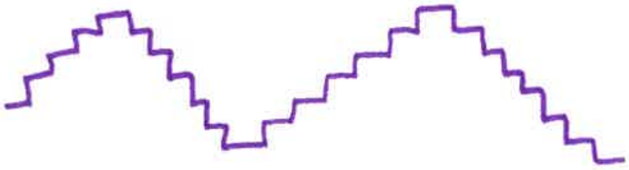
Analog

Quantized



Digital

1 0 0 0
0 0 0 0
1 1 0 1
1 1 1 0

Positional Notation

$$N = (a_{n-1}\, a_{n-2} \cdots a_1\, a_0 \bullet a_{-1}\, a_{-2} \cdots a_{-m})_r$$

$\bullet$: radix point

r : radix or base

n : number of integer digits to the left of the radix point

m : number of fractional digits to the right of the radix point

a_{n-1} : most significant digit (MSD)

a_{-m} : least significant digit (LSD)

Polynomial Notation

$$N = a_{n-1} \cdot r^{n-1} + a_{n-2} \cdot r^{n-2} + \cdots + a_0 \cdot r^0$$

$$+ a_{-1} \cdot r^{-1} + \cdots + a_{-m} \cdot r^{-m}$$

$$N = \sum_{i=m}^{n-1} a_i \cdot r^i$$

$N = (251.41)_{10}$ ← positional notation

$$N = 2 \times 10^2 + 5 \times 10^1 + 1 \times 10^0 + 4 \times 10^{-1} + 1 \times 10^{-2}$$

↑

polynomial notation

Binary

digits $= \{0, 1\}$

$(11010.11)_2$

$= 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-2}$

Octal

digits $= \{0, 1, 2, 3, 4, 5, 6, 7\}$

$(127.4)_8$

$= 1 \times 8^2 + 2 \times 8^1 + 7 \times 8^0 + 4 \times 8^{-1}$

Hexadecimal

digits $= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$

$(B65F)_{16}$

$= B \times 16^3 + 6 \times 16^2 + 5 \times 16^1 + F \times 16^0$

Decimal

digits $= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Binary Arithmetic

Addition

$$
\begin{array}{cccccccc}
 & 1 & 1 & 1 & & 1 & 1 & \\
 & & 1 & 0 & 1 & 0 & 1 & 1 \\
+ & & & 1 & 1 & 0 & 1 & 1 \\
\hline
 & 1 & 0 & 0 & 0 & 1 & 1 & 0
\end{array}
$$

Subtraction

$$
\begin{array}{ccccccc}
 & \overset{0}{\cancel{1}} & \overset{1}{\cancel{10}} & 10 & \overset{0}{\cancel{1}} & 10 & 1 \\
- & & 1 & 1 & 0 & 1 & 1 \\
\hline
 & & & 1 & 0 & 1 & 0
\end{array}
$$

Multiplication

$$
\begin{array}{cccccccccc}
 & & & & & 1 & 1 & 0 & 1 & 0 \\
\times & & & & & & 1 & 0 & 1 & 0 \\
\hline
 & & & & {}^{1} & 0 & 0 & 0 & 0 & 0 \\
 & & & {}^{1} & 1 & 1 & 0 & 1 & 0 & \\
 & {}^{1} & {}^{1} & 0 & 0 & 0 & 0 & 0 & ● & \\
+ & & 1 & 1 & 0 & 1 & 0 & & & \\
\hline
 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}
$$

Division

$$
\begin{array}{r}
110 \\
1001\overline{)\,111101} \\
-1001\downarrow \\
\hline
1\overset{0}{\cancel{1}}\overset{1}{\cancel{0}}{}^{1}0 \\
-1001\downarrow \\
\hline
111 \\
-\quad 0 \\
\hline
111
\end{array}
$$

110 R111

Octal Arithmetic

Addition

$$\begin{array}{rccccc}
 & {}^{1} & {}^{1} & {}^{1} & & \\
 & & 5 & 4 & 7 & 1 \\
+ & & 3 & 7 & 5 & 4 \\
\hline
 & 1 & 1 & 4 & 4 & 5
\end{array}$$

Subtraction

$$\begin{array}{rcccc}
 & \overset{6}{\cancel{7}} & 14 & \overset{4}{\cancel{5}} & 11 \\
- & 5 & 6 & 4 & 3 \\
\hline
 & 1 & 6 & 0 & 6
\end{array}$$

Hexadecimal Arithmetic

Addition

$$\begin{array}{rccccc}
 & {}^{1} & & {}^{1} & {}^{1} & \\
 & & 5 & B & A & 9 \\
+ & & D & 0 & 5 & 8 \\
\hline
 & 1 & 2 & C & 0 & 1
\end{array}$$

Subtraction

$$\begin{array}{rcccc}
 & \overset{9}{\cancel{A}} & 15 & \overset{A}{\cancel{B}} & 19 \\
- & 5 & 8 & 0 & D \\
\hline
 & 4 & D & A & C
\end{array}$$

Base Conversion (Base A to Base B)

Series Substitution Method

$$N = a_{n-1} r^{n-1} + \cdots + a_0 r^0 + a_{-1} r^{-1} + \cdots + a_{-m} r^{-m}$$

- Write Base A number in polynomial form
- Evaluate the polynomial series expression using Base B arithmetic

$(11010)_2 \rightarrow (?)_{10}$

$$N = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$$

$$N = 16 + 8 + 0 + 2 + 0 = (26)_{10}$$

$(627)_8 \rightarrow (?)_{10}$

$$N = 6 \times 8^2 + 2 \times 8^1 + 7 \times 8^0$$

$$N = 6 \times 64 + 2 \times 8 + 7 \times 1 = 384 + 16 + 7$$

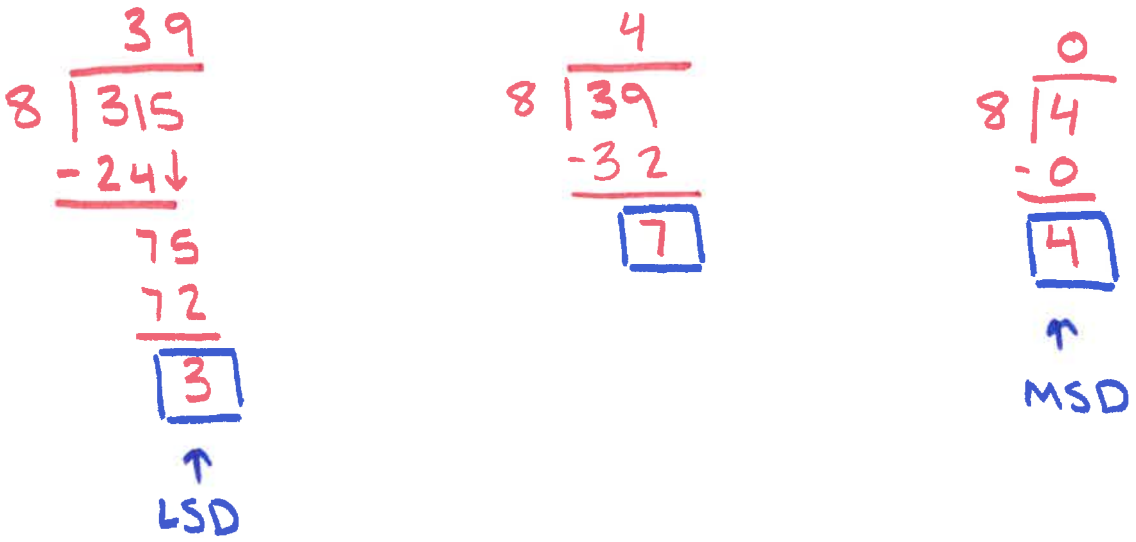
$$= (407)_{10}$$

Radix Divide Method

- Converting Base A to Base B integers

- Divide Base A number by Base B radix using Base A arithmetic
- Remainder becomes Base B LSD
- Repeat dividing quotient by Base B radix

$(315)_{10} \rightarrow (?)_8$



$(473)_8$

$(315)_{10} \rightarrow (?)_{16}$

19
$16 \,\overline{)\,315}$
$-16\downarrow$
155
-144
11
$\downarrow$
$\boxed{B}$
LSD

1
$16 \,\overline{)\,19}$
16
$\boxed{3}$

0
$16 \,\overline{)\,1}$
0
$\boxed{1}$
MSD

$(13B)_{16}$

Radix Multiply Method

- Converting Base A to Base B fractional digits

- Multiply Base A fractional number by Base B radix using Base A arithmetic

- Digit to left of radix point becomes Base B MSD

- Repeat with new fractional number

$(0.479)_{10} \rightarrow (?)_8$

$$\begin{array}{r} {}^{3\;6\;7} \\ 0.479 \\ \times \qquad 8 \\ \hline 3.832 \end{array} \qquad \begin{array}{r} {}^{6\;2\;1} \\ 0.832 \\ \times \qquad 8 \\ \hline 6.656 \end{array} \qquad \begin{array}{r} {}^{5\;4\;4} \\ 0.656 \\ \times \qquad 8 \\ \hline 5.248 \end{array} \qquad \begin{array}{r} {}^{1\;3\;6} \\ 0.248 \\ \times \qquad 8 \\ \hline 1.984 \end{array}$$

≡ ≡ ≡ ≡

MSD LSD

$$(0.3651)_8$$

$(0.479)_{10} \rightarrow (?)_2$

$$\begin{array}{r} 0.479 \\ \times \qquad 2 \\ \hline 0.958 \end{array} \qquad \begin{array}{r} 0.958 \\ \times \qquad 2 \\ \hline 1.916 \end{array} \qquad \begin{array}{r} 0.916 \\ \times \qquad 2 \\ \hline 1.832 \end{array} \qquad \begin{array}{r} 0.832 \\ \times \qquad 2 \\ \hline 1.664 \end{array}$$

≡ ≡ ≡ ≡

MSD LSD

$$(0.0111)_2$$

General Conversion Algorithm

To convert a number N from Base A to Base B

a) series substitution method with Base B arithmetic

b) radix multiply or divide method with Base A arithmetic

Alternate Conversion Algorithm

1) Use series substitution methot to convert from Base A to Base 10

2) Use radix multiply or divide to convert from Base 10 to Base B

$(18.6)_9 \rightarrow (?)_{11}$

$N = 1 \times 9^1 + 8 \times 9^0 + 6 \times 9^{-1}$

$= 9 + 8 + 0.666\ldots$

$= (17.666\ldots)_{10}$

$$\begin{array}{r} 1 \\ 11\overline{)17} \\ -11 \\ \hline 6 \end{array}$$

↑ least significant integer digit

$$\begin{array}{r} 0 \\ 11\overline{)1} \\ -0 \\ \hline 1 \end{array}$$

↑ most significant integer digit

$$\begin{array}{r} 0.666 \\ \times \quad 11 \\ \hline 7.326 \end{array}$$

↑ most significant fractional digit

$$\begin{array}{r} 0.326 \\ \times \quad 11 \\ \hline 3.586 \end{array}$$

$$\begin{array}{r} 0.586 \\ \times \quad 11 \\ \hline 6.446 \end{array}$$

↑ least significant fractional digit

$(16.736)_{11}$

When $B = A^k$

$2^k = 8$

$k = 3$

$(1010111.1)_2 \rightarrow (?)_8$

$(001\ 010\ 111.100)_2$

$= (1\ 2\ 7.4)_8$

$(1011011001011111)_2 \rightarrow (?)_{16}$

$2^k = 16$

$k = 4$

$(1011\ 0110\ 0101\ 1111)_2$

$= (B\ 6\ 5\ F)_{16}$

Signed Magnitude

$$N = \pm(a_{n-1} \cdots a_0 . a_{-1} \cdots a_{-m})$$

$$\downarrow$$

$$N = S\, a_{n-1} \cdots a_0 . a_{-1} \cdots a_{-m}$$

N positive → $S = 0$

N negative → $S = r - 1$

Binary

$S = 0$ N positive

$S = 1$ N negative

$N = -(15)_{10}$

signed magnitude binary

$N = -(15)_{10} = -(1111)_2 = (11111)_{2sm}$

$N = +(15)_{10} = (1111)_2 = (01111)_{2sm}$

signed magnitude decimal

$N = -(15)_{10} = (915)_{10sm}$

$N = +(15)_{10} = (015)_{10sm}$

Complementary Number Systems

- Radix Complements (r's complement)

$$[N]_r = r^n - (N)_r$$

where n is the number of digits in $(N)_r$

Positive full scale: $r^{n-1} - 1$

Negative full scale: $-r^n - 1$

Two's Complement (2's complement)

$$[N]_2 = 2^n - (N)_2$$

- Diminished Radix Complements (r-1's complement)

$$[N]_{r-1} = r^n - (N)_r - 1$$

Find the 2's complement $(N)_2 = (101001)_2$

$$[N]_2 = 2^6 - (101001)_2$$

$$= (1000000)_2 - (101001)_2$$

$$\begin{array}{r} 1000000 \\ -\ \ 101001 \\ \hline \end{array} \longrightarrow \begin{array}{rrrrrrr} 0 & 1 & 1 & 1 & 1 & 1 & 10 \\ - & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline & 0 & 1 & 0 & 1 & 1 & 1 \end{array}$$

$$[N]_2 = (010111)_{2cns}$$

$[N]_2 + (N)_2$

$$\begin{array}{rrrrrrr} & 1 & 1 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 0 & 1 & 1 & 1 \\ + & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline \cancel{1} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$n=6$ discard carry

$$\Rightarrow [N]_2 = -(N)_2$$

$$\left[[N]_2 \right]_2 = 2^6 - (010111)_{2cns}$$

$$= 1000000 - 010111$$

$$\begin{array}{r} 1000000 \\ -\ 010111 \\ \hline \end{array}$$

$$\downarrow$$

$$\begin{array}{r} 0\,1\,1\,1\,1\,1\,{}^{1}0 \\ -\ 0\,1\,0\,1\,1\,1 \\ \hline 1\,0\,1\,0\,0\,1 \end{array}$$

Find 2's complement of $(N)_2 = (1010)_2$ for $n=6$

$$[N]_2 = 2^6 - (001010)_2$$

$$= 1000000 - 001010$$

$$\begin{array}{r} \overset{0}{\cancel{1}}\;\overset{1}{{}^{1}\cancel{0}}\;\overset{1}{{}^{1}\cancel{0}}\;\overset{1}{{}^{1}\cancel{0}}\;\overset{1}{{}^{1}\cancel{0}}\;{}^{1}0\;0 \\ -\quad 0\;0\;1\;0\;1\;0 \\ \hline 1\;1\;0\;1\;1\;0 \end{array}$$

$$[N]_2 = (110110)_{2cns}$$

Find 10's complement of $(N)_{10} = (72092)_{10}$

$$[N]_{10} = 10^5 - (72092)_{10}$$

$$= 100000 - 72092$$

$$= (27908)_{10cns}$$

Methods to find Radix Complement

- * copy digits of N, starting with LSD until reaching first non-zero digit, a_i
- * replace a_i with $(r-a_i)$
- * replace remaining digits a_j with $(r-1)-a_j$

2's complement of $(10110)_2$

$$\begin{array}{ccccc}1&0&1&1&0\\\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\0&1&0&1&0\end{array}$$

$(01010)_{2cns}$

10's complement of $(56700)_{10}$

$$\begin{array}{ccccc}5&6&7&0&0\\\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\4&3&3&0&0\end{array}$$

$(43300)_{10cns}$

2's complement of $(101001)_2$

$$\begin{array}{cccccc}1&0&1&0&0&1\\\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\0&1&0&1&1&1\end{array}$$

$(010111)_{2cns}$

* • replace each digit a_k by $(r-1)-a_k$

* • add 1 to the result

2's complement of $N=(01100101)_2$

01100101

↓

10011010

\+ 1

10011011

$(10011011)_{2cns}$

10's complement of $N=(40960)_{10}$

40960

↓

59039

\+ 1

59040

$(59040)_{10cns}$

2's complement of $N=(101001)_2$

101001

↓

010110

\+ 1

010111

$(010111)_{2cns}$

2's complement representation of $\pm(N)_2$

when $(N)_2 = (1011001)_2$ for $n=8$

$+(N)_2 = (01011001)_{2cns}$

$-(N)_2 = [N]_2$

$$
\begin{array}{r}
01011001 \\
\downarrow \\
10100110 \\
+ \quad 1 \\
\hline
(10100111)_{2cns}
\end{array}
$$

2's complement representation of $-(18)_{10}$ for $n=8$

$$+(18)_{10} = (00010010)_{2cns}$$

$$-(18)_{10} = (11101110)_{2cns}$$

Decimal representation of $N = (11101110)_{2cns}$

$$(11101110)_{2cns} = -[11101110]_2$$

$$\begin{array}{c} 11101110 \\ \downarrow \\ 00010010 \end{array}$$

$$= -(00010010)_2 = -(2^4 + 2^1) = -(16+2)$$

$$= -(18)_{10}$$

$(21)_{10}$ $(18)_{10}$ $n=8$

21 → 16
4
1

18 → 16
2

$(00010101)_2$ $(00010010)_2$

$-(21)_{10}$ $(-18)_{10}$

(11101011) (11101110)

$21 - 18 \rightarrow 21 + (-18)$

$$\begin{array}{r}
1\;\;1\;1\;1\;1\;\;1\;\;\;\;\;\; \\
00010101 \\
+\;11101110 \\
\hline
\cancel{1}\underbrace{00000011}
\end{array}$$

$(7)_{10} + (4)_{10}$ using 5 bit 2's complemen
arithmetic

$(7)_{10} \rightarrow (00111)$

$(4)_{10} \rightarrow (00100)$

$$\begin{array}{r} 0\overset{1}{0}111 \\ +00100 \\ \hline 01011 \end{array}$$

$(9)_{10} + (8)_{10}$ using 5bit 2's complement
arithmetic

$(9)_{10} \rightarrow (01001)$

$(8)_{10} \rightarrow (01000)$

$$\begin{array}{r} \overset{1}{0}1001 \\ +01000 \\ \hline 10001 \\ \uparrow \end{array}$$

represents
overflow

$-(12)_{10} - (5)_{10}$ using 5bit 2's complement arithmetic

$(12)_{10} \rightarrow (01100)$ $-(12)_{10} \rightarrow (10100)$

$(5)_{10} \rightarrow (00101)$ $-(5)_{10} \rightarrow (11011)$

$$\begin{array}{r} {}^{1}\,10100 \\ +11011 \\ \hline \cancel{1}01111 \end{array}$$

↑

represents overflow

2's complement limits

$$-2^{n-1} \le N \le 2^{n-1} - 1$$

Binary Coded Decimal (BCD)

- represent decimal digits (0-9)
- use 4 bits to represent each digit

0: 0000

1: 0001

2: 0010

3: 0011

4: 0100

5: 0101

6: 0110

7: 0111

8: 1000

9: 1001

$$(9750)_{10} = (1001011101010000)_{BCD}$$