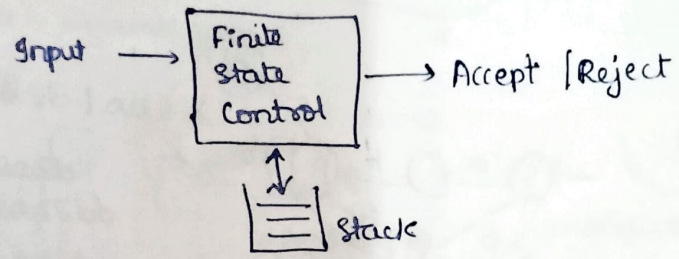
Pushdown Automata

It is an extention of ε-NFA. ~~with~~

PDA = ε-NFA + stack

Informal definition :

PDA is NFA with ε-transitions permitted and one additional capability: a stack on which it can store a string of stack symbols.



In one transition, the PDA

- Consumes from input symbol that it uses in transition. If ε is used for input, then no input symbol is consumed.
- Goes to new state which may or may not be same as previous state.
- Replaces the symbol at top of stack by any string.

Formal definition

PDA involves 7 components :

$$P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$

Q = Finite set of states

Σ = Finite set of input symbols

Γ = Finite stack alphabet

δ = Transition function

(takes $\delta(q, a, X)$)

q → state, a → input symbol, X → stack symbol

q_0 = start state

Z_0 = start symbol of stack

F = set of accepting states

PDA acceptance —— Final State, Empty stack

Instantaneous descriptions

ID or configuration of PDA by triple (q, w, γ)

q → state, w → remaining input, γ → stack contents

To prove the string is accepted or not by PDA, we use ID

① PDA $P = (\{q, p\}, \{0, 1\}, \{Z_0, X\}, \delta, q, Z_0, \{p\})$

$\delta(q, 0, Z_0) = \{(q, XZ_0)\}$

$\delta(q, 0, X) = (q, XX)$

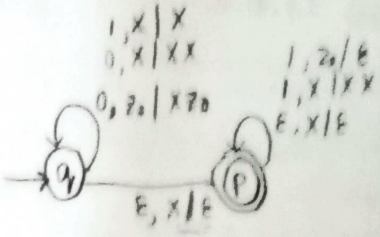
$\delta(q, 1, X) = (q, X)$

$\delta(q, \varepsilon, X) = (p, \varepsilon)$

$\delta(p, \varepsilon, X) = (p, \varepsilon)$

$\delta(p, 1, X) = (p, XX)$

$\delta(p, 1, Z_0) = (p, \varepsilon)$



a) 01

$(q, 01, Z_0)$ —no push→ ✗

↓ push

$(q, 1, XZ_0)$

↓ push → $(p, 1, Z_0) \longrightarrow (p, \varepsilon, \varepsilon)$

(q, ε, XZ_0)

↓

(p, ε, Z_0) Accepted

b) 0011

$(q, 0011, Z_0)$

↓

$(q, 011, XZ_0)$ → $(p, 011, Z_0)$ ✗

↓

$(q, 11, XX)$ → $(p, 11, X) \longrightarrow (p, 1, XX) \longrightarrow (p, \varepsilon, XXX)$

↓ → $(p, 1, Z_0)$

$(q, 1, X)$

↓

(q, ε, X)

↓

(p, ε, Z_0)

Accepted

(p, ε, XX)

↓

(p, ε, X)

↓

(p, ε, Z_0)

c) 010

$(q_1, 010, z_0)$ → ✗

↓

$(q_1, 10, Xz_0)$

↓

$(q_1, 0, Xz_0)$ → $(p, 10, z_0) \longrightarrow (p, 0, z_0) \longrightarrow$ ✗

↓

(q_1, ϵ, XXz_0) → $(p, 0, z_0)$ → ✗

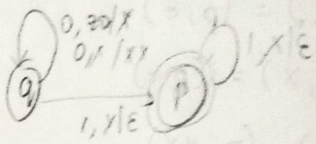
↓

✗

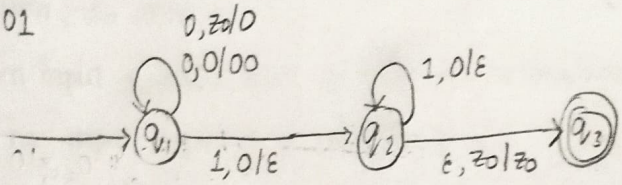
Rejected

② Design PDA

a) $\{0^n 1^n \mid n \geq 1\}$

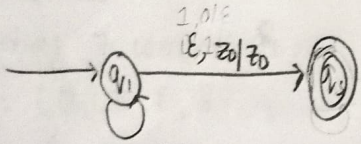


$S \rightarrow 0S1 \mid 01$



b) Set of strings of 0's and 1's are equal.

01, 10, 0101, 1010, 000111



0, z0/0z0

1, z0/1z0

0, 0/00

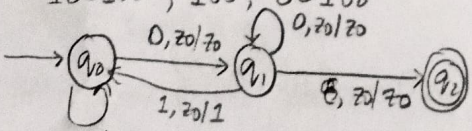
1, 1/11

1, 0/ε

0, 1/ε

c) Set of all strings of 0's and 1's such that no prefix has more 1's than 0's

100100, 100, 00100



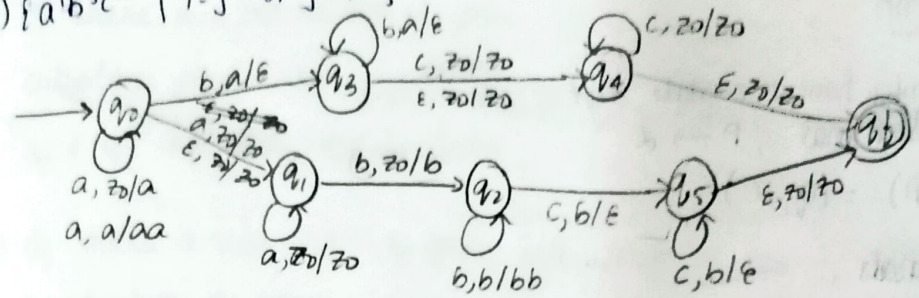
0, z0/z0

1, z0/1

1, 1/11

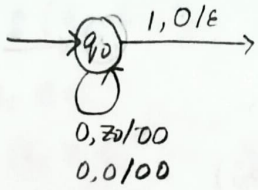
0, 1/ε

d) $\{a^i b^j c^k \mid i = j$ or $j = k\}$

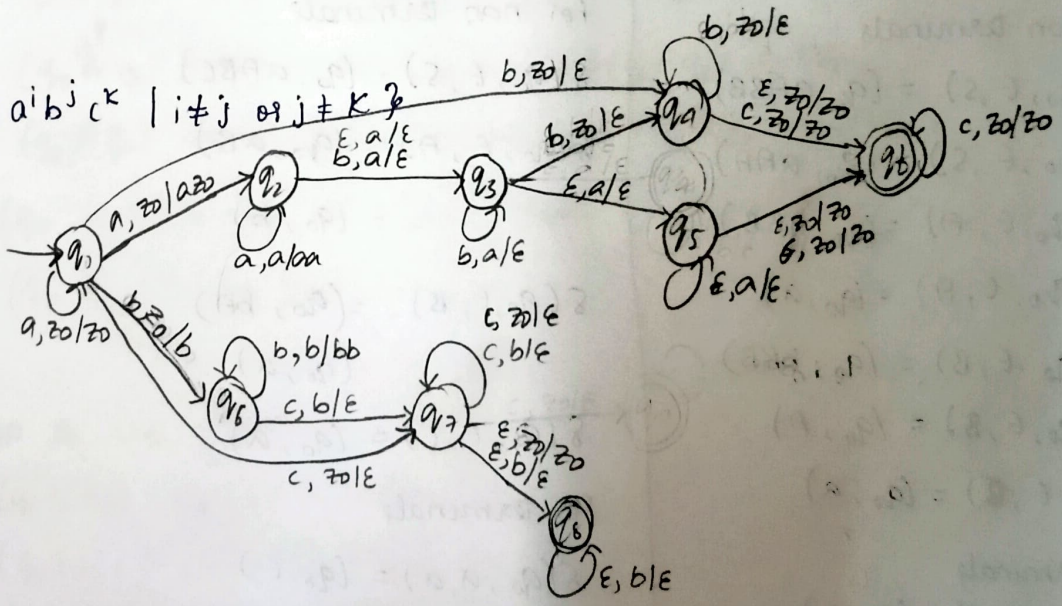


e) Set of strings with twice as many 0's as 1's

001, 100, 001100



f) $\{a^i b^j c^k \mid i \neq j$ or $j \neq k\}$



CFG → PDA

Steps: PDA will only have one state $\{q\}$

1) For non terminal, $A \to \alpha$

$$\delta(q, \epsilon, A) = (q, \alpha)$$

2) For terminals

$$\delta(q, a, a) = (q, \epsilon)$$

Ex: 1) $S \to aABB \mid aAA$

$A \to aBB \mid a$

$B \to bBB \mid A$

$C \to a$

$\delta(q, \epsilon, z_0) = (q_0, Sz_0)$

For non terminals

$\delta(q_0, \epsilon, S) = (q_0, aABB)$

$\delta(q_0, \epsilon, S) = (q_0, aAA)$

$\delta(q_0, \epsilon, A) = (q_0, aBB)$

$\delta(q_0, \epsilon, A) = (q_0, a)$

$\delta(q_0, \epsilon, B) = (q_0, bBB)$

$\delta(q_0, \epsilon, B) = (q_0, A)$

$\delta(q_0, \epsilon, C) = (q_0, a)$

For terminals

$\delta(q_0, a, a) = (q_0, \epsilon)$

$\delta(q_0, b, b) = (q_0, \epsilon)$

$\delta(q_0, \epsilon, z_0) = (q_f, z_0)$

2) $S \to aABC$

$A \to aB \mid b$

$B \to bA \mid a$

$C \to a$

$\delta(q, \epsilon, z_0) = (q_0, Sz_0)$

For non terminals

$\delta(q_0, \epsilon, S) = (q_0, aABC)$

$\delta(q_0, \epsilon, A) = (q_0, aB)$

$= (q_0, b)$

$\delta(q_0, \epsilon, B) = (q_0, bA)$

(q_0, a)

$\delta(q_0, \epsilon, C) = (q_0, a)$

For terminals

$\delta(q_0, a, a) = (q_0, \epsilon)$

$\delta(q_0, b, b) = (q_0, \epsilon)$

$\delta(q_0, \epsilon, z_0) = (q_f, z_0)$

PDA → CFG

Steps:

1) Assume there is a transition from q_i to q_j on input symbol 'a' on top of stack z_0, the non-terminal corresponding to this transition is given by $q_i z_0 q_j$

ii) If there is a transition of form
introduce production of form $(q_i$
q_2 & q_k are all possible states

iii) If there is transition of form
production of form $(q_i z q_j)$

ex: 1) $\delta(q_0, a, z_0) = (q_0, Az)$ —

$\delta(q_0, b, A) = (q_0, AA)$ →

$\delta(q_0, a, A) = (q_1, \epsilon)$ →

$S \to q_0 z q_1$

① $(q_0 z q_0) \to a(q_0 A q_0)(q$ [$q_k = 0$]

$(q_0 z q_1) \to a(q_0 A q_0)(q$ [$q_k = 1$]

② $(q_0 A q_0) \to b(q_0 A q_0)(q$

$(q_0 A q_1) \to b(q_0 A q_0)($

③ $\delta(q_0, a, A) = (q_1, \epsilon)$ - The

$(q_0 A q_1) \to a$

① PDA to CFG

$\delta(q_1, 1, z_0) = (q, Xz_0)$ — ①

$\delta(q, 1, X) = (q, XX)$ — ②

$\delta(q, 0, X) = (p, X)$ — ③

$\delta(q, \epsilon, X) = (q, \epsilon)$

$\delta(p, 1, X) = (p, \epsilon)$

$\delta(p, 0, z_0) = (q, z_0)$ — ④

$S \to q z q \mid q z p$

① $(q z q) \to 1(q X q)(q z q)$

$(q z p) \to 1(q X q)(q z p)$

② $(q X q) \to 1(q X q)(q X q)$

$(q X p) \to 1(q X q)(q X p)$

③ $(q X p) \to 0(p X q) \mid 0(p$

④ $(p z_0 q) \to 0(q z_0 q) \mid 0($

ii) If there is a transition of form $\delta(q_i, a, z_0) = (q_j, AB)$ then introduce production of form $(q_i z q_k) \rightarrow a(q_j A q_l)(q_l B q_k)$ where q_l & q_k are all possible states.

iii) If there is transition of form $\delta(q_i, a, z) = (q_j, \epsilon)$ then introduce production of form $(q_i z q_j) \rightarrow a$

Ex: 1) $\delta(q_0, a, z_0) = (q_0, Az) \rightarrow ①$

$\delta(q_0, b, A) = (q_0, AA) \rightarrow ②$

$\delta(q_0, a, A) = (q_1, \epsilon) \rightarrow ③$

$S \rightarrow q_0 z q_1$

Second step

① $(q_0 z q_0) \rightarrow a(q_0 A q_0)(q_0 z q_0) \mid a(q_0 A q_1)(q_1 z q_0)$ $\quad q_k = 0$

$(q_0 z q_1) \rightarrow a(q_0 A q_0)(q_0 z q_1) \mid a(q_0 A q_1)(q_1 z q_1)$ $\quad q_k = 1$

② $(q_0 A q_0) \rightarrow b(q_0 A q_0)(q_0 A q_0) \mid b(q_0 A q_1)(q_1 A q_0)$

$(q_0 A q_1) \rightarrow b(q_0 A q_0)(q_0 A q_1) \mid b(q_0 A q_1)(q_1 A q_1)$

③ $\delta(q_0, a, A) = (q_1, \epsilon)$ = Third step

$(q_0 A q_1) \rightarrow a$

① PDA to CFG

$\delta(q, 1, z_0) = (q, Xz_0) \rightarrow ①$

$\delta(q, 1, X) = (q, XX) \rightarrow ②$

$\delta(q, 0, X) = (p, X) \rightarrow ③$

$\delta(q, \epsilon, X) = (q, \epsilon)$

$\delta(p, 1, X) = (p, \epsilon)$

$\delta(p, 0, z_0) = (q, z_0) \rightarrow ④$

$S \rightarrow q z q \mid q z p$

① $(q z q) \rightarrow 1(q X q)(q z q) \mid 1(q X p)(p z q)$

$(q z p) \rightarrow 1(q X q)(q z p) \mid 1(q X p)(p z p)$

② $(q X q) \rightarrow 1(q X q)(q X q) \mid 1(q X p)(p X q)$

$(q X p) \rightarrow 1(q X q)(q X p) \mid 1(q X p)(p X p)$

③ $(q X p) \rightarrow 0(p X q) \mid 0(p X p)$

④ $(p z_0 q) \rightarrow 0(q z_0 q) \mid 0(q z_0 p)$

$(q \times q) \longrightarrow e$

$(p \times p) \longrightarrow 1$

② CFG $\longrightarrow$ PDA that accepts same lang by empty stack

a) $S \longrightarrow 0S1 \mid A$

$A \longrightarrow 1A0 \mid S \mid \varepsilon$

$\delta(q_0, \varepsilon, S) = (q_0, 0S1)$

$\delta(q_0, \varepsilon, S) = (q_0, A)$

$\delta(q_0, \varepsilon, A) = (q_0, 1A0)$

$\delta(q_0, \varepsilon, A) = (q_0, S)$

$\delta(q_0, \varepsilon, A) = (q_0, \varepsilon)$

$\delta(q_0, 0, 0) = (q_0, \underline{\varepsilon})$

$\delta(q_0, 1, 1) = (q_0, \underline{\varepsilon})$ empty

$\delta(q_0, \varepsilon, Z_0) = (q_f, \underline{\varepsilon})$ empty string

b) $S \rightarrow aAA$

$A \rightarrow aS \mid bS \mid a$

$\delta(q_0, \varepsilon, S) = (q_0, aAA)$

$\delta(q_0, \varepsilon, A) = (q_0, aS)$

$\delta(q_0, \varepsilon, A) = (q_0, bS)$

$\delta(q_0, \varepsilon, A) = (q_0, a)$

$\delta(q_0, a, a) = (q_0, \varepsilon)$

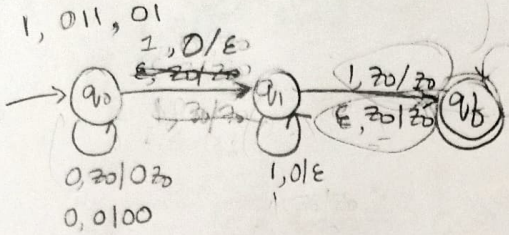
$\delta(q_0, \varepsilon, Z_0) = (q_f, \varepsilon)$

DPDA = DFA + PDA Ex: wcw^r

① Draw DPDA

a) $0^n 1^m \mid n \le m$

1, 011, 01



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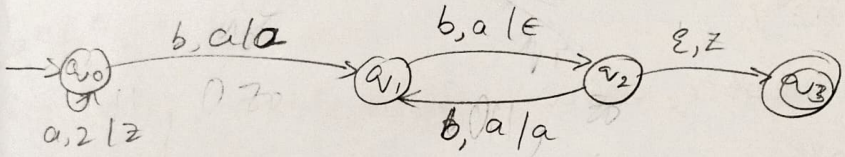
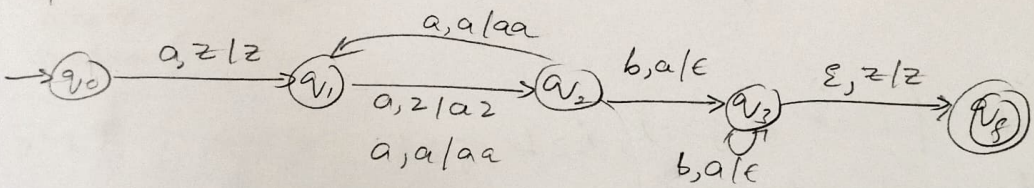
② $a^n b^{2n} \mid n > 0$

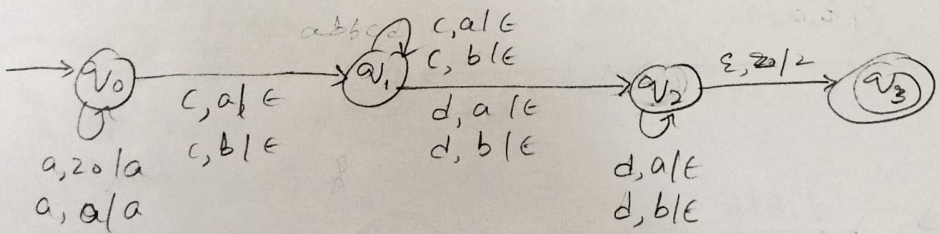
L = {abb, aabbbb, ...}

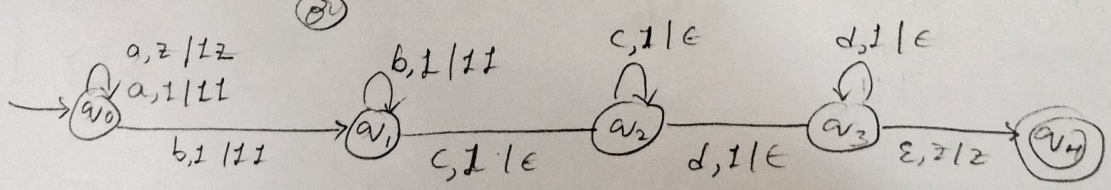


3) NPDA = $a^{2m} b^{3m} \mid m \ge 1$



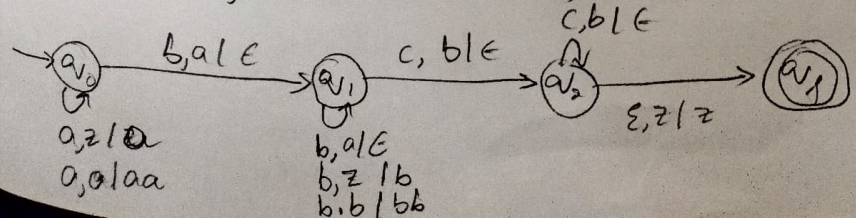
4) NPDA → $\{a^m b^n c^p d^q \mid m+n = p+q;\ m, n, p, q >= 1\}$





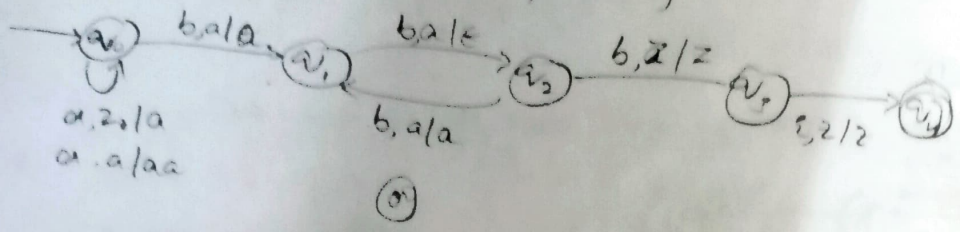
5) NPDA → $a^m b^{(m+n)} c^n \mid m, n \ge 1$

L = {abbc, abbbcc ....}

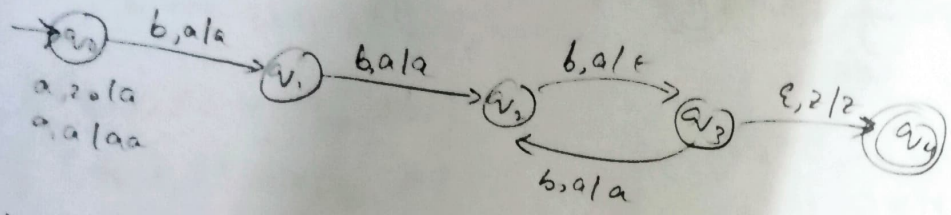


6) $a^m b^{2m+1} \mid m \geq 1$

$L = \{abbb, aabbbbbb, \ldots\}$

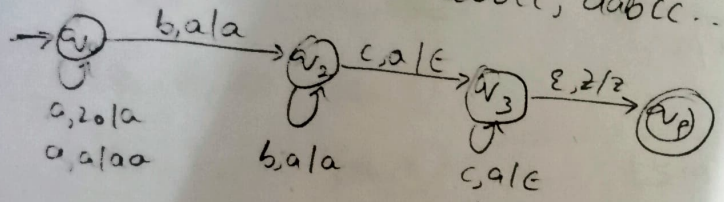


(or)



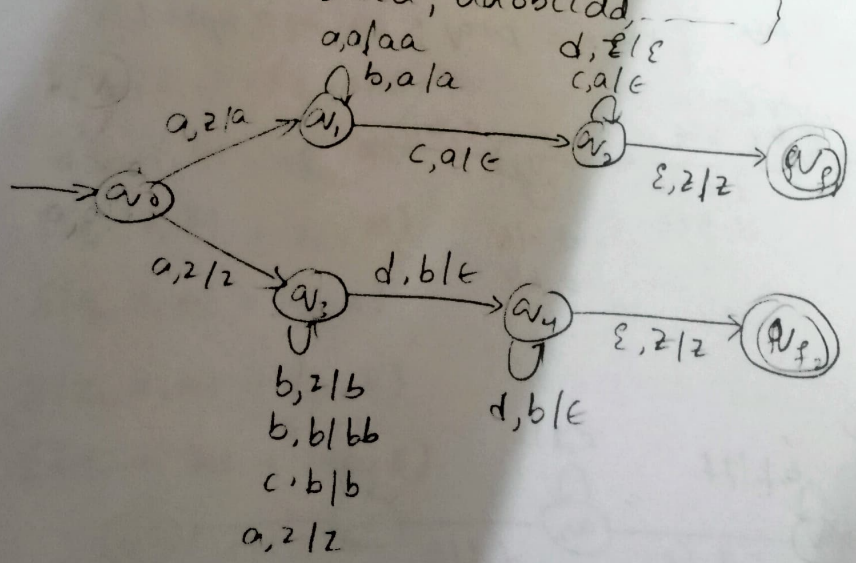
7) $a^n b^m c^n \mid m, n \geq 1$

$L = \{abc, abbc, aabbcc, aabcc \ldots\}$



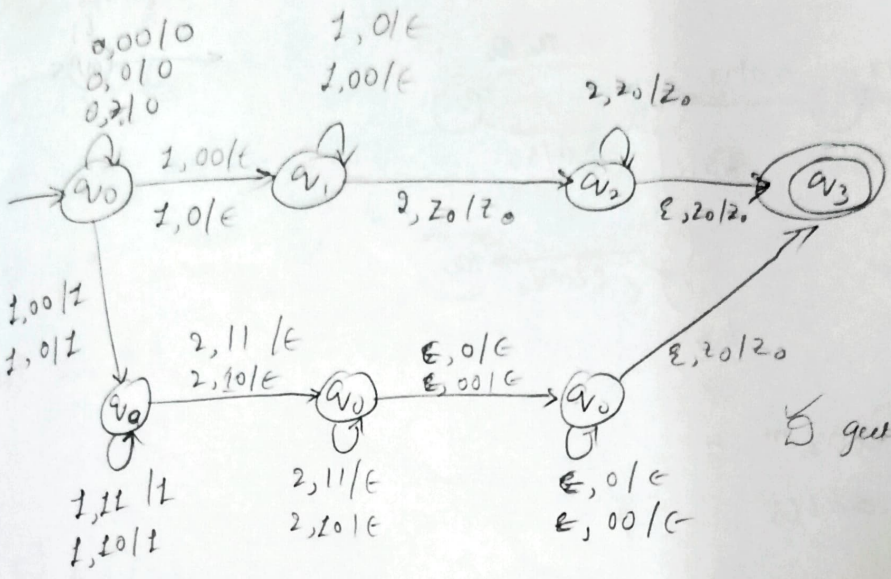
8) $a^i b^j c^k d^l \mid i==k$ or $j==l$, $i \geq 1$, $j \geq 1$

$L = \{abcd, aaabcccd, aabbccdd \ldots\}$



9) $0^i 1^j 2^k \mid i==j$ or $j==k$; $j \geq i, k >= 1$

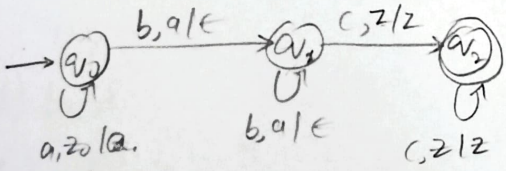
$L = \{00011122222, 00111222, 000111222\}$



5 gets for grade

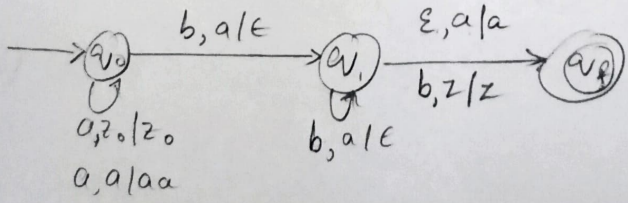
10) $a^n b^n c^m \mid m, n \geq 1$

$L = \{abc, aabbc, aabbccc, \ldots\}$



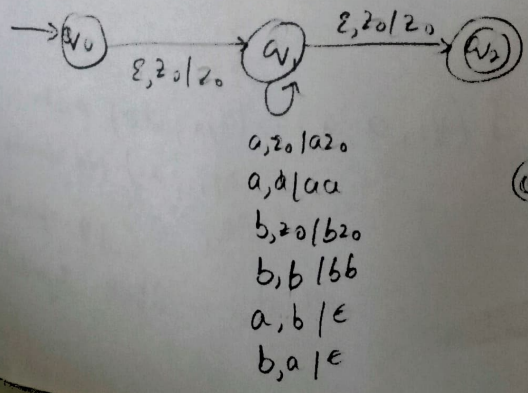
11) $a^n b^m \mid m, n \geq 1 \;\&\; n \neq m$

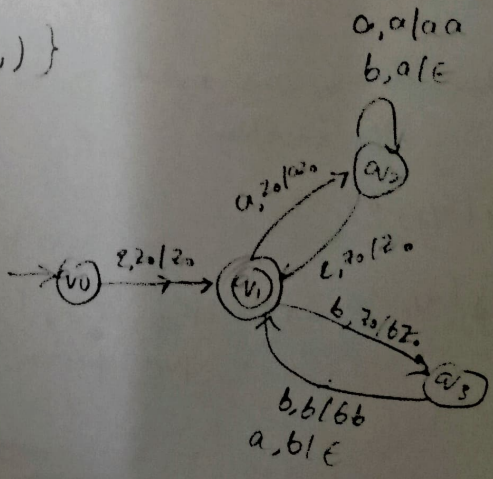
$L = \{abb, aab, aabbb, aaab, \ldots\}$



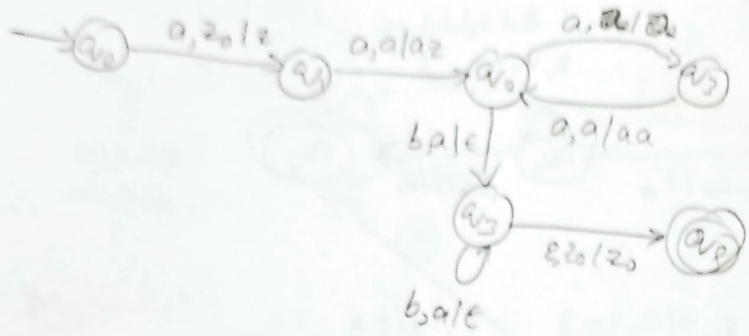
4) DPDA

1) $L = \{w \in \{a,b\}^* \mid n_a(w) = n_b(w)\}$



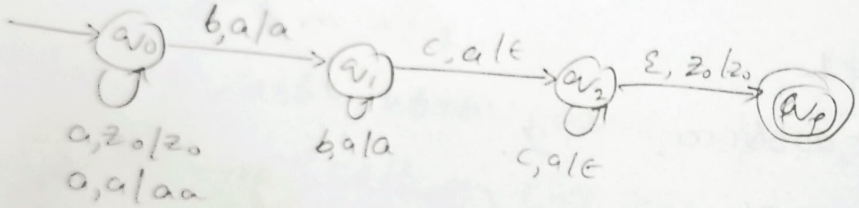


2) NPDA – $a^n b^n$



3) DPDA → $a^n b^m c^n$

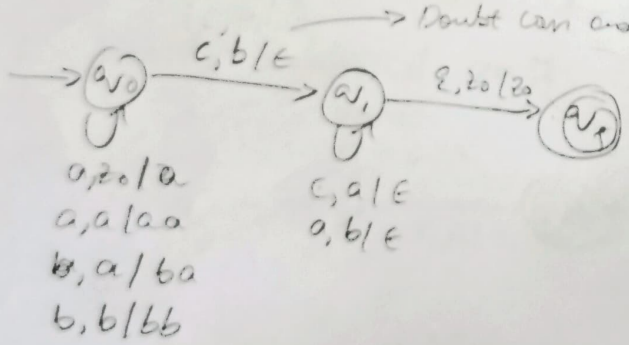
$L = \{abc, aabcc, \ldots\}$



4) DPDA → $a^n b^m c^{m+n}$

$L = \{abcc, aabccc, \ldots\}$

→ Doubt can add c, a/ε



DPDA

- centre symbol is known
  WCW^R, $a^n b^n$
- there is only one move in all situation
  $\delta(q_0, a) \to q_1$
- $\delta(q_0, a, z_0) \to (q_0, a z_0)$
  push operation

NPDA

- centre symbol not known
  LL^R
- multiple move & multiple op
  $\delta(q_0, a) \to q_0$
  q_1
  q_0
  $\delta(q_0, a, a) \to (q_0, aa)$ push
  (q_1, a) No operation
  (q_1, ε) pop operation
  with different input symbol

Closure property :

1) $L_1 \cup L_2$ -> Union (either or) $S \to S_1 \mid S_2$

2) $L_1 L_2$ -> concat (.) $S \to S_1 . S_2$

3) L^* -> Kleen. $L \to \{a\}$ $L^* \to \{\epsilon, a, aa, aaa\}$

4) $L_1 \cap L_2$ $L_1 \to a^n b^n c^m$ ->

$L_2 \to a^m b^n c^n$

$L_3 \to a^n b^n c^n$ $\to$ CSL

5) $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$ Assume $\overline{L_1}$ is CFL

$= \overline{CFL \cup CFL}$ Assume compliment is CFL

$= \overline{CFL} \to CFL \to$ so it contriduct above fact about intersection not CFL so $\overline{L}$ also not CFL.

Difference

6) $L_1 \cap L_2 = L_1 - (L_1 - L_2)$

7) closed under Reverse

$L_1 = 0^n 1^n, n \geq 1$

$S \to 0S1 \mid 01$

$L_1^R = S \to 1S0 \mid 10$

8) closed under Homomorphism

$S \to 0S1 \mid 01$

$h(0) = ab$ $h(1) = \varepsilon$

$S \to abS \mid ab$

* Converting CFG to CNF (Greeke for Greeks)