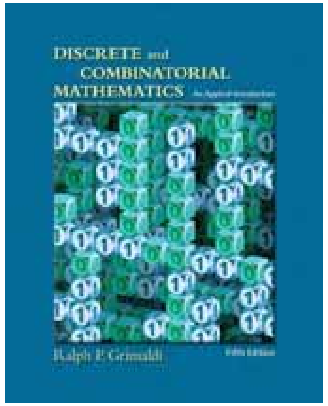
Discrete Mathematics

-- Chapter 8: The Principle of Inclusion and Exclusion



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Outline

- The Principle of Inclusion and Exclusion
- Generalization of the Principle
- Derangements: Nothing Is in Its Right Place
- Rook Polynomials
- Arrangements with Forbidden Positions

8.1 The principle of Inclusion and Exclusion

- For a given finite set S ($|S| = N$) with conditions C_i
- $N(\overline{c_1}\,\overline{c_2}) = N - [N(c_1) + N(c_2)] + N(c_1c_2)$ $\boxed{N(\overline{c_1} \text{ and } \overline{c_2})}$

$$= N(\overline{c_1}) - N(\overline{c_1}c_2)$$

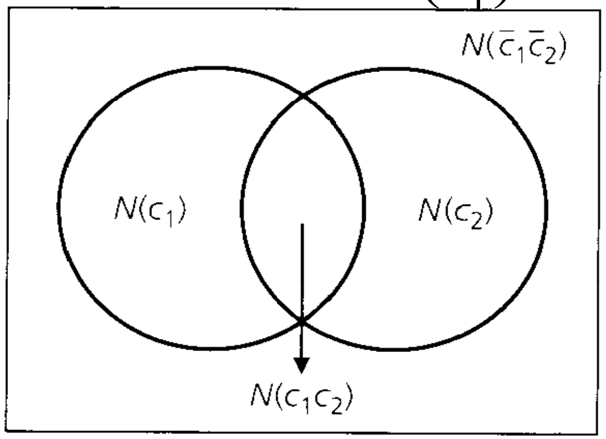


Figure 8.1

$$N(\overline{c_1c_2}) = N - N(c_1c_2)$$

$$\neq N(\overline{c_1}\,\overline{c_2}) \qquad \boxed{N(\overline{c_1} \text{ or } \overline{c_2})}$$

- $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}) = N - [N(c_1) + N(c_2) + N(c_3)]$

$$+ [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$

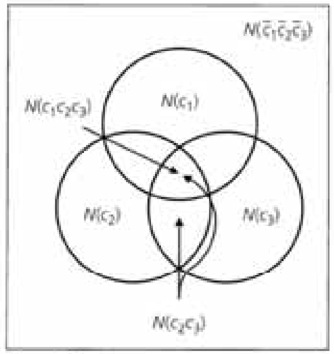


Figure 8.2

Four sets

- **Ex 8.3** :

$$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)]$$
$$+ [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)]$$
$$- [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] + N(c_1c_2c_3c_4)$$

- For each element $x \in S$, we have five cases:
  - (0) x satisfies none of the four conditions;
  - (1) x satisfies only one of the four conditions;
  - (2) x satisfies exactly two of the four conditions;
  - (3) x satisfies exactly three of the four conditions;
  - (4) x satisfies all the four conditions.

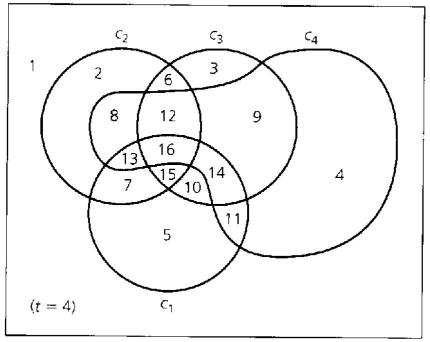


Figure 8.5

Four sets

$$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = N - [N(c_1)+N(c_2)+N(c_3)+N(c_4)]$$
$$+[N(c_1c_2)+N(c_1c_3)+N(c_1c_4)+N(c_2c_3)+N(c_2c_4)+N(c_3c_4)]$$
$$-[N(c_1c_2c_3)+N(c_1c_2c_4)+N(c_1c_3c_4)+N(c_2c_3c_4)]+N(c_1c_2c_3c_4)$$

1. x satisfies no condition. x is counted once in $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$ and once in N. [1=1]
2. x satisfies c_1. It is not counted on the left side. It is counted once in N and once in $N(c_1)$. [0 = 1-1 = 0]
3. x satisfies c_2 and c_4. It is not counted on the left side. It is counted once in N, $N(c_2)$, $N(c_4)$ and $N(c_2c_4)$.
   $$[0 = 1-(1+1)+1 = 1-\binom{2}{1}+\binom{2}{2} = 0]$$
4. x satisfies c_1, c_2 and c_4. It is not counted on the left side. It is counted once in N, $N(c_1)$, $N(c_2)$, $N(c_4)$, $N(c_1c_2)$, $N(c_1c_4)$, $N(c_2c_4)$ and $N(c_1c_2c_4)$.
   $$[0 = 1-(1+1+1)+(1+1+1)-1 = 1-\binom{3}{1}+\binom{3}{2}-\binom{3}{3} = 0]$$
5. x satisfies all conditions. It is not counted on the left side. It is counted once in all the subsets on the right side.
   $$[0 = 1-\binom{4}{1}+\binom{4}{2}-\binom{4}{3}+\binom{4}{4} = 0]$$

Four sets

$$N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) = \underline{N(\overline{c}_2\overline{c}_3\overline{c}_4) - N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)}$$

$$= \{N - [N(c_2) + N(c_3) + N(c_4)] + [N(c_2c_3) + N(c_2c_4) + N(c_3c_4)]$$

$$- N(c_2c_3c_4)\} - \{N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)]$$

$$+ [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)]$$

$$- [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] + N(c_1c_2c_3c_4)\}, \text{ or}$$

$$N(\underline{c_1}\overline{c}_2\overline{c}_3\overline{c}_4) = N(\underline{c_1}) - [N(\underline{c_1}c_2) + N(\underline{c_1}c_3) + N(\underline{c_1}c_4)]$$

$$+ [N(\underline{c_1}c_2c_3) + N(\underline{c_1}c_2c_4) + N(\underline{c_1}c_3c_4)] - N(\underline{c_1}c_2c_3c_4).$$

The Principle of Inclusion and Exclusion

- Theorem 8.1:
  - $|S| = N$, conditions c_i, $1 \le i \le t$
  - $\overline{N} = N(\overline{c_1}\,\overline{c_2} \cdots \overline{c_t})$ denote the number of elements of S that satisfy none of the conditions.

$$\overline{N} = N - \sum_{1 \le i \le t} N(c_i) + \sum_{1 \le i < j \le t} N(c_i c_j) - \sum_{1 \le i < j < k \le t} N(c_i c_j c_k) + \cdots \quad (2)$$

$$+ (-1)^t N(c_1 c_2 c_3 \cdots c_t)$$

The other possibility is that x satisfies *exactly* r of the conditions where $1 \le r \le t$. In this case x contributes nothing to $\overline{N}$. But on the right-hand side of Eq. (2), x is counted

(1) One time in N.

(2) r times in $\sum_{1\le i\le t} N(c_i)$. (Once for each of the r conditions.)

(3) $\binom{r}{2}$ times in $\sum_{1\le i<j\le t} N(c_i c_j)$. (Once for each pair of conditions selected from the r conditions it satisfies.)

(4) $\binom{r}{3}$ times in $\sum_{1\le i<j<k\le t} N(c_i c_j c_k)$. (Why?)

. . . .

$(r+1)$ $\binom{r}{r} = 1$ time in $\sum N(c_{i_1} c_{i_2} \cdots c_{i_r})$, where the summation is taken over all selections of size r from the t conditions.

Consequently, on the right-hand side of Eq. (2), x is counted

$$1 - r + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} = [1 + (-1)]^r = 0^r = 0 \text{ times},$$

The Principle of Inclusion and Exclusion

- Corollary 8.1: $N(c_1 \text{ or } c_2 \text{ or} \ldots \text{or } c_t) = N - \overline{N}.$
- Some notation for simplifying Theorem 8.1

$$
\begin{aligned}
S_0 &= N, \\
S_1 &= [N(c_1) + N(c_2) + \cdots + N(c_t)], \\
S_2 &= [N(c_1c_2) + N(c_1c_3) + \cdots + N(c_1c_t) + N(c_2c_3) + \cdots + N(c_{t-1}c_t)],
\end{aligned}
$$

and, in general,

$$S_k = \sum N(c_{i_1}c_{i_2}\cdots c_{i_k}),\ 1 \le k \le t,$$

where the summation is taken over all selections of size k from the collection of t conditions. Hence S_k has $\binom{t}{k}$ summands in it.

Using this notation we can rewrite the result in Eq. (2) as

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t.$$

The Principle of Inclusion and Exclusion

- **Ex 8.4** : Determine the number of positive integers n where $1 \le n \le 100$ and n is not divisible by 2, 3 or 5.
  - Condition c_1 if n is divisible by 2.
  - Condition c_2 if n is divisible by 3.
  - Condition c_3 if n is divisible by 5.
  - Then the answer to this problem is

$$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}) = S_0 - S_1 + S_2 - S_3 \qquad \lfloor 100/(2*3) \rfloor = 16$$

$$= N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$

$$= 100 - [50 + 33 + 20] + [16 + 10 + 6] - 3 = 26.$$

The Principle of Inclusion and Exclusion

- **Ex 8.5** : Determine the number of nonnegative integer solutions to the equation
  $x_1 + x_2 + x_3 + x_4 = 18$ and $x_i \leq 7$ for all i.
  - We say that a solution x_1, x_2, x_3, x_4 satisfies condition c_i if $x_i > 7$ (i.e., $x_i \geq 8$) .
  - Then the answer to this problem is

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = S_0 - S_1 + S_2 - S_3 + S_4 =$$

$$\binom{21}{18} - \binom{4}{1}\binom{13}{10} + \binom{4}{2}\binom{5}{2} - 0 + 0 = 246.$$

$$\binom{4+18-1}{18} \qquad \binom{4+10-1}{10} \qquad \binom{4+2-1}{2}$$

The Principle of Inclusion and Exclusion

- **Ex 8.6** : For finite sets A, B, where $|A| = m \geq n = |B|$, and function f: $A \rightarrow B$, determine the number of onto functions f.
  - Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$.
  - Let c_i be the condition that b_i is not in the range of f.
    Then $N(\overline{c_1})$ is the number of functions in S that have b_i in their range.
  - Then the answer to this problem is $N(\overline{c_1}\,\overline{c_2}\ldots\ldots\overline{c_n})$.

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\cdots\overline{c}_n) = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n$$

$$= n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m$$

$$+ \cdots + (-1)^n(n-n)^m = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^m$$

$$= \sum_{i=0}^{n}(-1)^i\binom{n}{n-i}(n-i)^m.$$

$N = S_0 = |S| = n^m$

$N(c_i) = (n-1)^m \Rightarrow S_1 = \binom{n}{1}(n-1)^m$

$N(c_i c_j) = (n-2)^m \Rightarrow S_2 = \binom{n}{2}(n-2)^m$

The Principle of Inclusion and Exclusion

- **Ex 8.7** : In how many ways can the 26 letters of the alphabet be permuted so that none of the patterns car, dog, pun, or byte occurs?

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = 26! - [3(24!) + 23!] + [3(22!) + 3(21!)] - [20! + 3(19!)] + 17!$$

- **Ex 8.8** : Let $\phi(n)$ be the number of positive integers m, where $1 \le m < n$ and $\gcd(m, n)=1$, i.e., m and n are relatively prime.
  - Consider $n = p_1^{e_1} p_2^{e_2} p_3^{e_3} p_4^{e_4}$
  - For $1 \le i \le 4$, let c_i denote that k is divisible by p_i.
    - $N = S_0 = n$; $N(c_i) = n/p_i$; $N(c_ic_j) = n/(p_ip_j)$; …
  - Then the answer to this problem is $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$.

$$
\begin{aligned}
\phi(n) &= S_0 - S_1 + S_2 - S_3 + S_4 \\
&= n - \left[\frac{n}{p_1} + \cdots + \frac{n}{p_4}\right] + \left[\frac{n}{p_1 p_2} + \frac{n}{p_1 p_3} + \cdots + \frac{n}{p_3 p_4}\right] \\
&\quad - \left[\frac{n}{p_1 p_2 p_3} + \cdots + \frac{n}{p_2 p_3 p_4}\right] + \frac{n}{p_1 p_2 p_3 p_4} \\
&= n\left[1 - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_4}\right) + \left(\frac{1}{p_1 p_2} + \frac{1}{p_1 p_3} + \cdots + \frac{1}{p_3 p_4}\right)\right. \\
&\quad \left. - \left(\frac{1}{p_1 p_2 p_3} + \cdots + \frac{1}{p_2 p_3 p_4}\right) + \frac{1}{p_1 p_2 p_3 p_4}\right] \\
&= \frac{n}{p_1 p_2 p_3 p_4}\left[p_1 p_2 p_3 p_4 - (p_2 p_3 p_4 + p_1 p_3 p_4 + p_1 p_2 p_4 + p_1 p_2 p_3)\right. \\
&\quad + (p_3 p_4 + p_2 p_4 + p_2 p_3 + p_1 p_4 + p_1 p_3 + p_1 p_2) \\
&\quad \left. - (p_4 + p_3 + p_2 + p_1) + 1\right] \\
&= \frac{n}{p_1 p_2 p_3 p_4}\left[(p_1 - 1)(p_2 - 1)(p_3 - 1)(p_4 - 1)\right] \\
&= n\left[\frac{p_1 - 1}{p_1} \cdot \frac{p_2 - 1}{p_2} \cdot \frac{p_3 - 1}{p_3} \cdot \frac{p_4 - 1}{p_4}\right] = n \prod_{i=1}^{4}\left(1 - \frac{1}{p_i}\right).
\end{aligned}
$$

$$\boxed{(p_1 - 1)p_1^{e_1 - 1}(p_2 - 1)p_2^{e_2 - 1}(p_3 - 1)p_3^{e_3 - 1}(p_4 - 1)p_4^{e_4 - 1}}$$

The Principle of Inclusion and Exclusion

- **Ex 8.9** : Six married couples are to be seated at a circular table. In how many ways can they arrange themselves so that no wife sits next to her husband?
  - For $1 \le i \le 6$, let c_i denote the condition where a seating arrangement has couple i seated next to each other. $N(c_i) = 2(11-1)!$
  - Then the answer to this problem is $N(\overline{c_1}\,\overline{c_2}\ldots\overline{c_6})$.

$N(c_1c_2c_3) = 2^3(8!)$, $S_3 = \binom{6}{3}2^3(8!)$ $\qquad$ $N(c_1c_2c_3c_4) = 2^4(7!)$, $S_4 = \binom{6}{4}2^4(7!)$

$N(c_1c_2c_3c_4c_5) = 2^5(6!)$, $S_5 = \binom{6}{5}2^5(6!)$ $\qquad$ $N(c_1c_2c_3c_4c_5c_6) = 2^6(5!)$, $S_6 = \binom{6}{6}2^6(5!)$.

The Principle of Inclusion and Exclusion

- **Ex 8.10** : In a certain area of the countryside are five villages. An engineer is to devise a system of two-way roads so that after the system is completed, no village will be isolated. In how many ways can he do this?
  - Let c_i denote the condition that a system of these roads isolates village a, b, c, d, and e, respectively.

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4\overline{c}_5) = 2^{10} - \binom{5}{1}2^6 + \binom{5}{2}2^3 - \binom{5}{3}2^1 + \binom{5}{4}2^0 - \binom{5}{5}2^0 = 768.$$

C(5,2) C(4,2)

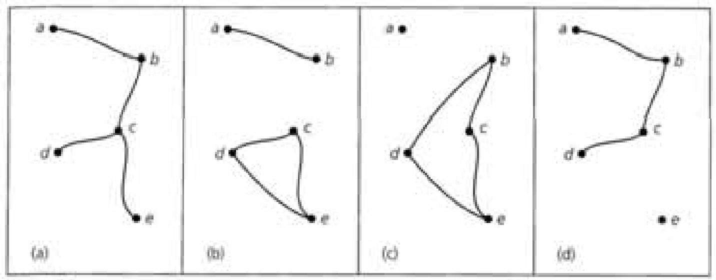


Figure 8.3

8.2 Generalizations of the Principle

- E_m denotes the number of elements in S that satisfy exactly m of the t conditions.
- $E_1 = N(c_1\overline{c_2}\,\overline{c_3}\cdots\overline{c_t}) + N(\overline{c_1}c_2\overline{c_3}\cdots\overline{c_t}) + \cdots + N(\overline{c_1}\,\overline{c_2}\cdots\overline{c_{t-1}}c_t).$

  $E_2 = N(c_1c_2\overline{c_3}\cdots\overline{c_t}) + N(c_1\overline{c_2}c_3\cdots\overline{c_t}) + \cdots + N(\overline{c_1}\,\overline{c_2}\cdots\overline{c_{t-2}}c_{t-1}c_t).$
- E_1= **2**+**3**+**4**=$N(c_1)+N(c_2)+N(c_3)$ - $2[N(c_1c_2)+ N(c_1c_3)+ N(c_2c_3)] + 3N(c_1c_2c_3)$

  $= S_1 - 2S_2 + 3S_3 = S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3$
- E_2= **5**+**6**+**7**=$N(c_1c_2)+N(c_1c_3)+N(c_2c_3)-3N(c_1c_2c_3)$

  $= S_2 - 3S_3 = S_2 - \binom{3}{1}S_3$
- E_3=**8**= $N(c_1c_2c_3) = S_3$

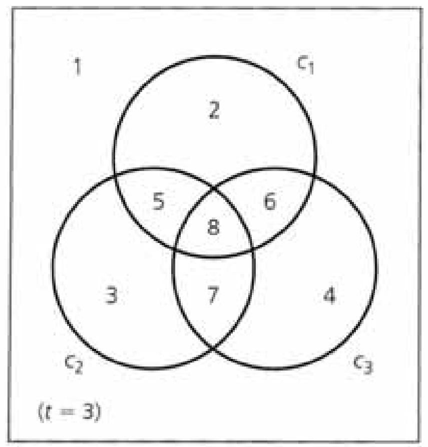


Figure 8.4

Generalizations of the Principle

$$E_1 = S_1 - 2S_2 + 3S_3 - 4S_4$$
$$= S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3 - \binom{4}{3}S_4$$

$$E_2 = S_2 - 3S_3 + 6S_4$$
$$= S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4$$

$$E_3 = S_3 - 4S_4 = S_3 + \binom{4}{1}S_4$$

$$E_4 = S_4$$

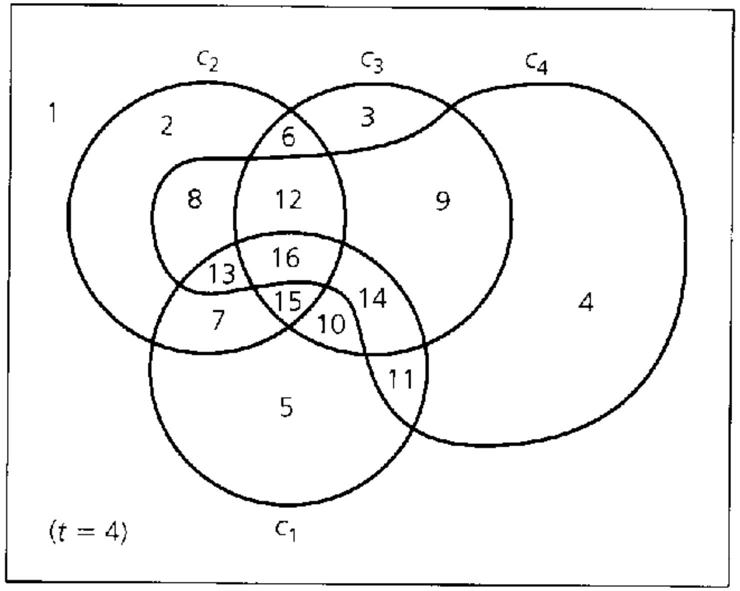


Figure 8.5

Table 8.1

S_2	S_3	S_4
$N(c_1c_2)$: 7, 13, 15, 16 $N(c_1c_3)$: 10, 14, 15, 16 $N(c_1c_4)$: 11, 13, 14, 16 $N(c_2c_3)$: 6, 12, 15, 16 $N(c_2c_4)$: 8, 12, 13, 16 $N(c_3c_4)$: 9, 12, 14, 16	$N(c_1c_2c_3)$: 15, 16 $N(c_1c_2c_4)$: 13, 16 $N(c_1c_3c_4)$: 14, 16 $N(c_2c_3c_4)$: 12, 16	$N(c_1c_2c_3c_4)$: 16

Theorem 8.2

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t. \quad (1)$$

(If $m = 0$, we obtain Theorem 8.1.)

Proof: Arguing as in Theorem 8.1, let $x \in S$ and consider the following three cases.

a) When x satisfies fewer than m conditions, it contributes a count of 0 to each of the terms $E_m, S_m, S_{m+1}, \ldots, S_t$, so it is not counted on either side of the equation.

b) When x satisfies exactly m of the conditions, it is counted once in E_m and once in S_m, but not in $S_{m+1}, \ldots, S_t$. Consequently, it is included once in the count for either side of the equation.

c) Suppose x satisfies r of the conditions, where $m < r \leq t$. Then x contributes nothing to E_m. Yet it is counted $\binom{r}{m}$ times in S_m, $\binom{r}{m+1}$ times in $S_{m+1}, \ldots,$ and $\binom{r}{r}$ times in S_r, but 0 times for any term beyond S_r. So on the right-hand side of the equation, x is counted $\binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} - \cdots + (-1)^{r-m}\binom{r}{r-m}\binom{r}{r}$ times.

For $0 \le k \le r - m$,

$$\binom{m+k}{k}\binom{r}{m+k} = \frac{(m+k)!}{k!\,m!} \cdot \frac{r!}{(m+k)!(r-m-k)!}$$

$$= \frac{r!}{m!} \cdot \frac{1}{k!(r-m-k)!} = \frac{r!}{m!(r-m)!} \cdot \frac{(r-m)!}{k!(r-m-k)!}$$

$$= \binom{r}{m}\binom{r-m}{k}.$$

Consequently, on the right-hand side of Eq. (1), x is counted

$$\binom{r}{m}\binom{r-m}{0} - \binom{r}{m}\binom{r-m}{1} + \binom{r}{m}\binom{r-m}{2} - \cdots + (-1)^{r-m}\binom{r}{m}\binom{r-m}{r-m}$$

$$= \binom{r}{m}\left[\binom{r-m}{0} - \binom{r-m}{1} + \binom{r-m}{2} - \cdots + (-1)^{r-m}\binom{r-m}{r-m}\right]$$

$$= \binom{r}{m}[1-1]^{r-m} = \binom{r}{m} \cdot 0 = 0 \text{ times,}$$

and the formula is verified.

Corollary 8.2

- Let L_m denotes the number of elements in S that satisfy at least m of the t conditions.

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t .$$

When $m = 1$, the result in Corollary 8.2 becomes

$$\begin{aligned} L_1 &= S_1 - \binom{1}{0} S_2 + \binom{2}{0} S_3 - \cdots + (-1)^{t-1} \binom{t-1}{0} S_t \\ &= S_1 - S_2 + S_3 - \cdots + (-1)^{t-1} S_t . \end{aligned}$$

Comparing this with the result in Theorem 8.1, we find that

$$L_1 = N - \overline{N} = |S| - \overline{N}.$$

8.3 Derangements: Nothing Is in Its Right Place

- $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots$$
- e^{-1}=0.36788, 1-1+(1/2!)-(1/3!)+…-(1/7!) ≈ 0.36786

- **Derangement** means that all numbers are in the wrong positions.
- **Ex 8.12** : Determine the number of derangements of 1, 2,…,10.
  Let c_i be the condition that integer i is in the ith place.

$$\begin{aligned} d_{10} &= N(\bar{c}_1\bar{c}_2\bar{c}_3 \cdots \bar{c}_{10}) = 10! - \binom{10}{1}9! + \binom{10}{2}8! - \binom{10}{3}7! + \cdots + \binom{10}{10}0! \\ &= 10!\left[1 - \binom{10}{1}(9!/10!) + \binom{10}{2}(8!/10!) - \binom{10}{3}(7!/10!) + \cdots + \binom{10}{10}(0!/10!)\right] \\ &= 10![1 - 1 + (1/2!) - (1/3!) + \cdots + (1/10!)] \doteq (10!)(e^{-1}). \end{aligned}$$

Derangements: Nothing Is in Its Right Place

- The general formula:

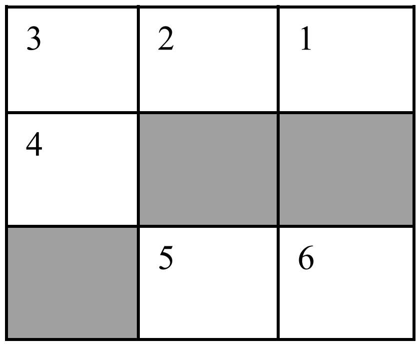
$$d_n = n!e^{-1} = n![1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots \frac{1}{n!}]$$

$P = d_n/n!$

- **Ex 8.14** : Peggy has seven books and hires seven reviewers. She wants two reviewers per book. In how many ways can she make the distributions?
  - The first time: 7! ways
  - The second time: $d_7 = 7! * e^{-1}$ Ways (different position)
  - Totally, we have $7! \times d_7$ ways

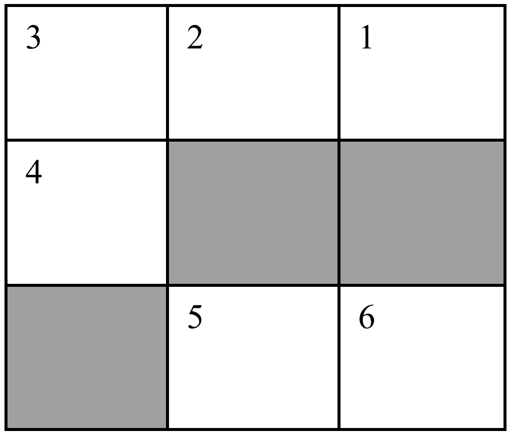
8.4 Rook Polynomials

- In Fig. 8.6, we want to determine *the number of ways* in which *k rooks* can be placed on the unshaded squares of this chessboard so that no two of them can take each other—that is, *no two of them are in the same row or column of the chessboard C*. This number is denoted as $r_k(C)$.



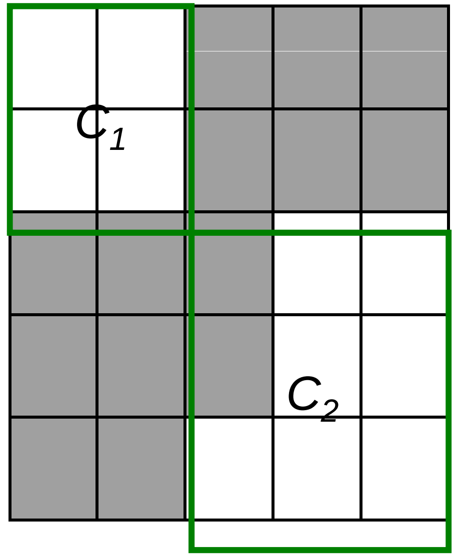
Rook Polynomials

- In Fig. 8.6, we have r_0=1, r_1= 6, r_2= 8, r_3= 2 and r_k= 0 for $k \geq 4$.
- Rook polynomial: $r(C, x) = 1+6x+8x^2+2x^3$. For each $k \geq 0$, the coefficient of x^k is the number of ways we can place k nontaking rooks on chessboard C.



Disjoint Subboards

- Break up a larger board into smaller subboards.
- In Fig. 8.7, the chessboard contains two disjoint subboards that have no squares in the same column or row of C.
- $r(C, x) = r(C_1, x) \cdot r(C_2, x)$



$$r(C_1, x) = 1 + 4x + 2x^2$$

$$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$$

$$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$$

$$= r(C_1, x) \cdot r(C_2, x)$$

Disjoint Subboards

- r_3 for C

a) All three rooks are on subboard C_2 (and none is on C_1): $(2)(1) = 2$ ways.

b) Two rooks are on subboard C_2 and one is on C_1: $(10)(4) = 40$ ways.

c) One rook is on subboard C_2 and two are on C_1: $(7)(2) = 14$ ways.

- In general, if C is a chessboard made up of pairwise disjoint subboards $C_1, C_2, \ldots, C_n$, then $r(C, x) = r(C_1, x)$. $r(C_2, x) \ldots . \ r(C_n, x)$.

Recursive Formula

- Fig. 8.8 (a), For a given designated square ($*$), we either (b) place one rook here, or (c) do not use this square.
- $r_k(C) = r_{k\text{-}1}(C_s)+r_k(C_e)$
  - C_s: denote the remaining smaller subboard (Fig. 8.8(b))
  - C_e: C with the one designed square eliminated (Fig. 8.8(c))
- $r_k(C)x^k = r_{k\text{-}1}(C_s)\, x^k + r_k(C_e)x^k$ for $1\leq k \leq$ n.

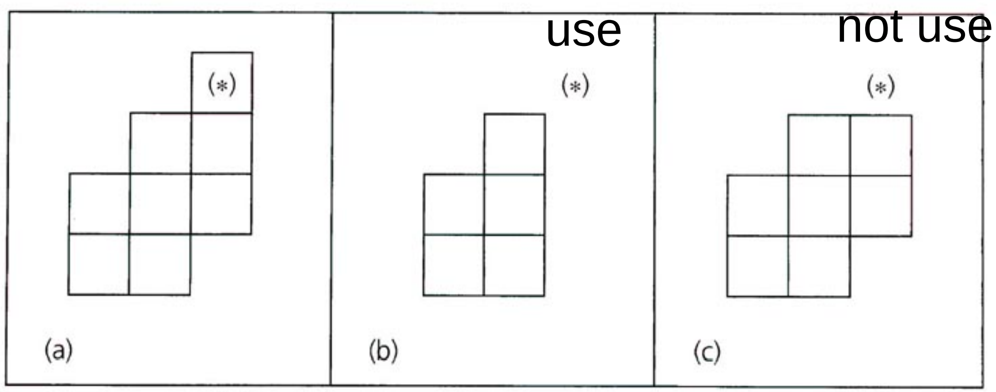


Figure 8.8

Recursive Formula

$$\sum_{k=1}^{n} r_k(C)x^k = \sum_{k=1}^{n} r_{k-1}(C_s)x^k + \sum_{k=1}^{n} r_k(C_e)x^k.$$

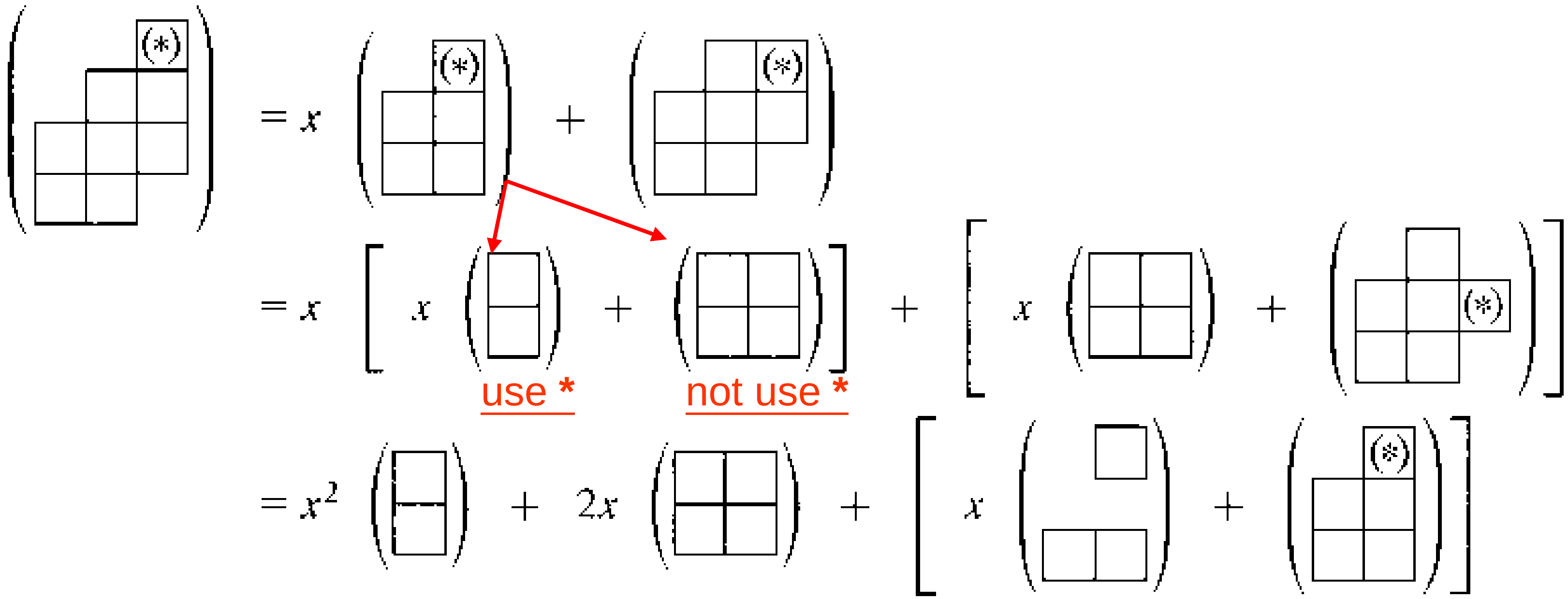
$r_0(C_s)x^0$
typo in p405

$$\sum_{k=1}^{n} r_k(C)x^k = x \sum_{k=1}^{n} r_{k-1}(C_s)x^{k-1} + \sum_{k=1}^{n} r_k(C_e)x^k$$

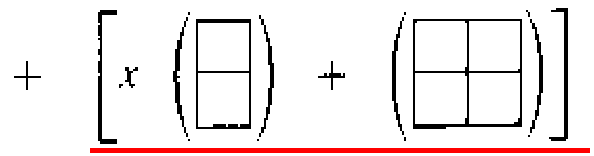
$$1 + \sum_{k=1}^{n} r_k(C)x^k = x \cdot r(C_s, x) + \sum_{k=1}^{n} r_k(C_e)x^k + 1,$$

$$r(C, x) = x \cdot r(C_s, x) + r(C_e, x).$$

Apply the Recursive Formula



$$= x^2(1+2x) + 2x(1+4x+2x^2) + x(1+3x+x^2)$$



$$= 3x + 12x^2 + 7x^3 + x(1+2x) + (1+4x+2x^2) = 1 + 8x + 16x^2 + 7x^3.$$

8.5 Arrangements with Forbidden Positions

- **Ex 8.15** : In making seating arrangements, the shaded square of the figure means relative R_i will not sit at table T_j.
  - Determine the number of ways that we can seat these four relatives at five different tables.
  - Let |S| be the total number of ways we can place the four relatives. (|S| = 5!)
  - Let c_i be the condition that R_i is seated in a forbidden position but at different tables.
    - $N(c_1) = 4! + 4!$ $(R_1 \rightarrow T_1$ or $R_1 \rightarrow T_2)$
    - $N(c_2) = 4!$ $(R_2 \rightarrow T_2)$
    - $N(c_3) = ?$ $4! + 4!$
    - $N(c_4) = ?$ $4! + 4!$
  - $S_1 = 7(4!)$

number of shaded squares

	T_1	T_2	T_3	T_4	T_5
R_1	shaded	shaded			
R_2		shaded			
R_3			shaded	shaded	
R_4				shaded	shaded

Arrangements with Forbidden Positions

- $S_2 = 16(3!)$
  - $N(c_1c_2) = 3!$ ($R_1 \rightarrow T_1$ or $R_2 \rightarrow T_2$)
  - $N(c_1c_3) = \mathbf{4}(3!)$
  - $N(c_1c_4) = ?$, $N(c_2c_3) = ?$, $N(c_2c_4) = ?$, $N(c_3c_4) = ?$
- Observation: 16 is the number of ways two nontaking rooks can be placed on the shaded chessboard.
- Let r_i be the number of ways in which it is possible to place i nontaking rooks on the shaded chessboard.
  - For all $0 \leq i \leq 4$, $S_i = r_i(5 - i)!$
- Decompose C into the disjoint subboards in the upper left and lower right corners.
  - $r(C, x) = (1 + 3x + x^2)(1 + 4x + 3x^2) = 1 + 7x + 16x^2 + 13x^3 + 3x^4$

$$\therefore N(\overline{c_1c_2c_3c_4}) = S_0 - S_1 + S_2 - S_3 + S_4 = 1(5!) - 7(4!) + 16(3!) - 13(2!) + 3(1!)$$

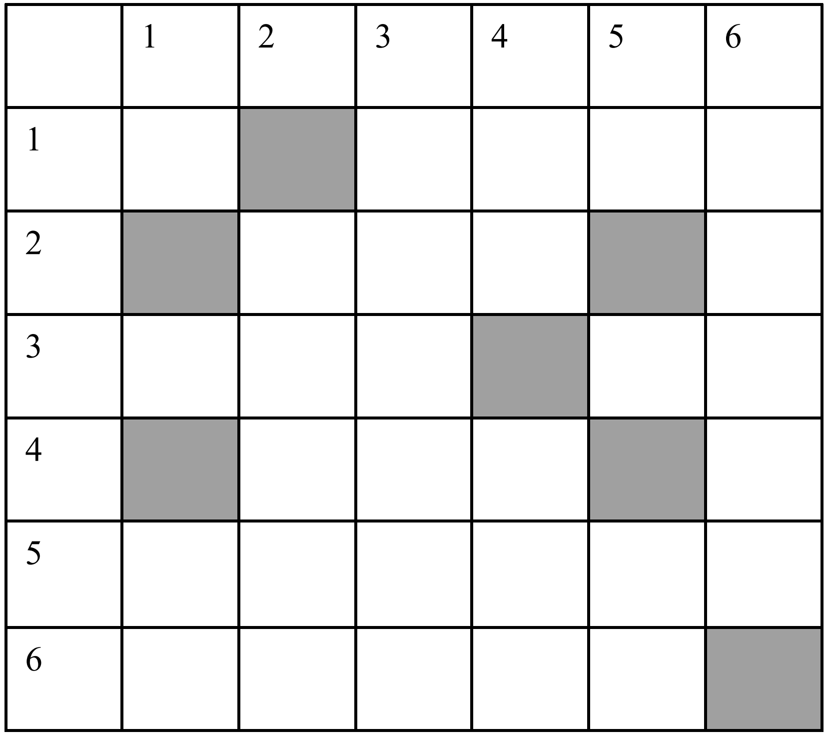
$$= \sum_{i=0}^{4} (-1)^i r_i (5-i)! = 25$$

Arrangements with Forbidden Positions

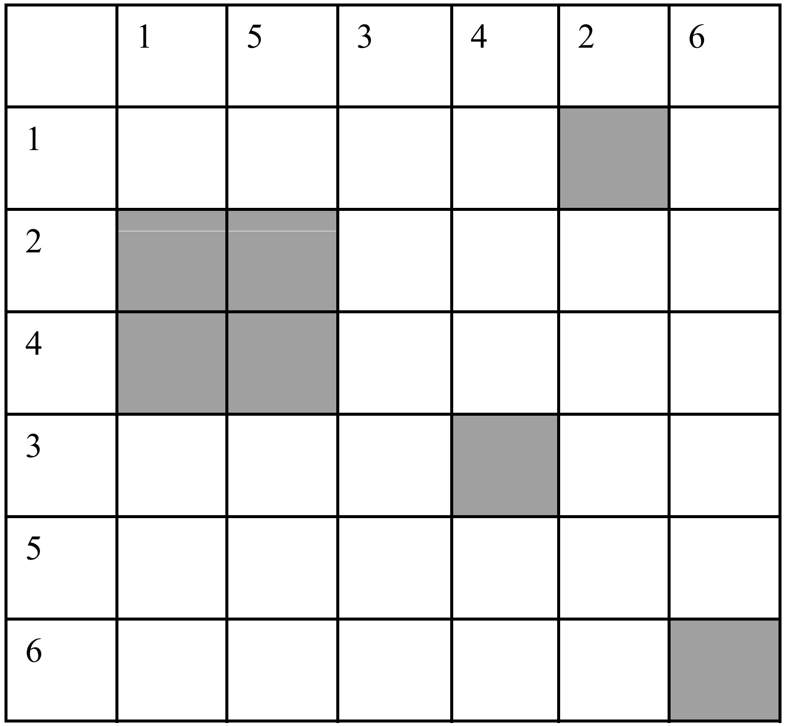
- **Ex 8.16** : We roll two dice six times, where one is red die and the other green die.
- We know the following pairs did not occur: (1, 2), (2, 1), (2, 5), (3, 4), (4, 1), (4, 5) and (6, 6).
- What is the probability that we obtain all six values both on red die and green die?
  - One of solutions is like (1, 1), (2, 3), (4, 4), (3, 2), (5, 6), (6, 5).
- In Fig. 8.10(b), chessboard C with seven shaded squares
  - $r(C, x) = (1+4x+2x^2)(1+x)^3 = 1+7x+17x^2+19x^3+10x^4+2x^5$
  - c_i denotes that all six values occur on both the red and green dies, but i on the red die is paired with one of the forbidden numbers on the green die.

Figure 8.10

(a)



(b)



Arrangements with Forbidden Positions

$$
\begin{aligned}
(6!)N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4\overline{c}_5\overline{c}_6) &= (6!)\sum_{i=0}^{6}(-1)^i S_i = (6!)\sum_{i=0}^{6}(-1)^i r_i(6-i)! \\
&= 6![6! - 7(5!) + 17(4!) - 19(3!) + 10(2!) - 2(1!) + 0(0!)] \\
&= 6![192] = 138{,}240.
\end{aligned}
$$

Since the sample space consists of all sequences of six ordered pairs selected with repetition from the 29 unshaded squares of the chessboard, the probability of this event is $138{,}240/(29)^6 \doteq 0.00023$.

Arrangements with Forbidden Positions

- **Ex 8.17** : How many one-to-one functions f: A $\to$ B satisfy none of the following conditions:

$$c_1 : f(1) = u \text{ or } v$$
$$c_2 : f(2) = w$$
$$c_3 : f(3) = w \text{ or } x$$
$$c_4 : f(4) = x, y, \text{or } z$$

B

A

	u	v	w	x	y	z
1						
2						
3						
4						

- $r(C, x) = (1+2x)(1+6x+9x^2+2x^3) = 1+8x+21x^2+20x^3+4x^4$
-

$$\begin{aligned} N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) &= S_0 - S_1 + S_2 - S_3 + S_4 \\ &= \underbrace{(6!/2!)}_{P(6,4)} - 8\underbrace{(5!/2!)}_{P(5,3)} + 21(4!/2!) - 20(3!/2!) + 4(2!/2!) \\ &= \sum_{i=0}^{4} (-1)^i r_i (6-i)!/2! = 76 \end{aligned}$$

Arrangements with Forbidden Positions

Finally, for $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, suppose we want to count the number of one-to-one functions $h: A \to A$ where $h(i) \neq i$ for all $i \in A$. Here the rook polynomial would be

$$r(C, x) = (1 + x)^8 = \sum_{k=0}^{8} \binom{8}{k} x^k$$

and we find that the number of such one-to-one functions h is

$$\binom{8}{0}8! - \binom{8}{1}7! + \binom{8}{2}6! - \binom{8}{3}5! + \cdots + \binom{8}{8}0!$$

$$= 8!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{8!}\right]$$

$= d_8$, the number of derangements of 1, 2, 3, . . . , 8.

	1	2	3	4	5	6	7	8
1								
2								
3								
4								
5								
6								
7								
8								

Exercises

- ***You should explain your answers in detail, only one-line answer will be rated 0***
- 8-1: 6, 13, 16
  8-2: 5
  8-3: 6, 10
  8-4,5: 5, 12 (add $f(5) \neq z$)
- How many integers n are such that $0 \leq n < 10{,}000$ and the sum of the digits is less than or equal to 27?