

# Solutions to Time Series Analysis: with Applications in R

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# Preface

This book contains solutions to the problems in the book *Time Series Analysis: with Applications in R*, third edition, by Cryer and Chan. It is provided as a github repository so that anybody may contribute to its development. Unlike the book, the solutions here use lattice graphics when possible instead of base graphics.



# Chapter 1

## Introduction

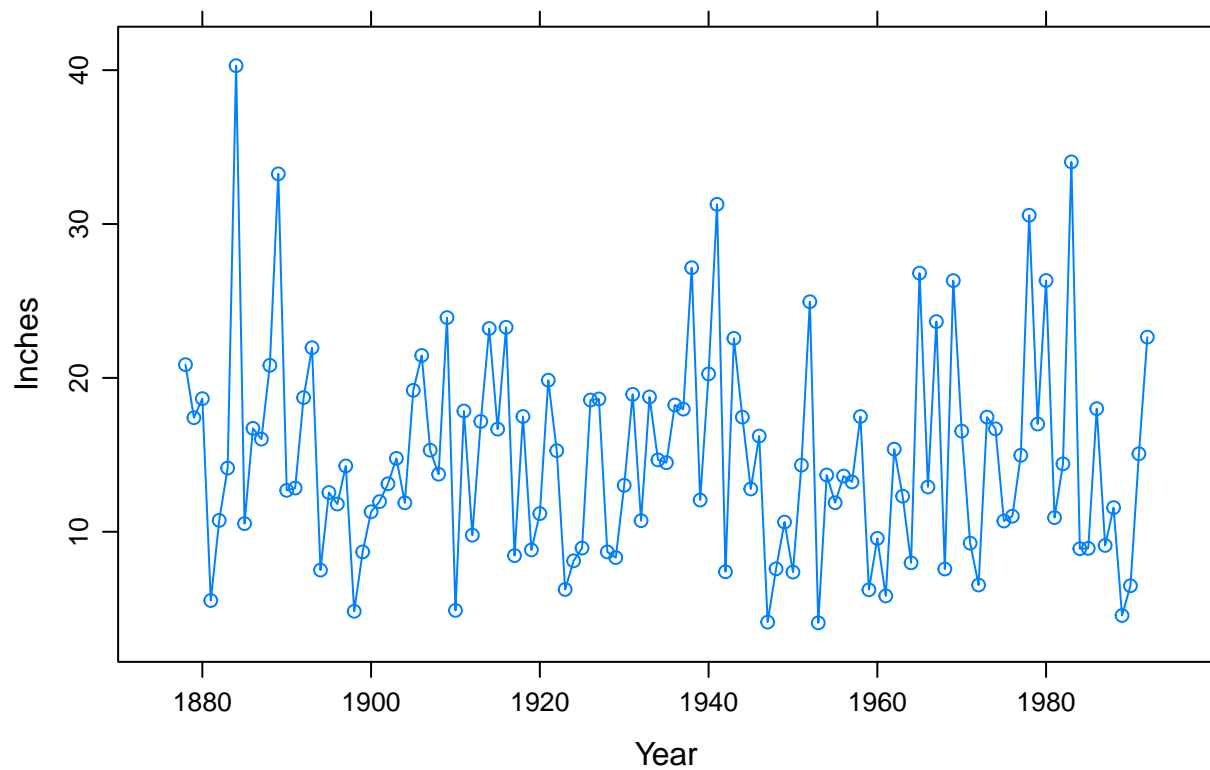
### 1.1 Larain

Use software to produce the time series plot shown in Exhibit 1.2, on page 2. The data are in the file named larain.

```
library(TSA)
library(latticeExtra)

data(larain, package = "TSA")
```

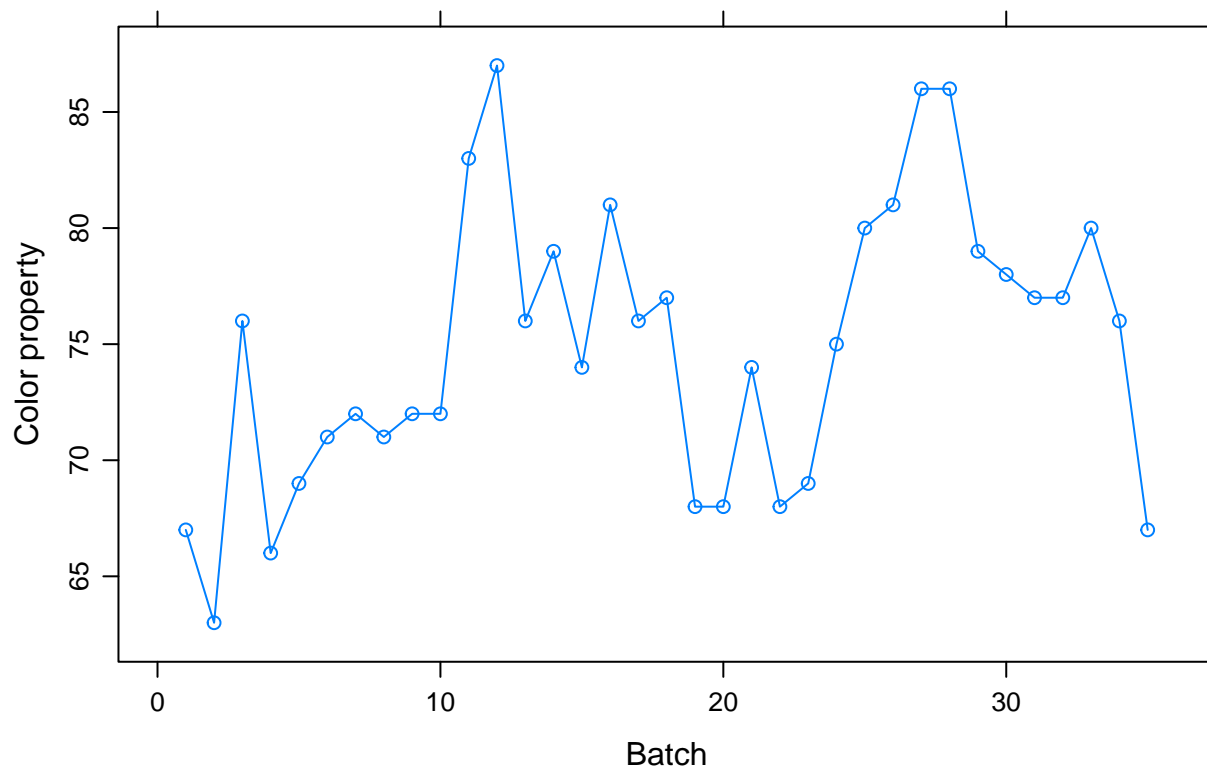
```
xyplot(larain, ylab = "Inches", xlab = "Year", type = "o")
```



## 1.2 Colors

Produce the time series plot displayed in Exhibit 1.3, on page 3. The data file is named color.

```
data(color)
xyplot(color, ylab = "Color property", xlab = "Batch", type = "o")
```

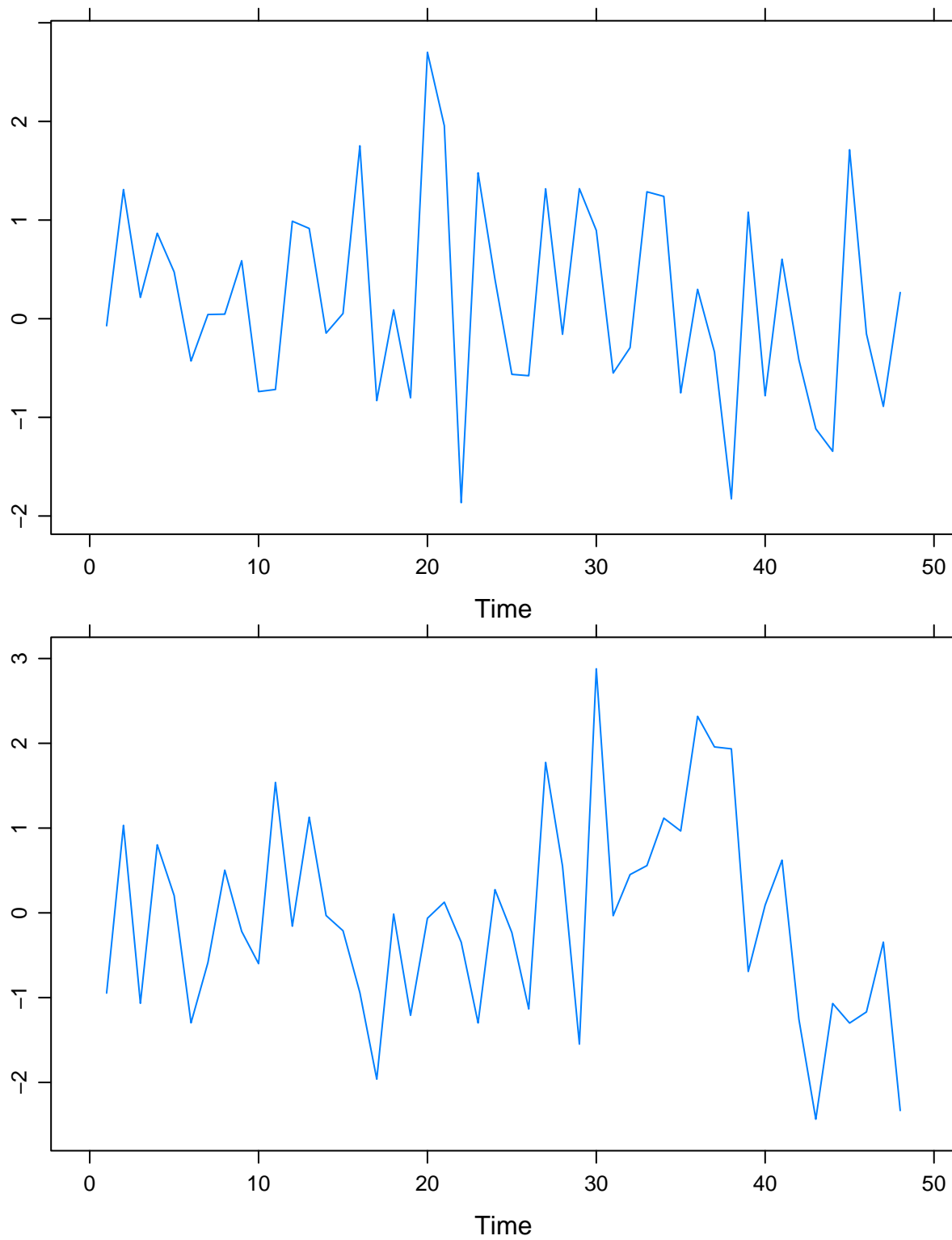


## 1.3 Random, normal time series

Simulate a completely random process of length 48 with independent, normal values. Plot the time series plot. Does it look “random”? Repeat this exercise several times with a new simulation each time.

```
xyplot(as.ts(rnorm(48)))
xyplot(as.ts(rnorm(48)))
```



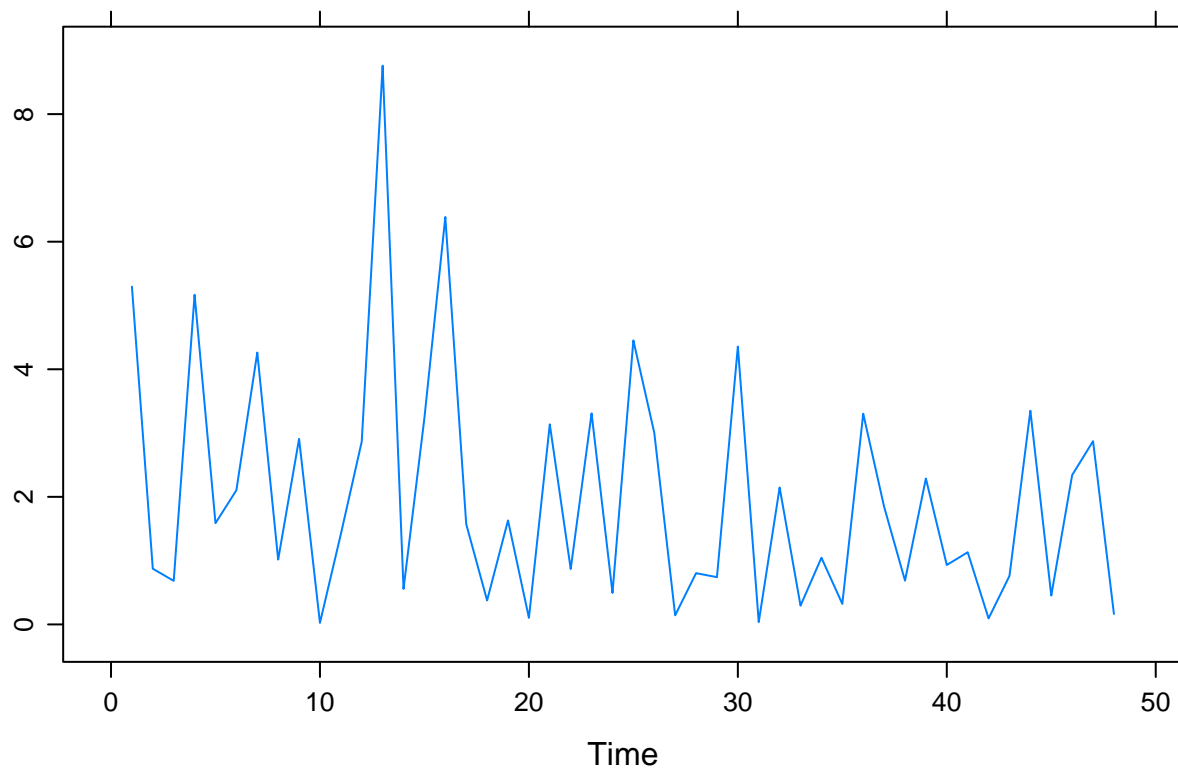


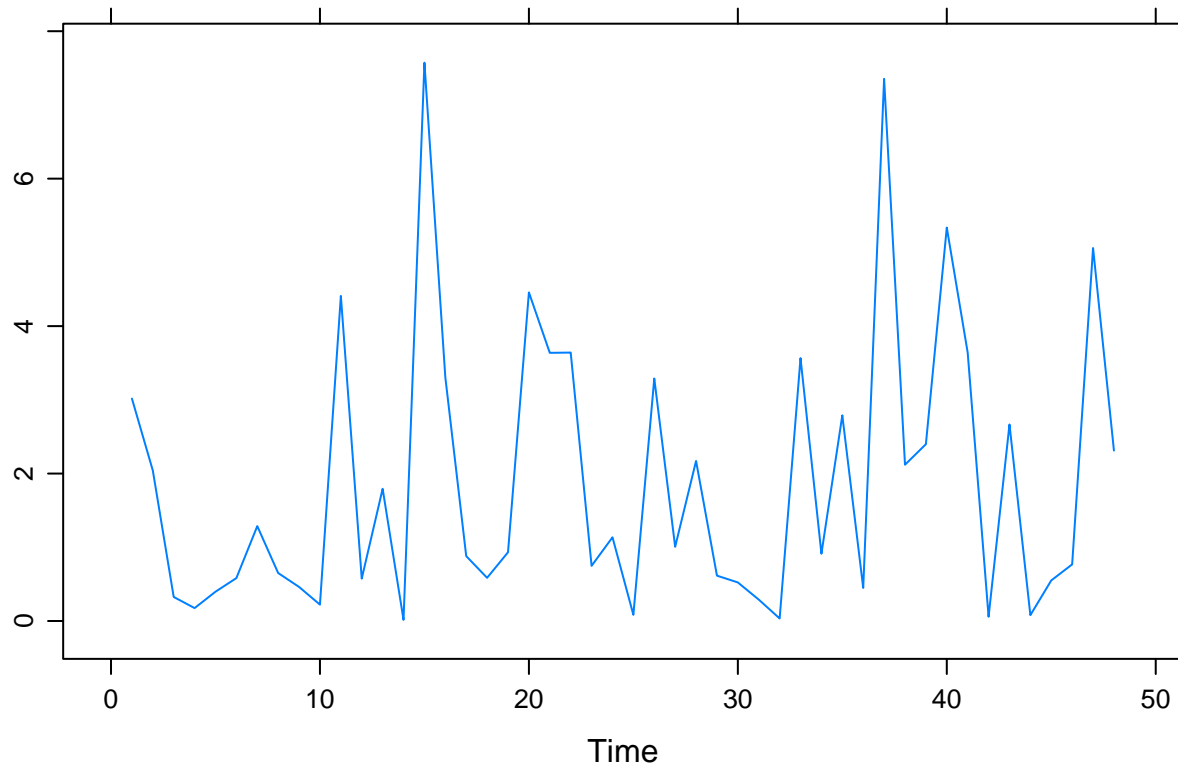
As far as we can tell there is no discernable pattern here.

## 1.4 Random, $\chi^2$ -distributed time series

Simulate a completely random process of length 48 with independent, chi-square distributed values, each with 2 degrees of freedom. Display the time series plot. Does it look “random” and nonnormal? Repeat this exercise several times with a new simulation each time.

```
xyplot(as.ts(rchisq(48, 2)))  
xyplot(as.ts(rchisq(48, 2)))
```



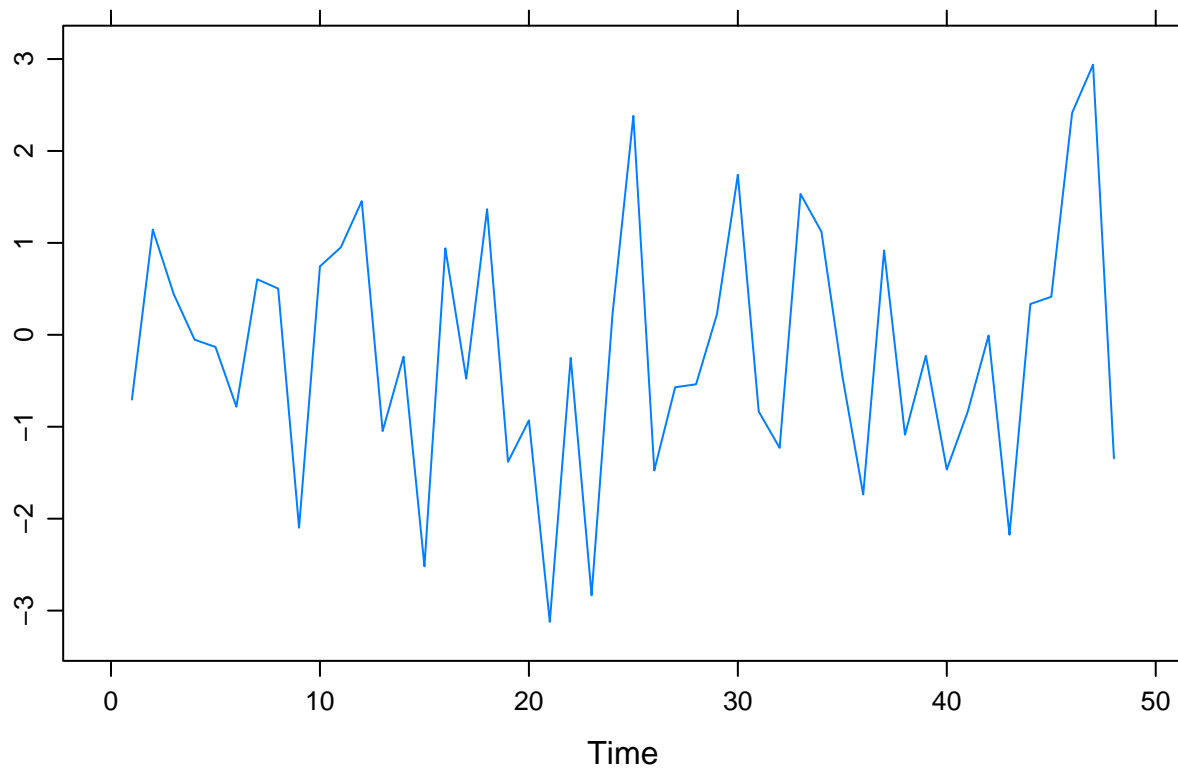


The process appears random, though non-normal.

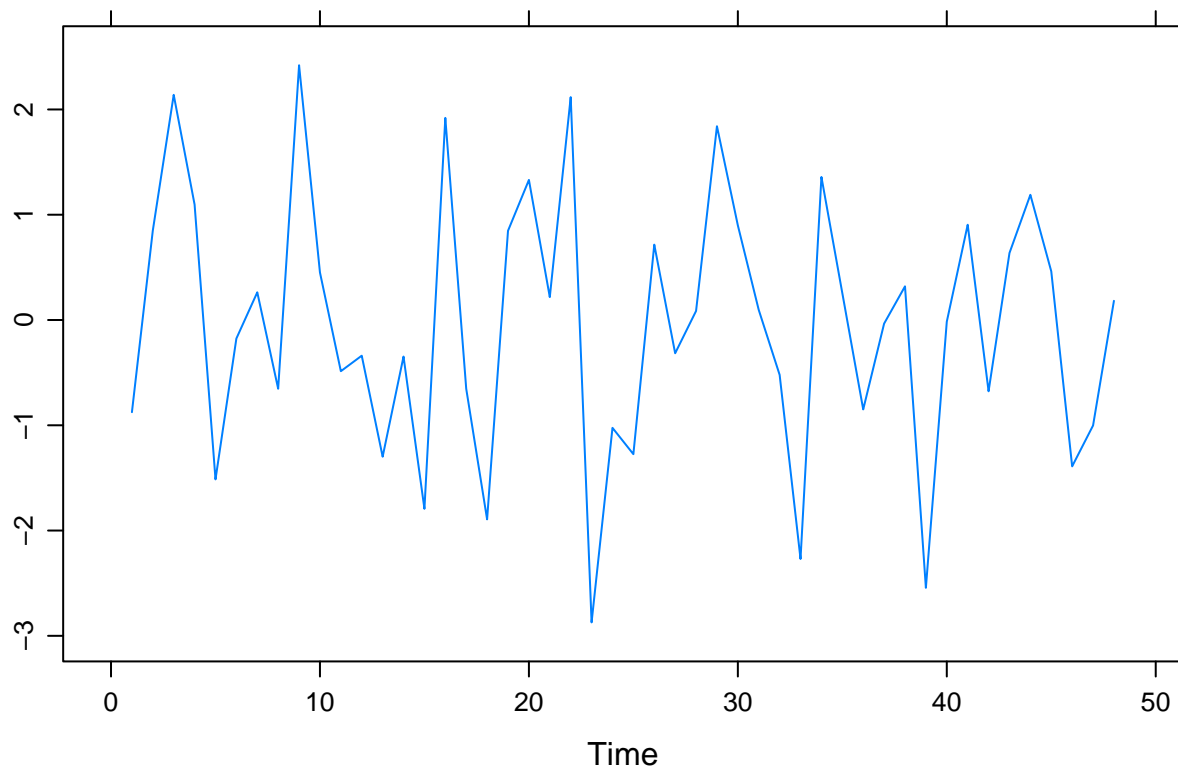
## 1.5 $t(5)$ -distributed, random values

Simulate a completely random process of length 48 with independent,  $t$ -distributed values each with 5 degrees of freedom. Construct the time series plot. Does it look “random” and nonnormal? Repeat this exercise several times with a new simulation each time.

```
xyplot(as.ts(rt(48, 5)))
```



```
xyplot(as.ts(rt(48, 5)))
```

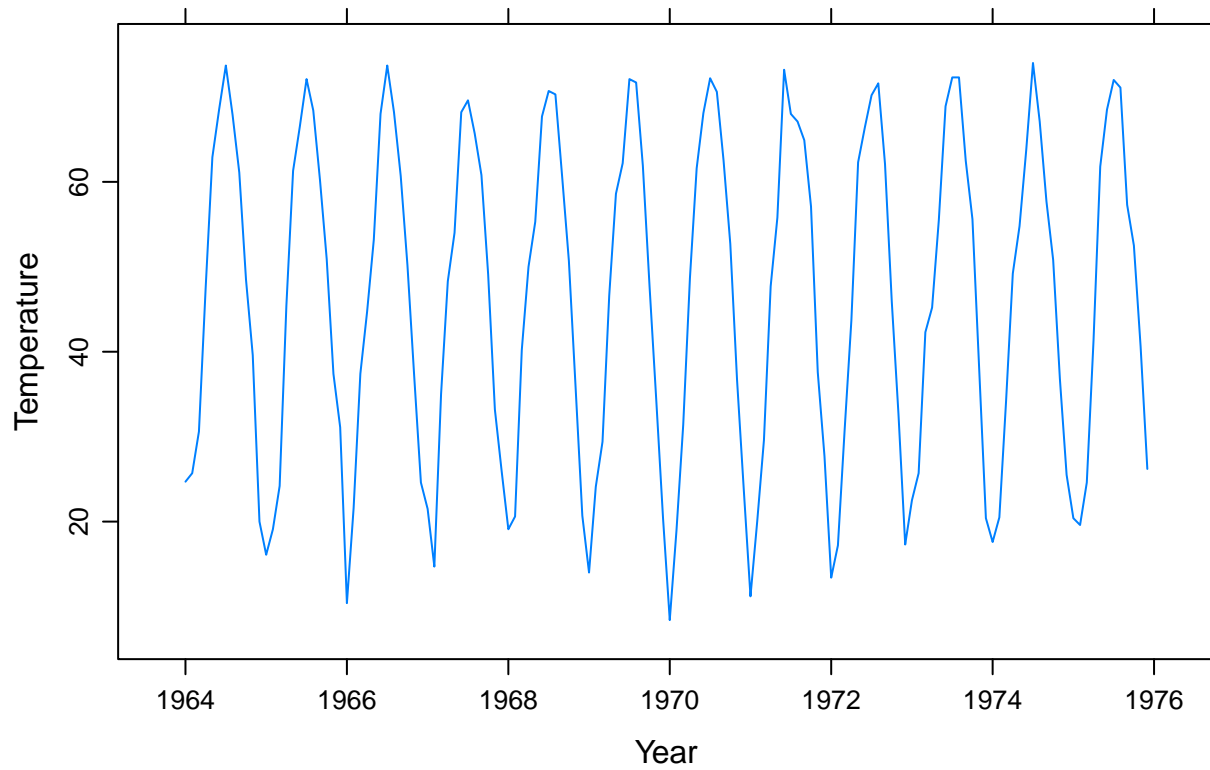


It looks random but not normal, though it should be approximately so, considering the distribution that we have sampled from.

## 1.6 Dubuque temperature series

Construct a time series plot with monthly plotting symbols for the Dubuque temperature series as in Exhibit 1.7, on page 6. The data are in the file named tempdub.

```
data(tempdub)
xyplot(tempdub, ylab = "Temperature", xlab = "Year")
```





## Chapter 2

# Fundamental concepts

### 2.1 Basic properties of expected value and covariance

(a)

$$\text{Cov}[X, Y] = \text{Corr}[X, Y] \sqrt{\text{Var}[X] \text{Var}[Y]} \quad (2.1)$$

$$= 0.25 \sqrt{9 \times 4} = 1.5 \quad (2.2)$$

$$\text{Var}[X, Y] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}[X, Y] \quad (2.3)$$

$$= 9 + 4 + 2 \times 3 = 16 \quad (2.4)$$

$$(2.5)$$

(b)

$$\text{Cov}[X, X + Y] = \text{Cov}[X, X] + \text{Cov}[X, Y] = \text{Var}[X] + \text{Cov}[X, Y] = 9 + 1.5 = 10.5$$

(c)

$$\text{Corr}[X + Y, X - Y] = \text{Corr}[X, X] + \text{Corr}[X, -Y] + \text{Corr}[Y, X] + \text{Corr}[Y, -Y] \quad (2.6)$$

$$= \text{Corr}[Y, X] + \text{Corr}[Y, -Y] \quad (2.7)$$

$$= 1 - 0.25 + 0.25 - 1 \quad (2.8)$$

$$= 0 \quad (2.9)$$

$$(2.10)$$

### 2.2 Dependence and covariance

$$\text{Cov}[X + Y, X - Y] = \text{Cov}[X, X] + \text{Cov}[X, -Y] + \text{Cov}[Y, X] + \text{Cov}[Y, -Y] = \text{Var}[X] - \text{Cov}[X, Y] + \text{Cov}[X, Y] - \text{Var}[Y] = 0$$

since  $\text{Var}[X] = \text{Var}[Y]$ .

### 2.3 Strict and weak stationarity

(a) We have that

$$P(Y_{t_1}, Y_{t_2}, \dots, Y_{t_n}) = P(X_1, X_2, \dots, X_n) = P(Y_{t_1-k}, Y_{t_2-k}, \dots, Y_{t_n-k}),$$

which satisfies our requirement for strict stationarity.

(b) The autocovariance is given by

$$\gamma_{t,s} = \text{Cov}[Y_t, Y_s] = \text{Cov}[X, X] = \text{Var}[X] = \sigma^2.$$

(c)

```
library(lattice)
tstest <- ts(runif(100))

lattice::xyplot(tstest,
  panel = function(x, y, ...) {
    panel.abline(h = mean(y), lty = 2)
    panel.xyplot(x, y, ...)
  })
```

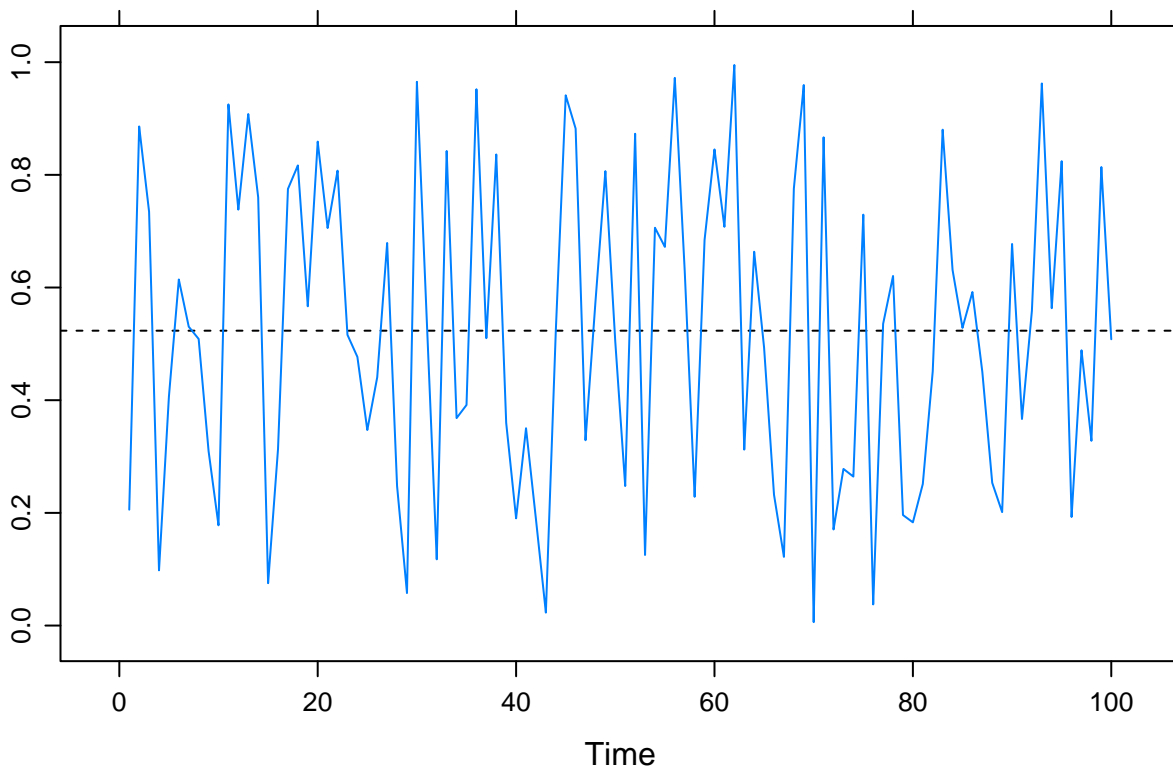


Figure 2.1: A white noise time series: no drift, independence between observations.

## 2.4 Zero-mean white noise

(a)

$$E[Y_t] = E[e_t + \theta e_{t-1}] = E[e_t] + \theta E[e_{t-1}] = 0 + 0 = 0 \quad V[Y_t] = V[e_t + \theta e_{t-1}] = V[e_t] + \theta^2 V[e_{t-1}] = \sigma_e^2 + \theta^2 \sigma_e^2 = \sigma_e^2(1 + \theta^2)$$

For  $k = 1$  we have

$$C[e_t + \theta e_{t-1}, e_{t-1} + \theta e_{t-2}] = C[e_t, e_{t-1}] + C[e_t, \theta e_{t-2}] + C[\theta e_{t-1}, e_{t-1}] + C[\theta e_{t-1}, \theta e_{t-2}] = 0 + 0 + \theta V[e_{t-1}] + 0 = \theta \sigma_e^2, \text{Corr}[Y_t$$



and for  $k = 0$  we get

$$\text{Corr}[Y_t, Y_{t-k}] = \text{Corr}[Y_t, Y_t] = 1$$

and, finally, for  $k > 0$ :

$$C[e_t + \theta e_{t-1}, e_{t-k} + \theta e_{t-k-1}] = C[e_t, e_{t-k}] + C[e_t, e_{t-1-k}] + C[\theta e_{t-1}, e_{t-k}] + C[\theta e_{t-1}, \theta e_{t-1-k}] = 0$$

given that all terms are independent. Taken together, we have that

$$\text{Corr}[Y_t, Y_{t-k}] = \begin{cases} 1 & \text{for } k = 0 \\ \frac{\theta}{1+\theta^2} & \text{for } k = 1 \\ 0 & \text{for } k > 1 \end{cases}.$$

And, as required,

$$\text{Corr}[Y_t, Y_{t-k}] = \begin{cases} \frac{3}{1+3^2} = \frac{3}{10} & \text{if } \theta = 3 \\ \frac{1/3}{1+(1/3)^2} = \frac{1}{10/3} = \frac{3}{10} & \text{if } \theta = 1/3 \end{cases}.$$

- (b) No, probably not. Given that  $\rho$  is standardized, we will not be able to detect any difference in the variance regardless of the values of  $k$ .

## 2.5 Zero-mean stationary series

(a)

$$\mu_t = E[Y_t] = E[5 + 2t + X_t] = 5 + 2E[t] + E[X_t] = 5 + 2t + 0 = 2t + 5$$

(b)

$$\gamma_k = \text{Corr}[5 + 2t + X_t, 5 + 2(t-k) + X_{t-k}] = \text{Corr}[X_t, X_{t-k}]$$

- (c) No, the mean function ( $\mu_t$ ) is constant and the autocovariance ( $\gamma_{t,t-k}$ ) free from  $t$ .

## 2.6 Stationary time series

(a)

$$\text{Cov}[a + X_t, b + X_{t-k}] = \text{Cov}[X_t, X_{t-k}],$$

which is free from  $t$  for all  $k$  because  $X_t$  is stationary.

(b)

$$\mu_t = E[Y_t] = \begin{cases} E[X_t] & \text{for odd } t \\ 3 + E[X_t] & \text{for even } t \end{cases}.$$

Since  $\mu_t$  varies depending on  $t$ ,  $Y_t$  is not stationary.

## 2.7 First and second-order difference series

(a)

$$\mu_t = E[W_t] = E[Y_t - Y_{t-1}] = E[Y_t] - E[Y_{t-1}] = 0$$

because  $Y_t$  is stationary.

$$\text{Cov}[W_t] = \text{Cov}[Y_t - Y_{t-1}, Y_t - Y_{t-1}] = \text{Cov}[Y_t, Y_t] + \text{Cov}[Y_t, Y_{t-1}] + \text{Cov}[Y_{t-1}, Y_t] + \text{Cov}[Y_{t-1}, Y_{t-1}] = \gamma_0 - \gamma_1 - \gamma_1 + \gamma_0 = 2\gamma_0 - 2\gamma_1$$

(b) In (a), we discovered that the difference between two stationary processes,  $\nabla Y_t$  itself was stationary. It follows that the difference between two of these differences,  $\nabla^2 Y_t$  is also stationary.

## 2.8 Generalized difference series

$$E[W_t] = c_1 E[Y_t] + c_2 E[Y_t] + \cdots + c_n E[Y_t] \quad (2.11)$$

$$= E[Y_t](c_1 + c_2 + \cdots + c_n), \quad (2.12)$$

and thus the expected value is constant. Moreover,

$$\text{Cov}[W_t] = \text{Cov}[c_1 Y_t + c_2 Y_{t-1} + \cdots + c_n Y_{t-k}, c_1 Y_{t-k} + c_2 Y_{t-k-1} + \cdots + c_n Y_{t-k-n}] \quad (2.13)$$

$$= \sum_{i=0}^n \sum_{j=0}^n c_i c_j \text{Cov}[Y_{t-j} Y_{t-i-k}] \quad (2.14)$$

$$= \sum_{i=0}^n \sum_{j=0}^n c_i c_j \gamma_{j-k-i}, \quad (2.15)$$

which is free of  $t$ ; consequently,  $W_t$  is stationary.

## 2.9 Zero-mean stationary difference series

(a)

$$E[Y_t] = \beta_0 + \beta_1 t + E[X_t] = \beta_0 + \beta_1 t + \mu_{t_x},$$

which is not free of  $t$  and hence *not* stationary.

$$\text{Cov}[Y_t] = \text{Cov}[X_t, X_t - 1] = \gamma_{t-1}$$

$$E[W_t] = E[Y_t - Y_{t-1}] = E[\beta_0 + \beta_1 t + X_t - (\beta_0 + \beta_1(t-1) + X_{t-1})] = \beta_0 + \beta_1 t - \beta_0 - \beta_1 t + \beta_1 = \beta_1,$$

is free of  $t$  and, furthermore, we have

$$\text{Cov}[W_t] = \text{Cov}[\beta_0 + \beta_1 t + X_t, \beta_0 + \beta_1(t-1) + X_{t-1}] = \text{Cov}[X_t, X_{t-1}] = \gamma_k,$$

which is also free of  $t$ , thereby proving that  $W_t$  is stationary.

(b)

$$E[Y_t] = E[\mu_t + X_t] = \mu_t + \mu_t = 0 + 0 = 0, \quad \text{and } \text{Cov}[Y_t] = \text{Cov}[\mu_t + X_t, \mu_{t-k} + X_{t-k}] = \text{Cov}[X_t, X_{t-k}] = \gamma_k$$

$$\nabla^m Y_t = \nabla(\nabla^{m-1} Y_t)$$

Currently unsolved. ## Zero-mean, unit-variance process

(c)

$$\mu_t = E[Y_t] = E[\mu_t + \sigma_t X_t] = \mu_t + \sigma_t E[X_t] = \mu_t + \sigma_t \times 0 = \mu_t \gamma_{t,t-k} = \text{Cov}[Y_t] = \text{Cov}[\mu_t + \sigma_t X_t, \mu_{t-k} + \sigma_{t-k} X_{t-k}] = \sigma_t \sigma_{t-k} \text{Cov}[X_t, X_{t-k}] = \sigma_t \sigma_{t-k} \gamma_k$$

(d) First, we have

$$\text{Var}[Y_t] = \text{Var}[\mu_t + \sigma_t X_t] = 0 + \sigma_t^2 \text{Var}[X_t] = \sigma_t^2 \times 1 = \sigma_t^2$$

since  $\{X_t\}$  has unit-variance. Furthermore,

$$\text{Corr}[Y_t, Y_{t-k}] = \frac{\sigma_t \sigma_{t-k} \rho_k}{\sqrt{\text{Var}[Y_t] \text{Var}[Y_{t-k}]}} = \frac{\sigma_t \sigma_{t-k} \rho_k}{\sigma_t \sigma_{t-k}} = \rho_k,$$

which depends only on the time lag,  $k$ . However,  $\{Y_t\}$  is not necessarily stationary since  $\mu_t$  may depend on  $t$ .

(e) Yes,  $\rho_k$  might be free from  $t$  but if  $\sigma_t$  is not, we will have a non-stationary time series with autocorrelation free from  $t$  and constant mean.

## 2.10 Drift

(a)

$$\text{Cov}[X_t, X_{t-k}] = \gamma_k E[X_t] = 3t$$

$\{X_t\}$  is not stationary because  $\mu_t$  varies with  $t$ .

(b)

$$E[Y_t] = 3 - 3t + E[X_t] = 7 - 3t - 3t = 7 \text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[7 - 3t + X_t, 7 - 3(t-k) + X_{t-k}] = \text{Cov}[X_t, X_{t-k}] = \gamma_k$$

Since the mean function of  $\{Y_t\}$  is constant (7) and its autocovariance free of  $t$ ,  $\{Y_t\}$  is stationary.

## 2.11 Periods

$$E[Y_t] = E[e_t - e_{t-12}] = E[e_t] - E[e_{t-12}] = 0 \text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[e_t - e_{t-12}, e_{t-k} - e_{t-12-k}] = \text{Cov}[e_t, e_{t-k}] - \text{Cov}[e_t, e_{t-12-k}] - \text{Cov}[e_{t-12}, e_{t-k}] + \text{Cov}[e_{t-12}, e_{t-12-k}]$$

Then, as required, we have

$$\text{Cov}[Y_t, Y_{t-k}] = \begin{cases} \text{Cov}[e_t, e_{t-12}] - \text{Cov}[e_t, e_t] - \text{Cov}[e_{t-12}, e_{t-12}] + \text{Cov}[e_{t-12}, e_t] = \text{Var}[e_t] - \text{Var}[e_{t-12}] \neq 0 & \text{for } k = 12 \\ \text{Cov}[e_t, e_{t-k}] - \text{Cov}[e_t, e_{t-12-k}] - \text{Cov}[e_{t-12}, e_{t-k}] + \text{Cov}[e_{t-12}, e_{t-12-k}] = 0 + 0 + 0 + 0 = 0 & \text{for } k \neq 12 \end{cases}$$

## 2.12 Drift, part 2

(a)

$$E[Y_t] = E[e_t - \theta e_{t-1}^2] = E[e_t] - \theta E[e_{t-1}^2] = 0 - \theta \text{Var}[e_{t-1}] = -\theta \sigma_e^2$$

And thus the requirement of constant variance is fulfilled. Moreover,

$$\text{Var}[Y_t] = \text{Var}[e_t - \theta e_{t-1}^2] = \text{Var}[e_t] + \theta^2 \text{Var}[e_{t-1}^2] = \sigma_e^2 + \theta^2 (E[e_{t-1}^4] - E[e_{t-1}^2]^2),$$

where

$$E[e_{t-1}^4] = 3\sigma_e^4 \quad \text{and} \quad E[e_{t-1}^2]^2 = \sigma_e^4,$$

gives us

$$\text{Var}[Y_t] = \sigma_e^2 + \theta(3\sigma_e^4 - \sigma_e^2) = \sigma_e^2 + 2\theta^2 \sigma_e^4$$

and

$$\text{Cov}[Y_t, Y_{t-1}] = \text{Cov}[e_t - \theta e_{t-1}^2, e_{t-1} - \theta e_{t-2}^2] = \text{Cov}[e_t, e_{t-1}] + \text{Cov}[e_t, -\theta e_{t-2}^2] + \text{Cov}[-\theta e_{t-1}^2, e_{t-1}] + \text{Cov}[-\theta e_{t-1}^2, -\theta e_{t-2}^2] = 0$$

which means that the autocorrelation function  $\gamma_{t,s}$  also has to be zero.

(b) The autocorrelation of  $\{Y_t\}$  is zero and its mean function is constant, thus  $\{Y_t\}$  must be stationary.

## 2.13 Stationarity, again

(a)

$$E[Y_t] = E[\theta_0 + te_t] = \theta_0 + E[te_t] = \theta_0 + t \times 0 = \theta_0 \quad \text{Var}[Y_t] = \text{Var}[\theta_0] + \text{Var}[te_t] = 0 + t^2 \sigma_e^2 = t^2 \sigma_e^2$$

So  $\{Y_t\}$  is not stationary.

(b)

$$E[W_t] = E[\nabla Y_t] = E[\theta_0 + te_t - \theta_0 - (t-1)e_{t-1}] = tE[e_t] - (t-1)E[e_{t-1}] = 0 \quad \text{Var}[\nabla Y_t] = \text{Var}[te_t] = -\text{Var}[(t-1)e_{t-1}] = t^2 \sigma_e^2$$

which varies with  $t$  and means that  $\{W_t\}$  is not stationary.

(c)

$$E[Y_t] = E[e_t e_{t-1}] = E[e_t]E[e_{t-1}] = 0 \quad \text{Cov}[Y_t, Y_{t-1}] = \text{Cov}[e_t e_{t-1}, e_{t-1} e_{t-2}] = E[(e_t e_{t-1} - \mu_t^2)(e_{t-1} e_{t-2} - \mu_{t-1}^2)] = E[e_t]E[e_{t-1}] = 0$$

Both the covariance and the mean function are zero, hence the process is stationary.

## 2.14 Random variable, zero mean

(a)  $E[Y_t] = (-1)^t E[X] = 0$ (b)  $\text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[(-1)^t X, (-1)^{t-k} X] = (-1)^{2t-k} \text{Cov}[X, X] = (-1)^k \text{Var}[X] = (-1)^k \sigma_t^2$ (c) Yes, the covariance is free of  $t$  and the mean is constant.

## 2.15 Mean and variance

$$E[Y_t] = E[A + X_t] = E[A] + E[X_t] = \mu_A + \mu_X \quad \text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[A + X_t, A + X_{t-k}] = \text{Cov}[A, A] + \text{Cov}[A, X_{t-k}] + \text{Cov}[X_t, A] + \text{Cov}[X_t, X_{t-k}]$$

## 2.16 Variance of sample mean

$$\text{Var}[\bar{Y}] = \text{Var}\left[\frac{1}{n} \sum_{t=1}^n Y_t\right] = \frac{1}{n^2} \text{Var}\left[\sum_{t=1}^n Y_t\right] = \frac{1}{n^2} \text{Cov}\left[\sum_{t=1}^n Y_t, \sum_{s=1}^n Y_s\right] = \frac{1}{n^2} \sum_{t=1}^n \sum_{s=1}^n \gamma_{t-s}$$

Setting  $k = t - s, j = t$  gives us

$$\text{Var}[\bar{Y}] = \frac{1}{n^2} \sum_{j=1}^n \sum_{j-k=1}^n \gamma_k = \frac{1}{n^2} \sum_{j=1}^n \sum_{j=k+1}^{n+k} \gamma_k = \frac{1}{n^2} \left( \sum_{k=1}^{n-1} \sum_{j=k+1}^n \gamma_k + \sum_{k=-n+1}^0 \sum_{j=1}^{n+k} \gamma_k \right) = \frac{1}{n^2} \left( \sum_{k=1}^{n-1} (n-k) \gamma_k + \sum_{k=-n+1}^0 (n+k) \gamma_k \right)$$

## 2.17 Sample variance

(a)

$$\sum_{t=1}^n (Y_t - \mu)^2 = \sum_{t=1}^n ((Y_t - \bar{Y}) + (\bar{Y} - \mu))^2 = \sum_{t=1}^n ((Y_t - \bar{Y})^2 - 2(Y_t - \bar{Y})(\bar{Y} - \mu) + (\bar{Y} - \mu)^2) = n(\bar{Y} - \mu)^2 + 2(\bar{Y} - \mu) \sum_{t=1}^n (Y_t - \bar{Y}) + \sum_{t=1}^n (Y_t - \bar{Y})^2$$

(b)

$$E[s^2] = E\left[\frac{n}{n-1} \sum_{t=1}^n (Y_t - \bar{Y})^2\right] = \frac{n}{n-1} E\left[\sum_{t=1}^n ((Y_t - \mu)^2 + n(\bar{Y} - \mu)^2)\right] = \frac{n}{n-1} \sum_{t=1}^n (E[(Y_t - \mu)^2] + nE[(\bar{Y} - \mu)^2])$$

(c) Since  $\gamma_k = 0$  for  $k \neq 0$ , in our case for all  $k$ , we have

$$E[s^2] = \gamma_0 - \frac{2}{n-1} \sum_{t=1}^n \left(1 - \frac{k}{n}\right) \times 0 = \gamma_0$$

## 2.18 Random walk with drift

(a)

$$Y_1 = \theta_0 + e_1, Y_2 = \theta_0 + \theta_0 + e_2 + e_1, Y_t = \theta_0 + \theta_0 + \dots + \theta_0 + e_t + e_{t-1} + \dots + e_1 = Y_t = t\theta_0 + e_t + e_{t-1} + \dots + e_1 \quad \square$$

(b)

$$\mu_t = E[Y_t] = E[t\theta_0 + e_t + e_{t-1} + \dots + e_1] = t\theta_0 + E[e_t] + E[e_{t-1}] + \dots + E[e_1] = t\theta_0 + 0 + 0 + \dots + 0 = t\theta_0$$

(c)

$$\gamma_{t,t-k} = \text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[t\theta_0 + e_t + e_{t-1} + \dots + e_1, (t-k)\theta_0 + e_{t-k} + e_{t-1-k} + \dots + e_1] = \text{Cov}[e_{t-k}, e_{t-k} + e_{t-1-k} + \dots + e_1, e_t + e_{t-1} + \dots + e_1]$$

## 2.19 Random walk

(a)

$$\mu_1 = E[Y_1] = E[e_1] = 0, \mu_2 = E[Y_2] = E[Y_1 + e_2] = E[Y_1] + E[e_2] = 0 + 0 = 0, \dots, \mu_{t-1} = E[Y_{t-1}] = E[Y_{t-2} + e_{t-1}] = E[Y_{t-2}] + E[e_{t-1}] = \mu_{t-2} + 0 = \mu_{t-2}$$

which implies  $\mu_t = \mu_{t-1}$  Q.E.D.

(b)

$$\text{Var}[Y_1] = \sigma_e^2 \text{Var}[Y_2] = \text{Var}[Y_1 - e_2] = \text{Var}[Y_1] + \text{Var}[e_1] = \sigma_e^2 + \sigma_e^2 = 2\sigma_e^2 \dots \text{Var}[Y_{t-1}] = \text{Var}[Y_{t-2} - e_{t-1}] = \text{Var}[Y_{t-2}] + \text{Var}[e_{t-1}]$$

(c)

$$\text{Cov}[Y_t, Y_s] = \text{Cov}[Y_t, Y_t + e_{t+1} + e_{t+2} + \dots + e_s] = \text{Cov}[Y_t, Y_t] = \text{Var}[Y_t] = t\sigma_e^2$$

## 2.20 Random walk with random starting value

(a)

$$E[Y_t] = E[Y_0 + e_t + e_{t-1} + \dots + e_1] = E[Y_0] + E[e_t] + E[e_{t-1}] + E[e_{t-2}] + \dots + E[e_1] = \mu_0 + 0 + \dots + 0 = \mu_0 \quad \square$$

(b)

$$\text{Var}[Y_t] = \text{Var}[Y_0 + e_t + e_{t-1} + \dots + e_1] = \text{Var}[Y_0] + \text{Var}[e_t] + \text{Var}[e_{t-1}] + \dots + \text{Var}[e_1] = \sigma_0^2 + t\sigma_e^2 \quad \square$$

(c)

$$\text{Cov}[Y_t, Y_s] = \text{Cov}[Y_t, Y_t + e_{t+1} + e_{t+2} + \dots + e_s] = \text{Cov}[Y_t, Y_t] = \text{Var}[Y_t] = \sigma_0^2 + t\sigma_e^2 \quad \square$$

(d)

$$\text{Corr}[Y_t, Y_s] = \frac{\sigma_0^2 + t\sigma_e^2}{\sqrt{(\sigma_0^2 + t\sigma_e^2)(\sigma_0^2 + s\sigma_e^2)}} = \sqrt{\frac{\sigma_0^2 + t\sigma_e^2}{\sigma_0^2 + s\sigma_e^2}} \quad \square$$

## 2.21 Asymptotic stationarity

(a)

$$E[Y_1] = E[e_1] = 0, E[Y_2] = E[cY_1 + e_2] = cE[Y_1] + E[e_2] = 0 \dots E[Y_t] = E[cY_{t-1} + e_t] = cE[Y_{t-1}] + E[e_t] = 0 \quad \square$$

(b)

$$\text{Var}[Y_1] = \text{Var}[e_1] = \sigma_e^2, \text{Var}[Y_2] = \text{Var}[cY_1 + e_2] = c^2 \text{Var}[Y_{t-1}] + \text{Var}[e_2] = c^2 \sigma_e^2 + \sigma_e^2 = \sigma_e^2(1 + c^2) \dots \text{Var}[Y_t] = \sigma_e^2(1 + c^2 + c^4 + \dots + c^{2(t-1)})$$

$\{Y_t\}$  is not stationary, given that its variance varies with  $t$ .

(c)

$$\text{Cov}[Y_t, Y_{t-1}] = \text{Cov}[cY_{t-1} + e_t, Y_{t-1}] = c\text{Cov}[Y_{t-1}, Y_{t-1}] = c\text{Var}[Y_{t-1}] \quad \text{giving } \text{Corr}[Y_t, Y_{t-1}] = \frac{c\text{Var}[Y_{t-1}]}{\sqrt{\text{Var}[Y_t]\text{Var}[Y_{t-1}]}} = c\sqrt{\frac{\text{Var}[Y_{t-1}]}{\text{Var}[Y_t]}}$$

And, in the general case,

$$\text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[cY_{t-1} + e_t, Y_{t-k}] = c\text{Cov}[cY_{t-2} + e_{t-1}, Y_{t-k}] = c^3 \text{Cov}[Y_{t-2} + e_{t-1}, Y_{t-k}] = \dots = c^k \text{Var}[Y_{t-k}]$$

giving

$$\text{Corr}[Y_t, Y_{t-k}] = \frac{c^k \text{Var}[Y_{t-k}]}{\sqrt{\text{Var}[Y_t]\text{Var}[Y_{t-k}]}} = c^k \sqrt{\frac{\text{Var}[Y_{t-k}]}{\text{Var}[Y_t]}} \quad \square$$

(d)

$$\text{Var}[Y_t] = \sigma_e^2(1 + c^2 + c^4 + \dots + c^{2(t-1)}) = \sigma_e^2 \sum_{t=1}^n c^{2(t-1)} = \sigma_e^2 \sum_{t=0}^{n-1} c^{2t} = \sigma_e^2 \frac{1 - c^{2n}}{1 - c^2}$$

And because

$$\lim_{t \rightarrow \infty} \sigma_e^2 \frac{1 - c^{2t}}{1 - c^2} = \sigma_e^2 \frac{1}{1 - c^2} \quad \text{since } |c| < 1,$$

which is free of  $t$ ,  $\{Y_t\}$  can be considered *asymptotically* stationary.

(e)

$$Y_t = c(cY_{t-2} + e_{t-1}) + e_t = \dots = e_t + ce_{t-1} + c^2e_{t-2} + \dots + c^{t-2}e_2 + \frac{c^{t-1}}{\sqrt{1-c^2}}e_1 \text{Var}[Y_t] = \text{Var}[e_t + ce_{t-1} + c^2e_{t-2} + \dots + c^{t-2}e_2 + \dots]$$

Futhermore,

$$E[Y_1] = E\left[\frac{e_1}{\sqrt{1-c^2}}\right] = \frac{E[e_1]}{\sqrt{1-c^2}} = 0, E[Y_2] = E[cY_1 + e_2] = cE[Y_1] = 0 \dots E[Y_t] = E[cY_{t-1} + e_t] = cE[Y_{t-1}] = 0,$$

which satisfies our first requirement for weak stationarity. Also,

$$\text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[cY_{t-1} + e_t, Y_{t-1}] = c^k \text{Var}[Y_{t-1}] = c^k \frac{\sigma_e^2}{1-c^2},$$

which is free of  $t$  and hence  $\{Y_t\}$  is now stationary.

## 2.22 Stationarity in sums of stochastic processes

$$E[W_t] = E[Z_t + Y_t] = E[Z_t] + E[Y_t] = \mu_{Z_t} + \mu_{Y_s}$$

Since both processes are stationary – and hence their sums are constant – the sum of both processes must also be constant.

$$\text{Cov}[W_t, W_{t-k}] = \text{Cov}[Z_t + Y_t, Z_{t-k} + Y_{t-k}] = \text{Cov}[Z_t, Z_{t-k}] + \text{Cov}[Z_t, Y_{t-k}] + \text{Cov}[Y_t, Z_{t-k}] + \text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[Z_t, Z_{t-k}] + \text{Cov}[Y_t, Y_{t-k}]$$

both free of  $t$ .

## 2.23 Measurement noise

$$E[Y_t] = E[Y_t + e_t] = E[X_t] + E[e_t] - \mu_t \text{Var}[Y_t] = \text{Var}[X_t + e_t] = \text{Var}[X_t] + \text{Var}[e_t] = \sigma_X^2 + \sigma_e^2 \text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[X_t + e_t, X_{t-k} + e_{t-k}]$$

## 2.24 Random cosine wave

$$E[Y_t] = E\left[\beta_0 + \sum_{i=1}^k (A_i \cos(2\pi f_i t) + B_i \sin(2\pi f_i t))\right] = \beta_0 + \sum_{i=1}^k (E[A_i] \cos(2\pi f_i t) + E[B_i] \sin(2\pi f_i t)) = \beta_0 \text{Cov}[Y_t, Y_s] = \text{Cov}\left[\sum_{i=1}^k (A_i \cos(2\pi f_i t) + B_i \sin(2\pi f_i t)), \sum_{i=1}^k (A_i \cos(2\pi f_i s) + B_i \sin(2\pi f_i s))\right]$$

and is thus free of  $t$  and  $s$ .

## 2.25 Semivariogram

(a)

$$\Gamma_{t,s} = \frac{1}{2}E[(Y_t - Y_s)^2] = \frac{1}{2}E[Y_t^2 - 2Y_t Y_s + Y_s^2] = \frac{1}{2}(E[Y_t^2] - 2E[Y_t Y_s] + E[Y_s^2]) = \frac{1}{2}\gamma_0 + \frac{1}{2}\gamma_0 - 2 \times \frac{1}{2}\gamma_{|t-s|} = \gamma_0 - \gamma_{|t-s|} \text{Cov}[Y_t, Y_s]$$

(b)

$$Y_t - Y_s = e_t + e_{t-1} + \dots + e_1 - e_s - e_{s-1} - \dots - e_1 = e_t + e_{t-1} + \dots + e_{s+1}, \quad \text{for } t > s \Gamma_{t,s} = \frac{1}{2}E[(Y_t - Y_s)^2] = \frac{1}{2}\text{Var}[e_t + e_{t-1} + \dots + e_{s+1}]$$

## 2.26 Polynomials

(a)

$$E[Y_t] = E[e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^r e_{t-r}] = 0 \text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[e_t + \phi e_{t-1} + \cdots + \phi^r e_{t-r}, e_{t-k} + \phi e_{t-1-k} + \cdots + \phi^r e_{t-r-k}]$$

Hence, because of the zero mean and covariance free of  $t$ , it is a stationary process.

(b)

$$\text{Var}[Y_t] = \text{Var}[e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^r e_{t-r}] = \sigma_e^2(1 + \phi + \phi^2 + \cdots + \phi^{2r}) \text{Corr}[Y_t, Y_{t-k}] = \frac{\sigma_e^2 \phi^k (1 + \phi^2 + \phi^4 + \cdots + \phi^{2(r-k)})}{\sqrt{(\sigma_e^2(1 + \phi + \phi^2 + \cdots + \phi^{2r}))}}$$

## 2.27 Random cosine wave extended

(a)

$$E[Y_t] = E[R \cos(2\pi(ft + \phi))] = E[R] \cos(2\pi(ft + \phi)) = E[R] \int_0^1 \cos(E[R \cos(2\pi(ft + \phi))]) d\phi = E[R] \left[ \frac{1}{2\pi} \sin(2\pi(ft + \phi)) \right]_0^1$$

(b)

$$\gamma_{t,s} = E[R \cos(2\pi(ft + \phi)) R \cos(2\pi(fs + \phi))] = \frac{1}{2} E[R^2] \int_0^1 \left( \cos(2\pi(f(t-s))) + \frac{1}{4\pi} \sin(2\pi(f(t+s) + 2\phi)) \right) d\phi = \frac{1}{2} E[R^2] \cos(2\pi(f(t-s)))$$

which is free of  $t$ .

## 2.28 Random cosine wave further

(a)

$$E[Y_t] = \sum_{j=1}^m E[R_j] E[\cos(2\pi(f_j t + \phi))] = \text{via 2.28} = \sum_{j=1}^m E[R_j] \times 0 = 0$$

(b)

$$\gamma_k = \sum_{j=1}^m E[R_j] \cos(2\pi f_j k), \text{ also from 2.28.}$$

## 2.29 Rayleigh distribution

$$Y = R \cos(2\pi(ft + \phi)), \quad X = R \sin(2\pi(ft + \phi)) \begin{bmatrix} \frac{\partial X}{\partial R} & \frac{\partial X}{\partial \Phi} \\ \frac{\partial Y}{\partial R} & \frac{\partial Y}{\partial \Phi} \end{bmatrix} = \begin{bmatrix} \cos(2\pi(ft + \Phi)) & 2\pi R \sin(2\pi(ft + \Phi)) \\ \sin(2\pi(ft + \Phi)) & 2\pi R \cos(2\pi(ft + \Phi)) \end{bmatrix},$$

with jacobian

$$-2\pi R = -2\pi \sqrt{X^2 + Y^2}$$

and inverse Jacobian

$$\frac{1}{-2\pi \sqrt{X^2 + Y^2}}.$$



Furthermore,

$$f(r, \Phi) = r e^{-r^2/2}$$

and

$$f(x, y) = \frac{e^{-(x^2+y^2)/2} \sqrt{x^2+y^2}}{2\pi \sqrt{x^2+y^2}} = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \quad \square$$



## Chapter 3

# Trends