

Challenging Problem 10

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Download all python codes from

<https://github.com/AmulyaTallamraju/ChallengingProblem10/blob/main/ChallengingProblem10/codes/ChallengingProblem10.py>

and latex-tikz codes from

<https://github.com/AmulyaTallamraju/ChallengingProblem10/blob/main/ChallengingProblem10/ChallengingProblem10.tex>

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Suppose that $X_1, X_2, X_3, \dots, X_{10}$ are i.i.d, $N(0,1)$. Which of the following statements are correct ?

- (A) $\Pr(X_1 > X_2 + X_3 + \dots + X_{10}) = \frac{1}{2}$
- (B) $\Pr(X_1 > X_2 X_3 \dots X_{10}) = \frac{1}{2}$
- (C) $\Pr(\sin X_1 > \sin X_2 + \sin X_3 + \dots + \sin X_{10}) = \frac{1}{2}$
- (D) $\Pr(\sin X_1 > \sin(X_2 + X_3 + \dots + X_{10})) = \frac{1}{2}$

SOLUTION

The random variables follow the standard normal distribution. If $X \sim N(0,1)$ then $Y = -X$ also follows standard normal distribution. This can be proved in the following way:

$$P(Y \leq u) = P(-X \leq u) \quad (0.0.1)$$

$$= P(X > -u) \quad (0.0.2)$$

$$= 1 - P(X \leq -u) \quad (0.0.3)$$

$$= 1 - (1 - P(X \leq u)) \quad (0.0.4)$$

$$= P(X \leq u) \quad (0.0.5)$$

As the distribution is symmetric,

$$P(X \leq -u) = P(X \geq u) = 1 - P(X \leq u) \quad (0.0.6)$$

Therefore, $-X_1, -X_2, \dots, -X_{10}$ follow the same distribution as X_1, X_2, \dots, X_{10}

If

$$X_1, X_2, \dots, X_n \sim N(0,1) \quad (0.0.7)$$

where n is an even number and $g(x)$ is an odd function, then

$$\begin{aligned} \Pr\left(g(X_1) > \sum_{k=2}^n g(X_k)\right) \\ &= \Pr\left(g(-X_1) < \sum_{k=2}^n g(-X_k)\right) \\ &= \Pr\left(g(X_1) < \sum_{k=2}^n g(X_k)\right) \quad (0.0.8) \end{aligned}$$

As they follow the same distribution, the above expression is true. Thus we have

$$\Pr\left(g(X_1) > \sum_{k=2}^n g(X_k)\right) = \Pr\left(g(X_1) < \sum_{k=2}^n g(X_k)\right) \quad (0.0.9)$$

Also, as X_1 is a continuous random variable

$$\Pr\left(g(X_1) = \sum_{k=2}^n g(X_k)\right) = 0 \quad (0.0.10)$$

As the cases

$$g(X_1) > \sum_{k=2}^n g(X_k) \quad (0.0.11)$$

and

$$g(X_1) < \sum_{k=2}^n g(X_k) \quad (0.0.12)$$

are complementary to each other and from (0.0.15) we have

$$\Pr\left(g(X_1) > \sum_{k=2}^n g(X_k)\right) = \frac{1}{2} \quad (0.0.13)$$

(D)

$$\begin{aligned}
\Pr\left(g(X_1) > \prod_{k=2}^n g(X_k)\right) \\
&= \Pr\left(g(-X_1) < \prod_{k=2}^n g(-X_k)\right) \\
&= \Pr\left(g(X_1) < \prod_{k=2}^n g(X_k)\right) \quad (0.0.14)
\end{aligned}$$

As they follow the same distribution, the above expression is true. Thus we have

$$\Pr\left(g(X_1) > \prod_{k=2}^n g(X_k)\right) = \Pr\left(g(X_1) < \prod_{k=2}^n g(X_k)\right) \quad (0.0.15)$$

Also, as X_1 is a continuous random variable

$$\Pr\left(g(X_1) = \prod_{k=2}^n g(X_k)\right) = 0 \quad (0.0.16)$$

As the cases

$$g(X_1) > \prod_{k=2}^n g(X_k) \quad (0.0.17)$$

and

$$g(X_1) < \prod_{k=2}^n g(X_k) \quad (0.0.18)$$

are complementary to each other and from (0.0.15) we have

$$\Pr\left(g(X_1) > \prod_{k=2}^n g(X_k)\right) = \frac{1}{2} \quad (0.0.19)$$

(A) From (0.0.13) , taking $g(x) = x$,

$$\Pr(X_1 > X_2 + \dots + X_{10}) = \frac{1}{2} \quad (0.0.20)$$

(B) From (0.0.19) taking $g(x) = x$

$$\Pr(X_1 > X_2 X_3 \dots X_{10}) = \frac{1}{2} \quad (0.0.21)$$

(C) From (0.0.13) taking $g(x) = \sin x$

$$\Pr(\sin X_1 > \sin X_2 + \dots + \sin X_{10}) = \frac{1}{2} \quad (0.0.22)$$

$$\begin{aligned}
&\Pr(\sin X_1 > \sin(X_2 + \dots + X_{10})) \\
&= \Pr(\sin(-X_1) < \sin(-X_2 - \dots - X_{10})) \\
&= \Pr(\sin X_1 < \sin(X_2 + \dots + X_{10})) \quad (0.0.23)
\end{aligned}$$

As they follow the same distribution, the above expression is true. Thus we have

$$\begin{aligned}
&\Pr(\sin X_1 > \sin(X_2 + \dots + X_{10})) \\
&= \Pr(\sin X_1 < \sin(X_2 + \dots + X_{10})) \quad (0.0.24)
\end{aligned}$$

Also, as X_1 is a continuous random variable

$$\Pr(\sin X_1 = \sin(X_2 + \dots + X_{10})) = 0 \quad (0.0.25)$$

As the cases

$$X_1 > X_2 + \dots + X_{10} \quad (0.0.26)$$

and

$$X_1 < X_2 + \dots + X_{10} \quad (0.0.27)$$

are complementary to each other and from , we have

$$\Pr(\sin X_1 > \sin(X_2 + \dots + X_{10})) = \frac{1}{2} \quad (0.0.28)$$