Assignment 6

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Download all python codes from

```
https://github.com/AmulyaTallamraju/
    ChallengingProblem10/blob/main/
    ChallengingProblem10/codes/
    ChallengingProblem10.py
```

and latex-tikz codes from

```
https://github.com/AmulyaTallamraju/
    ChallengingProblem10/blob/main/
    ChallengingProblem10/ChallengingProblem10.
    tex
```

UGC DEC 2018 MATH SET A Q 114

Suppose that $X_1, X_2, X_3, ..., X_{10}$ are i.i.d, N(0,1). Which of the following statements are correct ?

(A) $\Pr(X_1 > X_2 + X_3 + ... + X_{10}) = \frac{1}{2}$
(B) $\Pr(X_1 > X_2X_3...X_{10}) = \frac{1}{2}$
(C) $\Pr(\sin X_1 > \sin X_2 + \sin X_3 + ... + \sin X_{10}) = \frac{1}{2}$
(D) $\Pr(\sin X_1 > \sin(X_2 + X_3 + \ldots + X_{10})) = \frac{1}{2}$

Solution

The random variables follow the standard normal distribution. If $X \sim \mathcal{N}(0,1)$ then $Y = -X$ also follows standard normal distribution. This can be proved in the following way:

$$P(Y \le u) = P(-X \le u) \tag{0.0.1}$$
$$= P(X > -u) \tag{0.0.2}$$
$$= 1 - P(X \le -u) \tag{0.0.3}$$
$$= 1 - (1 - P(X \le u) \tag{0.0.4}$$
$$= P(X \le u) \tag{0.0.5}$$

As the distribution is symmetric,

$$P(X \le -u) = P(X \ge u) = 1 - P(X \le u) \tag{0.0.6}$$

Therefore, $-X_1, -X_2... - X_{10}$ follow the same distribution as $X_1, X_2...X_{10}$

(A)

$$\Pr(X_1 > X_2 + ... + X_{10}) = \Pr(-X_1 < -X_2 - ... - X_{10}) = \Pr(X_1 < X_2 + ... + X_{10}) \tag{0.0.7}$$

As they follow the same distribution, the above expression is true. Thus we have

$$\Pr(X_1 > X_2 + ... + X_{10}) = \Pr(X_1 < X_2 + ... + X_{10}) \tag{0.0.8}$$

Also, as X_1 is a continuous random variable

$$\Pr(X_1 = X_2 + ... + X_{10}) = 0 \tag{0.0.9}$$

As the cases

$$X_1 > X_2 + ... + X_{10} \tag{0.0.10}$$

and

$$X_1 < X_2 + ... + X_{10} \tag{0.0.11}$$

are complementary to each other and from (0.0.26), we have

$$\Pr(X_1 > X_2 + ... + X_{10}) = \frac{1}{2} \tag{0.0.12}$$

We can extend this argument to all the other options as well.

(B)

$$\Pr(X_1 > X_2...X_{10}) = \Pr(-X_1 < (-X_2)(-X_3)...(-X_{10})) = \Pr(X_1 < X_2X_3...X_{10}) \tag{0.0.13}$$

As they follow the same distribution, the above expression is true. Thus we have

$$\Pr(X_1 > X_2X_3...X_{10}) = \Pr(X_1 < X_2X_3...X_{10}) \tag{0.0.14}$$

Also, as X_1 is a continuous random variable

$$\Pr(X_1 = X_2X_3...X_{10}) = 0 \tag{0.0.15}$$

As the cases

$$X_1 > X_2X_3...X_{10} \tag{0.0.16}$$

and

$$X_1 < X_2X_3...X_{10} \tag{0.0.17}$$

are complementary to each other and from (0.0.26), we have

$$\Pr(X_1 > X_2X_3...X_{10}) = \frac{1}{2} \tag{0.0.18}$$

(C)

$$\begin{aligned}&\Pr(\sin X_1 > \sin X_2 + ... + \sin X_{10})\\ &= \Pr(\sin(-X_1) < \sin(-X_2)...\sin(-X_{10}))\\ &= \Pr(X_1 < X_2 + ... + X_{10})\end{aligned} \tag{0.0.19}$$

As they follow the same distribution, the above expression is true. Thus we have

$$\begin{aligned}&\Pr(\sin X_1 > \sin X_2 + ... + \sin X_{10})\\ &= \Pr(\sin X_1 < \sin X_2 + ... + \sin X_{10})\end{aligned} \tag{0.0.20}$$

Also, as X_1 is a continuous random variable

$$\Pr(\sin X_1 = \sin X_2 + ... + \sin X_{10}) = 0 \tag{0.0.21}$$

As the cases

$$\sin X_1 > \sin X_2 + ... + \sin X_{10} \tag{0.0.22}$$

and

$$\sin X_1 < \sin X_2 + ... + \sin X_{10} \tag{0.0.23}$$

are complementary to each other and from (0.0.26), we have

$$\Pr(\sin X_1 > \sin X_2 + ... + \sin X_{10}) = \frac{1}{2} \tag{0.0.24}$$

(D)

$$\begin{aligned}&\Pr(\sin X_1 > \sin(X_2 + ... + X_{10}))\\ &= \Pr(\sin(-X_1) < \sin(-X_2 - ... - X_{10}))\\ &= \Pr(\sin X_1 < \sin(X_2 + ... + X_{10}))\end{aligned} \tag{0.0.25}$$

As they follow the same distribution, the above expression is true. Thus we have

$$\begin{aligned}&\Pr(\sin X_1 > \sin(X_2 + ... + X_{10}))\\ &= \Pr(\sin X_1 < \sin(X_2 + ... + X_{10}))\end{aligned} \tag{0.0.26}$$

Also, as X_1 is a continuous random variable

$$\Pr(\sin X_1 = \sin(X_2 + ... + X_{10})) = 0 \tag{0.0.27}$$

As the cases

$$X_1 > X_2 + ... + X_{10} \tag{0.0.28}$$

and

$$X_1 < X_2 + ... + X_{10} \tag{0.0.29}$$

are complementary to each other and from (0.0.26), we have

$$\Pr(\sin X_1 > \sin(X_2 + ... + X_{10})) = \frac{1}{2} \tag{0.0.30}$$

(E) A more general case. If

$$X_1, X_2...X_n \sim \mathcal{N}(0,1) \tag{0.0.31}$$

where n is an even number, then

$$\Pr\left(g(X_i) > \sum_{k=0}^{n} g(X_k)\right) = \frac{1}{2} \tag{0.0.32}$$

and

$$\Pr\left(g(X_i) > \prod_{k=0}^{n} g(X_k)\right) = \frac{1}{2} \tag{0.0.33}$$

where g(x) is an odd function.