7.2-7.4

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0.1 Using Definition

0.1. Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{0.1.1}$$

for

$$Y = AX + N, \tag{0.1.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \le \gamma \le 10$ dB.

0.2. Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

Solution: The estimated value $\hat{X}$ is given by

$$\hat{X} = \begin{cases} +1 & Y > 0 \\ -1 & Y < 0 \end{cases} \tag{0.2.1}$$

For $X = 1$,

$$Y = A + N \tag{0.2.2}$$

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{0.2.3}$$

$$= \Pr\left(Y < 0 | X = 1\right) \tag{0.2.4}$$

$$= \Pr\left(A < -N\right) \tag{0.2.5}$$

$$= F_A(-N) \tag{0.2.6}$$

$$= \int_{-\infty}^{-N} f_A(x)dx \tag{0.2.7}$$

By definition

$$f_A(x) = \begin{cases} \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right) & x \ge 0 \\ 0 & otherwise \end{cases} \tag{0.2.8}$$

If $N > 0, f_A(x) = 0$. Then,

$$P_e = 0 \tag{0.2.9}$$

If $N < 0$. Then,

$$P_e(N) = \int_{-\infty}^{-N} f_A(x)dx \tag{0.2.10}$$

$$= \int_{-\infty}^{0} 0dx + \int_{0}^{-N} f_A(x)dx \tag{0.2.11}$$

$$= \int_{0}^{-N} \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)dx \tag{0.2.12}$$

$$= 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) \tag{0.2.13}$$

Therefore,

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) & N < 0 \\ 0 & otherwise \end{cases} \tag{0.2.14}$$

0.3. For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx \tag{0.3.1}$$

Find $P_e = E\left[P_e(N)\right]$.

Solution: Since $N \sim \mathcal{N}(0,1)$,

$$p_N(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \tag{0.3.2}$$

$$\tag{0.3.3}$$

And from (0.2.14)

$$P_e(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) & x < 0 \\ 0 & otherwise \end{cases} \tag{0.3.4}$$

$$P_e = E\left[P_e(N)\right] = \int_{-\infty}^{\infty} P_e(x)p_N(x)\,dx \tag{0.3.5}$$

If $x < 0, P_e(x) = 0$ and using the fact that for an even function

$$\int_{-\infty}^{\infty} f(x) = 2\int_{-\infty}^{0} f(x) \tag{0.3.6}$$

we get

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{x^2}{2}\right)\left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) dx \tag{0.3.7}$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx - \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(1+\sigma^2)x^2}{2\sigma^2}\right) dx \tag{0.3.8}$$

$$= \frac{\sqrt{2\pi} - \sqrt{\frac{\pi(2\sigma^2)}{1+\sigma^2}}}{2\sqrt{2\pi}} \tag{0.3.9}$$

$$= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\sigma^2}{1+\sigma^2}} \tag{0.3.10}$$

For a Rayleigh Distribution with scale $= \sigma$,

$$E\left[A^2\right] = 2\sigma^2 \tag{0.3.11}$$

$$\gamma = 2\sigma^2 \tag{0.3.12}$$

$$\therefore P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}} \tag{0.3.13}$$

0.4. Plot P_e in problems 0.1 and 0.3 on the same graph w.r.t γ. Comment.
Solution: P_e is plotted w.r.t γ in 0.4.1 using the code below.

https://github.com/AmulyaTallamraju/AI1103/blob/main/probman/codes/7.4.py

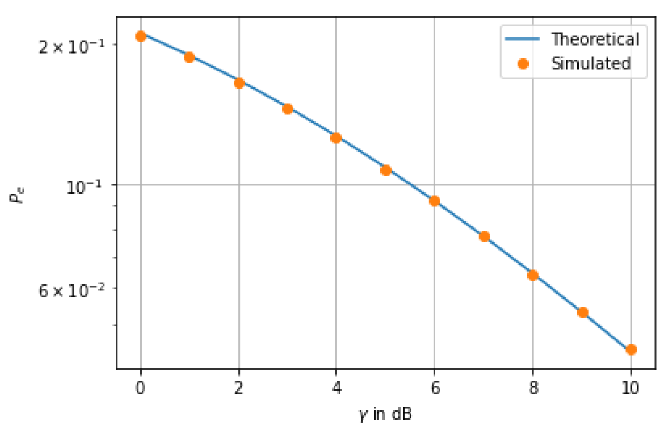

Fig. 0.4.1: P_e w.r.t γ