GATE EC 2021 Q.41

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Question

Consider the signals $x[n] = 2^{n-1}u[-n+2]$ and $y[n] = 2^{-n+2}u[n+1]$, where $u[n]$ is the unit step sequence. Let $X(e^{j\omega})$ and $Y(e^{j\omega})$ the discrete-time Fourier transform of $x[n]$ and $y[n]$, respectively. The value of the integral

$$\frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})\,d\omega \tag{1}$$

is

Convolution theorem

Theorem

Convolution theorem states that convolution in one domain (e.g., time domain) equals point-wise multiplication in the other domain (e.g., frequency domain). Let f and g be two functions with convolution $f * g$. Let F be the Fourier transform operator. Then

$$F(f * g) = F(f) \cdot F(g) \tag{2}$$

$$F(f \cdot g) = F(f) * F(g) \tag{3}$$

DTFT and IDTFT

DTFT

$H(e^{j\omega})$ is called the DTFT of $x[n]$ where

$$H(e^{j\omega}) = H(z = e^{j\omega}) \tag{4}$$

$$= \sum_{-\infty}^{\infty} x(n) z^{-n} \tag{5}$$

IDTFT

IDTFT is defined as

$$z[n] = \frac{1}{2\pi} \int_0^{2\pi} Z(e^{j\omega}) e^{j\omega n} \, d\omega \tag{6}$$

If the DTFT of $y[n]$ is $Y(e^{j\omega})$ then using the time reversal property, the DTFT of $y[-n]$ is

$$Y\left(\frac{1}{e^{j\omega}}\right) = Y(e^{-j\omega}) \tag{7}$$

Let

$$z[n] = x[n] * y[-n] \tag{8}$$

Let $Z(e^{j\omega})$ be the DTFT of $z[n]$. Using Theorem 1 we get

$$Z(e^{j\omega}) = X(e^{j\omega})Y(e^{-j\omega}) \tag{9}$$

By applying IDTFT, and substituting $n = 0$ we get the required expression:

$$z[0] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})\,d\omega \tag{10}$$

The $\mathcal{Z}$ transform for a sequence of the form $a^n u[n]$ is given by

$$\sum_{n=-\infty}^{\infty} \left(az^{-1}\right)^n u[n] = \left((1 - az^{-1})^{-1}\right) \tag{11}$$

provided, $|z| < |a|$, when $n < 0$ or $|a| < |z|$, when $n > 0$ which is the region of convergence. From (11), $\mathcal{Z}$ transform of $x[n]$ and $y[n]$ in the Region of Convergence are

$$X(z) = \mathcal{Z}\{x[n]\} = \frac{1}{2}\left(\left(2z^{-1}\right)^2 \left(1 - \frac{z}{2}\right)^{-1}\right) \tag{12}$$

$$Y(z) = \mathcal{Z}\{y[n]\} = 4\left(2z\left(1 - \frac{z^{-1}}{2}\right)^{-1}\right) \tag{13}$$

Now,

$$Z(z) = X(z)Y\left(\frac{1}{z}\right) \tag{14}$$

$$= 2\left(\left(2z^{-1}\right)^{3}\left(1-\frac{z}{2}\right)^{-2}\right) \tag{15}$$

The DTFT of $x[n]$ converges for all values of ω since,

$$z = e^{j\omega} \tag{16}$$

$$a = 2 \tag{17}$$

$$|za| = \frac{1}{2} \tag{18}$$

The DTFT of $y[n]$ also converges for all values of ω since

$$z = e^{j\omega} \tag{19}$$

$$a = \frac{1}{2} \tag{20}$$

$$\left|az^{-1}\right| = \frac{1}{2} \tag{21}$$

$$\sum_{n=-\infty}^{\infty} \left(az^{-1}\right)^{n} u[-n] = \sum_{n=-\infty}^{\infty} \left(az^{-1}\right)^{-n} u[n] \tag{22}$$

$$= \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^{n} \tag{23}$$

$$= \left(\left(1 - \frac{z}{a}\right)^{-1}\right) \tag{24}$$

Differentiating the above expression

$$\sum_{n=-\infty}^{\infty} na^{-n}z^{n-1}u[n] = \frac{1}{a}\left(\left(1-\frac{z}{a}\right)^{-2}\right) \tag{25}$$

$$\implies \sum_{n=-\infty}^{\infty} na^{-n}z^{n-4}u[n] = \frac{1}{a}\left(z^{-3}\left(1-\frac{z}{a}\right)^{-2}\right) \tag{26}$$

$$\implies \sum_{n=-\infty}^{\infty} na^{-n+5}z^{n-4}u[n] = a^{4}\left(z^{-3}\left(1-\frac{z}{a}\right)^{-2}\right) \tag{27}$$

$$\implies \sum_{n=-\infty}^{\infty} (4-n)a^{n+1}z^{-n}u[4-n] = a^{4}\left(z^{-3}\left(1-\frac{z}{a}\right)^{-2}\right) \tag{28}$$

Comparing the above equation with (14) we get $a = 2$. From the definition of DTFT we have

$$Z(z = e^{j\omega}) = \sum_{n=-\infty}^{\infty} z[n]z^{-n} \tag{29}$$

Thus

$$z[n] = (4 - n)2^{n+1}u[4 - n] \tag{30}$$

$$\therefore \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})\, d\omega = z[0] = 8 \tag{31}$$

Graphs: z[n]

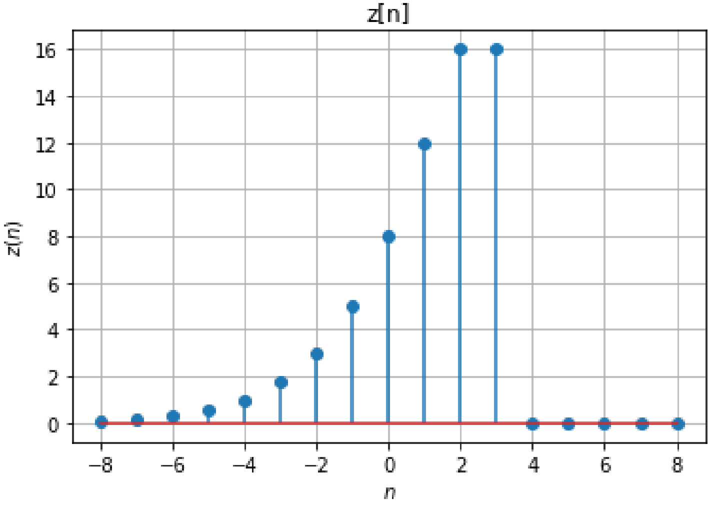

Figure: z[n]