Assignment 5

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Download all python codes from

https://github.com/AmulyaTallamraju/EE3900/blob/main/Assignment−5/codes/Assignment−5.py

and latex-tikz codes from

https://github.com/AmulyaTallamraju/EE3900/blob/main/Assignment−5/Assignment−5.tex

1 Quadratic Forms 2.77

Find the area enclosed by the parabola $4y = x^2$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = -2$.

2 Solution

Lemma 2.1. *The points of intersection of* ***Line*** L : $\mathbf{x} = \mathbf{q} + \mu\mathbf{m}$ *with* ***parabola***

$$y = ax^2 + bx + c \tag{2.0.1}$$

or

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} \frac{b}{2} \\ \frac{-1}{2} \end{pmatrix} \tag{2.0.4}$$

$$f = c \tag{2.0.5}$$

are given by:

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{2.0.6}$$

where,

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\Big(-\mathbf{m}^T\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\right]^2 - \left(\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\Big) \tag{2.0.7}$$

Proof. The points of intersection must satisfy the line and parabola equation. Thus,

$$(\mathbf{q} + \mu\mathbf{m})^{\mathbf{T}}\mathbf{V}(\mathbf{q} + \mu\mathbf{m}) + 2\mathbf{u}^{\mathbf{T}}(\mathbf{q} + \mu\mathbf{m}) + f = 0 \tag{2.0.8}$$

On expansion, we get

$$\mu^2\mathbf{m}^{\mathbf{T}}\mathbf{V}\mathbf{m} + \mu\left[\mathbf{m}^{\mathbf{T}}\mathbf{V}\mathbf{q} + \mathbf{q}^{\mathbf{T}}\mathbf{V}\mathbf{m} + 2\mathbf{u}^{\mathbf{T}}\mathbf{m}\right] + \mathbf{q}^{\mathbf{T}}\mathbf{V}\mathbf{q} + 2\mathbf{u}^{\mathbf{T}}\mathbf{m} + f = 0 \tag{2.0.9}$$

Since, $\mathbf{q}^{\mathbf{T}}\mathbf{V}\mathbf{m}, 2\mathbf{u}^{\mathbf{T}}\mathbf{m}$ are scalars

$$\mathbf{q}^{\mathbf{T}}\mathbf{V}\mathbf{m} = \mathbf{m}^{\mathbf{T}}\mathbf{V}^{\mathbf{T}}\mathbf{q} \tag{2.0.10}$$

$$2\mathbf{u}^{\mathbf{T}}\mathbf{m} = 2\mathbf{m}^{\mathbf{T}}\mathbf{u} \tag{2.0.11}$$

Solving the above quadratic equation we get

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\Big(-\mathbf{m}^T\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\right]^2 - \left(\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\Big) \tag{2.0.12}$$

□

The matrix parameters of the parabola are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{2.0.13}$$

with eigen parameters

$$\lambda_1 = 0, \lambda_2 = 1 \tag{2.0.14}$$

$$\mathbf{p_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{p_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.15}$$

The vertex of the parabola can be expressed as

$$\begin{pmatrix} \mathbf{u}^{\mathbf{T}} + \eta\mathbf{p}_{\mathbf{1}}^{\mathbf{T}} \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -\mathbf{f} \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.16}$$

where,

$$\eta = \mathbf{u}^{\mathbf{T}}\mathbf{p_1} = -2 \tag{2.0.17}$$

$$\implies \begin{pmatrix} 0 & -2 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.18}$$

or

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.19}$$

From (2.0.7),

$$\mu_1 = 4 + 2\sqrt{3}, \mu_2 = 4 - 2\sqrt{3} \tag{2.0.20}$$

The given line is

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2 \tag{2.0.21}$$

In parametric form, the given line can be written as:

$$L : \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \tag{2.0.22}$$

$$\implies \mathbf{x} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.23}$$

Substituting μ_1 and μ_2 in (2.0.23),the points of intersection

$$\mathbf{K} = \begin{pmatrix} 2+2\sqrt{3} \\ 4+2\sqrt{3} \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 2-2\sqrt{3} \\ 4-2\sqrt{3} \end{pmatrix} \tag{2.0.24}$$

1) Thus, from Fig. 4 the area enclosed by parabola and line can be given as

$$A = \text{Area under line} - \text{Area under parabola} \tag{2.0.25}$$

$$A = Ar(KLMNK) - Ar(KCLMCNK) \tag{2.0.26}$$

$$A = A_1 - A_2 \tag{2.0.27}$$

2) Area under the line y=x+2 i.e, A_1-

$$A_1 = \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} y dx \tag{2.0.28}$$

$$= \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} (x+2)\, dx \tag{2.0.29}$$

$$= \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} x dx + \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} 2 dx \tag{2.0.30}$$

$$= \frac{1}{2}\left((2+2\sqrt{3})^2 - (2-2\sqrt{3})^2\right) + 2\left(4\sqrt{3}\right) \tag{2.0.31}$$

$$= 16\sqrt{3} \text{ units} \tag{2.0.32}$$

3) Area under the parabola that is A_2-

$$A_2 = \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} y dx \tag{2.0.33}$$

$$= \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} \frac{1}{4} x^2 dx \tag{2.0.34}$$

$$= \frac{1}{12} \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} x^3 dx \tag{2.0.35}$$

$$= \frac{1}{12}\left((2+2\sqrt{3})^3 - \left(2-2\sqrt{3}\right)^3\right) \tag{2.0.36}$$

$$= 8\sqrt{3} \text{ units} \tag{2.0.37}$$

4) Putting (2.0.32) and (2.0.37) in (2.0.27) we get required area A as:

$$A = A_1 - A_2 \tag{2.0.38}$$

$$A = 8\sqrt{3} \text{ units} \tag{2.0.39}$$

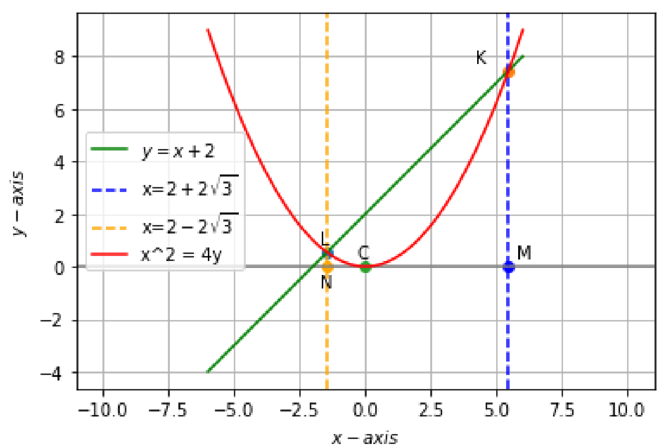

Fig. 4: Plot of the parabola and line