SI 211: Numerical Analysis
Homework 4

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Question 1

Solve the Gauss approximation problem

$$\min_{p \in \mathbf{P}_n} \int_a^b (f(x) - p(x))^2 dx$$

for

$(a) f(x) = x^2, a = 0, b = 1,$ and $n = 1$,
$(b) f(x) = x^{\frac{3}{2}}, a = 0, b = 1,$ and $n = 2$,
$(c) f(x) = \frac{1}{1+x^2}, a = -5, b = 5,$ and $n = 8$.

Plot the function f as well as the polynomial p on the interval $[a, b]$ for all problems above. (Hint: Problem (c) is a bit tricky-we won't ask you to solve such problems in the exam, but it is an interesting homework problem. We recommend to look up the Legendre polynomials on Wikipedia or in the Numerical Analysis book; of course, you can also use a computer algebra program to work out integrals—no need to do everything by hand)

Solution

(a)

$f(x) = x^2, a = 0, b = 1,$ and $n = 1$,

$$p(x) = b_0 + b_1 x = c_0 q_0(x) + c_1 q_1(x)$$

where

$$q_0 = \sqrt{\frac{1}{2}} \times \sqrt{2} = 1$$

$$q_1 = \sqrt{\frac{3}{2}} \times \sqrt{2} \times (2x - 1) = \sqrt{3} \times (2x - 1)$$

So

$$c_0 =< f, q_0 >= \int_0^1 (x^2 \cdot 1)dx = \frac{1}{3}$$

$$c_1 =< f, q_1 >= \int_0^1 (x^2 \cdot \sqrt{\frac{3}{2}} \cdot \sqrt{2} \cdot (2x-1))dx = \frac{\sqrt{3}}{6}$$

The polynomial p is

$$p = \frac{1}{3} \cdot 1 + \frac{\sqrt{3}}{6} \cdot \sqrt{3}(2x-1)$$

Plot the function f as well as the polynomial p on the interval $[a, b]$:

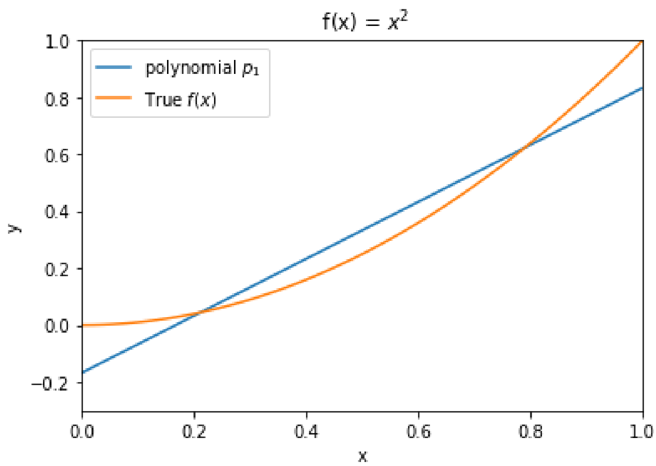

Figure 1: f VS p

(b)

$f(x) = x^{\frac{3}{2}}, a = 0, b = 1$, and $n = 2$,

$$p(x) = b_0 + b_1 x + b_2 x^2 = c_0 q_0(x) + c_1 q_1(x) + c_2 q_2(x)$$

The q_0, q_1 are the same as (a). And

$$q_2 = \sqrt{\frac{5}{8}} \times \sqrt{2} \times (3 \times (2x-1)^2 - 1) = \sqrt{5} \times (6x^2 - 6x + 1)$$

Therefore,

$$c_0 = < f, q_0 > = \frac{2}{5}$$
$$c_1 = < f, q_1 > = \frac{6\sqrt{3}}{35}$$
$$c_2 = < f, q_2 > = \frac{2\sqrt{5}}{105}$$

Plot the function f as well as the polynomial p on the interval $[a, b]$:

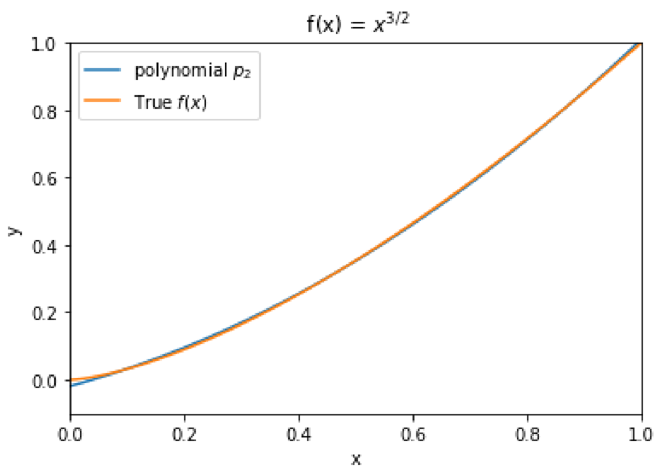

Figure 2: f VS p

(c)

$f(x) = \frac{1}{1+x^2}, a = -5, b = 5$, and $n = 8$,

I directly write a python program to get the polynomial p.
Here is the code.

```
import numpy as np
import matplotlib.pyplot as plt
from sympy import *
from sympy.abc import x,y
def Guass_appr(n,a,b,case):
    A,B,N = symbols("A B N", positive=True)
    coeff1 = sqrt((2*N+1)/2) * 1/(2**N * factorial(N))* sqrt(2/(B-A))
    values = {A: a, B: b, N: n}
    y = Rational(2,(b-a)) * x - Rational(2*a,(b-a)) - 1
    ff = (x**2-1)**n
```

```
    dff = diff(ff,x,n)
    dff = dff.subs(x,y)
    q = simplify( coeff1.subs(values)*dff)
    print ('q_'+ str(n) +':' ,q)
    if case == 0:
       f = x**2
    elif case == 1:
       f = x**(Rational(3,2))
    else:
       f = (1 + x**2)**(-1)
    coef = integrate(q*f,(x,a,b))
    print('c_' + str(n) +':', coef)
    return coef,q
#(c)
print ('solution of question 1(c):')
case_c = 2
ac = -5
bc = 5
nc = 8
Pc = 0
for i in range(nc+1):
    cc,qc= Guass_appr(i,ac,bc,case_c)
    Pc = Pc + cc * qc
Pcc = lambdify(x, Pc)
xc = np.linspace(-5, 5, 100)
# Plot
fig = plt.figure(3)
axes = fig.add_subplot(1, 1, 1)
axes.plot(xc,Pcc(xc), label="polynomial $p_8$")
axes.plot(xc, 1/(1+xc**2),label="True $f(x)$")
axes.set_title(" f(x) = $1/(1 + x^2)$")
axes.set_xlabel("x")
axes.set_ylabel("y")
axes.set_xlim([-5, 5])
axes.set_ylim([0, 1])
axes.legend(loc = 0)
plt.show()
```

The plot is as follows.

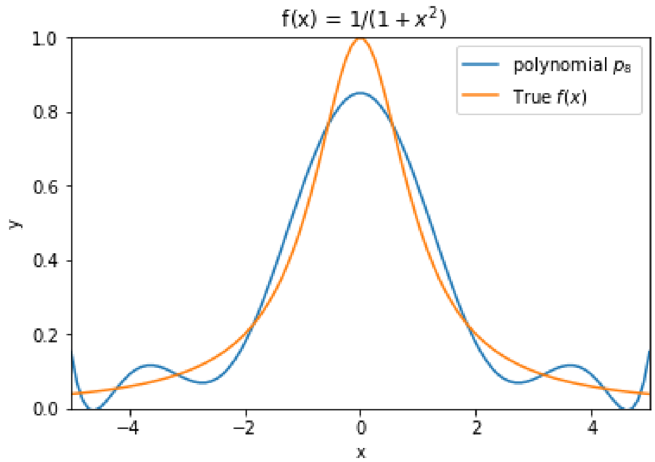

Figure 3: f VS p

Question 2

Implement a computer program that applies Simpson's rule for evaluating the integral

$$\int_a^b f(x)dx$$

Use your program to work out numerical approximations of the integrals

(a) $\int_0^1 x^3 dx$
(b) $\int_0^1 e^x dx$
(c) $\int_0^{10} e^x dx$
(d) $\int_0^{\pi} sin(x)dx$

How big is the numerical integration error in these cases?

Solution

Here is the program.

```
import numpy as np
from scipy.integrate import quad

def f1(x):
    return x**3
def f2(x):
    return np.exp(x)
def f3(x):
```

```
    return np.sin(x)

def Simpson_integral(x,f):
    H = (x[1]-x[0])/2.
    ff = H/3.*(f(x[0]) + 4. * f((x[0] + x[1])/2.) + f(x[1]))
    true_ff = quad(f,x[0],x[1])[0]
    error = np.abs(ff-true_ff)
    return ff,error

x1 = np.array([0,1])
x2 = np.array([0,10])
x3 = np.array([0,np.pi])

ff1,error1 = Simpson_integral(x1,f1)
print('the a integral:', ff1, 'the abs error :', error1)

ff2,error2 = Simpson_integral(x1,f2)
print('the b integral:', ff2,'the abs error :', error2)

ff3,error3 = Simpson_integral(x2,f2)
print('the c integral:', ff3,'the abs error :', error3)

ff4,error4 = Simpson_integral(x3,f3)
print('the d integral:', ff4,'the abs error :', error4)
```

The numerical approximations of the intergrals are :

$(a) \int_0^1 x^3 dx \approx 0.25$
$(b) \int_0^1 e^x dx \approx 1.7188611518765928$
$(c) \int_0^{10} e^x dx \approx 37701.86405202837$
$(d) \int_0^{\pi} sin(x) dx \approx 2.0943951023931953$

The numerical integration errors are :

$(a) |\int_0^1 x^3 dx - 0.25| = 0$
$(b) |\int_0^1 e^x dx - 1.7188611518765928| = 5.7932e-4$
$(c) |\int_0^{10} e^x dx - 37701.86405202837| = 1.5676e+4$
$(d) |\int_0^{\pi} sin(x) dx \approx 2.0943951023931953| = 9.4395e-2$

Question 3

Can you use Simpson's rule to find a numerical approximation of the integral

$$\int_1^{\infty} e^{-x^2} dx$$

Use a suitable variable transformation to replace the integral by an integral over $[0,1]$ rather than $[1,\infty]$. Also explain how you can find an upper bound on the numerical integration error.

Solution

The original intergration becomes

$$\begin{aligned}\int_1^\infty e^{-x^2}dx &= \int_0^\infty e^{-x^2}dx - \int_0^1 e^{-x^2}dx \\ &= \frac{\sqrt{\pi}}{2} - \int_0^1 e^{-x^2}dx\end{aligned}$$

Using Simpson's rule to solve the second term:

$$\begin{aligned}\int_0^1 e^{-x^2}dx &= \frac{H}{3}(f(0)+4f(\frac{1}{2})+f(1)) \\ &= 0.7472\end{aligned}$$

Therefore,

$$\int_1^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} - 0.7472$$

And the fourth of $f(x)$ is

$$f^{(4)}(x) = 4e^{-x^2}(4x^4 - 12x^2 + 3)$$

Here is a graph of the fourth derivative.

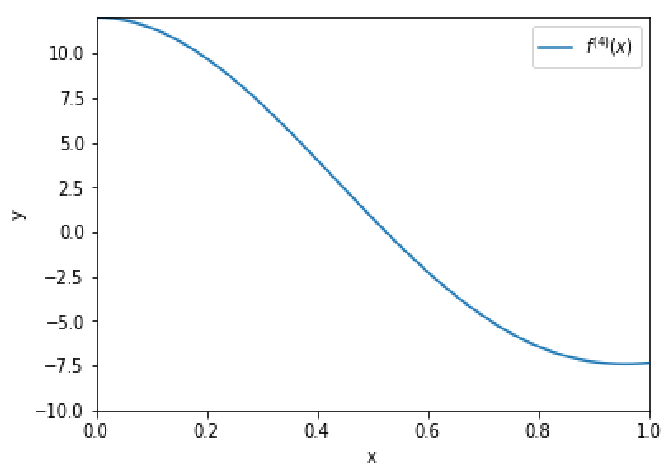

Figure 4: the fourth derivative

Thus,

$$\max_{\xi \in [0,1]} f^{(4)}(\xi) = f^{(4)}(0) = 12$$

Here is the upper bound.

$$|\int_0^1 (f(x) - p(x))dx| \leq \frac{1}{2880} \max_{\xi \in [0,1]} f^{(4)}(\xi) = \frac{1}{2880} f^{(4)}(0) = 0.004167$$