Assignment 4

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Download all python codes from

```
https://github.com/AmulyaTallamraju/Assignment
    −4/blob/main/Assignment4/codes/Assignment
    −4.py
```

and latex-tikz codes from

```
https://github.com/AmulyaTallamraju/Assignment
    −4/blob/main/Assignment4/Assignment−4.tex
```

GATE 2018 MA - Q.54

Let X_1 and X_2 be independent geometric random variables with the same probability mass function given by $\Pr(X = k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$ Then the value of $\Pr(X_1 = 2|X_1 + X_2 = 4)$ correct up to three decimal places is

Solution

Let

$$p_{X_i}(k) = \Pr(X_i = k) = \begin{cases} p(1-p)^{k-1} & n = 1, 2, ... \\ 0 & otherwise \end{cases} \tag{0.0.1}$$

where i=1,2

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{0.0.2}$$

$$(X_1 = 2) \cap (X_1 + X_2 = 4) = (X_1 = 2, X_2 = 2) \tag{0.0.3}$$

Thus,

$$\Pr(X_1 = 2|X_1 + X_2 = 4) = \frac{\Pr(X_1 = 2, X_2 = 2)}{\Pr(X_1 + X_2 = 4)} \tag{0.0.4}$$

Since the two events are independent,

$$\Pr(X_1 = 2|X_1 + X_2 = 4) = \frac{\Pr(X_1 = 2)\Pr(X_2 = 2)}{\Pr(X_1 + X_2 = 4)} \tag{0.0.5}$$

Let

$$X = X_1 + X_2 \tag{0.0.6}$$

From (0.0.6),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{0.0.7}$$

$$= \sum_k \Pr(X_1 = n - k|X_2 = k)\, p_{X_2}(k) \tag{0.0.8}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr(X_1 = n - k|X_2 = k) = \Pr(X_1 = n - k) = p_{X_1}(n - k) \tag{0.0.9}$$

From (0.0.8) and (0.0.9),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{0.0.10}$$

where $*$ denotes the convolution operation. Substituting from (0.0.1) in (0.0.10),

$$p_X(n) = \sum_{k=1}^{n-1} p_{X_1}(n - k) p_{X_2}(k) \tag{0.0.11}$$

$$= \sum_{k=1}^{n-1} (1-p)^{k-1} p \cdot (1-p)^{n-k-1} p \tag{0.0.12}$$

$$= (1-p)^{n-2} p^2 \sum_{k=1}^{n-1} 1 \tag{0.0.13}$$

$$= (n-1)(1-p)^{n-2} p^2 \tag{0.0.14}$$

From (0.0.14) and (0.0.1) we have

$$\Pr(X_1 = 2) = \Pr(X_2 = 2) = p(1-p) \tag{0.0.15}$$

$$\Pr(X_1 + X_2 = 4) = 3(1-p)^2 p^2 \tag{0.0.16}$$

Substituting in (0.0.5)

$$\Pr(X_1 = 2|X_1 + X_2 = 4) = \frac{(1-p)^2 p^2}{3(1-p)^2 p^2} \tag{0.0.17}$$

$$= \frac{1}{3} \tag{0.0.18}$$

Therefore, the value of $\Pr(X_1 = 2|X_1 + X_2 = 4)$ correct up to three decimal places is 0.333 [h]

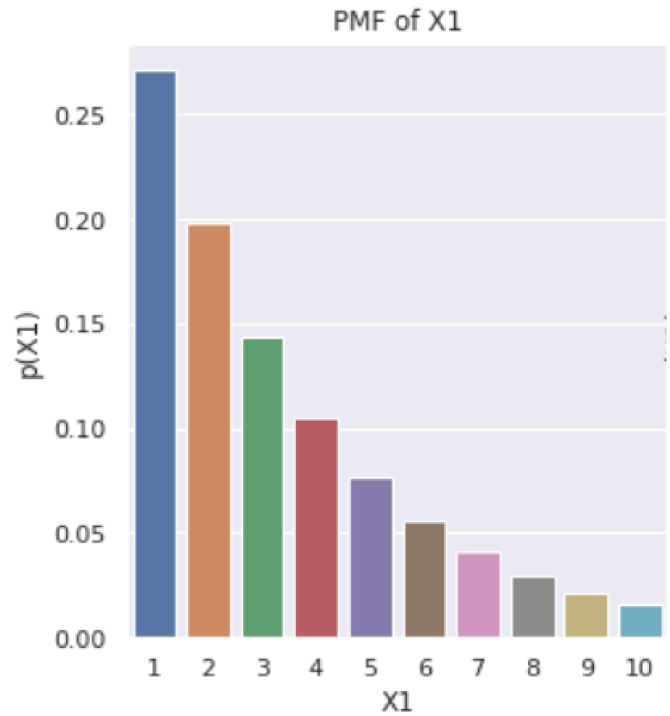

Fig. 0: PMF of X_1 when p=0.27047335

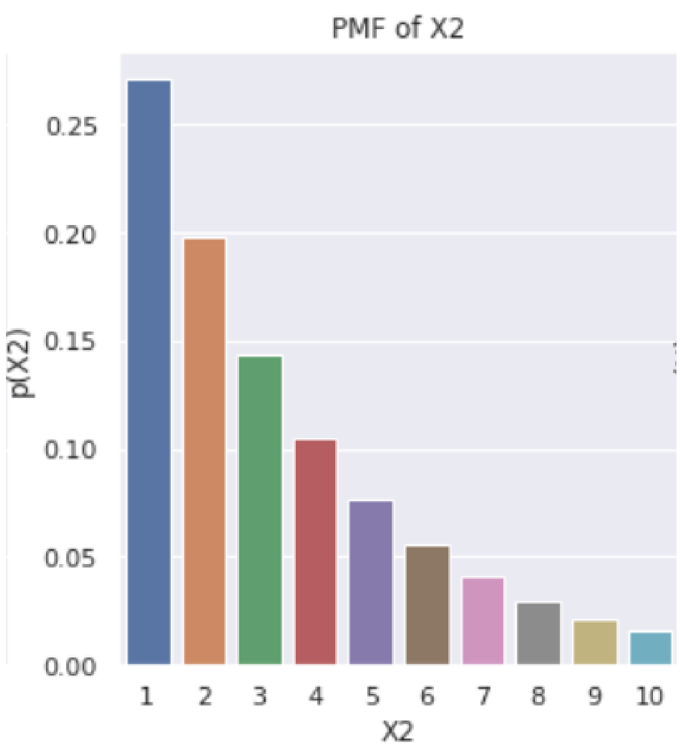

Fig. 0: PMF of X_2 when p=0.27047335

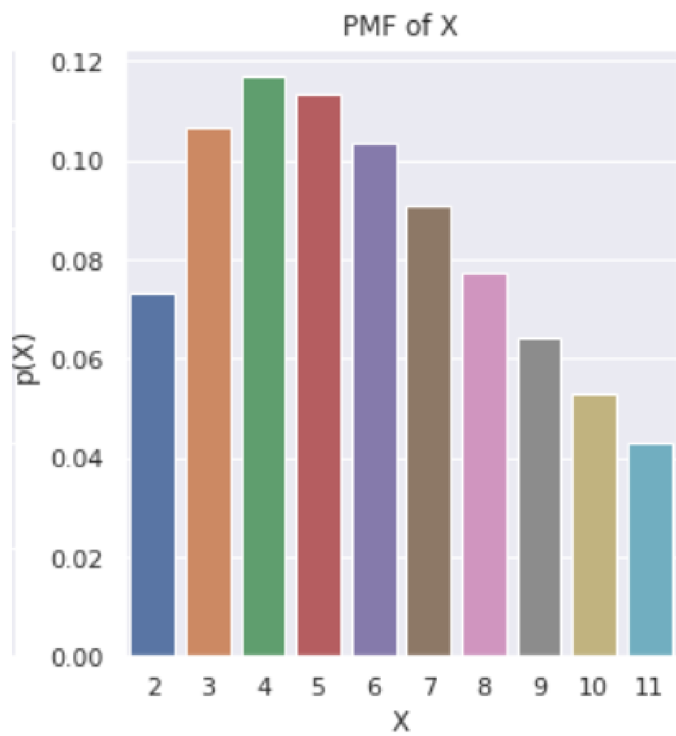

Fig. 0: PMF of X when p=0.27047335