REPORT

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Theorem 1. ***Statement:***

Let G be a connected AT-free graph. If there exists a vertex x in G such that the BFS levels of x are $H_0, H_1, H_2, \ldots$, then there exists a minimum cardinality dominating set D and a minimum cardinality total dominating set T in G such that:

For all $i \in \{0, 1, \ldots, l\}$ and $j \in \{0, 1, \ldots, l-i\}$, the set of all vertices common to D and BFS levels from i to $i+j$ is less than or equal to $j+3$.

For all $i \in \{0, 1, \ldots, l\}$ and $j \in \{0, 1, \ldots, l-i\}$, the set of all vertices common to T and BFS levels from i to $i+j$ is less than or equal to $j+3$.

Proof. This proof makes use of property (3) which states that for a graph with a dominating shortest path and BFS levels, there exists a minimum cardinality dominating set and a minimum cardinality total dominating set T such that for all $i \in \{0, 1, \ldots, l\}$ and $j \in \{0, 1, \ldots, l-i\}$, the set of all vertices common to D and BFS levels from i to $i+j$ is less than or equal to $j+4$.

We begin by calculating a dominating pair (x, y), which can be done in AT-free graphs, and constructing a path P based on the BFS levels of x, where P is the path corresponding to the dominating pair. Since $V(P)$ is a dominating set, each vertex in H_i is adjacent to x_{i-1}, x_i, or x_{i+1}.

Let G be a connected AT-free graph, and let D_r be a minimum cardinality dominating set of G, where r is a positive integer. Suppose D_r violates property (3). Then, the number of vertices in its BFS levels from H_i to H_{i+j} will be greater than $j+4$.

We define a subpath A as the set of vertices $\{x_{i'_r-2}, x_{i'_r-1}, \ldots, x_{i'_r+j'_r+1}\}$. The neighborhood of A is the superset of BFS levels from $i'_r - 1$ to $i'_r + j'_r + 1$.

The replacement of D_r by D_{r+1} is called an exchange step. If D_{r+1} satisfies property (3), then G has a minimum cardinality dominating set with property (3).

This process continues until a Q_r is obtained that does not violate the theorem. Hence, starting with a minimum cardinality dominating set D_1 of G, we ultimately obtain a minimum cardinality dominating set D such that it satisfies the theorem. □

Algorithm

The following algorithm computes a minimum cardinality dominating set for a given connected graph G. If the input graph is an AT-free graph, then the algorithm computes a minimum cardinality dominating set of G.

Input: Connected AT-free graph G
Output: Dominating set D

1. Initialize $D = V$.
2. For each vertex x in V:
   (a) Compute the BFS levels of x as $H_0, H_1, H_2, \ldots$.
3. Assume:
   (a) Queue A_1 is initialized to contain a tuple $(S, S, \mathrm{val}(S))$ for all non-empty subsets S of the closed neighborhood of x, such that $\mathrm{val}(S) = |S| \leq w$.
4. While A_i is not empty and $i < l$:
   (a) Increment i.

(b) For each triple $(S, S', \mathrm{val}(S'))$ in queue A_{i-1}, where S' is the subsolution set which is a subset of the union of all H_j with $j \in \{0, \ldots, i-1\}$:
    i. For every subset U of H_i where $|S \cup U| \leq w$, do:
        A. If the neighborhood of $S \cup U$ is a superset of H_{i-1}:
        B. Assign R as the set containing vertices of S and U, then remove vertices of H_{i-2} from it.
        C. Assign R' as the set containing vertices of S' and U.
        D. Compute $\mathrm{val}(R')$ as $\mathrm{val}(S') +$ cardinality of U.
        E. If there is no triple with R as the first entry in the queue, insert $(R, R', \mathrm{val}(R'))$ into queue A_i.
        F. If there is a triple $(R, R', \mathrm{val}(R'))$ where $\mathrm{val}(R') <$ newly computed $\mathrm{val}(R')$, replace the triple with the new values $(R, R', \mathrm{val}(R'))$.

5. Among all triples $(S, (S', \mathrm{val}(S')))$ in queue A_l that satisfy the conditions, find the triple with minimum $\mathrm{val}(S')$. If $\mathrm{val}(S') < |D|$, then $D = S'$.

Output: The minimum cardinality dominating set D. This is proved by the following theorem.

Algorithm Analysis

Theorem: Running Time Analysis of BFS-levels

The running time of the algorithm to check the BFS-levels of a fixed vertex is $O(n^{w+1})$ since it includes the time taken to test all subsets of S and U contained in three consecutive BFS-levels of x. The time taken to test each subset is $O(n)$, and there are $O(n^w)$ subsets to be tested in total. Also, to avoid duplicates, the triples $(S, S', \mathrm{val}(S'))$ are stored simultaneously in the queue A_i and according to S in a w-dimensional array. For any such triple, S' represents the subsolution corresponding to S and $\mathrm{val}(S')$. However, only S and $\mathrm{val}(S')$ are used in dynamic programming. The main purpose of storing S' is to find a dominating set B' corresponding to $\mathrm{val}(B')$ that has at most w vertices across any three consecutive BFS-levels of a vertex x.

We claim that for any triplet in the queue A, S is defined as $S' \cup H_{i-1} \cup H_i$, $\mathrm{val}(S') = |S'|$, and the neighborhood of S' is the superset of the union of all H_j from $j = 0$ to $i-1$. This is true for $i = 1$. By initializing A_1 for all triplets in A_1, $S = S'$ and S is a subset of the neighborhood of X. Hence, the closed neighborhood of S, which is a superset of H_0, is equal to x.

Suppose the claim is true for $i-1$ from 1 to $i-1$. By the algorithm, the triple $(R, R', \mathrm{val}(R'))$ is in A_i only if there is a triple $(S, S', \mathrm{val}(S'))$ in A_{i-1} and a subset $U \subseteq H_i$ such that $|S \cup U| \leq w$ and the closed neighborhood of $S \cup U$ is a superset of H_{i-1}, where $R = S \cup U - H_{i-1}$, $R' = S' \cup U$, and $\mathrm{val}(R') = \mathrm{val}(S') + |U|$. Consequently, $R = R' \cap (H_{i-1} \cup H_i)$, $\mathrm{val}(R') = |R'|$, and the closed neighborhood of R' is a superset of the union of all H_j for $j = 0$ to $i-1$. This hence proves the claim.

Hence, for any triple $(S, S', \mathrm{val}(S'))$ in A_l, where the closed neighborhood of S is a superset of all H_l, S' is a dominating set of the graph G. Also, for any minimum cardinality dominating set D of G such that it has at most w vertices across any three consecutive BFS levels of x, there exists a triple $(D \cap (H_{l-1} \cup H_l), D', |D'|)$ in A_l such that the closed neighborhood of D is a superset of all H_l when the algorithm checks all BFS levels of x. Hence, the output will be a minimum cardinality dominating set.

Theorem on Weakly Connected Graphs

Assume u and v are two vertices in R, where R and L denote the right and left halves of D_r. Let u_1 be the last vertex in R before entering L when traveling leftward from R to L. Let y_1 be in $H_{i'}$ or $H_{i'+j'}$, while u_1 is in $H_{i'-1}$ or $H_{i'+j'+1}$.

Let w_2 be a vertex in A that is adjacent to u_1, and let w_1 be a vertex adjacent to y_2. Also, x is adjacent to y, where y is in layers $H_{i'+1}$ to $H_{i'+j'}$ while x is in layers $H_{i-1}, H_i, H_{i'+j'+1}$. A dominates all these layers, and there is an edge from A to x.

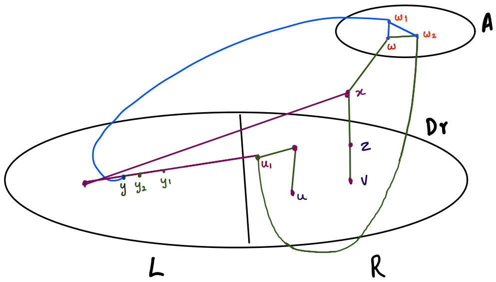

Figure 1: Representation of the figure as proposed