QUADRATIC FORMS 2.77

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Question

Find the area enclosed by the parabola $4y = x^2$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathrm{x} = -2$.

Lemma

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The points of intersection of Line $L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m}$ *with* parabola

$$\mathbf{x}^{\top} \mathbf{V} \mathbf{x} + 2\mathbf{u}^{\top} \mathbf{x} + f = 0 \tag{1}$$

are given by:

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{2}$$

where,

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} - \mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\right]^2 - \left(\mathbf{q}^T \mathbf{V}\mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f\right)\left(\mathbf{m}^T \mathbf{V} \mathbf{m}\right)} \tag{3}$$

Proof

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The points of intersection must satisfy the line and parabola equation. Thus,

$$(\mathrm{q}+\mu\mathrm{m})^{\mathrm{T}}\mathrm{V}(\mathrm{q}+\mu\mathrm{m})+2\mathrm{u}^{\mathrm{T}}(\mathrm{q}+\mu\mathrm{m})+f=0 \tag{4}$$

On expansion, we get

$$\mu^2\mathrm{m}^{\mathrm{T}}\mathrm{V}\mathrm{m}+\mu\left[\mathrm{m}^{\mathrm{T}}\mathrm{V}\mathrm{q}+\mathrm{q}^{\mathrm{T}}\mathrm{V}\mathrm{m}+2\mathrm{u}^{\mathrm{T}}\mathrm{m}\right]+\mathrm{q}^{\mathrm{T}}\mathrm{V}\mathrm{q}+2\mathrm{u}^{\mathrm{T}}\mathrm{m}+f=0 \tag{5}$$

$\square$

Proof cont.

Proof.

Since, $q^T Vm, 2u^T m$ are scalars

$$q^T Vm = m^T V^T q \tag{6}$$

$$2u^T m = 2m^T u \tag{7}$$

Solving the above quadratic equation we get

$$\mu_i = \frac{1}{m^T Vm} - m^T (Vq + u) \pm \sqrt{\left[m^T (Vq + u)\right]^2 - \left(q^T Vq + 2u^T q + f\right)\left(m^T Vm\right)} \tag{8}$$

□

The matrix parameters of the parabola are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{9}$$

with eigen parameters

$$\lambda_1 = 0, \lambda_2 = 1 \tag{10}$$

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11}$$

The vertex of the parabola can be expressed as

$$\begin{pmatrix} \mathbf{u}^\top + \eta \mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{12}$$

where,

$$\eta = \mathbf{u}^\top \mathbf{p}_1 = -2 \tag{13}$$

$$\implies \begin{pmatrix} 0 & -2 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{14}$$

or

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{15}$$

From (3),

$$\mu_1 = 4 + 2\sqrt{3}, \mu_2 = 4 - 2\sqrt{3} \tag{16}$$

The given line is

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2 \tag{17}$$

In parametric form, the given line can be written as:

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \tag{18}$$

$$\implies \mathbf{x} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{19}$$

Substituting μ_1 and μ_2 in (19),the points of intersection

$$\mathbf{K} = \begin{pmatrix} 2 + 2\sqrt{3} \\ 4 + 2\sqrt{3} \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 2 - 2\sqrt{3} \\ 4 - 2\sqrt{3} \end{pmatrix} \tag{20}$$

- Thus, from Fig. 1 the area enclosed by parabola and line can be given as

$$A = \text{Area under line} - \text{Area under parabola} \tag{21}$$
$$A = Ar(KLMNK) - Ar(KCLMCNK) \tag{22}$$
$$A = A_1 - A_2 \tag{23}$$

- Area under the line y=x+2 i.e, A_1-

$$A_1 = \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} ydx \tag{24}$$
$$= \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} (x+2)\, dx \tag{25}$$
$$= \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} xdx + \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} 2dx \tag{26}$$
$$= \frac{1}{2}\left((2+2\sqrt{3})^2 - (2-2\sqrt{3})^2\right) + 2\left(4\sqrt{3}\right) \tag{27}$$
$$= 16\sqrt{3} \text{ units} \tag{28}$$

- Area under the parabola that is A_2-

$$A_2 = \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} y dx \tag{29}$$

$$= \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} \frac{1}{4} x^2 dx \tag{30}$$

$$= \frac{1}{12} \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} x^3 dx \tag{31}$$

$$= \frac{1}{12} \left((2+2\sqrt{3})^3 - \left(2-2\sqrt{3}\right)^3 \right) \tag{32}$$

$$= 8\sqrt{3} \text{ units} \tag{33}$$

- Putting (28) and (33) in (23) we get required area A as:

$$A = A_1 - A_2 \tag{34}$$

$$A = 8\sqrt{3} \text{ units} \tag{35}$$

Parabola and line plot

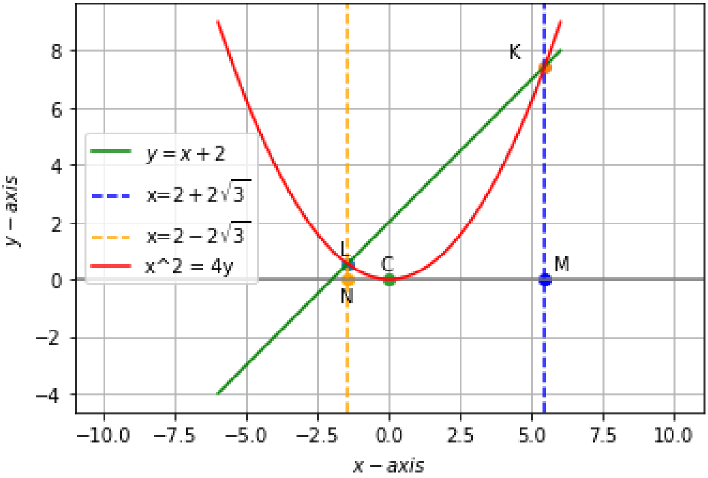

Figure: Plot of the Parabola and line