MAT-1005: Discrete mathematics

Assignment 1

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Exercise 1

i

a	$\neg a$	b	$\neg b$	$(a \wedge \neg a)$	$(b \vee \neg b)$	$(a \wedge \neg a) \rightarrow (b \vee \neg b)$
T	F	T	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	T	F	T	F	T	T

ii

a	$\neg a$	b	$\neg b$	$(a \wedge \neg a)$	$(b \vee \neg b)$	$(a \wedge \neg a) \leftrightarrow (b \vee \neg b)$
T	F	T	F	F	T	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	T	F	T	F	T	F

Exercise 2

i $(A \cap B) \cup C = A \cap (B \cup C)$

① $(A \cap B) \cup C \subseteq A \cap (B \cup C)$

② $(A \cap B) \cup C \supseteq A \cap (B \cup C)$

① assume $x \in C \cup (A \cap B)$

$x \in C$ or ($x \in A$ and $x \in B$)

$x \in C$ or $x \in A$ and $x \in C$ or $x \in B$

$x \in (C \cup A) \cap (C \cup B)$

② assume $x \in A \cap (B \cup C)$

$x \in A$ and ($x \in B$ or $x \in C$)

$x \in A$ and $x \in B$ or $x \in A$ and $x \in C$

$x \in (A \cap B) \cup (A \cap C)$

Therefor this is not true

Counter-example

$A = \{1\}$ $B = \{1, 2\}$ $C = \{1, 2, 3\}$

$A \cap B = \{1\}$

$(A \cap B) \cup C = \{1, 1, 2, 3\}$
$= \{1, 2, 3\}$

$B \cup C = \{1, 1, 2, 2, 3\}$
$= \{1, 2, 3\}$

$A \cap (B \cup C) = \{1\}$

Exercise 2

ii $(C \setminus A) \cap (C \setminus B) = C \setminus (A \cup B)$

1 $\subseteq$

2 $\supseteq$

1 assume: $x \in (C \setminus A) \cap (C \setminus B)$

So. $x \in C$ and $x \notin A$ and $x \in C$ and $x \notin B$

So. $x \in C$ and $x \notin A$ or $x \notin B$

So. $x \in C \setminus (A \cup B)$

2 assume: $x \in C \setminus (A \cup B)$

So. $x \in C$ and $x \notin A$ or $x \notin B$

So. $x \in C$ and $x \notin A$ and $x \in C$ and $x \notin B$

So. $x \in (C \setminus A) \cap (C \setminus B)$

Exercise 3

i $f: \mathbb{R} \to \mathbb{R}, \ x \mapsto x^2$

<u>not injective, not surjective</u>

<u>Proof: not injective</u>

$f(\frac{1}{2})^2 = \frac{1}{4}$

$f(-\frac{1}{2})^2 = \frac{1}{4}$

but $\frac{1}{2} \neq -\frac{1}{2}$

<u>Proof: not surjective</u>

There is no $x \in \mathbb{R}$

Such that $\Longrightarrow$

$f(x) = x^2 = -\frac{1}{2}$

ii $g: \mathbb{R} \to \mathbb{R}_{\geq 0}, \ x \longmapsto x^2$

<u>Surjective</u>

<u>Proof: Surjective</u>

$g(\frac{1}{2})^2 = \frac{1}{4}$

$g(-\frac{1}{2})^2 = \frac{1}{4}$

<u>Proof: not injective</u>

$g(\frac{1}{2})^2 = \frac{1}{4}$

$g(-\frac{1}{2})^2 = \frac{1}{4}$

but $-\frac{1}{2} \neq \frac{1}{2}$

Exercise 3

iii $h: \mathbb{Z} \to \mathbb{Z},\ x \mapsto x^3$

injective

Proof: injective

let $x_1, x_2 \in \mathbb{Z}$

$h(x_1) = h(x_2)$

$x_1^3 = x_2^3$

$x_1 = x_2$

Proof: not surjective

there is no $x \in \mathbb{Z}$

such that $\Rightarrow$

$h(x) = x^3 = 2$

iv $P: \mathbb{R} \to \mathbb{R},\ x \mapsto x^3$

Bijective

Proof: injective

let $x_1, x_2 \in \mathbb{R}$

$P(x_1) = P(x_2)$

$x_1^3 = x_2^3$

$x_1 \times x_2$

Proof: surjective

let $y \in \mathbb{R}$, define $x = \sqrt[3]{y}$

then $P(x) = P(\sqrt[3]{y})$

$= (\sqrt[3]{y})^3$

$= y$

Exercise 4

Let $n \in \mathbb{N}$

$$S_n : 0^2 + 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(1. Base)

$n = 1$

$$S_1 : 1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$$

$$1 = \frac{(2)(3)}{6}$$

$$1 = \frac{6}{6}$$

$$1 = 1$$

(2. Hypothesis) we assume that $n = k$ is true

$$S_k : 0^2 + 1^2 + \cdots + k^2 = \underbrace{\frac{k(k+1)(2k+1)}{6}}_{①}$$

(3. Inductive Step)

Show that $n = k$ is true for the next term $\Rightarrow k+1$

$$S_{k+1} : \underbrace{0^2 + 1^2 + \cdots + k^2}_{①\ \text{we replace}} + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\frac{k(2k^2+k+2k+1)}{6} + \frac{6(k+1)(k+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\frac{2k^3+k^2+2k^2+k}{6} + \frac{6(k^2+k+k+1)}{6} = \frac{(k+1)(2k^2+3k+4k+6)}{6}$$

$$\frac{2k^3+3k^2+k}{6} + \frac{6k^2+12k+6}{6} = \frac{2k^3+7k^2+6k+2k^2+7k+6}{6}$$

$$\frac{2k^3+9k^2+13k+6}{6} = \frac{2k^3+9k^2+13k+6}{6}$$

This proves that $0^2 + 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for any $n \in \mathbb{N}$

Exercise 4 ii

$n^3 + 5n$ is divisible by 6, let $n \in \mathbb{N}$

(1. Base) $G_n: \; n^3 + 5n = 6k$, let $k \in \mathbb{N}$

$G_1: \; 1^3 + 5 \cdot 1 = 6k$

$1 + 5 = 6k$

$6 = 6k$

$1 = k$

(2. hypothesis) We assume that $n = x$ is true

$G_x: \; \underbrace{x^3 + 5x}_{1} = 6q$ for some integer q

(3. Inductive Step)

Show that $n = x$ is true for the next term $\Rightarrow x + 1$

$G_{x+1}: \; (x+1)^3 + 5(x+1) = 6k$

$x^3 + 3x^2 + 3x + 1 + 5x + 5 = 6k$

$\underbrace{x^3 + 8x + 6}_{1} + 3x^2 = 6k$

1 we replace

$6(q+1) + 3x^2 + 3x = 6k$

We breaks this into 2 cases, odd and even

- case odd, $x = 2w + 1$

$6(q+1) + 3x^2 + 3x$

$6(q+1) + 3(2w+1)^2 + 3(2w+1)$

$6(q+1) + 3(4w^2 + 4w + 1) + 6w + 3$

$6(q+1) + 12w^2 + 12w + 3 + 6w + 3$

$6(q+1) + 6(2w^2 + 3w + 1)$

$6(q + 2w^2 + 3w + 2)$

- Case even, $x = 2r$

$6(q+1) + 3x^2 + 3x$

$6(q+1) + 3(2r)^2 + 3(2r)$

$6(q+1) + 3(4r^2) + 3(2r)$

$6(q+1) + 12r^2 + 6r$

$6(q+1) + 6(2r^2 + r)$

$6(q + 2r^2 + r + 1)$

This proves that $n^3 + 5n$ is divisible by 6, $n \in \mathbb{N}$