Boolean Algebra

Postulate 1 (Definition)

A Boolean Algebra is a closed algebraic system containing a set K of two or more elements and the two operators $(\cdot)$ and $(+)$

$+$ is called OR, $\cdot$ is called AND

Postulate 2 (Existance of 1 and 0 elements)

For every a in K, there exists unique elements one and zero

a) $a + 0 = a$ b) $a \cdot 1 = a$

Postulate 3 (Commutativity)

For every a and b in K

a) $a + b = b + a$ a) $a \cdot b = b \cdot a$

Postulate 4 (Associativity)

For every a, b, and c in K

a) $a + (b + c) = (a + b) + c$

b) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Postulate 5 (Distributivity)

For every a, b, and c in K

a) $a + (b \cdot c) = (a + b) \cdot (a + c)$

b) $a \cdot (b + c) = a \cdot b + a \cdot c$

Postulate 6 (Existence of a complement)

For every a in K there exists a unique element called $\bar{a}$ (complement of a)

a) $a + \bar{a} = 1$ b) $a \cdot \bar{a} = 0$

* Duality

If an expression is valid in Boolean Algebra, the dual of that expression is also valid

Dual is found by replacing + with •, • with +, 1 with 0, 0 with 1

Theorems of Boolean Algebra

Theorem 1 (Idempotency)

a) $a + a = a$ b) $a \cdot a = a$

Proof of part a:

$$
\begin{aligned}
a + a &= (a+a)\cdot 1 && \text{P2b} \\
&= (a+a)\cdot(a+\bar{a}) && \text{P6b} \\
&= a + a\cdot\bar{a} && \text{P5a} \\
&= a + 0 && \text{P6b} \\
&= a && \text{p2a}
\end{aligned}
$$

Proof of part b:

$$
\begin{aligned}
a \cdot a &= (a\cdot a) + 0 && \text{P2a} \\
&= (a\cdot a) + (a\cdot\bar{a}) && \text{P6b} \\
&= a\cdot(a+\bar{a}) && \text{P5b} \\
&= a\cdot 1 && \text{P6a} \\
&= a && \text{P2b}
\end{aligned}
$$

Theorem 2 (Null Element)

a) $a + 1 = 1$ b) $a \cdot 0 = 0$

Proof of part b:

$$
\begin{aligned}
a \cdot 0 &= a \cdot 0 + 0 && \text{P2a} \\
&= a \cdot 0 + a \cdot \bar{a} && \text{P6b} \\
&= a \cdot (0 + \bar{a}) && \text{P5b} \\
&= a \cdot \bar{a} && \text{P2a} \\
&= 0 && \text{P6b}
\end{aligned}
$$

Theorem 3 (Involution)

$$\bar{\bar{a}} = a$$

Properties of 0 and 1 elements

OR	AND	Complement
$a + 0 = a$	$a \cdot 0 = 0$	$\bar{0} = 1$
$a + 1 = 1$	$a \cdot 1 = a$	$\bar{1} = 0$

Theorem 4 (Absorption)

a) $a + a \cdot b = a$

b) $a \cdot (a+b) = a$

Proof of a

$$
\begin{aligned}
a + a \cdot b &= a \cdot 1 + a \cdot b && \text{P2b} \\
&= a \cdot (1+b) && \text{P5b} \\
&= a \cdot (b+1) && \text{P3a} \\
&= a \cdot 1 && \text{T2a} \\
&= a && \text{P2b}
\end{aligned}
$$

Example

$$(X+Y) + (X+Y) \cdot Z = X + Y$$

$$A \cdot \overline{B} \cdot (A \cdot \overline{B} + \overline{B} \cdot C) = A \cdot \overline{B}$$

Theorem 5

a) $a + \bar{a} \cdot b = a + b$

b) $= a \cdot (\bar{a} + b) = a \cdot b$

Proof of part a

$$
\begin{aligned}
a + \bar{a} \cdot b &= (a + \bar{a}) \cdot (a + b) && \text{P5a} \\
&= 1 \cdot (a + b) && \text{P6a} \\
&= (a + b) \cdot 1 && \text{P3b} \\
&= a + b && \text{P2b}
\end{aligned}
$$

Example

$B + A \cdot \bar{B} \cdot \bar{C} \cdot D = B + A \cdot \bar{C} \cdot D$

Theorem 6

a) $a \cdot b + a \cdot \overline{b} = a$

b) $(a+b) \cdot (a+\overline{b}) = a$

Proof of part a

$$
\begin{aligned}
a \cdot b + a \cdot \overline{b} &= a \cdot (b+\overline{b}) && \text{P5b} \\
&= a \cdot 1 && \text{P6a} \\
&= a && \text{P2b}
\end{aligned}
$$

Example

$$A \cdot B \cdot C + A \cdot \overline{B} \cdot C = A \cdot C$$

Theorem 7

a) $a \cdot b + a \cdot \bar{b} \cdot c = a \cdot b + a \cdot c$

b) $(a+b) \cdot (a+\bar{b}+c) = (a+b) \cdot (a+c)$

Proof of part a

$$a \cdot b + a \cdot \bar{b} \cdot c = a \cdot (b + \bar{b} \cdot c) \qquad \text{P5b}$$

$$= a \cdot (b + c) \qquad \text{T5a}$$

$$= a \cdot b + a \cdot c \qquad \text{P5b}$$

Theorem 8: DeMorgan's Law

a) $\overline{(a+b)} = \bar{a} \cdot \bar{b}$

b) $\overline{(a \cdot b)} = \bar{a} + \bar{b}$

Generalized DeMorgan's

a) $\overline{(a+b+\cdots+z)} = \bar{a} \cdot \bar{b} \cdot \cdots \cdot \bar{z}$

b) $\overline{(a \cdot b \cdot \cdots \cdot z)} = \bar{a} + \bar{b} + \cdots + \bar{z}$

Proof: Postulate 6 $x \cdot \bar{x} = 0$, $x + \bar{x} = 1$

Let $x = a+b$ $y = \bar{a} \cdot \bar{b}$

$$
\begin{aligned}
x \cdot y &= (a+b) \cdot (\bar{a} \cdot \bar{b}) \\
&= (\bar{a} \cdot \bar{b}) \cdot (a+b) && \text{P3b} \\
&= (\bar{a} \cdot \bar{b}) \cdot a + (\bar{a} \cdot \bar{b}) \cdot b && \text{P5b} \\
&= a \cdot (\bar{a} \cdot \bar{b}) + (\bar{a} \cdot \bar{b}) \cdot b && \text{P3b} \\
&= (a \cdot \bar{a}) \cdot \bar{b} + \bar{a} \cdot (\bar{b} \cdot b) && \text{P4b} \\
&= 0 \cdot \bar{b} + \bar{a} \cdot 0 && \text{P6b [P3b]} \\
&= 0 + 0 && \text{T2b} \\
&= 0 && \text{P2a}
\end{aligned}
$$

$$X + Y = (a+b) + (\bar{a} \cdot \bar{b})$$

$$= (b+a) + (\bar{a} \cdot \bar{b}) \quad \text{P3a}$$

$$= b + (a + \bar{a} \cdot \bar{b}) \quad \text{P4a}$$

$$= b + (a + \bar{b}) \quad \text{T5a}$$

$$= a + (\bar{b} + b) \quad \text{P3a, P4a}$$

$$= a + 1 \quad \text{P6a}$$

$$= 1 \quad \text{T2a}$$

Example:

$$\overline{(a \cdot (b + z \cdot (x+\bar{a})))}$$

$$= \bar{a} + \overline{(b+z \cdot (x+\bar{a}))} \qquad T8b$$

$$= \bar{a} + \bar{b} \cdot \overline{(z \cdot (x+\bar{a}))} \qquad T8a$$

$$= \bar{a} + \bar{b} \cdot (\bar{z} + \overline{(x+\bar{a})}) \qquad T8b$$

$$= \bar{a} + \bar{b} \cdot (\bar{z} + (\bar{x} \cdot \bar{\bar{a}})) \qquad T8a$$

$$= \bar{a} + \bar{b} \cdot (\bar{z} + (\bar{x} \cdot a)) \qquad T3$$

$$= \bar{a} + \bar{b} \cdot \bar{z} + \bar{b} \cdot \bar{x} \cdot a \qquad P5b$$

$$= \bar{a} + a \cdot \bar{b} \cdot \bar{x} + \bar{b} \cdot \bar{z} \qquad P3$$

$$= \bar{a} + \bar{b} \cdot \bar{x} + \bar{b} \cdot \bar{z} \qquad T5a$$

$$\overline{(a \cdot (b+c) + \bar{a} \cdot \bar{b})}$$

$$= \overline{(a \cdot (b+c))} \cdot \overline{(\bar{a} \cdot \bar{b})} \qquad \text{T8a}$$

$$= (\bar{a} + \overline{(b+c)}) \cdot (\bar{\bar{a}} + \bar{\bar{b}}) \qquad \text{T8b}$$

$$= (\bar{a} + (\bar{b} \cdot \bar{c})) \cdot (a+b) \qquad \text{T8a, T3}$$

$$= \bar{a} \cdot a + \bar{a} \cdot b + a \cdot \bar{b} \cdot \bar{c} + b \cdot \bar{b} \cdot \bar{c}$$

$$= 0 + \bar{a} \cdot b + a \cdot \bar{b} \cdot c + 0$$

$$= \bar{a} \cdot b + a \cdot \bar{b} \cdot \bar{c}$$

Theorem 9: Consensus

a) $a \cdot b + \bar{a} \cdot c + b \cdot c = a \cdot b + \bar{a} \cdot c$

b) $(a+b) \cdot (\bar{a}+c) \cdot (b+c) = (a+b) \cdot (\bar{a}+c)$

Proof of Part a

$a \cdot b + \bar{a} \cdot c + b \cdot c$

$= a \cdot b + \bar{a} \cdot c + 1 \cdot b \cdot c$ P2b

$= a \cdot b + \bar{a} \cdot c + (a+\bar{a}) \cdot b \cdot c$ P6a

$= a \cdot b + \bar{a} \cdot c + a \cdot b \cdot c + \bar{a} \cdot b \cdot c$ P5b

$= (a \cdot b + a \cdot b \cdot c) + (\bar{a} \cdot c + \bar{a} \cdot b \cdot c)$

$= a \cdot b + \bar{a} \cdot c$ T4a

Example:

$$A \cdot B + \bar{A} \cdot C \cdot D + B \cdot C \cdot D$$

$$= A \cdot B + \bar{A} \cdot C \cdot D \qquad T9a$$

$$A \cdot B \cdot C + \bar{A} \cdot D + \bar{B} \cdot D + C \cdot D$$

$$= A \cdot B \cdot C + (\bar{A} + \bar{B}) \cdot D + C \cdot D \qquad P5b$$

$$= A \cdot B \cdot C + \overline{(A \cdot B)} \cdot D + C \cdot D \qquad T8b$$

$$= A \cdot B \cdot C + \overline{(A \cdot B)} \cdot D \qquad T9a$$

$$= A \cdot B \cdot C + (\bar{A} + \bar{B}) \cdot D \qquad T8b$$

$$= A \cdot B \cdot C + \bar{A} \cdot D + \bar{B} \cdot D \qquad P5b$$

Switching Algebra:

Boolean Algebra with the set of elements

$$K = \{0, 1\}$$

For n variables $\rightarrow$ 2^{2^n} switching functions

2 variables: A B

A	B	f_0	f_1	f_2	f_3		f_{15}	
0	0	0	1	0	1		1	$\bar{A} \cdot \bar{B}$
0	1	0	0	1	1	...	1	$\bar{A} \cdot B$
1	0	0	0	0	0		1	$A \cdot \bar{B}$
1	1	0	0	0	0		1	$A \cdot B$

$$f_0(A,B) = 0$$

$$f_1(A,B) = \bar{A} \cdot \bar{B}$$

$$f_6(A,B) = \bar{A} \cdot B + A \cdot \bar{B}$$

$$f_{15}(A,B) = \bar{A} \cdot \bar{B} + \bar{A} \cdot B + A \cdot \bar{B} + A \cdot B$$

Truth Table

Tabular form for representing results from evaluating a switching function for all possible input values

$f(a) = \bar{a}$ NOT

a	$\bar{a}$
0	1
1	0

$f(a,b) = a+b$ OR

a	b	$a+b$
0	0	0
0	1	1
1	0	1
1	1	1

$f(a,b) = a \circ b$ AND

a	b	$a \circ b$
0	0	0
0	1	0
1	0	0
1	1	1

$$f(A,B,C) = A \cdot B \cdot \bar{C} + A \cdot B + B \cdot C$$

ABC	$A \cdot B$	$\bar{C}$	$A \cdot B \cdot \bar{C}$	$B \cdot C$	$A \cdot B \cdot \bar{C} + A \cdot B$	$f(A,B,C)$
000	0	1	0	0	0	0
001	0	0	0	0	0	0
010	0	1	0	0	0	0
011	0	0	0	1	0	1
100	0	1	0	0	0	0
101	0	0	0	0	0	0
110	1	1	1	0	1	1
111	1	0	0	1	1	1

ABC	$f(A,B,C)$
000	0
001	0
010	0
011	1
100	0
101	0
110	1
111	1

Literal: a variable, complemented or uncomplemented

Product Term: a literal or literals that are ANDed together

Sum Term: a literal or literals that are ORed together

Sum of Products (SOP)

ORing of product terms

$$f(A,B,C) = A \cdot B \cdot C + \bar{A} \cdot C + \bar{B} \cdot C$$

Product of Sums (POS)

ANDing of sum terms

$$f(A,B,C) = (\bar{A} + \bar{B} + \bar{C}) \cdot (A + \bar{C}) \cdot (B + \bar{C})$$

minterms are product terms in which all variables appear exactly once in either complemented or uncomplemented form

Canonical Sum of Products

expression is represented as a sum of minterms

uncomplemented variable : 1

complemented variable : 0

Minterms	Minterm Code	Minterm Number
$\bar{A} \cdot \bar{B} \cdot \bar{C}$	000	m_0
$\bar{A} \cdot \bar{B} \cdot C$	001	m_1
$\bar{A} \cdot B \cdot \bar{C}$	010	m_2
$\bar{A} \cdot B \cdot C$	011	m_3
$A \cdot \bar{B} \cdot \bar{C}$	100	m_4
$A \cdot \bar{B} \cdot C$	101	m_5
$A \cdot B \cdot \bar{C}$	110	m_6
$A \cdot B \cdot C$	111	m_7

SOP expression

$$f(A,B,C) = \bar{A}\cdot B\cdot \bar{C} + \bar{A}\cdot B\cdot C + A\cdot B\cdot \bar{C} + A\cdot B\cdot C$$

compact form of canonical SOP

$$f(A,B,C) = m_2 + m_3 + m_6 + m_7$$

another notation simplification

$$f(A,B,C) = \Sigma_m(2,3,6,7)$$

ABC	$f(A,B,C) = \Sigma_m(2,3,6,7)$		$f'(A,B,C) = \Sigma_m(0,1,4,5)$	
000	0		1	$\leftarrow m_0$
001	0		1	$\leftarrow m_1$
010	1	$\leftarrow m_2$	0	
011	1	$\leftarrow m_3$	0	
100	0		1	$\leftarrow m_4$
101	0		1	$\leftarrow m_5$
110	1	$\leftarrow m_6$	0	
111	1	$\leftarrow m_7$	0	

$$f(A,B,Q,Z) = \bar{A}\cdot\bar{B}\cdot\bar{Q}\cdot\bar{Z} + \bar{A}\cdot\bar{B}\cdot\bar{Q}\cdot Z + \bar{A}\cdot B\cdot Q\cdot\bar{Z} + \bar{A}\cdot B\cdot Q\cdot Z$$

Express $f(A,B,Q,Z)$ and $f'(A,B,Q,Z)$ in minterm form

$$f(A,B,Q,Z) = m_0 + m_1 + m_6 + m_7$$

$$= \Sigma_m(0,1,6,7)$$

$$f'(A,B,Q,Z) = m_2 + m_3 + m_4 + m_5 + m_8 + m_9 + m_{10} + m_{11} + m_{12} + m_{13} + m_{14} + m_{15}$$

$$= \Sigma_m(2,3,4,5,8,9,10,11,12,13,14,15)$$

Maxterms a sum terms in which all variables appear exactly once in either complemented or uncomplemented form

Canonical Product of Sums

expression is represented as a product of Maxterms

uncomplemented variables : 0

complemented variables : 1

Maxterms	Maxterm Code	Maxterm Number
$A+B+C$	0 0 0	M_0
$A+B+\bar{C}$	0 0 1	M_1
$A+\bar{B}+C$	0 1 0	M_2
$A+\bar{B}+\bar{C}$	0 1 1	M_3
$\bar{A}+B+C$	1 0 0	M_4
$\bar{A}+B+\bar{C}$	1 0 1	M_5
$\bar{A}+\bar{B}+C$	1 1 0	M_6
$\bar{A}+\bar{B}+\bar{C}$	1 1 1	M_7

$$f(A,B,C) = (A+B+C)\cdot(A+B+\bar{C})\cdot(\bar{A}+B+C)\cdot(\bar{A}+B+\bar{C})$$

$$= M_0 \cdot M_1 \cdot M_4 \cdot M_5$$

$$= \prod_M (0,1,4,5)$$

ABC	$A+B+C$ M_0	$A+B+\bar{C}$ M_1	$\bar{A}+B+C$ M_4	$\bar{A}+B+\bar{C}$ M_5	$f(A,B,C)$	
000	0	1	1	1	0	$\leftarrow M_0$
001	1	0	1	1	0	$\leftarrow M_1$
010	1	1	1	1	1	
011	1	1	1	1	1	
100	1	1	0	1	0	$\leftarrow M_4$
101	1	1	1	0	0	$\leftarrow M_5$
110	1	1	1	1	1	
111	1	1	1	1	1	

Since same truth table output as minterm example

$$\sum_m (2,3,6,7) = \prod_M (0,1,4,5)$$

$$\overline{m_i} = M_i$$

$$\overline{M_i} = m_i$$

Deriving Canonical Forms Using Switching Algebra

Theorem 10: Shannon's Expansion Theorem

a) $f(x_1, x_2, \ldots, x_n) = x_1 \cdot f(1, x_2, \ldots, x_n) + (\overline{x_1}) \cdot f(0, x_2, \ldots, x_n)$

b) $f(x_1, x_2, \ldots, x_n) = [x_1 + f(0, x_2, \ldots, x_n)] \cdot [\overline{x_1} + f(1, x_2, \ldots, x_n)]$

Example: $f(A,B,C) = A \cdot B + A \cdot \overline{C} + \overline{A} \cdot C$

$$= A \cdot (1 \cdot B + 1 \cdot \overline{C} + \overline{1} \cdot C) + \overline{A} \cdot (0 \cdot B + 0 \cdot \overline{C} + \overline{0} \cdot C)$$

$$= A \cdot (B + \overline{C} + 0) + \overline{A} \cdot (0 + 0 \cdot C)$$

$$= A \cdot B + A \cdot \overline{C} + \overline{A} \cdot C$$

$$= B \cdot (A \cdot 1 + A \cdot \overline{C} + \overline{A} \cdot C) + \overline{B}(A \cdot 0 + A \cdot \overline{C} + \overline{A} \cdot C)$$

$$= A \cdot B + A \cdot B \cdot \overline{C} + \overline{A} \cdot B \cdot C + A \cdot \overline{B} \cdot \overline{C} + \overline{A} \cdot \overline{B} \cdot C$$

$$= C \cdot (A \cdot B + A \cdot B \cdot \overline{1} + \overline{A} \cdot B \cdot 1 + A \cdot \overline{B} \cdot \overline{1} + \overline{A} \cdot \overline{B} \cdot 1)$$

$$+ \overline{C} \cdot (A \cdot B + A \cdot B \cdot \overline{0} + \overline{A} \cdot B \cdot 0 + A \cdot \overline{B} \cdot \overline{0} + \overline{A} \cdot \overline{B} \cdot 0)$$

$$= A \cdot B \cdot C + \overline{A} \cdot B \cdot C + \overline{A} \cdot \overline{B} \cdot C + A \cdot B \cdot \overline{C} + A \cdot B \cdot \overline{C} + A \cdot \overline{B} \cdot \overline{C}$$

$$= A \cdot B \cdot C + \overline{A} \cdot B \cdot C + \overline{A} \cdot \overline{B} \cdot C + A \cdot B \cdot \overline{C} + A \cdot \overline{B} \cdot \overline{C}$$

Alternatively Use Th 6

Th6: a) $a \cdot b + a \cdot \bar{b} = a$

$$f(A,B,C) = A \cdot B + A \cdot \bar{C} + \bar{A} \cdot C$$

$$A \cdot B = A \cdot B \cdot C + A \cdot B \cdot \bar{C}$$

$$A \cdot \bar{C} = A \cdot B \cdot \bar{C} + A \cdot \bar{B} \cdot \bar{C}$$

$$\bar{A} \cdot C = \bar{A} \cdot B \cdot C + \bar{A} \cdot \bar{B} \cdot C$$

$$f(A,B,C) = A \cdot B \cdot C + A \cdot B \cdot \bar{C} + A \cdot B \cdot \bar{C} + A \cdot \bar{B} \cdot \bar{C} + \bar{A} \cdot B \cdot C + \bar{A} \cdot \bar{B} \cdot C$$

$$f(A,B,C) = A \cdot B \cdot C + A \cdot B \cdot \bar{C} + A \cdot \bar{B} \cdot \bar{C} + \bar{A} \cdot B \cdot C + \bar{A} \cdot \bar{B} \cdot C$$

$$f(A,B,C) = m_7 + m_6 + m_4 + m_3 + m_1$$

$$= \Sigma_m(1,3,4,6,7)$$

Use Th 6 to find canonical POS

Th6: b) $(a+b)\cdot(a+\bar{b}) = a$

$$f(A,B,C) = A\cdot(A+\bar{C})$$

$$A = (A+B)\cdot(A+\bar{B}) = (A+B+C)\cdot(A+B+\bar{C})\cdot(A+\bar{B}+C)\cdot(A+\bar{B}+\bar{C}$$

$$A+\bar{C} = (A+B+\bar{C})\cdot(A+\bar{B}+\bar{C})$$

$$f(A,B,C) = (A+B+C)\cdot(A+B+\bar{C})\cdot(A+\bar{B}+C)\cdot(A+\bar{B}+\bar{C})$$
$$\cdot(A+B+\bar{C})\cdot(A+\bar{B}+\bar{C})$$

$$f(A,B,C) = (A+B+C)\cdot(A+B+\bar{C})\cdot(A+\bar{B}+C)\cdot(A+\bar{B}+\bar{C})$$

$$f(A,B,C) = M_0\cdot M_1\cdot M_2\cdot M_3$$
$$= \Pi_M(0,1,2,3)$$

Incompletely Specified Functions

→ some minterms (Maxterms) are omitted

↓

don't care minterms (Maxterms)

Why don't care?

↳ some input combinations can never occur

↳ only require the output to be a 1 or 0 for certain combinations

don't care minterm → d_i

don't care Maxterm → D_i

Example: $f(A,B,C)$ has minterms m_0, m_3, m_7 and don't cares d_4, d_5

minterm list: $f(A,B,C) = \Sigma_m(0,3,7) + d(4,5)$

Maxterm list: $f(A,B,C) = \Pi_M(1,2,6) \cdot D(4,5)$

$$f(A,B,C) = \overline{A}\cdot\overline{B}\cdot\overline{C} + \underbrace{\overline{A}\cdot B\cdot C + A\cdot B\cdot C}_{B\cdot C} + d(A\cdot\overline{B}\cdot\overline{C} + A\cdot\overline{B}\cdot C)$$

$$f(A,B,C) = \overline{A}\cdot\overline{B}\cdot\overline{C} + B\cdot C + d(A\cdot\overline{B}\cdot\overline{C} + A\cdot\overline{B}\cdot C)$$

$$\overline{B}\cdot\overline{C}$$

$$f(A,B,C) = \overline{B}\cdot\overline{C} + B\cdot C$$

Electronic Signals + Logic Values
(ES) (LV)

ES	LV	Negative Logic
High Voltage (H)	1	0
Low Voltage (L)	0	1

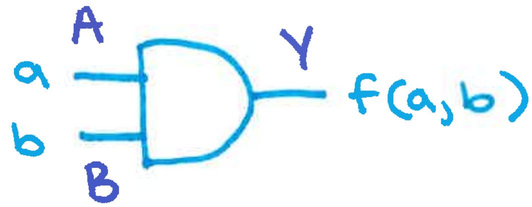
positive logic

a signal set to Logic 1 is asserted, active, true

an active-high signal is asserted when it is high (positive logic)

an active-low signal is asserted when it is low (negative logic)

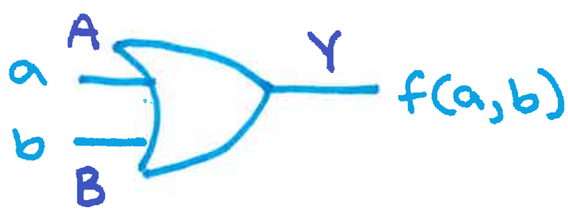
AND



a	b	f(a,b)
0	0	0
0	1	0
1	0	0
1	1	1

A	B	Y
L	L	L
L	H	L
H	L	L
H	H	H

OR



a	b	f(a,b)
0	0	0
0	1	1
1	0	1
1	1	1

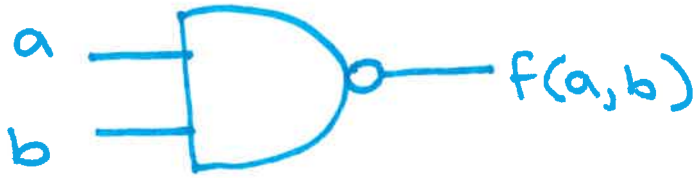
A	B	Y
L	L	L
L	H	H
H	L	H
H	H	H

NOT

a	f(a)
0	1
1	0

A	Y
L	H
H	L

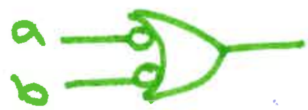
NAND



$$f(a,b) = \overline{a \cdot b} \Rightarrow \bar{a} + \bar{b}$$

alternative notation from book (convert to above)

a	b	f(a,b)
0	0	1
0	1	1
1	0	1
1	1	0

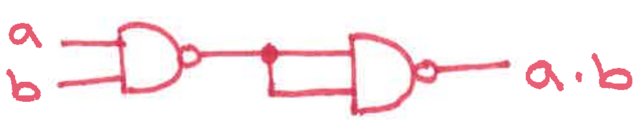


$$f(a,b) = \bar{a} + \bar{b} = \overline{a \cdot b}$$

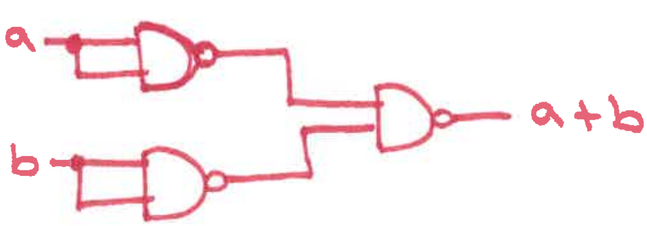
Basic Gates Using NAND Gates Only

$$\bar{a} = \bar{a} + \bar{a} = \overline{a \cdot a}$$

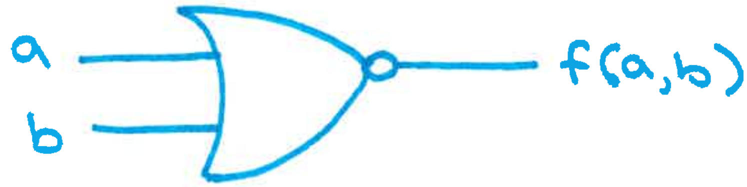
$$a \cdot b = \overline{\overline{a \cdot b}} = \overline{\overline{a \cdot b}} + \overline{\overline{a \cdot b}} = \overline{\overline{a \cdot b} \cdot \overline{a \cdot b}}$$



$$a + b = \bar{\bar{a}} + \bar{\bar{b}} = \overline{\overline{a \cdot a}} + \overline{\overline{b \cdot b}} = \overline{\overline{a \cdot a} \cdot \overline{b \cdot b}}$$



NOR



$$f(a,b) = \overline{a+b} \Rightarrow \bar{a} \cdot \bar{b}$$

alternative notation from book (convert to above)

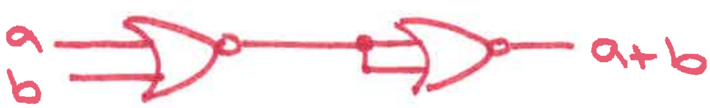
a	b	f(a,b)
0	0	1
0	1	0
1	0	0
1	1	0

$$f(a,b) = \bar{a} \cdot \bar{b} = \overline{a+b}$$

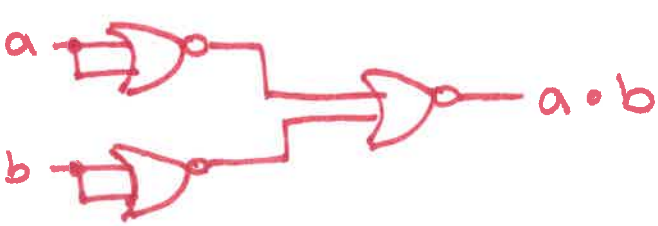
Basic Gates Using NOR Gates only

$$\bar{a} = \bar{a} \cdot \bar{a} = \overline{a+a}$$

$$a+b = \overline{\overline{a+b}} = \overline{\overline{a+b}} \cdot \overline{\overline{a+b}} = \overline{\overline{a+b} + \overline{a+b}}$$



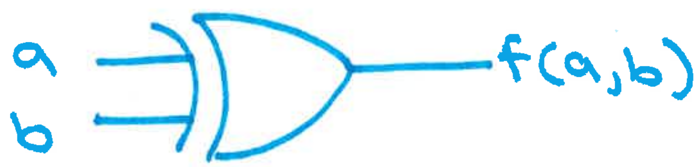
$$a \cdot b = \bar{\bar{a}} \cdot \bar{\bar{b}} = \overline{\overline{a+a}} \cdot \overline{\overline{b+b}} = \overline{\overline{a+a} + \overline{b+b}}$$



Exclusive - OR (XOR)

$$f_{XOR}(a,b) = a \oplus b = \bar{a} \cdot b + a \cdot \bar{b}$$

a	b	f(a,b)
0	0	0
0	1	1
1	0	1
1	1	0



XOR as POS

$$a \oplus b = \bar{a} \cdot b + a \cdot \bar{b}$$

$$= \bar{a} \cdot a + \bar{a} \cdot b + a \cdot \bar{b} + b \cdot \bar{b}$$

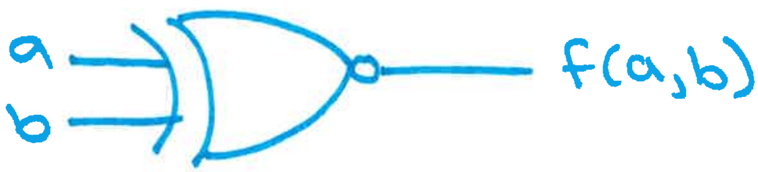
$$= \bar{a} \cdot (a+b) + \bar{b} \cdot (a+b)$$

$$= (\bar{a} + \bar{b}) \cdot (a+b)$$

Exclusive-NOR (XNOR)

$$f_{XNOR}(a,b) = \overline{a \oplus b} = a \odot b$$

a	b	f(a,b)
0	0	1
0	1	0
1	0	0
1	1	1



XNOR as SOP and POS

$$a \odot b = \overline{a \oplus b} = \overline{\bar{a} \cdot b + a \cdot \bar{b}} = \overline{\bar{a} \cdot b} \cdot \overline{a \cdot \bar{b}}$$

$$= (\bar{\bar{a}} + \bar{b}) \cdot (\bar{a} + \bar{\bar{b}}) = \underbrace{(a + \bar{b}) \cdot (\bar{a} + b)}_{\text{POS}}$$

$$= a \cdot \bar{a} + a \cdot b + \bar{a} \cdot \bar{b} + \bar{b} \cdot b$$

$$= \underbrace{a \cdot b + \bar{a} \cdot \bar{b}}_{\text{SOP}}$$

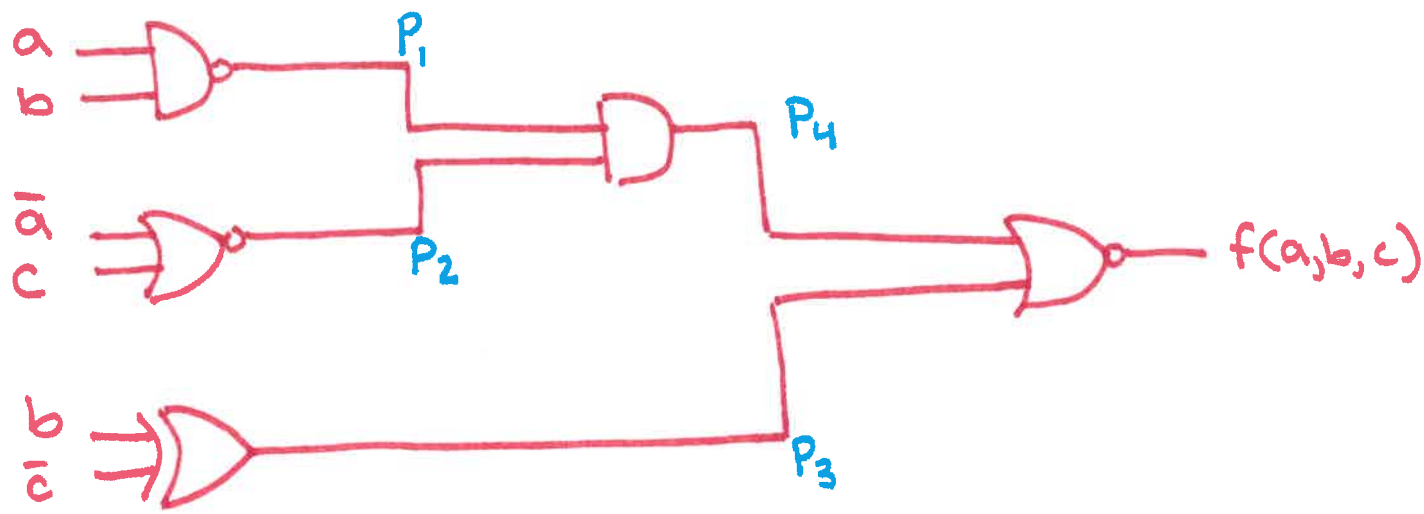
Digital Circuit Design

- Start from word description
- transform to switching expressions
- realize in hardware

Digital Circuit Analysis

- Start with hardware realization
- Find circuit description from various methods including switching expressions, truth tables, and timing diagrams

Digital Circuit Analysis – Using Switching Algebra



$$P_1 = \overline{a \cdot b}$$

$$P_2 = \overline{\bar{a} + c}$$

$$P_3 = b \oplus \bar{c}$$

$$P_4 = P_1 \cdot P_2 = \overline{a \cdot b} \cdot \overline{\bar{a} + c}$$

$$f(a,b,c) = \overline{P_3 + P_4}$$

$$= \overline{b \oplus \bar{c} + \overline{a \cdot b} \cdot \overline{\bar{a} + c}}$$

$$f'(a,b,c) = b \oplus \bar{c} + \overline{a \cdot b} \cdot \overline{\bar{a} + c}$$

$$f'(a,b,c) = b \oplus \bar{c} \quad + \quad \overline{a \cdot b} \cdot \overline{\bar{a} + c}$$

$$= b \cdot \bar{\bar{c}} + \bar{b} \cdot \bar{c} + (\bar{a} + \bar{b}) \cdot (\bar{\bar{a}} \cdot \bar{c})$$

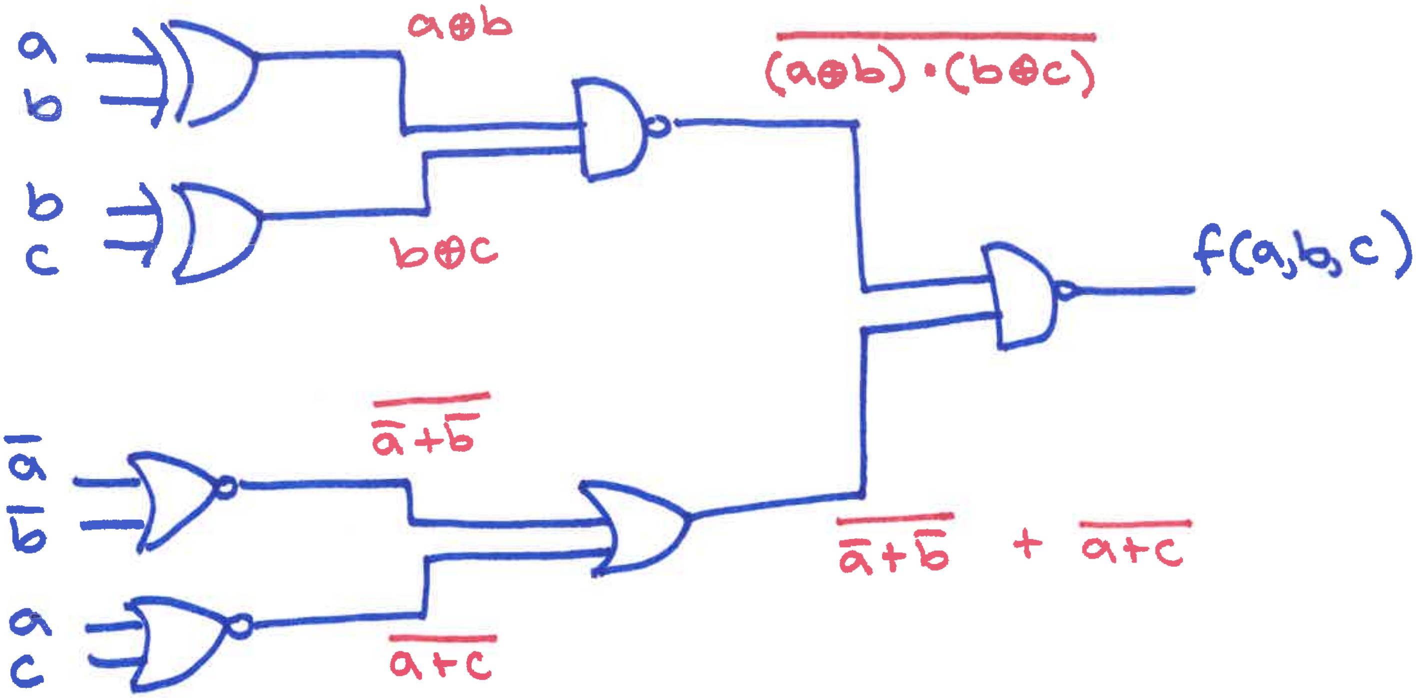
$$= b \cdot c + \bar{b} \cdot \bar{c} + (\bar{a} + \bar{b}) \cdot (a \cdot \bar{c})$$

$$= b \cdot c + \bar{b} \cdot \bar{c} + \bar{a} \cdot a \cdot \bar{c} + a \cdot \bar{b} \cdot \bar{c}$$

$$= b \cdot c + \bar{b} \cdot \bar{c} + a \cdot \bar{b} \cdot \bar{c}$$

$$= b \cdot c + \bar{b} \cdot \bar{c} = b \odot c$$

$$f(a,b,c) = \overline{(f'(a,b,c))} = \overline{b \odot c} = b \oplus c$$



$$f(a,b,c) = \overline{\overline{(a \oplus b) \cdot (b \oplus c)} \cdot \left(\overline{\bar{a} + \bar{b}} + \overline{a + c} \right)}$$

$$= \overline{\overline{(a \oplus b) \cdot (b \oplus c)}} + \overline{\left(\overline{\bar{a} + \bar{b}} + \overline{a + c} \right)}$$

$$= (a \oplus b) \cdot (b \oplus c) + \left(\overline{\overline{\bar{a} + \bar{b}}} \cdot \overline{\overline{a + c}} \right)$$

$$= (a \cdot \bar{b} + \bar{a} \cdot b) \cdot (b \cdot \bar{c} + \bar{b} \cdot c) + (\bar{a} + \bar{b}) \cdot (a + c)$$

$$= a \cdot \bar{b} \cdot b \cdot \bar{c} + a \cdot \bar{b} \cdot \bar{b} \cdot c + \bar{a} \cdot b \cdot b \cdot \bar{c} + \bar{a} \cdot b \cdot \bar{b} \cdot c$$
$$+ \bar{a} \cdot a + \bar{a} \cdot c + a \cdot \bar{b} + \bar{b} \cdot c$$

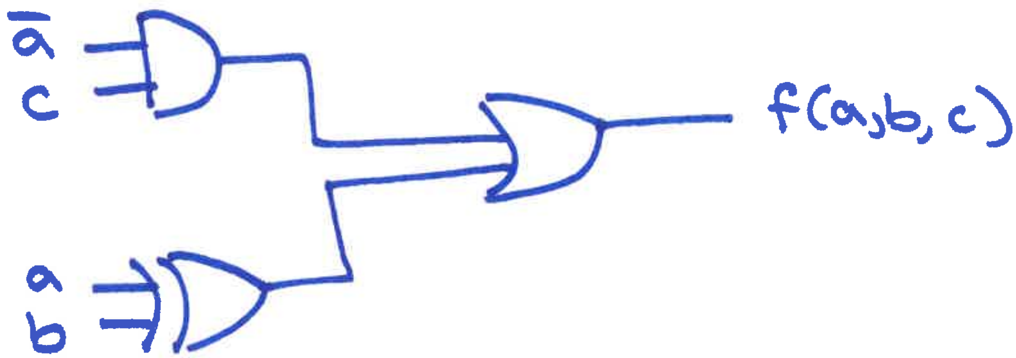
$$= \underline{a \cdot \bar{b} \cdot c} + \bar{a} \cdot b \cdot \bar{c} + \bar{a} \cdot c + a \cdot \bar{b} + \underline{\bar{b} \cdot c}$$

$$= \underline{\bar{a} \cdot b \cdot \bar{c}} + \underline{\bar{a} \cdot c} + a \cdot \bar{b} + \bar{b} \cdot c$$

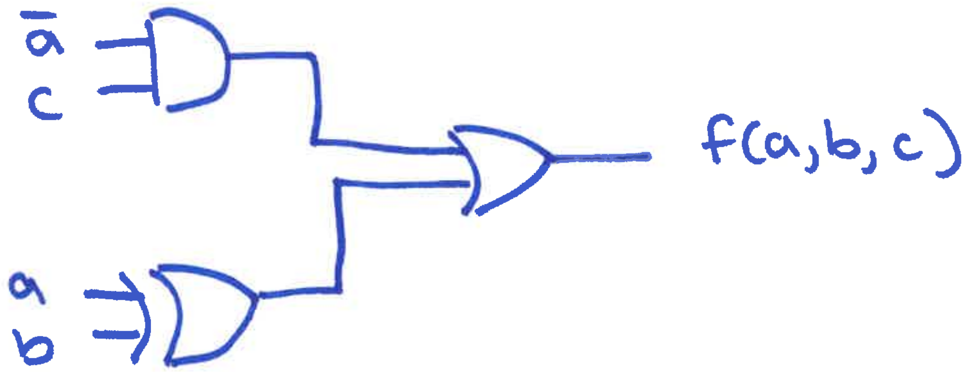
$$= \underline{\bar{a} \cdot c} + \bar{a} \cdot b + \underline{a \cdot \bar{b}} + \underline{\bar{b} \cdot c}$$

$$= \bar{a} \cdot c + \bar{a} \cdot b + a \cdot \bar{b} = \bar{a} \cdot c + a \oplus b$$

$f(a,b,c) = \bar{a} \cdot c + a \oplus b$



Truth Table Method

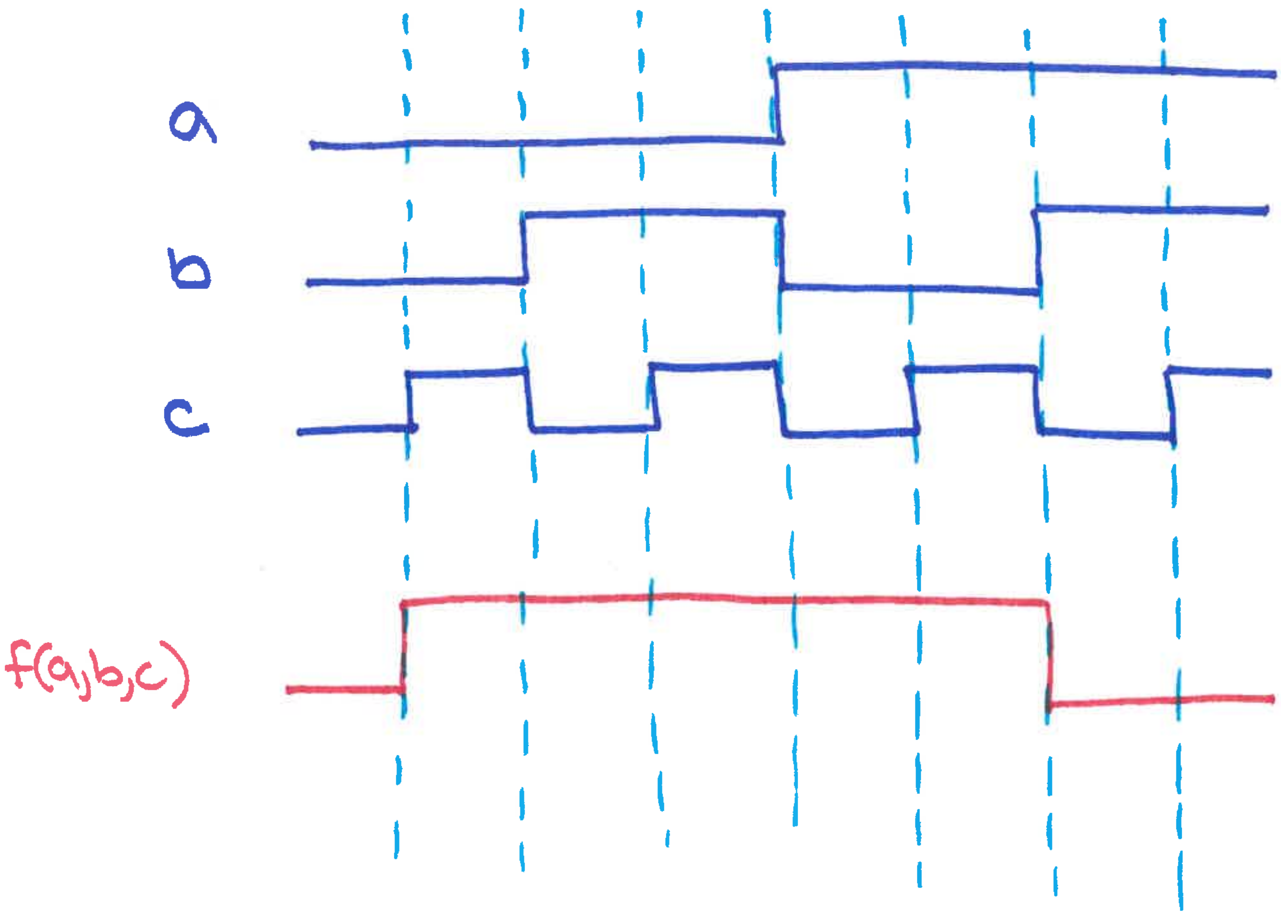


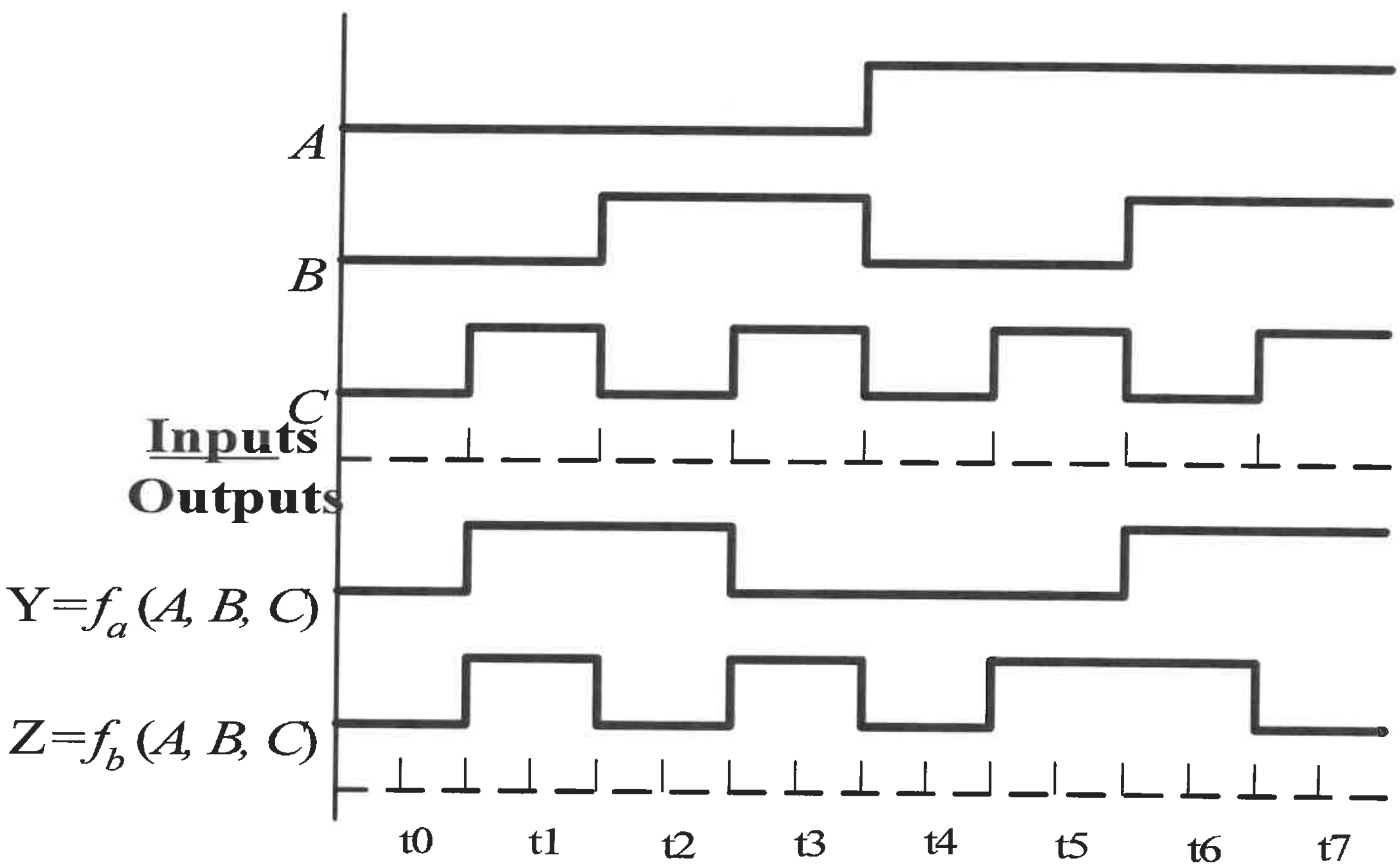
a	b	c	$\bar{a} \cdot c$	$a \oplus b$	f(a,b,c)
0	0	0	0	0	0
0	0	1	1	0	1
0	1	0	0	1	1
0	1	1	1	1	1
1	0	0	0	1	1
1	0	1	0	1	1
1	1	0	0	0	0
1	1	1	0	0	0

Analysis of Timing Diagrams

A timing diagram is a graphical representation of input and output signals and their relationships over time

It can also show intermediate signals and propagation delays


a
b
c
f(a,b,c)



A
B
C
Inputs
Outputs
Y=f_a(A, B, C)
Z=f_b(A, B, C)
t0
t1
t2
t3
t4
t5
t6
t7

A	B	C	$f_a(A,B,C)$	$f_b(A,B,C)$
0	0	0	0	0
0	0	1	1	1
0	1	0	1	0
0	1	1	0	1
1	0	0	0	0
1	0	1	0	1
1	1	0	1	1
1	1	1	1	0

Propagation Delay

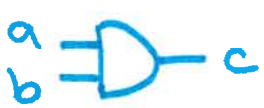
The delay between the time of an input change and the corresponding output change

t_{PLH} = propagation delay from low-to-high output

t_{PHL} = propagation delay from high-to-low output

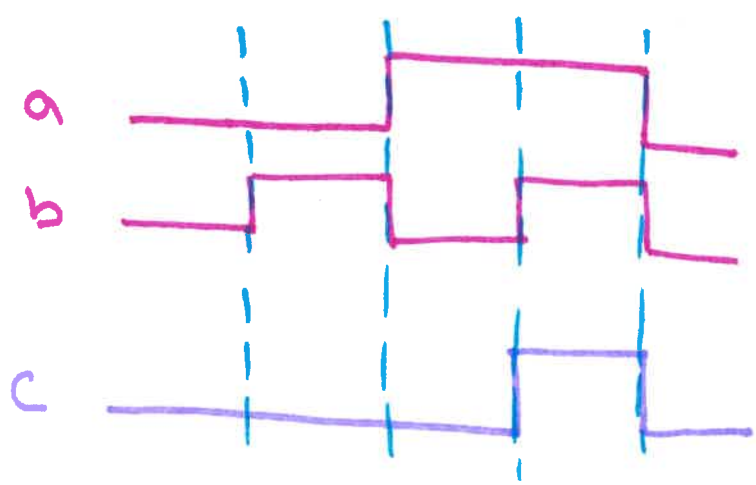
total propagation delay approximation

$$t_{PD} = \frac{t_{PLH} + t_{PHL}}{2}$$

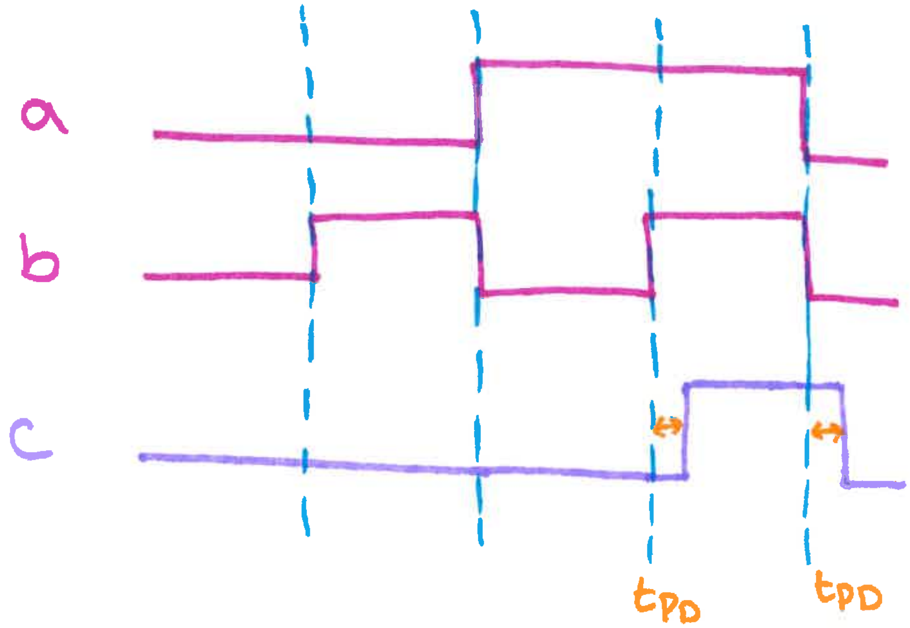


a	b	c
0	0	0
0	1	0
1	0	0
1	1	1

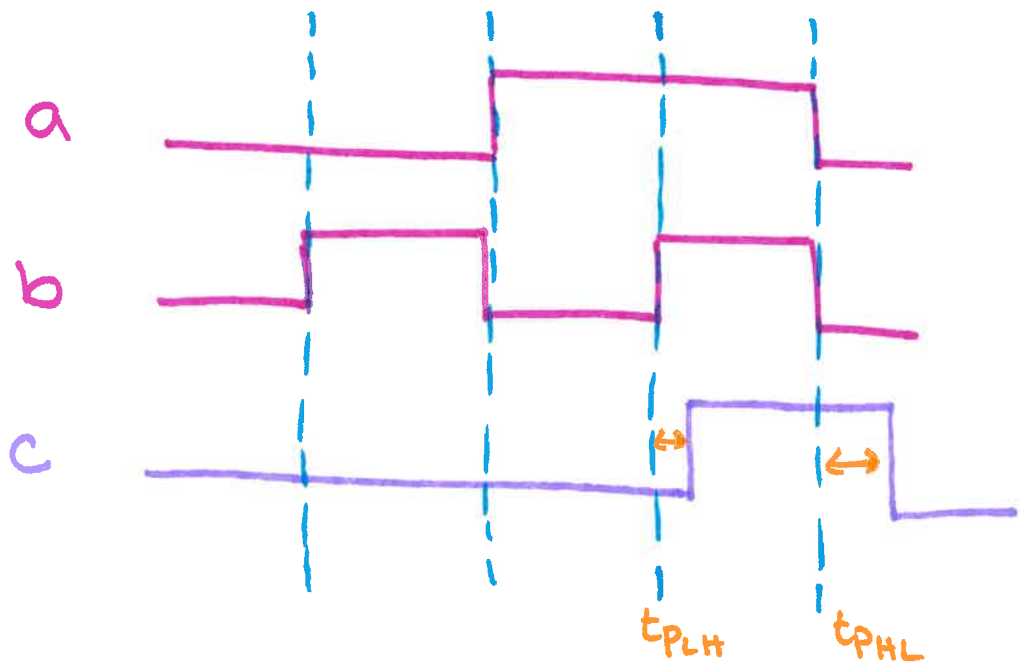
Ideal (no propagation delay)

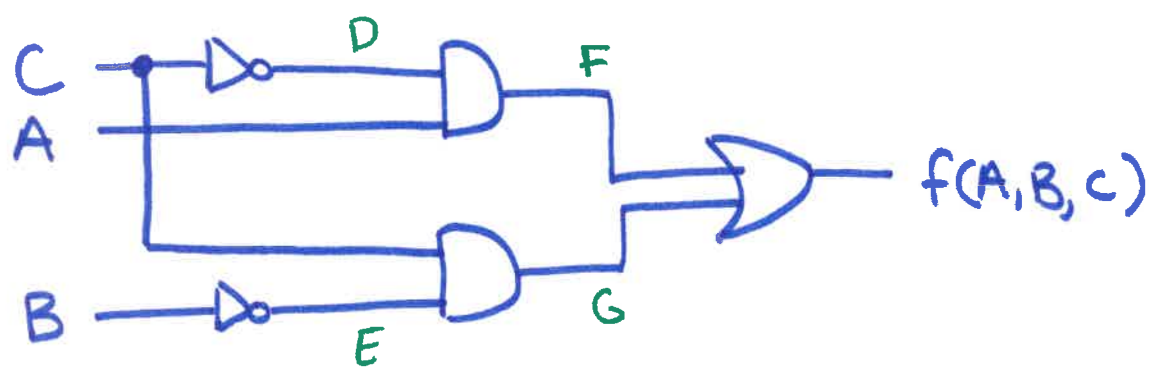


$t_{p_{LH}} = t_{p_{HL}} = t_{pD}$



$t_{p_{LH}} < t_{p_{HL}}$

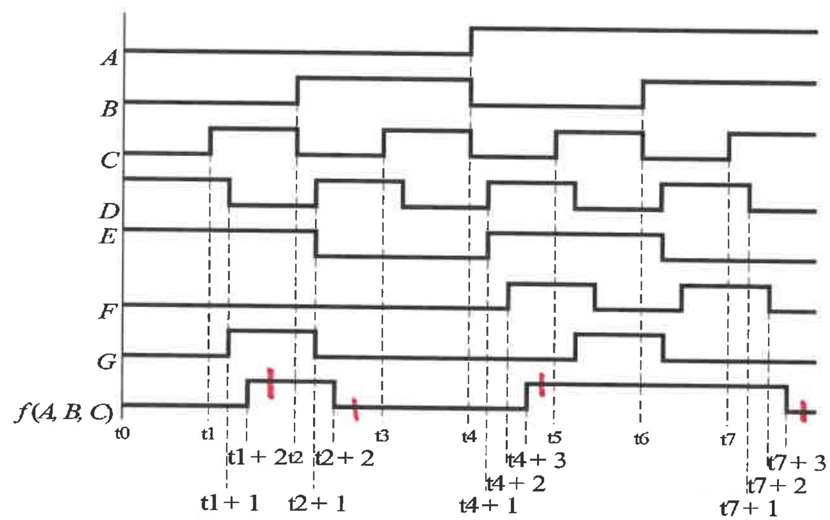




$D = \overline{C}$
$E = \overline{B}$

$F = A \cdot \overline{C}$
$G = \overline{B} \cdot C$

$f(A,B,C) = A \cdot \overline{C} + \overline{B} \cdot C$



ABC	f(A,B,C)
000	0
001	1
010	0
011	0
100	1
101	1
110	1
111	0

$f(A,B,C) = \Sigma_m(1,4,5,6)$

$= \overline{A} \cdot \overline{B} \cdot C + A \cdot \overline{B} \cdot \overline{C} + A \cdot \overline{B} \cdot C + A \cdot B \cdot \overline{C}$

$= \overline{B} \cdot C + A \cdot \overline{C}$

$f(A,B,C) = \Pi_M(0,2,3,7)$

$= (A+B+C) \cdot (A+\overline{B}+C) \cdot (A+\overline{B}+\overline{C}) \cdot (\overline{A}+\overline{B}$

$= (A+C) \cdot (\overline{B}+\overline{C})$

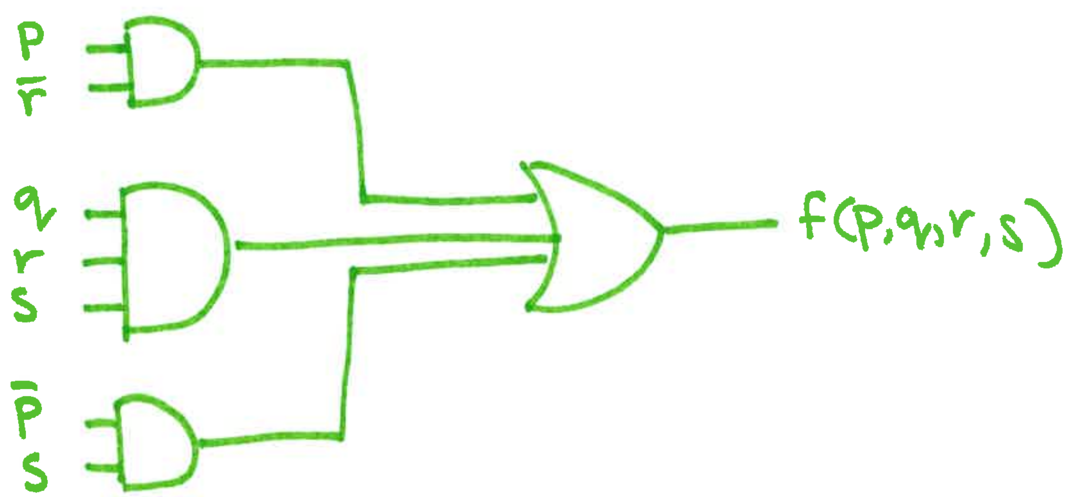
$= A \cdot \overline{B} + A \cdot \overline{C} + C \cdot \overline{B} + C \cdot \overline{C}$

$= A \cdot \overline{C} + \overline{B} \cdot C$

AND-OR/NAND Network

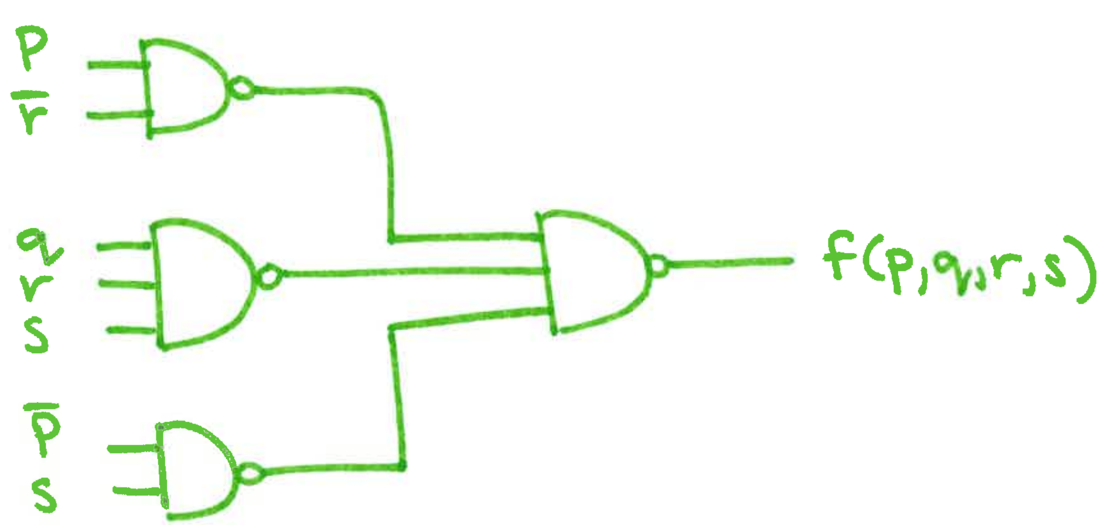
Switching expression in SOP form

$$f(p,q,r,s) = p\cdot\bar{r} + q\cdot r\cdot s + \bar{p}\cdot s$$



$$f(p,q,r,s) = \overline{\overline{p\cdot\bar{r} + q\cdot r\cdot s + \bar{p}\cdot s}}$$

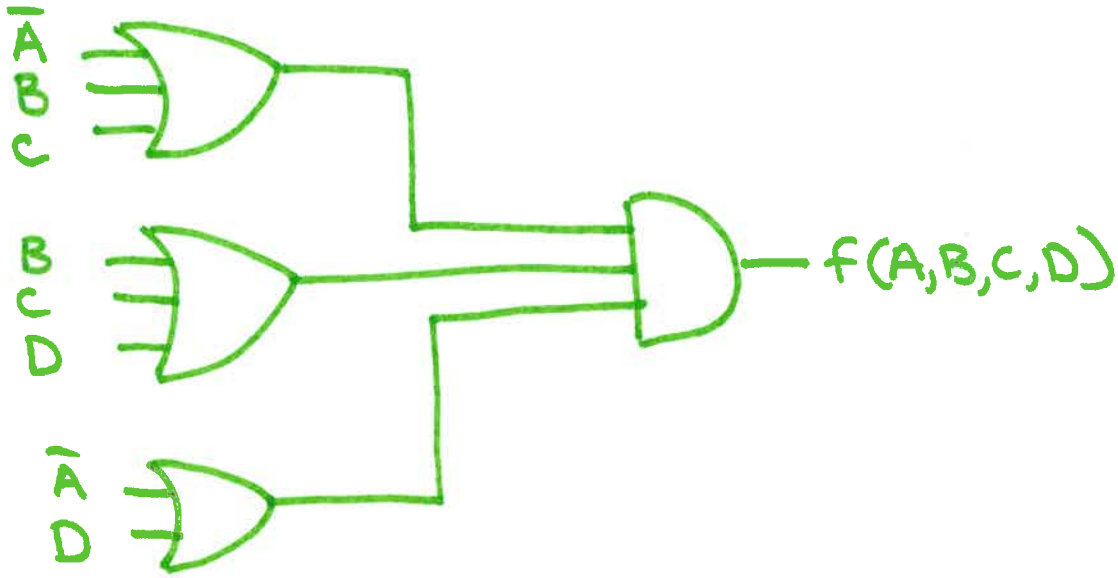
$$= \overline{\overline{p\cdot\bar{r}} \bullet \overline{q\cdot r\cdot s} \bullet \overline{\bar{p}\cdot s}}$$



OR-AND / NOR Network

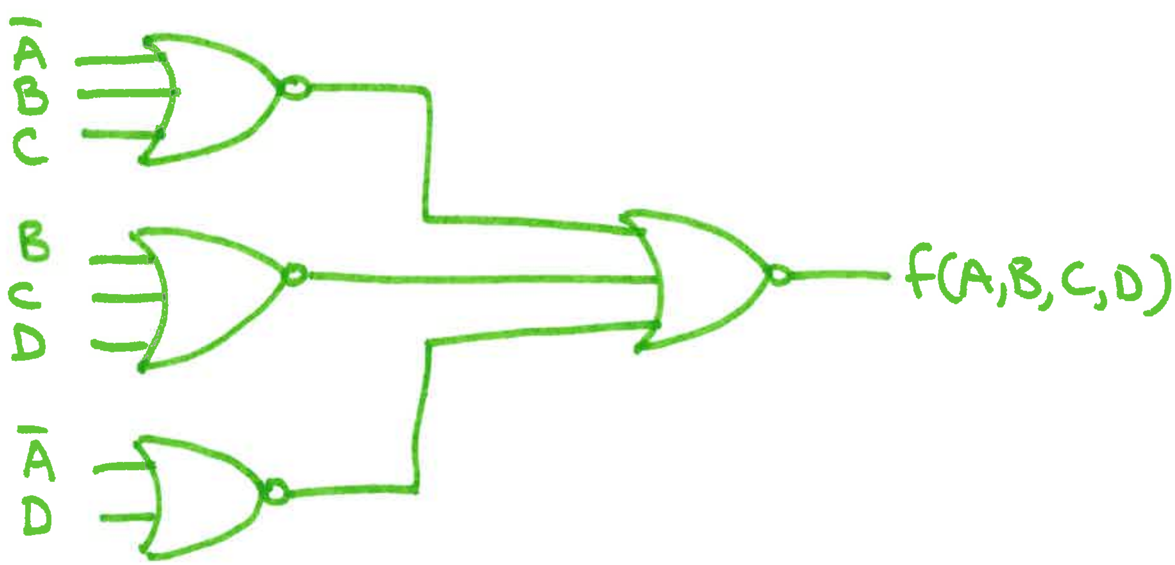
Switching expression in POS form

$$f(A,B,C,D) = (\bar{A}+B+C)\cdot(B+C+D)\cdot(\bar{A}+D)$$



$$f(A,B,C,D) = \overline{\overline{(\bar{A}+B+C)\cdot(B+C+D)\cdot(\bar{A}+D)}}$$

$$= \overline{\overline{(\bar{A}+B+C)} + \overline{(B+C+D)} + \overline{(\bar{A}+D)}}$$



Procedure for Implementing NAND/NOR Logic

1. Express the function in minterm/Maxterm form
2. Write out minterms/Maxterms in algebraic form
3. Simplify the function to SOP/POS form
4. Transform to NAND/NOR form
5. Draw the NAND/NOR logic diagram

$f(X,Y,Z) = \sum m(0,3,4,5,7)$

Find NAND implementation

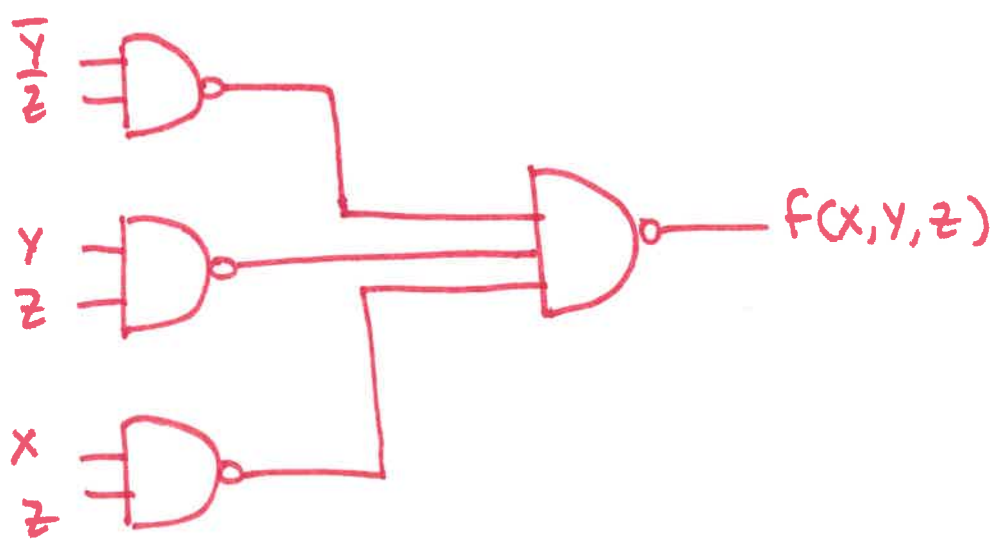
$$f(X,Y,Z) = \bar{X}\cdot\bar{Y}\cdot\bar{Z} + \bar{X}\cdot Y\cdot Z + X\cdot\bar{Y}\cdot\bar{Z} + X\cdot\bar{Y}\cdot Z + X\cdot Y\cdot Z$$

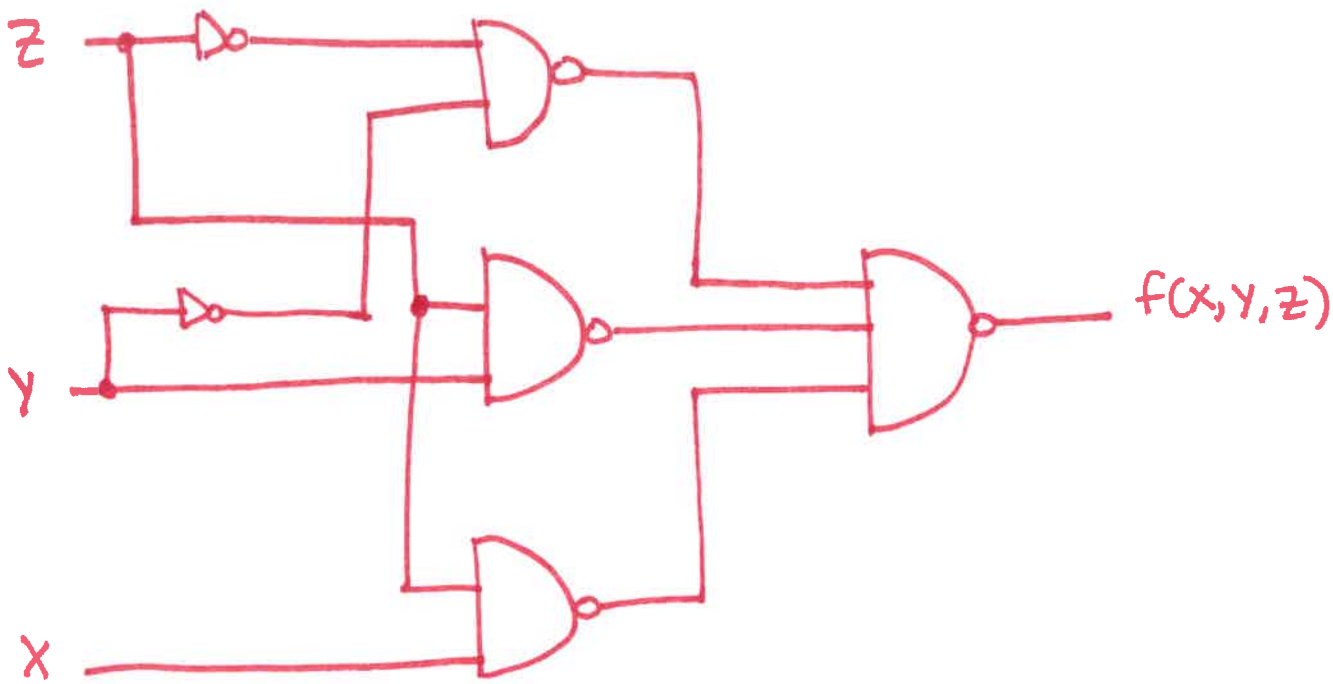
$$= \underbrace{\bar{X}\cdot\bar{Y}\cdot\bar{Z} + X\cdot\bar{Y}\cdot\bar{Z}}_{T6} + \underbrace{\bar{X}\cdot Y\cdot Z + X\cdot Y\cdot Z}_{T6} + \underbrace{X\cdot\bar{Y}\cdot Z + X\cdot Y\cdot Z}_{T6}$$

$$= \bar{Y}\cdot\bar{Z} + Y\cdot Z + X\cdot Z$$

$$f(X,Y,Z) = \overline{\overline{\bar{Y}\cdot\bar{Z} + Y\cdot Z + X\cdot Z}}$$

$$= \overline{\overline{\bar{Y}\cdot\bar{Z}} \bullet \overline{Y\cdot Z} \bullet \overline{X\cdot Z}}$$




z
y
x
f(x, y, z)

Sometimes more levels needed due to fan-in/ input constraints

If only allowed to use 2-input AND gates