Week-5

Data Science for Engineers (NPTEL)

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→ Session is being recorded

→ 3-5 min break every half hour & 10 min break every hour.

→ If you would like me to cover more topics in next session please let me know at the end.

→ Be respectful during the session, I will not hesitate to remove you from the session

* Topics

1. Constraint Optimization
2. Lagrange Multipliers
3. Inequality Constraint Optimization
4. Introduction to data science

Constraints Optimization

* Unconstraint Optimization — soft boundaries or No boundaries.

In Google Maps, let's say you want to travel from A to B

* Constraint Optimization

Constraint → hard limit placed on a variable that prevents us from going into certain directions.

Constraint → Equality
Constraint → Inequality

* Equality Constraint Optimization

$$\min_{x_1, x_2} \; 2x_1^2 + 4x_2^2 \longrightarrow \text{Objective function}$$

$$\text{Such that} \quad 3x_1 + 2x_2 = 12 \longrightarrow \text{Constraint}$$

Contour Lines

→ This was covered in lecture videos.

* Lagrange Multipliers

→ Optimization + constraint problems

→ helps finding optimal sol. of an obj f^n while satisfying one or more constraints.

$f(x, y) \longrightarrow$ Objective f^n

$g(x, y) \longrightarrow$ Constraint f^n.

$$\boxed{\nabla f(x,y) = -\lambda \nabla g(x,y)}$$

$\nabla \rightarrow$ gradient

$\lambda \rightarrow$ lagrange Multiplier

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

$\nabla \rightarrow$ gradient

If $f(x, y)$, then

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

Questions:

Time till 6:15PM.

find minima for $f(x_1, x_2) = 2x_1^2 + 4x_2^2$ with constraint

$g(x_1, x_2) \longrightarrow 3x_1 + 2x_2 = 12$

Clue: $\nabla f(x_1, x_2) = -\lambda \nabla g(x_1, x_2)$

Sol:

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = -\lambda \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4x_1 \\ 8x_2 \end{bmatrix} = -\lambda \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$4x_1 = -3\lambda \longrightarrow (1)$

$8x_2 = -2\lambda \longrightarrow (2)$

(1) & (2).

$\lambda = -\frac{4}{3}x_1$

(2) $\cancel{8}x_2 = +\cancel{2}\left(+\frac{\cancel{4}}{3}x_1\right)$

$\boxed{x_1 = 3x_2}$

$x_1 = 3(1.09)$

$x_1 = 3.27$

$3x_1 + 2x_2 = 12 \longrightarrow (3)$

$3(3x_2) + 2x_2 = 12$

$9x_2 + 2x_2 = 12$

$11x_2 = 12$

$x_2 = \frac{12}{11} \approx 1.09$

$(x_1, x_2) = (3.27, 1.09)$

$\downarrow$

$\boxed{\text{Minima}}$

Question

Time till 6:23.

For a rectangle whose perimeter is 20m.

find the dimensions that maximize the area?

Clue:

$$\max_{l,b} f(x) = A = l \times b$$

$$\text{st} \quad g_{l,b}(x) = 2l + 2b = 20$$

$$\nabla f(l,b) = -\lambda \nabla g(l,b)$$

$$\begin{bmatrix} b \\ l \end{bmatrix} = -\lambda \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\underbrace{b = -2\lambda \qquad l = -2\lambda}$$

$$\Rightarrow b = l.$$

$$2l + 2b = 20.$$

$$4l = 20$$

$$l = 5$$

$$\Rightarrow b = 5$$

$$\text{Area} = 5 \times 5$$

$$= 25 \,//.$$

Question

$\boxed{6:33\text{ PM}}$

Find the pt. on the circle $x^2+y^2=80$ which are closest to & farthest from the point (1,2)?

Clue: $d=\sqrt{(x-x_1)^2+(y-y_1)^2}$

$$\max_{x,y} f(x,y)=(x-1)^2+(y-2)^2$$
$$\text{s.t. } x^2+y^2=80$$

$$\frac{\partial f}{\partial x}=2(x-1) \qquad \frac{\partial g}{\partial x}=2x$$

$$\frac{\partial f}{\partial y}=2(y-2) \qquad \frac{\partial g}{\partial y}=2y$$

$$\begin{bmatrix}2(x-1)\\2(y-2)\end{bmatrix}=-\lambda\begin{bmatrix}2x\\2y\end{bmatrix} \Rightarrow \begin{array}{l}2(x-1)=-\lambda(2x) \to (1)\\ 2(y-2)=-\lambda(2y) \to (2)\end{array}$$

(1) $-\lambda=\dfrac{x-1}{x}$

Sub in (2)

$$\cancel{2}(y-2)=\frac{(x-1)}{x}(\cancel{2}y)$$

$$\cancel{xy}+2x=\cancel{xy}-y$$

$$\boxed{2x=y}$$

$$x^2+y^2=80$$
$$x^2+(2x)^2=80$$
$$x^2+4x^2=80$$
$$5x^2=80$$
$$x^2=80/5$$
$$=16$$

$$\boxed{x=\pm 4}$$

$y = 2x$

$y = \pm 8$

$(x, y) = (\pm 4, \pm 8)$

$(4, 8)$

$(-4, 8)$

$(4, -8)$

$(-4, -8)$

Question

6:49 PM

$f(x,y,z) = x+z$

$g(x,y,z) = x^2+y^2+z^2 = 1 \longrightarrow (A)$

Clue: $\nabla f(x,y,z) = -\lambda \nabla g(x,y,z)$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \nabla g = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

$1 = -\lambda(2x) \Rightarrow -\lambda = \frac{1}{2x} \hookrightarrow (1)$

$0 = -\lambda(2y) \hookrightarrow (2) \rightarrow y = 0$

$1 = -\lambda(2z) \Rightarrow -\lambda = \frac{1}{2z} \hookrightarrow (3)$

from (1), (3)

$2x = 2z$

$\Rightarrow x = z.$

Sub in (A)

$x^2+y^2+z^2 = 1$

$x^2 + (0) + (x)^2 = 1$

$2x^2 = 1$

$x^2 = 1/2$

$\boxed{x = \pm\frac{1}{\sqrt{2}}} \Rightarrow \boxed{z = \pm\frac{1}{\sqrt{2}}}$

$y = 0$

Points $\longrightarrow \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$

$\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ ✗ $\left(+\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$ ✗

$x \neq z.$

Which of the following methods is used to solve optimization problems with equality constraints?

a) Simplex Method
b) Gradient Descent
c) Lagrange Multipliers
d) Genetic Algorithm

In an optimization problem with equality constraints, the objective is to:

a) Minimize the objective function only.
b) Maximize the objective function only.
c) Satisfy the equality constraints only.
d) Optimize the objective function while satisfying the equality constraints.

Which of the following represents an equality constraint in a linear programming problem?

a) $3x+2y\leq 10$
b) $2x-5y=8$
c) $4x+6y\geq 20$
d) $7x+3y>15$

Consider the optimization problem: Minimize: f(x, y) = x^2 + y^2 Subject to: x + y = 5
What are the coordinates (x, y) of the optimal solution?

a) (2, 3)
b) (3, 2)
c) (2.5, 2.5)
d) (1, 4)

$f(x,y) = x^2 + y^2$
$g(x,y) = x+y=5$

* Inequality Constraint Optimization

$$\min_{x_1, x_2} \; 2x_1^2 + 4x_2^2 \quad \text{st} \quad 3x_1 + 2x_2 \le 12$$

$$\min_{x_1, x_2} \; 2x_1^2 + 4x_2^2 \quad \text{st} \quad 3x_1 + 2x_2 \ge 12$$

* Multi-variate Inequality constraint

$$\min_{x_1, x_2} \quad 2x_1^2 + 4x_2^2$$

$$\text{st} \quad 3x_1 + 2x_2 \le 12 \rightarrow 3x_1 + 2x_2 - 12 \le 0$$

$$2x_1 + 5x_2 \ge 10 \rightarrow 10 - 2x_1 + 5x_2 \le 0$$

$$x_1 \le 1 \rightarrow x_1 - 1 \le 0$$

$$\le 0$$

$$L(x_1, x_2, \cancel{\mu_1, \mu_2, \mu_3}) = 2x_1^2 + 4x_2^2 + \mu_1(3x_1 + 2x_2 - 12) + \mu_2(10 - 2x_1 - 5x_2) + \mu_3(x_1 - 1)$$

$$\min_{x_1, x_2} \quad 2x_1^2 + 4x_2^2$$

$$st$$

$$3x_1 + 2x_2 \le 12 \Rightarrow (a)$$

$$2x_1 + 5x_2 \ge 10 \Rightarrow (b)$$

$$x_1 \le 1 \Rightarrow (c)$$

- Lagrangian

$$L(x_1, x_2, \mu_1, \mu_2, \mu_3) = 2x_1^2 + 4x_2^2 + \mu_1(3x_1 + 2x_2 - 12) + \mu_2(10 - 2x_1 - 5x_2) + \mu_3(x_1 - 1)$$

- First order KKT conditions

diff wrt x_1 — 1 $\leftarrow$ $4x_1 + 3\mu_1 - 2\mu_2 + \mu_3 = 0$

diff wrt x_2 — 2 $\leftarrow$ $8x_2 + 2\mu_1 - 5\mu_2 = 0$

$\rightarrow$ differentiation

$$\mu_1(3x_1 + 2x_2 - 12) = 0$$

$$\mu_2(10 - 2x_1 - 5x_2) = 0$$

$$\mu_3(x_3 - 1) = 0$$

$$\mu_i \ge 0 \; *$$

3

$\rightarrow$ constraints

KKT $\rightarrow$ Karush – Kuhn Tucker

$$\min_{x_1, x_2} \quad 2x_1^2 + 4x_2^2 \longrightarrow \text{Obj } f^n$$

$$\text{St} \quad \left.\begin{aligned} 3x_1 + 2x_2 &\le 12 \\ 2x_1 + 5x_2 &\ge 10 \\ x_1 &\le 1 \end{aligned}\right\} \text{Constraints}$$

① Always make them ≤ 0

$$3x_1 + 2x_2 - 12 \le 0$$
$$10 - 2x_1 - 5x_2 \le 0$$
$$x_1 - 1 \le 0$$

$$\mathcal{L}(x_1, x_2, \mu_1, \mu_2, \mu_3) = 2x_1^2 + 4x_2^2 + \mu_1(3x_1 + 2x_2 - 12) + \mu_2(10 - 2x_1 - 5x_2) + \mu_3(x_1 - 1)$$

① Diff of $\mathcal{L}$ w.r.t x_1 & x_2.

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= 4x_1 + 3\mu_1 - 2\mu_2 + \mu_3 = 0 \longrightarrow ① \\ \frac{\partial \mathcal{L}}{\partial x_2} &= 8x_2 + 2\mu_1 - 5\mu_2 = 0 \longrightarrow ② \end{aligned}\right\}$$

② Constraints

$$\left.\begin{aligned} \mu_1(3x_1 + 2x_2 - 12) &= 0 \\ \mu_2(10 - 2x_1 - 5x_2) &= 0 \\ \mu_3(x_1 - 1) &= 0 \end{aligned}\right\}$$

③ $\mu_1, \mu_2, \mu_3 \ge 0$

Sl.no	Active (A) /Inactive (I) constraints (a)	(b)	(c)	Solution (x, μ)	Possible optima (Y/N)	Remark
1	A	A	A	Infeasible	N	Equations do not have a valid solution.
2	A	A	I	$x = [3.6364 \quad 0.5455]$ $\mu = [-5.2 \quad -1.45 \quad 0]$	N	$x_1 \leq 1$ is not satisfied, $\mu_1 < 0, \mu_2 < 0$
3	A	I	A	$x = [1 \quad 4.5]$ $\mu = [-18 \quad 0 \quad 50]$	N	$\mu_1 < 0$
4	I	A	A	$x = [1 \quad 1.6]$ $\mu = [0 \quad 2.56 \quad 1.12]$	Y	All constraints and KKT conditions satisfied
5	A	I	I	$x = [3.27 \quad 1.09]$ $\mu = [-4.36 \quad 0 \quad 0]$	N	$x_1 \leq 1$ is not satisfied
6	I	A	I	$x = [1.21 \quad 1.51]$ $\mu = [0 \quad 2.45 \quad 0]$	N	$x_1 \leq 1$ is not satisfied
7	I	I	A	$x = [1 \quad 0]$ $\mu = [0 \quad 0 \quad -4]$	N	$2x_1 + 5x_2 \geq 10$ is not satisfied
8	I	I	I	$x = [0 \quad 0]$ $\mu = [0 \quad 0 \quad 0]$	N	$2x_1 + 5x_2 \geq 10$ is not satisfied

Q) Which of the following represents an inequality constraint in optimization?
a) $x+y = 10$
b) $2x-3y \leq 5$
c) $4x-3y = 25$
d) $3x+2y = 12$

Q) For an optimization problem, what does the feasible region represent?
a) The region where the objective function is minimized
b) The region where the objective function is maximized
c) The region satisfying all the constraints
d) The region outside the constraints

If the feasible region of an optimization problem is empty, it means that:
a) The problem has no solution
b) The problem has multiple solutions
c) The problem is unbounded
d) The problem is well-posed

Q) In linear programming, a feasible solution that also satisfies all the constraints with equality is called:
a) An degenerate solution
b) A bounded solution
c) An infeasible solution
d) A optimal solution

Question:

For a given f^n $z = 4x^2 - 5xy + 6y$

constraints : $x + y = 9 \longrightarrow (x+y-9=0)$

$2x - 9y \leq 1 \longrightarrow 2x - 9y - 1 \leq 0$

$xy + x \geq 3 \longrightarrow 3 - xy + x \leq 0$

(A) $\mathcal{L}(x, y, \lambda, \mu_1, \mu_2) = 4x^2 - 5xy + 6y + \lambda(x+y-9) + \mu_1(1-2x+9y)$
$+ \mu_2(3 - xy - x)$

(B) $\mathcal{L}(x, y, \lambda, \mu_1, \mu_2) = 4x^2 - 5xy + 6y + \lambda(x+y-9) + \mu_1(2x-9y-1)$
$+ \mu_2(3 - xy - x)$

(C) $\mathcal{L}(x, y, \lambda, \mu_1, \mu_2) = 4x^2 - 5xy + 6y + \lambda(x+y-9) + \mu_1(1-2x+9y)$
$+ \mu_2(xy + x - 3)$

Data Science

This lecture is to give you a view of how one should think about general data science problems and how you convert a problem you know which is not well defined into something that is manageable using the techniques such you learn in this course.

Techniques we will cover overall.

1. Regression Analysis
2. K-Nearest Neighbour (K-NN)
3. K-Mean Clustering
4. Logistics Regression
5. Principal Components Analysis (PCA)
6. Predictive Modelling – you can learn.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}_{2\times 2}$$ → Doubt from a student

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$\lambda \rightarrow$ eigen values.

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} (2-\lambda) & 1 \\ 4 & (5-\lambda) \end{vmatrix} = 0$$

$$(2-\lambda)(5-\lambda) - 4 = 0$$

$$10 - 2\lambda - 5\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda^2 - 6\lambda - \lambda + 6 = 0$$

$$\lambda(\lambda - 6) - 1(\lambda - 6) = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$$\lambda = 1, 6 \quad //$$