GATE ASSIGNMENT 1

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Download all python codes from

```
https://github.com/AmulyaTallamraju/EE3900/blob/
    main/GATE_Assignment−1/codes/
    GATE_Assignment−1.py
```

and latex-tikz codes from

```
https://github.com/AmulyaTallamraju/EE3900/blob/
    main/GATE_Assignment−1/
    GATE_Assignment−1.tex
```

1 GATE EC 2021 Q.41

Consider the signals $x[n] = 2^{n-1}u[-n+2]$ and $y[n] = 2^{-n+2}u[n+1]$, where $u[n]$ is the unit step sequence. Let $X(e^{j\omega})$ and $Y(e^{j\omega})$ the discrete-time Fourier transform of $x[n]$ and $y[n]$, respectively. The value of the integral

$$\frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})\,d\omega \tag{1.0.1}$$

is

2 Solution

Theorem 2.1 (Convolution Theorem)**.** *Let f and g be two functions with convolution $f*g$. Let F be the Fourier transform operator. Then*

$$F(f*g) = F(f)\cdot F(g) \tag{2.0.1}$$
$$F(f\cdot g) = F(f)*F(g) \tag{2.0.2}$$

If the DTFT of $y[n]$ is $Y(e^{j\omega})$ then using the time reversal property, the DTFT of $y[-n]$ is

$$Y\left(\frac{1}{e^{j\omega}}\right) = Y(e^{-j\omega}) \tag{2.0.3}$$

Let

$$z[n] = x[n]*y[-n] \tag{2.0.4}$$

Let $Z(e^{j\omega})$ be the DTFT of $z[n]$. Using Theorem 2.1 we get

$$Z(e^{j\omega}) = X(e^{j\omega})Y(e^{-j\omega}) \tag{2.0.5}$$

By applying IDTFT, we can write:

$$z[n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})e^{j\omega n}\,d\omega \tag{2.0.6}$$

Putting $n = 0$, we get the required value which is

$$z[0] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})\,d\omega \tag{2.0.7}$$

The $\mathcal{Z}$ transform for a sequence of the form $a^{(n)}u[n]$ when $d = n$ is given by

$$\sum_{n=-\infty}^{\infty}\left(az^{-1}\right)^n u[n] = \left((1-az^{-1})^{-1}\right) \tag{2.0.8}$$

provided, $|z| < |a|$, when $n < 0$ or $|a| < |z|$, when $n > 0$ which is the region of convergence. From (2.0.8), $\mathcal{Z}$ transform of $x[n]$ in the Region of Convergence is

$$X(z) = \mathcal{Z}\{x[n]\} = \frac{1}{2}\left(\left(2z^{-1}\right)^2\left(1-\frac{z}{2}\right)^{-1}\right) \tag{2.0.9}$$

From (2.0.8) $\mathcal{Z}$ transform of $y[n]$ in the Region of Convergence is

$$Y(z) = \mathcal{Z}\{y[n]\} = 4\left(2z\left(1-\frac{z^{-1}}{2}\right)^{-1}\right) \tag{2.0.10}$$

Now,

$$Z(z) = X(z)Y\left(\frac{1}{z}\right) \tag{2.0.11}$$
$$= 2\left(\left(2z^{-1}\right)^3\left(1-\frac{z}{2}\right)^{-2}\right) \tag{2.0.12}$$

The DTFT of $x[n]$ converges for all values of ω since,

$$z = e^{j\omega} \tag{2.0.13}$$
$$a = 2 \tag{2.0.14}$$
$$|za| = \frac{1}{2} \tag{2.0.15}$$

The DTFT of $y[n]$ also converges for all values of

ω since

$$z = e^{j\omega} \tag{2.0.16}$$
$$a = \frac{1}{2} \tag{2.0.17}$$
$$|az^{-1}| = \frac{1}{2} \tag{2.0.18}$$

$$\sum_{n=-\infty}^{\infty} \left(az^{-1}\right)^{n} u[-n] = \sum_{n=-\infty}^{\infty} \left(az^{-1}\right)^{-n} u[n] \tag{2.0.19}$$
$$= \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^{n} \tag{2.0.20}$$
$$= \left(\left(1 - \frac{z}{a}\right)^{-1}\right) \tag{2.0.21}$$

Differentiating the above expression

$$\sum_{n=-\infty}^{\infty} na^{-n}z^{n-1}u[n] = \frac{1}{a}\left(\left(1 - \frac{z}{a}\right)^{-2}\right) \tag{2.0.22}$$
$$\implies \sum_{n=-\infty}^{\infty} na^{-n}z^{n-4}u[n] = \frac{1}{a}\left(z^{-3}\left(1 - \frac{z}{a}\right)^{-2}\right) \tag{2.0.23}$$
$$\implies \sum_{n=-\infty}^{\infty} na^{-n+5}z^{n-4}u[n] = a^{4}\left(z^{-3}\left(1 - \frac{z}{a}\right)^{-2}\right) \tag{2.0.24}$$
$$\implies \sum_{n=-\infty}^{\infty} (4-n)a^{n+1}z^{-n}u[4-n] = a^{4}\left(z^{-3}\left(1 - \frac{z}{a}\right)^{-2}\right) \tag{2.0.25}$$

Comparing the above equation with (2.0.11) we get $a = 2$. From the definition of DTFT we have

$$Z(z = e^{j\omega}) = \sum_{n=-\infty}^{\infty} z[n]z^{-n} \tag{2.0.26}$$

Thus

$$z[n] = (4-n)2^{n+1}u[4-n] \tag{2.0.27}$$

$$\therefore \frac{1}{2\pi}\int_{0}^{2\pi} X(e^{j\omega})Y(e^{-j\omega})\,d\omega = z[0] = 8 \tag{2.0.28}$$

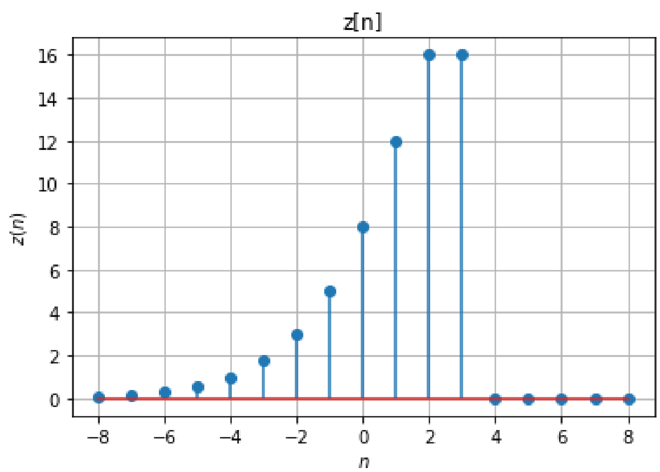

Fig. 0: Plot of z[n]