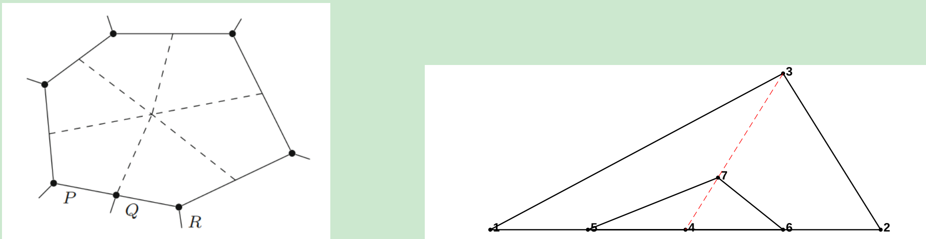
4 多角形网格加密

4.1 加密规则

在对单元进行加密时, 一个必要的规则是单元的边必须进行细分, 从而对多角形网格, 一个自然的加密方式是图 11 (a) 所示的四点划分. 这里, 目标单元通过连接重心和各边的中点划分为若干个四边形. 注意, 对有悬挂点 Q 的单元, 最好补充从重心到点 Q 的一个新边, 而不是对边 PQ 和 QR 进行二分, 否则可能出现退化三角形. 我们称这种处理为容许二分. 如图 11 (b) 所示, 三角形 $\Delta 123$ 由二分 PQ 和 QR 而产生, 新的三角形 $\Delta 567$ 由三角形 $\Delta 123$ 经类似方式获得. 根据三角形的性质, 重心 z_7 位于中线 e_{34} 上且满足 $|e_{47}| : |e_{73}| = 1 : 2$, 这意味着 $\angle 7 > \angle 3$. 这样, 如果继续 "双边二分", 那么退化三角形就会出现. 另一方面, 容许二分也导致更简单的多角形单元.



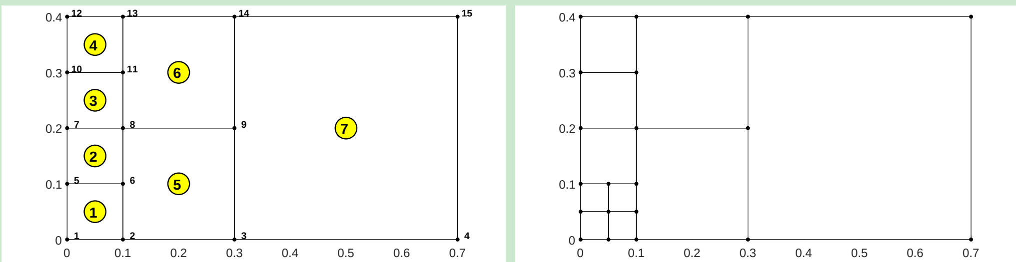
(c) (d)

图 11. 边二分. (a) 容许二分: 添加从悬挂点到重心的新边; (b) 双边二分: 将悬挂点连接的两条边均进行二分

一个网格加密函数应具有如下形式

```
[node,elem] = PolyMeshRefine(node,elem,idElemMarked).
```

这里, `node` 和 `elem` 是基本数据结构, 存储网格的节点坐标和顶点连通性, 而数组 `idElemMarked` 给出标记单元的索引. 一般而言, 待加密单元要多于标记单元. 若不对单条边上的悬挂点个数做出限制, 则很可能违反无短边假设. 尽管这种要求对 VEMs 不是必要的, 但我们仍期望获得没有短边的高质量网格. 为此, 一些额外的单元需要加入到待加密单元集合中.



(a) (b)

图 12. (a) Initial mesh; (b) Refinement of only the marked elements.

给定图 12 (a) 中的初始网格, 我们发现短边问题归结为两个相邻单元 ① 和 ⑤ 的加密这一典型情形.

- 假设 ① 是标记单元: `idElemMarked = 1`. 如果我们只按图 12 (b) 那样加密标记单元, 并且继续划分右下角的小单元, 那么单元 ⑤ 中就会出现短边 (由于不断地添加悬挂点). 这种现象也出现在图 11 (b) 中的 two-edge bisection 中, 后者通过使用容许二分得以解决.

- 为了避免短边的出现, 我们进一步加密相邻单元 ⑤, 见图 13. 也就是说, 对以 e 为公共边的两个单元 K_1 和 K_2, 若 K_1 在待加密集中且 e 的一个端点是 K_2 的悬挂点, 则同时加密 K_1 和 K_2.

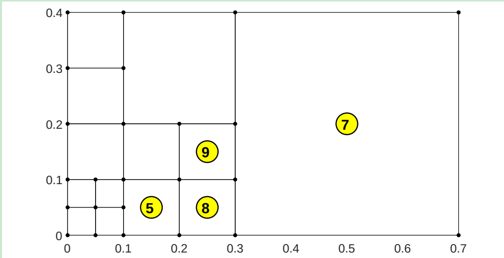


图 13. Refine both ① and ⑤

- 对图 12 (a) 中的单元, ⑤ 和 ⑦ 可分别视为 ① 和 ⑤, 基于同样的原因, 我们也需要划分单元 ⑦.

- 重复上面的过程, 我们就可以获得所有新的待加密单元.

注意, 标记单元本身可能出现上面指出的情况. 例如, 图 14 中红色的单元 9, 19 和 20 就是如此, 此时含悬挂点的单元 9 的地位和额外需要加密的单元的地位类似. 把这些地位等同的标记单元也加入额外单元中, 就可避免产生短边, 因为上面的处理确保每条边仅有一个悬挂点 (a midpoint of the collinear edge).

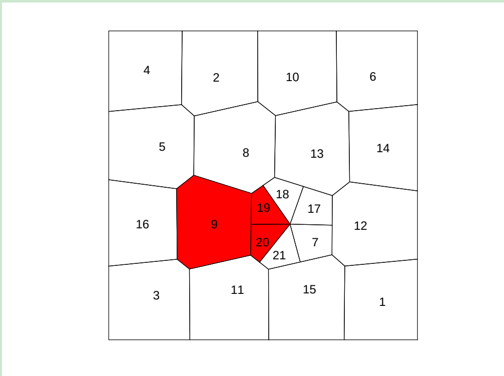


图 14. 悬挂点出现在标记单元加密后的子单元中

在下文中, 称一条边为非平凡的, 如果它的一个端点是某个单元的悬挂点; 称含悬挂点的单元为非平凡单元; 称新产生的待加密单元为额外待加密单元. 根据该规定, 待加密单元分为

$$\text{待加密单元}\begin{cases}\text{平凡待加密单元 : 恰为平凡标记单元}\\[1ex] \text{非平凡待加密单元}\begin{cases}\text{非平凡标记单元}\\ \text{额外待加密单元}\end{cases}\end{cases}\tag{4.5}$$

4.2 确定待加密单元

根据 (4.5) 的分类, 我们先确定标记单元中的平凡和非平凡单元. 根据 "单悬挂点规则", 悬挂点一定是单元共线边的中点, 为此可通过计算如下误差找到悬挂点

$$\mathrm{err}(i) = \left| z_i - \frac{1}{2}(z_{i-1} + z_{i+1}) \right|, \quad i = 1, \cdots, N_v,$$

式中, z_i 是单元顶点, N_v 是顶点数. 非平凡标记单元按如下代码确定

```
ismElem = cell(NT,1); % store the elementwise mid-point locations
%% Determine the trivial and nontrivial marked elements
% nontrivial marked elements: marked elements with hanging nodes
nMarked = length(idElemMarked);
isT = false(nMarked,1);
for s = 1:nMarked
    % current element
    iel = idElemMarked(s);
    % local logical index of elements with hanging nodes
    p0 = node(elem{iel},:); p1 = circshift(p0,1,1); p2 = circshift(p0,-1,1);
    err = vecnorm(p0-0.5*(p1+p2),2,2);
    ism = (err<tol);  ismElem{iel} = ism;  % is midpoint
    if sum(ism)<1, isT(s) = true; end  % trivial
end
idElemMarkedT = idElemMarked(isT);
idElemMarkedNT = idElemMarked(¬isT);
```

这里, `idElemMarkedT` 是平凡标记单元, `idElemMarkedNT` 是非平凡标记单元. 为了避免后面重复确定单元悬挂点, 用 `ismElem` 存储每个单元悬挂点的局部逻辑数组.

现在收集额外的待加密单元. 用 `idElemMarked` 记录标记单元, `idElemNew` 记录每一步新增加的单元. 向量 `idElemMarkedNew` 收集直到当前步的所有待加密单元 (包括给定的标记单元和所有的额外的待加密单元). 在循环找额外单元的过程中, `idElemMarkedNew` 的地位与初始标记单元的地位等同. 这是找额外单元的核心思想, 即把每一循环步归结到一开始的局面.

4.1 节的描述可总结为算法 2.

Algorithm 2 找到额外的待加密单元

- 初始化 `idElemMarkedNew` 和 `idElemNew` 为标记单元集合 `idElemMarked`.
- 找到 `idElemNew` 的相邻单元 (去除已找到的待加密单元), 编号存储为 `idElemNewNeighbor`, 并获得已找到的待加密单元集 `idElemMarkedNew` 中所有单元的边的序号.
- 对 `idElemNewNeighbor` 中单元循环, 并更新 `idElemNew`: 确定当前单元非平凡边的索引, 记为 `idEdgeDg`. 若 `idEdgeDg` 与 `idElemMarkedNew` 确定的边有交集 (实际上只要与新产生的单元 `idElemNew` 的边有交集即可), 则当前单元需要加密.
- 更新 `idElemMarkedNew`, 即添加 `idElemNew` 中单元.
- 若 `idElemNew` 是空集, 停止循环; 否则, 可视当前所有待加密单元 `idElemMarkedNew` 为初始的标记单元, 回到第一步.
- 所有额外待加密单元为

  `idElemRefineAddL = setdiff(idElemMarkedNew,idElemMarked);)`.

上面的算法对应的 MATLAB 代码如下.

```
%% Find the additional elements to be refined
% initialized as marked elements
idElemMarkedNew = idElemMarked; % marked and all new elements
idElemNew = idElemMarked; % new elements generated in current step
isEdgeMarked = false(NE,1);
while ¬isempty(idElemNew)
```

```
    % adjacent polygons of new elements
    for iel = idElemNew(:)'
        if ¬isempty(neighbor{iel}); continue; end
        index = elem2edge{iel};
        ia = edge2elem(index,1); ib = edge2elem(index,2);
        ia(ia==iel) = ib(ia==iel);
        neighbor{iel} = ia';
    end
    idElemNewNeighbor = unique(horzcat(neighbor{idElemNew}));
    idElemNewNeighbor = setdiff(idElemNewNeighbor,idElemMarkedNew); % delete the ...
        ones in the new marked set
    % edge set of new marked elements
    isEdgeMarked(horzcat(elem2edge{idElemNew})) = true;
    % find the adjacent elements to be refined
    nElemNewAdj = length(idElemNewNeighbor);
    isRefine = false(nElemNewAdj,1);
    for s = 1:nElemNewAdj
        % current element
        iel = idElemNewNeighbor(s);
        index = elem{iel};  indexEdge = elem2edge{iel}; Nv = length(index);
        % local logical index of elements with hanging nodes
        v1 = [Nv,1:Nv-1]; v0 = 1:Nv;  % left,current
        p0 = node(elem{iel},:); p1 = circshift(p0,1,1); p2 = circshift(p0,-1,1);
        err = vecnorm(p0-0.5*(p1+p2),2,2);
        ism = (err<tol);  ismElem{iel} = ism;  % is midpoint
        if sum(ism)<1, continue; end  % start the next loop if no hanging nodes exist
        % index numbers of edges connecting hanging nodes in the adjacent elements ...
            to be refined
        idEdgeDg = indexEdge([v1(ism),v0(ism)]);
        % whether or not the above edges are in the edge set of new marked elements
        if sum(isEdgeMarked(idEdgeDg)), isRefine(s) = true; end
    end
    idElemNew = idElemNewNeighbor(isRefine);
    idElemMarkedNew = unique([idElemMarkedNew(:); idElemNew(:)]);
end
idElemRefineAddL = setdiff(idElemMarkedNew,idElemMarked);
```

4.3 单元的加密与扩展

单元加密与扩展的预说明

先说明本文的单元扩展思路, 因为它影响单元的加密. 需要扩展的单元由三部分组成:

- 一是所有待加密单元的相邻单元;
- 二是额外待加密单元加密后的子单元;
- 三是某些标记单元加密后的子单元. 如图 (14) 所示, 单元 9, 19, 20 都是标记单元. 当单元 9 进行划分后, 因 19 和 20 要划分, 从而单元 9 的子单元会出现悬挂点.

简单分析一下扩展单元的特点.

1. 若某个单元需要扩展, 即该单元某条边需要二分, 则该条边必是某个待加密单元的平凡边.
2. 但要指出的是, 这种平凡边可能是另一个待加密单元的非平凡边, 如图 14 中单元 19 的左侧边是单元 9 的非平凡边.

3. 需要二分的边仍是原始剖分的边, 即悬挂点仅添加在原始剖分的边上.

为此, 我们先找到所有待加密单元中的平凡边集合, 记为 idEdgeCut. 注意, 这些平凡边只要是某个待加密单元的即可. 需扩展单元的特点是: 至少有一条边对应 idEdgeCut 中的索引.

本文的扩展方法是: 查看单元的边序号 elem2edge, 确定其中是否有位于 idEdgeCut 中的元素. 正因为如此, 非平凡待加密单元要先加密, 以获得子单元的数据结构 elem2edge.

由于标记单元中有一些单元不需要扩展 (它们直接加密), 但它们也有 idEdgeCut 中的边, 如单元 19 和 20 的左侧边. 为此, 单元加密时, 这部分单元要最后考虑.

非平凡待加密单元的加密

悬挂点可能出现在非平凡待加密单元加密后的子单元中, 且本文的单元扩展方法需要数据结构 elem2edge. 为此, 先对非平凡待加密单元进行加密.

某些边的中点和单元重心需要加入到节点矩阵 node 中. 我们将顶点、边和单元用单指标 $i = 1, 2, \cdots, \mathrm{N} + \mathrm{NE} + \mathrm{NT}$ 按如下顺序重新编号

$$z_1, \cdots, z_{\mathrm{N}};\ \ e_1, \cdots, e_{\mathrm{NE}};\ \ K_1, \cdots, K_{\mathrm{NT}},$$

称其为连接序号. 然而, 在大多数情形下, 使用 edge 的索引更加方便.

为了构造四点子单元, 首先考虑一个例子, 其顶点和边的连接序号为

$$z_1, e_1, z_2, e_2, z_3, e_3, z_4, e_4, z_5, e_5, \quad N_v = 5,$$

其中 z_2 和 z_5 是悬挂点, 下标表示局部索引. 接下来, 将非平凡边的连接序号用悬挂点序号如下代替

$$\begin{aligned} & z_1, e_1, z_2, e_2, z_3, e_3, z_4, e_4, z_5, e_5 \\ \hookrightarrow\ & z_1, z_2, z_2, z_2, z_3, e_3, z_4, z_5, z_5, z_5, \quad N_v = 5. \end{aligned}$$

这样可以用统一的方式构造数据结构 elem. 该方法也适用于标记单元, 无论它是否有悬挂点.

对数据结构 elem2edge, 子单元在内部的非平凡边仅用 0 进行标记, 而边界边用边的连接序号标记.

程序如下

```
%% Partition the nontrivial elements to be refined
% these elements are composed of:
%  - nontrivial marked elements
%  - additional elements to be refined
idElemRefineNT = [idElemMarkedNT; idElemRefineAddL];
nRefineNT = length(idElemRefineNT);
elemRefineNT = cell(nRefineNT,1);
elem2edgeRefineNT = cell(nRefineNT,1);
for s = 1:nRefineNT
    % current element
    iel = idElemRefineNT(s);
    index = elem{iel}; indexEdge = elem2edge{iel}; Nv = length(index);
    % find midpoint
    v1 = [Nv,1:Nv-1]; v0 = 1:Nv;
    ism = ismElem{iel};
```

```
16     % modify the edge number
17     ide = indexEdge+N;  % the connection number
18     ide(v1(ism)) = index(ism); ide(ism) = index(ism);
19     % elem (with or without hanging nodes)
20     nsub = Nv-sum(ism);
21     z1 = ide(v1(¬ism));  z0 = index(¬ism);
22     z2 = ide(¬ism);      zc = iel*ones(nsub,1)+N+NE;
23     elemRefineNT{s} = [z1(:), z0(:), z2(:), zc(:)];
24     % elem2edge
25     ise = false(Nv,1); ise([v1(ism),v0(ism)]) = true;
26     idg = zeros(Nv,1); idg(ise) = indexEdge(ise);
27     e1 = idg(v1(¬ism));     e0 = idg(¬ism);
28     elem2edgeRefineNT{s} = [e1(:), e0(:), zeros(nsub,2)]; % e2 = ec = 0
29 end
30 addElemRefineNT = num2cell(vertcat(elemRefineNT{:}), 2);
31 addElemRefineNT2edge = num2cell(vertcat(elem2edgeRefineNT{:}), 2);
```

注意, num2cell.m 对空元胞也可操作. Line 23 的 1, 0, 2, c 标记分别对应当前顶点的上一个顶点、当前顶点、当前顶点的下一个顶点和重心.

单元扩展: 增加悬挂点

需要扩展的单元由两部分组成: 一些是待加密单元的相邻单元, 另一些是非平凡单元加密后的子单元.

```
%% Determine the elements to be expanded
% these elements are composed of
% - adjacent polygonals of elements to be refined
% - subcells of nontrivial elements
% adjacent polygons of elements to be refined
idElemRefine = unique([idElemRefineAddL(:); idElemMarked(:)]); % ascending order ...
    for update
for iel = idElemRefine(:)'
    if ¬isempty(neighbor{iel}); continue; end
    index = elem2edge{iel};
    ia = edge2elem(index,1); ib = edge2elem(index,2);
    ia(ia==iel) = ib(ia==iel);
    neighbor{iel} = ia';
end
idElemRefineNeighbor = unique(horzcat(neighbor{idElemRefine}));
idElemRefineNeighbor = setdiff(idElemRefineNeighbor,idElemRefine);
% basic data structure of elements to be extended
elemExtend = [elem(idElemRefineNeighbor); addElemRefineNT];
elem2edgeExtend = [elem2edge(idElemRefineNeighbor); addElemRefineNT2edge];
```

注意, 悬挂点仅出现在原始剖分的边上, 且这些边是某个待加密单元的平凡边. 但要指出的是, 这种非平凡边可能是另一个待加密单元的非平凡边, 如图 14 中单元 19 的左侧边是单元 9 的非平凡边. 为此, 我们先找到所有待加密单元中的平凡边, 这些平凡边只要是某个待加密单元的即可. 为了方便, 我们用逻辑数组 `isEdgeCut` 对所有边的平凡和非平凡性质进行标记, `true` 的位置表示该条边要进行二分, 即为平凡边.

```
%% Extend elements by adding hanging nodes
% natural numbers of trivial edges w.r.t some element to be refined
isEdgeCut = false(NE,1);
```

```
for s = 1:length(idElemRefine)
    iel = idElemRefine(s);
    index = elem{iel}; indexEdge = elem2edge{iel}; Nv = length(index);
    v1 = [Nv,1:Nv-1];
    ism = ismElem{iel};
    idx = true(Nv,1); idx(v1(ism)) = false; idx(ism) = false;
    isEdgeCut(indexEdge(idx)) = true;
end
```

需扩展单元的特点是: 至少有一条边对应 `isEdgeCut` 中逻辑真的索引. 为了扩展单元, 先初始化一个长度为 $2N_v$ 的零向量, 在奇数位插入当前单元的顶点序号, 但仅在偶数位插入 `isEdgeCut` 中逻辑真的边索引. 这样, 删除零元素后即得连通性向量.

```
% extend the elements
for s = 1:length(elemExtend)
    index = elemExtend{s}; indexEdge = elem2edgeExtend{s};
    id = find(indexEdge>0);
    id = id(isEdgeCut(indexEdge(id)));
    idvec = zeros(1,2*length(index));
    idvec(1:2:end) = index;  idvec(2*id) = indexEdge(id)+N;
    elemExtend{s} = idvec(idvec>0);
end
```

划分平凡标记单元

剩下的未加密单元是平凡标记单元, 它们类似之前的方法进行划分.

```
%% Partition the trivial marked elements
nMarkedT = length(idElemMarkedT);
addElemMarkedT = cell(nMarkedT,1);
for s = 1:nMarkedT
    iel = idElemMarkedT(s);
    index = elem{iel}; indexEdge = elem2edge{iel}; Nv = length(index);
    ide = indexEdge+N;  % connection number
    z1 = ide([Nv,1:Nv-1]);   z0 = index;
    z2 = ide;                zc = iel*ones(Nv,1)+N+NE;
    addElemMarkedT{s} = [z1(:), z0(:), z2(:), zc(:)];
end
% addElem
addElemMarkedT = num2cell(vertcat(addElemMarkedT{:}), 2);
addElem = [elemExtend; addElemMarkedT];
```

更新基本数据结构

如下更新基本数据结构 `node` 和 `elem`:

```
%% Update node and elem
% node
nodeEdgeCut = (node(edge(isEdgeCut,1),:) + node(edge(isEdgeCut,2),:))/2;
nodeCenter = zeros(length(idElemRefine),2);
for s = 1:length(idElemRefine)
    iel = idElemRefine(s);  index = elem{iel};
    verts = node(index(¬ismElem{iel}),:); verts1 = verts([2:end,1],:);
    area_components = verts(:,1).*verts1(:,2)-verts1(:,1).*verts(:,2);
```

```
    ar = 0.5*abs(sum(area_components));
    nodeCenter(s,:) = sum((verts+verts1).*repmat(area_components,1,2))/(6*ar);
end
node = [node; nodeEdgeCut; nodeCenter];
% elem
elem([idElemRefine(:); idElemRefineNeighbor(:)]) = []; % delete old
elem = [elem; addElem];

%% Reorder the vertices
[¬,¬,totalid] = unique(horzcat(elem{:})');
elemLen = cellfun('length',elem);
elem = mat2cell(totalid', 1, elemLen)';
```

4.4 程序整理

完整的加密程序总结如下.

CODE 7. PolyMeshRefine.m (多角形网格加密)

```
function [node,elem] = PolyMeshRefine(node,elem,elemMarked)
%PolyMeshRefine refines a 2-D polygonal mesh satisfying one-hanging node rule
%
% We divide elements by connecting the midpoint of each edge to its
% barycenter.
% We remove small edges by further partitioning some adjacent elements.
%
% Copyright (C) Terence Yu.

idElemMarked = unique(elemMarked(:)); % in ascending order
tol = 1e-10; % accuracy for finding midpoint

%% Get auxiliary data
NT = size(elem,1);
if ¬iscell(elem), elem = num2cell(elem,2); end
% diameter
diameter = cellfun(@(index) max(pdist(node(index,:))), elem(idElemMarked));
if max(diameter)<tol
    disp('The mesh is too dense'); return;
end
% totalEdge
shiftfun = @(verts) [verts(2:end),verts(1)];
T1 = cellfun(shiftfun, elem, 'UniformOutput', false);
v0 = horzcat(elem{:})'; v1 = horzcat(T1{:})';
totalEdge = sort([v0,v1],2);
% edge, elem2edge
[edge, i1, totalJ] = unique(totalEdge,'rows');
elemLen = cellfun('length',elem);
elem2edge = mat2cell(totalJ',1,elemLen)';
% edge2elem
Num = num2cell((1:NT)');    Len = num2cell(elemLen);
totalJelem = cellfun(@(n1,n2) n1*ones(n2,1), Num, Len, 'UniformOutput', false);
totalJelem = vertcat(totalJelem{:});
i2(totalJ) = 1:length(totalJ); i2 = i2(:);
edge2elem = totalJelem([i1,i2]);
% neighbor
neighbor = cell(NT,1);
% number
```

```
N = size(node,1); NE = size(edge,1);

ismElem = cell(NT,1); % store the elementwise mid-point locations
%% Determine the trivial and nontrivial marked elements
% nontrivial marked elements: marked elements with hanging nodes
nMarked = length(idElemMarked);
isT = false(nMarked,1);
for s = 1:nMarked
    % current element
    iel = idElemMarked(s);
    % local logical index of elements with hanging nodes
    p0 = node(elem{iel},:); p1 = circshift(p0,1,1); p2 = circshift(p0,-1,1);
    err = vecnorm(p0-0.5*(p1+p2),2,2);
    ism = (err<tol);  ismElem{iel} = ism;  % is midpoint
    if sum(ism)<1, isT(s) = true; end  % trivial
end
idElemMarkedT = idElemMarked(isT);
idElemMarkedNT = idElemMarked(¬isT);

%% Find the additional elements to be refined
% initialized as marked elements
idElemMarkedNew = idElemMarked; % marked and all new elements
idElemNew = idElemMarked; % new elements generated in current step
isEdgeMarked = false(NE,1);
while ¬isempty(idElemNew)
    % adjacent polygons of new elements
    for iel = idElemNew(:)'
        if ¬isempty(neighbor{iel}); continue; end
        index = elem2edge{iel};
        ia = edge2elem(index,1); ib = edge2elem(index,2);
        ia(ia==iel) = ib(ia==iel);
        neighbor{iel} = ia';
    end
    idElemNewNeighbor = unique(horzcat(neighbor{idElemNew}));
    idElemNewNeighbor = setdiff(idElemNewNeighbor,idElemMarkedNew); % delete the ...
        ones in the new marked set
    % edge set of new marked elements
    isEdgeMarked(horzcat(elem2edge{idElemNew})) = true;
    % find the adjacent elements to be refined
    nElemNewAdj = length(idElemNewNeighbor);
    isRefine = false(nElemNewAdj,1);
    for s = 1:nElemNewAdj
        % current element
        iel = idElemNewNeighbor(s);
        index = elem{iel};  indexEdge = elem2edge{iel}; Nv = length(index);
        % local logical index of elements with hanging nodes
        v1 = [Nv,1:Nv-1]; v0 = 1:Nv;  % left,current
        p0 = node(elem{iel},:); p1 = circshift(p0,1,1); p2 = circshift(p0,-1,1);
        err = vecnorm(p0-0.5*(p1+p2),2,2);
        ism = (err<tol);  ismElem{iel} = ism;  % is midpoint
        if sum(ism)<1, continue; end  % start the next loop if no hanging nodes exist
        % index numbers of edges connecting hanging nodes in the adjacent elements ...
            to be refined
        idEdgeDg = indexEdge([v1(ism),v0(ism)]);
        % whether or not the above edges are in the edge set of new marked elements
        if sum(isEdgeMarked(idEdgeDg)), isRefine(s) = true; end
    end
    idElemNew = idElemNewNeighbor(isRefine);
```

```
    idElemMarkedNew = unique([idElemMarkedNew(:); idElemNew(:)]);
end
idElemRefineAddL = setdiff(idElemMarkedNew,idElemMarked);

%% Partition the nontrivial elements to be refined
% these elements are composed of:
%  - nontrivial marked elements
%  - additional elements to be refined
idElemRefineNT = [idElemMarkedNT; idElemRefineAddL];
nRefineNT = length(idElemRefineNT);
elemRefineNT = cell(nRefineNT,1);
elem2edgeRefineNT = cell(nRefineNT,1);
for s = 1:nRefineNT
    % current element
    iel = idElemRefineNT(s);
    index = elem{iel}; indexEdge = elem2edge{iel}; Nv = length(index);
    % find midpoint
    v1 = [Nv,1:Nv-1]; v0 = 1:Nv;
    ism = ismElem{iel};
    % modify the edge number
    ide = indexEdge+N;  % the connection number
    ide(v1(ism)) = index(ism); ide(ism) = index(ism);
    % elem (with or without hanging nodes)
    nsub = Nv-sum(ism);
    z1 = ide(v1(¬ism));  z0 = index(¬ism);
    z2 = ide(¬ism);      zc = iel*ones(nsub,1)+N+NE;
    elemRefineNT{s} = [z1(:), z0(:), z2(:), zc(:)];
    % elem2edge
    ise = false(Nv,1); ise([v1(ism),v0(ism)]) = true;
    idg = zeros(Nv,1); idg(ise) = indexEdge(ise);
    e1 = idg(v1(¬ism));     e0 = idg(¬ism);
    elem2edgeRefineNT{s} = [e1(:), e0(:), zeros(nsub,2)]; % e2 = ec = 0
end
addElemRefineNT = num2cell(vertcat(elemRefineNT{:}), 2);
addElemRefineNT2edge = num2cell(vertcat(elem2edgeRefineNT{:}), 2);

%% Determine the elements to be expanded
% these elements are composed of
% - adjacent polygonals of elements to be refined
% - subcells of nontrivial elements
% adjacent polygons of elements to be refined
idElemRefine = unique([idElemRefineAddL(:); idElemMarked(:)]); % ascending order ...
    for update
for iel = idElemRefine(:)'
    if ¬isempty(neighbor{iel}); continue; end
    index = elem2edge{iel};
    ia = edge2elem(index,1); ib = edge2elem(index,2);
    ia(ia==iel) = ib(ia==iel);
    neighbor{iel} = ia';
end
idElemRefineNeighbor = unique(horzcat(neighbor{idElemRefine}));
idElemRefineNeighbor = setdiff(idElemRefineNeighbor,idElemRefine);
% basic data structure of elements to be extended
elemExtend = [elem(idElemRefineNeighbor); addElemRefineNT];
elem2edgeExtend = [elem2edge(idElemRefineNeighbor); addElemRefineNT2edge];

%% Extend elements by adding hanging nodes
% natural numbers of trivial edges w.r.t some element to be refined
```

```
isEdgeCut = false(NE,1);
for s = 1:length(idElemRefine)
    iel = idElemRefine(s);
    index = elem{iel}; indexEdge = elem2edge{iel}; Nv = length(index);
    v1 = [Nv,1:Nv-1];
    ism = ismElem{iel};
    idx = true(Nv,1); idx(v1(ism)) = false; idx(ism) = false;
    isEdgeCut(indexEdge(idx)) = true;
end
% extend the elements
for s = 1:length(elemExtend)
    index = elemExtend{s}; indexEdge = elem2edgeExtend{s};
    id = find(indexEdge>0);
    id = id(isEdgeCut(indexEdge(id)));
    idvec = zeros(1,2*length(index));
    idvec(1:2:end) = index;  idvec(2*id) = indexEdge(id)+N;
    elemExtend{s} = idvec(idvec>0);
end

%% Partition the trivial marked elements
nMarkedT = length(idElemMarkedT);
addElemMarkedT = cell(nMarkedT,1);
for s = 1:nMarkedT
    iel = idElemMarkedT(s);
    index = elem{iel}; indexEdge = elem2edge{iel}; Nv = length(index);
    ide = indexEdge+N;  % connection number
    z1 = ide([Nv,1:Nv-1]);   z0 = index;
    z2 = ide;                zc = iel*ones(Nv,1)+N+NE;
    addElemMarkedT{s} = [z1(:), z0(:), z2(:), zc(:)];
end
% addElem
addElemMarkedT = num2cell(vertcat(addElemMarkedT{:}), 2);
addElem = [elemExtend; addElemMarkedT];

%% Update node and elem
% node
nodeEdgeCut = (node(edge(isEdgeCut,1),:) + node(edge(isEdgeCut,2),:))/2;
nodeCenter = zeros(length(idElemRefine),2);
for s = 1:length(idElemRefine)
    iel = idElemRefine(s);  index = elem{iel};
    verts = node(index(¬ismElem{iel}),:); verts1 = verts([2:end,1],:);
    area_components = verts(:,1).*verts1(:,2)-verts1(:,1).*verts(:,2);
    ar = 0.5*abs(sum(area_components));
    nodeCenter(s,:) = sum((verts+verts1).*repmat(area_components,1,2))/(6*ar);
end
node = [node; nodeEdgeCut; nodeCenter];
% elem
elem([idElemRefine(:); idElemRefineNeighbor(:)]) = []; % delete old
elem = [elem; addElem];

%% Reorder the vertices
[¬,¬,totalid] = unique(horzcat(elem{:})');
elemLen = cellfun('length',elem);
elem = mat2cell(totalid', 1, elemLen)';
```