Read theorems

Relations and Functions

Cartesian Products and Relations

For sets $A, B \in U$, the Cartesian product or cross product of A & B is denoted by $A \times B$ & equals $\{(a,b) \mid a \in A, b \in B\}$

The elements of $A \times B$ are ordered pairs

$|A \times B| = |A| \times |B| = |B \times A|$

But in general $A \times B \neq B \times A$ and

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots a_n) \mid a_i \in A_i, 1 \leq i \leq n\}$

For sets $A, B \in U$, any subset of $A \times B$ is called relation from A to B. Any subset of $A \times A$ is called binary relation on A. 2^{mn} or 2^{n^2}

In general, for finite sets A, B with $|A| = m$ & $|B| = n$, there are 2^{mn} relations from A to B, including the empty relation as well as the relation $A \times B$ itself.

$|A| = n$, $|A \times A| = n^2$

Relations on A $= 2^{n^2}$

Reflexive relations on A $= 2^{n^2 - n}$

Symmetric relations on A $= 2^n \times 2^{\frac{1}{2}(n^2 - n)}$

Reflexive & symmetric relations on A $= 2^{\frac{1}{2}(n^2 - n)}$

Antisymmetric relations on A $= 2^n \times 3^{\frac{1}{2}(n^2 - n)}$

Reflexive & antisymmetric relations on A $= 3^{\frac{1}{2}(n^2 - n)}$

(1,1) (2,2) = Reflexive

(2,1) (1,2) (2,3) (3,2) = Symmetric

(1) If $A = \{w, x, y, z\}$

Determine the no/. of relations on A that are

(a) reflexive

$n = 4$ $\quad \cancel{2^4 \cdot 2^{\frac{1}{}}}$ $\quad 2^{n^2-n}$ $= 2^{16-4}$ $= 2^{12}$

(b) reflexive & symmetric

$2^{\frac{1}{2}(n^2-n)}$

$= 2^{\frac{1}{2}(12)}$

$= 2^6$

(c) symmetric

$2^n \cdot 2^{\frac{1}{2}(n^2-n)}$

$= 2^4 \cdot 2^6$ $= 2^{10}$

(d) reflexive & contain (x,y)

$\cancel{2^{4-2} = 2^2}$ $\quad 2^{4-2-1}$ $= 2^1 = 2$

(e) antisymmetric

$2^n \cdot 3^{\frac{1}{2}(n^2-n)}$ $= 2^4 \cdot 3^{\frac{1}{2}(16-4)}$

$= 2^4 \cdot 3^6$

(f) symmetric & contain (x,y)

$2^4 \cdot 2^{\frac{16-4}{2}-1}$ $= 2^4 \cdot 2^5$ $= 2^9$

(g) antisymmetric & contain (x,y)

$2^4 \cdot 3^{\frac{16-4}{2}-1}$ $= 2^4 \cdot 3^5$

(h) symmetric & antisymmetric

2^4

(i) reflexive, symmetric & antisymmetric $= 1$

$\cancel{2^{n^2-n} \times 2^n \times 2^{\frac{1}{2}(n^2-n)} \times 2^n \times 3^{\frac{1}{2}(n^2-n)}}$

$\cancel{2^{12} \times 2^4 \times 2^6 \times 2^4 \times 3^6}$

1) If $A = \{1, 2, 3, 4\}$ give an example of a relation R on A that is

a) reflexive & symmetric, but not transitive

$R_1 = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$

b) reflexive & transitive, but not symmetric

$R_1 = \{(1,1)\ (2,2)\ (3,3)\ (4,4)$
$(2,3)\ (4,1)\}$

c) symmetric & transitive, but not reflexive

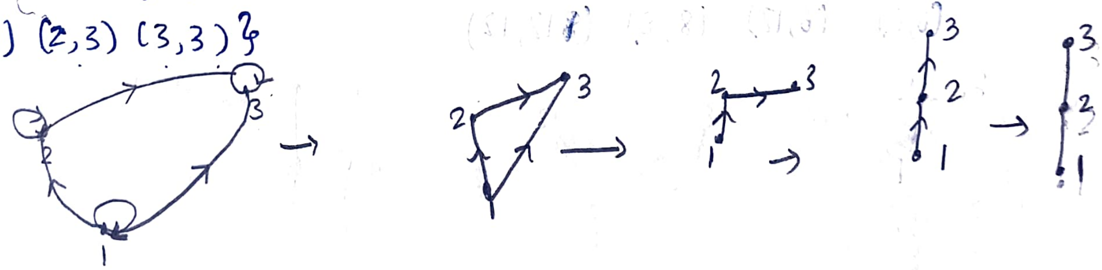
$R_1 = \{(1,2)\ (2,1)\ (1,1)$
$(2,3)\ (3,3)\}$

Poset = Should be reflexive, antisymmetric & transitive

Hasse Diagram = Graphical representation of Poset

Ex: construct for $(\{1,2,3\}, \leq)$

$R = \{(1,1)\ (1,2)\ (1,3)$
$(2,2)\ (2,3)\ (3,3)\}$

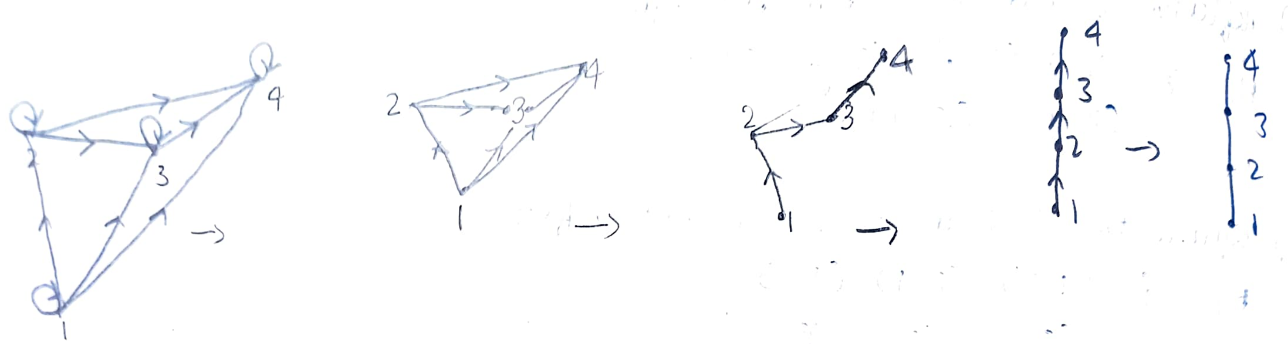


Procedure:

- Start with the digraph of poset
- Remove the loops at each vertex
- Remove all edges that must be present bcoz of transitivity.
- Arrange each edge so that all arrows point up
- Remove all arrowheads

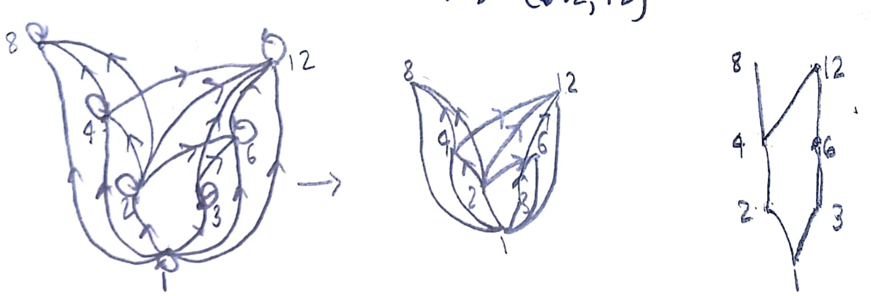
① Construct Hasse diagram for $(\{1,2,3,4\}, \leq)$

$R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)$
$(2,2)\ (2,3)\ (2,4)$
$(3,3)\ (3,4)$
$(4,4)\ \}$

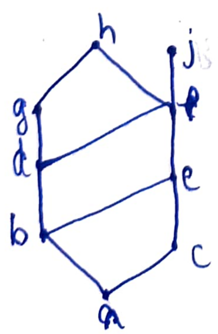


② Construct for $(\{1,2,3,4,6,8,12\}, |)$

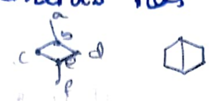
$R = (1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,6)\ (1,8)\ (1,12)$
$(2,2)\ (2,4)\ (2,6)\ (2,8)\ (2,12)$
$(3,3)\ (3,6)\ (3,12)\ (4,4)\ (4,8)\ (4,12)$
$(6,6)\ (6,12)\ (8,8)\ (12,12)$



(1,2) (2,4) ·· (1,3) (3,6)
(1,4) ✗ (1,6) x



Maximal :: h, j
Minimal : a
Greatest element : None
Least element : a
Upper bound of {a, b, c} : e, f, j, h, g, d
Least upper bound of {a, b, c} : e
Lower bound of {a, b, c} :
Least lower bound of {a, b, c} :

Poset in which every pair of elements has both least Upper bound & greatest lower bound is called lattice. 

Equality relation $\{(a_i, a_i)\ a_i \text{ in } A\}$ is both poset & equivalence relation.

Equivalence relation: Reflexive, symmetric & transitive

① Two integers a and b are congruent modulo m iff they have the same remainder when divided by m

$a \equiv b \pmod{m} \longrightarrow m \mid a-b$

a is congruent to b modulo m

Equivalence classes and partitions

Let R be an equivalence relation on a set A. Following statements for elements a & b of set A are equivalent

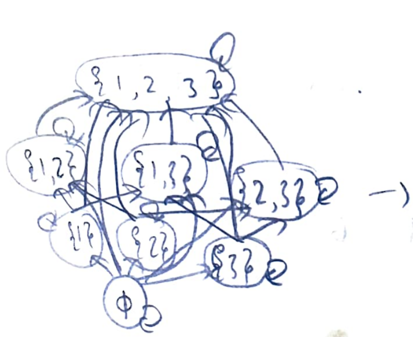
(i) aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \phi$

Let R be an equivalence relation on a set A. For any $x \in A$, the equivalence class of x, denoted $[x]$ is defined by $[x] = \{y \in A \mid yRx\}$

$R^n = R \circ R^{n-1}$

Draw hasse diagram for set $A = \{1,2,3\}$ of $P(A) = \{A, \subseteq\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$



Functions

floor function = the greatest ~~#~~ integer less than or equal to x

$\lfloor 3.8 \rfloor = 3$

$\lfloor -3.8 \rfloor = -4$ $\lfloor -3 \rfloor = 3$ $\cancel{\lfloor 3.8 \rfloor = 4}$

ceiling function = Least integer greather than or equal to x

$\lceil 3.8 \rceil = 4$

$\cancel{\lfloor 3.8 \rfloor = 4}$ $\lceil -3.8 \rceil = -3$

$\cancel{\lceil 3.8 \rceil = -3}$ $\cancel{\lceil 2.8 \rceil} = \lceil 3 \rceil = 3$

trunc function = integer part of x = trunc(x)

① Determine the following

(a) $\lfloor 2.3 - 1.6 \rfloor = \cancel{0.7} \lfloor 0.7 \rfloor = 0$

(b) $\lfloor 2.3 \rfloor - \lfloor 1.6 \rfloor = 2 - 1 = 1$

(c) $\lceil 3.4 \rceil \lfloor 6.2 \rfloor = 4 \times 6 = 24$

② Determine true or false

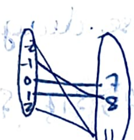
(a) $\lfloor a \rfloor = \lceil a \rceil$ for all $a \in Z$

(b) $\lfloor a \rfloor = \lceil a \rceil$ for all $a \in R$

(c) $\lfloor a \rfloor = \lceil a \rceil - 1$ for all $a \in R - Z$

(d) $-\lfloor a \rfloor = \lceil -a \rceil$ for

③ Check whether relation is a function. If so then find its range.



(a) $\{(x,y) \mid x,y \in Z, y = x^2+7\}$, a relation from Z to Z ✓ Range = $\{7, 8, 11, 16 \ldots\}$

(b) $\{(x,y) \mid x,y \in R, y^2 = x\}$, a relation from R to R ✗

(c) $\{(x,y) \mid x,y \in R, y = 3x+1\}$, a relation from R to R ✓ Range = set of all real no/.

(d) $\{(x,y) \mid x,y \in Q, x^2+y^2 = 1\}$, a relation from Q to Q ✗

(e) R is a relation from A to B where $|A| = 5$ $|B| = 6$ & $|R| = 6$ ✗

no/. of elements

A function $f: A \to B$ is called one-to-one or injective, if each element of B appears at most once as the image of an elements of A.

If $f: A \to B$ is one to one, with A,B finite, we must have $|A| \leq |B|$. For arbitary sets A, B if $f: A \to B$ is one to one, then for $a_1, a_2 \in A$, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

$f(x) = 3x+7$ is one to one

$g(x) = x^4 - x$ is not ; $g(0) = 0$ $g(1) = 0$

Let $A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4, 5\}$.

Then $f = \{(1,1)\ (2,3)\ (3,4)\}$ Function and One to One

$g = \{(1,1)\ (2,3)\ (3,3)\}$ Function but not one to one

Theorem: Let $f: A \to B$, with $A_1, A_2 \subseteq A$. Then
a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$
b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$
c) $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ when f is injective.

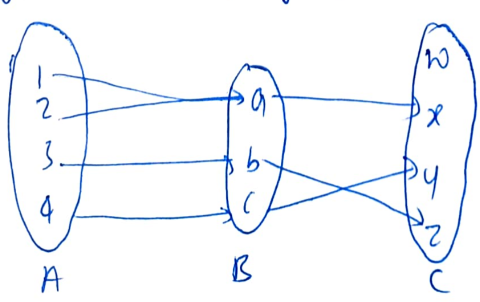
Def:

→ If $f: A \to B$, then f is said to be bijective or to be a one-to-one correspondance. If f is both one to one & onto

→ The function $1_A: A \to A$ defined by $1_A(a) = a$ for all $a \in A$ is called identity function for A

→ If $f, g: A \to B$, we say that f & g are equal & write as

→ If $f: A \to B$ & $g: B \to C$, we define the composite function, which is denoted $g \circ f: A \to C$, by $(g \circ f)(a) = g(f(a))$, for each $a \in A$



$(g \circ f)(1) = g(f(1)) = g(a) = x$

Ex: Let $f, g : R \to R$ be defined by $f(x) = x^2$, $g(x) = x+5$ then

1) $g \circ f(x) = g(f(x)) = g(x^2) = x^2+5$

2) $f \circ g(x) = f(x+5) = (x+5)^2$

Ex: let $f, g, h : R \to R$ where $f(x) = x^2$, $g(x) = x+5$, $h(x) = \sqrt{x^2+2}$

$((h \circ g) \circ f)(x) = (h \circ g) f(x) = h(g(x^2)) = h(x^2+5) = \sqrt{(x^2+5)^2+2}$

Theorem: If $f : A \to B$, $g : B \to C$; $h : C \to D$, then $(h \circ g) \circ f = h \circ (g \circ f)$

Def: If $f : A \to A$, we define $f^1 = f$ and for $n \in Z^+$, $f^{n+1} = f \circ f^n$

Ex: ① $A = \{1, 2, 3, 4\}$; $f : A \to A$ defined by $f = \{(1,2) (2,2) (3,1) (4,3)\}$

Then $f^2 = f \circ f = (1,2)\ (2,2)\ (3,2)\ (4,1)$

$f \circ f(1) = f(2) = 2$ $\quad f(f(2)) = f(2) = 2$ $\quad f(f(3)) = f(1) = 2$ $\quad f(f(4)) = f(3) = 1$

② $f, g : R \to R$, $f(x) = 2x+5$, $g(x) = \frac{1}{2}(x-5)$

Then $gf(x) = g(2x+5) = \frac{1}{\cancel{2}}(\cancel{2}x + \cancel{5} - \cancel{5}) = x$

$fg(x) = f\left(\frac{x-5}{2}\right) = \cancel{2}\left(\frac{x-\cancel{5}}{\cancel{2}}\right) + \cancel{5} = x$

$\therefore$ f & g are both invertible functions.

③ $f : R \to R^+$, $f(x) = e^x$. f is one to one & onto

$f^{-1} = \{(x, y) \mid y = e^x\}^c$

$= \{(y, x) \mid y = e^x\}$ $\qquad x = e^y$

$= \{(y, x) \mid x = e^y\}$ $\qquad \ln x = \ln(e^y)$

$\qquad \ln x = y$

$= \{(x, y) \mid y = \ln x\}$

$f^{-1}(x) = \ln x$

Onto Function

A function $f: A \to B$ is called onto or surjective, if $f(A) = B$ that is, if for all $b \in B$ there is at least one $a \in A$ with $f(a) = b$.

$$\text{Formula} = \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m$$

$|A| = m$

$|B| = n$

$A = 3 \quad B = 2$

$\sum_{k=0} (-1)^k \binom{n}{n-k} (n-k)^3$

Stirling number of second kind

$$\frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m$$

$f(x) = 3x - 5 \qquad x > 0$

$f^{-1}(x) = \{(x, y) \mid y = 3x - 5\}^c$

$= \{(x, y) \mid x = 3y - 5\}$

$= \{(x, y) \mid y = \frac{x+5}{3}\}$

$x = 3y - 5$

$x + 5 = 3y$

$y = \frac{x+5}{3}$

$x > 0$

$y > 0$

$\frac{x+5}{3} > 0.$

$x > -5$

$f^{-1}(x) = \frac{x+5}{3} \qquad x > 0$

$f(x) = -3x + 1 \qquad x \leq 0$

$f^{-1}(x) = \{(x, y) \mid y = -3x + 1\}^c$

$= \{(x, y) \mid x = -3y + 1\}$

$x = -3y + 1$

$x - 1 = -3y$

$y = \frac{x-1}{-3} = \frac{1-x}{3}$

$x \leq 0$

$y \leq 0$

$\frac{1-x}{3} \leq 0$

$-x \leq -1$

$x \leq 1$

$f^{-1}(x) = \frac{1-x}{3} \qquad x \leq 0$

$\frac{x+5}{3}$

$\frac{1-x}{3}$

$f^{-1}(0) = \frac{x+5}{3} \cup \frac{1-x}{3}$

$= \frac{5}{3} \cup \frac{1}{3}$

$f^{-1}(1) = \frac{x+5}{3} \cup \frac{1-x}{3} = 2 \cup 0 = \{0, 2\}$

$f^{-1}(-1) = \frac{4}{3} \cup \frac{2}{3} = \{\frac{2}{3}, \frac{4}{3}\}$