Solutions to Time Series Analysis: with Applications in R

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Preface

This book contains solutions to the problems in the book *Time Series Analysis: with Applications in R*, third edition, by Cryer and Chan. It is provided as a github repository so that anybody may contribute to its development. Unlike the book, the solutions here use lattice graphics when possible instead of base graphics.

Chapter 1

Introduction

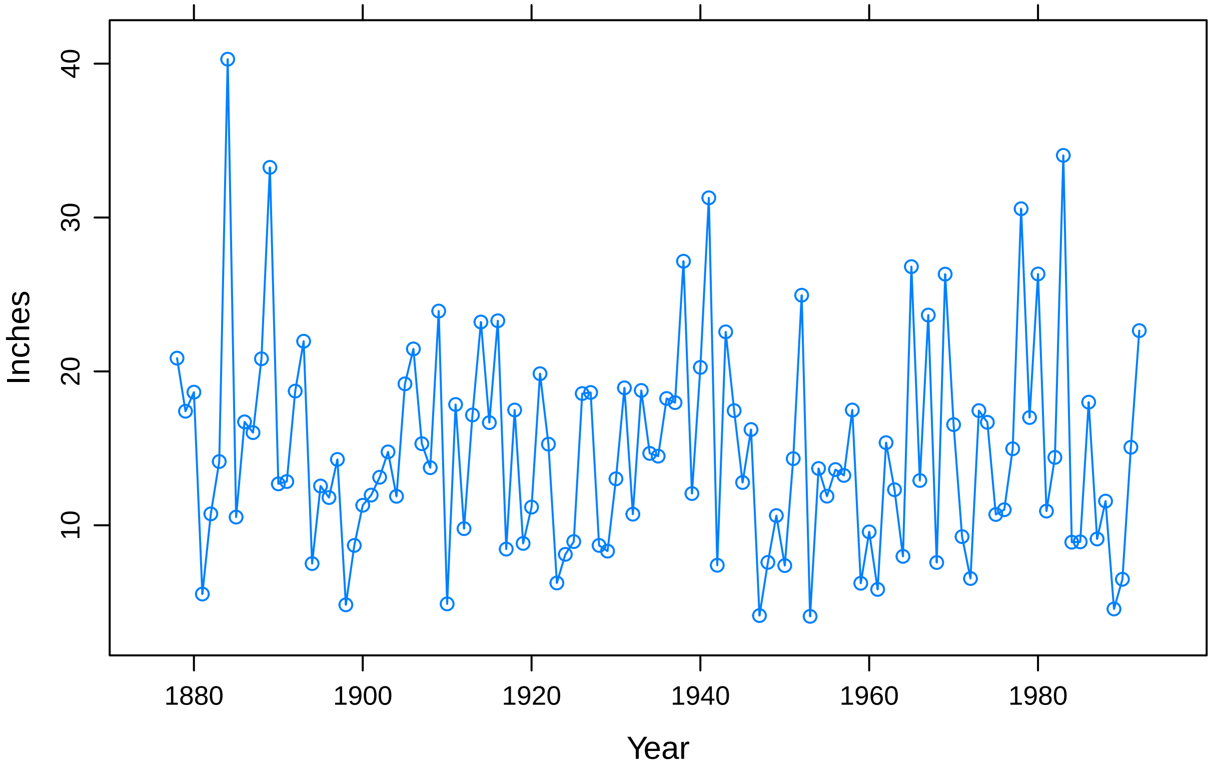
1.1 Larain

Use software to produce the time series plot shown in Exhibit 1.2, on page 2. The data are in the file named larain.

```r
library(TSA)
library(latticeExtra)

data(larain, package = "TSA")
```

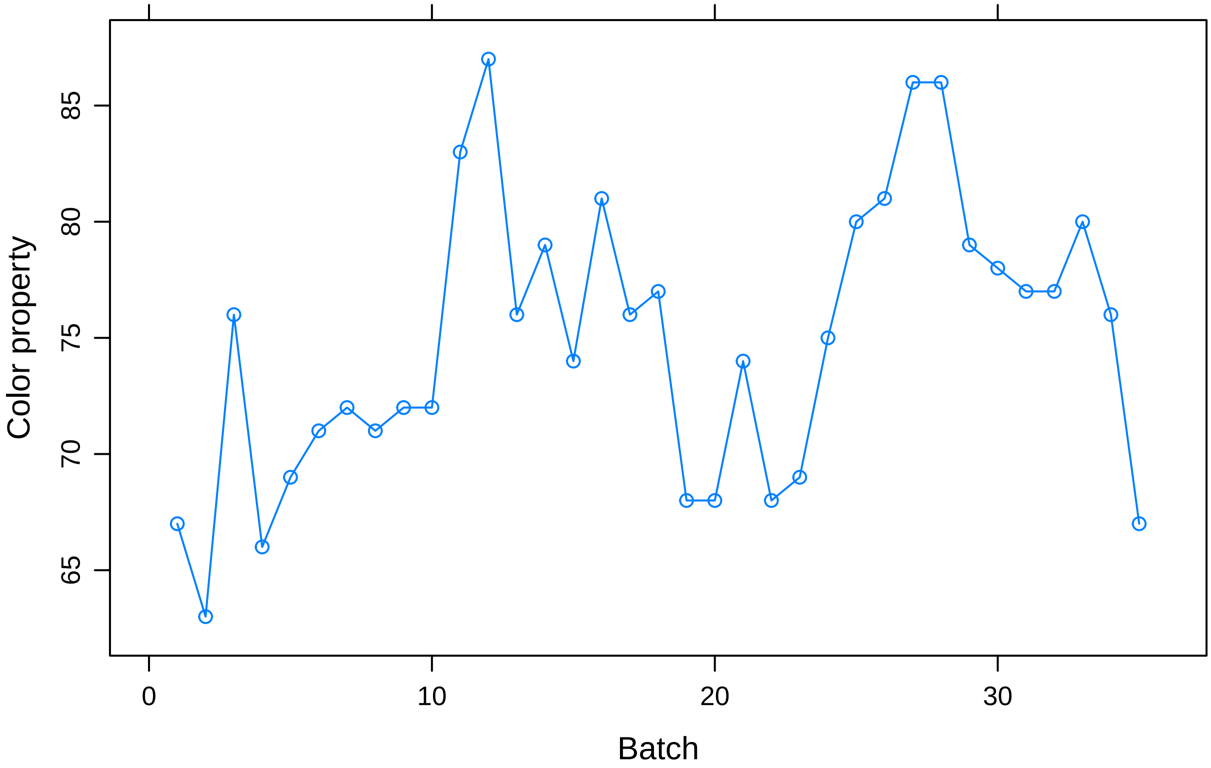
```r
xyplot(larain, ylab = "Inches", xlab = "Year", type = "o")
```



1.2 Colors

Produce the time series plot displayed in Exhibit 1.3, on page 3. The data file is named color.

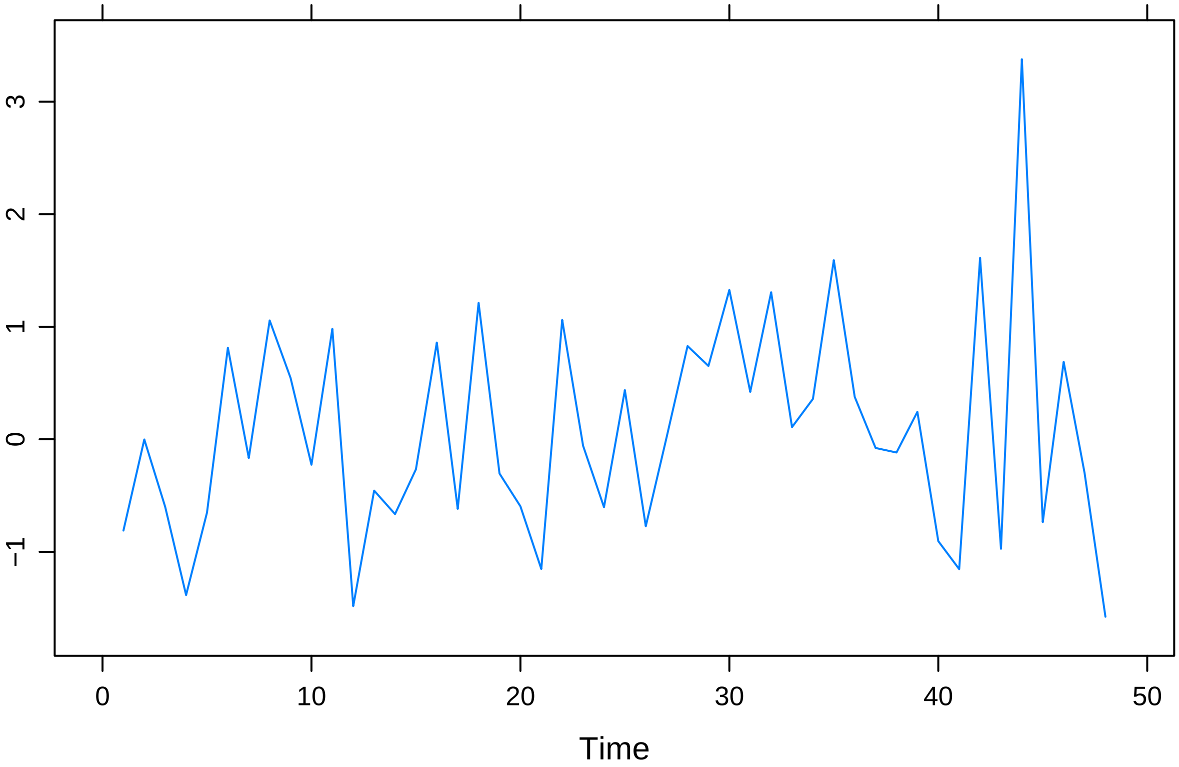
```
data(color)
xyplot(color, ylab = "Color property", xlab = "Batch", type = "o")
```

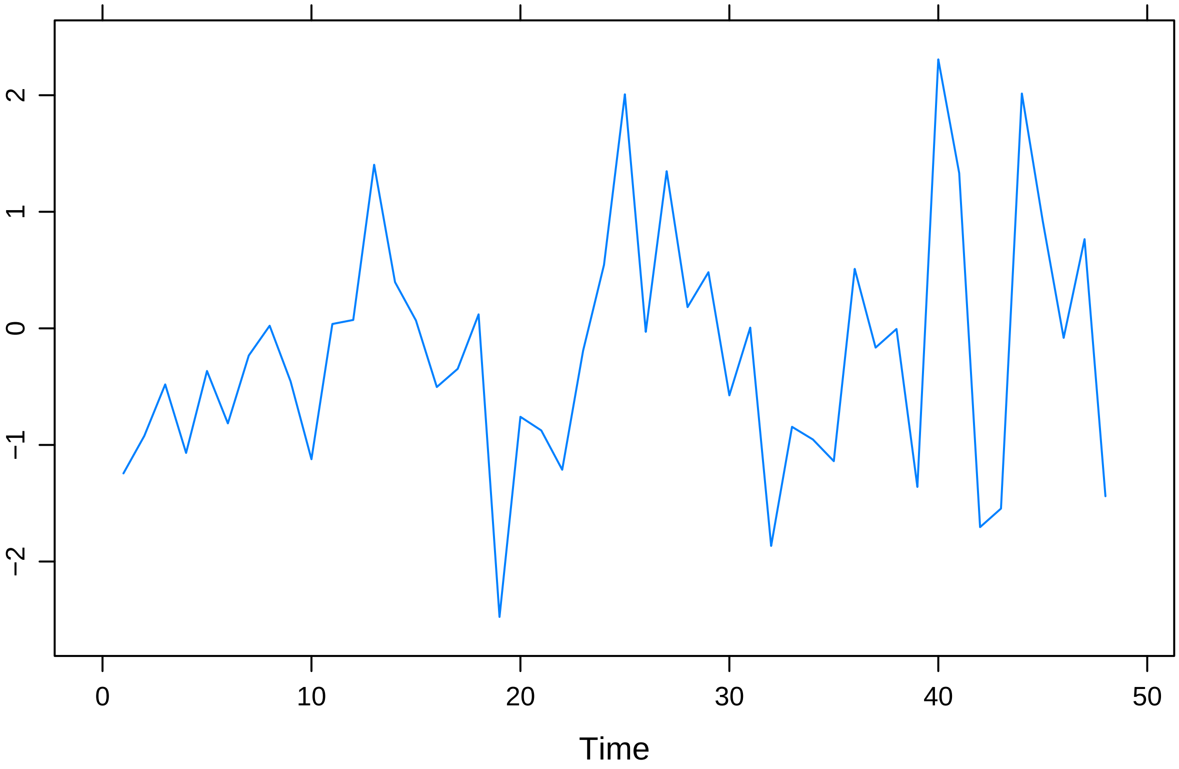


1.3 Random, normal time series

Simulate a completely random process of length 48 with independent, normal values. Plot the time series plot. Does it look “random”? Repeat this exercise several times with a new simulation each time.

```
xyplot(as.ts(rnorm(48)))
xyplot(as.ts(rnorm(48)))
```



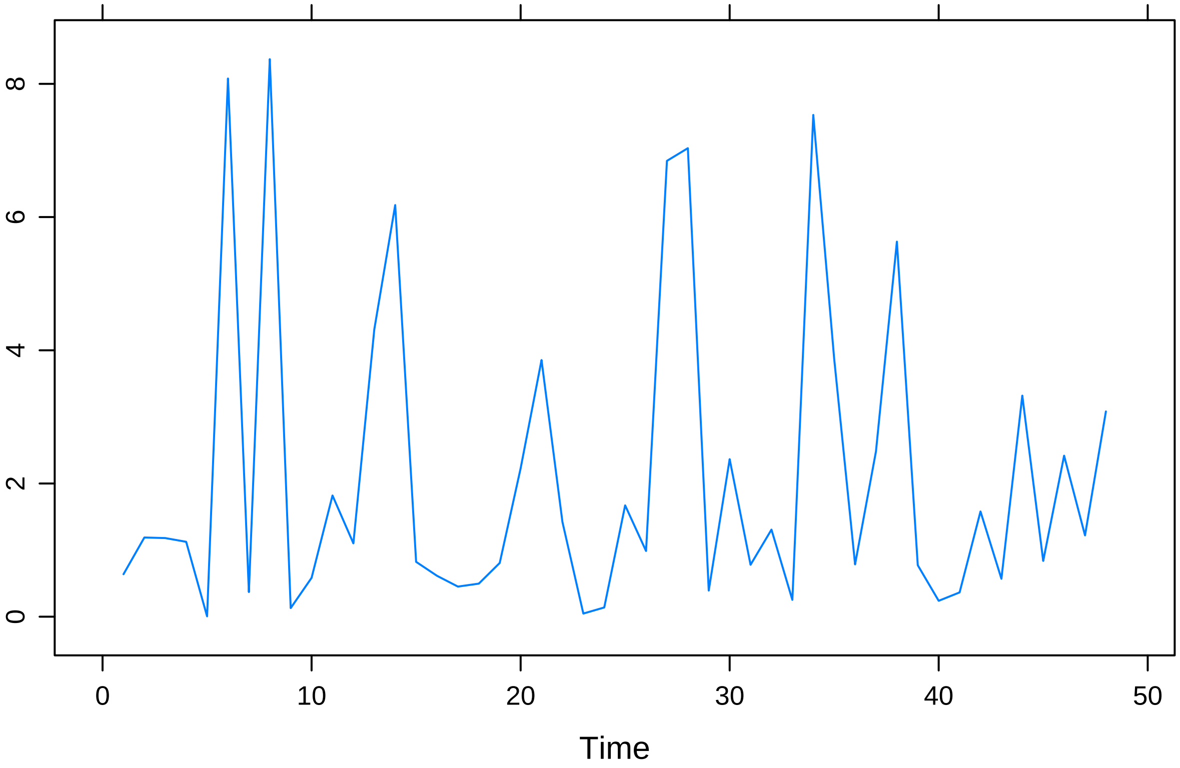


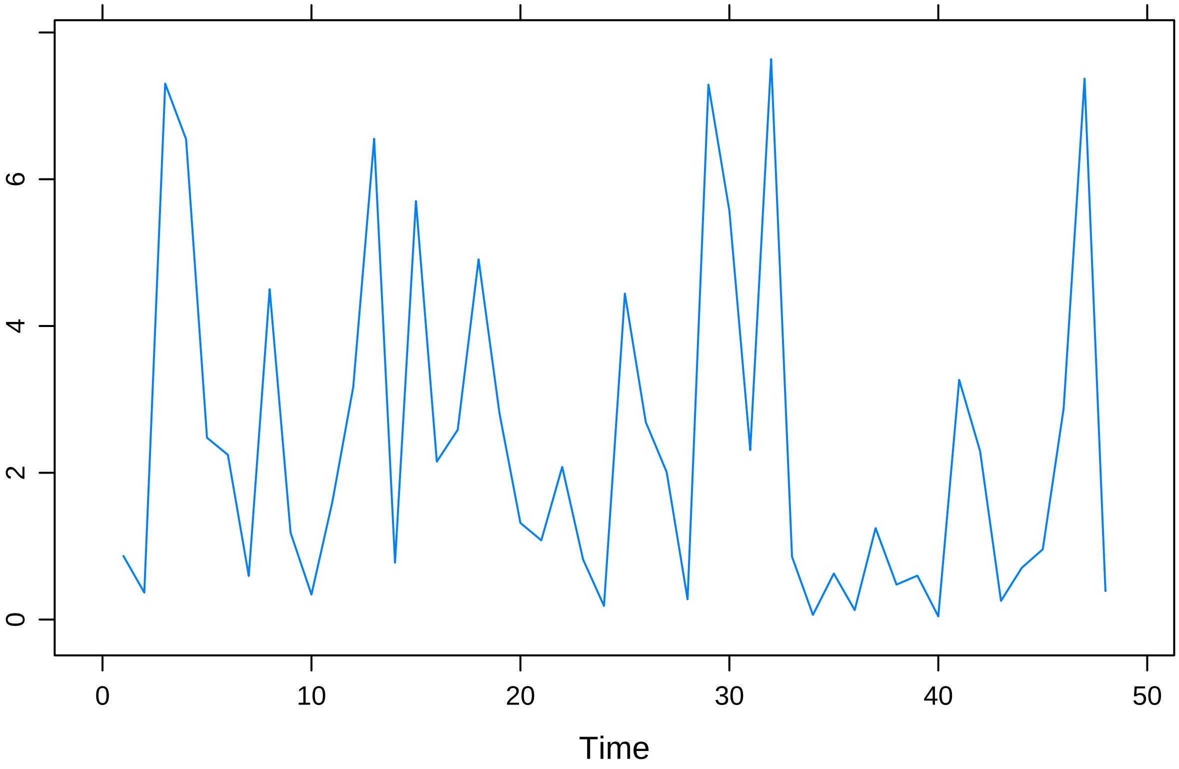
As far as we can tell there is no discernable pattern here.

1.4 Random, χ^2-distributed time series

Simulate a completely random process of length 48 with independent, chi-square distributed values, each with 2 degrees of freedom. Display the time series plot. Does it look “random” and nonnormal? Repeat this exercise several times with a new simulation each time.

```r
xyplot(as.ts(rchisq(48, 2)))
xyplot(as.ts(rchisq(48, 2)))
```



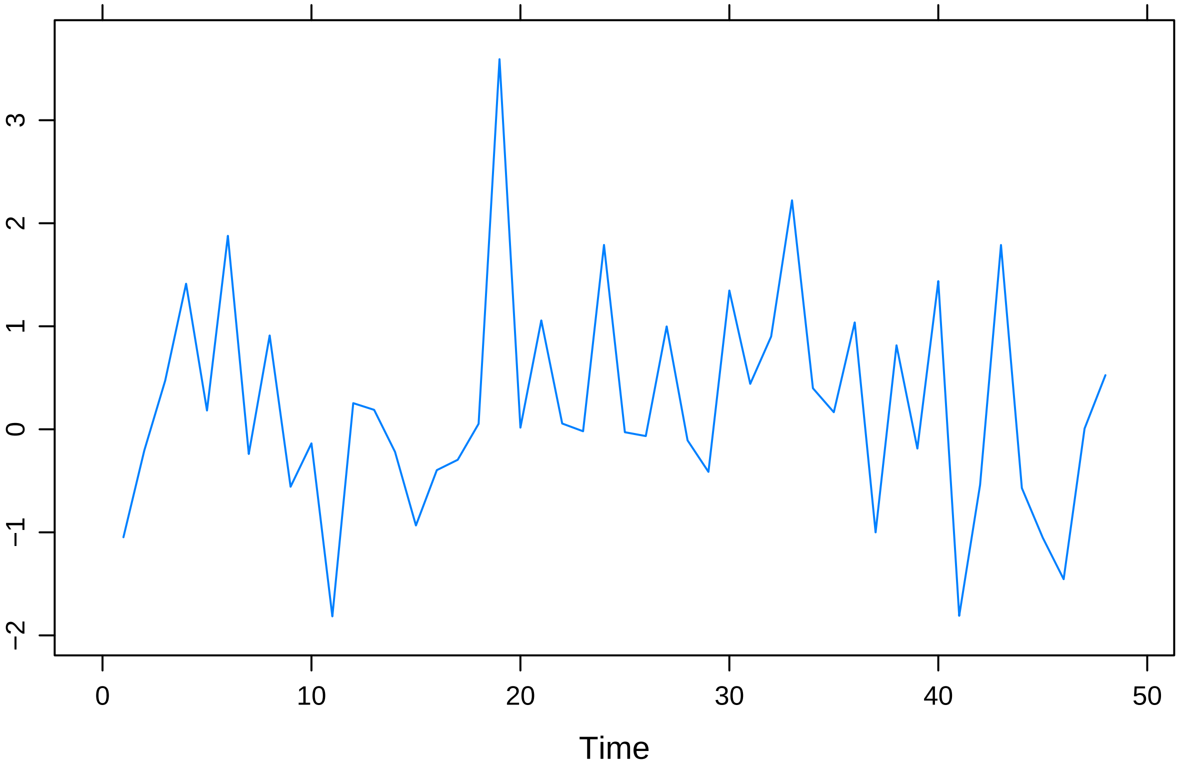


The process appears random, though non-normal.

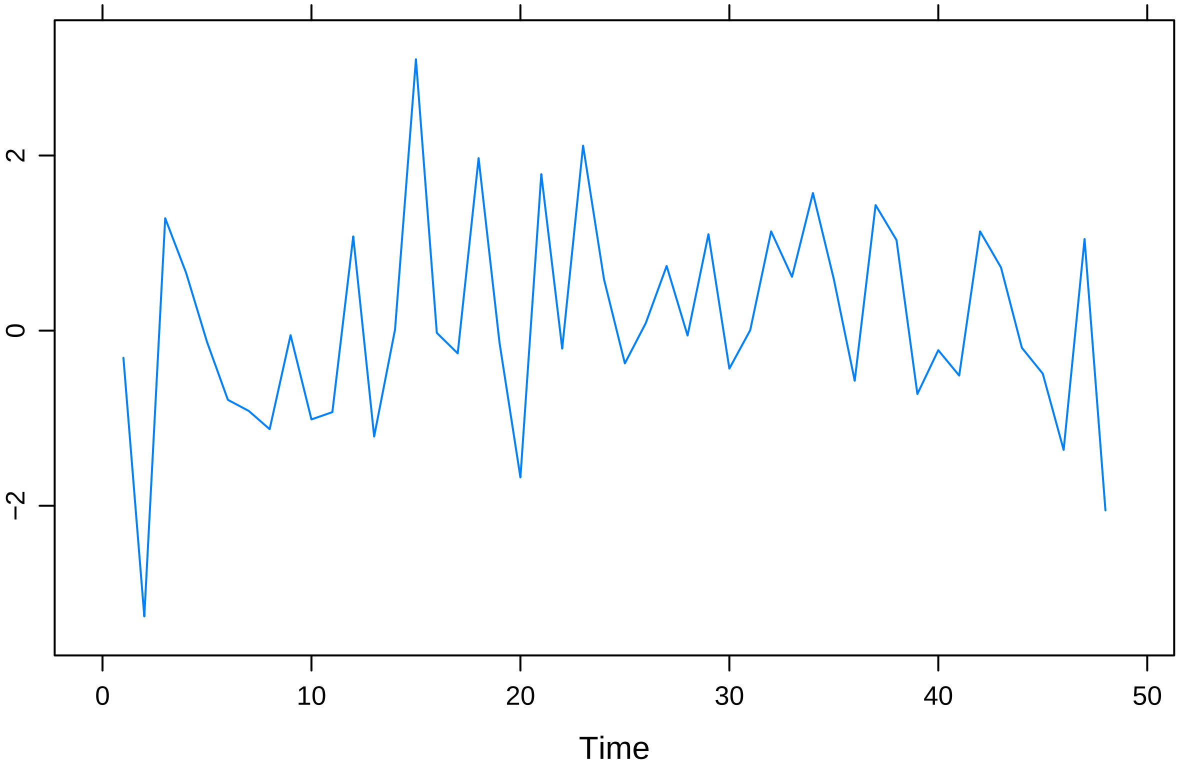
1.5 $t(5)$-distributed, random values

Simulate a completely random process of length 48 with independent, t-distributed values each with 5 degrees of freedom. Construct the time series plot. Does it look “random” and nonnormal? Repeat this exercise several times with a new simulation each time.

```
xyplot(as.ts(rt(48, 5)))
```



```
xyplot(as.ts(rt(48, 5)))
```

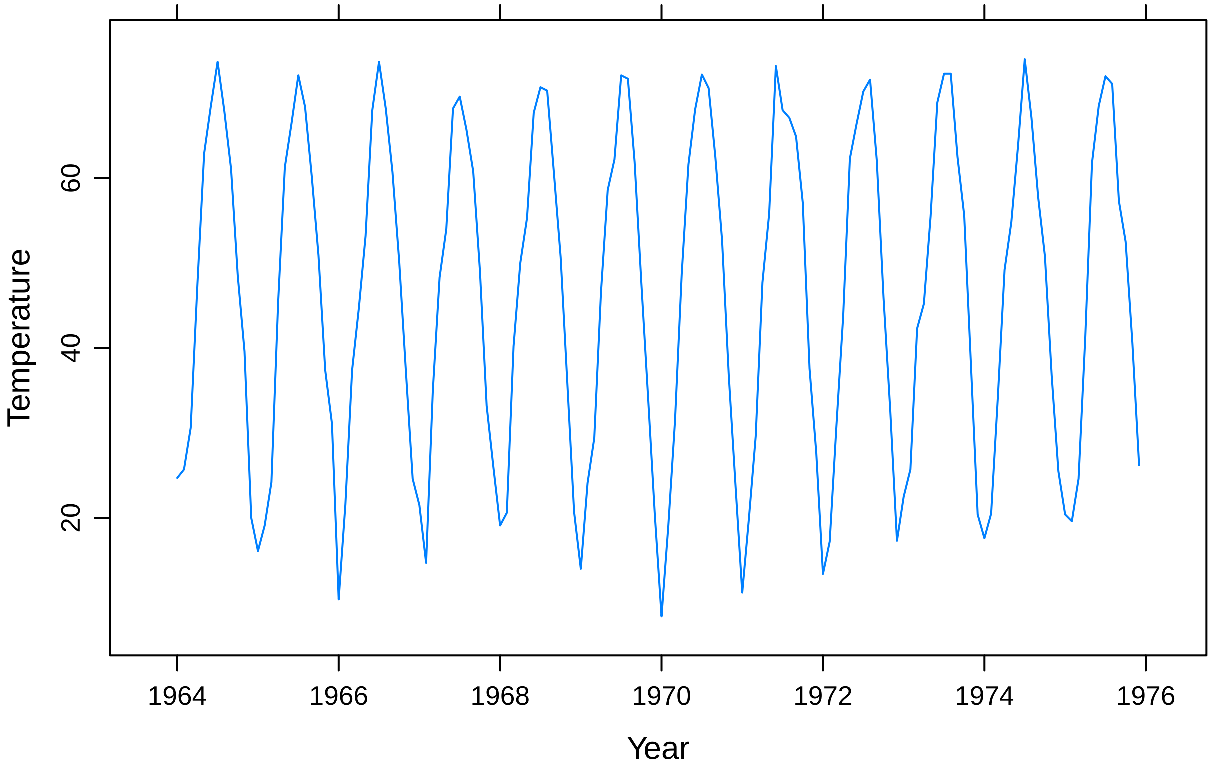


It looks random but not normal, though it should be approximately so, considering the distribution that we have sampled from.

1.6 Dubuque temperature series

Construct a time series plot with monthly plotting symbols for the Dubuque temperature series as in Exhibit 1.7, on page 6. The data are in the file named tempdub.

```
data(tempdub)
xyplot(tempdub, ylab = "Temperature", xlab = "Year")
```



Chapter 2

Fundamental concepts

2.1 Basic properties of expected value and covariance

(a)

$$
\begin{align}
\text{Cov}[X,Y] &= \text{Corr}[X,Y]\sqrt{Var[X]Var[Y]} \tag{2.1}\\
&= 0.25\sqrt{9\times 4} = 1.5 \tag{2.2}\\
\text{Var}[X,Y] &= Var[X] + Var[Y] + 2Cov[X,Y] \tag{2.3}\\
&= 9 + 4 + 2\times 3 = 16 \tag{2.4}\\
\tag{2.5}
\end{align}
$$

(b)

$$\text{Cov}[X, X+Y] = \text{Cov}[X,X] + \text{Cov}[X,Y] = \text{Var}[X] + \text{Cov}[X,Y] = 9 + 1.5 = 10.5$$

(c)

$$
\begin{align}
\text{Corr}[X+Y, X-Y] =& \text{Corr}[X,X] + \text{Corr}[X,-Y] + \text{Corr}[Y,X] + \text{Corr}[Y,-Y] \tag{2.6}\\
=& \text{Corr}[Y,X] + \text{Corr}[Y,-Y] \tag{2.7}\\
=& 1 - 0.25 + 0.25 - 1 \tag{2.8}\\
=& 0 \tag{2.9}\\
\tag{2.10}
\end{align}
$$

2.2 Dependence and covariance

$\text{Cov}[X+Y, X-Y] = \text{Cov}[X,X] + \text{Cov}[X,-Y] + \text{Cov}[Y,X] + \text{Cov}[Y,-Y] = Var[X] - Cov[X,Y] + Cov[X,Y] - Var[Y] = 0$ since $Var[X] = Var[Y]$.

2.3 Strict and weak stationarity

(a) We have that

$$P(Y_{t_1}, Y_{t_2}, \dots, Y_{t_n}) = P(X_1, X_2, \dots, X_n) = P(Y_{t_1-k}, Y_{t_2-k}, \dots, Y_{t_n-k}),$$

which satisfies our requirement for strict stationarity.

(b) The autocovariance is given by

$$\gamma_{t,s} = \text{Cov}[Y_t, Y_s] = \text{Cov}[X, X] = \text{Var}[X] = \sigma^2.$$

(c)

```r
library(lattice)
tstest <- ts(runif(100))

lattice::xyplot(tstest,
                panel = function(x, y, ...) {
                  panel.abline(h = mean(y), lty = 2)
                  panel.xyplot(x, y, ...)
                })
```

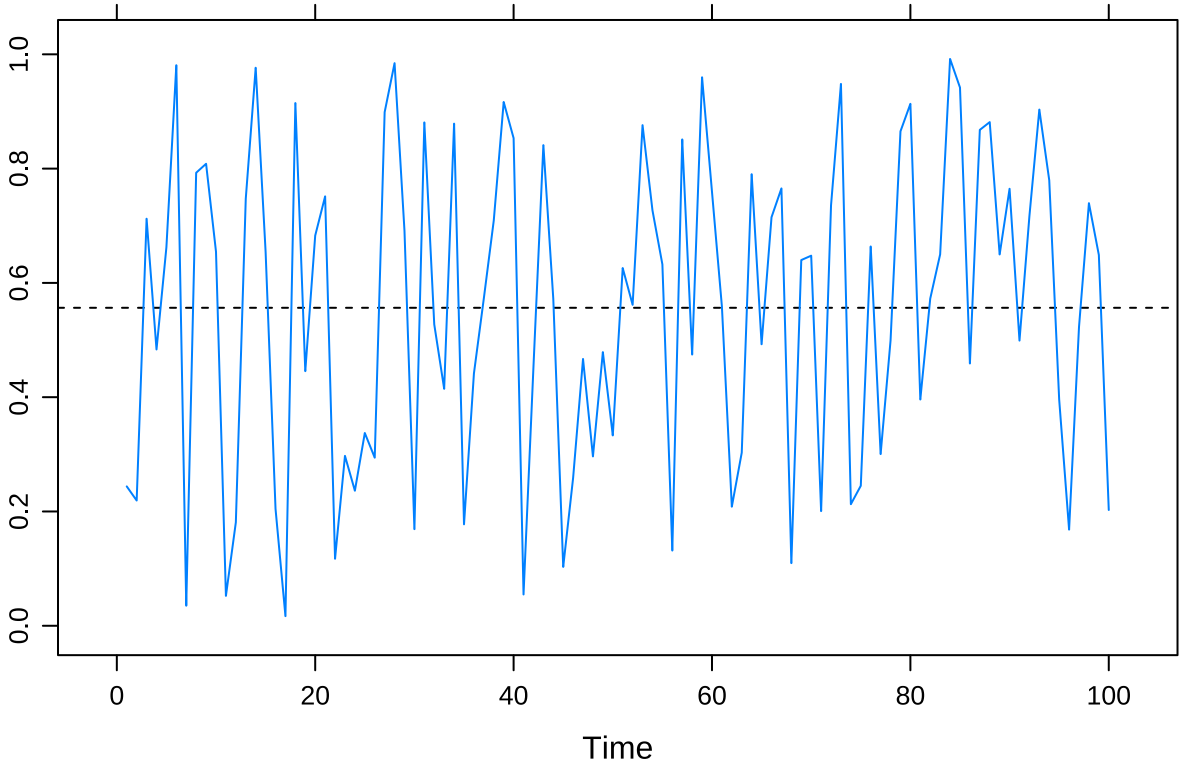


Figure 2.1: A white noise time series: no drift, independence between observations.

2.4 Zero-mean white noise

(a)

$$E[Y_t] = E[e_t+\theta e_{t-1}] = E[e_t]+\theta E[e_{t-1}] = 0+0 = 0 V[Y_t] = V[e_t+\theta e_{t-1}] = V[e_t]+\theta^2 V[e_{t-1}] = \sigma_e^2+\theta^2\sigma_e^2 = \sigma_2^2(1+\theta^2)$$

For $k = 1$ we have

$$C[e_t+\theta e_{t-1}, e_{t-1}+\theta e_{t-2}] = C[e_t, e_{t-1}]+C[e_t, \theta e_{t-2}]+C[\theta e_{t-1}, e_{t-1}]+C[\theta e_{t-1}, \theta e_{t-2}] = 0+0+\theta V[e_{t-1}]+0 = \theta\sigma_e^2, \text{Corr}[Y_t$$

and for $k = 0$ we get

$$\text{Corr}[Y_t, Y_{t-k}] = \text{Corr}[Y_t, Y_t] = 1$$

and, finally, for $k > 0$:

$$C[e_t + \theta e_{t-1}, e_{t-k} + \theta e_{t-k-1}] = C[e_t, e_{t-k}] + C[e_t, e_{t-1-k}] + C[\theta e_{t-1}, e_{t-k}] + C[\theta e_{t-1}, \theta e_{t-1-k}] = 0$$

given that all terms are independent. Taken together, we have that

$$\text{Corr}[Y_t, Y_{t-k}] = \begin{cases} 1 & \text{for } k = 0 \\ \frac{\theta}{1+\theta^2} & \text{for } k = 1 \\ 0 & \text{for } k > 1 \end{cases}.$$

And, as required,

$$\text{Corr}[Y_t, Y_{t-k}] = \begin{cases} \frac{3}{1+3^2} = \frac{3}{10} & \text{if } \theta = 3 \\ \frac{1/3}{1+(1/3)^2} = \frac{1}{10/3} = \frac{3}{10} & \text{if } \theta = 1/3 \end{cases}.$$

(b) No, probably not. Given that ρ is standardized, we will not be able to detect any difference in the variance regardless of the values of k.

2.5 Zero-mean stationary series

(a)

$$\mu_t = E[Y_t] = E[5 + 2t + X_t] = 5 + 2E[t] + E[X_t] = 5 + 2t + 0 = 2t + 5$$

(b)

$$\gamma_k = \text{Corr}[5 + 2t + X_t, 5 + 2(t-k) + X_{t-k}] = \text{Corr}[X_t, X_{t-k}]$$

(c) No, the mean function (μ_t) is constant and the aurocovariance ($\gamma_{t,t-k}$) free from t.

2.6 Stationary time series

(a)

$$\text{Cov}[a + X_t, b + X_{t-k}] = \text{Cov}[X_t, X_{t-k}],$$

which is free from t for all k because X_t is stationary.

(b)

$$\mu_t = E[Y_t] = \begin{cases} E[X_t] & \text{for odd } t \\ 3 + E[X_t] & \text{for even } t \end{cases}.$$

Since μ_t varies depending on t, Y_t is not stationary.

2.7 First and second-order difference series

(a)
$$\mu_t = E[W_t] = E[Y_t - Y_{t-1}] = E[Y_t] - E[Y_{t-1}] = 0$$
because Y_t is stationary.

$$\text{Cov}[W_t] = \text{Cov}[Y_t - Y_{t-1}, Y_{t-k} - Y_{t-1-k}] = \text{Cov}[Y_t, Y_{t-k}] + \text{Cov}[Y_t, Y_{t-1-k}] + \text{Cov}[-Y_{t-k}, Y_{t-k}] + \text{Cov}[-Y_{t-k}, -Y_{t-1-k}] = \gamma$$

(b) In (a), we discovered that the difference between two stationary processes, $\triangledown Y_t$ itself was stationary. It follows that the difference between two of these differences, $\triangledown^2 Y_t$ is also stationary.

2.8 Generalized difference series

$$E[W_t] = c_1 E[Y_t] + c_2 E[Y_t] + \cdots + c_n E[Y_t] \tag{2.11}$$
$$= E[Y_t](c_1 + c_2 + \cdots + c_n), \tag{2.12}$$

and thus the expected value is constant. Moreover,

$$\text{Cov}[W_t] = \text{Cov}[c_1 Y_t + c_2 Y_{t-1} + \cdots + c_n Y_{t-k}, c_1 Y_{t-k} + c_2 Y_{t-k-1} + \cdots + c_n Y_{t-k-n}] \tag{2.13}$$
$$= \sum_{i=0}^{n} \sum_{j=0}^{n} c_i c_j \text{Cov}[Y_{t-j} Y_{t-i-k}] \tag{2.14}$$
$$= \sum_{i=0}^{n} \sum_{j=0}^{n} c_i c_j \gamma_{j-k-i}, \tag{2.15}$$

which is free of t; consequently, W_t is stationary.

2.9 Zero-mean stationary difference series

(a)
$$E[Y_t] = \beta_0 + \beta_1 t + E[X_t] = \beta_0 + \beta_1 t + \mu_{t_x},$$
which is not free of t and hence *not* stationary.

$$\text{Cov}[Y_t] = \text{Cov}[X_t, X_t - 1] = \gamma_{t-1}$$

$$E[W_t] = E[Y_t - Y_{t-1}] = E[\beta_0 + \beta_1 t + X_t - (\beta_0 + \beta_1(t-1) + X_{t-1})] = \beta_0 + \beta_1 t - \beta_0 - \beta_1 t + \beta_1 = \beta_1,$$

is free of t and, furthermore, we have

$$\text{Cov}[W_t] = \text{Cov}[\beta_0 + \beta_1 t + X_t, \beta_0 + \beta_1(t-1) + X_{t-1}] = \text{Cov}[X_t, X_{t-1}] = \gamma_k,$$

which is also free of t, thereby proving that W_t is stationary.

(b)

$$E[Y_t] = E[\mu_t{+}X_t] = \mu_t{+}\mu_t = 0{+}0 = 0, \quad \text{and}\mathrm{Cov}[Y_t] = \mathrm{Cov}[\mu_t{+}X_t, \mu_{t-k}{+}X_{t-k}] = \mathrm{Cov}[X_t, X_{t-k}] = \gamma_k$$

$$\nabla^m Y_t = \nabla(\nabla^{m1} Y_t)$$

Currently unsolved.

2.10 Zero-mean, unit-variance process

(a)

$$\mu_t = E[Y_t] = E[\mu_t{+}\sigma_t X_t] = \mu_t{+}\sigma_t E[X_t] = \mu_t{+}\sigma_t{\times}0 = \mu_t \gamma_{t,t-k} = \mathrm{Cov}[Y_t] = \mathrm{Cov}[\mu_t{+}\sigma_t X_t, \mu_{t-k}{+}\sigma_{t-k} X_{t-k}] = \sigma_t \sigma_{t-k}$$

(b) First, we have

$$\mathrm{Var}[Y_t] = \mathrm{Var}[\mu_t + \sigma_t X_t] = 0 + \sigma_t^2 \mathrm{Var}[X_t] = \sigma_t^2 \times 1 = \sigma_t^2$$

since $\{X_t\}$ has unit-variance. Futhermore,

$$\mathrm{Corr}[Y_t, Y_{t-k}] = \frac{\sigma_t \sigma_{t-k} \rho_k}{\sqrt{\mathrm{Var}[Y_t]\mathrm{Var}[Y_{t-k}]}} = \frac{\sigma_t \sigma_{t-k} \rho_k}{\sigma_t \sigma_{t-k}} = \rho_k,$$

which depends only on the time lag, k. However, $\{Y_t\}$ is not necessarily stationary since μ_t may depend on t.

(c) Yes, ρ_k might be free from t but if σ_t is not, we will have a non-stationary time series with autocorrelation free from t and constant mean.

2.11 Drift

(a)

$$\mathrm{Cov}[X_t, X_{t-k}] = \gamma_k E[X_t] = 3t$$

$\{X_t\}$ is not stationary because μ_t varies with t.

(b)

$$E[Y_t] = 3{-}3t{+}E[X_t] = 7{-}3t{-}3t = 7\mathrm{Cov}[Y_t, Y_{t-k}] = \mathrm{Cov}[7{-}3t{+}X_t, 7{-}3(t{-}k){+}X_{t-k}] = \mathrm{Cov}[X_t, X_{t-k}] = \gamma_k$$

Since the mean function of $\{Y_t\}$ is constant (7) and its autocovariance free of t, $\{Y_t\}$ is stionary.

2.12 Periods

$$E[Y_t] = E[e_t{-}e_{t-12}] = E[e_t]{-}E[e_{t-12}] = 0\mathrm{Cov}[Y_t, Y_{t-k}] = \mathrm{Cov}[e_t{-}e_{t-12}, e_{t-k}{-}e_{t-12-k}] = \mathrm{Cov}[e_t, e_{t-k}]{-}\mathrm{Cov}[e_t, e_{t-12-k}]{-}\mathrm{Co}$$

Then, as required, we have

$$\mathrm{Cov}[Y_t, Y_{t-k}] = \begin{cases} \mathrm{Cov}[e_t, e_{t-12}] - \mathrm{Cov}[e_t, e_t] - \\ \mathrm{Cov}[e_{t-12}, e_{t-12}] + \mathrm{Cov}[e_{t-12}, e_t] = \\ \mathrm{Var}[e_t] - \mathrm{Var}[e_{t-12}] \neq 0 & \text{for } k = 12 \\ \\ \mathrm{Cov}[e_t, e_{t-k}] - \mathrm{Cov}[e_t, e_{t-12-k}] - \\ \mathrm{Cov}[e_{t-12}, e_{t-k}] + \mathrm{Cov}[e_{t-12}, e_{t-12-k}] = \\ 0 + 0 + 0 + 0 = 0 & \text{for } k \neq 12 \end{cases}$$

2.13 Drift, part 2

(a)

$$E[Y_t] = E[e_t - \theta e_{t-1}^2] = E[e_t] - \theta E[e_{t-1}^2] = 0 - \theta \text{Var}[e_{t-1}] = -\theta\sigma_e^2$$

And thus the requirement of constant variance is fulfilled. Moreover,

$$\text{Var}[Y_t] = \text{Var}[e_t - \theta e_{t-1}^2] = \text{Var}[e_t] + \theta^2 \text{Var}[e_{t-1}^2] = \sigma_e^2 + \theta^2 (E[e_{t-1}^4] - E[e_{t-1}^2]^2),$$

where

$$E[e_{t-1}^4] = 3\sigma_e^4 \quad \text{and} \quad E[e_{t-1}^2]^2 = \sigma_e^4,$$

gives us

$$\text{Var}[Y_t] = \sigma_e^2 + \theta(3\sigma_e^4 - \sigma_e^2) = \sigma_e^2 + 2\theta^2\sigma_e^4$$

and

$$\text{Cov}[Y_t, Y_{t-1}] = \text{Cov}[e_t - \theta e_{t-1}^2, e_{t-1} - \theta e_{t-2}^2] = \text{Cov}[e_t, e_{t-1}] + \text{Cov}[e_t, -\theta e_{t-2}^2] + \text{Cov}[-\theta e_{t-1}^2, e_{t-1}] \text{Cov}[-\theta e_{t-1}^2, -\theta e_{t-2}^2] =$$

which means that the autocorrelation function $\gamma_{t,s}$ also has to be zero.

(b) The autocorrelation of $\{Y_t\}$ is zeor and its mean function is constant, thus $\{Y_t\}$ must be stationary.

2.14 Stationarity, again

(a)

$$E[Y_t] = E[\theta_0 + te_t] = \theta_0 + E[e_t] = \theta_0 + t \times 0 = \theta_0 \text{Var}[Y_t] = \text{Var}[\theta_0] + \text{Var}[te_t] = 0 + t^2\sigma_e^2 = t^2\sigma_e^2$$

So $\{Y_t\}$ is not stationary.

(b)

$$E[W_t] = E[\triangledown Y_t] = E[\theta_0 + te_t - \theta_0 - (t-1)e_{t-1}] = tE[e_t] - tE[e_{t-1} + E[e_{t-1}] = 0 \text{Var}[\triangledown Y_t] = \text{Var}[te_t] = -\text{Var}[(t-1)e_{t-1}] = t$$

which varies with t and means that $\{W_t\}$ is not stationary.

(c)

$$E[Y_t] = E[e_t e_{t-1}] = E[e_t]E[e_{t-1}] = 0 \text{Cov}[Y_t, Y_{t-1}] = \text{Cov}[e_t e_{t-1}, e_{t-1} e_{t-2}] = E[(e_t e_{t-1} - \mu_t^2)(e_{t-1} e_{t-2} - \mu_t^2)] = E[e_t]E[e$$

Both the covariance and the mean function are zero, hence the process is stationary.

2.15 Random variable, zero mean

(a) $ E[Y_t] = (-1)^tE[X] = 0$
(b) $\text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[(-1)^t X, (-1)^{t-k} X] = (-1)^{2t-k} \text{Cov}[X, X] = (-1)^k \text{Var}[X] = (-1)^k \sigma_t^2$
(c) Yes, the covariance is free of t and the mean is constant.

2.16 Mean and variance

$$E[Y_t] = E[A + X_t] = E[A] + E[X_t] = \mu_A + \mu_X \text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[A + X_t, A + X_{t-k}] = \text{Cov}[A, A] + \text{Cov}[A, X_{t-k}] + \text{Cov}[X_t, A] +$$

2.17 Variance of sample mean

$$\text{Var}[\bar{Y}] = \text{Var}\left[\frac{1}{n}\sum_{t=1}^{n} Y_t\right] = \frac{1}{n^2}\text{Var}\left[\sum_{t=1}^{n} Y_t\right] = \frac{1}{n^2}\text{Cov}\left[\sum_{t=1}^{n} Y_t, \sum_{s=1}^{n} Y_s\right] = \frac{1}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n}\gamma_{t-s}$$

Setting $k = t - s, j = t$ gives us

$$\text{Var}[\bar{Y}] = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{j-k=1}^{n}\gamma_k = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{j=k+1}^{n+k}\gamma_k = \frac{1}{n^2}\left(\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}\gamma_k + \sum_{k=-n+1}^{0}\sum_{j=1}^{n+k}\gamma_k\right) = \frac{1}{n^2}\left(\sum_{k=1}^{n-1}(n-k)\gamma_k + \sum_{k=-n+1}^{0}(n+k)\gamma_k\right.$$

2.18 Sample variance

(a)

$$\sum_{t=1}^{n}(Y_t-\mu)^2 = \sum_{t=1}^{n}((Y_t-\bar{Y})+(\bar{Y}-\mu))^2 = \sum_{t=1}^{n}((Y_t-\bar{Y})^2-2(Y_t-\bar{Y})(\bar{Y}-\mu)+(\bar{Y}-\mu)^2) = n(\bar{Y}-\mu)^2+2(\bar{Y}-\mu)\sum_{t=1}^{n}(Y_t-\bar{Y})+\sum_{t=1}^{n}$$

(b)

$$E[s^2] = E\left[\frac{n}{n-1}\sum_{t=1}^{n}(Y_t-\bar{Y})^2\right] = \frac{n}{n-1}E\left[\sum_{t=1}^{n}\left((Y_t-\mu)^2+n(\bar{Y}-\mu)^2\right)\right] = \frac{n}{n-1}\sum_{t=1}^{n}\left(E[(Y_t-\mu)^2]+nE[(\bar{Y}-\mu)^2]\right) =$$

(c) Since $\gamma_k = 0$ for $k \neq 0$, in our case for all k, we have

$$E[s^2] = \gamma_0 - \frac{2}{n-1}\sum_{t=1}^{n}\left(1-\frac{k}{n}\right)\times 0 = \gamma_0$$

2.19 Random walk with drift

(a)

$$Y_1 = \theta_0+e_1 Y_2 = \theta_0+\theta_0+e_2+e_1 Y_t = \theta_0+\theta_0+\cdots+\theta_0+e_t+e_{t-1}+\cdots+e_1 = Y_t = t\theta_0+e_t+e_{t-1}+\cdots+e_1 \quad \square$$

(b)

$$\mu_t = E[Y_t] = E[t\theta_0+e_t+e_{t-1}+\cdots+e_1] = t\theta_0+E[e_t]+E[e_{t-1}]+\cdots+E[e_1] = t\theta_0+0+0+\cdots+0 = t\theta_0$$

(c)

$$\gamma_{t,t-k} = \text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[t\theta_0+e_t, +e_{t-1}+\cdots+e_1, (t-k)\theta_0+e_{t-k}, +e_{t-1-k}+\cdots+e_1] = \text{Cov}[e_{t-k}, +e_{t-1-k}+\cdots+e_1, e_{t-}$$

2.20 Random walk

(a)

$$\mu_1 = E[Y_1] = E[e_1] = 0\mu_2 = E[Y_2] = E[Y_1-e_2] = E[Y_1]-E[e_2] = 0-0 = 0\ldots\mu_{t-1} = E[Y_{t-1}] = E[Y_{t-2}-e_{t-1}] = E[Y_{t-}$$

which implies $\mu_t = \mu_{t-1}$ Q.E.D.

(b)

$$\text{Var}[Y_1] = \sigma_e^2 \text{Var}[Y_2] = \text{Var}[Y_1 - e_2] = \text{Var}[Y_1] + \text{Var}[e_1] = \sigma_e^2 + \sigma_e^2 = 2\sigma_e^2 \ldots \text{Var}[Y_{t-1}] = \text{Var}[Y_{t-2} - e_{t-1}] = \text{Var}[Y_{t-2}] + \text{Var}$$

(c)

$$\text{Cov}[Y_t, Y_s] = \text{Cov}[Y_t, Y_t + e_{t+1} + e_{t+2} + \cdots + e_s] = \text{Cov}[Y_t, Y_t] = \text{Var}[Y_t] = t\sigma_e^2$$

2.21 Random walk with random starting value

(a)

$$E[Y_t] = E[Y_0 + e_t + e_{t-1} + \cdots + e_1] = E[Y_0] + E[e_t] + E[e_{t-1}] + E[e_{t-2}] + \cdots + E[e_1] = \mu_0 + 0 + \cdots + 0 = \mu_0 \quad \square$$

(b)

$$\text{Var}[Y_t] = \text{Var}[Y_0 + e_t + e_{t-1} + \cdots + e_1] = \text{Var}[Y_0] + \text{Var}[e_t] + \text{Var}[e_{t-1}] + \cdots + \text{Var}[e_1] = \sigma_0^2 + t\sigma_e^2 \quad \square$$

(c)

$$\text{Cov}[Y_t, Y_s] = \text{Cov}[Y_t, Y_t + e_{t+1} + e_{t+2} + \cdots + e_s] = \text{Cov}[Y_t, Y_t] = \text{Var}[Y_t] = \sigma_0^2 + t\sigma_e^2 \quad \square$$

(d)

$$\text{Corr}[Y_t, Y_s] = \frac{\sigma_0^2 + t\sigma_e^2}{\sqrt{(\sigma_0^2 + t\sigma_e^2)(\sigma_0^2 + s\sigma_e^2)}} = \sqrt{\frac{\sigma_0^2 + t\sigma_e^2}{\sigma_0^2 + s\sigma_e^2}} \quad \square$$

2.22 Asymptotic stationarity

(a)

$$E[Y_1] = E[e_1] = 0 E[Y_2] = E[cY_1 + e_2] = cE[Y_1] + E[e_2] = 0 \ldots E[Y_t] = E[cY_{t-1} + e_t] = cE[Y_{t-1}] + E[e_t] = 0 \quad \square$$

(b)

$$\text{Var}[Y_1] = \text{Var}[e_1] = \sigma_e^2 \text{Var}[Y_2] = \text{Var}[cY_1 + e_2] = c^2 \text{Var}[Y_{t-1}] + \text{Var}[e_2] = c^2\sigma_e^2 + \sigma_e^2 = \sigma_e^2(1 + c^2) \ldots \text{Var}[Y_t] = \sigma_e^2(1 + c^2 + c^4$$

$\{Y_t\}$ is not stationary, given that its variance varies with t.

(c)

$$\text{Cov}[Y_t, Y_{t-1}] = \text{Cov}[cY_{t-1} + e_t, Y_{t-1}] = c\text{Cov}[Y_{t-1}, Y_{t-1}] = c\text{Var}[Y_{t-1}] \quad \text{giving} \text{Corr}[Y_t, Y_{t-1}] = \frac{c\text{Var}[Y_{t-1}]}{\sqrt{\text{Var}[Y_t]\text{Var}[Y_{t-1}]}} = c\sqrt{}$$

And, in the general case,

$$\text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[cY_{t-1} + e_t, Y_{t-k}] = c\text{Cov}[cY_{t-2} + e_{t-1}, Y_{t-k}] = c^3 \text{Cov}[Y_{t-2} + e_{t-1}, Y_{t-k}] = \ldots = c^k \text{Var}[Y_{t-k}]$$

giving

$$\text{Corr}[Y_t, Y_{t-k}] = \frac{c^k \text{Var}[Y_{t-k}]}{\sqrt{\text{Var}[Y_t]\text{Var}[Y_{t-k}]}} = c^k \sqrt{\frac{\text{Var}[Y_{t-k}]}{\text{Var}[Y_t]}} \quad \square$$

(d)

$$\text{Var}[Y_t] = \sigma_e^2(1 + c^2 + c^4 + \cdots + c^{2t-2}) = \sigma_e^2 \sum_{t=1}^{n} c^{2(t-1)} = \sigma_e^2 \sum_{t=0}^{n-1} c^{2t} = \sigma_e^2 \frac{1 - c^{2t}}{1 - c^2}$$

And because

$$\lim_{t \to \infty} \sigma_e^2 \frac{1 - c^{2t}}{1 - c^2} = \sigma_e^2 \frac{1}{1 - c^2} \quad \text{since } |c| < 1,$$

which is free of t, $\{Y_t\}$ can be considered *asymptotically* stationary.

(e)

$$Y_t = c(cY_{t-2}+e_{t-1})+e_t = \cdots = e_t+ce_{t-1}+c^2e_{t-2}+\cdots+c^{t-2}e_2+\frac{c^{t-1}}{\sqrt{1-c^2}}e_1\mathrm{Var}[Y_t] = \mathrm{Var}[e_t+ce_{t-1}+c^2e_{t-2}+\cdots+c^{t-2}e_2+$$

Futhermore,

$$E[Y_1] = E\left[\frac{e_1}{\sqrt{1-c^2}}\right] = \frac{E[e_1]}{\sqrt{1-c^2}} = 0E[Y_2] = E[cY_1+e_2] = cE[Y_1] = 0\ldots E[Y_t] = E[cY_{t-1}+e_2] = cE[Y_{t-1}] = 0,$$

which satisfies our first requirement for weak stationarity. Also,

$$\mathrm{Cov}[Y_t, Y_{t-k}] = \mathrm{Cov}[cY_{t-1} + e_t, Y_{t-1}] = c^k\mathrm{Var}[Y_{t-1}] = c^k\frac{\sigma_e^2}{1-c^2},$$

which is free of t and hence $\{Y_t\}$ is now stationary.

2.23 Stationarity in sums of stochastic processes

$$E[W_t] = E[Z_t + Y_t] = E[Z_t] + Y[Z_t] = \mu_{Z_t} + \mu_{Y_s}$$

Since both processes are stationary – and hence their sums are constant – the sum of both processes must also be constant.

$$\mathrm{Cov}[W_t, W_{t-k}] = \mathrm{Cov}[Z_t+Y_t, Z_{t-k}+Y_{t-k}] = \mathrm{Cov}[Z_t, Z_{t-k}]+\mathrm{Cov}[Z_t, Y_{t-k}]+\mathrm{Cov}[Y_t, Z_{t-k}]+\mathrm{Cov}[Y_t, Y_{t-k}] = \mathrm{Cov}[Z_t, Z_{t-k}]+\mathrm{Cov}$$

both free of t.

2.24 Measurement noise

$$E[Y_t] = E[Y_t+e_t] = E[X_t]+E[e_t]-\mu_t\mathrm{Var}[Y_t] = \mathrm{Var}[X_t+e_t] = \mathrm{Var}[X_t]+\mathrm{Var}[e_t] = \sigma_X^2+\sigma_e^2\mathrm{Cov}[Y_t, Y_{t-k}] = \mathrm{Cov}[X_t+e_t, X_{t-k}+e_t$$

2.25 Random cosine wave

$$E[Y_t] = E\left[\beta_0 + \sum_{i=1}^{k}(A_i\cos(2\pi f_it) + B_i\sin(2\pi f_it))\right] = \beta_0+\sum_{i=1}^{k}(E[A_i]\cos(2\pi f_it)+E[B_i]\sin(2\pi f_it) = \beta_0\mathrm{Cov}[Y_t, Y_s] = \mathrm{Cov}\Bigg[\sum_{i=1}^{k}$$

and is thus free of t and s.

2.26 Semivariogram

(a)

$$\Gamma_{t,s} = \frac{1}{2}E[(Y_t-Y_s)^2] = \frac{1}{2}E[Y_t^2-2Y_tY_s+Y_s^2] = \frac{1}{2}\left(E[Y_t^2] - 2E[Y_tY_s] + E[Y_s^2]\right) = \frac{1}{2}\gamma_0+\frac{1}{2}\gamma_0-2\times\frac{1}{2}\gamma_{|t-s|} = \gamma_0-\gamma_{|t-s|}\mathrm{Cov}[$$

(b)

$$Y_t-Y_s = e_t+e_{t-1}+\cdots+e_1-e_s-e_{s-1}-\cdots-e_1 = e_t+e_{t-1}+\cdots+e_{s+1}, \quad \text{for } t > s\Gamma_{t,s} = \frac{1}{2}E[(Y_t-Y_s)^2] = \frac{1}{2}\mathrm{Var}[e_t+e_{t-1}+\cdots$$

2.27 Polynomials

(a)

$$E[Y_t] = E[e_t+\phi e_{t-1}+\phi^2 e_{t-2}+\cdots+\phi^r e_{t-r}] = 0\text{Cov}[Y_t, Y_{t-k}] = \text{Cov}[e_t+\phi e_{t-1}+\cdots+\phi^r e_{t-r}, e_{t-k}+\phi e_{t-1-k}+\cdots+\phi^r e_{t-r-}$$

Hence, because of the zero mean and covariance free of t, it is a stationary process.

(b)

$$\text{Var}[Y_t] = \text{Var}[e_t+\phi e_{t-1}+\phi^2 e_{t-2}+\cdots+\phi^r e_{t-r}] = \sigma_e^2(1+\phi+\phi^2+\cdots+\phi^{2r})\text{Corr}[Y_t, Y_{t-k}] = \frac{\sigma_e^2\phi^k(1+\phi^2+\phi^4+\cdots+\phi^{2(r-}}{\sqrt{(\sigma_e^2(1+\phi+\phi^2+\cdots+\phi^{2r})}}$$

2.28 Random cosine wave extended

(a)

$$E[Y_t] = E[R\cos(2\pi(ft+\phi))] = E[R]\cos(2\pi(ft+\phi)) = E[R]\int_0^1 \cos(E[R\cos(2\pi(ft+\phi))])d\phi = E[R]\left[\frac{1}{2\pi}\sin(2\pi(ft+\right.$$

(b)

$$\gamma_{t,s} = E[R\cos(2\pi(ft+\phi))R\cos(2\pi(fs+\phi))] = \frac{1}{2}E[R^2]\int_0^1 \left(\cos(2\pi(f(t-s)) + \frac{1}{4\pi}\sin(2\pi(f(t+s)+2\phi))\right) = \frac{1}{2}E[R$$

which is free of t.

2.29 Random cosine wave further

(a)

$$E[Y_t] = \sum_{j=1}^{m} E[R_j]E[\cos(2\pi(f_j t+\phi))] = \text{via } 2.28 = \sum_{j=1}^{m} E[R_j]\times 0 = 0$$

(b)

$$\gamma_k = \sum_{j=1}^{m} E[R_j]\cos(2\pi f_j k),\ \text{also from 2.28.}$$

2.30 Rayleigh distribution

$$Y = R\cos(2\pi(ft+\phi)),\quad X = R\sin(2\pi(ft+\phi))\begin{bmatrix}\frac{\partial X}{\partial R} & \frac{\partial X}{\partial \Phi}\\ \frac{\partial Y}{\partial R} & \frac{\partial X}{\partial \Phi}\end{bmatrix} = \begin{bmatrix}\cos(2\pi(ft+\Phi)) & 2\pi R\sin(2\pi(ft+\Phi))\\ \sin(2\pi(ft+\Phi)) & 2\pi R\cos(2\pi(ft+\Phi))\end{bmatrix},$$

with jacobian

$$-2\pi R = -2\pi\sqrt{X^2+Y^2}$$

and inverse Jacobian

$$\frac{1}{-2\pi\sqrt{X^2+Y^2}}.$$

Furthermore,

$$f(r, \Phi) = re^{-r^2/2}$$

and

$$f(x, y) = \frac{e^{-(x^2+y^2)/2}\sqrt{x^2+y^2}}{2\pi\sqrt{x^2+y^2}} = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \quad \square$$

Chapter 3

Trends