UNIT-01

Enumeration or counting is a obvious process when first studying arithmetic
Applications of Enumeration : Coding Theory, Probability, Statistics, Analysis of algorithm.

Rules of Sum & Product

m + n   mn

6 & 8 + 8 & 3

R + D: $8 + 5 = 13$

$8 \times 5 = 40$

letters: 26 25

digits: 10 9 8 7

5 5 .   5 5 5 5 .

10 mem

$10 + 9 + 8 + 7 = 34$

3

10, 18, 80

$$P(7,2) = \frac{7!}{5!} = \frac{5040}{120} = 42 \qquad \frac{7\times 6\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}}{\cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}} = 42$$

① SOCIOLOGICAL

(a) All letters

S	O	C	I	L	G	A
1	3	2	2	2	1	1

$$\frac{12!}{1!\times 3!\times 2!\times 2!\times 2!}$$

(b) A & G are adjacent

S	O	C	I	L	$\boxed{GA}$
1	3	2	2	2	1

$$2\times\frac{11!}{2!\times 2!\times 2!\times 3!}$$

(c) all vowels adjacent

A	E	I	O	U
1	0	2	3	0

$\underbrace{\qquad}_{1}$

S~~O~~C~~I~~~~O~~L~~O~~G~~I~~C~~A~~L

$$\frac{6!}{1!\,2!\,3!} \times \frac{7!}{2!\,2!}$$

② a, b, c, d, e, e, e, e, e = no e is adjacent to other e

4!

③ How many +ve int n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5000000?

(_) _ _ _ _ _ _

5, 6, 7

case 1 : 5 _ _ _ _ _ _ $\frac{6!}{2!}$

case 2 : 6 _ _ _ _ _ _ $\frac{6!}{(2!)\times(2!)}$ + 720

case 3 : 7 _ _ _ _ _ _ $\frac{6!}{(2!)\times 2!}$

Permutation — Repetion
— Circular

$$\frac{n!}{(n-r)!}$$

$$nC_r = \binom{n}{r}$$

Binomial Coefficient

$(x+y)^0 = 1$

$(x+y)^1 = x+y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Binomial Theorem: If x & y are variables & n is +ve int, then

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \dots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^n y^0$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$\uparrow$

binomial coefficient

$$T_{r+1} = {}^nC_r \, x^{n-r} y^r$$

① Co-efficient of x^5y^2 in the expansion of $(x+y)^7$ is

$\binom{7}{5} = \binom{7}{2} = 21$ $\quad n=7 \quad r=2 \quad {}^7C_2\, x^5 y^2$

② Coefficient of a^5b^2 in expansion of $(2a-3b)^7$

$n=7$

$= {}^7C_2 (2a)^5 (-3b)^2$

$= \binom{7}{2} 2^5 a^5 (-3)^2 b^2$

$= \binom{7}{2} 2^5 (-3)^2 (a^5 b^2)$

③ Determine co-efficient of x^9y^3 in expansion of $(x+2y)^{12}$

$n=12$

$\binom{12}{3} x^9 (2y)^3$

$\binom{12}{3} 2^3$

Note:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + \binom{n}{n-1} - \binom{n}{n} = 0$$

Multinomial theorem:

For +ve int n, t

$$\frac{n!}{n_1!\, n_2! \ldots n_t!} = \binom{n}{n_1\ n_2 \ldots n_t}$$

Co-efficient

a) xyz^2 in $(x+y+z)^4$

$\binom{4}{1,1,2}$

b) xyz^2 in $(w+x+y+z)^4$

$\binom{4}{0,1,1,2}$

c) xyz^2 in $(2x-y-z)^4$

$\binom{4}{1,1,2}(2)(-1)(-1)^2$

d) xyz^{-2} in $(x-2y+3z^{-1})^4$ $\quad x^{1}y^{1}\frac{1}{z^{2}}$

$\binom{4}{1\ 1\ 2}(-2)(3)^2$

e) $w^3x^2yz^2$ in $(2w-x+3y-2z)^8$

$\binom{8}{3,2,1,2}(2)^3(-1)^2(3)(-2)^2$

Combinations with repetition

$$\frac{(n+r-1)!}{r!\,(n-1)!} = \binom{n+r-1}{r}$$

$x+y=3 \quad x,y \geq 0$

0	3
1	2
2	1
3	0

4

$n=2$, $r=3$ $\binom{2+3-1}{3}$

$x+y=4$

0	4
2	2
1	3
3	1
4	0

5

$\binom{2+4-1}{4}$

$n=2$, $r=4$

$x+y+z=3$

3	0	0
0	3	0
0	0	3
1	1	1
1	2	0
1	0	2
0	1	2
0	2	1
2	0	1
2	1	0

10

$n=3$, $r=3$ $\binom{5}{3}$

Apply formula

n = no. of distinct object

r = repetion

① In how many ways can 10 (identical) dimes be distributed among 5 children if (a) there are no restrictions?

(b) each child gets at least one dime?

(c) oldest child gets at least two dimes?

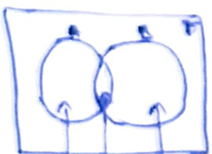
(a) $x_1 + x_2 + x_3 + x_4 + x_5 = 10$ $n = 5$ $r = 10$

$\binom{14}{10}$

(b) $x_1 + x_2 + x_3 + x_4 + x_5 = 5$ $n = 5$ $r = 5$ $\binom{9}{5}$

(c) $x_1 + x_2 + x_3 + x_4 + x_5 = 8$ $n = 5$ $r = 8$ $\binom{12}{8}$

Principle of Inclusion & Exclusion :

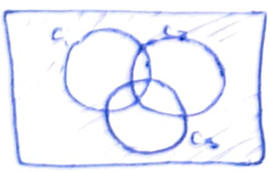


N(A) N(B)

N(A∩B)

$N(\bar{A}\bar{B}) = N - \{[N(A) + N(B)] - [N(A \cap B)]\}$

$= N - \{N(A) + N(B)\} + N(AB)$



$N(\bar{C_1}\bar{C_2}\bar{C_3}) = N - \{N(C_1) + N(C_2) + N(C_3)\} + \{N(C_1C_2) + N(C_1C_3) + N(C_2C_3)\} - N(C_1C_2C_3)$

① Find the numbers of +ve integers not exceeding 100 that are not divisible by 7 & 5

C_1 : Numbers divisible by 5

C_2 : Numbers divisible by 7

$N(C_1) = 100/5 = 20$

$N(C_2) = 100/7 = 14$

$N(C_1 \cap C_2) = 100/(5.7) = 2$

~~$N(C_1) = N(C_1 \cup C_2)$~~ . $N(\bar{C_1}\bar{C_2}) = N - [N(C_1) + N(C_2)] + \frac{N(C_1 \cap C_2)}{N(C_1C_2)}$

$= 100 - [34] + 2$

$= 100 - 36$

$= 64$

② Determine no. of +ve int n where $1 \le n \le 100$ & n is not divisible by 2, 3 & 5

C_1 : Numbers divisible by 2 $N(C_1) [100/2] = 50$

C_2 : —— " —— 3 $N(C_2) 100/3 = 33$

—— 5 $N(C_3) 100/5 = 20$

$N(C_1C_2) = \left[\frac{100}{6}\right] = 16$ $N(C_3C_1) = \left[\frac{100}{10}\right] = 10$

$N(C_2C_3) = \left[\frac{100}{15}\right] = 6$ $N(C_1C_2C_3) = \left[\frac{100}{30}\right] = 3$

$N(\bar{C}_1\bar{C}_2\bar{C}_3) = N - [N(C_1) + N(C_2) + N(C_3)] + [N(C_1C_2) + N(C_2C_3) + N(C_1C_3)] - N(C_1C_2C_3)$

$= 100 - [50 + 33 + 20] + [16 + 6 + 10] - 3$

$= 100 - 103 + 32 - 3$

$\cancel{= 100 - 68}\ -6 + 32$

$= \cancel{32}\ 26$

③ Determine no/: of +ve int n, $1 \le n \le 2000$, that are

(a) not divisible by 2, 3, 5, or 7

C_1: Divisible by 2,

C_2: — " — 3

C_3: — " — 5

C_4: — " — 7

$N(C_1C_2) = \left\lfloor \frac{2000}{6} \right\rfloor = 333$ $N(C_1C_4) = \left\lfloor \frac{2000}{14} \right\rfloor = 142$

$N(C_1C_3) = \left\lfloor \frac{2000}{10} \right\rfloor = 200$ $N(C_2C_3) = \left\lfloor \frac{2000}{15} \right\rfloor = 133$

$N(C_3C_4) = \left\lfloor \frac{2000}{35} \right\rfloor = 57$ $N(C_2C_4) = \left\lfloor \frac{2000}{21} \right\rfloor = 95$

~~Nf~~

$N(C_1C_2C_3) = \left\lfloor \frac{2000}{30} \right\rfloor = 66$ $N(C_1C_3C_4) = \left\lfloor \frac{2000}{70} \right\rfloor = 28$ $N(C_2C_3C_4) = \left\lfloor \frac{2000}{105} \right\rfloor = 19$

$N(C_1) = \left\lfloor \frac{2000}{2} \right\rfloor = 1000$ $N(C_2) = \left\lfloor \frac{2000}{3} \right\rfloor = 666$ $N(C_3) = \left\lfloor \frac{2000}{5} \right\rfloor = 400$ $N(C_4) = \left\lfloor \frac{2000}{7} \right\rfloor = 2$

$N(C_1C_2C_4) = \left\lfloor \frac{2000}{42} \right\rfloor = 47$ $N(C_1C_2C_3C_4) = \left\lfloor \frac{2000}{210} \right\rfloor = 9$

$N(\bar{C}_1\bar{C}_2\bar{C}_3\bar{C}_4) = 2000 - [2351] + [960] + 160 - 9$

$= 3120 - 2360$

$= 760$

Recurrence Relation

① $t^2 - 7t + 10 = 0$

$t = 5,\ t = 2$

$a_0 = 1$

$a_1 = 8$

$a_n = c_1 5^n + c_2 2^n$

$1 = c_1 + c_2$ $8 = 5c_1 + 2c_2$

$c_1 =$ ___ $c_2 =$ ___