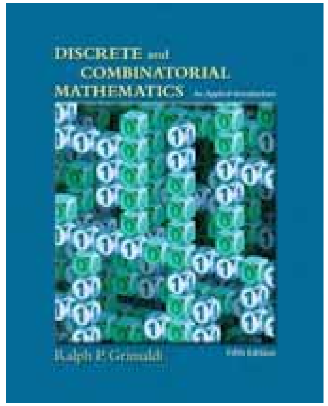
Discrete Mathematics

-- *Chapter 9: Generating Function*



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Outline

- Calculational Techniques
- Partitions of Integers
- The Exponential Generating Function
- The Summation Operator

Enumeration again

- Chapter 1: $c_1+c_2+c_3+c_4=25$, where $c_i >= 0$
- Chapter 8: $c_1+c_2+c_3+c_4=25$, where **10>**$c_i >= 0$
- In chapter 9, c_2 to be even and c_3 to be a multiple of 3

- the coefficient xy^2 in $(x+y)^3$
- the coefficient x^4 in $(x+x^2)(x^2+x^3+x^4)(1+x+2x^2)$

9.1 Introductory Examples

- **Ex 9.1** :
  - One mother buys 12 oranges for three children, Grace, Mary, and Frank.

Table 9.1

G	M	F	G	M	F
4	3	5	6	2	4
4	4	4	6	3	3
4	5	3	6	4	2
4	6	2	7	2	3
5	2	5	7	3	2
5	3	4	8	2	2
5	4	3			
5	5	2			

  - Grace gets at least four, and Mary and Frank gets at least two, but Frank gets no more than five.
  - **Solution**
    - $c_1 + c_2 + c_3 = 12$, where $4 \le c_1, 2 \le c_2$, and $2 \le c_3 \le 5$
    - Generating function:
      $f(x) = (x^4+ x^5+ x^6+ x^7+ x^8)(x^2+ x^3+x^4+ x^5+ x^6)(x^2+ x^3+x^4+ x^5)$
      product $x^i x^j x^k \rightarrow$ every triple (i, j, k)
    - The coefficient of x^{12} in $f(x)$ yields the solution.

Introductory Examples

- **Ex 9.2** :
  - There is an unlimited number of red, green, white, and black jelly beans.
  - In how many ways can we select 24 jelly beans so that we have an even number of white beans and at least six black ones?
  - **Solution**
    - red (green): $1+ x^1+ x^2+\dots+ x^{23}+ x^{24}$
    - white: $1+ x^2+ x^4+\dots+ x^{22}+ x^{24}$
    - black: $x^6+ x^7+\dots+ x^{23}+ x^{24}$
    - Generating function:
      $f(x) = (1+ x^1+ x^2+\dots+ x^{23}+ x^{24})^2(1+ x^2+ x^4+\dots+ x^{22}+ x^{24})$
      $(x^6+ x^7+\dots+ x^{23}+ x^{24})$
    - The coefficient of x^{24} in $f(x)$ is the answer.

Introductory Examples

- **Ex 9.3** : How many nonnegative integer solutions are there for $c_1+c_2+c_3+c_4 = 25$?
  - **Solution**
  - Alternatively, in how many ways 25 pennies can be distributed among four children?
  - Generating function:
    $f(x) = (1+ x^1+ x^2+\ldots+ x^{24}+ x^{25})^4$ (polynomial)
  - The coefficient of x^{25} is the solution.
  - **Note**:
  - $g(x) = (1+ x^1+ x^2+\ldots+ x^{24}+ x^{25} + x^{26}+\ldots)^4$ (**power series**)
    can also generate the answer

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

  - Easier to compute with a power series than with a polynomial

9.2 Definition and Examples: Calculational Techniques

- Definition 9.1:

  Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. The function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

  is called the *generating function* for the given sequence.

- **Ex 9.4** : $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$

  so, $(1 + x)^n$ is the generating function for the sequence

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{n}, 0, 0, 0, \cdots$$

Definition and Examples: Calculational Techniques

- **Ex 9.5** :

a) $(1 - x^{n+1})/(1 - x)$ is the generating function for the sequence 1, 1, 1,…, 1, 0, 0, 0,…, where the first n+1 terms are 1.

$$\because (1-x^{n+1}) = (1-x)(1+x+x^2+\cdots+x^n).$$

b) 1/(1-x) is the generating function for the sequence 1, 1, 1, 1,… $\because$ while $|x|<1, \frac{1}{1-x} = 1+x+x^2+x^3+\cdots$

c) $1/(1-x)^2$ is the generating function for the sequence 1, 2, 3, 4,…

$$\because \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = (-1)(1-x)^{-2}(-1)$$

$$= \frac{1}{(1-x)^2} = \frac{d}{dx}(1+x+x^2+x^3+\cdots) = 1+2x+3x^2+4x^3+\cdots$$

d) $x/(1-x)^2$ is the generating function for the sequence 0,1,2,3,….

$$\because \frac{x}{(1-x)^2} = 0+1x+2x^2+3x^3+4x^4+\cdots$$

Definition and Examples: Calculational Techniques

- **Ex 9.5** :

e) $(x+1)/(1-x)^3$ is the generating function for the sequence $1^2, 2^2, 3^2, 4^2, \ldots$

$$\because \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{d}{dx}(0 + x + 2x^2 + 3x^3 + \cdots)$$

$$\frac{x+1}{(1-x)^3} = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots$$

$$\because \frac{d}{dx}\frac{x}{(1-x)^2}$$

$$= \frac{d}{dx} x(1-x)^{-2}$$

$$= (1-x)^{-2} + x(-2)(1-x)^{-3}(-1)$$

$$= \frac{(1-x)+2x}{(1-x)^3} = \frac{x+1}{(1-x)^3}$$

f) $x(x+1)/(1-x)^3$ is the generating function for the sequence $0^2, 1^2, 2^2, 3^2, 4^2, \ldots$

$$\because \frac{x(x+1)}{(1-x)^3} = 0 + 1x + 2^2 x^2 + 3^2 x^3 + \cdots$$

Definition and Examples: Calculational Techniques

- **Ex 9.5** :
  - g) Further extensions:

$$f_0(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$f_1(x) = x\frac{d}{dx} f_0(x) = \frac{x}{(1-x)^2}$$

$$= 0 + x + 2x^2 + 3x^3 + \cdots$$

$$f_2(x) = x\frac{d}{dx} f_1(x) = \frac{x^2 + x}{(1-x)^3}$$

$$= 0^2 + 1^2 x + 2^2 x^2 + 3^2 x^3 + \cdots$$

$$f_3(x) = x\frac{d}{dx} f_2(x) = \frac{x^3 + 4x^2 + x}{(1-x)^4}$$

$$= 0^3 + 1^3 x + 2^3 x^2 + 3^3 x^3 + \cdots$$

$$f_4(x) = x\frac{d}{dx} f_3(x) = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-x)^5}$$

$$= 0^4 + 1^4 x + 2^4 x^2 + 3^4 x^3 + \cdots$$

Definition and Examples: Calculational Techniques

- **Ex 9.6**：
  a) $1/(1 - ax)$ is the generating function for the sequence $a^0, a^1, a^2, a^3, \ldots$
  b) $f(x) = 1/(1- x)$ is the generating function for the sequence 1, 1, 1, 1,… Then
     - $g(x) = f(x) - x^2$ is the generating function for the sequence 1, 1, 0, 1, 1, 1,…
     - $h(x) = f(x) + 2x^3$ is the generating function for the sequence 1, 1, 1, 3, 1, 1,…
  c) Can we find a generating function for the sequence 0, 2, 6, 12, 20, 30, 42,…?

Definition and Examples: Calculational Techniques

- **Ex 9.6** :

c) Observe $0, 2, 6, 12, 20, \ldots$

$$a_0 = 0 = 0^2 + 0, \qquad a_1 = 2 = 1^2 + 1,$$

$$a_2 = 6 = 2^2 + 2, \qquad a_3 = 12 = 3^2 + 3,$$

$$a_4 = 20 = 4^2 + 4, \cdots$$

$$\therefore a_n = n^2 + n$$

$$\frac{x(x+1)}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(x+1) + x(1-x)}{(1-x)^3} = \frac{2x}{(1-x)^3}$$

is the generating function.

Extension of Binomial Theorem

- Binomial theorem: $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$
  - When $n \in \mathbf{Z}^+$, we have $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$
  - If $n \in \mathbf{R}$, we define $\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$
  - If $n \in \mathbf{Z}^+$, we have $\binom{-n}{r} = \frac{(-n)(-n-1)(-n-2)\ldots(-n-r+1)}{r!}$

$$= \frac{(-1)^r (n)(n+1)\ldots(n+r-1)}{r!}$$

$$= \frac{(-1)^r (n+r-1)!}{(n-1)!\,r!} = (-1)^r \binom{n+r-1}{r}$$

Extension of Binomial Theorem

- **Ex 9.7** :

For $n \in \mathbf{Z}^+$, the Maclaurin series expansion for $(1+x)^{-n}$ is given by

$$\begin{aligned}(1+x)^{-n} &= 1 + (-n)x + (-n)(-n-1)x^2/2! \\ &\quad + (-n)(-n-1)(-n-2)x^3/3! + \cdots \\ &= 1 + \sum_{r=1}^{\infty} \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!}x^r \\ &= \sum_{r=0}^{\infty}(-1)^r\binom{n+r-1}{r}x^r.\end{aligned}$$

(1-x)$^{-n}$?

Hence $(1+x)^{-n} = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots = \sum_{r=0}^{\infty}\binom{-n}{r}x^r$. This generalizes the binomial theorem of Chapter 1 and shows us that $(1+x)^{-n}$ is the generating function for the sequence $\binom{-n}{0}, \binom{-n}{1}, \binom{-n}{2}, \binom{-n}{3}, \ldots$.

Extension of Binomial Theorem

- **Ex 9.8** : Find the coefficient of x^5 in $(1-2x)^{-7}$.
  - **Solution**

$$(1-2x)^{-7} = \sum_{r=0}^{\infty} \binom{-7}{r} (-2x)^r$$

The coefficient of x^5 :

$$\binom{-7}{5}(-2)^5 = (-1)^5 \binom{7+5-1}{5}(-32) = (32)\binom{11}{5}$$

- **Ex 9.9** : Find the coefficient of all x^i in $(1+3x)^{-1/3}$

$$(1+3x)^{-1/3} = 1 + \sum_{r=1}^{\infty} \frac{(-1/3)(-4/3)(-7/3)\cdots((-3r+2)/3)}{r!}(3x)^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1)(-4)(-7)\cdots(-3r+2)}{r!}x^r,$$

$$1 + \sum_{r=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r$$

Definition and Examples: Calculational Techniques

- **Ex 9.10** : Determine the coefficient of x^{15} in $f(x) = (x^2+x^3+x^4+\ldots)^4$.
  - **Solution**
    - $(x^2+x^3+x^4+\ldots) = x^2(1+x+x^2+\ldots) = x^2/(1-x)$
    - $f(x)=(x^2/(1-x))^4 = x^8/(1-x)^4$
    - Hence the solution is the coefficient of x^{7} in $(1-x)^{-4}$: $C(-4, 7)(-1)^7 = (-1)^7C(4+7-1, 7)(-1)^7 = C(10, 7) = 120$.

Table 9.2

For all $m, n \in \mathbf{Z}^+$, $a \in \mathbf{R}$,
1) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$
2) $(1+ax)^n = \binom{n}{0} + \binom{n}{1}ax + \binom{n}{2}a^2x^2 + \cdots + \binom{n}{n}a^nx^n$
3) $(1+x^m)^n = \binom{n}{0} + \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + \binom{n}{n}x^{nm}$
4) $(1-x^{n+1})/(1-x) = 1 + x + x^2 + \cdots + x^n$
5) $1/(1-x) = 1 + x + x^2 + x^3 + \cdots = \sum_{i=0}^{\infty} x^i$
6) $1/(1-ax) = 1 + (ax) + (ax)^2 + (ax)^3 + \cdots$
$= \sum_{i=0}^{\infty}(ax)^i = \sum_{i=0}^{\infty} a^i x^i$
$= 1 + ax + a^2x^2 + a^3x^3 + \cdots$

7) $1/(1+x)^n = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots$

$= \sum_{i=0}^{\infty} \binom{-n}{i} x^i$

$= 1 + (-1)\binom{n+1-1}{1}x + (-1)^2\binom{n+2-1}{2}x^2 + \cdots$

$= \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i} x^i$

8) $1/(1-x)^n = \binom{-n}{0} + \binom{-n}{1}(-x) + \binom{-n}{2}(-x)^2 + \cdots$

$= \sum_{i=0}^{\infty} \binom{-n}{i}(-x)^i$

$= 1 + (-1)\binom{n+1-1}{1}(-x) + (-1)^2\binom{n+2-1}{2}(-x)^2 + \cdots$

$= \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$

check n=1

If $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{i=0}^{\infty} b_i x^i$, and $h(x) = f(x)g(x)$, then $h(x) = \sum_{i=0}^{\infty} c_i x^i$, where for all $k \geq 0$,

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0 = \sum_{j=0}^{k} a_j b_{k-j}.$$

Definition and Examples: Calculational Techniques

- **Ex 9.11** : In how many ways can we select, with repetition allowed, r objects from n distinct objects?
  - **Solution**
  - For each object (with repetitions), $1+x+x^2+\ldots$ represents the possible choices for that object (namely none, one, two,…)
  - Consider all of the n distinct objects, the generating function is $f(x) = (1+x+x^2+\ldots)^n$

$$(1+x+x^2+\cdots)^n = \left(\frac{1}{1-x}\right)^n = \frac{1}{(1-x)^n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i.$$

  - The answer is the coefficient of x^r in $f(x)$, $\binom{n+r-1}{r}$.

Definition and Examples: Calculational Techniques

- **Ex 9.12** : Counting the compositions of a positive integer *n*.
  - **Solution**
  - E.g., $n = 4$
    - One-summand: $(x^1 + x^2 + x^3 + x^4 + \ldots) = [x/(1-x)]$, coefficient of $x^4 = 1$
    - Two-summand: $(x^1 + x^2 + x^3 + x^4 + \ldots)^2 = [x/(1-x)]^2$, coefficient of $x^4 = 3$
    - Three-summand: $(x^1 + x^2 + x^3 + x^4 + \ldots)^3 = [x/(1-x)]^3$, coefficient of $x^4 = 3$
    - Four-summand: $(x^1 + x^2 + x^3 + x^4 + \ldots)^4 = [x/(1-x)]^4$, coefficient of $x^4 = 1$
    - The number of compositions of 4: coefficient of x^4 in $\sum_{i=1}^{4} [x/(1-x)]^i$

How about n=5?

Definition and Examples: Calculational Techniques

- **Ex 9.12** : Counting the compositions of a positive integer *n*.
  - The number of ways to form an integer *n* is the coefficient of x^n in the following generating function.

$$\sum_{i=1}^{\infty}(x^1+x^2+x^3+\dots)^i=\sum_{i=1}^{\infty}[x/(1-x)]^i$$

$f(x)=\sum_{i=1}^{\infty}[x/(1-x)]^i$. But if we set $y=x/(1-x)$, it then follows that

$$f(x)=\sum_{i=1}^{\infty}y^i=y\sum_{i=0}^{\infty}y^i=y\left(\frac{1}{1-y}\right)=\left(\frac{x}{1-x}\right)\left[\frac{1}{1-\left(\frac{x}{1-x}\right)}\right]=\left(\frac{x}{1-x}\right)\left[\frac{1}{\frac{1-x-x}{1-x}}\right]$$
$$=x/(1-2x)=x[1+(2x)+(2x)^2+(2x)^3+\cdots]$$
$$=2^0x+2^1x^2+2^2x^3+2^3x^4+\cdots.$$

So the number of compositions of a positive integer n is the coefficient of x^n in $f(x)$ — and this is 2^{n-1} (as we found earlier in Examples 1.37, 3.11, and 4.12.)

Definition and Examples: Calculational Techniques

- **Ex 9.14** : In how many ways can a police captain distribute 24 rifle shells to four police officers, so that each officer gets at least three shells but not more than eight.
  - **Solution**
  - $f(x)= (x^3+x^4+ x^5+ x^6+x^7+x^8)^4$
    $= x^{12}(1+x+x^2+x^3+x^4+x^5)^4$
    $= x^{12}[(1-x^6)/(1-x)]^4$
  - The answer is the coefficient of x^{12} in $(1-x^6)^4(1-x)^{-4}$

$$=[1-\binom{4}{1}x^6+\binom{4}{2}x^{12}-\binom{4}{3}x^{18}+x^{24}][\binom{-4}{0}+\binom{-4}{1}(-x)+\binom{-4}{2}(-x)^2+\cdots]$$

$$[\binom{-4}{12}(-1)^{12}-\binom{4}{1}\binom{-4}{6}(-1)^6+\binom{4}{2}\binom{-4}{0}]=[\binom{15}{12}-\binom{4}{1}\binom{9}{6}+\binom{4}{2}]=125$$

Definition and Examples: Calculational Techniques

- **Ex 9.16** : Determine the coefficient of x^8 in $\dfrac{1}{(x-3)(x-2)^2}$.
  - **Solution**

Since $1/(x-a) = (-1/a)(1/(1-(x/a))) = (-1/a)[1+(x/a)+(x/a)^2+\cdots]$ for any $a \neq 0$, we could solve this problem by finding the coefficient of x^8 in $1/[(x-3)(x-2)^2]$ expressed as $(-1/3)[1+(x/3)+(x/3)^2+\cdots](1/4)\left[\binom{-2}{0}+\binom{-2}{1}(-x/2)+\binom{-2}{2}(-x/2)^2+\cdots\right]$.

$$\frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

$$1 = A(x-2)^2 + B(x-2)(x-3) + C(x-3),$$

$$\begin{aligned}
\frac{1}{(x-3)(x-2)^2} &= \frac{1}{x-3} - \frac{1}{x-2} - \frac{1}{(x-2)^2} \\
&= \left(\frac{-1}{3}\right)\frac{1}{1-(x/3)} + \left(\frac{1}{2}\right)\frac{1}{1-(x/2)} + \left(\frac{-1}{4}\right)\frac{1}{(1-(x/2))^2} \\
&= \left(\frac{-1}{3}\right)\sum_{i=0}^{\infty}\left(\frac{x}{3}\right)^i + \left(\frac{1}{2}\right)\sum_{i=0}^{\infty}\left(\frac{x}{2}\right)^i \\
&\quad + \left(\frac{-1}{4}\right)\left[\binom{-2}{0} + \binom{-2}{1}\left(\frac{-x}{2}\right) + \binom{-2}{2}\left(\frac{-x}{2}\right)^2 + \cdots\right].
\end{aligned}$$

The coefficient of x^8 is $(-1/3)(1/3)^8 + (1/2)(1/2)^8 + (-1/4)\binom{-2}{8}(-1/2)^8 = -[(1/3)^9 + 7(1/2)^{10}]$.

Definition and Examples: Calculational Techniques

- **Ex 9.17** : How many four-element subsets of $S = \{1, 2, \ldots, 15\}$ contains no consecutive integers?

  Solution

  - E.g., one subset {1, 3, 7, 10}, **1**≤1<3<7<10 ≤**15**, difference 0, 2, 4, 3, 5, difference sum = 14.
  - These suggest the integer solutions to $c_1+c_2+c_3+c_4+c_5 = 14$ where $0 \le c_1, c_5$ and $2 \le c_2, c_3, c_4$.
  - The answer is the coefficient of x^{14} in $f(x) = (1+x+x^2+x^3+\ldots)(x^2+x^3+x^4+\ldots)^3(1+x+x^2+x^3+\ldots)$
    $= x^6(1-x)^{-5}$
  - The coefficient of x^8 in $(1-x)^{-5}$.

$$\binom{-5}{8}(-1)^8 = \binom{5+8-1}{8} = \binom{12}{8} = 495$$

Convolution of Sequences

- **Ex 9.19** : Let
  - $f(x) = x/(1-x)^2 = 0+1x+2x^2+3x^3+\ldots$, for the sequence $a_k = k$
  - $g(x) = x(x+1)/(1-x)^3 = 0+1^2x+2^2x^2+3^2x^3+\ldots$, for the sequence $b_k = k^2$
  - $h(x) = f(x)g(x)$
    $= a_0b_0 + (a_0b_1+a_1b_0)x + (a_0b_2+a_1b_1+a_2b_0)x^2 + \ldots$, for the sequence $c_k = a_0b_k+a_1b_{k-1}+a_2b_{k-2}+\ldots+a_{k-2}b_2+a_{k-1}b_1+a_kb_0$
  - $$c_k = \sum_{i=0}^{k} i(k-i)^2.$$

$c_0 = 0\times 0^2$
$c_1 = 0\times 1^2 + 1\times 0^2 = 0$
$c_2 = 0\times 2^2 + 1\times 1^2 + 2\times 0^2 = 1$
$c_3 = 6$

- The sequence $c_0, c_1, c_2, \ldots$ is the convolution of the sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$

Convolution of Sequences

- **Ex 9.20** : Let
  - $f(x) = 1/(1-x) = 1+x+x^2+x^3+ \dots$
  - $g(x) = 1/(1+x) = 1-x+x^2-x^3+ \dots$
  - $h(x) = f(x)g(x)$
    $= 1/[(1-x)(1+x)] = 1/(1-x^2) = 1+x^2+x^4+x^6+ \dots$
- The sequence 1, 0, 1, 0, … is the convolution of the sequences 1, 1, 1, 1, … and 1, -1, 1, -1, …

9.3 Partition of Integers

- $p(n)$: the number of partitioning a positive integer n

$$
\begin{aligned}
p(1) &= 1: \quad 1 \\
p(2) &= 2: \quad 2 = 1 + 1 \\
p(3) &= 3: \quad 3 = 2 + 1 = 1 + 1 + 1 \\
p(4) &= 5: \quad 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 \\
p(5) &= 7: \quad 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 \\
&\qquad\quad = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1
\end{aligned}
$$

- The number of 1's is 0 or 1 or 2 or 3…. The power series is $1+x+x^2+x^3+x^4+\ldots$.
- The number of 2's can be kept tracked by the power series $1+x^2+x^4+x^6+x^8+\ldots$.
- For n, the number of 3's can be kept tracked by the power series $1+x^3+x^6+x^9+x^{12}+\ldots$.

Partition of Integers

- Determine $p(10)$
- The coefficient of x^{10} in $f(x)$ $=(1+x+x^2+x^3+\ldots)(1+x^2+x^4+x^6+\ldots)(1+x^3+x^6+x^9+\ldots)\ldots(1+x^{10}+x^{20}+\ldots)$

$$f(x) = \frac{1}{(1-x)}\frac{1}{(1-x^2)}\frac{1}{(1-x^3)}\cdots\frac{1}{(1-x^{10})} = \prod_{i=1}^{10}\frac{1}{(1-x^i)}$$

- By the coefficient of x^n in $P(x) = \prod_{i=1}^{\infty}\frac{1}{(1-x^i)}$, we get the sequence $p(0), p(1), p(2), p(3), \ldots$

Partition of Integers

- **Ex 9.21** : Find the generating function for the number of ways an advertising agent can purchase *n* minutes of air time if the time slots come in blocks of 30, 60, or 120 seconds.

Solution

- Let 30 seconds represent one time unit.
- Find integer solutions to $a+2b+4c = 2n$
- Generating function:
  $f(x)= (1+x+x^2+x^3+x^4+\ldots)(1+x^2+x^4+x^6+x^8+\ldots)(1+x^4+x^8+x^{12}+\ldots)$
  $$= \frac{1}{(1-x)}\frac{1}{(1-x^2)}\frac{1}{(1-x^4)}.$$
- Answer: the coefficient of x^{2n} is the number of partitions of $2n$ into 1's, 2's, and 4's.

Partition of Integers

- **Ex 9.22** : Find the generating function for $p_d(n)$, the number of partitions of a positive integer n into distinct summands.
  - One time of occurrence per summand
  - $P_d(x) = (1+x)(1+x^2)(1+x^3)\ldots\ldots$

6 = 1+5
6 = 1+2+3
6 = 2+4

- **Ex 9.23** : Find the generating function for $p_o(n)$, the number of partitions of a positive integer n into odd summands.
  - $P_o(x) = (1+x+x^2+x^3+\ldots)(1+x^3+x^6+\ldots)(1+x^5+x^{10}+\ldots)\ldots$
  - $= 1/(1-x) \times 1/(1-x^3) \times 1/(1-x^5) \times 1/(1-x^7) \times \ldots$
  - $P_d(x) = P_o(x)$?

Now because

$$1+x=\frac{1-x^2}{1-x},\qquad 1+x^2=\frac{1-x^4}{1-x^2},\qquad 1+x^3=\frac{1-x^6}{1-x^3},\qquad \ldots,$$

we have

$$P_d(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$$
$$= \frac{1-x^2}{1-x}\,\frac{1-x^4}{1-x^2}\,\frac{1-x^6}{1-x^3}\,\frac{1-x^8}{1-x^4}\cdots = \frac{1}{1-x}\,\frac{1}{1-x^3}\cdots = P_o(x).$$

6 = 1+1+1+3
6 = 1+5
6 = 3+3

Partition of Integers

- **Ex 9.24** : Find the generating function for the number of partitions of a positive integer n into odd summands and occurring an odd number of times.

Solution

$f(x) = (1+x+x^3+x^5+ \ldots)(1+x^3+x^9+x^{15}+ \ldots)$
$(1+x^5+x^{15}+x^{25}+ \ldots)\ldots$

$$= \prod_{k=0}^{\infty}\left(1+\sum_{i=0}^{\infty} x^{(2k+1)(2i+1)}\right).$$

6 = 1+1+1+3
6 = 1+5
6 = 3+3

Partition of Integers

- Ferrers graph uses rows of dots to represent a partition of an integer
- In fig. 9.2, two Ferrers graphs are transposed each other for the partitions of 14.
  - (a) 14 = 4+3+3+2+1+1
  - (b) 14 = 6+4+3+1

*The number of partitions of an integer **n** into **m summands** is equal to the number of partitions of n into summands where **m is the largest summand**.*

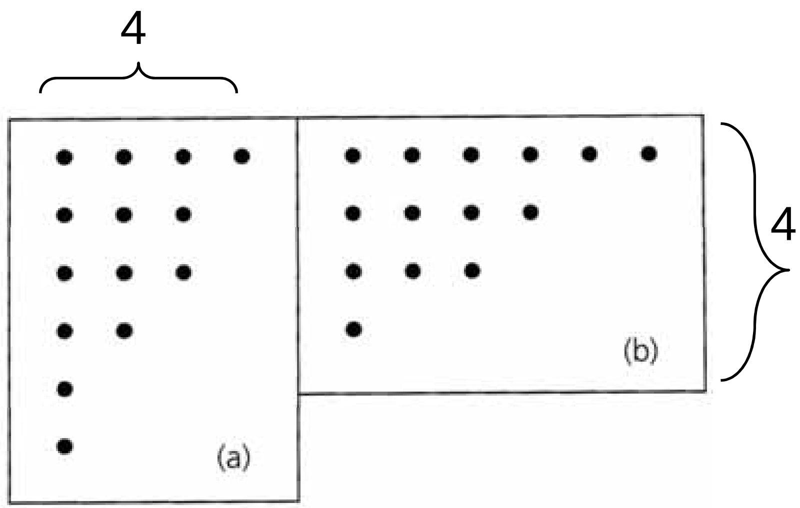

Figure 9.2

http://mathworld.wolfram.com/FerrersDiagram.html

9.4 The Exponential Generating Function

Now for all $0 \le r \le n$,

$$C(n, r) = \frac{n!}{r!(n-r)!} = \left(\frac{1}{r!}\right) P(n, r),$$

where $P(n, r)$ denotes the number of permutations of n objects taken r at a time. So

$$\begin{aligned}(1+x)^n &= C(n,0) + C(n,1)x + C(n,2)x^2 + C(n,3)x^3 + \cdots + C(n,n)x^n \\ &= P(n,0) + P(n,1)x + P(n,2)\frac{x^2}{2!} + P(n,3)\frac{x^3}{3!} + \cdots + P(n,n)\frac{x^n}{n!}.\end{aligned}$$

For a sequence $a_0, a_1, a_2, a_3, \ldots$ of real numbers,

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!},$$

is called the *exponential generating function* for the given sequence.

The Exponential Generating Function

- **Ex 9.25** : Examining the Maclaurin series expansion for e^x, we find

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

  so e^x is the exponential generating function for the sequence 1, 1, 1,….

The Exponential Generating Function

- **Ex 9.26** : In how many ways can four of the letters in ENGINE be arranged?

  Solution

  Table 9.4

E	E	N	N	4!/(2! 2!)	E	G	N	N	4!/2!
E	E	G	N	4!/2!	E	I	N	N	4!/2!
E	E	I	N	4!/2!	G	I	N	N	4!/2!
E	E	G	I	4!/2!	E	I	G	N	4!

  - Using exponential generating function: $f(x) = [1+x+(x^2/2!)]^2[1+x]^2$
    - E, N: $[1+x+(x^2/2!)]$
    - G, I: $[1+x]$
  - The answer is the coefficient of $x^4/4!$.

In the complete expansion of $f(x)$, the term involving x^4 [and, consequently, $x^4/4!$] is

$$\left(\frac{x^4}{2!\,2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+x^4\right)$$

$$=\left[\left(\frac{4!}{2!\,2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+4!\right]\left(\frac{x^4}{4!}\right),$$

The Exponential Generating Function

- **Ex 9.27** : Consider the Maclaurin series expansion of e^x and e^{-x}.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

The Exponential Generating Function

- **Ex 9.28** : A ship carries 48 flags, 12 each of the colors red, white, blue and black. Twelve flags are placed on a vertical pole to communicate signal to other ships.
  - How many of these signals use an even number of blue flags and an odd number of black flags?

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$f(x) = (e^x)^2 \left(\frac{e^x + e^{-x}}{2}\right) \left(\frac{e^x - e^{-x}}{2}\right) = \left(\frac{1}{4}\right)(e^{2x})(e^{2x} - e^{-2x}) = \frac{1}{4}(e^{4x} - 1)$$

$$= \frac{1}{4}\left(\sum_{i=0}^{\infty} \frac{(4x)^i}{i!} - 1\right) = \left(\frac{1}{4}\right) \sum_{i=1}^{\infty} \frac{(4x)^i}{i!},$$

the coefficient of $x^{12}/12!$ in $f(x)$ yields $(1/4)(4^{12}) = 4^{11}$ signals made up of 12 flags with an even number of blue flags and an odd number of black flags.

The Exponential Generating Function

- how many of these use at least three white flags or no white flag at all?

$$g(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\underline{\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2$$

$$= e^x\left(e^x - x - \frac{x^2}{2!}\right)(e^x)^2 = e^{3x}\left(e^x - x - \frac{x^2}{2!}\right) = e^{4x} - xe^{3x} - \left(\frac{1}{2}\right)x^2e^{3x}$$

$$= \sum_{i=0}^{\infty}\frac{(4x)^i}{i!} - x\sum_{i=0}^{\infty}\frac{(3x)^i}{i!} - \left(\frac{x^2}{2}\right)\left(\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}\right).$$

The Exponential Generating Function

i) $\sum_{i=0}^{\infty} \frac{(4x)^i}{i!}$ — Here we have the term $\frac{(4x)^{12}}{12!} = 4^{12}\left(\frac{x^{12}}{12!}\right)$, so the coefficient of $x^{12}/12!$ is 4^{12};

ii) $x\left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i!}\right)$ — Now we see that in order to get $x^{12}/12!$ we need to consider the term $x[(3x)^{11}/11!] = 3^{11}(x^{12}/11!) = (12)(3^{11})(x^{12}/12!)$, and here the coefficient of $x^{12}/12!$ is $(12)(3^{11})$; and

iii) $(x^2/2)\left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i!}\right)$ — For this last summand we observe that $(x^2/2)[(3x)^{10}/10!] = (1/2)(3^{10})(x^{12}/10!) = (1/2)(12)(11)(3^{10})(x^{12}/12!)$, where this time the coefficient of $x^{12}/12!$ is $(1/2)(12)(11)(3^{10})$.

Consequently, the number of 12 flag signals with at least three white flags, or none at all, is

$$4^{12} - 12(3^{11}) - (1/2)(12)(11)(3^{10}) = 10{,}754{,}218.$$

The Exponential Generating Function

- **Ex 9.29** : A company hires 11 new employees, and they will be assigned to four different subdivisions. Each subdivision has at least one new employee. In how many ways can these assignments be made?
  - **Solution**

$$f(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)^4 = (e^x - 1)^4 = e^{4x} - 4e^{3x} + 6e^{2x} - 4e^x + 1.$$

The answer then is the coefficient of $x^{11}/11!$ in $f(x)$:

$$4^{11} - 4(3^{11}) + 6(2^{11}) - 4(1^{11}) = \sum_{i=0}^{4}(-1)^i \binom{4}{i}(4-i)^{11}.$$

9.5 The Summation Operator

- Let $f(x) = a_0+a_1x+a_2x^2+a_3x^3+\ldots$. Then $f(x)/(1-x)$ generate the sequence of a_0, a_0+a_1, $a_0+a_1+a_2$, $a_0+a_1+a_2+a_3,\ldots$ So we refer to $1/(1-x)$ as the summation operator.

$$\frac{f(x)}{1-x} = f(x)\cdot\frac{1}{1-x} = [a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots][1 + x + x^2 + x^3 + \cdots]$$

$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + \cdots,$$

The Summation Operator

Summation operator

- **Ex 9.30** :
  - $1/(1-x)$ is the generating function for the sequence 1, 1, 1, 1, 1,…
  - $[1/(1-x)]\times [1/(1-x)]$ is the generating function for the sequence 1, 1+1, 1+1+1,… $\Rightarrow$ 1, 2, 3,…
  - $x+x^2$ is the generating function for the sequence 0, 1, 1, 0, 0, 0,…
  - $(x+x^2) \times [1/(1-x)]$ is the generating function for the sequence 0, 1, 2, 2, 2, 2, …
  - $(x+x^2)/(1-x)^2$ is the generating function for the sequence 0, 1, 3, 5, 7, 9, 11, …
  - $(x+x^2)/(1-x)^3$ is the generating function for the sequence 0, 1, 4, 9, 16, 25, 36, …

The Summation Operator

- **Ex 9.31** : Find a formula to express $0^2+1^2+2^2+\ldots+n^2$ as a function of n.

  Solution

$$f_0(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$f_1(x) = x\frac{d}{dx}f_0(x) = \frac{x}{(1-x)^2}$$

$$= 0 + x + 2x^2 + 3x^3 + \cdots$$

$$f_2(x) = x\frac{d}{dx}f_1(x) = \frac{x^2+x}{(1-x)^3}$$

$$= 0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \cdots$$

so $x(1+x)/(1-x)^3$ generates $0^2, 1^2, 2^2, 3^2, \ldots$. As a consequence of our earlier observations about the summation operator, we find that

$$\frac{x(1+x)}{(1-x)^3}\frac{1}{(1-x)} = \frac{x(1+x)}{(1-x)^4}$$

is the generating function for $0^2, 0^2+1^2, 0^2+1^2+2^2, 0^2+1^2+2^2+3^2, \ldots$.

The Summation Operator

Hence the coefficient of x^n in $[x(1+x)]/(1-x)^4$ is $\sum_{i=0}^{n} i^2$. But the coefficient of x^n in $[x(1+x)]/(1-x)^4$ can also be calculated as follows:

$$\frac{x(1+x)}{(1-x)^4} = (x+x^2)(1-x)^{-4} = (x+x^2)\left[\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \cdots\right],$$

so the coefficient of x^n is

$$\binom{-4}{n-1}(-1)^{n-1} + \binom{-4}{n-2}(-1)^{n-2}$$

$$= (-1)^{n-1}\binom{4+(n-1)-1}{n-1}(-1)^{n-1} + (-1)^{n-2}\binom{4+(n-2)-1}{n-2}(-1)^{n-2}$$

$$= \binom{n+2}{n-1} + \binom{n+1}{n-2} = \frac{(n+2)!}{3!(n-1)!} + \frac{(n+1)!}{3!(n-2)!}$$

$$= \frac{1}{6}[(n+2)(n+1)(n) + (n+1)(n)(n-1)]$$

$$= \frac{1}{6}(n)(n+1)[(n+2)+(n-1)] = \frac{n(n+1)(2n+1)}{6}.$$

Homework

- 9.1:
- 9.2:
- 9.3:
- 9.4:
- 9.5: