

7.2-7.4

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0.1 Using Definition

0.1. Plot

$$P_e = \Pr(\hat{X} = -1|X = 1) \quad (0.1.1)$$

for

$$Y = AX + N, \quad (0.1.2)$$

where A is Raleigh with $E[A^2] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

0.2. Assuming that N is a constant, find an expression for P_e . Call this $P_e(N)$

Solution: The estimated value \hat{X} is given by

$$\hat{X} = \begin{cases} +1 & Y > 0 \\ -1 & Y < 0 \end{cases} \quad (0.2.1)$$

For $X = 1$,

$$Y = A + N \quad (0.2.2)$$

$$P_e = \Pr(\hat{X} = -1|X = 1) \quad (0.2.3)$$

$$= \Pr(Y < 0|X = 1) \quad (0.2.4)$$

$$= \Pr(A < -N) \quad (0.2.5)$$

$$= F_A(-N) \quad (0.2.6)$$

$$= \int_{-\infty}^{-N} f_A(x) dx \quad (0.2.7)$$

By definition

$$f_A(x) = \begin{cases} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (0.2.8)$$

If $N > 0$, $f_A(x) = 0$. Then,

$$P_e = 0 \quad (0.2.9)$$

If $N < 0$. Then,

$$P_e(N) = \int_{-\infty}^{-N} f_A(x) dx \quad (0.2.10)$$

$$= \int_{-\infty}^0 0 dx + \int_0^{-N} f_A(x) dx \quad (0.2.11)$$

$$= \int_0^{-N} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \quad (0.2.12)$$

$$= 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) \quad (0.2.13)$$

Therefore,

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) & N < 0 \\ 0 & \text{otherwise} \end{cases} \quad (0.2.14)$$

0.3. For a function g ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x) dx \quad (0.3.1)$$

Find $P_e = E[P_e(N)]$.

Solution: Since $N \sim \mathcal{N}(0, 1)$,

$$p_N(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (0.3.2)$$

$$(0.3.3)$$

And from (0.2.14)

$$P_e(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) & x < 0 \\ 0 & \text{otherwise} \end{cases} \quad (0.3.4)$$

$$P_e = E[P_e(N)] = \int_{-\infty}^{\infty} P_e(x) p_N(x) dx \quad (0.3.5)$$

If $x < 0$, $P_e(x) = 0$ and using the fact that for an even function

$$\int_{-\infty}^{\infty} f(x) dx = 2 \int_{-\infty}^0 f(x) dx \quad (0.3.6)$$

we get

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \exp\left(-\frac{x^2}{2}\right) \left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) dx \quad (0.3.7)$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx - \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(1+\sigma^2)x^2}{2\sigma^2}\right) dx \quad (0.3.8)$$

$$= \frac{\sqrt{2\pi} - \sqrt{\frac{\pi(2\sigma^2)}{1+\sigma^2}}}{2\sqrt{2\pi}} \quad (0.3.9)$$

$$= \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\sigma^2}{1+\sigma^2}} \quad (0.3.10)$$

For a Rayleigh Distribution with scale = σ ,

$$E[A^2] = 2\sigma^2 \quad (0.3.11)$$

$$\gamma = 2\sigma^2 \quad (0.3.12)$$

$$\therefore P_e = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma}{1+\gamma}} \quad (0.3.13)$$

0.4. Plot P_e in problems 0.1 and 0.3 on the same graph w.r.t γ . Comment.

Solution: P_e is plotted w.r.t γ in 0.4.1 using the code below.

<https://github.com/AmulyaTallamraju/AI1103/blob/main/probman/codes/7.4.py>

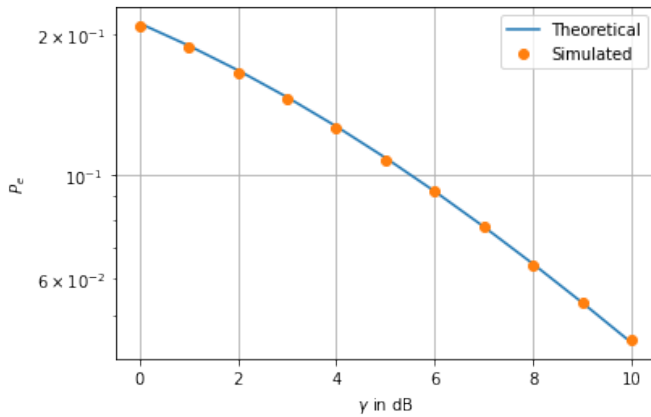


Fig. 0.4.1: P_e w.r.t γ