

Assignment 4

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Download all python codes from

<https://github.com/AmulyaTallamraju/Assignment-4/blob/main/Assignment4/codes/Assignment-4.py>

and latex-tikz codes from

<https://github.com/AmulyaTallamraju/Assignment-4/blob/main/Assignment4/Assignment-4.tex>

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Let X_1 and X_2 be independent geometric random variables with the same probability mass function given by $\Pr(X = k) = p(1 - p)^{k-1}$, $k = 1, 2, \dots$. Then the value of $\Pr(X_1 = 2 | X_1 + X_2 = 4)$ correct up to three decimal places is

SOLUTION

Let

$$p_{X_i}(k) = \Pr(X_i = k) = \begin{cases} p(1 - p)^{k-1} & n = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (0.0.1)$$

where $i=1,2$

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \quad (0.0.2)$$

$$(X_1 = 2) \cap (X_1 + X_2 = 4) = (X_1 = 2, X_2 = 2) \quad (0.0.3)$$

Thus,

$$\Pr(X_1 = 2 | X_1 + X_2 = 4) = \frac{\Pr(X_1 = 2, X_2 = 2)}{\Pr(X_1 + X_2 = 4)} \quad (0.0.4)$$

Since the two events are independent,

$$\Pr(X_1 = 2 | X_1 + X_2 = 4) = \frac{\Pr(X_1 = 2) \Pr(X_2 = 2)}{\Pr(X_1 + X_2 = 4)} \quad (0.0.5)$$

Let

$$X = X_1 + X_2 \quad (0.0.6)$$

From (0.0.6),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \quad (0.0.7)$$

$$= \sum_k \Pr(X_1 = n - k | X_2 = k) p_{X_2}(k) \quad (0.0.8)$$

after unconditioning. $\because X_1$ and X_2 are independent,

$$\begin{aligned} \Pr(X_1 = n - k | X_2 = k) \\ = \Pr(X_1 = n - k) = p_{X_1}(n - k) \end{aligned} \quad (0.0.9)$$

From (0.0.8) and (0.0.9),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \quad (0.0.10)$$

where $*$ denotes the convolution operation. Substituting from (0.0.1) in (0.0.10),

$$p_X(n) = \sum_{k=1}^{n-1} p_{X_1}(n - k) p_{X_2}(k) \quad (0.0.11)$$

$$= \sum_{k=1}^{n-1} (1 - p)^{k-1} p \cdot (1 - p)^{n-k-1} p \quad (0.0.12)$$

$$= (1 - p)^{n-2} p^2 \sum_{k=1}^{n-1} 1 \quad (0.0.13)$$

$$= (n - 1)(1 - p)^{n-2} p^2 \quad (0.0.14)$$

From (0.0.14) and (0.0.1) we have

$$\Pr(X_1 = 2) = \Pr(X_2 = 2) = p(1 - p) \quad (0.0.15)$$

$$\Pr(X_1 + X_2 = 4) = 3(1 - p)^2 p^2 \quad (0.0.16)$$

Substituting in (0.0.5)

$$\Pr(X_1 = 2 | X_1 + X_2 = 4) = \frac{(1 - p)^2 p^2}{3(1 - p)^2 p^2} \quad (0.0.17)$$

$$= \frac{1}{3} \quad (0.0.18)$$

Therefore, the value of $\Pr(X_1 = 2 | X_1 + X_2 = 4)$ correct up to three decimal places is 0.333