GATE ASSIGNMENT 1

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Download all python codes from

```
https://github.com/AmulyaTallamraju/EE3900/blob/
    main/GATE_Assignment-1/codes/
    GATE_Assignment-1.py
```

and latex-tikz codes from

```
https://github.com/AmulyaTallamraju/EE3900/blob/
    main/GATE_Assignment-1/
    GATE_Assignment-1.tex
```

1 GATE EC 2021 Q.41

Consider the signals $x[n] = 2^{n-1}u[-n+2]$ and $y[n] = 2^{-n+2}u[n+1]$, where $u[n]$ is the unit step sequence. Let $X(e^{j\omega})$ and $Y(e^{j\omega})$ the discrete-time Fourier transform of $x[n]$ and $y[n]$, respectively. The value of the integral

$$\frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})\,d\omega \tag{1.0.1}$$

is

2 Solution

Theorem 2.1 (Convolution Theorem)**.** *Let f and g be two functions with convolution $f * g$. Let F be the Fourier transform operator. Then*

$$F(f * g) = F(f)\cdot F(g) \tag{2.0.1}$$

$$F(f\cdot g) = F(f) * F(g) \tag{2.0.2}$$

If the DTFT of $y[n]$ is $Y(e^{j\omega})$ then using the time reversal property, the DTFT of $y[-n]$ is

$$Y\left(\frac{1}{e^{j\omega}}\right) = Y(e^{-j\omega}) \tag{2.0.3}$$

Let

$$z[n] = x[n] * y[-n] \tag{2.0.4}$$

Let $Z(e^{j\omega})$ be the DTFT of $z[n]$. Using Theorem 2.1 we get

$$Z(e^{j\omega}) = X(e^{j\omega})Y(e^{-j\omega}) \tag{2.0.5}$$

By applying IDTFT, we can write:

$$z[n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})e^{j\omega n}\,d\omega \tag{2.0.6}$$

Putting $n = 0$, we get the required value which is

$$z[0] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})\,d\omega \tag{2.0.7}$$

The $\mathcal{Z}$ transform for a sequence of the form $a^{(n+b)}u[d+f]$ when $d = n$ is given by

$$a^b\sum_{n=-f}^{\infty}\left(\frac{a}{z}\right)^n = a^b\left(\frac{\left(\frac{z}{a}\right)^f}{1-\frac{a}{z}}\right) \tag{2.0.8}$$

provided, $|a| < |z|$, which is the region of convergence.When $d = -n$ is given by

$$a^b\sum_{n=-\infty}^{f}\left(\frac{a}{z}\right)^n = a^b\sum_{n=-f}^{\infty}\left(\frac{a}{z}\right)^{-n} \tag{2.0.9}$$

$$= a^b\sum_{n=-f}^{\infty}\left(\frac{z}{a}\right)^n \tag{2.0.10}$$

$$= a^b\left(\frac{\left(\frac{a}{z}\right)^f}{1-\frac{z}{a}}\right) \tag{2.0.11}$$

provided, $|z| < |a|$, which is the region of convergence. The DTFT of $x[n]$ converges for all values of ω since,

$$z = e^{j\omega} \tag{2.0.12}$$

$$a = 2 \tag{2.0.13}$$

$$|\frac{z}{a}| = \frac{1}{2} \tag{2.0.14}$$

From (2.0.9), DTFT of $x[n]$ is

$$X(z = e^{j\omega}) = \mathcal{Z}\{x[n]\} = \frac{1}{2}\left(\frac{\left(\frac{2}{z}\right)^2}{1-\frac{z}{2}}\right) \tag{2.0.15}$$

The DTFT of $y[n]$ also converges for all values of

ω since

$$z = e^{j\omega} \tag{2.0.16}$$

$$a = \frac{1}{2} \tag{2.0.17}$$

$$|\frac{a}{z}| = \frac{1}{2} \tag{2.0.18}$$

From (2.0.8) we have

$$Y(z = e^{j\omega}) = \mathcal{Z}\{y[n]\} = 4\left(\frac{2z}{1 - \frac{1}{2z}}\right) \tag{2.0.19}$$

Now,

$$Z(z = e^{j\omega}) = X(z)Y\left(\frac{1}{z}\right) \tag{2.0.20}$$

$$= 2\left(\frac{\left(\frac{2}{z}\right)^3}{\left(1 - \frac{z}{2}\right)^2}\right) \tag{2.0.21}$$

Since, $|z| \leq 1$, using the binomial expansion

$$Z(z = e^{j\omega}) = 2\left(\frac{2}{z}\right)^3 \left(1 + 2\left(\frac{z}{2}\right) + 3\left(\frac{z}{2}\right)^2 + 4\left(\frac{z}{2}\right)^3 \dots\right) \tag{2.0.22}$$

From the definition of Z transform we have

$$Z(z = e^{j\omega}) = \sum_{n=-\infty}^{\infty} z[n]z^{-n} \tag{2.0.23}$$

Comparing (2.0.22) and (2.0.23) we get

$$z[0] = 8 \tag{2.0.24}$$

$$\therefore \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})\, d\omega = 8 \tag{2.0.25}$$

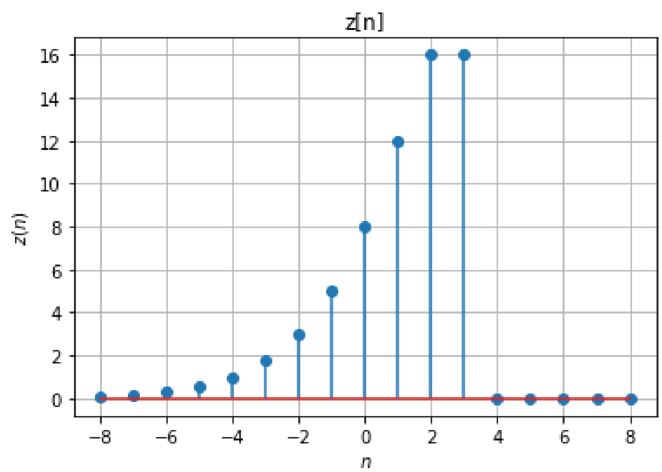

Fig. 0: Plot of z[n]