Challenging Problem 10

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Download all python codes from

```
https://github.com/AmulyaTallamraju/
    ChallengingProblem10/blob/main/
    ChallengingProblem10/codes/
    ChallengingProblem10.py
```

and latex-tikz codes from

```
https://github.com/AmulyaTallamraju/
    ChallengingProblem10/blob/main/
    ChallengingProblem10/ChallengingProblem10.
    tex
```

UGC DEC 2018 MATH SET A Q 114

Suppose that $X_1, X_2, X_3, ..., X_{10}$ are i.i.d, N(0,1). Which of the following statements are correct ?

(A) $\Pr(X_1 > X_2 + X_3 + ... + X_{10}) = \frac{1}{2}$
(B) $\Pr(X_1 > X_2 X_3 ... X_{10}) = \frac{1}{2}$
(C) $\Pr(\sin X_1 > \sin X_2 + \sin X_3 + ... + \sin X_{10}) = \frac{1}{2}$
(D) $\Pr(\sin X_1 > \sin(X_2 + X_3 + \ldots + X_{10})) = \frac{1}{2}$

Solution

The random variables follow the standard normal distribution. If $X \sim \mathcal{N}(0, 1)$ then $Y = -X$ also follows standard normal distribution. This can be proved in the following way:

$$P(Y \leq u) = P(-X \leq u) \tag{0.0.1}$$
$$= P(X > -u) \tag{0.0.2}$$
$$= 1 - P(X \leq -u) \tag{0.0.3}$$
$$= 1 - (1 - P(X \leq u) \tag{0.0.4}$$
$$= P(X \leq u) \tag{0.0.5}$$

As the distribution is symmetric,

$$P(X \leq -u) = P(X \geq u) = 1 - P(X \leq u) \tag{0.0.6}$$

Therefore, $-X_1, -X_2 ... - X_{10}$ follow the same distribution as $X_1, X_2 ... X_{10}$
If

$$X_1, X_2 ... X_n \sim \mathcal{N}(0, 1) \tag{0.0.7}$$

where n is an even number and g(x) is an odd function, then

$$\Pr\left(g(X_i) > \sum_{k=0}^{n} g(X_k)\right) = \Pr\left(g(-X_i) < \sum_{k=0}^{n} g(-X_k)\right) = \Pr\left(g(X_i) < \sum_{k=0}^{n} g(X_k)\right) \tag{0.0.8}$$

As they follow the same distribution, the above expression is true. Thus we have

$$\Pr\left(g(X_i) > \sum_{k=0}^{n} g(X_k)\right) = \Pr\left(g(X_i) < \sum_{k=0}^{n} g(X_k)\right) \tag{0.0.9}$$

Also, as X_i is a continuous random variable

$$\Pr\left(g(X_i) = \sum_{k=0}^{n} g(X_k)\right) = 0 \tag{0.0.10}$$

As the cases

$$g(X_i) > \sum_{k=0}^{n} g(X_k) \tag{0.0.11}$$

and

$$g(X_i) < \sum_{k=0}^{n} g(X_k) \tag{0.0.12}$$

are complementary to each other and from and from (0.0.15) we have

$$\Pr\left(g(X_i) > \sum_{k=0}^{n} g(X_k)\right) = \frac{1}{2} \tag{0.0.13}$$

$$\Pr\left(g(X_i) > \prod_{k=0}^{n} g(X_k)\right) = \Pr\left(g(-X_i) < \prod_{k=0}^{n} g(-X_k)\right) = \Pr\left(g(X_i) < \prod_{k=0}^{n} g(X_k)\right) \quad (0.0.14)$$

As they follow the same distribution, the above expression is true. Thus we have

$$\Pr\left(g(X_i) > \prod_{k=0}^{n} g(X_k)\right) = \Pr\left(g(X_i) < \prod_{k=0}^{n} g(X_k)\right) \quad (0.0.15)$$

Also, as X_i is a continuous random variable

$$\Pr\left(g(X_i) = \prod_{k=0}^{n} g(X_k)\right) = 0 \quad (0.0.16)$$

AS the cases

$$g(X_i) > \prod_{k=0}^{n} g(X_k) \quad (0.0.17)$$

and

$$g(X_i) < \prod_{k=0}^{n} g(X_k) \quad (0.0.18)$$

are complementary to each other and from and from (0.0.15) we have

$$\Pr\left(g(X_i) > \prod_{k=0}^{n} g(X_k)\right) = \frac{1}{2} \quad (0.0.19)$$

(A) From (0.0.13) , taking g(x)=x,

$$\Pr(X_1 > X_2 + ... + X_{10}) = \frac{1}{2} \quad (0.0.20)$$

(B) From (0.0.19) taking g(x)=x

$$\Pr(X_1 > X_2X_3...X_{10}) = \frac{1}{2} \quad (0.0.21)$$

(C) From (0.0.13) taking $g(x) = \sin x$

$$\Pr(\sin X_1 > \sin X_2 + ... + \sin X_{10}) = \frac{1}{2} \quad (0.0.22)$$

(D)

$$\Pr(\sin X_1 > \sin(X_2 + ... + X_{10})) = \Pr(\sin(-X_1) < \sin(-X_2 - ... - X_{10})) = \Pr(\sin X_1 < \sin(X_2 + ... + X_{10})) \quad (0.0.23)$$

As they follow the same distribution, the above expression is true. Thus we have

$$\Pr(\sin X_1 > \sin(X_2 + ... + X_{10})) = \Pr(\sin X_1 < \sin(X_2 + ... + X_{10})) \quad (0.0.24)$$

Also, as X_1 is a continuous random variable

$$\Pr(\sin X_1 = \sin(X_2 + ... + X_{10})) = 0 \quad (0.0.25)$$

As the cases

$$X_1 > X_2 + ... + X_{10} \quad (0.0.26)$$

and

$$X_1 < X_2 + ... + X_{10} \quad (0.0.27)$$

are complementary to each other and from , we have

$$\Pr(\sin X_1 > \sin(X_2 + ... + X_{10})) = \frac{1}{2} \quad (0.0.28)$$