Solutions to Time Series Analysis: with Applications in R

Johan Larsson

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Contents

Preface

This book contains solutions to the problems in the book *Time Series Analysis: with Applications in R*, third edition, by Cryer and Chan. It is provided as a github repository so that anybody may contribute to its development. Unlike the book, the solutions here use lattice graphics when possible instead of base graphics.

Chapter 1

Introduction

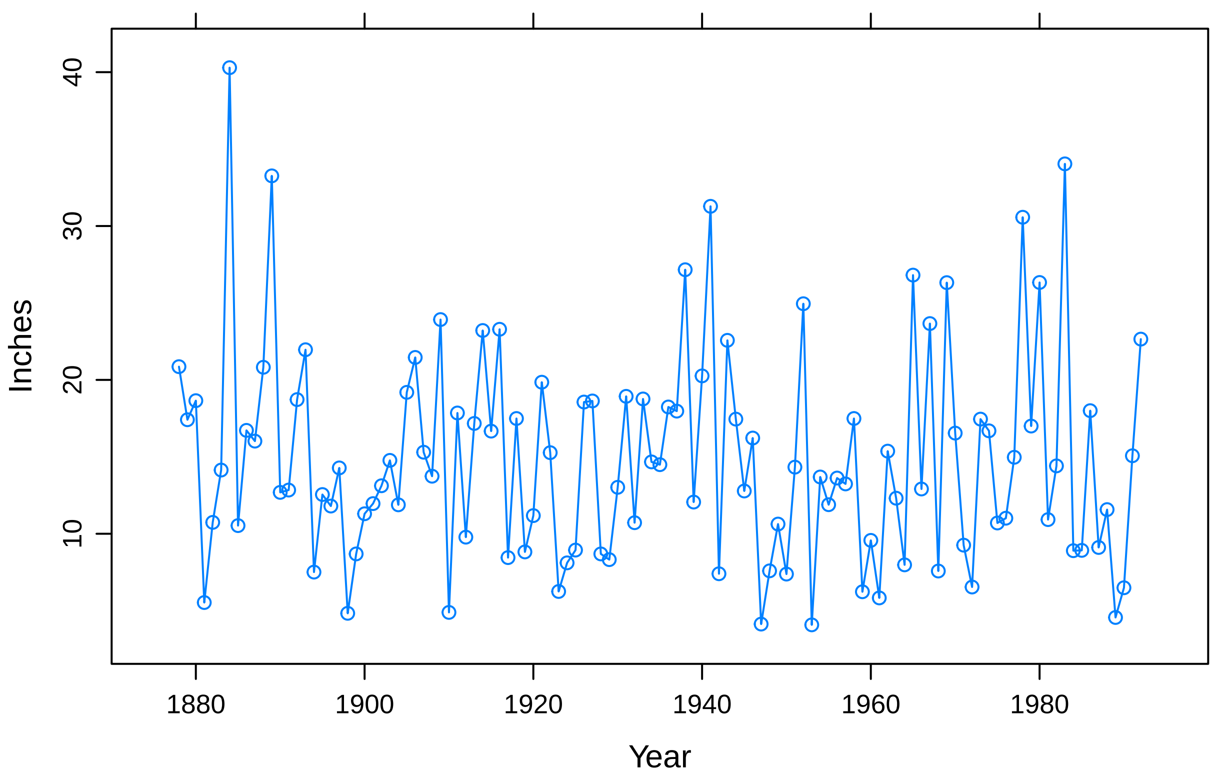
1.1 Larain

Use software to produce the time series plot shown in Exhibit 1.2, on page 2. The data are in the file named larain.

```
library(TSA)
library(latticeExtra)

data(larain, package = "TSA")
```

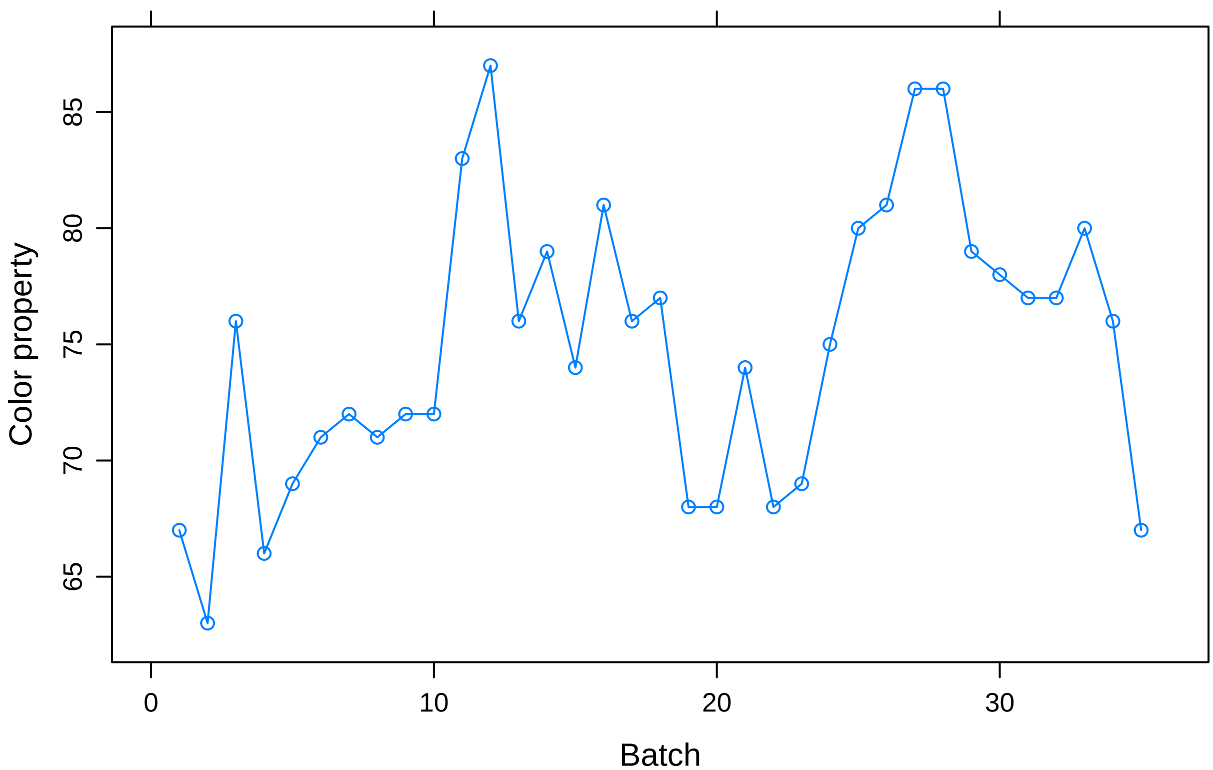
```
xyplot(larain, ylab = "Inches", xlab = "Year", type = "o")
```



1.2 Colors

Produce the time series plot displayed in Exhibit 1.3, on page 3. The data file is named color.

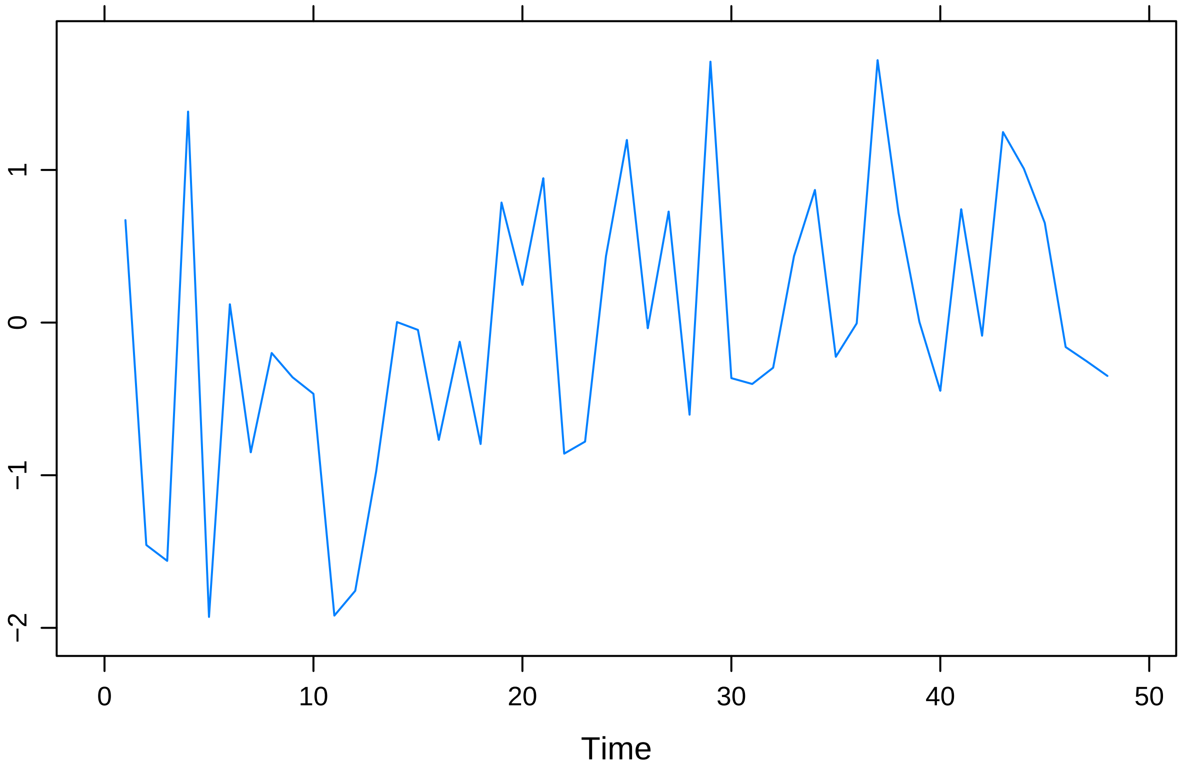
```
data(color)
xyplot(color, ylab = "Color property", xlab = "Batch", type = "o")
```

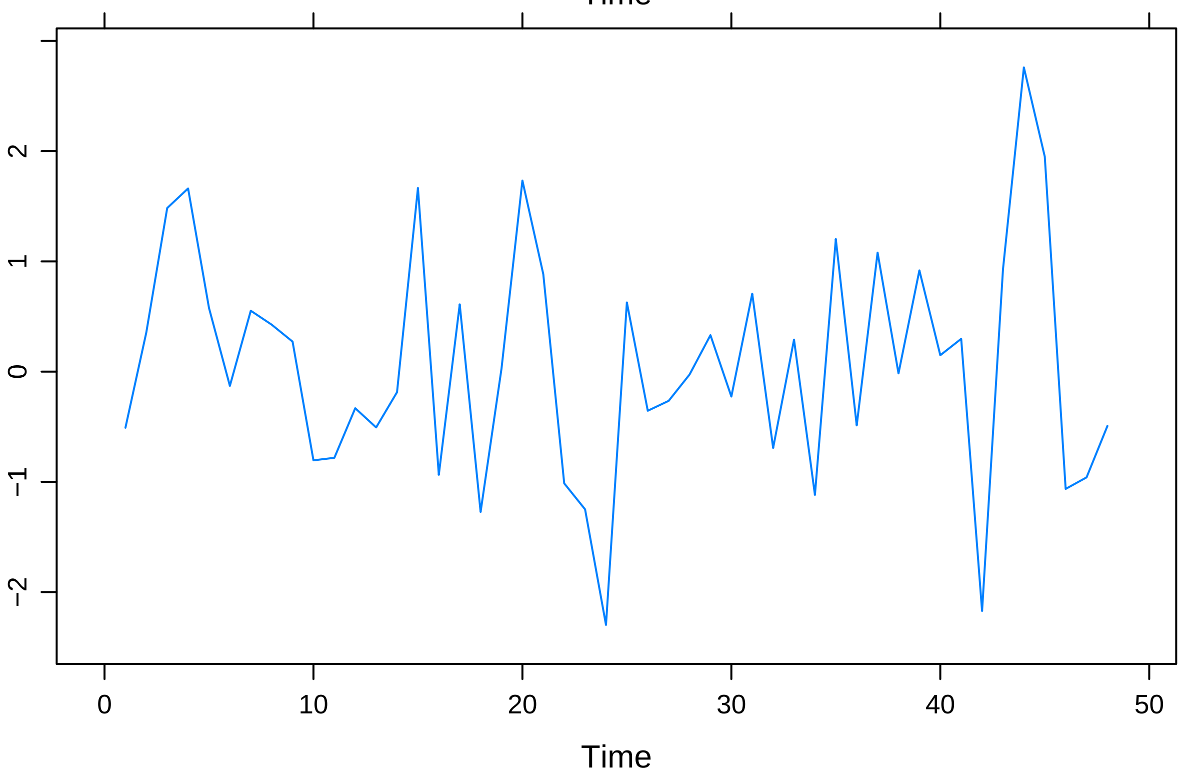


1.3 Random, normal time series

Simulate a completely random process of length 48 with independent, normal values. Plot the time series plot. Does it look “random”? Repeat this exercise several times with a new simulation each time.

```
xyplot(as.ts(rnorm(48)))
xyplot(as.ts(rnorm(48)))
```



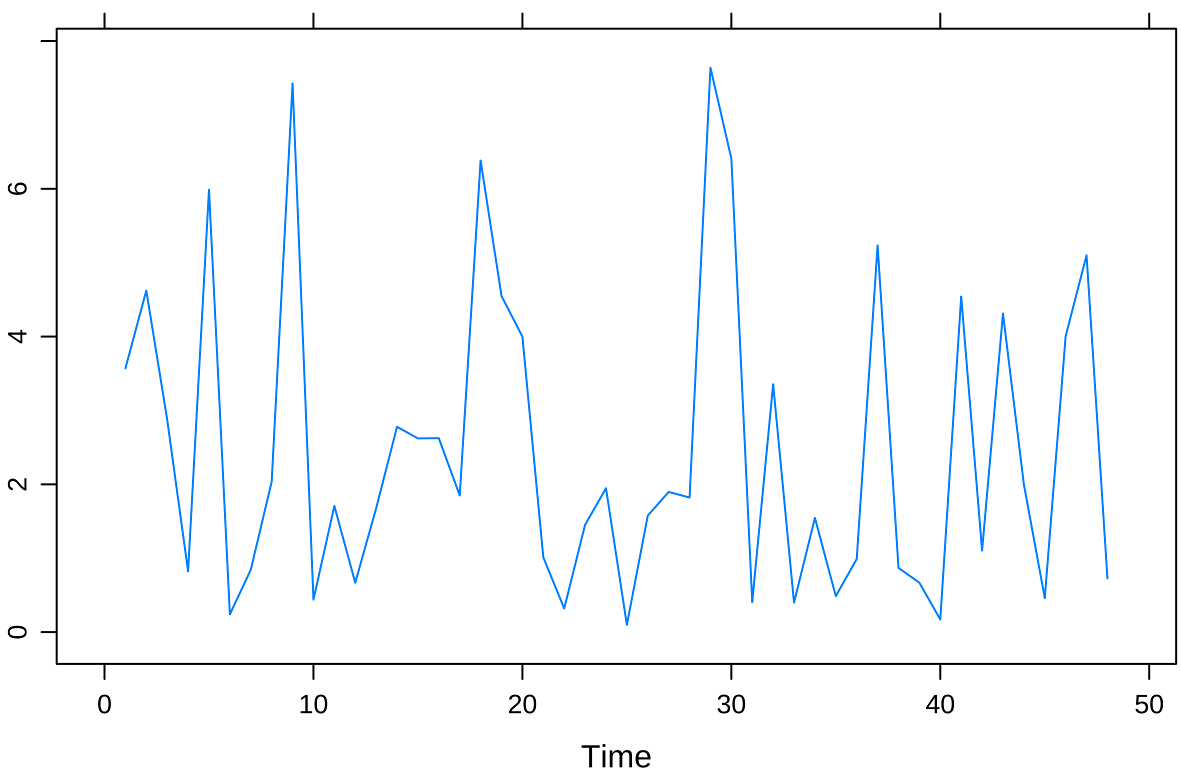


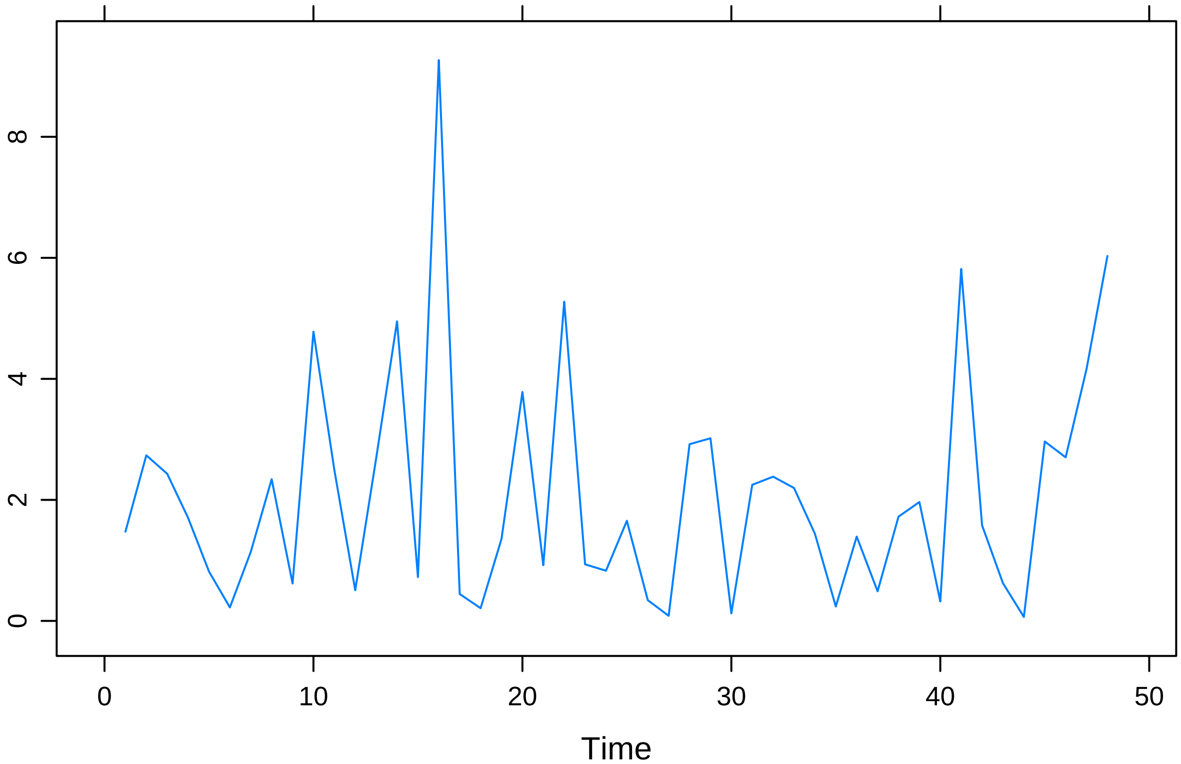
As far as we can tell there is no discernable pattern here.

1.4 Random, χ^2-distributed time series

Simulate a completely random process of length 48 with independent, chi-square distributed values, each with 2 degrees of freedom. Display the time series plot. Does it look "random" and nonnormal? Repeat this exercise several times with a new simulation each time.

```
xyplot(as.ts(rchisq(48, 2)))
xyplot(as.ts(rchisq(48, 2)))
```



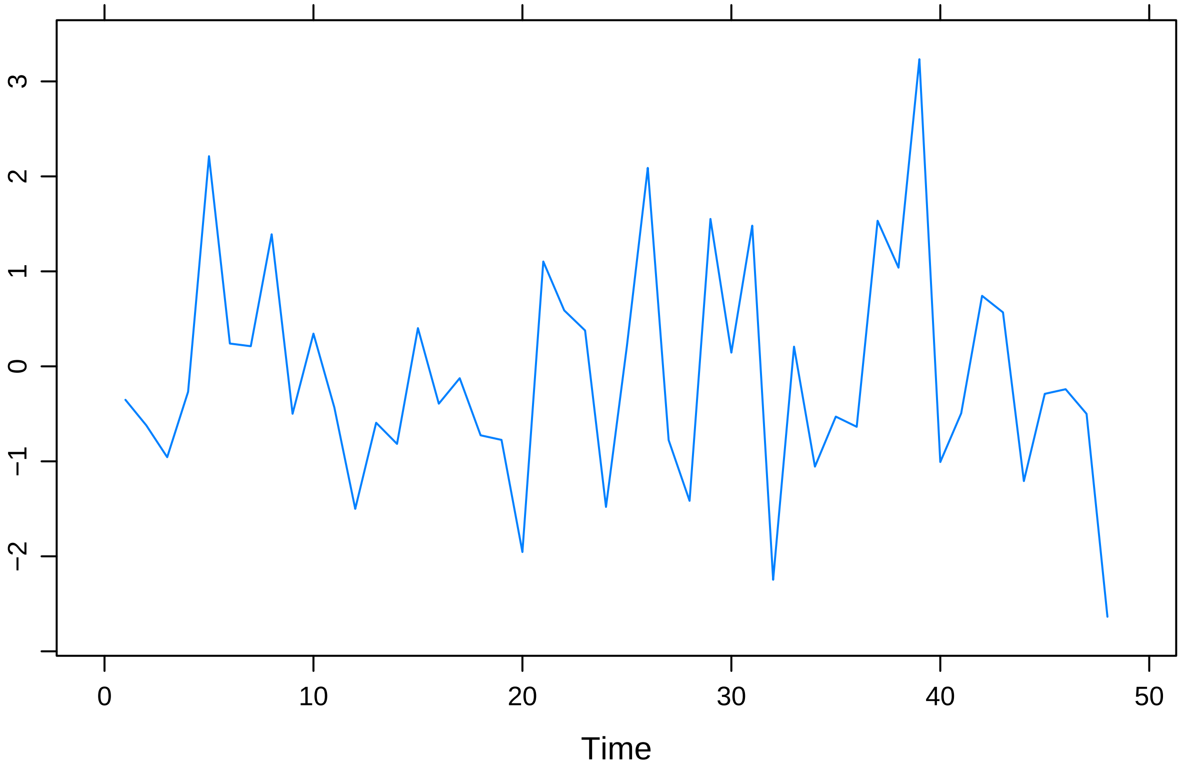


The process appears random, though non-normal.

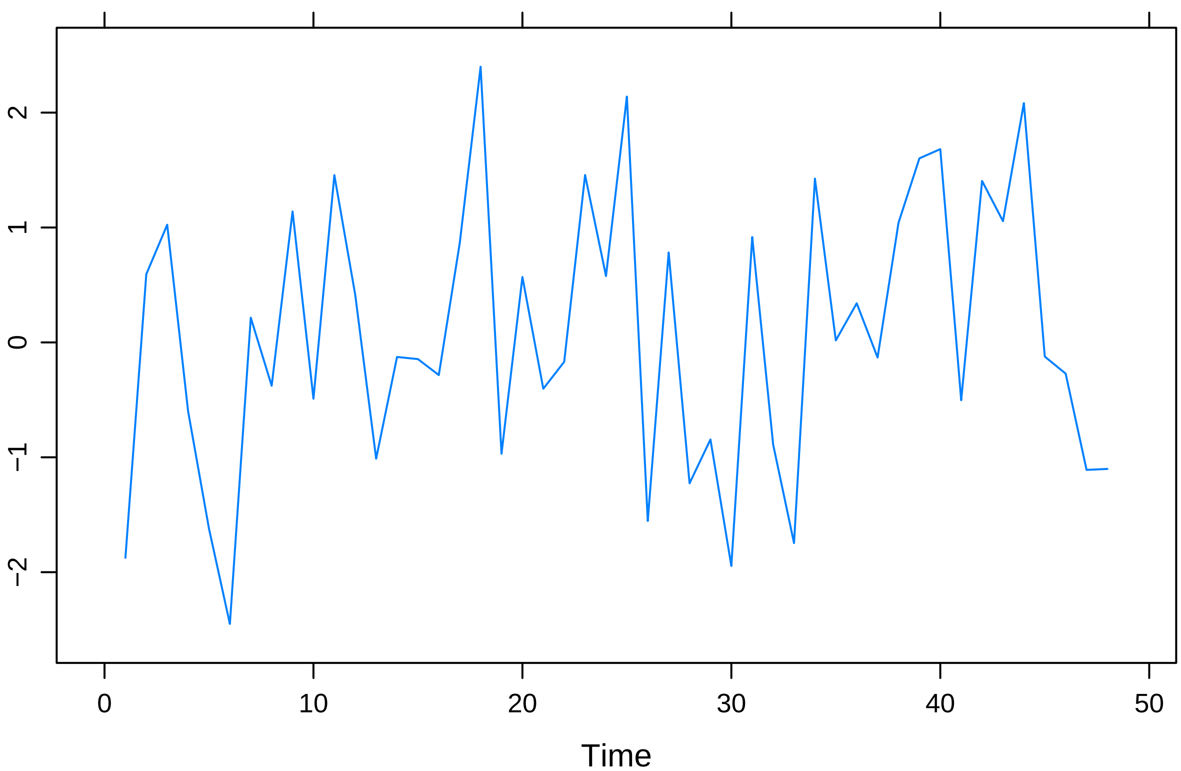
1.5 $t(5)$-distributed, random values

Simulate a completely random process of length 48 with independent, t-distributed values each with 5 degrees of freedom. Construct the time series plot. Does it look “random” and nonnormal? Repeat this exercise several times with a new simulation each time.

```
xyplot(as.ts(rt(48, 5)))
```




```
xyplot(as.ts(rt(48, 5)))
```

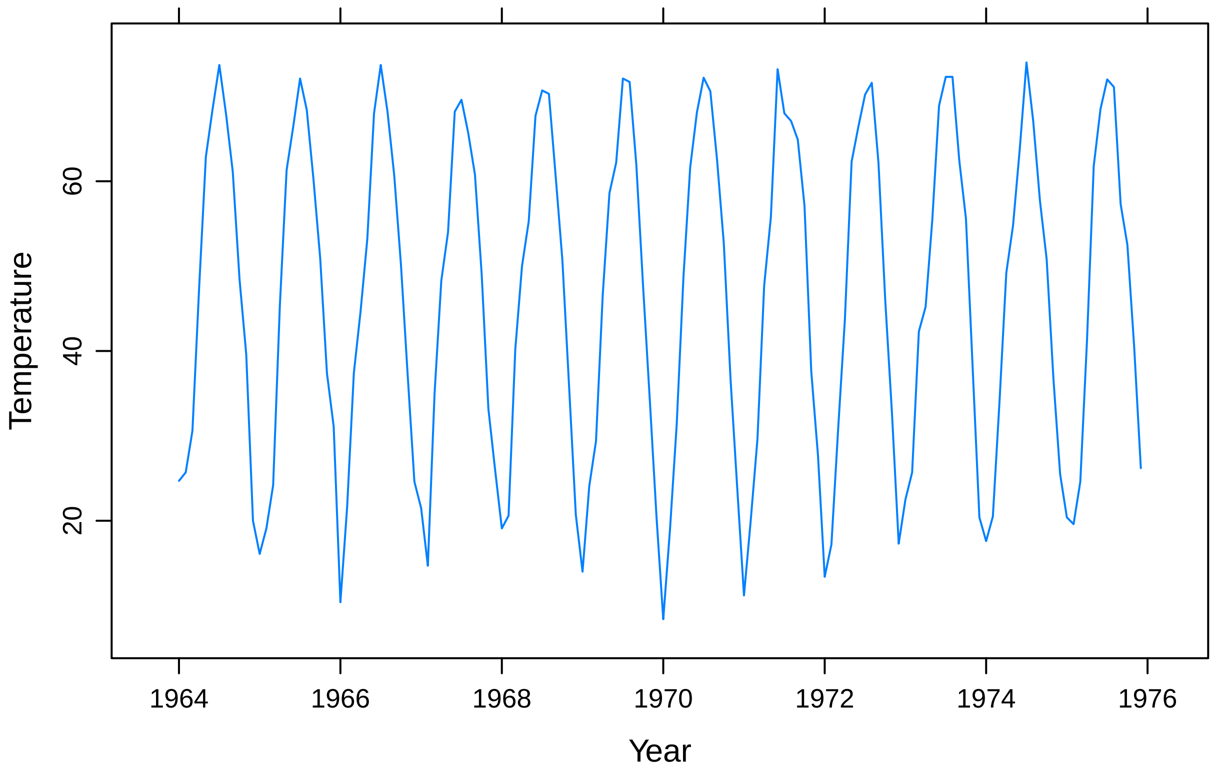


It looks random but not normal, though it should be approximately so, considering the distribution that we have sampled from.

1.6 Dubuque temperature series

Construct a time series plot with monthly plotting symbols for the Dubuque temperature series as in Exhibit 1.7, on page 6. The data are in the file named tempdub.

```
data(tempdub)
xyplot(tempdub, ylab = "Temperature", xlab = "Year")
```



Chapter 2

Fundamental concepts

2.1 Basic properties of expected value and covariance

a

$$
\begin{aligned}
\text{Cov}[X, Y] &= \text{Corr}[X, Y]\sqrt{Var[X]Var[Y]} && (2.1)\\
&= 0.25\sqrt{9 \times 4} = 1.5 && (2.2)\\
\text{Var}[X, Y] &= Var[X] + Var[Y] + 2Cov[X, Y] && (2.3)\\
&= 9 + 4 + 2 \times 3 = 16 && (2.4)\\
& && (2.5)
\end{aligned}
$$

b

$$\text{Cov}[X, X + Y] = \text{Cov}[X, X] + \text{Cov}[X, Y] = \text{Var}[X] + \text{Cov}[X, Y] = 9 + 1.5 = 10.5$$

c

$$
\begin{aligned}
\text{Corr}[X + Y, X - Y] =& \text{Corr}[X, X] + \text{Corr}[X, -Y] + \text{Corr}[Y, X] + \text{Corr}[Y, -Y] && (2.6)\\
=& \text{Corr}[Y, X] + \text{Corr}[Y, -Y] && (2.7)\\
=& 1 - 0.25 + 0.25 - 1 && (2.8)\\
=& 0 && (2.9)\\
& && (2.10)
\end{aligned}
$$

2.2 Dependence and covariance

$$\text{Cov}[X + Y, X - Y] = \text{Cov}[X, X] + \text{Cov}[X, -Y] + \text{Cov}[Y, X] + \text{Cov}[Y, -Y] =$$
$$Var[X] - Cov[X, Y] + Cov[X, Y] - Var[Y] = 0$$

since $Var[X] = Var[Y]$.

2.3 Strict and weak stationarity

a

We have that

$$
\begin{gathered}
P(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}) = \\
P(X_1, X_2, \ldots, X_n) = \\
P(Y_{t_1-k}, Y_{t_2-k}, \ldots, Y_{t_n-k}),
\end{gathered}
$$

which satisfies our requirement for strict stationarity.

b

The autocovariance is given by

$$\gamma_{t,s} = \mathrm{Cov}[Y_t, Y_s] = \mathrm{Cov}[X, X] = \mathrm{Var}[X] = \sigma^2.$$

c

```
library(lattice)
tstest <- ts(runif(100))

lattice::xyplot(tstest,
                panel = function(x, y, ...) {
                  panel.abline(h = mean(y), lty = 2)
                  panel.xyplot(x, y, ...)
                })
```

2.4 Zero-mean white noise

a

$$
\begin{gathered}
E[Y_t] = E[e_t + \theta e_{t-1}] = E[e_t] + \theta E[e_{t-1}] = 0 + 0 = 0 \\
V[Y_t] = V[e_t + \theta e_{t-1}] = V[e_t] + \theta^2 V[e_{t-1}] = \sigma_e^2 + \theta^2\sigma_e^2 = \sigma_2^2(1 + \theta^2)
\end{gathered}
$$

For $k = 1$ we have

$$
\begin{gathered}
C[e_t + \theta e_{t-1}, e_{t-1} + \theta e_{t-2}] = \\
C[e_t, e_{t-1}] + C[e_t, \theta e_{t-2}] + C[\theta e_{t-1}, e_{t-1}] + C[\theta e_{t-1}, \theta e_{t-2}] = \\
0 + 0 + \theta V[e_{t-1}] + 0 = \theta\sigma_e^2, \\
\mathrm{Corr}[Y_t, Y_{t-k}] = \frac{\theta\sigma_e^2}{\sqrt{(\sigma_e^2(1 + \theta^2))^2}} = \frac{\theta}{1 + \theta^2}
\end{gathered}
$$

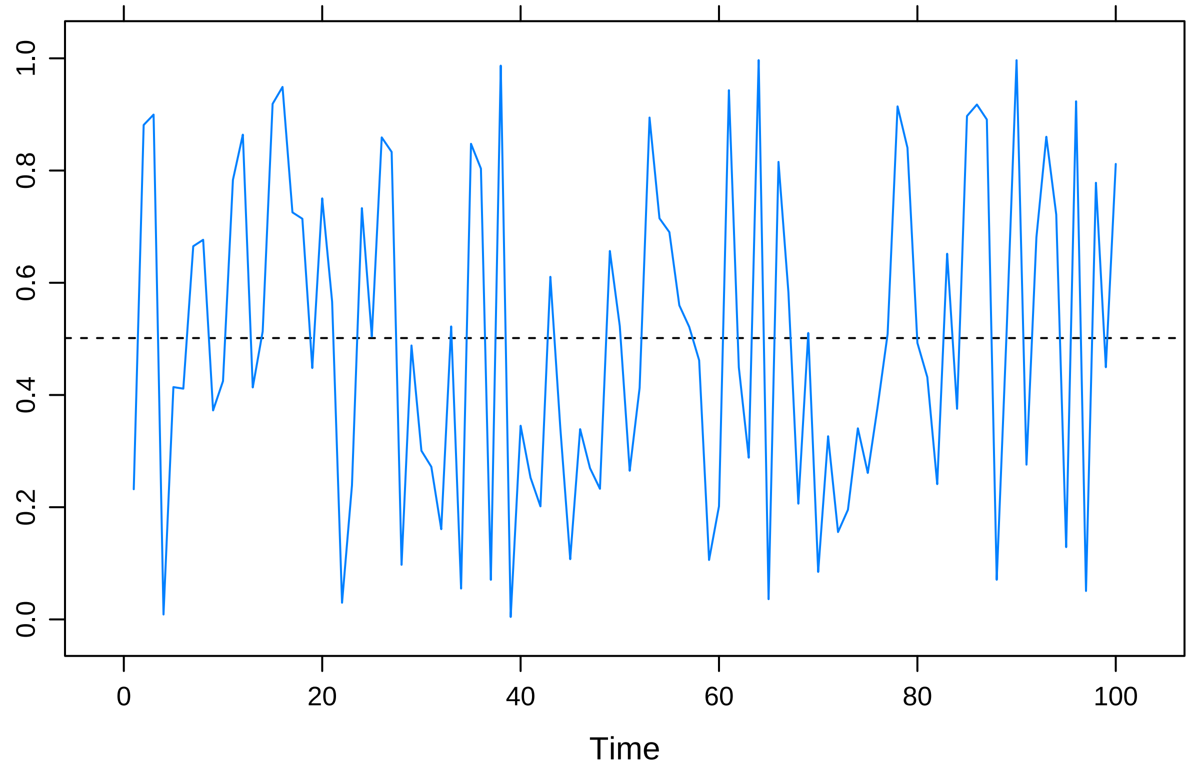


Figure 2.1: A white noise time series: no drift, independence between observations.

and for $k = 0$ we get

$$\mathrm{Corr}[Y_t, Y_{t-k}] = \mathrm{Corr}[Y_t, Y_t] = 1$$

and, finally, for $k > 0$:

$$C[e_t + \theta e_{t-1}, e_{t-k} + \theta e_{t-k-1}] =$$
$$C[e_t, e_{t-k}] + C[e_t, e_{t-1-k}] + C[\theta e_{t-1}, e_{t-k}] + C[\theta e_{t-1}, \theta e_{t-1-k}] = 0$$

given that all terms are independent. Taken together, we have that

$$\mathrm{Corr}[Y_t, Y_{t-k}] = \begin{cases} 1 & \text{for } k = 0 \\ \frac{\theta}{1+\theta^2} & \text{for } k = 1 \\ 0 & \text{for } k > 1 \end{cases}.$$

And, as required,

$$\mathrm{Corr}[Y_t, Y_{t-k}] = \begin{cases} \frac{3}{1+3^2} = \frac{3}{10} & \text{if } \theta = 3 \\ \frac{1/3}{1+(1/3)^2} = \frac{1}{10/3} = \frac{3}{10} & \text{if } \theta = 1/3 \end{cases}.$$

b

No, probably not. Given that ρ is standardized, we will not be able to detect any difference in the variance regardless of the values of k.

2.5 Zero-mean stationary series

a

$$\mu_t = E[Y_t] = E[5 + 2t + X_t] = 5 + 2E[t] + E[X_t] = 5 + 2t + 0 = 2t + 5$$

b

$$\gamma_k = \text{Corr}[5 + 2t + X_t, 5 + 2(t-k) + X_{t-k}] = \text{Corr}[X_t, X_{t-k}]$$

c

No, the mean function (μ_t) is constant and the aurocovariance ($\gamma_{t,t-k}$) free from t.

2.6 Stationary time series

a

$$\text{Cov}[a + X_t, b + X_{t-k}] = \text{Cov}[X_t, X_{t-k}],$$

which is free from t for all k because X_t is stationary.

b

$$\mu_t = E[Y_t] = \begin{cases} E[X_t] & \text{for odd } t \\ 3 + E[X_t] & \text{for even } t \end{cases}.$$

Since μ_t varies depending on t, Y_t is not stationary.

2.7 First and second-order difference series

a

$$\mu_t = E[W_t] = E[Y_t - Y_{t-1}] = E[Y_t] - E[Y_{t-1}] = 0$$

because Y_t is stationary.

$$\text{Cov}[W_t] = \text{Cov}[Y_t - Y_{t-1}, Y_{t-k} - Y_{t-1-k}] =$$
$$\text{Cov}[Y_t, Y_{t-k}] + \text{Cov}[Y_t, Y_{t-1-k}] + \text{Cov}[-Y_{t-k}, Y_{t-k}] + \text{Cov}[-Y_{t-k}, -Y_{t-1-k}] =$$
$$\gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k = 2\gamma_k - \gamma_{k+1} - \gamma_{k-1}. \quad \square$$

b

In (a), we discovered that the difference between two stationary processes, ∇Y_t itself was stationary. It follows that the difference between two of these differences, $\nabla^2 Y_t$ is also stationary.

2.8 Generalized difference series

$$
\begin{aligned}
E[W_t] &= c_1 E[Y_t] + c_2 E[Y_t] + \cdots + c_n E[Y_t] && (2.11)\\
&= E[Y_t](c_1 + c_2 + \cdots + c_n), && (2.12)
\end{aligned}
$$

and thus the expected value is constant. Moreover,

$$
\begin{aligned}
\mathrm{Cov}[W_t] &= \mathrm{Cov}[c_1 Y_t + c_2 Y_{t-1} + \cdots + c_n Y_{t-k}, c_1 Y_{t-k} + c_2 Y_{t-k-1} + \cdots + c_n Y_{t-k-n}] && (2.13)\\
&= \sum_{i=0}^{n} \sum_{j=0}^{n} c_i c_j \mathrm{Cov}[Y_{t-j} Y_{t-i-k}] && (2.14)\\
&= \sum_{i=0}^{n} \sum_{j=0}^{n} c_i c_j \gamma_{j-k-i}, && (2.15)
\end{aligned}
$$

which is free of t; consequently, W_t is stationary.

2.9 Zero-mean stationary difference series

a

$$E[Y_t] = \beta_0 + \beta_1 t + E[X_t] = \beta_0 + \beta_1 t + \mu_{t_x},$$

which is not free of t and hence *not* stationary.

$$\mathrm{Cov}[Y_t] = \mathrm{Cov}[X_t, X_t - 1] = \gamma_{t-1}$$

$$E[W_t] = E[Y_t - Y_{t-1}] = E[\beta_0 + \beta_1 t + X_t - (\beta_0 + \beta_1(t-1) + X_{t-1})] = \\ \beta_0 + \beta_1 t - \beta_0 - \beta_1 t + \beta_1 = \beta_1,$$

is free of t and, furthermore, we have

$$\mathrm{Cov}[W_t] = \mathrm{Cov}[\beta_0 + \beta_1 t + X_t, \beta_0 + \beta_1(t-1) + X_{t-1}] = \\ \mathrm{Cov}[X_t, X_{t-1}] = \gamma_k,$$

which is also free of t, thereby proving that W_t is stationary.

b

$$E[Y_t] = E[\mu_t + X_t] = \mu_t + \mu_t = 0 + 0 = 0, \quad \text{and}$$
$$\mathrm{Cov}[Y_t] = \mathrm{Cov}[\mu_t + X_t, \mu_{t-k} + X_{t-k}] = \mathrm{Cov}[X_t, X_{t-k}] = \gamma_k$$

$$\triangledown^m Y_t = \triangledown(\triangledown^{m1} Y_t)$$

Currently unsolved.

2.10 Zero-mean, unit-variance process

a

$$\mu_t = E[Y_t] = E[\mu_t + \sigma_t X_t] = \mu_t + \sigma_t E[X_t] = \mu_t + \sigma_t \times 0 = \mu_t$$
$$\gamma_{t,t-k} = \mathrm{Cov}[Y_t] = \mathrm{Cov}[\mu_t + \sigma_t X_t, \mu_{t-k} + \sigma_{t-k} X_{t-k}] = \sigma_t \sigma_{t-k} \mathrm{Cov}[X_t, X_{t-k}] = \sigma_t \sigma_{t-k} \rho_k$$

b

First, we have

$$\mathrm{Var}[Y_t] = \mathrm{Var}[\mu_t + \sigma_t X_t] = 0 + \sigma_t^2 \mathrm{Var}[X_t] = \sigma_t^2 \times 1 = \sigma_t^2$$

since $\{X_t\}$ has unit-variance. Futhermore,

$$\mathrm{Corr}[Y_t, Y_{t-k}] = \frac{\sigma_t \sigma_{t-k} \rho_k}{\sqrt{\mathrm{Var}[Y_t]\mathrm{Var}[Y_{t-k}]}} = \frac{\sigma_t \sigma_{t-k} \rho_k}{\sigma_t \sigma_{t-k}} = \rho_k,$$

which depends only on the time lag, k. However, $\{Y_t\}$ is not necessarily stationary since μ_t may depend on t.

c

Yes, ρ_k might be free from t but if σ_t is not, we will have a non-stationary time series with autocorrelation free from t and constant mean.

2.11 Drift

a

$$\mathrm{Cov}[X_t, X_{t-k}] = \gamma_k$$
$$E[X_t] = 3t$$

$\{X_t\}$ is not stationary because μ_t varies with t.

b

$$E[Y_t] = 3 - 3t + E[X_t] = 7 - 3t - 3t = 7$$
$$\text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[7 - 3t + X_t, 7 - 3(t-k) + X_{t-k}] = \text{Cov}[X_t, X_{t-k}] = \gamma_k$$

Since the mean function of $\{Y_t\}$ is constant (7) and its autocovariance free of t, $\{Y_t\}$ is stionary.

2.12 Periods

$$E[Y_t] = E[e_t - e_{t-12}] = E[e_t] - E[e_{t-12}] = 0$$
$$\text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[e_t - e_{t-12}, e_{t-k} - e_{t-12-k}] =$$
$$\text{Cov}[e_t, e_{t-k}] - \text{Cov}[e_t, e_{t-12-k}] - \text{Cov}[e_{t-12}, e_{t-k}] + \text{Cov}[e_{t-12}, e_{t-12-k}]$$

Then, as required, we have

$$\text{Cov}[Y_t, Y_{t-k}] = \begin{cases} \text{Cov}[e_t, e_{t-12}] - \text{Cov}[e_t, e_t] - \\ \text{Cov}[e_{t-12}, e_{t-12}] + \text{Cov}[e_{t-12}, e_t] = \\ \text{Var}[e_t] - \text{Var}[e_{t-12}] \neq 0 & \text{for } k = 12 \\ \\ \text{Cov}[e_t, e_{t-k}] - \text{Cov}[e_t, e_{t-12-k}] - \\ \text{Cov}[e_{t-12}, e_{t-k}] + \text{Cov}[e_{t-12}, e_{t-12-k}] = \\ 0 + 0 + 0 + 0 = 0 & \text{for } k \neq 12 \end{cases}$$

2.13 Drift, part 2

a

$$E[Y_t] = E[e_t - \theta e_{t-1}^2] = E[e_t] - \theta E[e_{t-1}^2] = 0 - \theta \text{Var}[e_{t-1}] = -\theta \sigma_e^2$$

And thus the requirement of constant variance is fulfilled. Moreover,

$$\text{Var}[Y_t] = \text{Var}[e_t - \theta e_{t-1}^2] = \text{Var}[e_t] + \theta^2 \text{Var}[e_{t-1}^2] = \sigma_e^2 + \theta^2 (E[e_{t-1}^4] - E[e_{t-1}^2]^2),$$

where

$$E[e_{t-1}^4] = 3\sigma_e^4 \quad \text{and} \quad E[e_{t-1}^2]^2 = \sigma_e^4,$$

gives us

$$\text{Var}[Y_t] = \sigma_e^2 + \theta(3\sigma_e^4 - \sigma_e^2) = \sigma_e^2 + 2\theta^2\sigma_e^4$$

and

$$
\begin{gathered}
\text{Cov}[Y_t, Y_{t-1}] = \text{Cov}[e_t - \theta e_{t-1}^2, e_{t-1} - \theta e_{t-2}^2] = \\
\text{Cov}[e_t, e_{t-1}] + \text{Cov}[e_t, -\theta e_{t-2}^2] + \text{Cov}[-\theta e_{t-1}^2, e_{t-1}]\text{Cov}[-\theta e_{t-1}^2, -\theta e_{t-2}^2] = \\
\text{Cov}[e_t, e_{t-1}] - \theta\text{Cov}[e_t, e_{t-2}^2] - \theta\text{Cov}[e_{t-1}^2, e_{t-1}] + \theta^2\text{Cov}[e_{t-1}^2, e_{t-2}^2] = \\
-\theta\text{Cov}[e_{t-1}^2, e_{t-1}] = -\theta(E[e_{t-1}^3] + \mu_{t-1} + \mu_t) = 0
\end{gathered}
$$

which means that the autocorrelation function $\gamma_{t,s}$ also has to be zero.

b

The autocorrelation of $\{Y_t\}$ is zeor and its mean function is constant, thus $\{Y_t\}$ must be stationary.

2.14 Stationarity, again

a

$$
\begin{gathered}
E[Y_t] = E[\theta_0 + te_t] = \theta_0 + E[e_t] = \theta_0 + t \times 0 = \theta_0 \\
\text{Var}[Y_t] = \text{Var}[\theta_0] + \text{Var}[te_t] = 0 + t^2\sigma_e^2 = t^2\sigma_e^2
\end{gathered}
$$

So $\{Y_t\}$ is not stationary.

b

$$
\begin{gathered}
E[W_t] = E[\triangledown Y_t] = E[\theta_0 + te_t - \theta_0 - (t-1)e_{t-1}] = tE[e_t] - tE[e_{t-1} + E[e_{t-1}] = 0 \\
\text{Var}[\triangledown Y_t] = \text{Var}[te_t] = -\text{Var}[(t-1)e_{t-1}] = t^2\sigma_e^2 - (t-1)^2\sigma_e^2 = \sigma_e^2(t^2 - t^2 + 2t - 1) = (2t-1)\sigma_e^2,
\end{gathered}
$$

which varies with t and means that $\{W_t\}$ is not stationary.

c

$$
\begin{gathered}
E[Y_t] = E[e_t e_{t-1}] = E[e_t]E[e_{t-1}] = 0 \\
\text{Cov}[Y_t, Y_{t-1}] = \text{Cov}[e_t e_{t-1}, e_{t-1}e_{t-2}] = E[(e_t e_{t-1} - \mu_t^2)(e_{t-1}e_{t-2} - \mu_t^2)] = \\
E[e_t]E[e_{t-1}]E[e_{t-1}]E[e_{t-2}] = 0
\end{gathered}
$$

Both the covariance and the mean function are zero, hence the process is stationary.

2.15 Random variable, zero mean

a

$E[Y_t] = (-1)^t E[X] = 0$

b

$$\text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[(-1)^t X, (-1)^{t-k} X] = (-1)^{2t-k}\text{Cov}[X, X] = (-1)^k \text{Var}[X] = (-1)^k \sigma_t^2$$

c

Yes, the covariance is free of t and the mean is constant.

2.16 Mean and variance

$$E[Y_t] = E[A + X_t] = E[A] + E[X_t] = \mu_A + \mu_X$$
$$\text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[A + X_t, A + X_{t-k}] =$$
$$\text{Cov}[A, A] + \text{Cov}[A, X_{t-k}] + \text{Cov}[X_t, A] + \text{Cov}[X_t, X_{t-k}] = \sigma_A^2 + \gamma_{k_k}$$

2.17 Variance of sample mean

$$\text{Var}[\bar{Y}] = \text{Var}\left[\frac{1}{n}\sum_{t=1}^{n} Y_t\right] = \frac{1}{n^2}\text{Var}\left[\sum_{t=1}^{n} Y_t\right] =$$
$$\frac{1}{n^2}\text{Cov}\left[\sum_{t=1}^{n} Y_t, \sum_{s=1}^{n} Y_s\right] = \frac{1}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n}\gamma_{t-s}$$

Setting $k = t - s, j = t$ gives us

$$\text{Var}[\bar{Y}] = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{j-k=1}^{n}\gamma_k = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{j=k+1}^{n+k}\gamma_k =$$
$$\frac{1}{n^2}\left(\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}\gamma_k + \sum_{k=-n+1}^{0}\sum_{j=1}^{n+k}\gamma_k\right) =$$
$$\frac{1}{n^2}\left(\sum_{k=1}^{n-1}(n-k)\gamma_k + \sum_{k=-n+1}^{0}(n+k)\gamma_k\right) =$$
$$\frac{1}{n^2}\sum_{k=-n+1}^{n-1}((n-k)\gamma_k + (n+k)\gamma_k) =$$
$$\frac{1}{n^2}\sum_{k=-n+1}^{n-1}(n-|k|)\gamma_k = \frac{1}{n}\sum_{k=-n+1}^{n-1}\left(1 - \frac{|k|}{n}\right)\gamma_k \quad \square$$

2.18 Sample variance

a

$$\sum_{t=1}^{n}(Y_t-\mu)^2=\sum_{t=1}^{n}((Y_t-\bar{Y})+(\bar{Y}-\mu))^2=$$
$$\sum_{t=1}^{n}((Y_t-\bar{Y})^2-2(Y_t-\bar{Y})(\bar{Y}-\mu)+(\bar{Y}-\mu)^2)=$$
$$n(\bar{Y}-\mu)^2+2(\bar{Y}-\mu)\sum_{t=1}^{n}(Y_t-\bar{Y})+\sum_{t=1}^{n}(Y_t-\bar{Y})^2=$$
$$n(\bar{Y}-\mu)^2+\sum_{t=1}^{n}(Y_t-\bar{Y})^2 \quad \square$$

b

$$E[s^2]=E\left[\frac{n}{n-1}\sum_{t=1}^{n}(Y_t-\bar{Y})^2\right]=\frac{n}{n-1}E\left[\sum_{t=1}^{n}\left((Y_t-\mu)^2+n(\bar{Y}-\mu)^2\right)\right]=$$
$$\frac{n}{n-1}\sum_{t=1}^{n}\left(E[(Y_t-\mu)^2]+nE[(\bar{Y}-\mu)^2]\right)=\frac{1}{n-1}\left(n\mathrm{Var}[Y_t]-n\mathrm{Var}[\bar{Y}]\right)=$$
$$\frac{n}{n-1}\gamma_0-\frac{n}{n-1}\mathrm{Var}[\bar{Y}]=\frac{1}{n-1}\left(n\gamma_0-n\left(\frac{\gamma_0}{n}+\frac{2}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k\right)\right)=$$
$$\frac{1}{n-1}\left(n\gamma_0-\gamma_0+2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k\right)=\frac{1}{n-1}\left(\gamma_0(n-1)+2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k\right)=$$
$$\gamma_0+\frac{2}{n-1}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k \quad \square$$

c

Since $\gamma_k=0$ for $k\neq 0$, in our case for all k, we have

$$E[s^2]=\gamma_0-\frac{2}{n-1}\sum_{t=1}^{n}\left(1-\frac{k}{n}\right)\times 0=\gamma_0$$

2.19 Random walk with drift

a

$$Y_1=\theta_0+e_1$$
$$Y_2=\theta_0+\theta_0+e_2+e_1$$
$$Y_t=\theta_0+\theta_0+\cdots+\theta_0+e_t+e_{t-1}+\cdots+e_1=$$
$$Y_t=t\theta_0+e_t+e_{t-1}+\cdots+e_1 \quad \square$$

b

$$\mu_t = E[Y_t] = E[t\theta_0 + e_t + e_{t-1} + \cdots + e_1] = t\theta_0 + E[e_t] + E[e_{t-1}] + \cdots + E[e_1] =$$
$$t\theta_0 + 0 + 0 + \cdots + 0 = t\theta_0$$

c

$$\gamma_{t,t-k} = \mathrm{Cov}[Y_t, Y_{t-k}] = \mathrm{Cov}[t\theta_0 + e_t, +e_{t-1} + \cdots + e_1, (t-k)\theta_0 + e_{t-k}, +e_{t-1-k} + \cdots + e_1] =$$
$$\mathrm{Cov}[e_{t-k}, +e_{t-1-k} + \cdots + e_1, e_{t-k}, +e_{t-1-k} + \cdots + e_1] \quad \text{(since all other terms are 0)} =$$
$$\mathrm{Var}[e_{t-k}, +e_{t-1-k} + \cdots + e_1, e_{t-k}, +e_{t-1-k} + \cdots + e_1] = (t-k)\sigma_e^2$$

2.20 Random walk

a

$$\mu_1 = E[Y_1] = E[e_1] = 0$$
$$\mu_2 = E[Y_2] = E[Y_1 - e_2] = E[Y_1] - E[e_2] = 0 - 0 = 0$$
$$\ldots$$
$$\mu_{t-1} = E[Y_{t-1}] = E[Y_{t-2} - e_{t-1}] = E[Y_{t-2}] - E[e_{t-1}] = 0$$
$$\mu_t = E[Y_t] = E[Y_{t-1} - e_t] = E[Y_t] - E[e_t] = 0,$$

which implies $\mu_t = \mu_{t-1}$ Q.E.D.

b

$$\mathrm{Var}[Y_1] = \sigma_e^2$$
$$\mathrm{Var}[Y_2] = \mathrm{Var}[Y_1 - e_2] = \mathrm{Var}[Y_1] + \mathrm{Var}[e_1] = \sigma_e^2 + \sigma_e^2 = 2\sigma_e^2$$
$$\ldots$$
$$\mathrm{Var}[Y_{t-1}] = \mathrm{Var}[Y_{t-2} - e_{t-1}] = \mathrm{Var}[Y_{t-2}] + \mathrm{Var}[e_{t-1}] = (t-1)\sigma_e^2$$
$$\mathrm{Var}[Y_t] = \mathrm{Var}[Y_{t-1} - e_t] = \mathrm{Var}[Y_{t-1}] + \mathrm{Var}[e_t] = (t-1)\sigma_e^2 + \sigma_e^2 = t\sigma_e^2 \quad \square$$

c

$$\mathrm{Cov}[Y_t, Y_s] = \mathrm{Cov}[Y_t, Y_t + e_{t+1} + e_{t+2} + \cdots + e_s] = \mathrm{Cov}[Y_t, Y_t] = \mathrm{Var}[Y_t] = t\sigma_e^2$$

2.21 Random walk with random starting value

a

$$E[Y_t] = E[Y_0 + e_t + e_{t-1} + \cdots + e_1] =$$
$$E[Y_0] + E[e_t] + E[e_{t-1}] + E[e_{t-2}] + \cdots + E[e_1] =$$
$$\mu_0 + 0 + \cdots + 0 = \mu_0 \quad \square$$

b

$$\begin{gathered}\mathrm{Var}[Y_t] = \mathrm{Var}[Y_0 + e_t + e_{t-1} + \cdots + e_1] = \\ \mathrm{Var}[Y_0] + \mathrm{Var}[e_t] + \mathrm{Var}[e_{t-1}] + \cdots + \mathrm{Var}[e_1] = \\ \sigma_0^2 + t\sigma_e^2 \quad \square\end{gathered}$$

c

$$\begin{gathered}\mathrm{Cov}[Y_t, Y_s] = \mathrm{Cov}[Y_t, Y_t + e_{t+1} + e_{t+2} + \cdots + e_s] = \\ \mathrm{Cov}[Y_t, Y_t] = \mathrm{Var}[Y_t] = \sigma_0^2 + t\sigma_e^2 \quad \square\end{gathered}$$

d

$$\mathrm{Corr}[Y_t, Y_s] = \frac{\sigma_0^2 + t\sigma_e^2}{\sqrt{(\sigma_0^2 + t\sigma_e^2)(\sigma_0^2 + s\sigma_e^2)}} = \sqrt{\frac{\sigma_0^2 + t\sigma_e^2}{\sigma_0^2 + s\sigma_e^2}} \quad \square$$

2.22 Asymptotic stationarity

a

$$\begin{gathered}E[Y_1] = E[e_1] = 0 \\ E[Y_2] = E[cY_1 + e_2] = cE[Y_1] + E[e_2] = 0 \\ \cdots \\ E[Y_t] = E[cY_{t-1} + e_t] = cE[Y_{t-1}] + E[e_t] = 0 \quad \square\end{gathered}$$

b

$$\begin{gathered}\mathrm{Var}[Y_1] = \mathrm{Var}[e_1] = \sigma_e^2 \\ \mathrm{Var}[Y_2] = \mathrm{Var}[cY_1 + e_2] = c^2\mathrm{Var}[Y_{t-1}] + \mathrm{Var}[e_2] = c^2\sigma_e^2 + \sigma_e^2 = \sigma_e^2(1 + c^2) \\ \cdots \\ \mathrm{Var}[Y_t] = \sigma_e^2(1 + c^2 + c^4 + \cdots + c^{2t-2}) \quad \square\end{gathered}$$

$\{Y_t\}$ is not stationary, given that its variance varies with t.

c

$$\begin{gathered}\mathrm{Cov}[Y_t, Y_{t-1}] = \mathrm{Cov}[cY_{t-1} + e_t, Y_{t-1}] = c\mathrm{Cov}[Y_{t-1}, Y_{t-1}] = c\mathrm{Var}[Y_{t-1}] \quad \text{giving} \\ \mathrm{Corr}[Y_t, Y_{t-1}] = \frac{c\mathrm{Var}[Y_{t-1}]}{\sqrt{\mathrm{Var}[Y_t]\mathrm{Var}[Y_{t-1}]}} = c\sqrt{\frac{\mathrm{Var}[Y_{t-1}]}{\mathrm{Var}[Y_t]}} \quad \square\end{gathered}$$

And, in the general case,

$$\begin{gathered}
\mathrm{Cov}[Y_t, Y_{t-k}] = \mathrm{Cov}[cY_{t-1} + e_t, Y_{t-k}] = \\
c\mathrm{Cov}[cY_{t-2} + e_{t-1}, Y_{t-k}] = \\
c^3\mathrm{Cov}[Y_{t-2} + e_{t-1}, Y_{t-k}] = \dots \\
= c^k \mathrm{Var}[Y_{t-k}]
\end{gathered}$$

giving

$$\mathrm{Corr}[Y_t, Y_{t-k}] = \frac{c^k \mathrm{Var}[Y_{t-k}]}{\sqrt{\mathrm{Var}[Y_t]\mathrm{Var}[Y_{t-k}]}} = c^k \sqrt{\frac{\mathrm{Var}[Y_{t-k}]}{\mathrm{Var}[Y_t]}} \quad \square$$

d

$$\mathrm{Var}[Y_t] = \sigma_e^2(1 + c^2 + c^4 + \dots + c^{2t-2}) = \sigma_e^2 \sum_{t=1}^{n} c^{2(t-1)} = \sigma_e^2 \sum_{t=0}^{n-1} c^{2t} = \sigma_e^2 \frac{1-c^{2t}}{1-c^2}$$

And because

$$\lim_{t\to\infty} \sigma_e^2 \frac{1-c^{2t}}{1-c^2} = \sigma_e^2 \frac{1}{1-c^2} \quad \text{since } |c| < 1,$$

which is free of t, $\{Y_t\}$ can be considered *asymptotically* stationary.

e

$$\begin{gathered}
Y_t = c(cY_{t-2} + e_{t-1}) + e_t = \dots = e_t + ce_{t-1} + c^2 e_{t-2} + \dots + c^{t-2} e_2 + \frac{c^{t-1}}{\sqrt{1-c^2}} e_1 \\
\mathrm{Var}[Y_t] = \mathrm{Var}[e_t + ce_{t-1} + c^2 e_{t-2} + \dots + c^{t-2} e_2 + \frac{c^{t-1}}{\sqrt{1-c^2}} e_1] = \\
\mathrm{Var}[e_t] + c^2\mathrm{Var}[e_{t-1}] + c^4\mathrm{Var}[e_{t-2}] + \dots + c^{2(t-2)}\mathrm{Var}[e_2] + \frac{c^{2(t-1)}}{1-c^2}\mathrm{Var}[e_1] = \\
\sigma_e^2(1 + c^2 + c^4 + \dots + c^{2(t-2)} + \frac{c^{2(t-1)}}{1-c^2}) = \sigma_e^2 \left(\sum_{t=1}^{n} c^{2(t-1)} - c^{2(t-1)} + \frac{c^{2(t-1)}}{1-c^2} \right) = \\
\sigma_e^2 \frac{1 - c^{2t} + c^{2t-2+2}}{1-c^2} = \sigma_e^2 \frac{1}{1-c^2} \quad \square
\end{gathered}$$

Futhermore,

$$\begin{gathered}
E[Y_1] = E\left[\frac{e_1}{\sqrt{1-c^2}}\right] = \frac{E[e_1]}{\sqrt{1-c^2}} = 0 \\
E[Y_2] = E[cY_1 + e_2] = cE[Y_1] = 0 \\
\dots \\
E[Y_t] = E[cY_{t-1} + e_2] = cE[Y_{t-1}] = 0,
\end{gathered}$$

which satisfies our first requirement for weak stationarity. Also,

$$\mathrm{Cov}[Y_t, Y_{t-k}] = \mathrm{Cov}[cY_{t-1} + e_t, Y_{t-1}] = c^k \mathrm{Var}[Y_{t-1}] =$$
$$c^k \frac{\sigma_e^2}{1 - c^2},$$

which is free of t and hence $\{Y_t\}$ is now stationary.

2.23 Stationarity in sums of stochastic processes

$$E[W_t] = E[Z_t + Y_t] = E[Z_t] + Y[Z_t] = \mu_{Z_t} + \mu_{Y_s}$$

Since both processes are stationary – and hence their sums are constant – the sum of both processes must also be constant.

$$\begin{gathered}
\mathrm{Cov}[W_t, W_{t-k}] = \mathrm{Cov}[Z_t + Y_t, Z_{t-k} + Y_{t-k}] = \\
\mathrm{Cov}[Z_t, Z_{t-k}] + \mathrm{Cov}[Z_t, Y_{t-k}] + \mathrm{Cov}[Y_t, Z_{t-k}] + \mathrm{Cov}[Y_t, Y_{t-k}] = \\
\mathrm{Cov}[Z_t, Z_{t-k}] + \mathrm{Cov}[Z_t, Y_{t-k}] + \mathrm{Cov}[Y_t, Z_{t-k}] + \mathrm{Cov}[Y_t, Y_{t-k}] = \mathrm{Cov}[Z_t, Z_{t-k}] + \mathrm{Cov}[Y_t, Y_{t-k}] = \gamma_{Z_k} + \gamma_{Y_k},
\end{gathered}$$

both free of t.

2.24 Measurement noise

$$\begin{gathered}
E[Y_t] = E[Y_t + e_t] = E[X_t] + E[e_t] - \mu_t \\
\mathrm{Var}[Y_t] = \mathrm{Var}[X_t + e_t] = \mathrm{Var}[X_t] + \mathrm{Var}[e_t] = \sigma_X^2 + \sigma_e^2 \\
\mathrm{Cov}[Y_t, Y_{t-k}] = \mathrm{Cov}[X_t + e_t, X_{t-k} + e_{t-k}] = \mathrm{Cov}[X_t, X_{t-k}] = \rho_k \\
\mathrm{Corr}[Y_t, Y_{t-k}] = \frac{\rho_k}{\sqrt{(\sigma_X^2 + \sigma_e^2)(\sigma_X^2 + \sigma_e^2)}} = \frac{\rho_k}{\sigma_X^2 + \sigma_e^2} = \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_X^2}} \quad \square
\end{gathered}$$

2.25 Random cosine wave

$$
\begin{gathered}
E[Y_t] = E\left[\beta_0 + \sum_{i=1}^{k}(A_i \cos(2\pi f_i t) + B_i \sin(2\pi f_i t))\right] = \\
\beta_0 + \sum_{i=1}^{k}(E[A_i]\cos(2\pi f_i t) + E[B_i]\sin(2\pi f_i t) = \beta_0 \\
\operatorname{Cov}[Y_t, Y_s] = \operatorname{Cov}\left[\sum_{i=1}^{k} A_i \cos(2\pi f_i t) + B_i \sin(2\pi f_i t), \sum_{j=1}^{k} A_j \cos(2\pi f_j s) + B_j \sin(2\pi f_j s)\right] = \\
\sum_{i=1}^{k}\operatorname{Cov}[A_i \cos(2\pi f_i t) + A_i \sin(2\pi f_i s)] + \sum_{i=1}^{k}\operatorname{Cov}[B_i \cos(2\pi f_j t) + B_i \sin(2\pi f_j s)] = \\
\sum_{i=1}^{k}\operatorname{Var}[A_i](\cos(2\pi f_i t) + \sin(2\pi f_i s)) + \sum_{i=1}^{k}\operatorname{Var}[B_i](\cos(2\pi f_j t) + \sin(2\pi f_j s)) = \\
\frac{\sigma_i^2}{2}\sum_{i=1}^{k}(\cos(2\pi f_i(t-s)) + \sin(2\pi f_i(t+s))) + \frac{\sigma_i^2}{2}\sum_{i=1}^{k}(\cos(2\pi f_j(t-s)) + \sin(2\pi f_j(t+s))) = \\
\sigma_i^2 \sum_{i=1}^{k}\cos(2\pi f_i(t-s)) = \sigma_i^2 \sum_{i=1}^{k}\cos(2\pi f_i k),
\end{gathered}
$$

and is thus free of t and s.

2.26 Semivariogram

a

$$
\begin{gathered}
\Gamma_{t,s} = \frac{1}{2}E[(Y_t - Y_s)^2] = \frac{1}{2}E[Y_t^2 - 2Y_tY_s + Y_s^2] = \\
\frac{1}{2}\left(E[Y_t^2] - 2E[Y_tY_s] + E[Y_s^2]\right) = \frac{1}{2}\gamma_0 + \frac{1}{2}\gamma_0 - 2\times\frac{1}{2}\gamma_{|t-s|} = \gamma_0 - \gamma_{|t-s|} \\
\operatorname{Cov}[Y_t, Y_s] = E[Y_tY_s] - \mu_t\mu_s = E[Y_tY_s] = \gamma_{|t-s|} \quad \square
\end{gathered}
$$

b

$$
\begin{gathered}
Y_t - Y_s = e_t + e_{t-1} + \cdots + e_1 - e_s - e_{s-1} - \cdots - e_1 = \\
e_t + e_{t-1} + \cdots + e_{s+1}, \quad \text{for } t > s \\
\Gamma_{t,s} = \frac{1}{2}E[(Y_t - Y_s)^2] = \frac{1}{2}\operatorname{Var}[e_t + e_{t-1} + \cdots + e_{s-1}] = \\
\frac{1}{2}\sigma_e^2(t-s) \quad \square
\end{gathered}
$$

2.27 Polynomials

a

$$
\begin{gathered}
E[Y_t] = E[e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^r e_{t-r}] = 0 \\
\mathrm{Cov}[Y_t, Y_{t-k}] = \mathrm{Cov}[e_t + \phi e_{t-1} + \cdots + \phi^r e_{t-r}, e_{t-k} + \phi e_{t-1-k} + \cdots + \phi^r e_{t-r-k}] = \\
\mathrm{Cov}[e_1 + \cdots + \phi^k e_{t-k} + \phi^{k+1} e_{t-k-1} + \cdots + \phi^r e_{t-r}, e_{t-r}, e_{t-k} + \cdots + \phi^k e_{t-k-1} + \cdots + \phi^r e_{t-k-r}] = \\
\sigma_e^2(\phi^k + \phi^{k+2} + \phi^{k+4} + \cdots + \phi^{k+2(r-k)}) = \sigma_e^2 \phi^k (1 + \phi^2 + \phi^4 + \cdots + \phi^{2(r-k)})
\end{gathered}
$$

Hence, because of the zero mean and covariance free of t, it is a stationary process.

b

$$
\begin{gathered}
\mathrm{Var}[Y_t] = \mathrm{Var}[e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^r e_{t-r}] = \sigma_e^2(1 + \phi + \phi^2 + \cdots + \phi^{2r}) \\
\mathrm{Corr}[Y_t, Y_{t-k}] = \frac{\sigma_e^2 \phi^k (1 + \phi^2 + \phi^4 + \cdots + \phi^{2(r-k)})}{\sqrt{(\sigma_e^2(1 + \phi + \phi^2 + \cdots + \phi^{2r}))^2}} = \frac{\phi^k (1 + \phi^2 + \phi^4 + \cdots + \phi^{2(r-k)})}{(1 + \phi + \phi^2 + \cdots + \phi^{2r})} \quad \square
\end{gathered}
$$

2.28 Random cosine wave extended

a

$$
\begin{gathered}
E[Y_t] = E[R\cos(2\pi(ft + \phi))] = E[R]\cos(2\pi(ft + \phi)) = \\
E[R]\int_0^1 \cos(E[R\cos(2\pi(ft + \phi))])d\phi = E[R]\left[\frac{1}{2\pi}\sin(2\pi(ft + \phi))\right]_0^1 = \\
E[R]\left(\frac{1}{2\pi}(\sin(2\pi(ft + 1)) - \sin(2\pi(ft)))\right) = \\
E[R]\left(\frac{1}{2\pi}(\sin(2\pi ft + 2\pi) - \sin(2\pi ft + 1))\right) = \\
E[R]\,(0) = 0
\end{gathered}
$$

b

$$
\begin{gathered}
\gamma_{t,s} = E[R\cos(2\pi(ft + \phi))R\cos(2\pi(fs + \phi))] = \\
\frac{1}{2}E[R^2]\int_0^1 \left(\cos(2\pi(f(t-s)) + \frac{1}{4\pi}\sin(2\pi(f(t+s) + 2\phi))\right) = \\
\frac{1}{2}E[R^2]\left[\cos(2\pi f(t-s)) + \frac{1}{4\pi}\sin(2\pi(f(t+s) + 2\phi))\right]_0^1 = \\
\frac{1}{2}E[R^2]\left(\cos(2\pi(f|t-s|))\right),
\end{gathered}
$$

which is free of t.

2.29 Random cosine wave further

a

$$E[Y_t] = \sum_{j=1}^{m} E[R_j] E[\cos(2\pi(f_j t + \phi))] = \text{via } 2.28 = \sum_{j=1}^{m} E[R_j] \times 0 = 0$$

b

$$\gamma_k = \sum_{j=1}^{m} E[R_j] \cos(2\pi f_j k), \text{ also from } 2.28.$$

2.30 Rayleigh distribution

$$Y = R\cos(2\pi(ft+\phi)), \quad X = R\sin(2\pi(ft+\phi))$$
$$\begin{bmatrix} \frac{\partial X}{\partial R} & \frac{\partial X}{\partial \Phi} \\ \frac{\partial Y}{\partial R} & \frac{\partial X}{\partial \Phi} \end{bmatrix} = \begin{bmatrix} \cos(2\pi(ft+\Phi)) & 2\pi R\sin(2\pi(ft+\Phi)) \\ \sin(2\pi(ft+\Phi)) & 2\pi R\cos(2\pi(ft+\Phi)) \end{bmatrix},$$

with jacobian

$$-2\pi R = -2\pi\sqrt{X^2+Y^2}$$

and inverse Jacobian

$$\frac{1}{-2\pi\sqrt{X^2+Y^2}}.$$

Furthermore,

$$f(r,\Phi) = re^{-r^2/2}$$

and

$$f(x,y) = \frac{e^{-(x^2+y^2)/2}\sqrt{x^2+y^2}}{2\pi\sqrt{x^2+y^2}} = \frac{e^{-x^2/2}}{\sqrt{2\pi}}\frac{e^{-y^2/2}}{\sqrt{2\pi}} \quad \square$$

Chapter 3

Trends