frac{2}{s_{\overline{k}|j}}\right) v_j^{2k-1} + \cdots + \left(\frac{n/k}{s_{\overline{k}|j}}\right) v_j^{n-1} \\
& + \left(\frac{1}{s_{\overline{k}|j}}\right) v_j^{k} + \left(\frac{2}{s_{\overline{k}|j}}\right) v_j^{2k} + \cdots + \left(\frac{n/k}{s_{\overline{k}|j}}\right) v_j^{n} \\
\\
= {} & (v_j + v_j^2 + \cdots + v_j^k) \left[\left(\frac{1}{s_{\overline{k}|j}}\right) + \left(\frac{2}{s_{\overline{k}|j}}\right) v_j^k + \cdots + \left(\frac{n/k}{s_{\overline{k}|j}}\right) v_j^{n-k} \right] \\
= {} & \left(\frac{a_{\overline{k}|j}}{s_{\overline{k}|j}}\right) \left[\left((I\ddot{a})_{\overline{\frac{n}{k}}|i} \right) \right] = \left(\frac{a_{\overline{k}|j}}{s_{\overline{k}|j}}\right) \left[\frac{\ddot{a}_{\overline{\frac{n}{k}}|i} - \left(\frac{n}{k}\right) v_i^{\frac{n}{k}}}{d} \right] = \left(\frac{a_{\overline{k}|j}}{s_{\overline{k}|j}}\right) \left[\frac{\frac{a_{\overline{n}|j}}{a_{\overline{k}|j}} - \left(\frac{n}{k}\right) v_j^n}{i} \right] (1+i) \\
= {} & \left(\frac{a_{\overline{k}|j} \cdot (1+j)^k}{s_{\overline{k}|j}}\right) \left[\frac{\frac{a_{\overline{n}|j}}{a_{\overline{k}|j}} - \left(\frac{n}{k}\right) \cdot v_j^n}{(1+j)^k - 1} \right] = \frac{\frac{a_{\overline{n}|j}}{a_{\overline{k}|j}} - \left(\frac{n}{k}\right) \cdot v_j^n}{j \cdot s_{\overline{k}|j}}
\end{aligned}
$$

Analysis of Annuities Payable More Frequency Than Interest Is Convertible

– payments of $\frac{1}{m}$ are made at the end of every $\frac{1}{m}$th of year for the first year, at $\frac{2}{m}$ for the second year and up to $\frac{n}{m}$ for the last year.



– the present value (at $t=0$) of an m^{th}ly increasing annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $(Ia)^{(m)}_{\overline{n}|i}$ and is calculated as follows:

$$
\begin{aligned}
(Ia)^{(m)}_{\overline{n}|i} =& (\frac{1}{m})v_i^{\frac{1}{m}} + (\frac{1}{m})v_i^{\frac{2}{m}} + \cdots + (\frac{1}{m})v_i^{\frac{m-1}{m}} + (\frac{1}{m})v_i^{\frac{m}{m}} && (1st \text{ year})\\
&+(\frac{2}{m})v_i^{\frac{m+1}{m}} + (\frac{2}{m})v_i^{\frac{m+2}{m}} + \cdots + (\frac{2}{m})v_i^{\frac{2m-1}{m}} + (\frac{2}{m})v_i^{\frac{2m}{m}} && (2nd \text{ year})\\
&\vdots\\
&+(\frac{n}{m})v_i^{\frac{(n-1)m+1}{m}} + (\frac{n}{m})v_i^{\frac{(n-1)m+2}{m}} + \cdots + (\frac{n}{m})v_i^{\frac{nm-1}{m}} + (\frac{n}{m})v_i^{\frac{nm}{m}} && (\text{last year})\\
\\
=&(\frac{1}{m})v_i^{\frac{1}{m}}\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right] && (1st \text{ year})\\
&+(\frac{2}{m})v_i^{\frac{m+1}{m}}\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right] && (2nd \text{ year})\\
&\vdots\\
&+(\frac{n}{m})v_i^{\frac{(n-1)m+1}{m}}\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right] && (\text{last year})\\
\\
=&\left((\frac{1}{m})v_i^{\frac{1}{m}} + (\frac{2}{m})v_i^{\frac{m+1}{m}} + \cdots + (\frac{n}{m})v_i^{\frac{(n-1)m+1}{m}})\right)\left[1 + v_i^{\frac{1}{m}} + \cdots + v_i^{\frac{m-2}{m}} + v_i^{\frac{m-1}{m}}\right]\\
=&(\frac{1}{m})v_i^{\frac{1}{m}}\left(1 + (2)v_i^{\frac{m}{m}} + \cdots + (n)v_i^{(n-1)}\right)\left[\frac{1-(v_i^{\frac{1}{m}})^m}{1-v_i^{\frac{1}{m}}}\right]\\
=&(\frac{1}{m})\frac{1}{(1+i)^{\frac{1}{m}}}\cdot(I\ddot{a})_{\overline{n}|i}\cdot\left[\frac{1-v_i^1}{1-v_i^{\frac{1}{m}}}\right]\\
=&(\frac{1}{m})\frac{1}{(1+i)^{\frac{1}{m}}}\left(\frac{\ddot{a}_{\overline{n}|i}-nv^n}{d}\right)\cdot\left[\frac{1-v_i^1}{1-v_i^{\frac{1}{m}}}\right]\\
=&(\frac{1}{m})\left(\frac{\ddot{a}_{\overline{n}|i}-nv^n}{1-v_i^1}\right)\cdot\left[\frac{1-v_i^1}{(1+i)^{\frac{1}{m}}-1}\right]\\
=&\frac{\ddot{a}_{\overline{n}|i}-nv^n}{m\left[(1+i)^{\frac{1}{m}}-1\right]}\\
=&\frac{\ddot{a}_{\overline{n}|i}-nv^n}{i^{(m)}} = \left(\frac{1}{m}\times m\right)\cdot\frac{\ddot{a}_{\overline{n}|i}-nv^n}{i^{(m)}}
\end{aligned}
$$

– payments of $\frac{1}{m^2}$ are made at the end of the first $\frac{1}{m}$th of the year and increase to $\frac{2}{m^2}$ at the second $\frac{1}{m}$th of the year, to $\frac{3}{m^2}$ at the end of the third $\frac{1}{m}$th of the year and eventually increase to $\frac{n \cdot m}{m^2}$ by the last conversion at the end of the last year.

– the present value (at $t = 0$) of this m^{th}ly increasing annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $(I^{(m)}a)^{(m)}_{\overline{n}|i}$ and is equal to:

$$
\begin{aligned}
(I^{(m)}a)^{(m)}_{\overline{n}|i} =& (\frac{1}{m^2})v_i^{\frac{1}{m}} + (\frac{2}{m^2})v_i^{\frac{2}{m}} + \cdots + (\frac{m-1}{m^2})v_i^{\frac{m-1}{m}} + (\frac{m}{m^2})v_i^{\frac{m}{m}} \\
&+ (\frac{m+1}{m^2})v_i^{\frac{m+1}{m}} + (\frac{m+2}{m^2})v_i^{\frac{m+2}{m}} + \cdots + (\frac{2m-1}{m^2})v_i^{\frac{2m-1}{m}} + (\frac{2m}{m^2})v_i^{\frac{2m}{m}} \\
&\vdots \\
&+ (\frac{(n-1)m+1}{m^2})v_i^{\frac{(n-1)m+1}{m}} + (\frac{(n-1)m+2}{m^2})v_i^{\frac{(n-1)m+2}{m}} + \cdots + (\frac{nm-1}{m^2})v_i^{\frac{nm-1}{m}} + (\frac{nm}{m^2})v_i^{\frac{nm}{m}} \\
=& \frac{\ddot{a}^{(m)}_{\overline{n}|i} - nv^n}{i^{(m)}} = \left(\frac{1}{m} \times m\right) \cdot \frac{\ddot{a}^{(m)}_{\overline{n}|i} - nv^n}{i^{(m)}}
\end{aligned}
$$

4.8 Continuous Varying Annuities

– payments are made continuously at a varying rate every year for the next n years

– the present value (at $t = 0$) of an increasing annuity, where payments are being made continuously at annual rate t at time t and where the annual effective rate of interest is i, shall be denoted as $\left(\bar{I}\bar{a}\right)_{\overline{n|}_i}$ and is calculated as follows (using integration by parts):

$$\begin{aligned}
\left(\bar{I}\bar{a}\right)_{\overline{n|}_i} &= \int_0^n tv^t dt \\
&= \int_0^n \underbrace{t}_{u} \underbrace{e^{-\delta t} dt}_{dv} \\
&= \underbrace{t\frac{e^{-\delta t}}{-\delta}}_{u \cdot v}\Bigg]_0^n - \int_0^n \underbrace{\frac{e^{-\delta t}}{-\delta} dt}_{v \cdot du} \\
&= \frac{n \cdot e^{-\delta n}}{-\delta} + \frac{\bar{a}_{\overline{n|}_i}}{\delta} \\
&= \frac{\bar{a}_{\overline{n|}_i} - n \cdot v_i^n}{\delta}
\end{aligned}$$

– one could also derive the above formula by simply substituting $m = \infty$ into one of the original m^{th}ly present value formulas:

$$\begin{aligned}
\left(\bar{I}\bar{a}\right)_{\overline{n|}_i} &= \left(I^{(\infty)}a^{(\infty)}\right)_{\overline{n|}_i} = \frac{\bar{a}_{\overline{n|}_i} - n \cdot v_i^n}{i^{(\infty)}} = \frac{\bar{a}_{\overline{n|}_i} - n \cdot v_i^n}{\delta} \\
&= \left(I^{(\infty)}\ddot{a}^{(\infty)}\right)_{\overline{n|}_i} = \frac{\bar{a}_{\overline{n|}_i} - n \cdot v_i^n}{d^{(\infty)}} = \frac{\bar{a}_{\overline{n|}_i} - n \cdot v_i^n}{\delta}
\end{aligned}$$

– the present value (at $t = 0$) of an annuity, where payment at time t is defined as $f(t)dt$ and where the annual effective rate of interest is i, shall be calculated as follows:

$$\int_0^n f(t) \cdot v_i^t dt$$

– if the force of interest becomes variable then the above formula becomes:

$$\int_0^n f(t) \cdot e^{\int_0^t \delta_s ds} dt$$

4.9 Summary Of Results

– material not TESTED in SoA Exam FM

Summary of Relationships for Level Annuities in Chapter 4

Interest conversion period	Payment period			
	Annual	Quarterly	Monthly	Continous
Annual	$60a_{\overline{10}\rceil .12}$	$60a^{(4)}_{\overline{10}\rceil .12}$	$60a^{(12)}_{\overline{10}\rceil .12}$	$60\bar{a}_{\overline{10}\rceil .12}$
Quarterly	$60\frac{a_{\overline{40}\rceil .03}}{s_{\overline{4}\rceil .03}}$	$15a_{\overline{40}\rceil .03}$	$15a^{(3)}_{\overline{40}\rceil .03}$	$15\bar{a}_{\overline{40}\rceil .03}$
Monthly	$60\frac{a_{\overline{120}\rceil .01}}{s_{\overline{12}\rceil .01}}$	$15\frac{a_{\overline{120}\rceil .01}}{s_{\overline{3}\rceil .01}}$	$5a_{\overline{120}\rceil .01}$	$5\bar{a}_{\overline{120}\rceil .01}$
Continous	$60\frac{1-e^{-1.2}}{e^{.12}-1}$	$15\frac{1-e^{-1.2}}{e^{.03}-1}$	$5\frac{1-e^{-1.2}}{e^{.01}-1}$	$60\frac{1-e^{-1.2}}{0.12}$

5 Yield Rates

5.1 Introduction

- this chapter extends the concepts and techniques covered to date and applies them to common financial situations
- simple financial transactions such as borrowing and lending are now replaced by a broader range of business/financial transactions
- taxes and investment expenses are to be ignored unless stated

5.2 Discounted Cash Flow Analysis

- by taking the present value of any pattern of future payments, you are performing a *discounted cash flow analysis*
- let C_k represent contributions by an investor that are made at time k
- let R_k represent returns made back to the investor at time k (these returns can also be considered withdrawals)
- in this case, we have $C_k = -R_k$
- the contributions can also be referred to as cash-flows-in while the returns can be referred to as cash-flows-out
- note that at certain times, contributions can be being made at the same time as returns are being made. At these time intervals, C_k (or $-R_k$) will be the difference between the contributions and the returns (sometimes referred to as <u>net</u> cash-flows)

  <u>Example</u>
  A 10–year investment project requires an initial investment of \$1,000,000 and subsequent beginning-of-year payments of \$100,000 for the following 9 years. The project is expected to produce 5 annual investment returns of \$600,000 commencing 6 years after the initial investment.

$$\begin{aligned}
C_0 &= 1,000,000 = -R_0 \\
C_1 &= C_2 = \cdots = C_5 = 100,000 = -R_1 = -R_2 = \cdots = -R_5 \\
C_6 &= C_7 = \cdots = C_9 = 100,000 - 600,000 = -500,000 = -R_6 = -R_7 = \cdots = -R_9 \\
C_{10} &= -600,000 = -R_{10}
\end{aligned}$$

  Given an investment rate i, the present value (sometimes called the net present value) of the returns is determined as follows:

$$\begin{aligned}
PV_0 &= \sum_{k=0}^{n} v_i^k \cdot R_k \\
&= -1,000,000 - 100,000v_i^1 - 100,000v_i^2 - \cdots - 100,000v_i^5 \\
&+ 500,000v_i^6 + 500,000v_i^7 + \cdots + 500,000v_i^9 \\
&+ 600,000v_i^{10} \\
&= -1,000,000 - 100,000v_i^1 \ddot{a}_{\overline{5}|i} + 500,000v_i^6 \ddot{a}_{\overline{4}|i} + 600,000v_i^{10}
\end{aligned}$$

From a cash flow perspective, we have

$$\begin{aligned}
CF_0^{in} &= 1,000,000 \\
CF_1^{in} &= CF_2^{in} = \cdots = CF_9^{in} = 100,000 \\
CF_6^{out} &= CF_7^{out} = \cdots = CF_{10}^{out} = 600,000
\end{aligned}$$

The net present value can also be calculated by taking the present value of the cash-flows-out and reducing them by the present value of the cash-flows-in.

$$\begin{aligned}
PV_0 &= \sum_{k=0}^{n} R_k \cdot v_i^k \\
&= \sum_{k=0}^{n} \left(CF_k^{out} - CF_k^{in}\right) v_i^k \\
PV_0 &= \sum_{k=0}^{n} CF_k^{out} \cdot v_i^k - \sum_{k=0}^{n} CF_k^{in} \cdot v_i^k \\
&= 600,000v_i^6 + 600,000v_i^7 + \cdots + 600,000v_i^{10} \\
&\quad - \left[1,000,000 + 100,000v_i^1 + 100,000v_i^2 + \cdots + 100,000v_i^9\right] \\
PV_0 &= 600,000v_i^6 \ddot{a}_{\overline{5}|i} - 1,000,000 - 100,000v_i^1 \ddot{a}_{\overline{9}|i} \\
&= -1,000,000 - 100,000v_i^1 \ddot{a}_{\overline{5}|i} + 500,000v_i^6 \ddot{a}_{\overline{4}|i} + 600,000v_i^{10}
\end{aligned}$$

The net present value depends on the annual effective rate of interest, i, that is adopted. Under either approach, if the net present value is negative, then the investment project would not be a desirable pursuit.

There exists a certain interest rate where the net present value is equal to 0. For example, under the first net present value formula:

$$PV_0 = \sum_{k=0}^{n} R_k \cdot v_i^k = 0$$

Under the cash flow approach,

$$PV_0 = \sum_{k=0}^{n} CF_k^{out} \cdot v_i^k - \sum_{k=0}^{n} CF_k^{in} \cdot v_i^k = 0$$

$$\sum_{k=0}^{n} CF_k^{out} \cdot v_i^k = \sum_{k=0}^{n} CF_k^{in} \cdot v_i^k$$

In other words, there is an effective rate of interest that exists such that the present value of cash-flows-in will yield the same present value of cash-flows out. This interest rate is called a yield rate or an internal rate of return (IRR) as it indicates the rate of return that the investor can expect to earn on their investment (i.e. on their contributions or cash-flows-in).

This yield rate for the above investment project can be determined as follows:

$$\sum_{k=0}^{n} CF_k^{out} \cdot v_i^k = \sum_{k=0}^{n} CF_k^{in} \cdot v_i^k$$

$$600,000v_i^6 \ddot{a}_{\overline{5}|i} = 1,000,000 + 100,000v_i^1 \ddot{a}_{\overline{9}|i}$$

$$600,000v_i^6 \ddot{a}_{\overline{5}|i} - 100,000v_i^1 \ddot{a}_{\overline{9}|i} - 1,000,000 = 0$$

$$600,000v_i^6 \cdot \frac{1-v_i^5}{1-v_i^1} - 100,000v_i^1 \cdot \frac{1-v_i^9}{1-v_i^1} - 1,000,000 = 0$$

$$600,000v_i^6 - 600,000v_i^{11} - 100,000v_i^1 + 100,000v_i^{10} - 1,000,000(1-v_i^1) = 0$$

$$600,000v_i^6 - 600,000v_i^{11} + 900,000v_i^1 + 100,000v_i^{10} - 1,000,000 = 0$$

The yield rate (solved by using a pocket calculator with advanced financial functions or by using Excel with its Goal Seek function) is 8.062%.

Using this yield rate, investment projects with a 10 year life can be compared to the above project. In general, those investment opportunities that have higher(lower) expected returns than the 8.062% may be more(less) desirable.

5.3 Uniqueness Of The Yield Rate

– quite often when solving for the yield rate, you can be left with a polynomial equation as was just shown. The problem with polynomials is that they can have multiple solutions.

Example Find the yield rate which will produce a return of \$230 at time 1 in exchange for contributions of \$100 immediately and \$132 at time 2.

$$100 + 132v_i^2 = 230v_i^1$$
$$132v_i^2 - 230v_i^1 + 100 = 0$$

$$v_i = \frac{-(-230) \pm \sqrt{(-230)^2 - 4(132)(100)}}{2(132)}$$
$$v_i = 0.909091 \text{ or } 0.833333$$
$$(1+i) = \frac{1}{v} = 1.1 \text{ or } 1.2$$
$$i = 10\% \text{ or } 20\%$$

Question:
How can you tell if you will get a unique yield rate?

Answer 1:
If there are m changes in going from cash-flows-in to cash-flows-out or visa-versa, then there are a maximum of m solutions to the polynomial (Descartes's Rule of Signs). Therefore, if there is only one change, then there is only one yield rate. The above example had 2 changes.

Answer 2:
A unique yield rate will always be produced as long as the outstanding balance at all times during the investment period is positive. In the 10-year investment project example, there was always a positive amount of money left in the project.

– it is possible for no yield rate to exist or for all multiple yield rates to be imaginary

5.4 Reinvestment Rates

– the yield rate that is calculated assumes that the positive returns (or cash-flows-out) will be reinvested at the same yield rate

– the actual rate of return can be higher or lower than the calculated yield rate depending on the reinvestment rates

Example

– an investment of 1 is invested for n years and earns an annual effective rate of i. The interest payments are reinvested in an account that credits an annual effective rate of interest of j.

– if the interest is payable at the end of every year and earns a rate of j, then the accumulated value at time n of the interest payments and the original investment is:

$$FV_n = (i \times 1)s_{\overline{n}|j} + 1$$

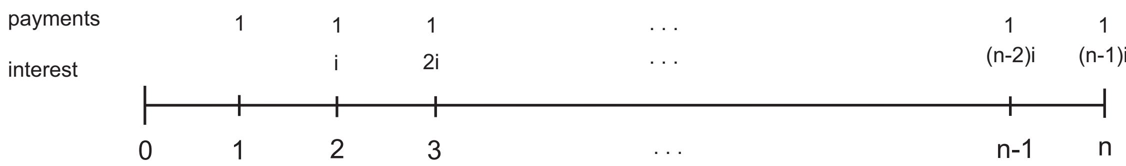
– if $i = j$, then the accumulated value at time n is:

$$FV_n = i \times \frac{(1+i)^n - 1}{i} + 1 = (1+i)^n$$

Example

– an investment of 1 is made at the end of every year for n years and earns an annual effective rate of interest of i. The interest payments are reinvested in an account that credits an annual effective rate of interest of j.

– note that interest payments increase every year by $i \times 1$ as each extra dollar is deposited each year into the original account



– if the interest is payable at the end of every year and earns a rate of j, then the accumulated value at time n of the interest payments and the original investment is:

$$FV_n = i \times (Is)_{\overline{n-1}|j} + n \times 1$$

– if $i = j$, then the accumulated value at time n is:

$$\begin{aligned} FV_n &= i \times \frac{\ddot{s}_{\overline{n-1}|i} - (n-1)}{i} + n \\ &= \ddot{s}_{\overline{n-1}|i} - n + 1 + n \\ &= \ddot{s}_{\overline{n-1}|i} + 1 \\ &= s_{\overline{n}|i} \end{aligned}$$

5.5 Interest Measurement Of A Fund

- investment funds typically experience multiple contributions and withdrawals during its life
- interest payments are often made at irregular periods rather than only at the end of the year
- let A and B represent the beginning-of-year and end-of-year balance, respectively of an investment fund
- let I represent the amount of interest earned during the one-year period where the interest rate earned from time b to time $a+b$ ($a+b \leq 1$) is denoted as ${}_a i_b$.
- let C_t be the net cash-flow contributed at time t ($0 \leq t \leq 1$) and let C represent the total net cashflow for the one-year period, $C = \sum_t C_t$.
- the fund at the end of the year, B, is equal to the fund at the beginning of the year, A, plus the contributions, C, and the investment income earned, I

$$B = A + C + I$$

- the investment income, I, is based on the fund at the beginning of the year, A, and on the net cash flows made during the year, C_t.

$$I = i \cdot A + \sum_t C_t \cdot {}_{1-t} i_t$$

- there are two approaches in determining the interest rate earned during the year

  (i) Exact Approach
  (ii) Approximate Approach

Exact Approach

- assume that the interest rate function is as follows:

$${}_{1-t} i_t = (1+i)^{1-t} - 1 \quad \text{(i.e. compound interest)}$$

- the total amount of investment income is then

$$I = i \cdot A + \sum_t C_t \cdot \left[(1+i)^{1-t} - 1\right]$$

$$I = i \cdot A + \sum_t C_t \cdot (1+i)^{1-t} - C$$

- this will again produce a polynomial
- the effective rate of interest, i, can be solved by using a pocket calculator with advanced financial functions or by using Excel with its Goal Seek function.

Approximate Approach

- assume that ${}_{1-t}i_t = (1-t)i$,   (a "pseudo" simple interest approach)
- the total amount of investment income is then

$$I = i \cdot A + \sum_t C_t \cdot [(1-t)i]$$

- solving for the effective rate of interest gives us

$$i = \frac{I}{A + \sum_t C_t(1-t)}$$

- the interest rate earned is estimated by computing the amount of interest earned to the average amount of principal invested. The denominator is often referred to as the "fund exposure".
- this approach can produce results that are fairly close to the exact approach as long as the cash-flows are small relative to A
- the approximate approach can be further simplified if one assumes that the cash-flows are made uniformally during the year. In other words, on average, C is contributed at time $t = \frac{1}{2}$

$$\begin{aligned} i &= \frac{I}{A + \sum_t C_t \left(1 - \frac{1}{2}\right)} \\ &= \frac{I}{A + \frac{1}{2} \cdot C} \\ &= \frac{2I}{2A + C} \\ &= \frac{2I}{2A + (B - A - I)} \\ &= \frac{2I}{A + B - I} \end{aligned}$$

- if the cash flows are made, on average, at time k, then the estimated interest rate is

$$\begin{aligned} i &= \frac{I}{A + \sum_t C_t(1-k)} \\ &= \frac{I}{A + (1-k)C} \\ &= \frac{I}{A + (1-k)(B - A - I)} \\ &= \frac{I}{k \cdot A + (1-K)B - (1-k)I} \end{aligned}$$

Continuous Approach

– A general model can be developed for an investment period of n years if we let B_t represent the outstanding fund balance at time t.

$$B_{t+n} = B_t(1+i)^n + \int_0^n C_{t+s}(1+i)^{n-s}ds$$

– If B_0 represents the fund balance at time 0, then the above formula can be simplified to:

$$B_n = B_0(1+i)^n + \int_0^n C_s(1+i)^{n-s}ds$$

– If a varying force of interest is introduced, then the general formula will be:

$$B_{t+n} = B_t e^{\int_t^{t+n} \delta_s ds} + \int_0^n C_{t+s}\left(e^{\int_{t+s}^{n} \delta_r dr}\right)ds$$

and if B_0 represents the fund balance at time 0, we have:

$$B_n = B_0 e^{\int_0^n \delta_s ds} + \int_0^n C_s\left(e^{\int_s^n \delta_r dr}\right)ds$$

– Also, note that

$$dB_t = (\delta_t dt)B_t + C_t dt$$

which shows that a fund changes instantaneously by the interest it earns on the fund and by the contribution made

5.6 Time-Weighted Rates Of Interest

– the techniques introduced in Section 5.5 show that the yield rate calculation is sensitive to the amount and the timing of the contribution

Example

A fund loses 50% of its original investment of 1 during the first six months, but earns 100% (i.e. doubles) over the last six months.

(i) If no other contributions are made into the fund, then the yield rate is determined as follows:

$$\begin{aligned} 1(1+i)^1 &= 1(1-50\%)(1+100\%) \\ (1+i) &= 1 \\ i &= 0\% \end{aligned}$$

(ii) If a contribution of 0.50 is made into the fund at $t = \frac{1}{2}$, then the yield rate is determined as follows:

$$\begin{aligned} 1(1+i)^1 + .5(1+i)^{\frac{1}{2}} &= 1(1-50\%)(1+100\%) + .5(1+100\%) \\ (1+i) + .5(1+i)^{\frac{1}{2}} &= 2 \\ i &= 40.69\% \end{aligned}$$

(iii) If a withdrawal (negative contribution) of 0.25 is made from the fund at $t = \frac{1}{2}$, then the yield rate is determined as follows:

$$\begin{aligned} 1(1+i)^1 - .25(1+i)^{\frac{1}{2}} &= 1(1-50\%)(1+100\%) - .25(1+100\%) \\ (1+i) - .25(1+i)^{\frac{1}{2}} &= .5 \\ i &= -28.92\% \end{aligned}$$

Scenario (i) suggests that the one-year performance of the fund produces a yield rate of 0%.

Scenario (ii) produces a higher yield rate due to the fact that more money was invested into the fund just at the time when money was ready to double.

Scenario (iii) produces a lower yield rate due to the fact that money was withdrawn from the fund just at the time when money was ready to double. In other words, the opportunity to earn 100% was lost.

– since the Section 5.5 yield rate is influenced by the dollar amount of the contribution, it often referred to as a *dollar-weighted rate of interest*.

– the dollar-weighted rate of interest is the actual return that the investor experiences over the year

– for the investment manager, who picked the fund for the investor, this dollar-weighted method will make the fund look great (40.69% return) or bad (28.92% loss) depending on the independent contribution behaviour of the investor.

– to measure annual fund performance without the influence of the contributions, one must look at the fund's performance over a variety of sub-periods. These sub-periods are triggered whenever a contribution takes place and end just before the next contribution (or when the end of the year is finally met).

– in general, the interest rate earned for the sub-period is determined by taking the ratio of the fund at the beginning of the sub-period versus the fund at the end of the sub-period.

– The yield rate derived from this method is called the *time-weighted rate of interest* and is determined using the following formula:

$$1 + i = \prod_{k=1}^{n} (1 + j_k)$$

where n is the number of sub-periods and,

$$1 + j_k = \frac{\text{fund at end of sub-period } k}{\text{fund at beginning of sub-period } k}.$$

Example
A fund loses 50% of its original investment of 1 during the first six months, but earns 100% (i.e. doubles) over the last six months.

(i) If no other contributions are made into the fund, then there is one sub-period, 0 to 1. The yield rate is determined as follows:

$$\begin{aligned} 1(1+i)^1 &= (1 + j_1) \\ (1+i) &= \frac{1(1-50\%)(1+100\%)}{1} \\ (1+i) &= 1 \\ i &= 0\% \end{aligned}$$

(ii) If a contribution of 0.50 is made into the fund at $t = \frac{1}{2}$, then there are two sub-periods, 0 to $\frac{1}{2}$ and $\frac{1}{2}$ to 1. The yield rate is determined as follows:

$$\begin{aligned} 1(1+i)^1 &= (1 + j_1)(1 + j_2) \\ (1+i) &= \left(\frac{1(1-50\%)}{1}\right) \cdot \left(\frac{1(1-50\%)(1+100\%) + .50(1+100\%)}{1(1-50\%) + .50}\right) \\ (1+i) &= \underbrace{\left(\frac{.50}{1}\right)}_{1+j_1=1-50\%} \cdot \underbrace{\left(\frac{2}{1}\right)}_{1+j_2=1+100\%} \\ (1+i) &= 1 \\ i &= 0\% \end{aligned}$$

(iii) If a withdrawal (negative contribution) of 0.25 is made from the fund at $t = \frac{1}{2}$, then there are <u>two</u> sub-periods, 0 to $\frac{1}{2}$ and $\frac{1}{2}$ to 1. The yield rate is determined as follows:

$$
\begin{aligned}
1(1+i)^1 &= (1+j_1)(1+j_2) \\
(1+i) &= \left(\frac{1(1-50\%)}{1}\right) \cdot \left(\frac{1(1-50\%)(1+100\%) - .25(1+100\%)}{1(1-50\%) - .25}\right) \\
(1+i) &= \underbrace{\left(\frac{.50}{1}\right)}_{1+j_1=1-50\%} \cdot \underbrace{\left(\frac{.50}{.25}\right)}_{1+j_2=1+100\%} \\
(1+i) &= 1 \\
i &= 0\%
\end{aligned}
$$

5.7 Portfolio Methods and Investment Year Methods

- you are given a fund for which there are many investors. Each investor holds a share of the fund expressed as a percentage. For example, investor k might hold 5% of the fund at the beginning of the year
- over a one-year period, the fund will earn investment income, I
- there are two ways in which the fund's investment income can be distributed to the investors at the end of the year

  (i) Portfolio Method
  (ii) Investment Year Method

  Portfolio Method

  - if investor k owns 5% of the fund at the beginning of the year, then investor k gets 5% of the investment income ($5\% \times I$).
  - this is the same approach as if the fund's *dollar-weighted* rate of return was calculated and all the investors were credited with that same yield rate
  - the disadvantage of the portfolio method is that it doesn't reward those investors who make good decisions
  - For example, investor k may have contributed large amounts of money during the last six months of the year when the fund was earning, say 100%. Investor c may have withdrawn money during that same period, and yet both would be credited with the same rate of return.
  - If the fund's overall yield rate was say, 0%, obviously, investor k would rather have their contributions credited with the actual 100% as opposed to $(1 + 0\%)^{\frac{1}{2}}$.

  Investment Year Method

  - an investors's contribution will be credited during the year with the interest rate that was in effect at the time of the contribution.
  - this interest rate is often referred to as the *new-money* rate.
  - Reinvestment rates can be handled in one of two ways:
    (a) Declining Index System - only principal is credited at new money rates
    (b) Fixed Index System - principal and interest is credited at new money rates
  - this "earmarking" of money for new money rates rates will only go on for a specified period before the portfolio method commences

5.8 Capital Budgeting

- **material not tested in SoA Exam FM**

5.9 More General Borrowing/Lending Models

- **material not tested in SoA Exam FM**

6 Amortization Schedules and Sinking Funds

6.1 Introduction

– there are two methods for paying off a loan

(i) Amortization Method - borrower makes installment payments at periodic intervals

(ii) Sinking Fund Method - borrower makes installment payments as the annual interest comes due and pays back the original loan as a lump-sum at the end. The lump-sum is built up with periodic payments going into a fund called a "sinking fund".

– this chapter also discusses how to calculate:

(a) the outstanding loan balance once the repayment schedule has begun, and

(b) what portion of an annual payment is made up of the interest payment and the principal repayment

6.2 Finding The Outstanding Loan

– There are two methods for determining the outstanding loan once the payment process commences

(i) Prospective Method

(ii) Retrospective Method

Prospective Method (see the future)

– the original loan at time 0 represents the present value of future repayments. If the repayments, P, are to be level and payable at the end of each year, then the original loan can be represented as follows:

$$\text{Loan} = P \cdot a_{\overline{n}|i}$$

– the outstanding loan at time t, O/S Loan$_t$, represents the present value of the remaining future repayments

$$O/S \text{ Loan}_t = P \cdot a_{\overline{n-t}|i}$$

– this also assumes that the repayment schedule determined at time 0 has been adhered to; otherwise, the prospective method will not work

Retrospective Method (see the past)

– If the repayments, P, are to be level and payable at the end of each year, then the outstanding loan at time t is equal to the accumulated value of the loan *less* the accumulated value of the payments made to date

$$O/S \text{ Loan}_t = \text{Loan} \cdot (1+i)^t - P \cdot s_{\overline{t}|i}$$

– this also assumes that the repayment schedule determined at time 0 has been adhered to; otherwise, the accumulated value of past payments will need to be adjusted to reflect what the actual payments were, with interest

Basic Relationship 1: Prospective Method = Retrospective Method

- let a loan be repaid with end-of-year payments of 1 over the next n years:

$$\text{Present Value of Payments} = \text{Present Value of Loan}$$
$$(1)a_{\overline{n}|i} = \text{Loan}$$

$$\text{Accumulated Value of Payments} = \text{Accumulated Value of Loan}$$
$$(1)a_{\overline{n}|i} \cdot (1+i)^t = \text{Loan} \cdot (1+i)^t$$

$$\begin{aligned} &\text{Accumulated Value of Past Payments} \\ &+\text{Present Value Future Payments} = \text{Accumulated Value of Loan} \\ &(1)s_{\overline{t}|i} + (1)a_{\overline{n-t}|i} = \text{Loan} \cdot (1+i)^t = a_{\overline{n}|i} \cdot (1+i)^t \end{aligned}$$

$$\begin{aligned} \text{Present Value Future Payments} &= \text{Accumulated Value of Loan} \\ &\quad - \text{Accumulated Value of Past Payments} \\ (1)a_{\overline{n-t}|i} &= a_{\overline{n}|i} \cdot (1+i)^t - (1)s_{\overline{t}|i} \end{aligned}$$

$$\text{Prospective Method} = \text{ Retrospective Method}$$

- the prospective method is preferable when the size of each level payment and the number of remaining payments is known
- the retrospective method is preferable when the number of remaining payments or a final irregular payment is unknown.

6.3 Amortization Schedules

- let a loan be repaid with end-of-year payments of 1 over the next n years
- the loan at time 0 (beginning of year 1) is $(1)a_{\overline{n}|i}$
- an annual end-of-year payment of 1 using the amortization method will contain an interest payment, I_t, and a principal repayment, P_t
- in other words, $1 = I_t + P_t$

Interest Payment

- I_t is intended to cover the interest obligation that is payable at the end of year t. The interest is based on the outstanding loan balance at the beginning of year t.
- using the prospective method for evaluating the outstanding loan balance, the interest payment is derived as follows:

$$\begin{aligned} I_t &= i \cdot \left(1 \cdot a_{\overline{n-(t-1)}|i}\right) \\ &= i \cdot \left(\frac{1 - v^{n-(t-1)}}{i}\right) \\ I_t &= 1 - v^{n-(t-1)} \end{aligned}$$

Principal Repayment

– once the interest owed for the year is paid off, then the remaining portion of the amortization payment goes towards paying back the principal:

$$\begin{aligned} P_t &= 1 - I_t \\ &= 1 - [1 - v^{n-(t-1)}] \\ P_t &= v^{n-(t-1)} \end{aligned}$$

Outstanding Loan Balance

– The outstanding loan balance is calculated using the prospective method

– However, the outstanding loan at the end of year t can also be viewed as the outstanding loan at the beginning of year t *less* the principal repayment that has just occured

$$\begin{aligned} O/S \text{ Loan}_t &= O/S \text{ Loan}_{t-1} - P_t \\ &= 1 \cdot a_{\overline{n-(t-1)}|i} - v^{n-(t-1)} \\ &= v + v^2 + \cdots + v^{n-t} + v^{n-(t-1)} - v^{n-(t-1)} \\ &= v + v^2 + \cdots + v^{n-t} \\ &= a_{\overline{n-t}|i} \end{aligned}$$

– The following amortization schedule illustrates the progression of the loan repayments

Year (t)	Payment	I_t	P_t	O/S Loan$_t$
1	1	$1 - v_i^n$	v_i^n	$a_{\overline{n-1}|i}$
2	1	$1 - v_i^{n-1}$	v_i^{n-1}	$a_{\overline{n-2}|i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	1	$1 - v_i^{n-(t-1)}$	$v_i^{n-(t-1)}$	$a_{\overline{n-t}|i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	1	$1 - v_i^2$	v_i^2	$a_{\overline{1}|i}$
n	1	$1 - v_i$	v_i	0
Total	n	$n - a_{\overline{n}|i}$	$a_{\overline{n}|i}$	

- note that the total of all the interest payments is represented by the total of all amortization payments *less* the original loan

$$\sum_{k=1}^{n} I_k = \sum_{k=1}^{n} 1 - v_i^{n-(k-1)} = \sum_{k=1}^{n} 1 - \sum_{k=1}^{n} v_i^{n-(k-1)} = n - a_{\overline{n}|i}$$

- note that the total of all the principal payments must equal the original loan

$$\sum_{k=1}^{n} P_k = \sum_{k=1}^{n} v_i^{n-(k-1)} = a_{\overline{n}|i}$$

- note that the outstanding loan at $t = n$ is equal to 0 (the whole point of amortizing is to reduce the loan to 0 within n years)
- note that the principal repayments increase geometrically by (1+i) i.e. $P_{t+n} = P_t \cdot (1 + i)^n$. This is to be expected since the outstanding loan gets smaller with each principal repayment and as a result, there is less interest accruing which leaves of the amortization payment left to pay off principal.

- remember that the above example is based on an annual payment of 1. The above formulas need to be multiplied by the actual payment, $\frac{\text{Loan}}{a_{\overline{n}|i}}$, if the original loan is not equal to $1 \cdot a_{\overline{n}|i}$.

6.4 Sinking Funds

- let a loan of $(1)a_{\overline{n}|i}$ be repaid with single lump-sum payment at time n. If annual end-of-year interest payments of $i \cdot a_{\overline{n}|i}$ are being met each year, then the lump-sum required at $t = n$, is the original loan amount. (i.e. the interest on the loan never gets to grow with interest)
- let the lump-sum that is to be built up in a "sinking fund" be credited with interest rate i

Sinking Fund Payment

- if the lump-sum is to be built up with annual end-of-year payments for the next n years, then the sinking fund payment is calculated as:

$$\frac{\text{Loan}}{s_{\overline{n}|i}} = \frac{a_{\overline{n}|i}}{s_{\overline{n}|i}}$$

- the total annual payment for year t made by the borrower is the annual interest due on the loan *plus* the sinking fund payment:

$$i \cdot a_{\overline{n}|i} + \frac{a_{\overline{n}|i}}{s_{\overline{n}|i}}$$

- this can be reduced down to:

$$a_{\overline{n}|i} \cdot \left(i + \frac{1}{s_{\overline{n}|i}}\right) = a_{\overline{n}|i} \cdot \left(\frac{1}{a_{\overline{n}|i}}\right) = 1$$

- in other words, the annual payment under the sinking fund method is the same annual payment under the amortization method (recall the basic relationship from Chapter 3, $\frac{1}{a_{\overline{n}|i}} = i + \frac{1}{a_{\overline{n}|i}}$).

Net Amount of Loan

- the accumulated value of the sinking fund at time t is the accumulated value of the sinking fund payments made to date and is calculated as follows:

$$\begin{aligned} SF_t &= \left(\frac{a_{\overline{n}|i}}{s_{\overline{n}|i}}\right) \cdot s_{\overline{t}|i} \\ &= \left(\frac{v^n \cdot s_{\overline{n}|i}}{s_{\overline{n}|i}}\right) \cdot s_{\overline{t}|i} \\ SF_t &= v^n \cdot s_{\overline{t}|i} \end{aligned}$$

- the loan itself will never grow as long as the annual interest growth,$i \cdot a_{\overline{n}|i}$, is paid off at the end of each year
- we define the loan amount that is not covered by the balance in the sinking fund as a "net" amount of loan outstanding and is equal to Loan - SF_t.

$$\begin{aligned} \text{net Loan}_t &= \text{Loan} - SF_t \\ &= (1)a_{\overline{n}|i} - v^n \cdot s_{\overline{t}|i} \\ &= \frac{1-v^n}{i} - v^n \cdot \frac{(1+i)^t - 1}{i} \\ &= \frac{1 - v^n - v^n \cdot (1+i)^t + v^n}{i} \\ &= \frac{1 - v^{n-t}}{i} \\ \text{net Loan}_t &= a_{\overline{n-t}|i} \end{aligned}$$

- in other words, the "net" amount of the loan outstanding under the sinking fund method is the same value as the outstanding loan balance under the amortization method

Net Amount of Interest

- each year, the borrower pays interest to the lender in the amount of $i \cdot a_{\overline{n}|i}$ and each year the borrower earns interest in the sinking fund of $i \cdot \ SF_{t-1}$
- the actual interest cost to the borrower for year t is referred to as the "net" amount of interest and is the difference between what amount of interest has been paid and what amount of interest has been earned

$$\begin{aligned} i \cdot a_{\overline{n}|i} - i \cdot \ SF_{t-1} &= i \cdot a_{\overline{n}|i} - i \cdot \left(\frac{a_{\overline{n}|i}}{s_{\overline{n}|i}}\right) \cdot s_{\overline{t-1}|i} \\ &= i \cdot \frac{1-v^n}{i} - i \cdot v^n \cdot \frac{(1+i)^t - 1}{i} \\ &= 1 - v^n - v^n(1+i)^{t-1} + v^n \\ &= 1 - v^{n-(t-1)} \end{aligned}$$

- in other words, the "net" amount of interest under the sinking fund method is the same value as the interest payment under the amortization method

Sinking Fund Increase

– the sinking fund grows each year by the amount of interest that it earns and by the end-of-year contribution that it receives

– the increase in the sinking fund, $SF_t - SF_{t-1}$ can be calculated as follows:

$$
\begin{aligned}
SF_t &= SF_{t-1} \cdot (1+i) + \left(\frac{a_{\overline{n}|i}}{s_{\overline{n}|i}}\right) \\
SF_t - SF_{t-1} &= i \cdot \ SF_{t-1} + \left(\frac{a_{\overline{n}|i}}{s_{\overline{n}|i}}\right) \\
&= i \cdot v^n \cdot s_{\overline{t}|i} + \frac{v^n \cdot s_{\overline{n}|i}}{s_{\overline{n}|i}} \\
&= i \cdot v^n \cdot s_{\overline{t}|i} + v^n \\
&= v^n \cdot \left[i \cdot \frac{(1+i)^t - 1}{i} + 1\right] \\
&= v^n \cdot \left[(1+i)^t - 1 + 1\right] \\
&= v^n \cdot (1+i)^t \\
&= v^{n-(t-1)}
\end{aligned}
$$

– in other words, the annual increase in the sinking fund is the same value as the principal repayment under the amortization method

– both methods are committed to paying back the principal; it's just that the amortization method does it every year and the sinking fund method waits until the very end but puts aside a little bit each year to meet this single payment obligation

What Happens When The Sinking Fund Earns Rate j, not i

– usually, the interest rate on borrowing, i, is greater than the interest rate offered by investing in a fund, j

– the total payment under the sinking fund approach is then

$$
i \cdot \text{Loan} + \frac{\text{Loan}}{s_{\overline{n}|j}}
$$

– we now wish to determine at what interest rate the amortization method would provide for the same level payment

$$
\frac{\text{Loan}}{a_{\overline{n}|i'}} = i \cdot \text{Loan} + \frac{\text{Loan}}{s_{\overline{n}|j}}
$$

where i' represents the annual effective rate of interest that produces this equality

$$
\begin{aligned}
\frac{\text{Loan}}{a_{\overline{n}|i'}} &= i \cdot \text{Loan} + \frac{\text{Loan}}{s_{\overline{n}|j}} \\
&= \text{Loan} \cdot \left(i + \frac{1}{s_{\overline{n}|j}}\right) \\
&= \text{Loan} \cdot \left(i + \frac{1}{a_{\overline{n}|j}} - j\right) \\
&= \frac{\text{Loan}}{a_{\overline{n}|j}} + \text{Loan} \cdot \ (i - j)
\end{aligned}
$$

– therefore, the amortization payment, using this "blended" interest rate, will cover the smaller amortization payment at rate j and the interest rate shortfall, $i - j$, that the smaller payment doesn't recognize

– the "blended" interest rate can now be determined using a pocket calculator with advanced financial functions or by using Excel with its Goal Seek function.

– the "blended" interest rate can be approximated by using

$$i' = i + \frac{i - j}{2}$$

6.5 Differing Payment Periods and Interest Conversion Periods

When Payments Are Made $\boxed{\text{Less}}$ Frequent Than Interest Is Converted

– **material not tested in SoA Exam FM**

– let a loan be repaid with end-of-year payments of 1 over the next n years and let interest be convertible k times a year

– the loan, which is $1 \cdot a_{\overline{n}|i}$, can also be expressed as a series of payments that coincide with each conversion period where the interest rate is $j = \frac{i^{(k)}}{k}$

$$\text{Loan} = \left(\frac{1}{s_{\overline{k}|j}} \right) \cdot a_{\overline{nk}|j}$$

– if the loan is to be paid off in n years (or nk conversion periods), then the level annual payments can be broken down into interest and principal payments

<u>Outstanding Loan Balance at time t</u>

– the prospective version of the outstanding loan balance at time t (or tk conversion periods) is:

$$O/S \text{ Loan}_t = \left(\frac{1}{s_{\overline{k}|j}} \right) \cdot a_{\overline{nk - tk}|j}$$

– the retrospective version of the outstanding loan balance at time t (or tk conversion periods) is:

$$O/S \text{ Loan}_t = \text{Loan} \cdot (1 + i)^t - \left(\frac{1}{s_{\overline{k}|j}} \right) \cdot s_{\overline{tk}|j}$$

Interest Payment

– I_t is intended to cover the interest obligation that is payable at the end of year t (or tk conversion periods). The interest is based on the outstanding loan balance at the beginning of year t.

– using the prospective method for evaluating the outstanding loan balance, the interest payment is derived as follows:

$$\begin{aligned}
I_t &= i \cdot \left(\frac{1}{s_{\overline{k}|j}} \right) \cdot a_{\overline{nk - tk}|j} \\
&= i \cdot \left(\frac{j}{(1+j)^k - 1} \right) \cdot \left(\frac{1 - v_j^{nk-(t-1)k}}{j} \right) \\
&= i \cdot \left(\frac{j}{i} \right) \cdot \left(\frac{1 - v_j^{nk-(t-1)k}}{j} \right) \\
I_t &= 1 - v_j^{nk-(t-1)k}
\end{aligned}$$

Principal Repayment

– once the interest owed for the year is paid off, then the remaining portion of the amortization payment goes towards paying back the principal:

$$\begin{aligned}
P_t &= 1 - I_t \\
&= 1 - [1 - v_j^{nk-(t-1)k}] \\
P_t &= v_j^{nk-(t-1)k}
\end{aligned}$$

Outstanding Loan Balance (again)

– The outstanding loan balance in the above formulas has been calculated using the prospective method

– However, the outstanding loan at the end of year t (or after tk conversion periods) can also be viewed as the outstanding loan at the beginning of year t *less* the principal repayment that has just occured

$$\begin{aligned}
O/S \text{ Loan}_t &= O/S \text{ Loan}_{t-1} - P_t \\
&= a_{\overline{n-(t-1)}|i} - v_j^{nk-(t-1)k} \\
&= a_{\overline{nk-(t-1)k}|j} - v_j^{nk-(t-1)k} \\
&= v_j^k + v_j^{2k} + \cdots + v_j^{(n-t)k} + v_j^{nk-(t-1)k} - v_j^{nk-(t-1)k} \\
&= v_j^k + v_j^{2k} + \cdots + v_j^{(n-t)k} \\
&= a_{\overline{nk-tk}|j} \\
&= a_{\overline{n-t}|i}
\end{aligned}$$

– The following amortization schedule illustrates the progression of the loan repayments. We let $j = \frac{i^{(k)}}{k}$.

Year/Conversions	Payment	I_t	P_t	O/S Loan$_t$
1=k	1	$1 - v_j^{nk}$	v_j^{nk}	$a_{\overline{nk-k}\rvert j}$
2=$2k$	1	$1 - v_j^{nk-k}$	v_j^{nk-k}	$a_{\overline{nk-2k}\rvert j}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t = tk$	1	$1 - v_j^{nk-(t-1)k}$	$v_j^{nk-(t-1)k}$	$a_{\overline{nk-tk}\rvert j}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1 = (n-1)k$	1	$1 - v_j^{2k}$	v_j^{2k}	$a_{\overline{k}\rvert j}$
$n = nk$	1	$1 - v_j^{k}$	v_j^{k}	0
Total	n	$n - a_{\overline{nk}\rvert j}$	$a_{\overline{nk}\rvert j}$	

– note that the total of all the interest payments is represented by the total of all amortization payments *less* the original loan

– note that the total of all the principal payments must equal the original loan

– note that the outstanding loan at $t = n$ is equal to 0 (the whole point of amortizing is to reduce the loan to 0 within n years)

– note that the principal repayments increase geometrically by $(1+i)$. In other words, $P_{t+n} = P_t \cdot (1+i)^n$. This is to be expected since the outstanding loan gets smaller with each principal repayment and as a result, there is less interest accruing which leaves of the amortization payment left to pay off principal.

– remember that the above example is based on an annual payment of 1. The above formulas need to be multiplied by the actual payment, $\frac{\text{Loan}}{a_{\overline{n}\rvert i}}$, if the original loan is not equal to $1 \cdot a_{\overline{n}\rvert i}$.

When Payments Are Made $\boxed{\text{More}}$ Frequent Than Interest Is Converted

– let a loan be repaid with *mth*ly end-of-conversion-period payments of $\frac{1}{m}$ over the next n years and let interest be convertible once a year

– the loan is then equal to:

$$\text{Loan} = \left(\frac{1}{m} \times m\right) \cdot a^{(m)}_{\overline{n}|i}$$

– if the loan is to be paid off in n years (or in nm payments), then the level *mth*ly payments of $\frac{1}{m}$ can be broken down into interest and principal payments

Outstanding Loan Balance at time t

– the prospective version of the outstanding loan balance at time t (or tm payments later) is:

$$O/S\ \text{Loan}_t = 1 \cdot a^{(m)}_{\overline{n-t}|i}$$

– the retrospective version of the outstanding loan balance at time t (or tm payments later) is:

$$O/S\ \text{Loan}_t = \text{Loan} \cdot (1+i)^t - 1 \cdot s^{(m)}_{\overline{t}|i}$$

Interest Payment

– $I_{\frac{t}{m}}$ is intended to cover the interest obligation that is payable at the end of each mth of a year. The interest is based on the outstanding loan balance at the beginning of the *mth*ly period.

– using the prospective method for evaluating the outstanding loan balance, the interest payment is derived as follows:

$$\begin{aligned} I_{\frac{t}{m}} &= \frac{i^{(m)}}{m} \cdot a^{(m)}_{\overline{n-\left(\frac{t-1}{m}\right)}|i} \\ &= \frac{i^{(m)}}{m} \cdot \left[\frac{1 - v^{n-\left(\frac{t-1}{m}\right)}}{i^{(m)}}\right] \\ &= \frac{1}{m} \cdot \left[1 - v^{n-\left(\frac{t-1}{m}\right)}\right] \\ I_{\frac{t}{m}} &= \frac{1}{m} - \frac{1}{m} \cdot v^{n-\left(\frac{t-1}{m}\right)} \end{aligned}$$

Principal Repayment

– once the interest owed for the year is paid off, then the remaining portion of the amortization payment goes towards paying back the principal:

$$\begin{aligned} P_{\frac{t}{m}} &= \frac{1}{m} - I_{\frac{t}{m}} \\ &= \frac{1}{m} - \left[\frac{1}{m} - \frac{1}{m} \cdot v^{n-\left(\frac{t-1}{m}\right)}\right] \\ P_{\frac{t}{m}} &= \frac{1}{m} \cdot v^{n-\left(\frac{t-1}{m}\right)} \end{aligned}$$

Outstanding Loan Balance (again)

- The outstanding loan balance in the above formulas has been calculated using the prospective method
- However, the outstanding loan at the end of period $\frac{t}{m}$ (or after t payments) can also be viewed as the outstanding loan at the beginning of period $\frac{t}{m}$ *less* the principal repayment that has just occured

$$\begin{aligned}
O/S \text{ Loan}_{\frac{t}{m}} &= O/S \text{ Loan}_{\frac{t-1}{m}} - P_{\frac{t}{m}} \\
&= a^{(m)}_{\overline{n-\left(\frac{t-1}{m}\right)}|i} - \frac{1}{m} \cdot v^{n-\left(\frac{t-1}{m}\right)} \\
&= \frac{1}{m}v^{\frac{1}{m}} + \frac{1}{m}v^{\frac{2}{m}} + \cdots + \frac{1}{m}v^{n-\frac{t}{m}} + \frac{1}{m}v^{n-\frac{t-1}{m}} - \frac{1}{m}v^{n-\frac{t-1}{m}} \\
&= \frac{1}{m}v^{\frac{1}{m}} + \frac{1}{m}v^{\frac{2}{m}} + \cdots + \frac{1}{m}v^{n-\frac{t}{m}} \\
&= a^{(m)}_{\overline{n-\left(\frac{t}{m}\right)}|i}
\end{aligned}$$

– The following amortization schedule illustrates the progression of the loan repayments.

Period (k)	Payment	I_k	P_k	O/S Loan$_k$
$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m} - \frac{1}{m} \cdot v^n$	$\frac{1}{m} \cdot v^n$	$a^{(m)}_{\overline{n-(\frac{1}{m})}\rvert i}$
$\frac{2}{m}$	$\frac{1}{m}$	$\frac{1}{m} - \frac{1}{m} \cdot v^{n-\left(\frac{1}{m}\right)}$	$\frac{1}{m} \cdot v^{n-\left(\frac{1}{m}\right)}$	$a^{(m)}_{\overline{n-(\frac{2}{m})}\rvert i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{t}{m}$	$\frac{1}{m}$	$\frac{1}{m} - \frac{1}{m} \cdot v^{n-\left(\frac{t-1}{m}\right)}$	$\frac{1}{m} \cdot v^{n-\left(\frac{t-1}{m}\right)}$	$a^{(m)}_{\overline{n-(\frac{t}{m})}\rvert i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n - \frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m} - \frac{1}{m} \cdot v^{n-\left(\frac{2}{m}\right)}$	$\frac{1}{m} \cdot v^{n-\left(\frac{2}{m}\right)}$	$a^{(m)}_{\overline{\frac{1}{m}}\rvert i}$
n	$\frac{1}{m}$	$\frac{1}{m} - \frac{1}{m} \cdot v^{n-\left(\frac{1}{m}\right)}$	$\frac{1}{m} \cdot v^{n-\left(\frac{1}{m}\right)}$	0
Total	$nm \cdot \frac{1}{m} = n$	$n - a^{(m)}_{\overline{n}\rvert i}$	$a^{(m)}_{\overline{n}\rvert i}$	

– note that the total of all the interest payments is represented by the total of all amortization payments *less* the original loan

– note that the total of all the principal payments must equal the original loan

– note that the outstanding loan at $t = n$ is equal to 0 (the whole point of amortizing is to reduce the loan to 0 within n years)

– note that the principal repayments increase geometrically by $(1+i)^{\frac{1}{m}}$ i.e. $P_{\frac{t}{m}+n} = P_{\frac{t}{m}} \cdot (1+i)^{\frac{n}{m}}$. This is to be expected since the outstanding loan gets smaller with each principal repayment and as a result, there is less interest accruing which leaves of the amortization payment left to pay off principal.

– remember that the above example is based on an annual payment of 1. The above formulas need to be multiplied by the actual payment, $\frac{\text{Loan}}{12 \times a^{(12)}_{\overline{n}\rvert i}}$, if the original loan is not equal to $a^{(m)}_{\overline{n}\rvert i}$.

– the above loan transactions could also be reproduced if an interest rate of $\frac{i^{(m)}}{m}$ is used to coincide with the payment frequency. The monthly amortization schedule would look similar to the annual amortization schedule that was developed earlier.

6.6 Varying Series of Payments

– what happens when the payments on a loan are not level

– assume that the payment frequency and interest conversion rate still coincide

– one has to resort to using general principals in order to evaluate the interest payment, principal payment, the outstanding loan and the amortization schedule

– consider the following 4 scenarios:

(i) payments increase/decrease arithmetically
(ii) payments increase/decrease geometrically
(iii) equal amounts of principal are paid each period
(iv) payments randomly vary

Payments Increase/Decrease Arithmetically

Example

A loan is to be paid off over 10 years with the first payment at \$200, the second payment at \$190, and so on. Assuming a 5% effective rate of interest:

(a) Calculate The Loan Amount

The loan represents the present value of future payments which are $\$100+100$, $\$100+90$, $\$100+80$, etc.

$$\text{Loan } = 100a_{\overline{10}|} + 10(Da)_{\overline{10}|} = \$1,227.83$$

(b) How Much Principal and Interest is Paid In The 5^{th} Payment?

Interest payment depends on the outstanding loan balance at $t=4$ where the future payments will now be $\$100+\$60$, $\$100+\$50$, $\$100+\$40$, etc.

$$I_5 = 5\%[100a_{\overline{6}|} + 10(Da)_{\overline{6}|}] = 5\%[\$692.43] = \$34.62$$

Principal payment is payment less interest payment:

$$P_5 = \$100 + \$60 - \$34.62 = \$125.38$$

(c) Assume that the same payment pattern as above is being used to pay off 6% interest on the loan as it becomes due and the remaining payments are placed in a sinking fund that credits 5%. What is the loan amount?

In general, a loan will be equal to the accumulate value of the sinking fund payments:

$$\text{Loan} = \sum_{k=1}^{n} \left[\text{Pymt}_k - i \cdot \text{Loan}\right](1+j)^{n-k}$$

$$\text{Loan} = \sum_{k=1}^{n} \text{Pymt}_k (1+j)^{n-k} - (i \cdot \text{Loan}) s_{\overline{n|}j}$$

$$\text{Loan} = \frac{\sum_{k=1}^{n} \text{Pymt}_k (1+j)^{n-k}}{1 + i \cdot s_{\overline{n|}j}}$$

Notice how the numerator reflects the accumulated value of the actual payments while the denominator reflects the accumulated value of the interest payments.

Payments Increase/Decrease Geometrically

Example

A 10,000 loan at 10% interest is paid off with 10 end-of-year payments that increase each year by 20%.

What is the total amount of principal repaid under the first three payments?

The payments start at $\dfrac{10,000}{v_i \ddot{a}_{\overline{10}|j}} = \$720.89$ ($i = 10\%, j = \frac{1.10}{1.20} - 1$) and increase 20% each year.

Principal payment is total payment less interest payment:

$$P_1 = \text{Pymt}_1 - I_1$$
$$P_1 = \$720.84 - 10\% \times 10,000 = -\$279.11$$

In other words, the first payment could not even meet the interest that due and as a result, the loan grows.

$$\begin{aligned}
P_2 &= \text{Pymt}_2 - I_2 \\
&= 720.89(1.20) - 10\%(10,000 - P_1) \\
&= 865.07 - 10\%(10,000 + 279.11) \\
&= -\$162.84 \quad \text{loan is still growing} \\
P_3 &= \text{Pymt}_3 - I_3 \\
&= 720.89(1.20)^2 - 10\%(10,000 - P_1 - P_2) \\
&= \$1,03.08 - 10\%(10,000 + 279.11 + 162.84) \\
&= -\$6.12 \quad \text{and the loan is still growing}
\end{aligned}$$

Total principal repaid is $P_1 + P_2 + P_3 = -279.11 - 162.84 - 6.12 = -448.07$

Note the principal payments could also be calculated by simply looking at the outstanding loan at time 3 and comparing it to the original loan amount.

$$\begin{aligned}
P_1 + P_2 + P_3 &= \text{Loan} - \text{O/S Loan}_3 \\
&= 10,000 - 720.89(1.20)^3 v_i \ddot{a}_{\overline{7}|i} \\
&= 10,000 - 10,448.15 \\
&= -448.15
\end{aligned}$$

Equal Amounts of Principal Paid Each Period

Example

A 20,000 loan is repaid with 20 equal end-of-year principal payments where the principal payments are $P_1 = P_2 = \cdots = P_{20} = \dfrac{20,000}{20} = 1000$ and with 20 interest payments at 3%, $I_t = 3\% \times [20,000 - 1,000(t-1)]$.

(a) What is accumulated value of the 11^{th} to 20^{th} payments if 5% can be earned for the next 5 years and 4% can be earned, thereafter?

$$\begin{aligned}
P_{11} &= P_{12} = \cdots = P_{20} = 1000 \\
I_{11} &= 3\% \times 10,000 = 300,\ I_{12} = 270,\ \cdots I_{20} = 30 \\
FV &= 1000 s_{\overline{5}|5\%}(1.04)^5 + 1000 s_{\overline{5}|4\%} + \left[150 s_{\overline{5}|5\%} + 30(Ds)_{\overline{5}|5\%}\right](1.04)^5 + 30(Ds)_{\overline{5}|4\%} \\
&= 12,139.10 + 1,836.99 \\
&= 13,976.09
\end{aligned}$$

(b) What is the value at $t = 10$?

$$13,976.09(v^5_{4\%} \cdot v^5_{5\%}) = \$9,000.62$$

Payments Randomly Vary

Example

You pay back a loan at 12% with four end-of-year payments of $100, 100, 1000$ and 1000, respectively. These payments are to meet the annual interest due on the loan and the remainder is to go into a sinking fund earning 8%.

How much should the loan be?

$$\begin{aligned}
I_t &= \min(\text{Pymt}_t, 12\% \times \text{O/S loan}_{t-1}) \\
D_t &= \text{Pymt}_t - I_t \\
\\
I_1 &= \min(100, 12\% \times \text{Loan}) \\
D_1 &= 100 - \min(100, 12\% \times \text{Loan}) \\
I_2 &= \min\left[100, 12\% \times \text{O/S Loan}_1\right] \\
&= \min\left[100, 12\%(\text{Loan}(1.12) - I_1)\right] \\
D_2 &= 100 - \min\left[100, 12\%(\text{Loan}(1.12) - I_1)\right] \\
I_3 &= \min\left[1000, 12\% \times \text{O/S Loan}_2\right] \\
&= \min\left[1000, 12\% \times (\text{Loan}(1.12)^2 - I_1(1.12) - I_2)\right] \\
D_3 &= 1000 - \min\left[1000, 12\%(\text{loan}(1.12)^2 - I_1(1.12) - I_2)\right] \\
I_4 &= \min[1000, 12\% \times \text{O/S Loan}_3] \\
&= \min[1000, 12\% \times (\text{loan}(1.12)^3 - I_1(1.12)^2 - I_2(1+i) - I_2 - I_3)] \\
D_4 &= 1000 - \min[1000, 12\%(\text{loan}(1.12)^3 - I_1(1.12)^2 - I_2(1+i) - I_3)] \\
\text{Loan} &= D_1(1.08)^3 + D_2(1.08)^2 + D_3(1.08) + D_4
\end{aligned}$$

- note that according to I_1, the loan will have to be less than 833.33 for the interest payment to be sufficient. ($12\% \times \text{loan} < 100 \rightarrow \text{loan} < 833.33$)
- intuitively, if the interest payments were adequate, then the deposits will accumulate to the value of the loan in the sinking fund. The loan will be:

$$\text{Loan} = \frac{100 s_{\overline{2}|8\%}(1.08)^2 + 1000 s_{\overline{2}|8\%}}{1 + 12\% \cdot S_{\overline{4}|8\%}} = 1507.47$$

- therefore, it appears that the interest payments at $t = 1$ and 2 will be inadequate and that there will be no sinking fund deposits for the first 2 years.
- the loan grows for 2 years until the interest payments become adequate. Therefore the loan at $t = 2$ is equal to

$$\text{Loan}(1.12)^2 - 100 s_{\overline{2}|12\%}$$

- at $t = 4$, the loan comes due and is now determined as follows:

$$\text{Loan}(1.12)^2 - 100 s_{\overline{2}|12\%} = \frac{0 \cdot s_{\overline{2}|8\%}(1.08)^2 + 1000 s_{\overline{2}|8\%}}{1 + 12\% s_{\overline{2}|8\%}}$$

$$\text{Loan}(1.12)^2 - 100 s_{\overline{2}|12\%} = \frac{2,080.00}{1.2496}$$

$$\text{Loan} = 1,459.96$$

6.7 Amortization With Continuous Payments

– material not tested in SoA Exam FM

6.8 Step-Rate Amounts Of Principal

– material not tested in SoA Exam FM

7 Bonds and Other Securities

7.1 Introduction

– interest theory is used to evaluate the prices and values of:

1. bonds
2. equity (common stock, preferred stock)

– this chapter will show how to:

1. calculate the price of a security, given a yield rate
2. calculate the yield rate of a security, give the price

7.2 Types Of Securities

There are three common types of securities available in the financial markets:

(1) **Bonds**

– promise to pay interest over a specific term at stated future dates and then pay lump sum at the end of the term (similar qualities to an amortized loan approach).

– issued by corporations and governments as a way to raise money (i.e. borrowing).

– the end of the term is called a maturity date; some bonds can be repaid at the discretion of the bond issuer at any redemption date (callable bonds).

– interest payments from bonds are called coupon payments.

– bonds without coupons and that pay out a lump sum in the future are called accumulation bonds or zero-coupon bonds.

– if the bond is registered, then the bond owner is listed in the issuing companies' records; change in ownership must be reported.

– a mortgage bond is a bond backed by collateral; in this case, by a mortgage on real property (more secure)

– an income bond pays coupons if company had sufficient funds; no threat of bankruptcy for missed coupon payment

– junk bonds have a high risk of defaulting on payments and therefore need to offer higher interest rates to encourage investment

– a convertible bond can be converted into the common stock of the company at the option of the bond owner

– borrowers in need of a large amount money may choose to issue serial bonds that have staggered maturity dates

– Government issued bonds are called:

- Treasury Bills (if less than one year)
- Treasury Notes (if in between one year and long-term)

- Treasury Bonds (if long-term)

– Treasury Bills are valued on a discount yield basis using actual/360

Question: Find the price of a 13-week T-bill that matures for 10,000 and is bought at discount to yield 7.5%.

Solution:

$$10,000\left[1-\frac{91}{360}(7.5\%)\right]=\$9,810.42$$

(2) **Preferred Stock**

– provides a fixed rate of return (similiar to bonds); called a dividend

– ownership of stock means ownership of company (not borrowing)

– no maturity date

– for creditor purpose, preferred stock is second in line, behind bond owners (common stock is third)

– failure to pay dividend does not result in default

– cumulative preferred stock will make up for any missed dividends; regular preferred stock does not have to

– convertible preferred stock gives the owner the option of converting to common stock under certain conditions

(3) **Common Stock**

– is an ownership security, like preferred stock, but does not have fixed dividends

– level of dividend is determined by company's directors

– common stock dividends are paid after interest payments for bonds and preferred stock are paid out

– variable dividend rates means prices are more volatile than bonds and preferred stock

7.3 Price of A Bond

– like any loan, the price(value) of a bond can be determined by taking the present value of its future payments

– prices will be calculated immediately after a coupon payment has been made, or alternatively, at issue date if the bond is brand new

– let P represent the price of a bond that offers coupon payments of (Fr) and a final lump sum payment of C. The present value is calculated as follows:

$$P=(Fr)\cdot a_{\overline{n}|i}+Cv_i^n$$

– F represents the *face amount(par value)* of a bond. It is used to define the coupon payments that are to be made by the bond.

- C represents the *redemption value* of a bond. This is the amount that is returned to the bond-holder(lender) at the end of the bond's term (i.e. at the maturity date).
- r represents the *coupon rate* of a bond. This is used with F to define the bond's coupon payments. This "interest rate" is usually quoted first on a nominal basis, convertible semi-annually since the coupon payments are often paid on a semi-annual basis. This interest rate will need to be converted before it can be applied.
- (Fr) represents the semi-annual *coupon payment* of a bond.
- n is the number of coupon payments remaining or the time until maturity.
- i is the bond's *yield rate* or *yield-to-maturity.* It is the IRR to the bond-holder for acquiring this investment. Recall that the yield rate for an investment is determined by setting the present value of cash-flows-in (the purchase price, P) equal to the present value of the cash-flows-out (the coupon payments, (Fr), and the redemption value, C).
- there are four formulas that can be used in order to determine the price of a bond:
  (i) Basic Formula
  (ii) Premium/Discount Formula
  (iii) Base Amount Formula
  (iv) Makeham Formula

<u>(i) Basic Formula</u>

$$P = (Fr) \cdot a_{\overline{n}|i} + Cv_i^n$$

As stated before, a bond's price is equal to the present value of its future payments.

<u>(ii) Premium/Discount Formula</u>

$$\begin{aligned} P &= (Fr) \cdot a_{\overline{n}|i} + Cv_i^n \\ &= (Fr) \cdot a_{\overline{n}|i} + C\left(1 - i \cdot a_{\overline{n}|i}\right) \\ &= C + (Fr - Ci)\, a_{\overline{n}|i} \end{aligned}$$

If we let C loosely represent the loan amount that the bond-holder gets back, then $(Fr - Ci)$ represents how much better the actual payments, Fr, are relative to the "expected" interest payments, Ci.

When $(Fr - Ci) > 0$ the bond will pay out a "superior" interest payment than what the yield rate says to expect. As a result, the bond-buyer is willing to pay(lend) a bit more, $P - C$, than what will be returned at maturity. This extra amount is referred to as a "premium". On the other hand, if the coupon payments are less than what is expected according to the yield rate, $(Fr - Ci) < 0$, then the bond-buyer won't buy the bond unless it is offered at a price less than C or, in other words, at a "discount".

(iii) Base Amount Formula

Let G represent the *base amount* of the bond such that if multiplied by the yield rate, it would produce the same coupon payments that the bond is providing: $Gi = Fr \to G = \frac{Fr}{i}$.

The price of the bond is then calculated as follows:

$$\begin{aligned} P &= (Fr) \cdot a_{\overline{n}|i} + Cv_i^n \\ &= (Gi) \cdot a_{\overline{n}|i} + C\left(1 - i \cdot a_{\overline{n}|i}\right) \\ &= G \cdot (1 - v_i^n) + Cv_i^n \\ &= G + (C - G)\, v_i^n \end{aligned}$$

If we now let G loosely represent the loan, then the amount that the bond-holder receives at maturity, in excess of the loan, would be a bonus. As a result, the bond becomes more valuable and a bond-buyer would be willing to pay a higher price than G. On the other hand, if the payout at maturity is perceived to be less than the loan G, then the bond-buyer will not purchase the bond unless the price is less than the loan amount.

(iv) Makeham Formula

Let K represent the present value of the redemption value C ($K = Cv_i^n$). Also, let g represent the *modified coupon rate* of the bond such that if multiplied by the redemption value, C, it would produce the same coupon payments that the bond is providing: $Cg = Fr \to g = \frac{Fr}{C}$.

The price of the bond is then calculated as follows:

$$\begin{aligned} P &= (Fr) \cdot a_{\overline{n}|i} + Cv_i^n \\ &= (Cg) \cdot a_{\overline{n}|i} + K \\ &= K + (Cg)\,(1 - v_i^n) \\ &= K + \frac{g}{i}\,(C - Cv_i^n) \\ &= K + \frac{g}{i}\,(C - K) \end{aligned}$$

The difference between the future value of a payment, C and the present value of that payment, K, is the interest, or rather the present value of that interest. If the modified coupon rate is better than the yield rate, $\frac{g}{i} > 1$, then the bond's interest payments are better than what the yield rate suggests to expect. This will increase the present value of the coupon rates accordingly. On the other hand, if the modified coupon rate is less than the yield rate, $\frac{g}{i} < 1$, then the coupon payments will not meet yield rate expectations and the bond will have to be sold for less.

7.4 Premium And Discount Pricing Of A Bond

- the redemption value, C, loosely represents a loan that is returned back to the lender after a certain period of time
- the coupon payments, (Fr), loosely represent the interest payments that the borrower makes so that the outstanding loan does not grow
- let the price of a bond, P, loosely represent the original value of the loan that the bond-buyer(lender) gives to the bond-issuer(borrower)
- the difference between what is lent and what is eventually returned, $P - C$, will represent the extra value (if $P - C > 0$) or the shortfall in value (if $P - C < 0$) that the bond offers
- the bond-buyer is willing to pay(lend) more than C if he/she perceives that the coupon payments, (Fr), are better than what the yield rate says to expect, which is (Ci).
- the bond-buyer will pay less than C if the coupon payments are perceived to be inferior to the expected interest returns, $Fr < Ci$.
- the bond is priced at a premium if $P > C$ (or $Fr > Ci$) or at a discount if $P < C$ (or $Fr < Ci$).
- recall the Premium/Discount Formula from the prior section:

$$\begin{aligned} P &= C + (Fr - Ci)\, a_{\overline{n}|i} \\ P - C &= (Fr - Ci)\, a_{\overline{n}|i} \\ &= (Cg - Ci)\, a_{\overline{n}|i} \\ &= C \cdot (g - i)\, a_{\overline{n}|i} \end{aligned}$$

- in other words, if the modified coupon rate, g, is better than the yield rate, i, the bond sells at a premium. Otherwise, if $g < i$, then the bond will have to be priced at a discount.
- $g = \frac{Fr}{C}$ represents the "true" interest rate that the bond-holder enjoys and is based on what the lump sum will be returned at maturity
- i represents the "expected" interest rate (the yield rate) and depends on the price of the bond
- the yield rate, i, is inversely related to the price of the bond.
- if the yield rate, i, is low, then the modified coupon rate, g, looks better and one is willing to pay a higher price.
- if the yield rate, i, is high, then the modified coupon rate, g, is not as attractive and one is not willing to pay a higher price (the price will have to come down).

Example
Let there exist a bond such that $C = 1$. The coupon payments are therefore equal to g ($Fr = Cg = g$).

Let the price of the bond be denoted as $1 + p$ where p is the premium (if $p > 0$) or the discount (if $p < 0$).

$$\begin{aligned}
P &= (Fr) \cdot a_{\overline{n}|i} + Cv_i^n \\
1 + p &= g \cdot a_{\overline{n}|i} + (1)v_i^n \\
&= g \cdot a_{\overline{n}|i} + 1 - i \cdot a_{\overline{n}|i} \\
&= 1 + (g - i) \cdot a_{\overline{n}|i}
\end{aligned}$$

Interest Payment (Again): I_t

– using the prospective method for evaluating the value (current price) of the bond, the interest payment is derived as follows:

$$\begin{aligned}
I_t &= i \cdot \ Price_{t-1} \\
&= i \left[g \cdot a_{\overline{n-(t-1)}|i} + (1)v_i^{n-(t-1)} \right] \\
&= g \cdot \left(1 - v_i^{n-(t-1)} \right) + v_i^{n-(t-1)} \\
&= g - (g - i) \cdot v_i^{n-(t-1)}
\end{aligned}$$

– therefore, each coupon payment, g, is intended to represent a periodic interest payment plus(less) a return of portion of the premium(discount) that was made at purchase.

$$\begin{aligned}
I_t &= g - (g - i) \cdot v_i^{n-(t-1)} \\
g &= I_t + (g - i) \cdot v_i^{n-(t-1)}
\end{aligned}$$

Principal Repayment (Again): P_t

– the principal repayment in this case is equal to the coupon payment less the interest payment, $g - I_t$:

$$\begin{aligned}
P_t &= g - I_t \\
&= g - [g - (g - i) \cdot v_i^{n-(t-1)}] \\
P_t &= (g - i) \cdot v_i^{n-(t-1)}
\end{aligned}$$

Outstanding Loan Balance (Again): Book Value Of The Bond

- the bond's value starts at price P (or $1+p$, in our example) and eventually, it will become value C (or 1, in our example) at maturity.
- this value, or current price of the bond, at any time between issue date and maturity date can be determined using the prospective approach:

$$\begin{aligned} Price_t &= (Fr) \cdot a_{\overline{n-t}|i} + Cv_i^{n-t} \\ 1+p &= g \cdot a_{\overline{n-t}|i} + (1)v_i^{n-t} \\ &= g \cdot a_{\overline{n-t}|i} + 1 - i \cdot a_{\overline{n-t}|i} \\ &= 1 + (g-i) \cdot a_{\overline{n-t}|i} \end{aligned}$$

- when this asset is first acquired by the bond-buyer, its value is recorded into the accounting records, or the "books", at its purchase price. Since it is assumed that the bond will now be held until maturity, the bond's future value will continue to be calculated using the original expected rate of return (the yield rate that was used at purchase).
- the current value of the bond is often referred to as a "book value".
- the value of the bond will eventually drop or rise to value C depending if it was originally purchased at a premium or at a discount.
- an amortization schedule for this type of loan can also be developed that will show how the bond is being *written down*, if it was purchased at a premium, or how it is being *written up*, if it was purchased at a discount.
- the following bond amortization schedule illustrates the progression of the coupon payments when $C=1$ and when the original price of the bond was $1+p = 1+(g-i) \cdot a_{\overline{n}|i}$:

Period (t)	Payment	I_t	P_t	Book Value$_t$
1	g	$g-(g-i) \cdot v_i^n$	$(g-i) \cdot v_i^n$	$1+(g-i) \cdot a_{\overline{n-1}|i}$
2	g	$g-(g-i) \cdot v_i^{n-1}$	$(g-i) \cdot v_i^{n-1}$	$1+(g-i) \cdot a_{\overline{n-2}|i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	g	$g-(g-i) \cdot v_i^{n-(t-1)}$	$(g-i) \cdot v_i^{n-(t-1)}$	$1+(g-i) \cdot a_{\overline{n-t}|i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	g	$g-(g-i) \cdot v_i^2$	$(g-i) \cdot v_i^2$	$1+(g-i) \cdot a_{\overline{1}|i}$
n	g	$g-(g-i) \cdot v_i^1$	$(g-i) \cdot v_i$	1
Total	$n \cdot g$	$n \cdot g-(g-i)a_{\overline{n}|i}$	$(g-i) \cdot a_{\overline{n}|i} = p$	

- note that the total of all the interest payments is represented by the total of all coupon payments *less* what is being returned as the premium (or *plus* what is being removed as discount since the loan is appreciating to C).

$$\begin{aligned}\sum_{k=1}^{n} I_k &= \sum_{k=1}^{n} g - (g-i)\cdot v_i^{n-(t-1)} \\ &= \sum_{k=1}^{n} g - \sum_{k=1}^{n}(g-i)\cdot v_i^{n-(k-1)} \\ &= ng - (g-i)a_{\overline{n}|i} \\ &= ng - p\end{aligned}$$

- note that the total of all the principal payments must equal the premium/discount

$$\sum_{k=1}^{n} P_k = \sum_{k=1}^{n}(g-i)v_i^{n-(k-1)} = (g-i)a_{\overline{n}|i} = p$$

- note that the book value at $t = n$ is equal to 1, the last and final payment back to the bond-holder.
- note that the principal (premium) repayments increase geometrically by $\left(1+\frac{i^{(2)}}{2}\right)$ i.e. $P_{t+n} = P_t\left(1+\frac{i^{(2)}}{2}\right)^n$.

- remember that the above example is based on a redemption value of $C = 1$. The above formulas need to be multiplied by the actual redemption value if C is not equal to 1

Straight Line Method

If an approximation for writing up or writing down the bond is acceptable, then the principal payments can be defined as follows:

$$P_t = \frac{P - C}{n}.$$

or $P_t = \dfrac{1 + p - 1}{n} = \dfrac{p}{n}$, if $C = 1$.

I_t will then be the coupon payment less the interest payment:

$$I_t = (Fr) - P_t = (Fr) - \frac{P - C}{n}.$$

or $I_t = g - \dfrac{p}{n}$, if $C = 1$.

The book value of the bond, $Price_t$, will then be equal the original price *less* the sum of the premium repayments made to date:

$$Price_t = P - \sum_{k=1}^{t} P_t = P - t\left(\frac{P - C}{n}\right).$$

or $Price_t = 1 + p - t\dfrac{p}{n} = 1 + \left(\dfrac{n - t}{n}\right)p$, if $C = 1$.

The bond amortization table would then be developed as follows:

Period (t)	Payment	I_t	P_t	Book Value$_t$
1	g	$g - \frac{p}{n}$	$\frac{p}{n}$	$1 + \left(\frac{n-1}{n}\right)p$
2	g	$g - \frac{p}{n}$	$\frac{p}{n}$	$1 + \left(\frac{n-2}{n}\right)p$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	g	$g - \frac{p}{n}$	$\frac{p}{n}$	$1 + \left(\frac{n-t}{n}\right)p$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	g	$g - \frac{p}{n}$	$\frac{p}{n}$	$1 + \left(\frac{1}{n}\right)p$
n	g	$g - \frac{p}{n}$	$\frac{p}{n}$	1
Total	ng	$ng - p$	$\frac{p}{n} \cdot n = p$	

7.5 Valuation Between Coupon Payment Dates

- up to now, bond prices and book values have been calculated assuming the coupon has just been paid.
- let B_t be the bond price (book value) just after a coupon payment has been made.

$$B_t = Fr.a_{\overline{n-t}|i} + C\, v_i^{n-t} = B_{t-1}(1+i) - Fr$$

- when buying an existing bond between its coupon dates, one must decide how to split up the coupon between the prior owner and the new owner.
- let Fr_k represent the amount of coupon (accrued coupon) that the prior owner is due where $0 < k < 1$
- the price of the bond (flat price) to be paid to the prior owner at time $t+k$ should be based on the market price of the bond and the accrued coupon

$$B^f_{t+k} = B^m_{t+k} + Fr_k$$

- notice that the market value of the bond, B^m_{t+k}, will only recognize future coupon payments; hence, the reason for a separate calculation to account for the accrued coupon, Fr_k.
- k may be calculated on an *actual/actual* or 30/360 basis if days are to be used.
- there are three methods used to compute the market price:

$$B^m_{t+k} = B^f_{t+k} - Fr_k$$

- (1) **<u>Theoretical Method</u>**
  - flat price is equal to the bond value as at the last coupon payment date, carried forward with compound interest.

$$B^f_{t+k} = B_t(1+i)^k$$

  - accrued coupon is equal to the coupon payment, prorated by the compound interest earned to date versus the coupon paying period

$$Fr_k = Fr \cdot \left[\frac{(1+i)^k - 1}{i}\right]$$

  - market price is equal to:

$$B^m_{t+k} = B_t(1+i)^k - Fr \cdot \left[\frac{(1+i)^k - 1}{i}\right]$$

- (2) **<u>Practical Method</u>**
  - flat price is is equal to the bond value as at the last coupon payment date, carried forward with simple interest.

$$B^f_{t+k} = B_t(1 + k \cdot i) = [(1-k)B_t + k \cdot B_{t+1}] + k \cdot Fr$$

- accrued coupon is equal to the face coupon, prorated by the simple interest earned to date versus the coupon paying period.

$$Fr_k = \left[\frac{1+ik-1}{i}\right] \cdot Fr = k \cdot Fr$$

- market price is equal to:

$$B^m_{t+k} = B_t(1+ki) - k \cdot Fr = (1-k)B_t + k \cdot B_{t+1}$$

- (3) **Semi-Theoretical Method**
  - flat price is the same as (1) Theoretical Method.

$$B^f_{t+k} = B_t(1+i)^k$$

  - accrued coupon is the same as (2) Practical Method.

$$Fr_k = k \cdot Fr$$

  - market price is equal to:

$$B^m_{t+k} = B_t(1+i)^k - k \cdot Fr$$

  - small problem with this method is that when $i = g$ and $P = C$, it should produce constant bond prices (it does not).
  - most widely used method in practice.

- the amount of premium or discount that exists between coupon payment dates is calculated using the market price (book value), not the flat price.

$$\begin{aligned} \text{Premium} &= B^m_{t+k} - C, \;\; \text{if } g > i \\ \text{Disount} &= C - B^m_{t+k}, \;\; \text{if } i > g \end{aligned}$$

7.6 Determination Of Yield Rates

- given a purchase price of a security, the yield rate can be determined under a number of methods.
  Problem:
  What is the yield rate convertible semi-annually for a \$100 par value 10-year bond with 8% semi-annual coupons that is currently selling for \$90?
  Solution:

1. Use an Society of Actuaries recommended calculator with built in financial functions:

$$\boxed{\text{PV}} = 90, \; \boxed{\text{N}} = 2 \times 10 = 20, \; \boxed{\text{FV}} = 100, \; \boxed{\text{PMT}} = \frac{8\%}{2} \times 100 = 4,$$

$$\boxed{\text{CPT}}\boxed{\%\text{i}} \rightarrow 4.788\% \times 2 = 9.5676\%$$

2. Do a linear interpolation with bond tables (not a very popular method anymore).

3. Develop an appropriate formula for the yield rate.

$$
\begin{aligned}
P &= C + (Fr - C\,i)a_{\overline{n}|i} \\
i &= \frac{Fr}{C} - \frac{P-C}{C \cdot a_{\overline{n}|i}} \\
i &= \frac{Fr}{C} - \frac{P-C}{C}\left[\frac{1}{n}\left(1 + \frac{n+1}{2}i + \frac{n^2-1}{12}i^2 + \cdots\right)\right] \\
i &\approx \frac{Fr}{C} - \frac{P-C}{C}\left[\frac{1}{n} + \frac{n+1}{2n}i\right] \\
i &\approx \frac{Fr - \frac{P-C}{n}}{C + \left(\frac{n+1}{2n}\right)(P-C)} \\
&= \frac{g - \frac{k}{n}}{1 + \left(\frac{n+1}{2n}\right)k} \quad \text{where } k = \frac{P-C}{C}
\end{aligned}
$$

$$\therefore \quad \frac{4 - \left(\frac{-10}{20}\right)}{100 + \frac{21}{40}(-10)} = 4.749\% \times 2 = 9.498\%$$

- remember that interest rates are determined by dividing the interest earned for the period by the average value of the investment
- the numerator represents the earned interest for the period, which will be the coupon less (plus) the average premium (discount) repayment (appreciation).
- the denominator represents the estimated value of the investment for the period which is the redemption value plus (minus) the average remaining premium (discount)
- the bond salesmen method replaces $\frac{n+1}{2n}$ with $\frac{1}{2}$, but will not be as accurate

4. Use an iteration method as developed in Section 2.7
   - rearrange the bond price formula to solve for i

$$
\begin{aligned}
P &= C + C\,(g - i)\,a_{\overline{n}|i} \\
i &= g - \left(\frac{P-C}{C}\right)/a_{\overline{n}|i} \\
f(i) &= i - g + \left(\frac{P-C}{C}\right)/a_{\overline{n}|i} = 0
\end{aligned}
$$

   - find $f(i^+) > 0$ and $f(i^-) < 0$; use the appropriate formula in (3) to find starting value, $i_0 = 0.0475$

$$
\begin{aligned}
i = 0.0475 &\rightarrow f(i) = -0.00036 \\
i = 0.04790 &\rightarrow f(i) = 0.00002
\end{aligned}
$$

$$
\begin{aligned}
&f(0.04749) = -0.00036 \\
&f(i) = 0 \\
&f(0.04790) = 0.00002
\end{aligned}
$$

$$i_1 = i_0 + (0.04790 - 0.04749)\left(\frac{0 - (-0.00036)}{0.00002 - (-0.00036)}\right) = 0.04788$$

–

$$\begin{aligned}
f(0.04749) &= -0.00036 \\
f(i) &= 0 \\
f(0.04788) &= 0.00000 \\
i_2 &= 0.04788
\end{aligned}$$

5. Use Newton-Raphson iteration method
   - rearrange the bond price formula such that $f(i) = 0$

$$\begin{aligned}
P &= C + (Fr - C\,i)\,a_{\overline{n}|i} \\
f(i) &= P - C - (Fr - C\,i)\,a_{\overline{n}|i} = P - Fr \cdot a_{\overline{n}|i} - C \cdot v_i^n \\
f'(i) &= \frac{d}{di}(P - Fr \cdot a_{\overline{n}|i} - C \cdot v_i^n) = -Fr \cdot \left(\frac{n\,v_i^{n+1} - a_{\overline{n}|i}}{i}\right) + n \cdot C \cdot v_i^{n+1}
\end{aligned}$$

   - apply Newton-Raphson formula

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
i_{s+1} &= i_s - \frac{P - Fr \cdot a_{\overline{n}|i_s} - C \cdot v_{i_s}^n}{-Fr \cdot \left(\frac{n\,v_{i_s}^{n+1} - a_{\overline{n}|i_s}}{i_s}\right) + n \cdot C \cdot v_{i_s}^{n+1}} \\
&= i_s \left[1 - \frac{P - Fr \cdot a_{\overline{n}|i_s} - C \cdot v_{i_s}^n}{-Fr \cdot \left(n\,v_{i_s}^{n+1} - a_{\overline{n}|i_s}\right) + n \cdot C \cdot i_s \cdot v_{i_s}^{n+1}}\right] \\
&= i_s \left[1 + \frac{Fr \cdot a_{\overline{n}|i_s} + C \cdot v_{i_s}^n - P}{-Fr \cdot \left(n\,v_{i_s}^{n+1} - a_{\overline{n}|i_s}\right) + n \cdot C \cdot i_s \cdot v_{i_s}^{n+1}}\right]
\end{aligned}$$

7.7 Callable Bonds

- this a bond where the issuer can redeem the bond prior to the maturity date if they so choose to; this is called a *call date*.
- the challenge in pricing callable bonds is trying to determine the most likely call date
- assuming that the redemption date is the same at any call date, then
  - (i) the call date will most likely be at the earliest date possible if the bond was sold at a premium (issuer would like to stop repaying the premium via the coupon payments as soon as possible).
  - (ii) the call date will most likely be at the latest date possible if the bond was sold at a discount (issuer is in no rush to pay out the redeption value).
- when the redemption date is not the same at every call date, then one needs to examine all possible call dates.

Example
A \$100 par value 4% bond with semi-annual coupons is callable at the following times:

\$109.00, 5 to 9 years after issue

\$104.50, 10 to 14 years after issue

\$100.00, 15 years after issue.

Question: What price should an investor pay for the callable bond if they wish to realize a yield rate of (1) 5% payable semi-annually and (2) 3% payable semi-annually?

Solution:

(1) Since the market rate is better than the coupon rate, the bond would have to be sold at a discount and as a result, the issuer will wait until the last possible date to redeem the bond:

$$P = \$2.00 \cdot a_{\overline{30}|2.5\%} + \$100.00 \cdot v^{3}_{2.5\%}0 = \$89.53$$

(2) Since the coupon rate is better than the market rate, the bond would sell at a premium and as a result, the issuer will redeem at the earliest possible date for each of the three different redemption values:

$$P = \$2.00 \cdot a_{\overline{10}|1.5\%} + \$109.00 \cdot v^{1}_{1.5\%}0 = \$112.37$$
$$P = \$2.00 \cdot a_{\overline{10}|1.5\%} + \$104.50 \cdot v^{1}_{1.5\%}0 = \$111.93$$
$$P = \$2.00 \cdot a_{\overline{10}|1.5\%} + \$100.00 \cdot v^{1}_{1.5\%}0 = \$112.01$$

In this case, the investor would only be willing to pay \$111.93.

Note that the excess of the redemption value over the par value is referred to as a *call premium* and starts at \$9.00, before dropping to \$4.50, before dropping to \$0.00.

7.8 Serial Bonds

– **material not tested in SoA Exam FM**

7.9 Some Generalizations

– **material not tested in SoA Exam FM**

7.10 Other Securities

Other types of securities are available in the financial markets which do not offer redemption values:
(1) **Preferred Stock and Perpetual Bonds**

– issued by corporations i.e. borrowing, but not paying back the principal.

– promises to pay interest forever at stated future dates.

– interest payments can be considered coupon payments.

– Price is equal to the present value of future coupon payments at a given yield rate i.

$$P = Fr \cdot a_{\overline{\infty}|i} = \frac{Fr}{i}$$

(2) **Common Stock**

– issued by corporations i.e. borrowing, but not paying back the principal.

– pays out annually a dividend, D, rather than interest with no requirement to guarantee payments. Also the dividend can be in any amount (not fixed income).

– an assumption is required with respect to the annual growth rate of dividends, k.

– Price is equal to the present value of future dividends at a given yield rate i and as given growth rate, k.

– the techqiques to be used are exactly the same as those methods presented in Section 4.6, Payments Varying In Geometric Progressing.

$$P = v_i \left(D \cdot \ddot{a}_{\overline{\infty}|i'=\frac{1+i}{1+k}-1} \right) = \frac{D}{i-k}$$

Example
Assuming an annual effective yield rate of 10%, calculate the price of a common stock that pays a \$2 annual dividend at the end of every year and grows at 5% for the first 5 years, 2.5% for the next 5 years and 0%, thereafter:

$$\begin{aligned} P &= v_{10\%} \left(\$2 \cdot \ddot{a}_{\overline{5}|i'=\frac{1+10\%}{1+5\%}-1} \right) \\ &\quad + v_{10\%}^{5} \left[v_{10\%} \left(\$2(1+5\%)^5 \cdot \ddot{a}_{\overline{5}|i'=\frac{1+10\%}{1+2.5\%}-1} \right) \right] \\ &\quad + v_{10\%}^{10} \left[v_{10\%} \left(\$2(1+5\%)^5(1+2.5\%)^5 \cdot \ddot{a}_{\overline{\infty}|i'=\frac{1+10\%}{1+0\%}-1} \right) \right] \\ P &= 8.30 + 6.29 + 11.13 = 25.72 \end{aligned}$$

7.11 Valuation Of Securities

– **material not tested in SoA Exam FM**

8 Practical Applications

8.1 Introduction

– this chapter will look at 3 three practical applications of interest theory:

(i) Depreciation Of A Fixed Asset

(ii) Capitalized Cost Of A Fixed Asset

(iii) Short Selling

8.2 Truth In Lending

– **material not tested in SoA Exam FM**

8.3 Real Estate Mortgages

– **material not tested in SoA Exam FM**

8.4 Approximate Methods

– **material not tested in SoA Exam FM**

8.5 Depreciation Methods

– **material not tested in SoA Exam FM**

– a fixed asset is purchased by an individual or firm for business and/or investment purposes i.e. manufacturing equipment, real estate, etc..

– therefore, this asset will have a yield rate

– a fixed asset will usually have a salvage value at the end of its useful life.

– therefore, there is an annual cost that will be incurred for holding this asset. We will call this annual cost *depreciation* since the value of the asset drops from its original price to it salvage value

– assume that the fixed asset will be replaced at the end of its useful life. The capital required to make a new purchase will have been saved up via a sinking fund. Note that a sinking fund does not have to actually exist.

Asset Returns
Let R represent the asset's annual return which will reflect the yield rate, i, on the original investment, A, and the cost for having to save up for another asset

$$R = i \cdot A + \frac{A - S}{s_{\overline{n}|j}}$$

Book Value
The asset loses its value over time until it reaches its salvage value. We define the book value of the asset at time t as B_t and examine how the book value changes over time.

Depreciation

Let D_t represent the asset's drop in book value during year t, or in other words, its depreciation:

$$D_t = B_{t-1} - B_t$$

There are 4 different ways in which the pattern of depreciation (and as a result, the book value) can be defined:

(i) Sinking Fund (Compound Interest) Method
(ii) Straight Line Method
(iii) Declining Balance (Constant Percentage, Compound Discount) Method
(iv) Sum Of The Years Digits Method

(i) Sinking Fund Method

- book value equals the initial value of the asset *less* the amount in the sinking fund

$$B_t = A - \left(\frac{A-S}{s_{\overline{n}|j}}\right) s_{\overline{t}|j}$$

- depreciation for year t is defined as the change in book value during year t

$$\begin{aligned} D_t &= B_{t-1} - B_t \\ &= A - \left(\frac{A-S}{s_{\overline{n}|j}}\right) s_{\overline{t-1}|j} - \left[A - \left(\frac{A-S}{s_{\overline{n}|j}}\right) s_{\overline{t}|j}\right] \\ &= \left(\frac{A-S}{s_{\overline{n}|j}}\right)\left(s_{\overline{t}|j} - s_{\overline{t-1}|j}\right) \\ &= \left(\frac{A-S}{s_{\overline{n}|j}}\right)(1+j)^{t-1} \end{aligned}$$

- as can be seen from the above formula, depreciation increases over time, slowly at first

(ii) Straight Line Method

- let depreciation be the same fixed amount for every year and let n represent the useful life of the fixed asset

$$D_t = \frac{A-S}{n}$$

- depreciation for year t is defined as the change in book value during year t and as a result, the book value now becomes a linear function of the depreciation cost

$$\begin{aligned} D_t &= B_{t-1} - B_t \\ \frac{A-S}{n} &= A - (t-1)D_{t-1} - B_t \\ &= A - (t-1)\left(\frac{A-S}{n}\right) - B_t \\ B_t &= A - \frac{t-1}{n}\cdot A + \frac{t-1}{n}\cdot S - \frac{A}{n} + \frac{S}{n} \\ B_t &= \left(1-\frac{t}{n}\right)A + \left(\frac{t}{n}\right)S \end{aligned}$$

- note that if $j = 0$ under the Sinking Fund Method, then you would have the Straight Line Method

(iii) Declining Balance Method

- let the rate of depreciation be a constant percentage of the book value and define this rate as d.

$$D_t = d \cdot B_{t-1}$$

- in this case, the annual depreciation amount is large at first and decreases with time
- the salvage value can now be defined as:

$$S = A(1-d)^n$$

and the book value can now be defined as:

$$B_t = A(1-d)^t$$

- the constant rate of depreciation, d can be determined as follows:

$$\begin{aligned} S &= A(1-d)^n \\ \frac{S}{A} &= (1-d)^n \\ d &= 1 - \left(\frac{S}{A}\right)^{\frac{1}{n}} \end{aligned}$$

- a variation of d is sometimes used:

$$d' = \frac{k}{n}$$

where k represents a multiple of the Straight Line Method but ignores the salvage value. For example, an asset has a 5-year life, which means on a straight line basis it loses 20% ($\frac{1}{5}$) of its original value each year. If k were set at 200%, then $d' = 40\%$.

(iv) Sum Of The Years Digits Method

– we now wish for depreciation to decrease arithmetically over time such that:

$$\begin{aligned}
D_1 &= \frac{n}{S_n}(A-S) \\
D_2 &= \frac{n-1}{S_n}(A-S) \\
&\vdots \\
D_t &= \frac{n-(t-1)}{S_n}(A-S) \\
&\vdots \\
D_{n-1} &= \frac{2}{S_n}(A-S) \\
D_n &= \frac{1}{S_n}(A-S)
\end{aligned}$$

– since the sum of the depreciation charges must equal $A-S$, S_n can be determined as follows:

$$\begin{aligned}
\sum_{k=1}^{n} D_k &= (A-S) \\
\sum_{k=1}^{n} \frac{n-(k-1)}{S_n}(A-S) &= (A-S) \\
\sum_{k=1}^{n} \frac{n-(k-1)}{S_n} &= 1 \\
n^2 - [0+1+2+\cdot+n-1] &= S_n \\
n^2 - \frac{(n-1)n}{2} &= S_n \\
n^2 - \frac{n^2}{2} + \frac{n}{2} &= S_n \\
\frac{n^2}{2} + \frac{n}{2} &= S_n \\
\frac{n^2+n}{2} &= S_n \\
\frac{n(n+1)}{2} &= S_n
\end{aligned}$$

– let the book value be defined as the original value of the asset *less* the total depreciation to date:

$$
\begin{aligned}
B_t &= A - \sum_{k=1}^{t} D_k \\
&= A - \sum_{k=1}^{t} D_k + \sum_{k=t+1}^{n} D_k - \sum_{k=t+1}^{n} D_k \\
&= A - \sum_{k=1}^{n} D_k + \sum_{k=t+1}^{n} D_k \\
&= S + \sum_{k=t+1}^{n} D_k \\
&= S + \sum_{k=t+1}^{n} \frac{n-(k-1)}{S_n}(A-S) \\
&= S + \frac{(n-t)+(n-t-1)+\cdots+2+1}{S_n}(A-S) \\
&= S + \frac{S_{n-t}}{S_n}(A-S)
\end{aligned}
$$

8.6 Capitalized Cost

– **material not tested in SoA Exam FM**

– there are 3 annual costs when owning a fixed asset

(i) opportunity costs: lost interest $= i \cdot A$

(ii) depreciation costs: $\dfrac{A-S}{s_{\overline{n}|j}}$ (if Sinking Fund Method)

(i) maintenance costs: M

– let H represent the periodic charge (i.e. the annual cost) for owning a fixed asset

$$H = i \cdot A + \frac{A-S}{s_{\overline{n}|j}} + M$$

– if we assume that the fixed asset will always be replaced, then the periodic charge will be experienced in perpetuity. Let K be defined as the *capitalized cost* of the fixed asset where K represents the present value of future periodic charges:

$$\begin{aligned} K &= H \cdot a_{\overline{\infty}|i} \\ K &= H \cdot \frac{1}{i} \\ K &= A + \frac{A-S}{i \cdot s_{\overline{n}|j}} + \frac{M}{i} \end{aligned}$$

– to compare the capitalized costs of alternative fixed assets, one needs to compare the capitalized costs on a per unit basis. For example, if machine 1 produces U_1 units per time period and if machine 2 produces U_2 units per time period, then the machines are considered equivalent if:

$$\frac{K_1}{U_1} = \frac{K_2}{U_2}$$

8.7 Short Sales

- supposed that you are an equity investor and you think that a stock is over-priced and will start to drop in the future
- you offer to sell the stock today at a certain price and promise to deliver the stock at a later date. However, you never owned the stock when you made your sale and now you hope that the you will be able to buy the stock later at a lower price
- your profit would be equal to the selling price *less* the purchase price
- this type of sale is called a "short sale" because you sell stock that you don't hold i.e. you are short of it (a "long sale" would be where you go and buy the stock now and sell it later, hoping that the price will increase)
- the risk under a short-sale is that if the future price goes up, you might find that you do not have enough money to pay for the stock and delivery may not occur. Government regulators typically require that the short-seller to put up collateral (i.e. 50% of the selling price) to help back their promise.
- the collateral is called a *margin* and will be returned to the short-seller when the delivery is made. Usually, the margin is deposited into an interest bearing account; however, the original sale proceeds are placed in a non-interest bearing account
- if the stock to be delivered has also paid a dividend during the waiting period, then the short-seller must also deliver the dividend

Example

An investor sells a stock short at the beginning of the year for 1000 and at the end of the year is able to buy it for 800 and deliver it. A margin of 50% was required and was placed into an account that credited interest at 8%. The stock itself paid out a dividend of 60 at the end of the year.

What was the yield rate to the short-seller?

The yield rate is defined as $\frac{\text{profit}}{\text{investment}}$.

The investment, is the margin that the short-seller had to provide:

$$\text{investment} = 50\% \times 1000 = 500$$

while the profit, is the gain on sale *plus* the interest earned on the margin *less* the dividend:

$$\begin{aligned} \text{profit} &= (1000 - 800) + 8\% \times (50\% \times 1000) - 60 \\ \text{profit} &= 180 \end{aligned}$$

The yield rate is therefore, $\frac{180}{500} = 36\%$.

Had there been no dividend payable, then the yield rate would have been $\frac{240}{500} = 48\%$.

8.8 Modern Financial Instruments

Introduction

- this section describes a number of investment alternatives versus the bond and equity investments that were described in Chapter 7

Money Market Funds (MMF)

- provide high liquidity and attactive yields; some allow cheque writing
- contains a variety of short-term, fixed-income securities issued by governments and private firms
- credited rates fluctuate frequently with movements in short-term interest rates
- investors will "park" their money in an MMF while contemplating their investment options

Certificate of Deposits (CD)

- rates are guaranteed for a fixed period of time ranging from 30 days to 6 months
- higher denominations will usually credit higher rates of interest
- yield rates are usually more stable than MMF's but less liquid
- withdrawal penalties tend to encourage a secondary trading market rather than cashing out

Guaranteed Investment Contracts (GIC)

- issued by insurance companies to large investors
- similiar to CD's; market value does not change with interest rate movements
- GIC might allow for additional deposits and can offer insurance contract features i.e. annuity purchase options
- interest rates higher than CD's; closer to Treasury securities
- banks compete with their "bank investment contracts" (BIC)

Mutual Funds

- pooled investment accounts; an investor buys shares in the fund
- offers more diversification than what an individual can achieve on their own

Mortgage Backed Securities (MBS)

- pooled real estate mortgages owned by government mortgage associations or corporations
- periodic payments comprise both principal and interest; principal payments depend on how much principal from underlying mortgage is being paid back

Collateralized Mortgage Obligations (CMO)

- similiar to MBS, but are designed to lessen the cash flow uncertainty that come with MBS's
- no specific maturity dates; based on "average life" that assumes a reasonable prepayment schedule

– offers higher yields than corporate bonds due to uncertainty of cash flows (also makes payments more frequently than bonds)

– very active trading market (very liquid)

– prices vary inversely with interest rates (just like bonds)

Options (Derivative Instrument: its value depends on the marketplace)

– a contract that allows the owner to buy or sell a security at a fixed price at a future date

– call option gives the owner the right to buy; put option gives the owner the right to sell

– European option can be used on a fixed date; American option can be used any time until its expiry date

– investors will buy(sell) call options or sell(buy) put options if they think a security's price is going to rise (fall)

– one motivation for buying or selling options is speculation; option prices depend on the value of the underlying asset (leverage)

– another motivation (and quite opposite to speculation) is developing hedging strategies to reduce investment risk (see Section 8.7 Short Sales)

– a warrant is similiar to a call option, but has more distinct expiry dates; the issuing firm also has to own the underlying security

– a convertible bond may be considered the combination of a regular bond and a warrant

Futures (Derivative Instrument: its value depends on the marketplace)

– this is a contract where the investor agrees, at issue, to buy or sell an asset at a fixed date (delivery date) at a fixed price (futures price)

– the current price of the asset is called the spot price

– an investor has two investment alternatives:

  (i) buy the asset immediately and pay the spot price now, or

  (ii) buy a futures contract and pay the futures price at the delivery date; earn interest on the money deferred, but lose the opportunity to receive dividends/interest payments

Forwards (Derivative Instrument: its value depends on the marketplace)

– similiar to futures except that forwards are tailored made between two parties (no active market to trade in)

– banks will buy and sell forwards with investors who want protection for currency rate fluctuations for one year or longer

– banks also sell futures to investors who wish to lock-in now a borrowing interest rate that will be applied to a future loan

– risk is that interest rates drop in the future and the investor is stuck with the higher interest rate

Swaps (Derivative Instrument: its value depends on the marketplace)

– a swap is an exchange of two similiar financial quantities

– for example, a change in loan repayments from Canadian dollars to American dollars is called a currency swap (risk depends on the exchange rate)

– an interest rate swap is where you agree to make interest payments based on a variable loan rate (floating rate) instead of on a predetermined loan rate (fixed rate)

9 More Advanced Financial Analysis

9.1 Introduction

- interest/inflation rates based on actual past experience are often called ex post rates.
- interest/inflation rates expected to occur in the future are often called ex ante rates.
- interest/inflation rates in existence at the present time are often called current or market rates

9.2 An Economic Rationale for Interest

- **material not tested in SoA Exam MF**

9.3 Determinants of the Level of Interest Rates

- **material not tested in SoA Exam MF**

9.4 Recognition of Inflation

Interest Rates and Inflation

- are assumed to move in the same direction over time since lenders will charge higher interest rates to make up for the loss of purchasing power due to higher inflation.
- the relationship is actually between the current rate of interest and the expected (not current) rate of inflation.

Real Rate of Interest

- let i' represent the real rate of interest and let r represent the rate of inflation where

$$i' = \frac{1+i}{1+r} - 1$$

Calculating Present and Future Values

- the techqiques to be used are exactly the same as those methods presented in Section 4.6, Payments Varying In Geometric Progressing.
- in this case, geometric increasing rate k is replaced by annual inflation rate, r, and adjusted interest rate j is replaced by real rate of interest, i'.

9.5 Reflecting Risk and Uncertainty

- **material not tested in SoA Exam MF**

9.6 Yield Curves

- usually long-term market interest rates are higher than short-term market interest rates.
- there are four theories that try to explain why the market requires interest rates to increase with the investment period.

  (1) **Expectations Theory**

  - there are more individuals and businesses that think interest rates will rise in the future than whose who think that they will fall.

  (2) **Liquidity Preference Theory**

  - investors prefer short-term periods as it keeps their money accessible (liquid) for possible opportunities.
  - higher interest rates for longer-term investments must be offered in order to entice investors to commit their funds for longer periods.

  (3) **Inflation Premium Theory**

  - investors are uncertain about future inflation rates and will want higher interest rates on longer-term investments to reduce their fears.
  - longer-term assets are more affected by interest rate movements than shorter-term assets (interest rates respond to expected inflation rates).

Spot Rates

- are the annual interest rates that make up the yield curve.
- let i_t represent the spot rate for period t.
- let $P(i_*)$ represent the net present value of a series of future payments (positive or negative) discounted using spot rates:

$$P(i_*) = \sum_{t=0}^{n} (1 + i_t)^{-t} \cdot R_t$$

Forward Rates

- are considered to be future reinvestment rates.

Example:
A firm wishes to borrow money repayable in two years, where the one-year and two-year spot rates are 8% and 7%, respectively.

The estimated one-year deferred one-year spot rate is called the forward rate, f, and is calculated by equating the two interest rates such that

$$(1.08)^2 = (1.07)(1 + f) \rightarrow f = 9.01\%$$

If the borrower thinks that the spot rate for the 2^{nd} year will be greater(less) than the forward rate, then they will select (reject) the 2-year borrowing rate.

9.7 Interest Rate Assumptions

– **material not tested in SoA Exam MF**

9.8 Duration

– how do you value two bonds that have the same maturity date and same yield rate, but have different coupon payments? How do you determine the average payment period?

– there are three indices that can be used to measure the average length of time of an investment in order to consider reinvestment risk:

(1) **Term-to-Maturity**

– crude index that states how long a bond has to mature.

– does not distinguish between the bonds other than by maturity.

(2) **Method of Equated Time**

– calculates the weighted average of the time payments for each bond by weighting with the individual payment amounts.

– in other words, solve for $\bar{t} = \dfrac{\sum_{t=1}^{n} t \cdot R_t}{\sum_{t=1}^{n} R_t}$ as described in Section 2.6.

(3) **Duration**

– calculates the weighted average of time payments for each bond by weighting with the present value of the individual payment amounts.

– let $\bar{d}$ represent the duration of a series of payments and be calculated as:

$$\bar{d} = \frac{\sum_{t=1}^{n} t \cdot v^t \cdot R_t}{\sum_{t=1}^{n} v^t \cdot R_t}$$

- <u>Theorem:</u> duration is a decreasing function of i.

- <u>Proof:</u> (where $\frac{d}{di}v^t = \frac{d}{di}(1+i)^{-t} = -t \cdot (1+i)^{-t-1} = -t \cdot v^{t+1}$)

$$
\begin{aligned}
\frac{d}{di}\bar{d} &= \frac{d}{di}\frac{\sum_{t=1}^{n} t \cdot v^t \cdot R_t}{\sum_{t=1}^{n} v^t \cdot R_t} \\
&= \frac{\sum_{t=1}^{n} v^t \cdot R_t \cdot \left[\sum_{t=1}^{n} -t^2 \cdot v^{t+1} \cdot R_t\right] - \sum_{t=1}^{n} t \cdot v^t \cdot R_t \cdot \left[\sum_{t=1}^{n} -t \cdot v^{t+1} \cdot R_t\right]}{\left[\sum_{t=1}^{n} v^t \cdot R_t\right]^2} \\
&= -v \cdot \left[\underbrace{\frac{\sum_{t=1}^{n} t^2 \cdot v^t \cdot R_t}{\sum_{t=1}^{n} v^t \cdot R_t}}_{E(\bar{d}^2)} - \underbrace{\left(\frac{\sum_{t=1}^{n} t \cdot v^t \cdot R_t}{\sum_{t=1}^{n} v^t \cdot R_t}\right)^2}_{[E(\bar{d})]^2} \right] \\
&= -v \cdot \sigma_{\bar{d}}^2 < 0
\end{aligned}
$$

- $\bar{d}$, regular duration, is sometimes referred to as *Macaulay Duration*.

- we should pay attention as to how a bond's price can change given an interest rate change. In other words, how volatile is a bond's price?
- volatility of the present value of future payments can be denoted as $\bar{v}$ where

$$
\bar{v} = -\frac{\frac{d}{di}Price}{Price} = -\frac{\frac{d}{di}\sum_{t=1}^{n} v^t \cdot R_t}{\sum_{t=1}^{n} v^t \cdot R_t}
$$

- $\bar{v}$ simplifies down to :

$$
\bar{v} = \frac{\bar{d}}{1+i}
$$

- volatility, $\bar{v}$, is also referred to as *Modified Duration*.

Example of Duration:

Assuming an interest rate of 8%, calculate the duration of:

(1) An n-year zero coupon bond:

$$\bar{d} = \frac{\sum_{t=n}^{n} t \cdot v^t \cdot R_t}{\sum_{t=n}^{n} v^t \cdot R_t} = \frac{n \cdot v^n \cdot R_n}{v^n \cdot R_n} = n$$

(2) An n-year bond with 8% coupons:

$$\bar{d} = \frac{\sum_{t=1}^{n} t \cdot v^t \cdot R_t}{\sum_{t=1}^{n} v^t \cdot R_t} = \frac{8\% \cdot (Ia)_{\overline{n}|i} + 10v_i^n}{8\% \cdot a_{\overline{n}|i} + v_i^n}$$

(3) An n-year mortgage repaid with level payments of principal and interest:

$$\bar{d} = \frac{\sum_{t=1}^{n} t \cdot v^t \cdot R_t}{\sum_{t=1}^{n} v^t \cdot R_t} = \frac{(Ia)_{\overline{n}|i}}{a_{\overline{n}|i}}$$

(4) A preferred stock paying level dividends into perpetuity:

$$\bar{d} = \frac{\sum_{t=1}^{\infty} t \cdot v^t \cdot R_t}{\sum_{t=1}^{\infty} v^t \cdot R_t} = \frac{(Ia)_{\overline{\infty}|i}}{a_{\overline{\infty}|i}}$$

9.9 Immunization

– it is very diffcult for a financial enterprise to match the cash flows of their assets to the cash flows of their liabilities.

– especially when the cash flows can change due to changes in interest rates.

A Problem with Interest Rates

– a bank issues a one-year deposit and guarantees a certain rate of return.

– if interest rates have gone up by the end of the year, then the deposit holder will not renew if the guaranteed rate is too low versus the new interest rate.

– the bank will need to pay out to the deposit holder and if the original proceeds were invested in long-duration assets ("going long"), then the bank needs to sell off its own assets (that have declined in value) in order to pay.

– if interest rates have gone down by the end of the year, then it is possible that the backing assets may not be able to meet the guaranteed rate; this becomes a greater possibility with short-duration assets ("going short").

– the bank may have to sell off some its assets to meet the guarantee.

A Solution to the Interest Rate Problem

– structure the assets so that their cash flows move at least the same amount as the liabilities' cash flows move when interest rates change.

– let A_t and L_t represent the cash flows at time t from an institution's assets and liabilites, respectively.

– let R_t represent the institution's net cash flows at time t such that $R_t = A_t - L_t$.

– if $P(i) = \sum_{t=1}^{n} v^t \cdot R_t = \sum_{t=1}^{n} v^t \cdot (A_t - L_t)$, then we would like the present value of asset cash flows to equal the present value of liability cash flows i.e. $P(i) = 0$:

$$\sum_{t=1}^{n} v^t \cdot A_t = \sum_{t=1}^{n} v^t \cdot L_t$$

– we'd also like the interest sensitivity (modified duration $\bar{v}$) of the asset cash flows to be equal to the interest sensitivity of the liabilites i.e. $\dfrac{P'(i)}{P(i)} = 0 \rightarrow P'(i) = 0$.

– in addition, we'd also like the convexity ($\bar{c}$) of the asset cash flows to be equal to the convexity of the liabilites i.e. $\dfrac{P''(i)}{P(i)} = 0 \rightarrow P''(i) > 0$.

– convexity is described as the rate of change in interest sensitivity. It is desirable to have positive (negative) changes in the asset values to be greater (less) than positive (negative) changes in liability values. If the changes were plotted on a curve against interest changes, you'd like the curve to be convex.

– We determine convexity by taking the 1^{st} derivative of $\bar{v}$:

$$\bar{c} = \frac{d}{di}\left(\frac{\bar{d}}{1+i}\right) = \frac{\dfrac{d^2}{d^2 i}\left(\displaystyle\sum_{t=1}^{n} v^t \cdot R_t\right)}{\displaystyle\sum_{t=1}^{n} v^t \cdot R_t}$$

– note that the forces that control liability cash flows are often out of the control of the financial institution.

– as a result, immunization will tend to focus more on the structure of the assets and how to match its volatility and convexity to that of the liabilities.

– Immunization is a three-step process:

(1) the present value of cash inflows (assets) should be equal to the present value of cash outflows (liabilities).

(2) the interest rate sensitivity ($\bar{v}$) of the present value of cash inflows (assets) should be equal to the interest rate sensitivity of the present value of cash outflows (liabilities).

(3) the convexity ($\bar{c}$) of the present value of cash inflows (assets) should be greater than the convexity of the present value of cash outflows(liabilities). In other words, asset growth (decline) should be greater (less) than liability growth(decline).

Difficulties/Limitations of Immunization

(a) choice of i is not always clear.

(b) doesn't work well for large changes in i.

(c) yield curve is assumed to change with Δi; actually, short-term rates are more volatile than long-term rates.

(d) frequent rebalancing is required in order to keep the modified duration of the assets and liabilities equal.

(e) exact cash flows may not be known and may have to be estimated.

(f) convexity suggests that profit can be achieved or that arbitrage is possible.

(g) assets may not have long enough maturities of duration to match liabilities.

Example:

A bank is required to pay $1,100 in one year. There are two investment options available with respect to how monies can be invested now in order to provide for the $1,100 payback:

(i) a non-interest bearing cash fund, for which x will be invested, and

(ii) a two-year zero-coupon bond earning 10% per year, for which y will be invested.

- Question: based on immunization theory, develop an asset portfolio that will minimize the risk that liability cash flows will exceed asset cash flows.
- Solution:
  - it is desirable to have the present value of the asset cash flows equal to that of the liability cash flows:
$$x + y(1.10)^2 \cdot v_i^2 = 1100v_i^1$$
  - it is desirable to have the modified duration ($\bar{v} = \frac{\bar{d}}{1+i}$) of the asset cash flows equal to that of the liability cash flows so that they are equally sensitive to interest rate changes:
$$\frac{x}{x+y}\left(\frac{0}{1+i}\right) + \frac{y}{x+y}\left(\frac{2}{1+i}\right) = \frac{1}{1+i}$$
$$\frac{2y}{x+y} = 1$$
  - it is desirable for the convexity ($\bar{c}$) of the asset cash flows to be greater than that of the liabilities:
$$\frac{\frac{d^2}{d^2i}\left(x + y(1.10)^2 \cdot v_i^2\right)}{x + y(1.10)^2 \cdot v_i^2} > \frac{\frac{d^2}{d^2i}\left(1100v_i^1\right)}{1100v_i^1}$$
$$\frac{y(1.10)^2(-2)(-3) \cdot v_i^4}{x + y(1.10)^2 \cdot v_i^2} > \frac{-1100(-2)v_i^3}{1100v_i^1}$$
  - if an effective rate of interest of 10% is assumed, then:
$$\left.\begin{array}{r} x + \frac{y(1.10)^2}{1.10^2} = \frac{1100}{1.1} \rightarrow x + y = 1000 \\ \frac{2y}{x+y} = 1 \end{array}\right\} \quad x = 500, y = 500$$
  - and the convexity of the assets is greater than convexity of the liabilities:
$$\frac{500(1.10)^2(-2)(-3) \cdot v_{10\%}^4}{500 + 500(1.10)^2 \cdot v_{10\%}^2} > \frac{-1100(-2)v_{10\%}^3}{1100v_{10\%}^1}$$
$$2.479 > 1.653$$
  - the interest volatility (modified duration $\bar{v} = \frac{\bar{d}}{1+i}$) of the assets and liability are:
    - $\bar{v}_A = \left(\frac{x}{x+y}\right) \cdot \left(\frac{0}{1+i}\right) + \left(\frac{y}{x+y}\right) \cdot \left(\frac{2}{1+i}\right) = \left(\frac{500}{1000}\right) \cdot \left(\frac{2}{1.1}\right) = .90909$
    - $\bar{v}_L = \left(\frac{1}{1+i}\right) = \left(\frac{1}{1.1}\right) = .90909$

9.10 Matching Assets and Liabilities

- the objective is to invest assets in a manner that will minimize the risk associated with the movements of the interest rates.
- the most preferable approach would be to match asset cash flows to liability cash flows (absolute matching or dedication).
- it is difficult to achieve absolute matching.
  (i) cash flows are hard to predict
  (ii) reinvestment risk increases for long-term investments
  (iii) yield rates may have to be compromised in an effort to match cash flows
- interest rate risk occurs when either:
  (i) rates fall, resulting in reinvestment at lower rates (incentive is to invest long-term), or
  (ii) rates rise, resulting in missed opportunities to reinvest (incentive is to invest short-term). Interest-sensitive assets drop in value also.

Example:

- a bank issues a two-year term deposit and guarantees 8% per year; a deposit holder can withdraw their money at the end of the first or second year without penalty.
- the bank can invest in :
  (1) one-year deposits that yeild 8% per year, and
  (2) two-year deposits that yeild 8.5% per year
- let f be the forward rate on one-year rate of returns for the second year
- let S_1 and S_2 represent the withdrawal amounts at the end of year 1 and 2, respectively and
- the present value of each dollar invested is equal to:

$$1 = S_1 \cdot v^1_{8\%} + S_2 \cdot v^2_{8\%}$$

- let p_1 and p_2 represent the proportion of \$1 invested by the bank in the one- and two-year deposits, respectively; $p_2 = 1 - p_1$
- if A_2 represents the bank's asset value at time 2, then

$$A_2 = [(p_1 \times 1)(1.08) - (S_1 \times 1)] \cdot (1 + f) + (p_2 \times 1)(1.085)^2 - (S_2 \times 1)$$

- A_2 will depend on how much is invested in the one and two-year deposits and that will depend on the forward rate.
  - if interest rates are expected to decrease ($f = 7\%$), then withdrawal rates at time 1 will be lower; let $S_1 = 10\%$ and if we desire that $A_2 > 0$, then p_1 will need to be less than 55%.
  - if interest rates are expected to increase ($f = 9.5\%$), then withdrawal rates at time 1 will be higher; let $S_1 = 90\%$ and if we desire that $A_2 > 0$, then p_1 will need to be greater than 50%.
- therefore, the recommended investment strategy is to place 50% to 55% into one-year deposits and 45% to 50% into two-year investments.
- the above calculation is highly sensitive to the assumed forward rate (f) and level of withdrawal rates.