Assignment 8: Dynamic Game

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Simulate data

Suppose that there are $m = 1, \cdots, M$ markets and in each market there are $i = 1, \cdots, N$ firms and each firm makes decisions for $t = 1, \cdots, \infty$. In the following, I suppress the index of market, m. We solve the model under the infinite-horizon assumption, but generate data only for $t = 1, \cdots, T$. There are $L = 3$ state $\{1, 2, 3\}$ states for each firm. Each firm can choose $K + 1 = 2$ actions $\{0, 1\}$. Thus, $m_a := (K + 1)^N$ and $m_s = L^N$. Let a_i and s_i be firm i's action and state and a and s are vectors of individual actions and states.

The mean period payoff to firm i is:

$$\pi_i(a, s) := \tilde{\pi}(a_i, s_i, \overline{s}_{-i}) := \alpha \ln s_i - \eta \ln s_i \sum_{j \neq i} \ln s_j - \beta a_i,$$

where $\alpha, \beta, \eta > 0$, and $\alpha > \eta$. The term η means that the returns to investment decreases as rival's average state profile improves. The period payoff is:

$$\tilde{\pi}(a_i, s_i, \overline{s}_{-i}) + \epsilon_i(a_i),$$

and $\epsilon_i(a_i)$ is an i.i.d. type-I extreme random variable that is independent of all the other variables.

At the beginning of each period, the state s is realized and publicly observed. Then choice-specific shocks $\epsilon_i(a_i), a_i = 0, 1$ are realized and privately observed by firm $i = 1, \cdots, N$. Then each firm simultaneously chooses her action. Then, the game moves to next period.

State transition is independent across firms conditional on individual state and action.

Suppose that $s_i > 1$ and $s_i < L$. If $a_i = 0$, the state stays at the same state with probability $1 - \kappa$ and moves down by 1 with probability κ. If $a = 1$, the state moves up by 1 with probability γ, moves down by 1 with probability κ, and stays at the same with probability $1 - \kappa - \gamma$.

Suppose that $s_i = 1$. If $a_i = 0$, the state stays at the same state with probability 1. If $a_i = 1$, the state moves up by 1 with probability γ and stays at the same with probability $1 - \gamma$.

Suppose that $s_i = L$. If $a_i = 0$, the state stays at the same state with probability $1 - \kappa$ and moves down by 1 with probability κ. If $a = 1$, the state moves down by 1 with probability κ, and stays at the same with probability $1 - \kappa$.

The mean period profit is summarized in Π as:

$$\Pi := \begin{pmatrix} \pi(1,1) \\ \vdots \\ \pi(m_a, 1) \\ \vdots \\ \pi(1, m_s) \\ \vdots \\ \pi(m_a, m_s) \end{pmatrix}$$

The transition law is summarized in G as:

$$g(a, s, s') := \mathbb{P}\{s_{t+1} = s' | s_t = s, a_t = a\},$$

$$G := \begin{pmatrix} g(1,1,1) & \cdots & g(1,1,m_s) \\ \vdots & & \vdots \\ g(m_a,1,1) & \cdots & g(m_a,1,m_s) \\ & \vdots & \\ g(1,m_s,1) & \cdots & g(1,m_s,m_s) \\ \vdots & & \vdots \\ g(m_a,m_s,1) & \cdots & g(m_a,m_s,m_s) \end{pmatrix}.$$

The discount factor is denoted by δ. We simulate data for M markets with N firms for T periods.

1. Set constants and parameters as follows:

```r
# set seed
set.seed(1)
# set constants
L <- 5
K <- 1
T <- 100
N <- 3
M <- 1000
lambda <- 1e-10
# set parameters
alpha <- 1
eta <- 0.3
beta <- 2
kappa <- 0.1
gamma <- 0.6
delta <- 0.95
```

2. Write a function `compute_action_state_space(K, L, N)` that returns a data frame for action and state space. Returned objects are list of data frame `A` and `S`. In `A`, column `k` is the index of an action profile, `i` is the index of a firm, and `a` is the action of the firm. In `S`, column `l` is the index of an state profile, `i` is the index of a firm, and `s` is the state of the firm.

```r
# possible actions and states
compute_action_state_space <-
  function(L, K, N) {
    # action profile
    A_i <- seq(0, K)
    A <- rep(list(A_i), N)
    A <- expand.grid(A)
    A <- A %>%
      dplyr::mutate(k = 1:dim(A)[1]) %>%
      reshape2::melt(id.vars = "k") %>%
      dplyr::rename(i = variable, a = value) %>%
      dplyr::mutate(i = gsub("Var", "", i),
                    i = as.integer(i)) %>%
      tibble::as_tibble() %>%
      dplyr::arrange(k, i)

    # state profile
```

```r
    S_i <- seq(1, L)
    S <- rep(list(S_i), N)
    S <- expand.grid(S)
    S <- S %>%
      dplyr::mutate(l = 1:dim(S)[1]) %>%
      reshape2::melt(id.vars = "l") %>%
      dplyr::rename(i = variable, s = value) %>%
      dplyr::mutate(i = gsub("Var", "", i),
                    i = as.integer(i)) %>%
      tibble::as_tibble() %>%
      dplyr::arrange(l, i)

    # return
    return(list(A = A, S = S))
  }
output <- compute_action_state_space(L, K, N)
A <- output$A
head(A)
```

```
## # A tibble: 6 x 3
##       k     i     a
##   <int> <int> <int>
## 1     1     1     0
## 2     1     2     0
## 3     1     3     0
## 4     2     1     1
## 5     2     2     0
## 6     2     3     0
```

```r
tail(A)
```

```
## # A tibble: 6 x 3
##       k     i     a
##   <int> <int> <int>
## 1     7     1     0
## 2     7     2     1
## 3     7     3     1
## 4     8     1     1
## 5     8     2     1
## 6     8     3     1
```

```r
S <- output$S
head(S)
```

```
## # A tibble: 6 x 3
##       l     i     s
##   <int> <int> <int>
## 1     1     1     1
## 2     1     2     1
## 3     1     3     1
## 4     2     1     2
## 5     2     2     1
## 6     2     3     1
```

```r
tail(S)
```

```
## # A tibble: 6 x 3
##       l     i     s
##   <int> <int> <int>
## 1   124     1     4
## 2   124     2     5
## 3   124     3     5
## 4   125     1     5
## 5   125     2     5
## 6   125     3     5
```

```
# dimension
m_a <- max(A$k); m_a
```

```
## [1] 8
```

```
m_s <- max(S$l); m_s
```

```
## [1] 125
```

3. Write function `compute_PI_game(alpha, beta, eta, L, K, N)` that returns a list of Π_i.

```
# compute PI for a game
compute_PI_game <-
  function(alpha, beta, eta, A, S) {
    # action and state space
    m_a <- max(A$k)
    m_s <- max(S$l)
    N <- max(A$i)
    # baseline PI
    PI <-
      expand.grid(i = 1:N, l = 1:m_s, k = 1:m_a) %>%
      tibble::as_tibble() %>%
      dplyr::arrange(i, l, k) %>%
      dplyr::left_join(S, by = c("i", "l")) %>%
      dplyr::left_join(A, by = c("i", "k"))
    # average s
    PI <-
      PI %>%
      dplyr::group_by(l, k) %>%
      dplyr::mutate(s_bar = sum(log(s)) - log(s)) %>%
      dplyr::ungroup()
    # mean profit
    PI <-
      PI %>%
      dplyr::mutate(pi = alpha * log(s) - eta * log(s) * s_bar - beta * a)
    # to list
    PI <-
      foreach (ii = 1:N) %do% {
        PI_i <-
          PI %>%
          dplyr::filter(i == ii) %>%
          dplyr::arrange(l, k)
        PI_i <- matrix(PI_i$pi)
        return(PI_i)
      }
    # return
    return(PI)
```

```
  }
PI <- compute_PI_game(alpha, beta, eta, A, S)
head(PI[[N]])
```

```
##      [,1]
## [1,]    0
## [2,]    0
## [3,]    0
## [4,]    0
## [5,]   -2
## [6,]   -2
```

```
dim(PI[[N]])[1] == m_s * m_a
```

```
## [1] TRUE
```

4. Write function `compute_G_game(g, A, S)` that converts an individual transition probability matrix into a joint transition probability matrix G.

```
# compute G for game
compute_G_game <-
  function(G_marginal, A, S) {
    # convert to a data frame
    G_marginal <- G_marginal %>%
      reshape2::melt() %>%
      tibble::as_tibble() %>%
      dplyr::mutate(a = gsub("_.*", "", Var1),
                    a = gsub("k", "", a),
                    a = as.integer(a),
                    s = gsub(".*_l", "", Var1),
                    s = as.integer(s),
                    s_next = gsub("l", "", Var2),
                    s_next = as.integer(s_next)) %>%
      dplyr::rename(g = value) %>%
      dplyr::select(s_next, s, a, g)
    # action and state space
    m_a <- max(A$k)
    m_s <- max(S$l)
    N <- max(A$i)
    # baseline G
    G <-
      expand.grid(l_next = 1:m_s, l = 1:m_s, k = 1:m_a, i = 1:N) %>%
      tibble::as_tibble() %>%
      dplyr::left_join(S, by = c("l_next" = "l", "i")) %>%
      dplyr::rename("s_next" = "s") %>%
      dplyr::left_join(S, by = c("l", "i")) %>%
      dplyr::left_join(A, by = c("k", "i")) %>%
      dplyr::left_join(G_marginal, by = c("s_next", "s", "a"))
    # joint probability
    G <-
      G %>%
      dplyr::group_by(l_next, l, k) %>%
      dplyr::summarise(g = prod(g)) %>%
      dplyr::ungroup()
    # to matrix
    G <-
```

```r
      G %>%
      reshape2::dcast(formula = l + k ~ l_next, value.var = "g") %>%
      dplyr::select(-l, -k) %>%
      as.matrix()

    # return
    return(G)
  }

G_marginal <- compute_G(kappa, gamma, L, K)
G <- compute_G_game(G_marginal, A, S)
head(G)
```

```
##         1    2 3 4 5    6    7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22
## [1,] 1.00 0.00 0 0 0 0.00 0.00 0 0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [2,] 0.40 0.60 0 0 0 0.00 0.00 0 0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [3,] 0.40 0.00 0 0 0 0.60 0.00 0 0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [4,] 0.16 0.24 0 0 0 0.24 0.36 0 0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [5,] 0.40 0.00 0 0 0 0.00 0.00 0 0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [6,] 0.16 0.24 0 0 0 0.00 0.00 0 0  0  0  0  0  0  0  0  0  0  0  0  0  0
##      23 24 25   26   27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44
## [1,]  0  0  0 0.00 0.00  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [2,]  0  0  0 0.00 0.00  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [3,]  0  0  0 0.00 0.00  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [4,]  0  0  0 0.00 0.00  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [5,]  0  0  0 0.60 0.00  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [6,]  0  0  0 0.24 0.36  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
##      45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67
## [1,]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [2,]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [3,]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [4,]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [5,]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [6,]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
##      68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90
## [1,]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [2,]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [3,]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [4,]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [5,]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
## [6,]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
##      91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109
## [1,]  0  0  0  0  0  0  0  0  0   0   0   0   0   0   0   0   0   0   0
## [2,]  0  0  0  0  0  0  0  0  0   0   0   0   0   0   0   0   0   0   0
## [3,]  0  0  0  0  0  0  0  0  0   0   0   0   0   0   0   0   0   0   0
## [4,]  0  0  0  0  0  0  0  0  0   0   0   0   0   0   0   0   0   0   0
## [5,]  0  0  0  0  0  0  0  0  0   0   0   0   0   0   0   0   0   0   0
## [6,]  0  0  0  0  0  0  0  0  0   0   0   0   0   0   0   0   0   0   0
##      110 111 112 113 114 115 116 117 118 119 120 121 122 123 124 125
## [1,]   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
## [2,]   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
## [3,]   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
## [4,]   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
## [5,]   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
```

```
## [6,]    0    0    0    0    0    0    0    0    0    0    0    0    0    0    0    0
```

```r
dim(G)[1] == m_s * m_a
```

```
## [1] TRUE
```

```r
dim(G)[2] == m_s
```

```
## [1] TRUE
```

The ex-ante-value function for a firm is written as a function of a conditional choice probability as follows:

$$\varphi_i^{(\theta_1, \theta_2)}(p) := [I - \delta \Sigma_i(p) G]^{-1} [\Sigma_i(p) \Pi_i + D_i(p)],$$

where $\theta_1 = (\alpha, \beta, \eta)$ and $\theta_2 = (\kappa, \gamma)$, $p_i(a_i|s)$ is the probability that firm i choose action a_i when the state profile is s, and:

$$p(a|s) = \prod_{i=1}^N p_i(a_i|s),$$

$$p(s) = \begin{pmatrix} p(1|s) \\ \vdots \\ p(m_a|s) \end{pmatrix},$$

$$p = \begin{pmatrix} p(1) \\ \vdots \\ p(m_s) \end{pmatrix},$$

$$\Sigma(p) = \begin{pmatrix} p(1)' & & \\ & \ddots & \\ & & p(L)' \end{pmatrix}$$

and:

$$D_i(p) = \begin{pmatrix} \sum_{k=0}^K \mathbb{E}\{\epsilon_i^k | a_i = k, 1\} p_i(a_i = k|1) \\ \vdots \\ \sum_{k=0}^K \mathbb{E}\{\epsilon_i^k | a_i = k, m_s\} p_i(a_i = k|m_s) \end{pmatrix}.$$

5. Write a function `initialize_p_marginal(A, S)` that defines an initial marginal condition choice probability. In the output `p_marginal`, `p` is the probability for firm `i` to take action `a` conditional on the state profile being `l`. Next, write a function `compute_p_joint(p_marginal, A, S)` that computes a corresponding joint conditional choice probability from a marginal conditional choice probability. In the output `p_joint`, `p` is the joint probability that firms take action profile `k` condition on the state profile being `l`. Finally, write a function `compute_p_marginal(p_joint, A, S)` that compute a corresponding marginal conditional choice probability from a joint conditional choice probability.

```r
# define a conditional choice probability for each firm
initialize_p_marginal <-
  function(A, S) {
    m_s <- max(S$l)
    N <- max(S$i)
    K <- max(A$a)
    p_marginal <-
      expand.grid(i = 1:N, l = 1:m_s, a = 0:K) %>%
      tibble::as_tibble() %>%
```

```r
      dplyr::mutate(p = 1/(K + 1)) %>%
      dplyr::arrange(i, l)
    return(p_marginal)
  }
# compute joint conitional choice probability
compute_p_joint <-
  function(p_marginal, A, S) {
    m_s <- max(S$l)
    m_a <- max(A$k)
    N <- max(A$i)
    # p_joint baseline
    p_joint <-
      expand.grid(i = 1:N, l = 1:m_s, k = 1:m_a) %>%
      tibble::as_tibble() %>%
      dplyr::arrange(i, l, k) %>%
      dplyr::left_join(A, by = c("i", "k")) %>%
      dplyr::left_join(p_marginal, by = c("i", "l", "a"))
    # joint probability
    p_joint <-
      p_joint %>%
      dplyr::group_by(l, k) %>%
      dplyr::summarise(p = prod(p)) %>%
      dplyr::ungroup()
    # return
    return(p_joint)
  }
# compute marginal conditional choice probability
compute_p_marginal <-
  function(p_joint, A, S) {
    N <- max(S$i)
    m_a <- max(A$k)
    m_s <- max(S$l)
    p_joint <-
      expand.grid(i = 1:N, l = 1:m_s, k = 1:m_a) %>%
      tibble::as_tibble() %>%
      dplyr::left_join(p_joint, by = c("l", "k")) %>%
      dplyr::left_join(A, by = c("i", "k")) %>%
      dplyr::group_by(i, l, a) %>%
      dplyr::summarise(p = sum(p)) %>%
      dplyr::ungroup()
    return(p_joint)
  }

# define a conditional choice probability for each firm
p_marginal <- initialize_p_marginal(A, S)
p_marginal
```

```
## # A tibble: 750 x 4
##         i     l     a     p
##     <int> <int> <int> <dbl>
##   1     1     1     0   0.5
##   2     1     1     1   0.5
##   3     1     2     0   0.5
##   4     1     2     1   0.5
```

```
## 5      1     3     0   0.5
## 6      1     3     1   0.5
## 7      1     4     0   0.5
## 8      1     4     1   0.5
## 9      1     5     0   0.5
## 10     1     5     1   0.5
## # ... with 740 more rows
```

```r
dim(p_marginal)[1] == N * m_s * (K + 1)
```

```
## [1] TRUE
```

```r
# compute joint conitional choice probability from marginal probability
p_joint <- compute_p_joint(p_marginal, A, S)
p_joint
```

```
## # A tibble: 1,000 x 3
##         l     k     p
##     <int> <int> <dbl>
##  1      1     1 0.125
##  2      1     2 0.125
##  3      1     3 0.125
##  4      1     4 0.125
##  5      1     5 0.125
##  6      1     6 0.125
##  7      1     7 0.125
##  8      1     8 0.125
##  9      2     1 0.125
## 10      2     2 0.125
## # ... with 990 more rows
```

```r
dim(p_joint)[1] == m_s * m_a
```

```
## [1] TRUE
```

```r
# compute marginal conditional chocie probability from joint probability
p_marginal_2 <- compute_p_marginal(p_joint, A, S)
max(abs(p_marginal - p_marginal_2))
```

```
## [1] 0
```

6. Write a function `compute_Sigma(p_marginal, A, S)` that computes $\Sigma(p)$ given a joint conditional choice probability. Then, write a function `compute_D(p_marginal)` that returns a list of $D_i(p)$.

```r
# compute Sigma for ex-ante value function calculation
compute_Sigma <-
  function(p_marginal, A, S) {
    p_joint <- compute_p_joint(p_marginal, A, S)
    m_s <- max(p_joint$l)
    p_joint <-
      foreach (l = 1:m_s) %do% {
        p_joint_l <- p_joint %>%
          dplyr::filter(l == 1) %>%
          dplyr::arrange(k)
        p_joint_l <- t(matrix(p_joint_l$p))
        return(p_joint_l)
      }
    Sigma <- Matrix::bdiag(p_joint)
```

```r
    return(Sigma)
  }

# compute D for ex-ante value function calculation
compute_D <-
  function(p_marginal) {
    N <- max(p_marginal$i)
    D <-
      p_marginal %>%
      dplyr::mutate(
        E_i = - digamma(1) - log(p),
        D_i = E_i * p
      ) %>%
      dplyr::group_by(i, l) %>%
      dplyr::summarise(D_i = sum(D_i)) %>%
      dplyr::ungroup()
    D <-
      foreach (ii = 1:N) %do% {
        D_i <- D %>%
          dplyr::filter(i == ii)
        D_i <- matrix(D_i$D_i)
        return(D_i)
      }
    # return
    return(D)
  }

# compute Sigma for ex-ante value function calculation
Sigma <- compute_Sigma(p_marginal, A, S)
head(Sigma)
```

```
## [1] 0.125 0.000 0.000 0.000 0.000 0.000
```

```r
dim(Sigma)[1] == m_s
```

```
## [1] TRUE
```

```r
dim(Sigma)[2] == m_s * m_a
```

```
## [1] TRUE
```

```r
# compute D for ex-ante value function calculation
D <- compute_D(p_marginal)
head(D[[N]])
```

```
##          [,1]
## [1,] 1.270363
## [2,] 1.270363
## [3,] 1.270363
## [4,] 1.270363
## [5,] 1.270363
## [6,] 1.270363
```

```r
dim(D[[N]])[1] == m_s
```

```
## [1] TRUE
```

7. Write a function `compute_exante_value_game(p_marginal, A, S, PI, G, delta)` that returns a list of matrices whose i-th element represents the ex-ante value function given a conditional choice probability for firm i.

```r
# compute ex-ante value funciton for each firm
compute_exante_value_game <-
  function(p_marginal, A, S, PI, G, delta) {
    # compute Sigma for ex-ante value function calculation
    Sigma <- compute_Sigma(p_marginal, A, S)
    # compute D for ex-ante value function calculation
    D <- compute_D(p_marginal)
    # compute exante value function
    term_1 <- diag(dim(Sigma)[1]) - delta * Sigma %*% G
    V <-
      foreach (i = 1:length(D)) %do% {
        PI_i <- PI[[i]]
        D_i <- D[[i]]
        term_2_i <- Sigma %*% PI_i + D_i
        V_i <-
          Matrix::solve(term_1, term_2_i)
        # name
        rownames(V_i) <- paste("l", 1:dim(V_i)[1], sep = "")
        # return
        return(V_i)
      }
    # return
    return(V)
  }

# compute ex-ante value funciton for each firm
V <- compute_exante_value_game(p_marginal, A, S, PI, G, delta)
head(V[[N]])
```

```
## [1] 10.786330 10.175982  9.606812  9.255459  9.115332 10.175982
```

```r
dim(V[[N]])[1] == m_s
```

```
## [1] TRUE
```

The optimal conditional choice probability is written as a function of an ex-ante value function and a conditional choice probability of others as follows:

$$\Lambda_i^{(\theta_1,\theta_2)}(V_i, p_{-i})(a_i, s) := \frac{\exp\{\sum_{a_{-i}} p_{-i}(a_{-i}|s)[\pi_i(a_i, a_{-i}, s) + \delta \sum_{s'} V_i(s')g(a_i, a_{-i}, s, s')]\}}{\sum_{a_i'} \exp\{\sum_{a_{-i}} p_{-i}(a_{-i}|s)[\pi_i(a_i', a_{-i}, s) + \delta \sum_{s'} V_i(s')g(a_i', a_{-i}, s, s')]\}},$$

where V is an ex-ante value function.

8. Write a function `compute_profile_value_game(V, PI, G, delta, S, A)` that returns a data frame that contains information on value function at a state and action profile for each firm. In the output `value`, `i` is the index of a firm, `l` is the index of a state profile, `k` is the index of an action profile, and `value` is the value for the firm at the state and action profile.

```r
# compute state-action-profile value function
compute_profile_value_game <-
  function(V, PI, G, delta, S, A) {
    m_s <- max(S$l)
    m_a <- max(A$k)
    value <-
```

```r
      foreach (ii = 1:length(V), .combine = "rbind") %do% {
        # extract firm i-th data
        PI_i <- PI[[ii]]
        V_i <- V[[ii]]
        # compute action-profile specific value function
        value_i <- PI_i + delta * G %*% V_i
        # to data frame
        header <-
          expand.grid(i = ii, l = 1:m_s, k = 1:m_a) %>%
          tibble::as_tibble() %>%
          dplyr::arrange(i, l, k)
        value_i <-
          data.frame(header, value = as.numeric(value_i))
        # return
        return(value_i)
      }
    value <- value %>%
      tibble::as_tibble()
    # return
    return(value)
  }

# compute state-action-profile value function
value <- compute_profile_value_game(V, PI, G, delta, S, A)
value
```

```
## # A tibble: 3,000 x 4
##        i     l     k value
##    <int> <int> <int> <dbl>
##  1     1     1     1 10.2
##  2     1     1     2  9.63
##  3     1     1     3  9.90
##  4     1     1     4  9.13
##  5     1     1     5  9.90
##  6     1     1     6  9.13
##  7     1     1     7  9.55
##  8     1     1     8  8.64
##  9     1     2     1 13.0
## 10     1     2     2 12.1
## # ... with 2,990 more rows
```

```r
dim(value)[1] == N * m_s * m_a
```

```
## [1] TRUE
```

9. Write a function `compute_choice_value_game(p_marginal, V, PI, G, delta, A, S)` that computes a data frame that contains information on a choice-specific value function given an ex-ante value function and a conditional choice probability of others.

```r
# compute choice-specific value function
compute_choice_value_game <-
  function(p_marginal, V, PI, G, delta, A, S) {
    # compute joint conitional choice probability
    p_joint <- compute_p_joint(p_marginal, A, S)
    # compute state-action-profile value function
    value <- compute_profile_value_game(V, PI, G, delta, S, A)
```

```r
    # compute choice specific value function
    value <- value %>%
      # joint information
      dplyr::left_join(A, by = c("i", "k")) %>%
      dplyr::left_join(p_marginal, by = c("i", "l", "a")) %>%
      dplyr::rename(p_marginal = p) %>%
      dplyr::left_join(p_joint, by = c("l", "k")) %>%
      # probability that an action profile realizes conditional on i having a
      dplyr::mutate(p_others = p/p_marginal) %>%
      # choice-specific value on each state profiel for each firm
      dplyr::group_by(i, l, a) %>%
      dplyr::summarise(value = sum(value * p_others)) %>%
      dplyr::ungroup()
    # return
    return(value)
  }

# compute choice-specific value function
value <- compute_choice_value_game(p_marginal, V, PI, G, delta, A, S)
value
```

```
## # A tibble: 750 x 4
##        i     l     a value
##    <int> <int> <int> <dbl>
##  1     1     1     0  9.90
##  2     1     1     1  9.13
##  3     1     2     0 12.4
##  4     1     2     1 11.4
##  5     1     3     0 14.5
##  6     1     3     1 13.2
##  7     1     4     0 16.0
##  8     1     4     1 14.3
##  9     1     5     0 16.8
## 10     1     5     1 14.8
## # ... with 740 more rows
```

10. Write a function `compute_ccp_game(p_marginal, V, PI, G, delta, A, S)` that computes a data frame that contains information on a conditional choice probability given an ex-ante value function and a conditional choice probability of others.

```r
# compute conditional choice probability
compute_ccp_game <-
  function(p_marginal, V, PI, G, delta, A, S) {
    # compute choice-specific value function
    value <- compute_choice_value_game(p_marginal, V, PI, G, delta, A, S)
    # compute conditional choice probabiltiy
    p_marginal <-
      value %>%
      dplyr::group_by(i, l) %>%
      dplyr::mutate(p = exp(value) / sum(exp(value))) %>%
      dplyr::ungroup() %>%
      dplyr::select(i, l, a, p)
    # return
    return(p_marginal)
  }
```

```r
# compute conditional choice probability
p_marginal <- compute_ccp_game(p_marginal, V, PI, G, delta, A, S)
p_marginal
```

```
## # A tibble: 750 x 4
##        i     l     a     p
##    <int> <int> <int> <dbl>
##  1     1     1     0 0.683
##  2     1     1     1 0.317
##  3     1     2     0 0.734
##  4     1     2     1 0.266
##  5     1     3     0 0.794
##  6     1     3     1 0.206
##  7     1     4     0 0.840
##  8     1     4     1 0.160
##  9     1     5     0 0.881
## 10     1     5     1 0.119
## # ... with 740 more rows
```

11. Write a function `solve_dynamic_game(PI, G, L, K, delta, lambda, A, S)` that find the equilibrium conditional choice probability and ex-ante value function by iterating the update of an ex-ante value function and a best-response conditional choice probability. The iteration should stop when $\max_s |V^{(r+1)}(s) - V^{(r)}(s)| < \lambda$ with $\lambda = 10^{-10}$. There is no theoretical guarantee for the convergence.

```r
# solve the dynamic game model
solve_dynamic_game <-
  function(PI, G, L, K, delta, lambda, A, S) {
    # define a conditional choice probability for each firm
    p_marginal <- initialize_p_marginal(A, S)
    # compute ex-ante value funciton for each firm
    V <- compute_exante_value_game(p_marginal, A, S, PI, G, delta)
    distance <- 10000
    while (distance > lambda) {
      V_old <- V
      # compute conditional choice probability
      p_marginal <- compute_ccp_game(p_marginal, V, PI, G, delta, A, S)
      V <- compute_exante_value_game(p_marginal, A, S, PI, G, delta)
      V_check <- purrr::reduce(V, rbind)
      V_old_check <- purrr::reduce(V_old, rbind)
      distance <- max(abs(unlist(V_check) - unlist(V_old_check)))
    }
    return(list(p_marginal = p_marginal, V = V))
  }

# solve the dynamic game model
output <-
  solve_dynamic_game(PI, G, L, K, delta, lambda, A, S)
save(output, file = "data/A8_equilibrium.RData")
```

```r
load(file = "data/A8_equilibrium.RData")
p_marginal <- output$p_marginal; head(p_marginal)
```

```
## # A tibble: 6 x 4
##       i     l     a     p
##   <int> <int> <int> <dbl>
```

```
## 1     1     1     0 0.650
## 2     1     1     1 0.350
## 3     1     2     0 0.712
## 4     1     2     1 0.288
## 5     1     3     0 0.785
## 6     1     3     1 0.215
```

```r
V <- output$V[[N]]; head(V)
```

```
## [1] 13.25670 12.39394 11.47346 10.82808 10.53018 12.39394
```

```r
# compute joint conitional choice probability
p_joint <- compute_p_joint(p_marginal, A, S); head(p_joint)
```

```
## # A tibble: 6 x 3
##       l     k      p
##   <int> <int>  <dbl>
## 1     1     1 0.275
## 2     1     2 0.148
## 3     1     3 0.148
## 4     1     4 0.0795
## 5     1     5 0.148
## 6     1     6 0.0795
```

12. Write a function `simulate_dynamic_game(p_joint, l, G, N, T, S, A, seed)` that simulate the data for a market starting from an initial state for T periods. The function should accept a value of seed and set the seed at the beginning of the procedure inside the function, because the process is stochastic.

```r
# simulate a dynamic game
simulate_dynamic_game <-
  function(p_joint, l, G, N, T, S, A, seed) {
    set.seed(seed)
    m_a <- max(A$k)
    df_base <-
      expand.grid(t = 1:T, l = 1, k = 1) %>%
      tibble::as_tibble() %>%
      dplyr::arrange(t)
    df_base[df_base$t == 1, "l"] <- l
    for (tt in 1:T) {
      # state
      l_t <- df_base %>%
        dplyr::filter(t == tt)
      l_t <- l_t$l
      # draw action
      p_t <-
        p_joint %>%
        dplyr::filter(l == l_t)
      k_t <-
        p_t %>%
        dplyr::sample_n(1, weight = p)
      k_t <- k_t$k
      df_base[df_base$t == tt, "k"] <- k_t
      # draw next state
      if (tt < T) {
        g_t <- G[m_a * (l_t - 1) + k_t, ]
```

```r
        l_t_1 <-
          rmultinom(1, 1, prob = g_t)
        l_t_1 <- which(as.logical(l_t_1))
        df_base[tt + 1, "l"] <- l_t_1
      }
    }
    # augment information
    df <- foreach (ii = 1:N, .combine = "rbind") %do% {
      df_i <- data.frame(i = ii, df_base)
      return(df_i)
    }
    df <-
      df %>%
      tibble::as_tibble() %>%
      dplyr::left_join(S, by = c("l", "i")) %>%
      dplyr::left_join(A, by = c("k", "i")) %>%
      dplyr::select(t, i, dplyr::everything()) %>%
      dplyr::arrange(t, i)
    # return
    return(df)
  }

# simulate a dynamic game
# set initial state profile
l <- 1
# draw simulation for a firm
seed <- 1
df <- simulate_dynamic_game(p_joint, l, G, N, T, S, A, seed)
df
```

```
## # A tibble: 300 x 6
##        t     i     l     k     s     a
##    <int> <int> <dbl> <dbl> <int> <int>
##  1     1     1     1     1     1     0
##  2     1     2     1     1     1     0
##  3     1     3     1     1     1     0
##  4     2     1     1     5     1     0
##  5     2     2     1     5     1     0
##  6     2     3     1     5     1     1
##  7     3     1    26     6     1     1
##  8     3     2    26     6     1     0
##  9     3     3    26     6     2     1
## 10     4     1    26     5     1     0
## # ... with 290 more rows
```

13. Write a function `simulate_dynamic_decision_across_firms(p_joint, l, G, N, T, M, S, A, seed)` that returns simulation data for N firm. For firm i, set the seed at i

```r
# simulate data across markets
simulate_dynamic_decision_across_markets <-
  function(p_joint, l, G, N, T, M, S, A) {
    df <-
      foreach (mm = 1:M, .combine = "rbind") %dopar% {
        seed <- mm
        df_m <- simulate_dynamic_game(p_joint, l, G, N, T, S, A, seed)
```

```r
        df_m <- data.frame(m = mm, df_m)
        return(df_m)
      }
    df <- tibble::as_tibble(df)
    return(df)
  }

# simulate data across markets
df <- simulate_dynamic_decision_across_markets(p_joint, l, G, N, T, M, S, A)
save(df, file = "data/A8_df.RData")
```

```r
load(file = "data/A8_df.RData")
df
```

```
## # A tibble: 300,000 x 7
##        m     t     i     l     k     s     a
##    <int> <int> <int> <dbl> <dbl> <int> <int>
##  1     1     1     1     1     1     1     0
##  2     1     1     2     1     1     1     0
##  3     1     1     3     1     1     1     0
##  4     1     2     1     1     5     1     0
##  5     1     2     2     1     5     1     0
##  6     1     2     3     1     5     1     1
##  7     1     3     1    26     6     1     1
##  8     1     3     2    26     6     1     0
##  9     1     3     3    26     6     2     1
## 10     1     4     1    26     5     1     0
## # ... with 299,990 more rows
```

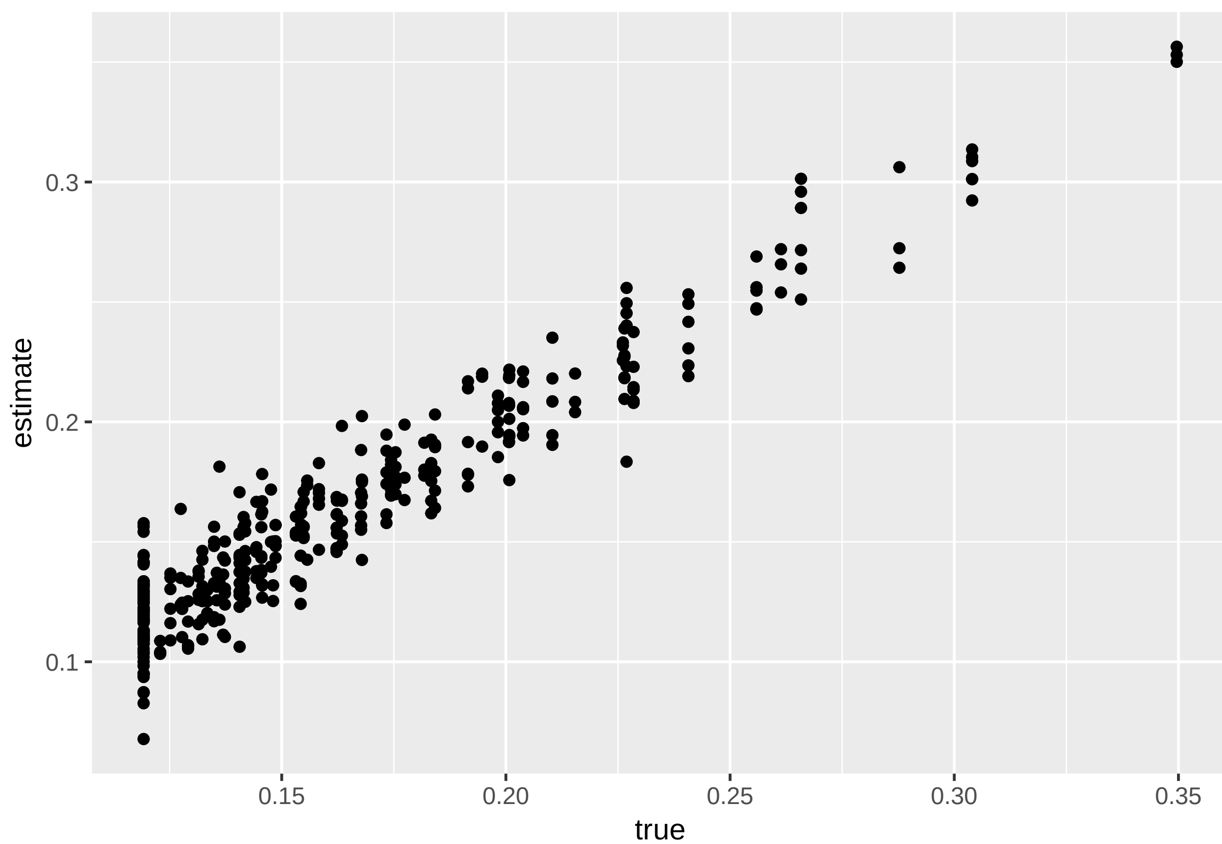
```r
summary(df)
```

```
##        m                t                 i           l         
##  Min.   :   1.0   Min.   :  1.00   Min.   :1   Min.   :  1.00  
##  1st Qu.: 250.8   1st Qu.: 25.75   1st Qu.:1   1st Qu.: 28.00  
##  Median : 500.5   Median : 50.50   Median :2   Median : 53.00  
##  Mean   : 500.5   Mean   : 50.50   Mean   :2   Mean   : 55.08  
##  3rd Qu.: 750.2   3rd Qu.: 75.25   3rd Qu.:3   3rd Qu.: 83.00  
##  Max.   :1000.0   Max.   :100.00   Max.   :3   Max.   :125.00  
##        k               s               a         
##  Min.   :1.000   Min.   :1.000   Min.   :0.0000  
##  1st Qu.:1.000   1st Qu.:2.000   1st Qu.:0.0000  
##  Median :1.000   Median :3.000   Median :0.0000  
##  Mean   :2.244   Mean   :2.753   Mean   :0.1776  
##  3rd Qu.:3.000   3rd Qu.:4.000   3rd Qu.:0.0000  
##  Max.   :8.000   Max.   :5.000   Max.   :1.0000  
```

14. Write a function `estimate_ccp_marginal_game(df)` that returns a non-parametric estimate of the marginal conditional choice probability for each firm in the data. Compare the estimated conditional choice probability and the true conditional choice probability by a bar plot.

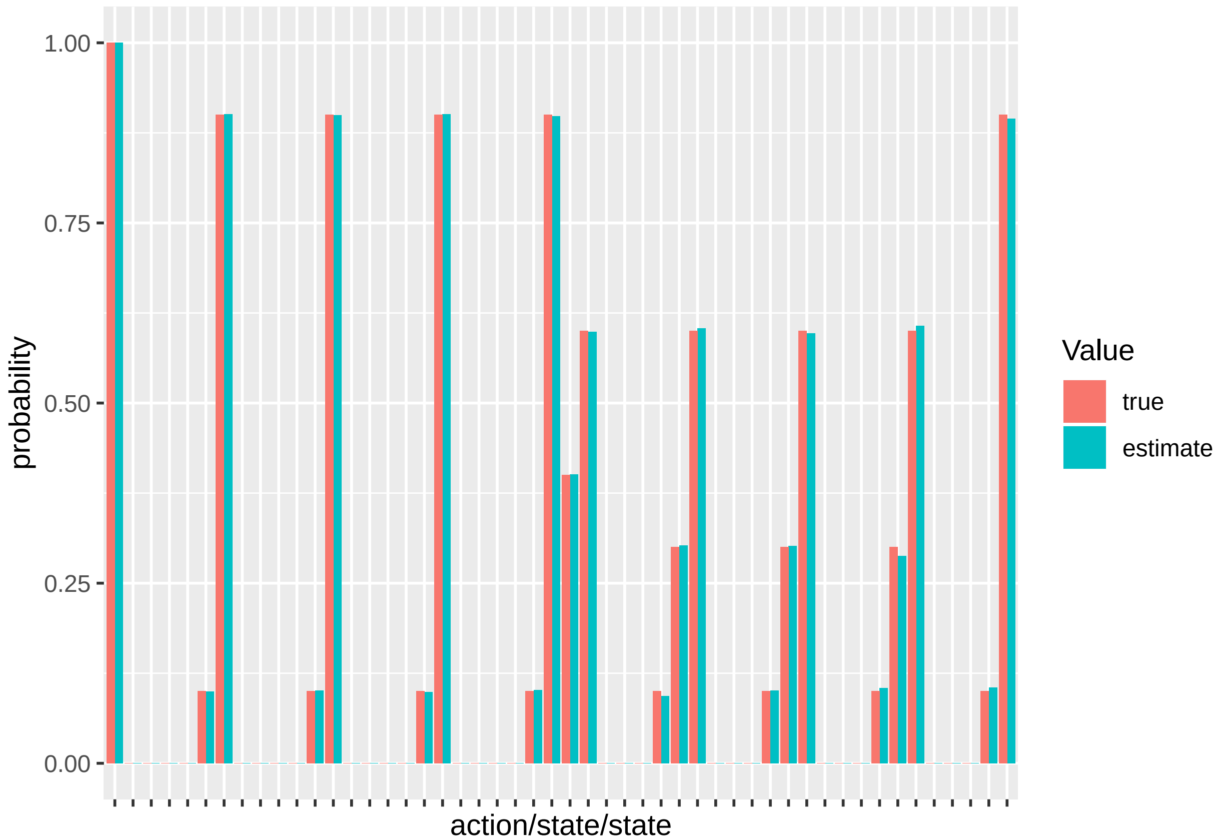
```r
# non-parametrically estimate the conditional choice probability
estimate_ccp_marginal_game <-
  function(df) {
    p_marginal_est <- df %>%
      dplyr::group_by(i, l, a) %>%
      dplyr::summarise(p = length(a)) %>%
```

```
      dplyr::ungroup()
    p_marginal_est <- p_marginal_est %>%
      dplyr::group_by(i, l) %>%
      dplyr::mutate(p = p / sum(p)) %>%
      dplyr::ungroup()
    p_marginal_est <-
      p_marginal_est %>%
      tidyr::complete(i, l, a, fill = list(p = 0)) %>%
      dplyr::arrange(i, l, a)
    return(p_marginal_est)
  }

# non-parametrically estimate the conditional choice probability
p_marginal_est <- estimate_ccp_marginal_game(df)
check_ccp <- p_marginal_est %>%
  dplyr::rename(estimate = p) %>%
  dplyr::left_join(p_marginal, by = c("i", "l", "a")) %>%
  dplyr::rename(true = p) %>%
  dplyr::filter(a == 1)
ggplot(data = check_ccp, aes(x = true, y = estimate)) +
  geom_point() +
  labs(fill = "Value") + xlab("true") + ylab("estimate")
```



15. Write a function `estimate_G_marginal(df)` that returns a non-parametric estimate of the marginal transition probability matrix. Compare the estimated transition matrix and the true transition matrix by a bar plot.

```r
# non-parametrically estimate individual transition probability
estimate_G_marginal <-
  function(df) {
    L <- max(df$s)
    # estimate individual transition probability matrixz
    G_marginal_est <- df %>%
      dplyr::arrange(m, i, t) %>%
      dplyr::group_by(m, i) %>%
      dplyr::mutate(s_lead = dplyr::lead(s, 1)) %>%
      dplyr::filter(!is.na(s_lead)) %>%
      dplyr::ungroup() %>%
      dplyr::group_by(s, a, s_lead) %>%
      dplyr::summarise(g = length(s_lead)) %>%
      dplyr::ungroup()
    G_marginal_est <-
      G_marginal_est %>%
      dplyr::group_by(s, a) %>%
      dplyr::mutate(g = g / sum(g)) %>%
      dplyr::ungroup() %>%
      tidyr::complete(s, a, s_lead, fill = list(g = 0)) %>%
      dplyr::arrange(s, a, s_lead) %>%
      reshape2::dcast(formula = s + a ~ s_lead, value.var = "g")
    rownames(G_marginal_est) <- paste("k", G_marginal_est[, "a"], "_l", G_marginal_est[, "s"], sep = "")
    G_marginal_est <- G_marginal_est[, !colnames(G_marginal_est) %in% c("s", "a")]
    colnames(G_marginal_est) <- paste("l", 1:L, sep = "")
    G_marginal_est <- as.matrix(G_marginal_est)
    return(G_marginal_est)
  }

# non-parametrically estimate individual transition probability
G_marginal_est <- estimate_G_marginal(df)
check_G <- data.frame(type = "true", reshape2::melt(G_marginal))
check_G_est <- data.frame(type = "estimate", reshape2::melt(G_marginal_est))
check_G <- rbind(check_G, check_G_est)
check_G$variable = paste(check_G$Var1, check_G$Var2, sep = "_")
ggplot(data = check_G, aes(x = variable, y = value,
                           fill = type)) +
    geom_bar(stat = "identity", position = "dodge") +
  labs(fill = "Value") + xlab("action/state/state") + ylab("probability") +
  theme(axis.text.x = element_blank())
```



Estimate parameters

1. Vectorize the parameters as follows:

```
theta_1 <- c(alpha, beta, eta)
theta_2 <- c(kappa, gamma)
theta <- c(theta_1, theta_2)
```

We estimate the parameters by a CCP approach.

1. Write a function `estimate_theta_2_game(df)` that returns the estimates of κ and γ directly from data by counting relevant events.

```
# estimate theta_2
estimate_theta_2_game <-
  function(df) {
    # estimate kappa
    kappa_est <- df %>%
      dplyr::arrange(m, i, t) %>%
      dplyr::group_by(m, i) %>%
      dplyr::mutate(s_lead = dplyr::lead(s, 1)) %>%
      dplyr::filter(!is.na(s_lead)) %>%
      dplyr::ungroup() %>%
      dplyr::mutate(move_down = ifelse(s_lead < s, 1, 0)) %>%
      dplyr::filter(s > 1) %>%
      dplyr::group_by(move_down) %>%
      dplyr::summarise(kappa = length(move_down)) %>%
      dplyr::ungroup() %>%
```

```r
      dplyr::mutate(kappa = kappa / sum(kappa)) %>%
      dplyr::filter(move_down == 1)
    kappa_est <- kappa_est$kappa
    # estimate gamma
    gamma_est <- df %>%
      dplyr::arrange(m, i, t) %>%
      dplyr::group_by(m, i) %>%
      dplyr::mutate(s_lead = dplyr::lead(s, 1)) %>%
      dplyr::filter(!is.na(s_lead)) %>%
      dplyr::ungroup() %>%
      dplyr::mutate(move_up = ifelse(s_lead > s, 1, 0)) %>%
      dplyr::filter(s < L, a == 1) %>%
      dplyr::group_by(move_up) %>%
      dplyr::summarise(gamma = length(move_up)) %>%
      dplyr::ungroup() %>%
      dplyr::mutate(gamma = gamma / sum(gamma)) %>%
      dplyr::filter(move_up == 1)
    gamma_est <- gamma_est$gamma
    # theta_2
    theta_2_est <- c(kappa_est, gamma_est)
    # return
    return(theta_2_est)
  }

# estimate theta_2
theta_2_est <- estimate_theta_2_game(df); theta_2_est
```

```
## [1] 0.09995377 0.60136442
```

The objective function of the minimum distance estimator based on the conditional choice probability approach is:

$$\frac{1}{NKm_s}\sum_{i=1}^{N}\sum_{l=1}^{m_s}\sum_{k=1}^{K}\{\hat{p}_i(a_k|s_l) - p_i^{(\theta_1,\theta_2)}(a_k|s_l)\}^2,$$

where $\hat{p}_i$ is the non-parametric estimate of the marginal conditional choice probability and $p_i^{(\theta_1,\theta_2)}$ is the marginal conditional choice probability under parameters θ_1 and θ_2 given $\hat{p}_i$. a_k is k-th action for a firm and s_l is l-th state profile.

2. Write a function `compute_CCP_objective_game(theta_1, theta_2, p_est, L, K, delta)` that returns the objective function of the above minimum distance estimator given a non-parametric estimate of the conditional choice probability and θ_1 and θ_2.

```r
# compute the objective function of the minimum distance estimator based on the CCP approach
compute_CCP_objective_game <-
  function(theta_1, theta_2, p_marginal_est, A, S, delta, lambda) {
    # extract parameters
    alpha <- theta_1[1]
    beta <- theta_1[2]
    eta <- theta_1[3]
    kappa <- theta_2[1]
    gamma <- theta_2[2]
    L <- max(S$s)
    K <- max(A$a)
    # construct PI
    PI <- compute_PI_game(alpha, beta, eta, A, S)
```

```r
    # construct G
    G_marginal <- compute_G(kappa, gamma, L, K)
    G <- compute_G_game(G_marginal, A, S)

    # update ccp
    V <- compute_exante_value_game(p_marginal_est, A, S, PI, G, delta)
    p_marginal <- compute_ccp_game(p_marginal_est, V, PI, G, delta, A, S)

    # minimum distance
    distance <- p_marginal %>%
      dplyr::rename(ccp = p) %>%
      dplyr::left_join(p_marginal_est, by = c("i", "l", "a")) %>%
      dplyr::filter(a > 0) %>%
      dplyr::mutate(x = (ccp - p)^2) %>%
      dplyr::summarise(mean(x, na.rm = TRUE)) %>%
      as.numeric()
    # return
    return(distance)
  }

# compute the objective function of the minimum distance estimator based on the CCP approach
objective <- compute_CCP_objective_game(theta_1, theta_2, p_marginal_est, A, S, delta, lambda)
save(objective, file = "data/A8_objective.RData")
```

```r
load(file = "data/A8_objective.RData")
objective
```

```
## [1] 0.0002737567
```

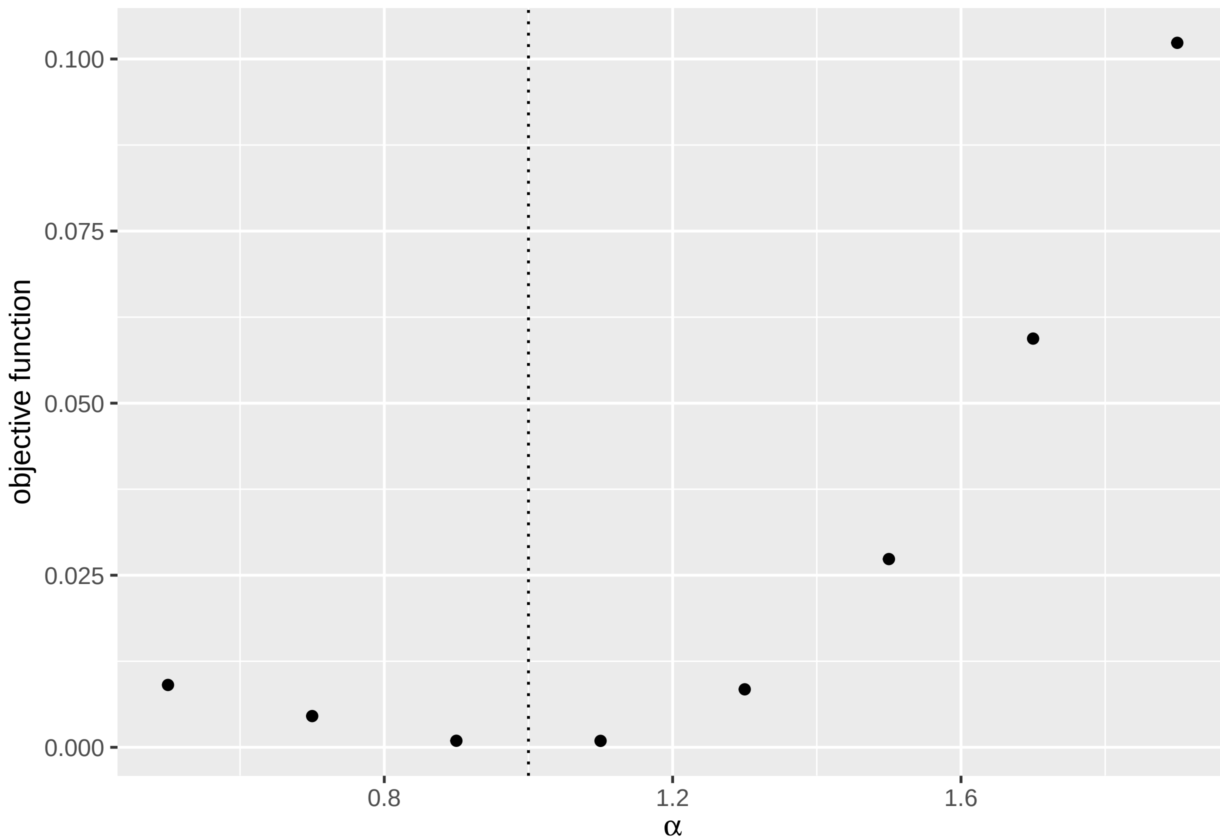
3. Check the value of the objective function around the true parameter.

```r
# label
label <- c("\\alpha", "\\beta", "\\eta")
label <- paste("$", label, "$", sep = "")
# compute the graph
graph <- foreach (i = 1:length(theta_1)) %do% {
  theta_i <- theta_1[i]
  theta_i_list <- theta_i * seq(0.5, 2, by = 0.2)
  objective_i <-
    foreach (j = 1:length(theta_i_list),
             .combine = "rbind") %dopar% {
               theta_ij <- theta_i_list[j]
               theta_j <- theta_1
               theta_j[i] <- theta_ij
               objective_ij <-
                 compute_CCP_objective_game(theta_j, theta_2, p_marginal_est, A, S, delta, lambda)
               return(objective_ij)
             }
  df_graph <- data.frame(x = theta_i_list, y = objective_i)
  g <- ggplot(data = df_graph, aes(x = x, y = y)) +
    geom_point() +
    geom_vline(xintercept = theta_i, linetype = "dotted") +
    ylab("objective function") + xlab(TeX(label[i]))
  return(g)
}
```

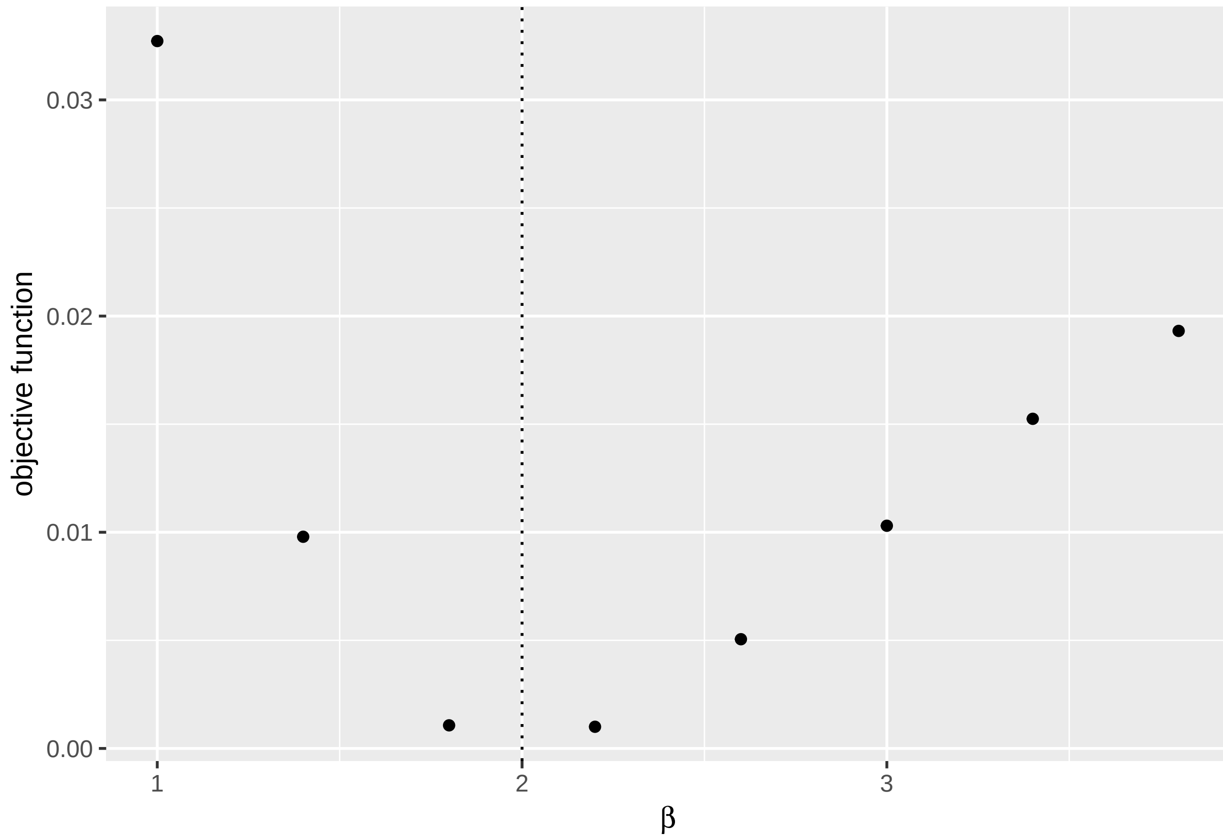
```
save(graph, file = "data/A8_CCP_graph.RData")
```

```
load(file = "data/A8_CCP_graph.RData")
graph
```

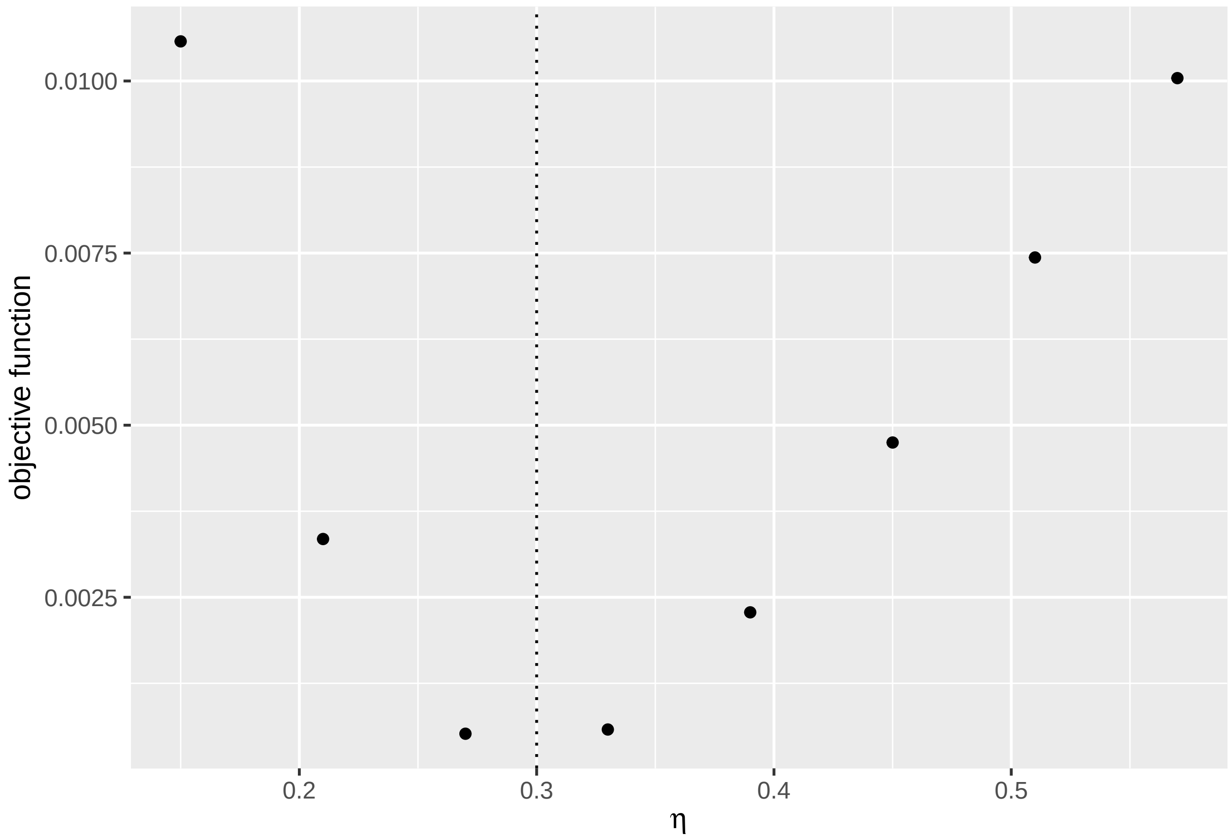
```
## [[1]]
```




```
##
## [[2]]
```




```
##
## [[3]]
```



4. Estimate the parameters by minimizing the objective function. To keep the model to be well-defined, impose an ad hoc lower and upper bounds such that $\alpha \in [0, 1], \beta \in [0, 5], \delta \in [0, 1]$.

```
lower <- rep(0, length(theta_1))
upper <- c(1, 5, 0.3)
CCP_result <-
  optim(par = theta_1,
        fn = compute_CCP_objective_game,
        method = "L-BFGS-B",
        lower = lower,
        upper = upper,
        theta_2 = theta_2_est,
        p_marginal_est = p_marginal_est,
        A = A,
        S = S,
        delta = delta,
        lambda = lambda)
save(CCP_result, file = "data/A8_CCP_result.RData")
```

```
load(file = "data/A8_CCP_result.RData")
CCP_result
```

```
## $par
## [1] 1.000000 2.011446 0.294446
##
## $value
## [1] 0.0002702126
##
```

```
## $counts
## function gradient
##       12       12
##
## $convergence
## [1] 0
##
## $message
## [1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"
```

```
compare <-
  data.frame(
    true = theta_1,
    estimate = CCP_result$par
  ); compare
```

```
##   true estimate
## 1  1.0 1.000000
## 2  2.0 2.011446
## 3  0.3 0.294446
```