VI Graph Algorithms

Introduction

Graph problems pervade computer science, and algorithms for working with them are fundamental to the field. Hundreds of interesting computational problems are couched in terms of graphs. In this part, we touch on a few of the more significant ones.

Chapter 22 shows how we can represent a graph in a computer and then discusses algorithms based on searching a graph using either breadth-first search or depth-first search. The chapter gives two applications of depth-first search: topologically sorting a directed acyclic graph and decomposing a directed graph into its strongly connected components.

Chapter 23 describes how to compute a minimum-weight spanning tree of a graph: the least-weight way of connecting all of the vertices together when each edge has an associated weight. The algorithms for computing minimum spanning trees serve as good examples of greedy algorithms (see Chapter 16).

Chapters 24 and 25 consider how to compute shortest paths between vertices when each edge has an associated length or "weight." Chapter 24 shows how to find shortest paths from a given source vertex to all other vertices, and Chapter 25 examines methods to compute shortest paths between every pair of vertices.

Finally, Chapter 26 shows how to compute a maximum flow of material in a flow network, which is a directed graph having a specified source vertex of material, a specified sink vertex, and specified capacities for the amount of material that can traverse each directed edge. This general problem arises in many forms, and a good algorithm for computing maximum flows can help solve a variety of related problems efficiently.

When we characterize the running time of a graph algorithm on a given graph $G = (V, E)$, we usually measure the size of the input in terms of the number of vertices $|V|$ and the number of edges $|E|$ of the graph. That is, we describe the size of the input with two parameters, not just one. We adopt a common notational convention for these parameters. Inside asymptotic notation (such as O-notation or Θ-notation), and *only* inside such notation, the symbol V denotes $|V|$ and the symbol E denotes $|E|$. For example, we might say, "the algorithm runs in time $O(VE)$," meaning that the algorithm runs in time $O(|V||E|)$. This convention makes the running-time formulas easier to read, without risk of ambiguity.

Another convention we adopt appears in pseudocode. We denote the vertex set of a graph G by $G.V$ and its edge set by $G.E$. That is, the pseudocode views vertex and edge sets as attributes of a graph.

22 Elementary Graph Algorithms

This chapter presents methods for representing a graph and for searching a graph. Searching a graph means systematically following the edges of the graph so as to visit the vertices of the graph. A graph-searching algorithm can discover much about the structure of a graph. Many algorithms begin by searching their input graph to obtain this structural information. Several other graph algorithms elaborate on basic graph searching. Techniques for searching a graph lie at the heart of the field of graph algorithms.

Section 22.1 discusses the two most common computational representations of graphs: as adjacency lists and as adjacency matrices. Section 22.2 presents a simple graph-searching algorithm called breadth-first search and shows how to create a breadth-first tree. Section 22.3 presents depth-first search and proves some standard results about the order in which depth-first search visits vertices. Section 22.4 provides our first real application of depth-first search: topologically sorting a directed acyclic graph. A second application of depth-first search, finding the strongly connected components of a directed graph, is the topic of Section 22.5.

22.1 Representations of graphs

We can choose between two standard ways to represent a graph $G = (V, E)$: as a collection of adjacency lists or as an adjacency matrix. Either way applies to both directed and undirected graphs. Because the adjacency-list representation provides a compact way to represent ***sparse*** graphs—those for which $|E|$ is much less than $|V|^2$—it is usually the method of choice. Most of the graph algorithms presented in this book assume that an input graph is represented in adjacency-list form. We may prefer an adjacency-matrix representation, however, when the graph is ***dense***—$|E|$ is close to $|V|^2$—or when we need to be able to tell quickly if there is an edge connecting two given vertices. For example, two of the all-pairs

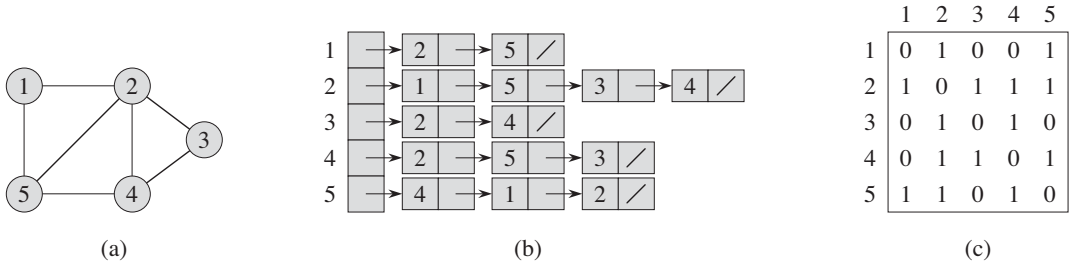

Figure 22.1 Two representations of an undirected graph. **(a)** An undirected graph G with 5 vertices and 7 edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.

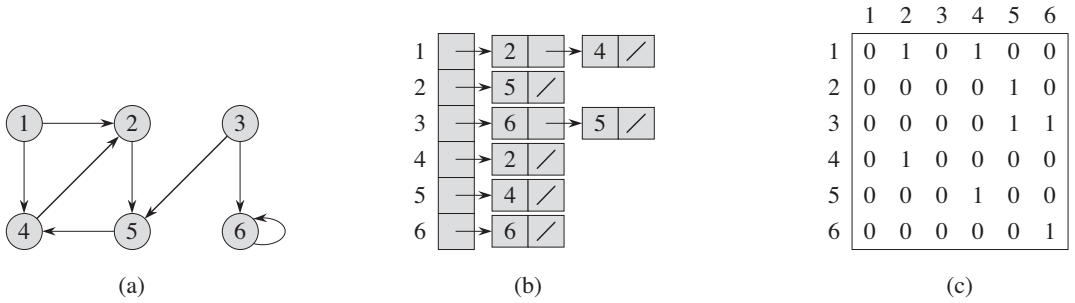

Figure 22.2 Two representations of a directed graph. **(a)** A directed graph G with 6 vertices and 8 edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.

shortest-paths algorithms presented in Chapter 25 assume that their input graphs are represented by adjacency matrices.

The ***adjacency-list representation*** of a graph $G = (V, E)$ consists of an array Adj of $|V|$ lists, one for each vertex in V. For each $u \in V$, the adjacency list $Adj[u]$ contains all the vertices v such that there is an edge $(u, v) \in E$. That is, $Adj[u]$ consists of all the vertices adjacent to u in G. (Alternatively, it may contain pointers to these vertices.) Since the adjacency lists represent the edges of a graph, in pseudocode we treat the array Adj as an attribute of the graph, just as we treat the edge set E. In pseudocode, therefore, we will see notation such as $G.Adj[u]$. Figure 22.1(b) is an adjacency-list representation of the undirected graph in Figure 22.1(a). Similarly, Figure 22.2(b) is an adjacency-list representation of the directed graph in Figure 22.2(a).

If G is a directed graph, the sum of the lengths of all the adjacency lists is $|E|$, since an edge of the form (u, v) is represented by having v appear in $Adj[u]$. If G is

an undirected graph, the sum of the lengths of all the adjacency lists is $2|E|$, since if (u, v) is an undirected edge, then u appears in v's adjacency list and vice versa. For both directed and undirected graphs, the adjacency-list representation has the desirable property that the amount of memory it requires is $\Theta(V + E)$.

We can readily adapt adjacency lists to represent ***weighted graphs***, that is, graphs for which each edge has an associated ***weight***, typically given by a ***weight function*** $w : E \to \mathbb{R}$. For example, let $G = (V, E)$ be a weighted graph with weight function w. We simply store the weight $w(u, v)$ of the edge $(u, v) \in E$ with vertex v in u's adjacency list. The adjacency-list representation is quite robust in that we can modify it to support many other graph variants.

A potential disadvantage of the adjacency-list representation is that it provides no quicker way to determine whether a given edge (u, v) is present in the graph than to search for v in the adjacency list $Adj[u]$. An adjacency-matrix representation of the graph remedies this disadvantage, but at the cost of using asymptotically more memory. (See Exercise 22.1-8 for suggestions of variations on adjacency lists that permit faster edge lookup.)

For the ***adjacency-matrix representation*** of a graph $G = (V, E)$, we assume that the vertices are numbered $1, 2, \ldots, |V|$ in some arbitrary manner. Then the adjacency-matrix representation of a graph G consists of a $|V| \times |V|$ matrix $A = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Figures 22.1(c) and 22.2(c) are the adjacency matrices of the undirected and directed graphs in Figures 22.1(a) and 22.2(a), respectively. The adjacency matrix of a graph requires $\Theta(V^2)$ memory, independent of the number of edges in the graph.

Observe the symmetry along the main diagonal of the adjacency matrix in Figure 22.1(c). Since in an undirected graph, (u, v) and (v, u) represent the same edge, the adjacency matrix A of an undirected graph is its own transpose: $A = A^{\mathrm{T}}$. In some applications, it pays to store only the entries on and above the diagonal of the adjacency matrix, thereby cutting the memory needed to store the graph almost in half.

Like the adjacency-list representation of a graph, an adjacency matrix can represent a weighted graph. For example, if $G = (V, E)$ is a weighted graph with edge-weight function w, we can simply store the weight $w(u, v)$ of the edge $(u, v) \in E$ as the entry in row u and column v of the adjacency matrix. If an edge does not exist, we can store a NIL value as its corresponding matrix entry, though for many problems it is convenient to use a value such as 0 or ∞.

Although the adjacency-list representation is asymptotically at least as space-efficient as the adjacency-matrix representation, adjacency matrices are simpler, and so we may prefer them when graphs are reasonably small. Moreover, adja-

cency matrices carry a further advantage for unweighted graphs: they require only one bit per entry.

Representing attributes

Most algorithms that operate on graphs need to maintain attributes for vertices and/or edges. We indicate these attributes using our usual notation, such as $v.d$ for an attribute d of a vertex v. When we indicate edges as pairs of vertices, we use the same style of notation. For example, if edges have an attribute f, then we denote this attribute for edge (u, v) by $(u, v).f$. For the purpose of presenting and understanding algorithms, our attribute notation suffices.

Implementing vertex and edge attributes in real programs can be another story entirely. There is no one best way to store and access vertex and edge attributes. For a given situation, your decision will likely depend on the programming language you are using, the algorithm you are implementing, and how the rest of your program uses the graph. If you represent a graph using adjacency lists, one design represents vertex attributes in additional arrays, such as an array $d[1..|V|]$ that parallels the *Adj* array. If the vertices adjacent to u are in $Adj[u]$, then what we call the attribute $u.d$ would actually be stored in the array entry $d[u]$. Many other ways of implementing attributes are possible. For example, in an object-oriented programming language, vertex attributes might be represented as instance variables within a subclass of a `Vertex` class.

Exercises

22.1-1

Given an adjacency-list representation of a directed graph, how long does it take to compute the out-degree of every vertex? How long does it take to compute the in-degrees?

22.1-2

Give an adjacency-list representation for a complete binary tree on 7 vertices. Give an equivalent adjacency-matrix representation. Assume that vertices are numbered from 1 to 7 as in a binary heap.

22.1-3

The ***transpose*** of a directed graph $G = (V, E)$ is the graph $G^T = (V, E^T)$, where $E^T = \{(v, u) \in V \times V : (u, v) \in E\}$. Thus, G^T is G with all its edges reversed. Describe efficient algorithms for computing G^T from G, for both the adjacency-list and adjacency-matrix representations of G. Analyze the running times of your algorithms.

24.1 The Bellman-Ford algorithm

The ***Bellman-Ford algorithm*** solves the single-source shortest-paths problem in the general case in which edge weights may be negative. Given a weighted, directed graph $G = (V, E)$ with source s and weight function $w : E \to \mathbb{R}$, the Bellman-Ford algorithm returns a boolean value indicating whether or not there is a negative-weight cycle that is reachable from the source. If there is such a cycle, the algorithm indicates that no solution exists. If there is no such cycle, the algorithm produces the shortest paths and their weights.

The algorithm relaxes edges, progressively decreasing an estimate $v.d$ on the weight of a shortest path from the source s to each vertex $v \in V$ until it achieves the actual shortest-path weight $\delta(s, v)$. The algorithm returns TRUE if and only if the graph contains no negative-weight cycles that are reachable from the source.

```
BELLMAN-FORD(G, w, s)
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  for i = 1 to |G.V| − 1
3       for each edge (u, v) ∈ G.E
4            RELAX(u, v, w)
5  for each edge (u, v) ∈ G.E
6       if v.d > u.d + w(u, v)
7            return FALSE
8  return TRUE
```

Figure 24.4 shows the execution of the Bellman-Ford algorithm on a graph with 5 vertices. After initializing the d and π values of all vertices in line 1, the algorithm makes $|V| - 1$ passes over the edges of the graph. Each pass is one iteration of the **for** loop of lines 2–4 and consists of relaxing each edge of the graph once. Figures 24.4(b)–(e) show the state of the algorithm after each of the four passes over the edges. After making $|V| - 1$ passes, lines 5–8 check for a negative-weight cycle and return the appropriate boolean value. (We'll see a little later why this check works.)

The Bellman-Ford algorithm runs in time $O(VE)$, since the initialization in line 1 takes $\Theta(V)$ time, each of the $|V| - 1$ passes over the edges in lines 2–4 takes $\Theta(E)$ time, and the **for** loop of lines 5–7 takes $O(E)$ time.

To prove the correctness of the Bellman-Ford algorithm, we start by showing that if there are no negative-weight cycles, the algorithm computes correct shortest-path weights for all vertices reachable from the source.

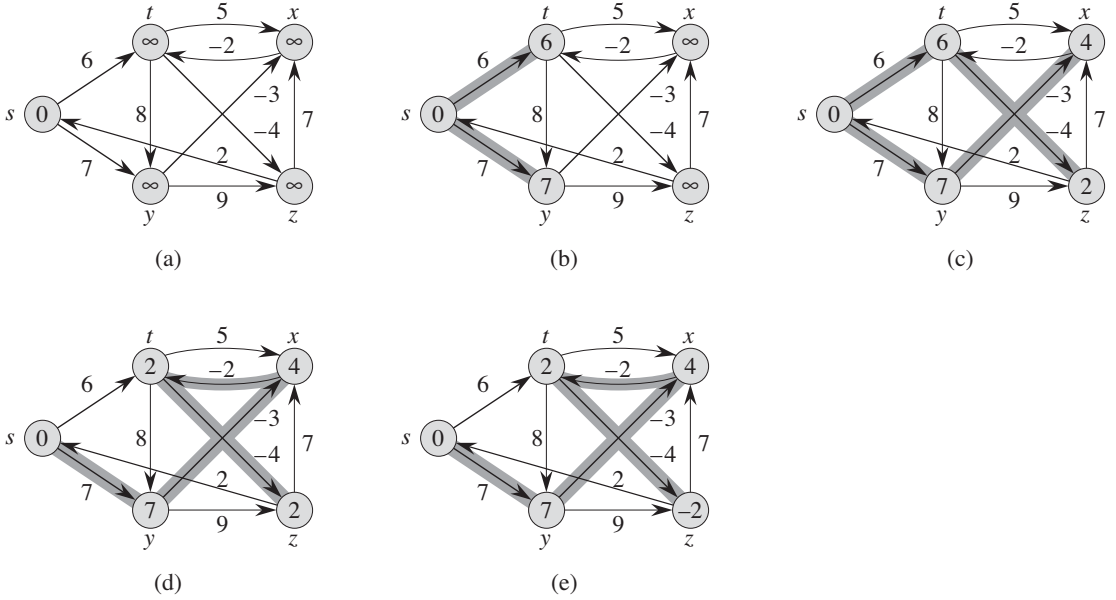

Figure 24.4 The execution of the Bellman-Ford algorithm. The source is vertex s. The d values appear within the vertices, and shaded edges indicate predecessor values: if edge (u, v) is shaded, then $v.\pi = u$. In this particular example, each pass relaxes the edges in the order $(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)$. **(a)** The situation just before the first pass over the edges. **(b)–(e)** The situation after each successive pass over the edges. The d and π values in part (e) are the final values. The Bellman-Ford algorithm returns TRUE in this example.

Lemma 24.2

Let $G = (V, E)$ be a weighted, directed graph with source s and weight function $w : E \to \mathbb{R}$, and assume that G contains no negative-weight cycles that are reachable from s. Then, after the $|V| - 1$ iterations of the **for** loop of lines 2–4 of BELLMAN-FORD, we have $v.d = \delta(s, v)$ for all vertices v that are reachable from s.

Proof We prove the lemma by appealing to the path-relaxation property. Consider any vertex v that is reachable from s, and let $p = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = s$ and $v_k = v$, be any shortest path from s to v. Because shortest paths are simple, p has at most $|V| - 1$ edges, and so $k \le |V| - 1$. Each of the $|V| - 1$ iterations of the **for** loop of lines 2–4 relaxes all $|E|$ edges. Among the edges relaxed in the ith iteration, for $i = 1, 2, \ldots, k$, is (v_{i-1}, v_i). By the path-relaxation property, therefore, $v.d = v_k.d = \delta(s, v_k) = \delta(s, v)$. ■

Corollary 24.3
Let $G = (V, E)$ be a weighted, directed graph with source vertex s and weight function $w : E \to \mathbb{R}$, and assume that G contains no negative-weight cycles that are reachable from s. Then, for each vertex $v \in V$, there is a path from s to v if and only if BELLMAN-FORD terminates with $v.d < \infty$ when it is run on G.

Proof The proof is left as Exercise 24.1-2. ■

Theorem 24.4 (Correctness of the Bellman-Ford algorithm)
Let BELLMAN-FORD be run on a weighted, directed graph $G = (V, E)$ with source s and weight function $w : E \to \mathbb{R}$. If G contains no negative-weight cycles that are reachable from s, then the algorithm returns TRUE, we have $v.d = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest-paths tree rooted at s. If G does contain a negative-weight cycle reachable from s, then the algorithm returns FALSE.

Proof Suppose that graph G contains no negative-weight cycles that are reachable from the source s. We first prove the claim that at termination, $v.d = \delta(s, v)$ for all vertices $v \in V$. If vertex v is reachable from s, then Lemma 24.2 proves this claim. If v is not reachable from s, then the claim follows from the no-path property. Thus, the claim is proven. The predecessor-subgraph property, along with the claim, implies that G_π is a shortest-paths tree. Now we use the claim to show that BELLMAN-FORD returns TRUE. At termination, we have for all edges $(u, v) \in E$,

$$\begin{aligned} v.d &= \delta(s, v) \\ &\le \delta(s, u) + w(u, v) \quad \text{(by the triangle inequality)} \\ &= u.d + w(u, v) , \end{aligned}$$

and so none of the tests in line 6 causes BELLMAN-FORD to return FALSE. Therefore, it returns TRUE.

Now, suppose that graph G contains a negative-weight cycle that is reachable from the source s; let this cycle be $c = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = v_k$. Then,

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0 . \tag{24.1}$$

Assume for the purpose of contradiction that the Bellman-Ford algorithm returns TRUE. Thus, $v_i.d \le v_{i-1}.d + w(v_{i-1}, v_i)$ for $i = 1, 2, \ldots, k$. Summing the inequalities around cycle c gives us

$$\begin{aligned}\sum_{i=1}^{k} v_i.d &\leq \sum_{i=1}^{k}(v_{i-1}.d + w(v_{i-1}, v_i)) \\ &= \sum_{i=1}^{k} v_{i-1}.d + \sum_{i=1}^{k} w(v_{i-1}, v_i) \ .\end{aligned}$$

Since $v_0 = v_k$, each vertex in c appears exactly once in each of the summations $\sum_{i=1}^{k} v_i.d$ and $\sum_{i=1}^{k} v_{i-1}.d$, and so

$$\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d \ .$$

Moreover, by Corollary 24.3, $v_i.d$ is finite for $i = 1, 2, \ldots, k$. Thus,

$$0 \leq \sum_{i=1}^{k} w(v_{i-1}, v_i) \ ,$$

which contradicts inequality (24.1). We conclude that the Bellman-Ford algorithm returns TRUE if graph G contains no negative-weight cycles reachable from the source, and FALSE otherwise. ■

Exercises

24.1-1

Run the Bellman-Ford algorithm on the directed graph of Figure 24.4, using vertex z as the source. In each pass, relax edges in the same order as in the figure, and show the d and π values after each pass. Now, change the weight of edge (z, x) to 4 and run the algorithm again, using s as the source.

24.1-2

Prove Corollary 24.3.

24.1-3

Given a weighted, directed graph $G = (V, E)$ with no negative-weight cycles, let m be the maximum over all vertices $v \in V$ of the minimum number of edges in a shortest path from the source s to v. (Here, the shortest path is by weight, not the number of edges.) Suggest a simple change to the Bellman-Ford algorithm that allows it to terminate in $m + 1$ passes, even if m is not known in advance.

24.1-4

Modify the Bellman-Ford algorithm so that it sets $v.d$ to $-\infty$ for all vertices v for which there is a negative-weight cycle on some path from the source to v.

24.1-5 ⋆

Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$. Give an $O(VE)$-time algorithm to find, for each vertex $v \in V$, the value $\delta^*(v) = \min_{u \in V} \{\delta(u, v)\}$.

24.1-6 ⋆

Suppose that a weighted, directed graph $G = (V, E)$ has a negative-weight cycle. Give an efficient algorithm to list the vertices of one such cycle. Prove that your algorithm is correct.

24.2 Single-source shortest paths in directed acyclic graphs

By relaxing the edges of a weighted dag (directed acyclic graph) $G = (V, E)$ according to a topological sort of its vertices, we can compute shortest paths from a single source in $\Theta(V + E)$ time. Shortest paths are always well defined in a dag, since even if there are negative-weight edges, no negative-weight cycles can exist.

The algorithm starts by topologically sorting the dag (see Section 22.4) to impose a linear ordering on the vertices. If the dag contains a path from vertex u to vertex v, then u precedes v in the topological sort. We make just one pass over the vertices in the topologically sorted order. As we process each vertex, we relax each edge that leaves the vertex.

```
DAG-SHORTEST-PATHS(G, w, s)
1  topologically sort the vertices of G
2  INITIALIZE-SINGLE-SOURCE(G, s)
3  for each vertex u, taken in topologically sorted order
4      for each vertex v ∈ G.Adj[u]
5          RELAX(u, v, w)
```

Figure 24.5 shows the execution of this algorithm.

The running time of this algorithm is easy to analyze. As shown in Section 22.4, the topological sort of line 1 takes $\Theta(V + E)$ time. The call of INITIALIZE-SINGLE-SOURCE in line 2 takes $\Theta(V)$ time. The **for** loop of lines 3–5 makes one iteration per vertex. Altogether, the **for** loop of lines 4–5 relaxes each edge exactly once. (We have used an aggregate analysis here.) Because each iteration of the inner **for** loop takes $\Theta(1)$ time, the total running time is $\Theta(V + E)$, which is linear in the size of an adjacency-list representation of the graph.

The following theorem shows that the DAG-SHORTEST-PATHS procedure correctly computes the shortest paths.

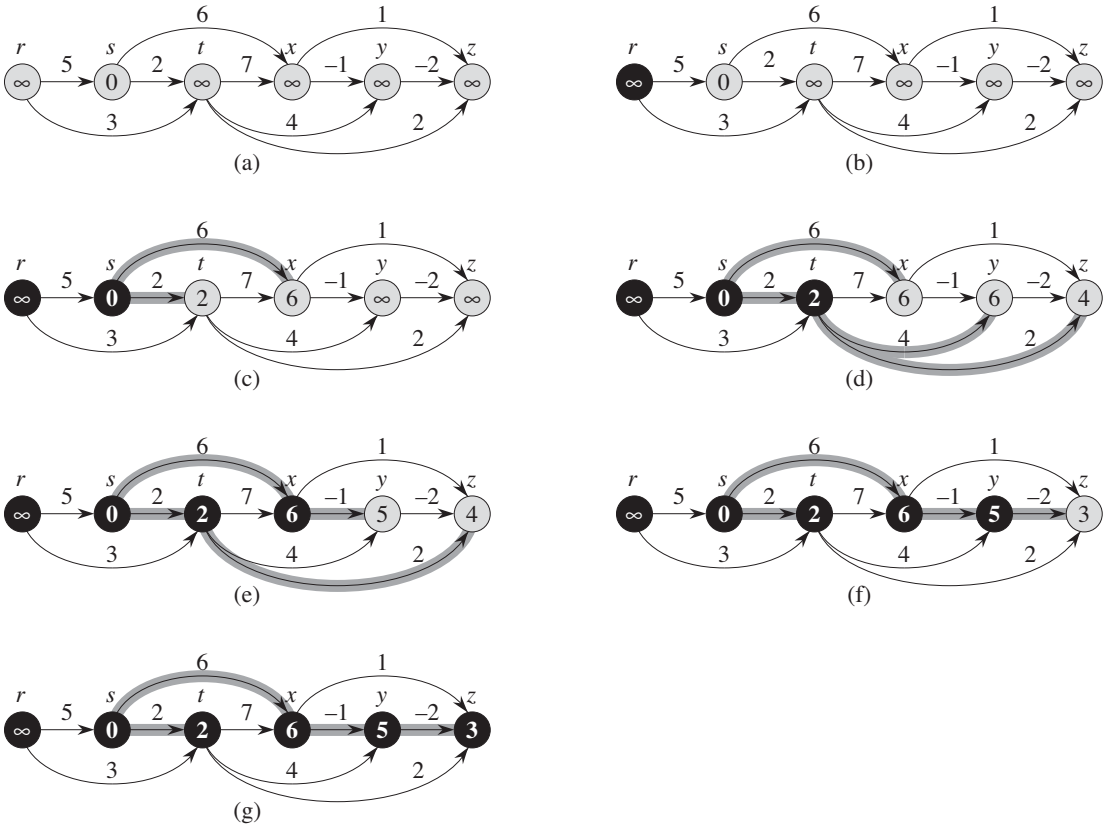

Figure 24.5 The execution of the algorithm for shortest paths in a directed acyclic graph. The vertices are topologically sorted from left to right. The source vertex is s. The d values appear within the vertices, and shaded edges indicate the π values. **(a)** The situation before the first iteration of the **for** loop of lines 3–5. **(b)–(g)** The situation after each iteration of the **for** loop of lines 3–5. The newly blackened vertex in each iteration was used as u in that iteration. The values shown in part (g) are the final values.

Theorem 24.5

If a weighted, directed graph $G = (V, E)$ has source vertex s and no cycles, then at the termination of the DAG-SHORTEST-PATHS procedure, $v.d = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest-paths tree.

Proof We first show that $v.d = \delta(s, v)$ for all vertices $v \in V$ at termination. If v is not reachable from s, then $v.d = \delta(s, v) = \infty$ by the no-path property. Now, suppose that v is reachable from s, so that there is a shortest path $p = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = s$ and $v_k = v$. Because we pro-

cess the vertices in topologically sorted order, we relax the edges on p in the order $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$. The path-relaxation property implies that $v_i.d = \delta(s, v_i)$ at termination for $i = 0, 1, \ldots, k$. Finally, by the predecessor-subgraph property, G_π is a shortest-paths tree. ■

An interesting application of this algorithm arises in determining critical paths in ***PERT chart***[2] analysis. Edges represent jobs to be performed, and edge weights represent the times required to perform particular jobs. If edge (u, v) enters vertex v and edge (v, x) leaves v, then job (u, v) must be performed before job (v, x). A path through this dag represents a sequence of jobs that must be performed in a particular order. A ***critical path*** is a *longest* path through the dag, corresponding to the longest time to perform any sequence of jobs. Thus, the weight of a critical path provides a lower bound on the total time to perform all the jobs. We can find a critical path by either

- negating the edge weights and running DAG-SHORTEST-PATHS, or
- running DAG-SHORTEST-PATHS, with the modification that we replace "∞" by "$-\infty$" in line 2 of INITIALIZE-SINGLE-SOURCE and "$>$" by "$<$" in the RELAX procedure.

Exercises

24.2-1
Run DAG-SHORTEST-PATHS on the directed graph of Figure 24.5, using vertex r as the source.

24.2-2
Suppose we change line 3 of DAG-SHORTEST-PATHS to read

3 **for** the first $|V| - 1$ vertices, taken in topologically sorted order

Show that the procedure would remain correct.

24.2-3
The PERT chart formulation given above is somewhat unnatural. In a more natural structure, vertices would represent jobs and edges would represent sequencing constraints; that is, edge (u, v) would indicate that job u must be performed before job v. We would then assign weights to vertices, not edges. Modify the DAG-SHORTEST-PATHS procedure so that it finds a longest path in a directed acyclic graph with weighted vertices in linear time.

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[2] "PERT" is an acronym for "program evaluation and review technique."

25.2-6
How can we use the output of the Floyd-Warshall algorithm to detect the presence of a negative-weight cycle?

25.2-7
Another way to reconstruct shortest paths in the Floyd-Warshall algorithm uses values $\phi_{ij}^{(k)}$ for $i, j, k = 1, 2, \ldots, n$, where $\phi_{ij}^{(k)}$ is the highest-numbered intermediate vertex of a shortest path from i to j in which all intermediate vertices are in the set $\{1, 2, \ldots, k\}$. Give a recursive formulation for $\phi_{ij}^{(k)}$, modify the FLOYD-WARSHALL procedure to compute the $\phi_{ij}^{(k)}$ values, and rewrite the PRINT-ALL-PAIRS-SHORTEST-PATH procedure to take the matrix $\Phi = \left(\phi_{ij}^{(n)}\right)$ as an input. How is the matrix Φ like the s table in the matrix-chain multiplication problem of Section 15.2?

25.2-8
Give an $O(VE)$-time algorithm for computing the transitive closure of a directed graph $G = (V, E)$.

25.2-9
Suppose that we can compute the transitive closure of a directed acyclic graph in $f(|V|, |E|)$ time, where f is a monotonically increasing function of $|V|$ and $|E|$. Show that the time to compute the transitive closure $G^* = (V, E^*)$ of a general directed graph $G = (V, E)$ is then $f(|V|, |E|) + O(V + E^*)$.

25.3 Johnson's algorithm for sparse graphs

Johnson's algorithm finds shortest paths between all pairs in $O(V^2 \lg V + VE)$ time. For sparse graphs, it is asymptotically faster than either repeated squaring of matrices or the Floyd-Warshall algorithm. The algorithm either returns a matrix of shortest-path weights for all pairs of vertices or reports that the input graph contains a negative-weight cycle. Johnson's algorithm uses as subroutines both Dijkstra's algorithm and the Bellman-Ford algorithm, which Chapter 24 describes.

Johnson's algorithm uses the technique of ***reweighting***, which works as follows. If all edge weights w in a graph $G = (V, E)$ are nonnegative, we can find shortest paths between all pairs of vertices by running Dijkstra's algorithm once from each vertex; with the Fibonacci-heap min-priority queue, the running time of this all-pairs algorithm is $O(V^2 \lg V + VE)$. If G has negative-weight edges but no negative-weight cycles, we simply compute a new set of nonnegative edge weights

that allows us to use the same method. The new set of edge weights $\hat{w}$ must satisfy two important properties:

1. For all pairs of vertices $u, v \in V$, a path p is a shortest path from u to v using weight function w if and only if p is also a shortest path from u to v using weight function $\hat{w}$.
2. For all edges (u, v), the new weight $\hat{w}(u, v)$ is nonnegative.

As we shall see in a moment, we can preprocess G to determine the new weight function $\hat{w}$ in $O(VE)$ time.

Preserving shortest paths by reweighting

The following lemma shows how easily we can reweight the edges to satisfy the first property above. We use δ to denote shortest-path weights derived from weight function w and $\hat{\delta}$ to denote shortest-path weights derived from weight function $\hat{w}$.

Lemma 25.1 (Reweighting does not change shortest paths)
Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$, let $h : V \to \mathbb{R}$ be any function mapping vertices to real numbers. For each edge $(u, v) \in E$, define

$$\hat{w}(u, v) = w(u, v) + h(u) - h(v) . \tag{25.9}$$

Let $p = \langle v_0, v_1, \ldots, v_k \rangle$ be any path from vertex v_0 to vertex v_k. Then p is a shortest path from v_0 to v_k with weight function w if and only if it is a shortest path with weight function $\hat{w}$. That is, $w(p) = \delta(v_0, v_k)$ if and only if $\hat{w}(p) = \hat{\delta}(v_0, v_k)$. Furthermore, G has a negative-weight cycle using weight function w if and only if G has a negative-weight cycle using weight function $\hat{w}$.

Proof We start by showing that

$$\hat{w}(p) = w(p) + h(v_0) - h(v_k) . \tag{25.10}$$

We have

$$\begin{aligned}
\hat{w}(p) &= \sum_{i=1}^{k} \hat{w}(v_{i-1}, v_i) \\
&= \sum_{i=1}^{k} (w(v_{i-1}, v_i) + h(v_{i-1}) - h(v_i)) \\
&= \sum_{i=1}^{k} w(v_{i-1}, v_i) + h(v_0) - h(v_k) \qquad \text{(because the sum telescopes)} \\
&= w(p) + h(v_0) - h(v_k) .
\end{aligned}$$

Therefore, any path p from v_0 to v_k has $\hat{w}(p) = w(p) + h(v_0) - h(v_k)$. Because $h(v_0)$ and $h(v_k)$ do not depend on the path, if one path from v_0 to v_k is shorter than another using weight function w, then it is also shorter using $\hat{w}$. Thus, $w(p) = \delta(v_0, v_k)$ if and only if $\hat{w}(p) = \hat{\delta}(v_0, v_k)$.

Finally, we show that G has a negative-weight cycle using weight function w if and only if G has a negative-weight cycle using weight function $\hat{w}$. Consider any cycle $c = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = v_k$. By equation (25.10),

$$\begin{aligned} \hat{w}(c) &= w(c) + h(v_0) - h(v_k) \\ &= w(c) , \end{aligned}$$

and thus c has negative weight using w if and only if it has negative weight using $\hat{w}$. ■

Producing nonnegative weights by reweighting

Our next goal is to ensure that the second property holds: we want $\hat{w}(u, v)$ to be nonnegative for all edges $(u, v) \in E$. Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$, we make a new graph $G' = (V', E')$, where $V' = V \cup \{s\}$ for some new vertex $s \notin V$ and $E' = E \cup \{(s, v) : v \in V\}$. We extend the weight function w so that $w(s, v) = 0$ for all $v \in V$. Note that because s has no edges that enter it, no shortest paths in G', other than those with source s, contain s. Moreover, G' has no negative-weight cycles if and only if G has no negative-weight cycles. Figure 25.6(a) shows the graph G' corresponding to the graph G of Figure 25.1.

Now suppose that G and G' have no negative-weight cycles. Let us define $h(v) = \delta(s, v)$ for all $v \in V'$. By the triangle inequality (Lemma 24.10), we have $h(v) \leq h(u) + w(u, v)$ for all edges $(u, v) \in E'$. Thus, if we define the new weights $\hat{w}$ by reweighting according to equation (25.9), we have $\hat{w}(u, v) = w(u, v) + h(u) - h(v) \geq 0$, and we have satisfied the second property. Figure 25.6(b) shows the graph G' from Figure 25.6(a) with reweighted edges.

Computing all-pairs shortest paths

Johnson's algorithm to compute all-pairs shortest paths uses the Bellman-Ford algorithm (Section 24.1) and Dijkstra's algorithm (Section 24.3) as subroutines. It assumes implicitly that the edges are stored in adjacency lists. The algorithm returns the usual $|V| \times |V|$ matrix $D = d_{ij}$, where $d_{ij} = \delta(i, j)$, or it reports that the input graph contains a negative-weight cycle. As is typical for an all-pairs shortest-paths algorithm, we assume that the vertices are numbered from 1 to $|V|$.

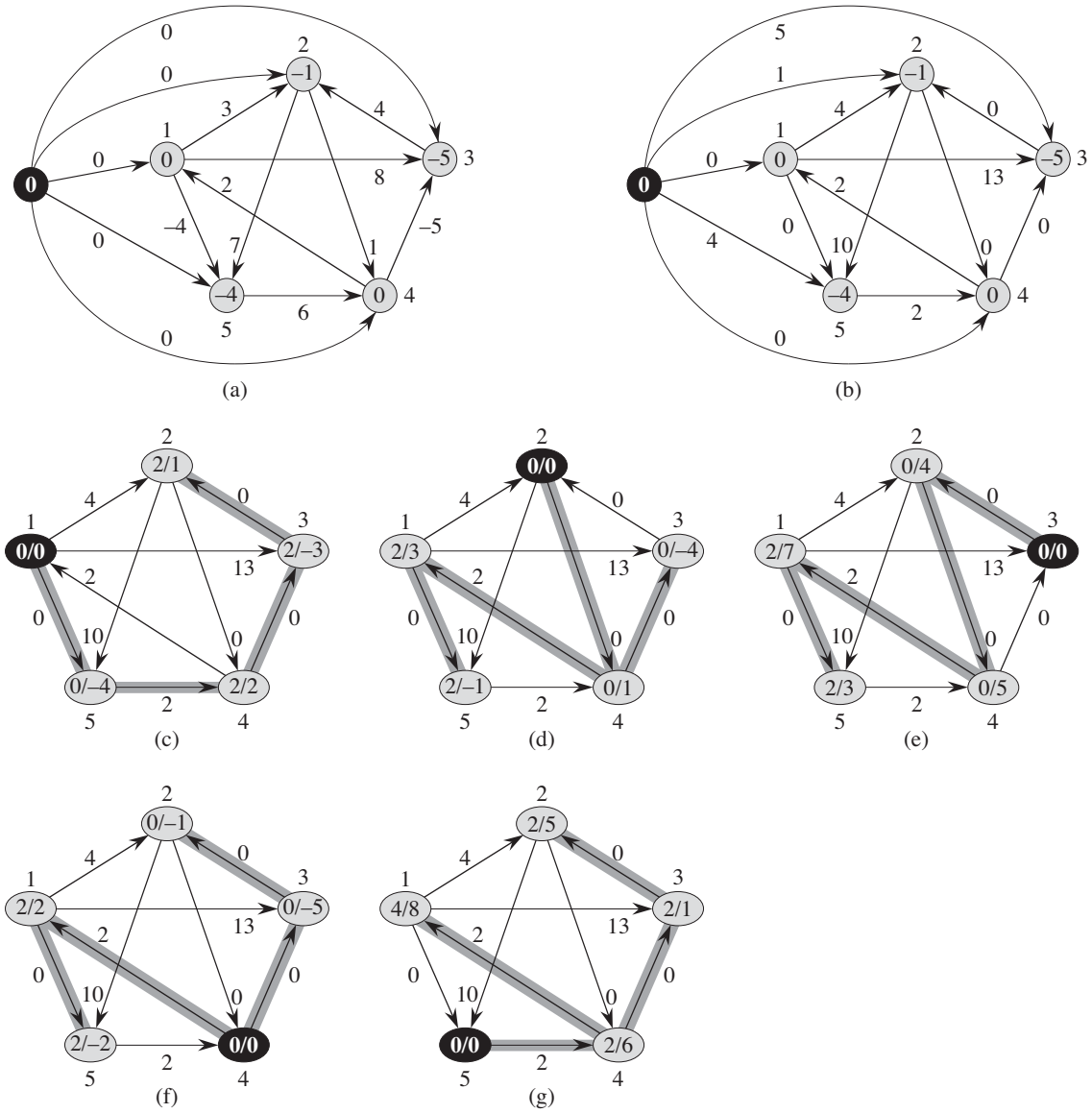

Figure 25.6 Johnson's all-pairs shortest-paths algorithm run on the graph of Figure 25.1. Vertex numbers appear outside the vertices. **(a)** The graph G' with the original weight function w. The new vertex s is black. Within each vertex v is $h(v) = \delta(s, v)$. **(b)** After reweighting each edge (u, v) with weight function $\hat{w}(u, v) = w(u, v) + h(u) - h(v)$. **(c)–(g)** The result of running Dijkstra's algorithm on each vertex of G using weight function $\hat{w}$. In each part, the source vertex u is black, and shaded edges are in the shortest-paths tree computed by the algorithm. Within each vertex v are the values $\hat{\delta}(u, v)$ and $\delta(u, v)$, separated by a slash. The value $d_{uv} = \delta(u, v)$ is equal to $\hat{\delta}(u, v) + h(v) - h(u)$.

```
JOHNSON(G, w)
 1  compute G', where G'.V = G.V ∪ {s},
        G'.E = G.E ∪ {(s, v) : v ∈ G.V}, and
        w(s, v) = 0 for all v ∈ G.V
 2  if BELLMAN-FORD(G', w, s) == FALSE
 3      print "the input graph contains a negative-weight cycle"
 4  else for each vertex v ∈ G'.V
 5          set h(v) to the value of δ(s, v)
                computed by the Bellman-Ford algorithm
 6      for each edge (u, v) ∈ G'.E
 7          ŵ(u, v) = w(u, v) + h(u) − h(v)
 8      let D = (d_uv) be a new n × n matrix
 9      for each vertex u ∈ G.V
10          run DIJKSTRA(G, ŵ, u) to compute δ̂(u, v) for all v ∈ G.V
11          for each vertex v ∈ G.V
12              d_uv = δ̂(u, v) + h(v) − h(u)
13      return D
```

This code simply performs the actions we specified earlier. Line 1 produces G'. Line 2 runs the Bellman-Ford algorithm on G' with weight function w and source vertex s. If G', and hence G, contains a negative-weight cycle, line 3 reports the problem. Lines 4–12 assume that G' contains no negative-weight cycles. Lines 4–5 set $h(v)$ to the shortest-path weight $\delta(s, v)$ computed by the Bellman-Ford algorithm for all $v \in V'$. Lines 6–7 compute the new weights $\hat{w}$. For each pair of vertices $u, v \in V$, the **for** loop of lines 9–12 computes the shortest-path weight $\hat{\delta}(u, v)$ by calling Dijkstra's algorithm once from each vertex in V. Line 12 stores in matrix entry d_{uv} the correct shortest-path weight $\delta(u, v)$, calculated using equation (25.10). Finally, line 13 returns the completed D matrix. Figure 25.6 depicts the execution of Johnson's algorithm.

If we implement the min-priority queue in Dijkstra's algorithm by a Fibonacci heap, Johnson's algorithm runs in $O(V^2 \lg V + VE)$ time. The simpler binary min-heap implementation yields a running time of $O(VE \lg V)$, which is still asymptotically faster than the Floyd-Warshall algorithm if the graph is sparse.

Exercises

25.3-1

Use Johnson's algorithm to find the shortest paths between all pairs of vertices in the graph of Figure 25.2. Show the values of h and $\hat{w}$ computed by the algorithm.