

Assignment 8: Dynamic Game

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Simulate data

Suppose that there are $m = 1, \dots, M$ markets and in each market there are $i = 1, \dots, N$ firms and each firm makes decisions for $t = 1, \dots, \infty$. In the following, I suppress the index of market, m . We solve the model under the infinite-horizon assumption, but generate data only for $t = 1, \dots, T$. There are $L = 3$ state $\{1, 2, 3\}$ states for each firm. Each firm can choose $K + 1 = 2$ actions $\{0, 1\}$. Thus, $m_a := (K + 1)^N$ and $m_s = L^N$. Let a_i and s_i be firm i 's action and state and a and s are vectors of individual actions and states.

The mean period payoff to firm i is:

$$\pi_i(a, s) := \tilde{\pi}(a_i, s_i, \bar{s}_{-i}) := \alpha \ln s_i - \eta \ln s_i \sum_{j \neq i} \ln s_j - \beta a_i,$$

where $\alpha, \beta, \eta > 0$, and $\alpha > \eta$. The term η means that the returns to investment decreases as rival's average state profile improves. The period payoff is:

$$\tilde{\pi}(a_i, s_i, \bar{s}_{-i}) + \epsilon_i(a_i),$$

and $\epsilon_i(a_i)$ is an i.i.d. type-I extreme random variable that is independent of all the other variables.

At the beginning of each period, the state s is realized and publicly observed. Then choice-specific shocks $\epsilon_i(a_i), a_i = 0, 1$ are realized and privately observed by firm $i = 1, \dots, N$. Then each firm simultaneously chooses her action. Then, the game moves to next period.

State transition is independent across firms conditional on individual state and action.

Suppose that $s_i > 1$ and $s_i < L$. If $a_i = 0$, the state stays at the same state with probability $1 - \kappa$ and moves down by 1 with probability κ . If $a_i = 1$, the state moves up by 1 with probability γ , moves down by 1 with probability κ , and stays at the same with probability $1 - \kappa - \gamma$.

Suppose that $s_i = 1$. If $a_i = 0$, the state stays at the same state with probability 1. If $a_i = 1$, the state moves up by 1 with probability γ and stays at the same with probability $1 - \gamma$.

Suppose that $s_i = L$. If $a_i = 0$, the state stays at the same state with probability $1 - \kappa$ and moves down by 1 with probability κ . If $a_i = 1$, the state moves down by 1 with probability κ , and stays at the same with probability $1 - \kappa$.

The mean period profit is summarized in Π as:

$$\Pi := \begin{pmatrix} \pi(1, 1) \\ \vdots \\ \pi(m_a, 1) \\ \vdots \\ \pi(1, m_s) \\ \vdots \\ \pi(m_a, m_s) \end{pmatrix}$$

The transition law is summarized in G as:

$$g(a, s, s') := \mathbb{P}\{s_{t+1} = s' | s_t = s, a_t = a\},$$

$$G := \begin{pmatrix} g(1, 1, 1) & \cdots & g(1, 1, m_s) \\ \vdots & & \vdots \\ g(m_a, 1, 1) & \cdots & g(m_a, 1, m_s) \\ \vdots & & \vdots \\ g(1, m_s, 1) & \cdots & g(1, m_s, m_s) \\ \vdots & & \vdots \\ g(m_a, m_s, 1) & \cdots & g(m_a, m_s, m_s) \end{pmatrix}.$$

The discount factor is denoted by δ . We simulate data for M markets with N firms for T periods.

1. Set constants and parameters as follows:

```
# set seed
set.seed(1)
# set constants
L <- 5
K <- 1
T <- 100
N <- 3
M <- 1000
lambda <- 1e-10
# set parameters
alpha <- 1
eta <- 0.3
beta <- 2
kappa <- 0.1
gamma <- 0.6
delta <- 0.95
```

2. Write a function `compute_action_state_space(K, L, N)` that returns a data frame for action and state space. Returned objects are list of data frame **A** and **S**. In **A**, column **k** is the index of an action profile, **i** is the index of a firm, and **a** is the action of the firm. In **S**, column **l** is the index of an state profile, **i** is the index of a firm, and **s** is the state of the firm.

```
# possible actions and states
compute_action_state_space <-
function(L, K, N) {
  # action profile
  A_i <- seq(0, K)
  A <- rep(list(A_i), N)
  A <- expand.grid(A)
  A <- A %>%
    dplyr::mutate(k = 1:dim(A)[1]) %>%
    reshape2::melt(id.vars = "k") %>%
    dplyr::rename(i = variable, a = value) %>%
    dplyr::mutate(i = gsub("Var", "", i),
                  i = as.integer(i)) %>%
    tibble::as_tibble() %>%
    dplyr::arrange(k, i)
```

```

# state profile
S_i <- seq(1, L)
S <- rep(list(S_i), N)
S <- expand.grid(S)
S <- S %>%
  dplyr::mutate(l = 1:dim(S)[1]) %>%
  reshape2::melt(id.vars = "l") %>%
  dplyr::rename(i = variable, s = value) %>%
  dplyr::mutate(i = gsub("Var", "", i),
               i = as.integer(i)) %>%
  tibble::as_tibble() %>%
  dplyr::arrange(l, i)

# return
return(list(A = A, S = S))
}
output <- compute_action_state_space(L, K, N)
A <- output$A
head(A)

```

```

## # A tibble: 6 x 3
##       k     i     a
##   <int> <int> <int>
## 1     1     1     0
## 2     1     2     0
## 3     1     3     0
## 4     2     1     1
## 5     2     2     0
## 6     2     3     0

```

```
tail(A)
```

```

## # A tibble: 6 x 3
##       k     i     a
##   <int> <int> <int>
## 1     7     1     0
## 2     7     2     1
## 3     7     3     1
## 4     8     1     1
## 5     8     2     1
## 6     8     3     1

```

```

S <- output$S
head(S)

```

```

## # A tibble: 6 x 3
##       l     i     s
##   <int> <int> <int>
## 1     1     1     1
## 2     1     2     1
## 3     1     3     1
## 4     2     1     2
## 5     2     2     1
## 6     2     3     1

```

```
tail(S)
```

```
## # A tibble: 6 x 3
##       l     i     s
##   <int> <int> <int>
## 1   124     1     4
## 2   124     2     5
## 3   124     3     5
## 4   125     1     5
## 5   125     2     5
## 6   125     3     5
```

```
# dimension
m_a <- max(A$k); m_a
```

```
## [1] 8
```

```
m_s <- max(S$l); m_s
```

```
## [1] 125
```

3. Write function `compute_PI_game(alpha, beta, eta, L, K, N)` that returns a list of Π_i .

```
# compute PI for a game
compute_PI_game <-
function(alpha, beta, eta, A, S) {
  # action and state space
  m_a <- max(A$k)
  m_s <- max(S$l)
  N <- max(A$i)
  # baseline PI
  PI <-
    expand.grid(i = 1:N, l = 1:m_s, k = 1:m_a) %>%
    tibble::as_tibble() %>%
    dplyr::arrange(i, l, k) %>%
    dplyr::left_join(S, by = c("i", "l")) %>%
    dplyr::left_join(A, by = c("i", "k"))
  # average s
  PI <-
    PI %>%
    dplyr::group_by(l, k) %>%
    dplyr::mutate(s_bar = sum(log(s)) - log(s)) %>%
    dplyr::ungroup()
  # mean profit
  PI <-
    PI %>%
    dplyr::mutate(pi = alpha * log(s) - eta * log(s) * s_bar - beta * a)
  # to list
  PI <-
    foreach (ii = 1:N) %do% {
      PI_i <-
        PI %>%
        dplyr::filter(i == ii) %>%
        dplyr::arrange(l, k)
      PI_i <- matrix(PI_i$pi)
      return(PI_i)
    }
}
```

```

    }
    # return
    return(PI)
  }
PI <- compute_PI_game(alpha, beta, eta, A, S)
head(PI[[N]])

```

```

##      [,1]
## [1,]    0
## [2,]    0
## [3,]    0
## [4,]    0
## [5,]   -2
## [6,]   -2

```

```
dim(PI[[N]])[1] == m_s * m_a
```

```
## [1] TRUE
```

4. Write function `compute_G_game(g, A, S)` that converts an individual transition probability matrix into a joint transition probability matrix G .

```

# compute G for game
compute_G_game <-
function(G_marginal, A, S) {
  # convert to a data frame
  G_marginal <- G_marginal %>%
    reshape2::melt() %>%
    tibble::as_tibble() %>%
    dplyr::mutate(a = gsub("_.*", "", Var1),
                  a = gsub("k", "", a),
                  a = as.integer(a),
                  s = gsub(".*_l", "", Var1),
                  s = as.integer(s),
                  s_next = gsub("l", "", Var2),
                  s_next = as.integer(s_next)) %>%
    dplyr::rename(g = value) %>%
    dplyr::select(s_next, s, a, g)
  # action and state space
  m_a <- max(A$k)
  m_s <- max(S$l)
  N <- max(A$i)
  # baseline G
  G <-
    expand.grid(l_next = 1:m_s, l = 1:m_s, k = 1:m_a, i = 1:N) %>%
    tibble::as_tibble() %>%
    dplyr::left_join(S, by = c("l_next" = "l", "i")) %>%
    dplyr::rename("s_next" = "s") %>%
    dplyr::left_join(S, by = c("l", "i")) %>%
    dplyr::left_join(A, by = c("k", "i")) %>%
    dplyr::left_join(G_marginal, by = c("s_next", "s", "a"))
  # joint probability
  G <-
    G %>%
    dplyr::group_by(l_next, l, k) %>%
    dplyr::summarise(g = prod(g)) %>%

```

```

    dplyr::ungroup()
    # to matrix
    G <-
    G %>%
    reshape2::dcast(formula = l + k ~ l_next, value.var = "g") %>%
    dplyr::select(-l, -k) %>%
    as.matrix()

    # return
    return(G)
  }

G_marginal <- compute_G(kappa, gamma, L, K)
G <- compute_G_game(G_marginal, A, S)
head(G)

##           1      2 3 4 5      6      7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
## [1,] 1.00 0.00 0 0 0 0.00 0.00 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [2,] 0.40 0.60 0 0 0 0.00 0.00 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [3,] 0.40 0.00 0 0 0 0.60 0.00 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [4,] 0.16 0.24 0 0 0 0.24 0.36 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [5,] 0.40 0.00 0 0 0 0.00 0.00 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [6,] 0.16 0.24 0 0 0 0.00 0.00 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
##           25      26      27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47
## [1,] 0 0.00 0.00 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [2,] 0 0.00 0.00 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [3,] 0 0.00 0.00 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [4,] 0 0.00 0.00 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [5,] 0 0.60 0.00 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [6,] 0 0.24 0.36 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
##           48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72
## [1,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [2,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [3,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [4,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [5,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [6,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
##           73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97
## [1,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [2,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [3,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [4,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [5,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [6,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
##           98 99 100 101 102 103 104 105 106 107 108 109 110 111 112 113 114 115 116
## [1,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [2,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [3,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [4,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [5,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [6,] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
##           117 118 119 120 121 122 123 124 125
## [1,] 0 0 0 0 0 0 0 0
## [2,] 0 0 0 0 0 0 0 0

```

```
## [3,] 0 0 0 0 0 0 0 0 0
## [4,] 0 0 0 0 0 0 0 0 0
## [5,] 0 0 0 0 0 0 0 0 0
## [6,] 0 0 0 0 0 0 0 0 0
```

```
dim(G)[1] == m_s * m_a
```

```
## [1] TRUE
```

```
dim(G)[2] == m_s
```

```
## [1] TRUE
```

The ex-ante-value function for a firm is written as a function of a conditional choice probability as follows:

$$\varphi_i^{(\theta_1, \theta_2)}(p) := [I - \delta \Sigma_i(p)G]^{-1}[\Sigma_i(p)\Pi_i + D_i(p)],$$

where $\theta_1 = (\alpha, \beta, \eta)$ and $\theta_2 = (\kappa, \gamma)$, $p_i(a_i|s)$ is the probability that firm i choose action a_i when the state profile is s , and:

$$p(a|s) = \prod_{i=1}^N p_i(a_i|s),$$

$$p(s) = \begin{pmatrix} p(1|s) \\ \vdots \\ p(m_a|s) \end{pmatrix},$$

$$p = \begin{pmatrix} p(1) \\ \vdots \\ p(m_s) \end{pmatrix},$$

$$\Sigma(p) = \begin{pmatrix} p(1)' & & \\ & \ddots & \\ & & p(L)' \end{pmatrix}$$

and:

$$D_i(p) = \begin{pmatrix} \sum_{k=0}^K \mathbb{E}\{\epsilon_i^k | a_i = k, 1\} p_i(a_i = k|1) \\ \vdots \\ \sum_{k=0}^K \mathbb{E}\{\epsilon_i^k | a_i = k, m_s\} p_i(a_i = k|m_s) \end{pmatrix}.$$

5. Write a function `initialize_p_marginal(A, S)` that defines an initial marginal condition choice probability. In the output `p_marginal`, `p` is the probability for firm `i` to take action `a` conditional on the state profile being 1. Next, write a function `compute_p_joint(p_marginal, A, S)` that computes a corresponding joint conditional choice probability from a marginal conditional choice probability. In the output `p_joint`, `p` is the joint probability that firms take action profile `k` condition on the state profile being 1. Finally, write a function `compute_p_marginal(p_joint, A, S)` that compute a corresponding marginal conditional choice probability from a joint conditional choice probability.

```
# define a conditional choice probability for each firm
initialize_p_marginal <-
function(A, S) {
  m_s <- max(S$1)
  N <- max(S$i)
  K <- max(A$a)
```

```

    p_marginal <-
      expand.grid(i = 1:N, l = 1:m_s, a = 0:K) %>%
      tibble::as_tibble() %>%
      dplyr::mutate(p = 1/(K + 1)) %>%
      dplyr::arrange(i, l)
    return(p_marginal)
  }
# compute joint conitional choice probability
compute_p_joint <-
  function(p_marginal, A, S) {
    m_s <- max(S$l)
    m_a <- max(A$k)
    N <- max(A$i)
    # p_joint baseline
    p_joint <-
      expand.grid(i = 1:N, l = 1:m_s, k = 1:m_a) %>%
      tibble::as_tibble() %>%
      dplyr::arrange(i, l, k) %>%
      dplyr::left_join(A, by = c("i", "k")) %>%
      dplyr::left_join(p_marginal, by = c("i", "l", "a"))
    # joint probability
    p_joint <-
      p_joint %>%
      dplyr::group_by(l, k) %>%
      dplyr::summarise(p = prod(p)) %>%
      dplyr::ungroup()
    # return
    return(p_joint)
  }
# compute marginal conditional choice probability
compute_p_marginal <-
  function(p_joint, A, S) {
    N <- max(S$i)
    m_a <- max(A$k)
    m_s <- max(S$l)
    p_joint <-
      expand.grid(i = 1:N, l = 1:m_s, k = 1:m_a) %>%
      tibble::as_tibble() %>%
      dplyr::left_join(p_joint, by = c("l", "k")) %>%
      dplyr::left_join(A, by = c("i", "k")) %>%
      dplyr::group_by(i, l, a) %>%
      dplyr::summarise(p = sum(p)) %>%
      dplyr::ungroup()
    return(p_joint)
  }

# define a conditional choice probability for each firm
p_marginal <- initialize_p_marginal(A, S)
p_marginal

```

```

## # A tibble: 750 x 4
##       i     l     a     p
##   <int> <int> <int> <dbl>
## 1     1     1     1     0  0.5

```



```
## 2      1      1      1      0.5
## 3      1      2      0      0.5
## 4      1      2      1      0.5
## 5      1      3      0      0.5
## 6      1      3      1      0.5
## 7      1      4      0      0.5
## 8      1      4      1      0.5
## 9      1      5      0      0.5
## 10     1      5      1      0.5
## # ... with 740 more rows
```

```
dim(p_marginal)[1] == N * m_s * (K + 1)
```

```
## [1] TRUE
```

```
# compute joint conditional choice probability from marginal probability
p_joint <- compute_p_joint(p_marginal, A, S)
p_joint
```

```
## # A tibble: 1,000 x 3
##       l      k      p
##   <int> <int> <dbl>
## 1     1     1  0.125
## 2     1     2  0.125
## 3     1     3  0.125
## 4     1     4  0.125
## 5     1     5  0.125
## 6     1     6  0.125
## 7     1     7  0.125
## 8     1     8  0.125
## 9     2     1  0.125
## 10    2     2  0.125
## # ... with 990 more rows
```

```
dim(p_joint)[1] == m_s * m_a
```

```
## [1] TRUE
```

```
# compute marginal conditional choice probability from joint probability
p_marginal_2 <- compute_p_marginal(p_joint, A, S)
max(abs(p_marginal - p_marginal_2))
```

```
## [1] 0
```

6. Write a function `compute_Sigma(p_marginal, A, S)` that computes $\Sigma(p)$ given a joint conditional choice probability. Then, write a function `compute_D(p_marginal)` that returns a list of $D_i(p)$.

```
# compute Sigma for ex-ante value function calculation
compute_Sigma <-
function(p_marginal, A, S) {
  p_joint <- compute_p_joint(p_marginal, A, S)
  m_s <- max(p_joint$l)
  p_joint <-
    foreach (ll = 1:m_s) %do% {
      p_joint_l <- p_joint %>%
        dplyr::filter(ll == 1) %>%
        dplyr::arrange(k)
      p_joint_l <- t(matrix(p_joint_l$p))
    }
}
```

```

    return(p_joint_l)
  }
  Sigma <- Matrix::bdiag(p_joint)
  return(Sigma)
}

# compute D for ex-ante value function calculation
compute_D <-
function(p_marginal) {
  N <- max(p_marginal$i)
  D <-
    p_marginal %>%
    dplyr::mutate(
      E_i = - digamma(1) - log(p),
      D_i = E_i * p
    ) %>%
    dplyr::group_by(i, l) %>%
    dplyr::summarise(D_i = sum(D_i)) %>%
    dplyr::ungroup()
  D <-
    foreach (ii = 1:N) %do% {
      D_i <- D %>%
        dplyr::filter(i == ii)
      D_i <- matrix(D_i$D_i)
      return(D_i)
    }
  # return
  return(D)
}

# compute Sigma for ex-ante value function calculation
Sigma <- compute_Sigma(p_marginal, A, S)
head(Sigma)

```

```

## 6 x 1000 sparse Matrix of class "dgCMatrix"
##
## [1,] 0.125 0.125 0.125 0.125 0.125 0.125 0.125 0.125 0.125 . . .
## [2,] . . . . . . . . . 0.125 0.125 0.125 0.125
## [3,] . . . . . . . . . . . . .
## [4,] . . . . . . . . . . . . .
## [5,] . . . . . . . . . . . . .
## [6,] . . . . . . . . . . . . .
##
## [1,] . . . . . . . . . . . . .
## [2,] 0.125 0.125 0.125 0.125 . . . . . . .
## [3,] . . . . 0.125 0.125 0.125 0.125 0.125 0.125 0.125 0.125
## [4,] . . . . . . . . . . . . .
## [5,] . . . . . . . . . . . . .
## [6,] . . . . . . . . . . . . .
##
## [1,] . . . . . . . . . . . . .
## [2,] . . . . . . . . . . . . .
## [3,] . . . . . . . . . . . . .
## [4,] 0.125 0.125 0.125 0.125 0.125 0.125 0.125 0.125 . . .

```

[illegible]

[illegible]

[illegible]

```

## [6,] . . . . .
##
## [1,] . . . . .
## [2,] . . . . .
## [3,] . . . . .
## [4,] . . . . .
## [5,] . . . . .
## [6,] . . . . .
##
## [1,] . . . . .
## [2,] . . . . .
## [3,] . . . . .
## [4,] . . . . .
## [5,] . . . . .
## [6,] . . . . .
##
## [1,] . . . . .
## [2,] . . . . .
## [3,] . . . . .
## [4,] . . . . .
## [5,] . . . . .
## [6,] . . . . .
##
## [1,] . . . . .
## [2,] . . . . .
## [3,] . . . . .
## [4,] . . . . .
## [5,] . . . . .
## [6,] . . . . .

dim(Sigma)[1] == m_s

## [1] TRUE

dim(Sigma)[2] == m_s * m_a

## [1] TRUE

# compute D for ex-ante value function calculation
D <- compute_D(p_marginal)
head(D[[N]])

##           [,1]
## [1,] 1.270363
## [2,] 1.270363
## [3,] 1.270363
## [4,] 1.270363
## [5,] 1.270363
## [6,] 1.270363

dim(D[[N]])[1] == m_s

## [1] TRUE

```

7. Write a function `compute_exante_value_game(p_marginal, A, S, PI, G, delta)` that returns a list of matrices whose i -th element represents the ex-ante value function given a conditional choice probability for firm i .

```

# compute ex-ante value function for each firm
compute_exante_value_game <-
function(p_marginal, A, S, PI, G, delta) {
  # compute Sigma for ex-ante value function calculation
  Sigma <- compute_Sigma(p_marginal, A, S)
  # compute D for ex-ante value function calculation
  D <- compute_D(p_marginal)
  # compute exante value function
  term_1 <- diag(dim(Sigma)[1]) - delta * Sigma %*% G
  V <-
  foreach (i = 1:length(D)) %do% {
    PI_i <- PI[[i]]
    D_i <- D[[i]]
    term_2_i <- Sigma %*% PI_i + D_i
    V_i <-
      Matrix::solve(term_1, term_2_i)
    # name
    rownames(V_i) <- paste("l", 1:dim(V_i)[1], sep = "")
    # return
    return(V_i)
  }
  # return
  return(V)
}

# compute ex-ante value function for each firm
V <- compute_exante_value_game(p_marginal, A, S, PI, G, delta)
head(V[[N]])

## 6 x 1 Matrix of class "dgeMatrix"
##      [,1]
## 11 10.786330
## 12 10.175982
## 13  9.606812
## 14  9.255459
## 15  9.115332
## 16 10.175982

dim(V[[N]])[1] == m_s

```

```
## [1] TRUE
```

The optimal conditional choice probability is written as a function of an ex-ante value function and a conditional choice probability of others as follows:

$$\Lambda_i^{(\theta_1, \theta_2)}(V_i, p_{-i})(a_i, s) := \frac{\exp\{\sum_{a_{-i}} p_{-i}(a_{-i}|s) [\pi_i(a_i, a_{-i}, s) + \delta \sum_{s'} V_i(s') g(a_i, a_{-i}, s, s')]\}}{\sum_{a'_i} \exp\{\sum_{a_{-i}} p_{-i}(a_{-i}|s) [\pi_i(a'_i, a_{-i}, s) + \delta \sum_{s'} V_i(s') g(a'_i, a_{-i}, s, s')]\}}},$$

where V is an ex-ante value function.

8. Write a function `compute_profile_value_game(V, PI, G, delta, S, A)` that returns a data frame that contains information on value function at a state and action profile for each firm. In the output value, i is the index of a firm, l is the index of a state profile, k is the index of an action profile, and value is the value for the firm at the state and action profile.

```

# compute state-action-profile value function
compute_profile_value_game <-
function(V, PI, G, delta, S, A) {
  m_s <- max(S$l)
  m_a <- max(A$k)
  value <-
    foreach (ii = 1:length(V), .combine = "rbind") %do% {
      # extract firm i-th data
      PI_i <- PI[[ii]]
      V_i <- V[[ii]]
      # compute action-profile specific value function
      value_i <- PI_i + delta * G %*% V_i
      # to data frame
      header <-
        expand.grid(i = ii, l = 1:m_s, k = 1:m_a) %>%
        tibble::as_tibble() %>%
        dplyr::arrange(i, l, k)
      value_i <-
        data.frame(header, value = as.numeric(value_i))
      # return
      return(value_i)
    }
  value <- value %>%
    tibble::as_tibble()
  # return
  return(value)
}

# compute state-action-profile value function
value <- compute_profile_value_game(V, PI, G, delta, S, A)
value

```

```

## # A tibble: 3,000 x 4
##       i     l     k value
##   <int> <int> <int> <dbl>
## 1     1     1     1  10.2
## 2     1     1     2   9.63
## 3     1     1     3   9.90
## 4     1     1     4   9.13
## 5     1     1     5   9.90
## 6     1     1     6   9.13
## 7     1     1     7   9.55
## 8     1     1     8   8.64
## 9     1     2     1  13.0
## 10    1     2     2  12.1
## # ... with 2,990 more rows

dim(value)[1] == N * m_s * m_a

```

```
## [1] TRUE
```

9. Write a function `compute_choice_value_game(p_marginal, V, PI, G, delta, A, S)` that computes a data frame that contains information on a choice-specific value function given an ex-ante value function and a conditional choice probability of others.


```

# compute choice-specific value function
compute_choice_value_game <-
function(p_marginal, V, PI, G, delta, A, S) {
  # compute joint conditional choice probability
  p_joint <- compute_p_joint(p_marginal, A, S)
  # compute state-action-profile value function
  value <- compute_profile_value_game(V, PI, G, delta, S, A)
  # compute choice specific value function
  value <- value %>%
    # joint information
    dplyr::left_join(A, by = c("i", "k")) %>%
    dplyr::left_join(p_marginal, by = c("i", "l", "a")) %>%
    dplyr::rename(p_marginal = p) %>%
    dplyr::left_join(p_joint, by = c("l", "k")) %>%
    # probability that an action profile realizes conditional on i having a
    dplyr::mutate(p_others = p/p_marginal) %>%
    # choice-specific value on each state profile for each firm
    dplyr::group_by(i, l, a) %>%
    dplyr::summarise(value = sum(value * p_others)) %>%
    dplyr::ungroup()
  # return
  return(value)
}

# compute choice-specific value function
value <- compute_choice_value_game(p_marginal, V, PI, G, delta, A, S)
value

```

```

## # A tibble: 750 x 4
##       i     l     a value
##   <int> <int> <int> <dbl>
## 1     1     1     0  9.90
## 2     1     1     1  9.13
## 3     1     2     0 12.4
## 4     1     2     1 11.4
## 5     1     3     0 14.5
## 6     1     3     1 13.2
## 7     1     4     0 16.0
## 8     1     4     1 14.3
## 9     1     5     0 16.8
## 10    1     5     1 14.8
## # ... with 740 more rows

```

10. Write a function `compute_ccp_game(p_marginal, V, PI, G, delta, A, S)` that computes a data frame that contains information on a conditional choice probability given an ex-ante value function and a conditional choice probability of others.

```

# compute conditional choice probability
compute_ccp_game <-
function(p_marginal, V, PI, G, delta, A, S) {
  # compute choice-specific value function
  value <- compute_choice_value_game(p_marginal, V, PI, G, delta, A, S)
  # compute conditional choice probability
  p_marginal <-
    value %>%

```

```

    dplyr::group_by(i, l) %>%
    dplyr::mutate(p = exp(value) / sum(exp(value))) %>%
    dplyr::ungroup() %>%
    dplyr::select(i, l, a, p)
  # return
  return(p_marginal)
}

# compute conditional choice probability
p_marginal <- compute_ccp_game(p_marginal, V, PI, G, delta, A, S)
p_marginal

```

```

## # A tibble: 750 x 4
##       i     l     a     p
##   <int> <int> <int> <dbl>
## 1     1     1     0 0.683
## 2     1     1     1 0.317
## 3     1     2     0 0.734
## 4     1     2     1 0.266
## 5     1     3     0 0.794
## 6     1     3     1 0.206
## 7     1     4     0 0.840
## 8     1     4     1 0.160
## 9     1     5     0 0.881
## 10    1     5     1 0.119
## # ... with 740 more rows

```

11. Write a function `solve_dynamic_game(PI, G, L, K, delta, lambda, A, S)` that find the equilibrium conditional choice probability and ex-ante value function by iterating the update of an ex-ante value function and a best-response conditional choice probability. The iteration should stop when $\max_s |V^{(r+1)}(s) - V^{(r)}(s)| < \lambda$ with $\lambda = 10^{-10}$. There is no theoretical guarantee for the convergence.

```

# solve the dynamic game model
solve_dynamic_game <-
function(PI, G, L, K, delta, lambda, A, S) {
  # define a conditional choice probability for each firm
  p_marginal <- initialize_p_marginal(A, S)
  # compute ex-ante value function for each firm
  V <- compute_exante_value_game(p_marginal, A, S, PI, G, delta)
  distance <- 10000
  while (distance > lambda) {
    V_old <- V
    # compute conditional choice probability
    p_marginal <- compute_ccp_game(p_marginal, V, PI, G, delta, A, S)
    V <- compute_exante_value_game(p_marginal, A, S, PI, G, delta)
    V_check <- purrr::reduce(V, rbind)
    V_old_check <- purrr::reduce(V_old, rbind)
    distance <- max(abs(unlist(V_check) - unlist(V_old_check)))
  }
  return(list(p_marginal = p_marginal, V = V))
}

# solve the dynamic game model
output <-
  solve_dynamic_game(PI, G, L, K, delta, lambda, A, S)

```

```
save(output, file = "data/A8_equilibrium.RData")
```

```
load(file = "data/A8_equilibrium.RData")
p_marginal <- output$p_marginal; head(p_marginal)
```

```
## # A tibble: 6 x 4
##       i     l     a     p
##   <int> <int> <int> <dbl>
## 1     1     1     0 0.534
## 2     1     1     1 0.466
## 3     1     2     0 0.545
## 4     1     2     1 0.455
## 5     1     3     0 0.629
## 6     1     3     1 0.371
```

```
V <- output$V[[N]]; head(V)
```

```
## 6 x 1 Matrix of class "dgeMatrix"
##      [,1]
## 11 18.98883
## 12 18.51236
## 13 18.08141
## 14 17.77417
## 15 17.59426
## 16 18.51236
```

```
# compute joint conitional choice probability
p_joint <- compute_p_joint(p_marginal, A, S); head(p_joint)
```

```
## # A tibble: 6 x 3
##       l     k     p
##   <int> <int> <dbl>
## 1     1     1 0.152
## 2     1     2 0.133
## 3     1     3 0.133
## 4     1     4 0.116
## 5     1     5 0.133
## 6     1     6 0.116
```

12. Write a function `simulate_dynamic_game(p_joint, l, G, N, T, S, A, seed)` that simulate the data for a market starting from an initial state for T periods. The function should accept a value of seed and set the seed at the beginning of the procedure inside the function, because the process is stochastic.

```
# simulate a dynamic game
simulate_dynamic_game <-
function(p_joint, l, G, N, T, S, A, seed) {
  set.seed(seed)
  m_a <- max(A$k)
  df_base <-
    expand.grid(t = 1:T, l = 1, k = 1) %>%
    tibble::as_tibble() %>%
    dplyr::arrange(t)
  df_base[df_base$t == 1, "l"] <- 1
  for (tt in 1:T) {
    # state
```

```

l_t <- df_base %>%
  dplyr::filter(t == tt)
l_t <- l_t$l
# draw action
p_t <-
  p_joint %>%
  dplyr::filter(l == l_t)
k_t <-
  p_t %>%
  dplyr::sample_n(1, weight = p)
k_t <- k_t$k
df_base[df_base$t == tt, "k"] <- k_t
# draw next state
if (tt < T) {
  g_t <- G[m_a * (l_t - 1) + k_t, ]
  l_t_1 <-
    rmultinom(1, 1, prob = g_t)
  l_t_1 <- which(as.logical(l_t_1))
  df_base[tt + 1, "l"] <- l_t_1
}
}
# augment information
df <- foreach(ii = 1:N, .combine = "rbind") %do% {
  df_i <- data.frame(i = ii, df_base)
  return(df_i)
}
df <-
  df %>%
  tibble::as_tibble() %>%
  dplyr::left_join(S, by = c("l", "i")) %>%
  dplyr::left_join(A, by = c("k", "i")) %>%
  dplyr::select(t, i, dplyr::everything()) %>%
  dplyr::arrange(t, i)
# return
return(df)
}

# simulate a dynamic game
# set initial state profile
l <- 1
# draw simulation for a firm
seed <- 1
df <- simulate_dynamic_game(p_joint, l, G, N, T, S, A, seed)
df

```

```

## # A tibble: 300 x 6
##       t     i     l     k     s     a
##   <int> <int> <dbl> <dbl> <int> <int>
## 1     1     1     1     5     1     0
## 2     1     2     1     5     1     0
## 3     1     3     1     5     1     1
## 4     2     1    26     2     1     1
## 5     2     2    26     2     1     0
## 6     2     3    26     2     2     0

```

```
## 7      3      1      26      8      1      1
## 8      3      2      26      8      1      1
## 9      3      3      26      8      2      1
## 10     4      1     52      8      2      1
## # ... with 290 more rows
```

13. Write a function `simulate_dynamic_decision_across_firms(p_joint, l, G, N, T, M, S, A, seed)` that returns simulation data for N firm. For firm i , set the seed at i

```
# simulate data across markets
simulate_dynamic_decision_across_markets <-
function(p_joint, l, G, N, T, M, S, A) {
  df <-
    foreach (mm = 1:M, .combine = "rbind",
             .packages = c("EmpiricalIO", "foreach", "magrittr")) %dopar% {
      seed <- mm
      df_m <- simulate_dynamic_game(p_joint, l, G, N, T, S, A, seed)
      df_m <- data.frame(m = mm, df_m)
      return(df_m)
    }
  df <- tibble::as_tibble(df)
  return(df)
}
```

```
# simulate data across markets
df <- simulate_dynamic_decision_across_markets(p_joint, l, G, N, T, M, S, A)
save(df, file = "data/A8_df.RData")
```

```
load(file = "data/A8_df.RData")
df
```

```
## # A tibble: 300,000 x 7
##       m      t      i      l      k      s      a
##   <int> <int> <int> <dbl> <dbl> <int> <int>
## 1      1      1      1      1      5      1      0
## 2      1      1      2      1      5      1      0
## 3      1      1      3      1      5      1      1
## 4      1      2      1     26      2      1      1
## 5      1      2      2     26      2      1      0
## 6      1      2      3     26      2      2      0
## 7      1      3      1     26      8      1      1
## 8      1      3      2     26      8      1      1
## 9      1      3      3     26      8      2      1
## 10     1      4      1     52      8      2      1
## # ... with 299,990 more rows
```

```
summary(df)
```

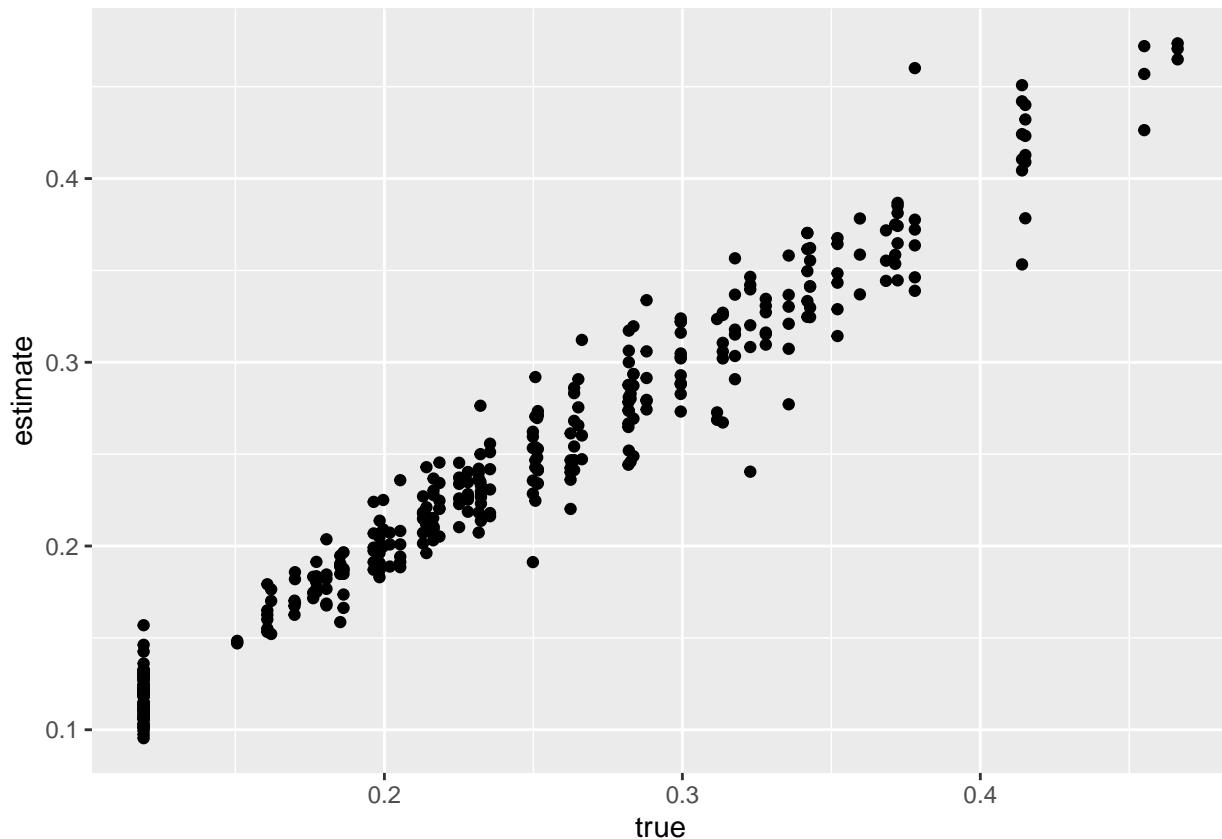
```
##           m           t           i           l           k
## Min.   : 1.0   Min.   : 1.00   Min.   :1   Min.   : 1.00   Min.   :1.000
## 1st Qu.: 250.8 1st Qu.: 25.75 1st Qu.:1 1st Qu.: 43.00 1st Qu.:1.000
## Median : 500.5 Median : 50.50 Median :2  Median : 74.00 Median :2.000
## Mean   : 500.5 Mean   : 50.50 Mean   :2   Mean   : 71.02 Mean   :2.497
## 3rd Qu.: 750.2 3rd Qu.: 75.25 3rd Qu.:3 3rd Qu.:100.00 3rd Qu.:4.000
## Max.   :1000.0 Max.   :100.00 Max.   :3   Max.   :125.00 Max.   :8.000
##           s           a
```

```
## Min.      :1.000   Min.      :0.0000
## 1st Qu.:2.000   1st Qu.:0.0000
## Median :3.000   Median :0.0000
## Mean    :3.273   Mean    :0.2141
## 3rd Qu.:5.000   3rd Qu.:0.0000
## Max.    :5.000   Max.    :1.0000
```

14. Write a function `estimate_ccp_marginal_game(df)` that returns a non-parametric estimate of the marginal conditional choice probability for each firm in the data. Compare the estimated conditional choice probability and the true conditional choice probability by a bar plot.

```
# non-parametrically estimate the conditional choice probability
estimate_ccp_marginal_game <-
function(df) {
  p_marginal_est <- df %>%
    dplyr::group_by(i, l, a) %>%
    dplyr::summarise(p = length(a)) %>%
    dplyr::ungroup()
  p_marginal_est <- p_marginal_est %>%
    dplyr::group_by(i, l) %>%
    dplyr::mutate(p = p / sum(p)) %>%
    dplyr::ungroup()
  p_marginal_est <-
    p_marginal_est %>%
    tidyr::complete(i, l, a, fill = list(p = 0)) %>%
    dplyr::arrange(i, l, a)
  return(p_marginal_est)
}

# non-parametrically estimate the conditional choice probability
p_marginal_est <- estimate_ccp_marginal_game(df)
check_ccp <- p_marginal_est %>%
  dplyr::rename(estimate = p) %>%
  dplyr::left_join(p_marginal, by = c("i", "l", "a")) %>%
  dplyr::rename(true = p) %>%
  dplyr::filter(a == 1)
ggplot(data = check_ccp, aes(x = true, y = estimate)) +
  geom_point() +
  labs(fill = "Value") + xlab("true") + ylab("estimate")
```



15. Write a function `estimate_G_marginal(df)` that returns a non-parametric estimate of the marginal transition probability matrix. Compare the estimated transition matrix and the true transition matrix by a bar plot.

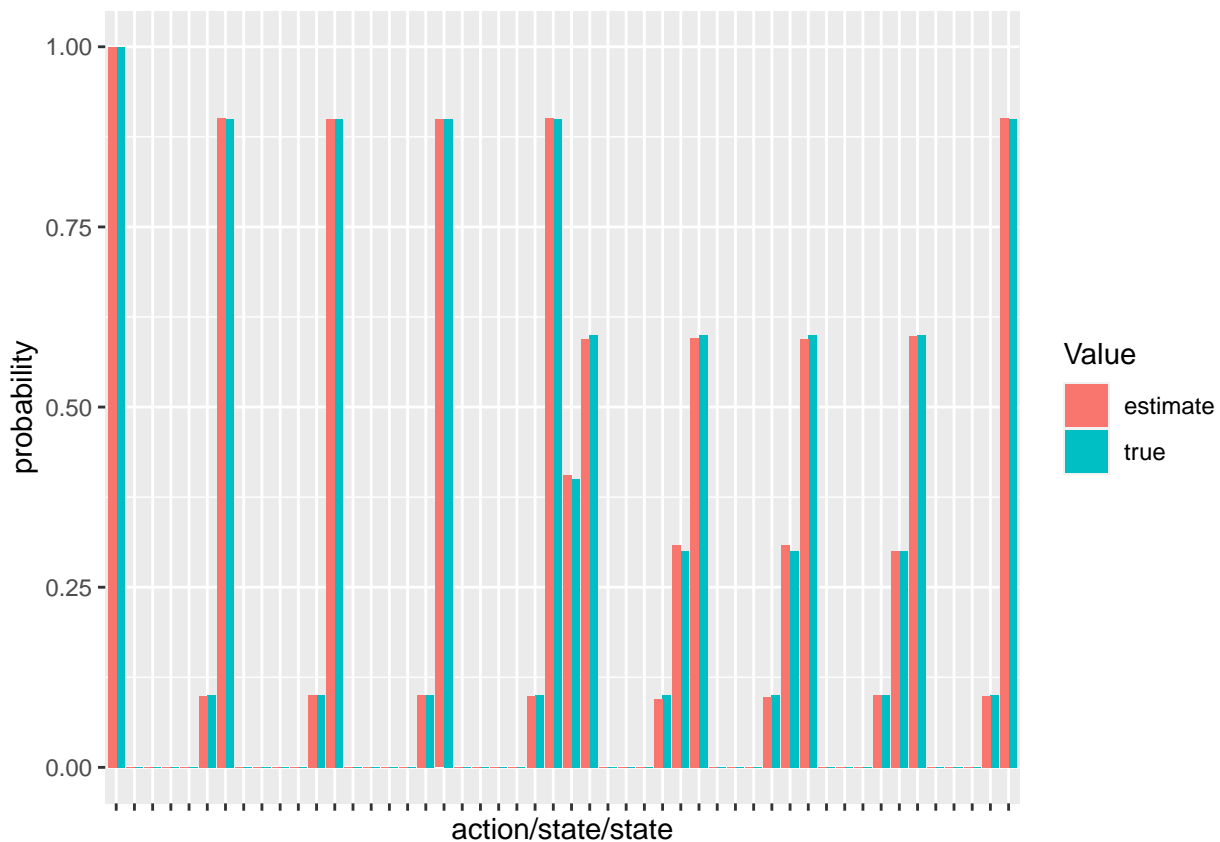
```
# non-parametrically estimate individual transition probability
estimate_G_marginal <-
function(df) {
  L <- max(df$s)
  # estimate individual transition probability matrix
  G_marginal_est <- df %>%
    dplyr::arrange(m, i, t) %>%
    dplyr::group_by(m, i) %>%
    dplyr::mutate(s_lead = dplyr::lead(s, 1)) %>%
    dplyr::filter(!is.na(s_lead)) %>%
    dplyr::ungroup() %>%
    dplyr::group_by(s, a, s_lead) %>%
    dplyr::summarise(g = length(s_lead)) %>%
    dplyr::ungroup()
  G_marginal_est <-
    G_marginal_est %>%
    dplyr::group_by(s, a) %>%
    dplyr::mutate(g = g / sum(g)) %>%
    dplyr::ungroup() %>%
    tidyr::complete(s, a, s_lead, fill = list(g = 0)) %>%
    dplyr::arrange(s, a, s_lead) %>%
    reshape2::dcast(formula = s + a ~ s_lead, value.var = "g")
  rownames(G_marginal_est) <- paste("k", G_marginal_est[, "a"], "_l", G_marginal_est[, "s"], sep = "k")
}
```

```

G_marginal_est <- G_marginal_est[, !colnames(G_marginal_est) %in% c("s", "a")]
colnames(G_marginal_est) <- paste("l", 1:L, sep = "")
G_marginal_est <- as.matrix(G_marginal_est)
return(G_marginal_est)
}

# non-parametrically estimate individual transition probability
G_marginal_est <- estimate_G_marginal(df)
check_G <- data.frame(type = "true", reshape2::melt(G_marginal))
check_G_est <- data.frame(type = "estimate", reshape2::melt(G_marginal_est))
check_G <- rbind(check_G, check_G_est)
check_G$variable = paste(check_G$Var1, check_G$Var2, sep = "_")
ggplot(data = check_G, aes(x = variable, y = value,
                           fill = type)) +
  geom_bar(stat = "identity", position = "dodge") +
  labs(fill = "Value") + xlab("action/state/state") + ylab("probability") +
  theme(axis.text.x = element_blank())

```



Estimate parameters

1. Vectorize the parameters as follows:

```

theta_1 <- c(alpha, beta, eta)
theta_2 <- c(kappa, gamma)
theta <- c(theta_1, theta_2)

```


We estimate the parameters by a CCP approach.

1. Write a function `estimate_theta_2_game(df)` that returns the estimates of κ and γ directly from data by counting relevant events.

```
# estimate theta_2
estimate_theta_2_game <-
function(df) {
  # estimate kappa
  kappa_est <- df %>%
    dplyr::arrange(m, i, t) %>%
    dplyr::group_by(m, i) %>%
    dplyr::mutate(s_lead = dplyr::lead(s, 1)) %>%
    dplyr::filter(!is.na(s_lead)) %>%
    dplyr::ungroup() %>%
    dplyr::mutate(move_down = ifelse(s_lead < s, 1, 0)) %>%
    dplyr::filter(s > 1) %>%
    dplyr::group_by(move_down) %>%
    dplyr::summarise(kappa = length(move_down)) %>%
    dplyr::ungroup() %>%
    dplyr::mutate(kappa = kappa / sum(kappa)) %>%
    dplyr::filter(move_down == 1)
  kappa_est <- kappa_est$kappa
  # estimate gamma
  gamma_est <- df %>%
    dplyr::arrange(m, i, t) %>%
    dplyr::group_by(m, i) %>%
    dplyr::mutate(s_lead = dplyr::lead(s, 1)) %>%
    dplyr::filter(!is.na(s_lead)) %>%
    dplyr::ungroup() %>%
    dplyr::mutate(move_up = ifelse(s_lead > s, 1, 0)) %>%
    dplyr::filter(s < L, a == 1) %>%
    dplyr::group_by(move_up) %>%
    dplyr::summarise(gamma = length(move_up)) %>%
    dplyr::ungroup() %>%
    dplyr::mutate(gamma = gamma / sum(gamma)) %>%
    dplyr::filter(move_up == 1)
  gamma_est <- gamma_est$gamma
  # theta_2
  theta_2_est <- c(kappa_est, gamma_est)
  # return
  return(theta_2_est)
}

# estimate theta_2
theta_2_est <- estimate_theta_2_game(df); theta_2_est
```

```
## [1] 0.09929994 0.59579550
```

The objective function of the minimum distance estimator based on the conditional choice probability approach is:

$$\frac{1}{NKm_s} \sum_{i=1}^N \sum_{l=1}^{m_s} \sum_{k=1}^K \{\hat{p}_i(a_k|s_l) - p_i^{(\theta_1, \theta_2)}(a_k|s_l)\}^2,$$

where \hat{p}_i is the non-parametric estimate of the marginal conditional choice probability and $p_i^{(\theta_1, \theta_2)}$ is the marginal conditional choice probability under parameters θ_1 and θ_2 given \hat{p}_i . a_k is k -th action for a firm and

s_l is l -th state profile.

2. Write a function `compute_CCP_objective_game(theta_1, theta_2, p_est, L, K, delta)` that returns the objective function of the above minimum distance estimator given a non-parametric estimate of the conditional choice probability and θ_1 and θ_2 .

compute the objective function of the minimum distance estimator based on the CCP approach

```
compute_CCP_objective_game <-
```

```
function(theta_1, theta_2, p_marginal_est, A, S, delta, lambda) {
  # extract parameters
  alpha <- theta_1[1]
  beta <- theta_1[2]
  eta <- theta_1[3]
  kappa <- theta_2[1]
  gamma <- theta_2[2]
  L <- max(S$s)
  K <- max(A$a)
  # construct PI
  PI <- compute_PI_game(alpha, beta, eta, A, S)
  # construct G
  G_marginal <- compute_G(kappa, gamma, L, K)
  G <- compute_G_game(G_marginal, A, S)

  # update ccp
  V <- compute_exante_value_game(p_marginal_est, A, S, PI, G, delta)
  p_marginal <- compute_ccp_game(p_marginal_est, V, PI, G, delta, A, S)

  # minimum distance
  distance <- p_marginal %>%
    dplyr::rename(ccp = p) %>%
    dplyr::left_join(p_marginal_est, by = c("i", "l", "a")) %>%
    dplyr::filter(a > 0) %>%
    dplyr::mutate(x = (ccp - p)^2) %>%
    dplyr::summarise(mean(x, na.rm = TRUE)) %>%
    as.numeric()
  # return
  return(distance)
}
```

compute the objective function of the minimum distance estimator based on the CCP approach

```
objective <- compute_CCP_objective_game(theta_1, theta_2, p_marginal_est, A, S, delta, lambda)
save(objective, file = "data/A8_objective.RData")
```

```
load(file = "data/A8_objective.RData")
objective
```

```
## [1] 0.0003059688
```

3. Check the value of the objective function around the true parameter.

label

```
label <- c("\\alpha", "\\beta", "\\eta")
label <- paste("$", label, "$", sep = "")
```

compute the graph

```
graph <- foreach (i = 1:length(theta_1)) %do% {
  theta_i <- theta_1[i]
```

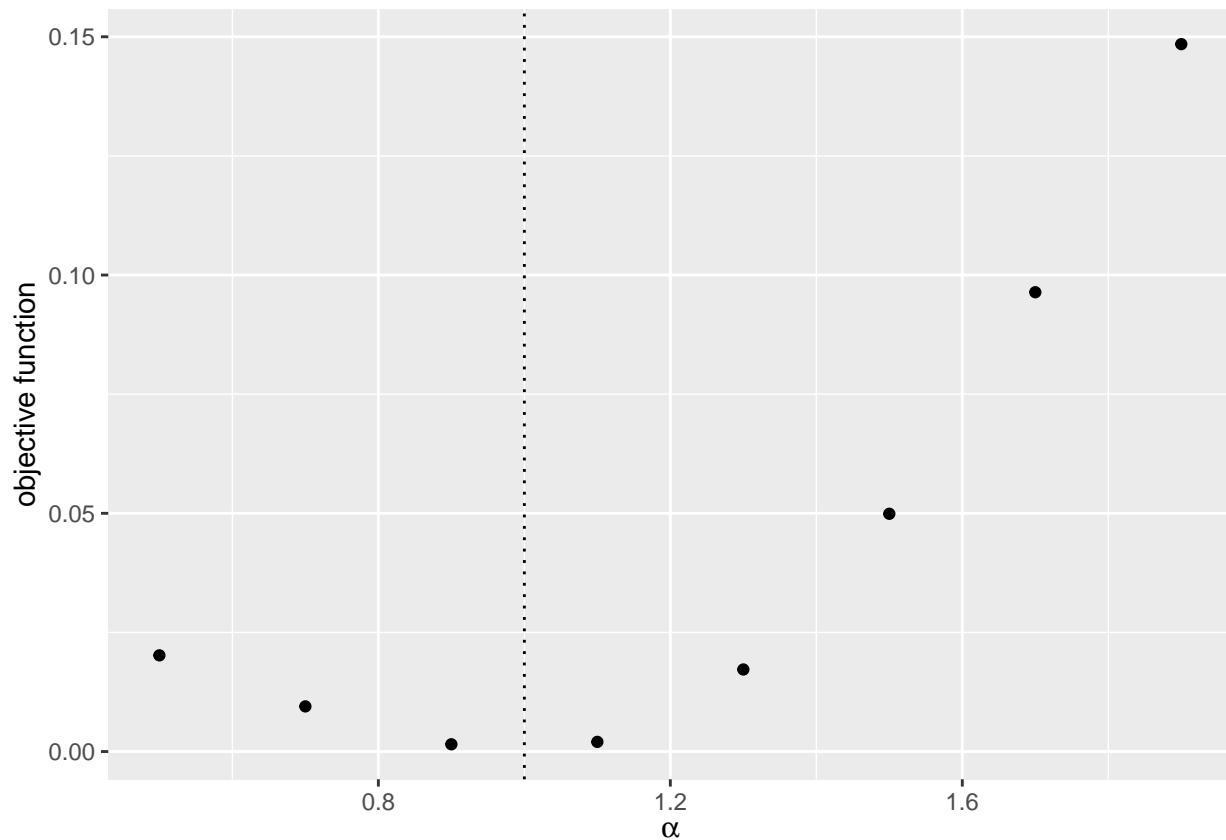
```

theta_i_list <- theta_i * seq(0.5, 2, by = 0.2)
objective_i <-
  foreach (j = 1:length(theta_i_list),
    .combine = "rbind") %dopar% {
    theta_ij <- theta_i_list[j]
    theta_j <- theta_1
    theta_j[i] <- theta_ij
    objective_ij <-
      compute_CCP_objective_game(theta_j, theta_2, p_marginal_est, A, S, delta, lambda)
    return(objective_ij)
  }
df_graph <- data.frame(x = theta_i_list, y = objective_i)
g <- ggplot(data = df_graph, aes(x = x, y = y)) +
  geom_point() +
  geom_vline(xintercept = theta_i, linetype = "dotted") +
  ylab("objective function") + xlab(TeX(label[i]))
return(g)
}
save(graph, file = "data/A8_CCP_graph.RData")

load(file = "data/A8_CCP_graph.RData")
graph

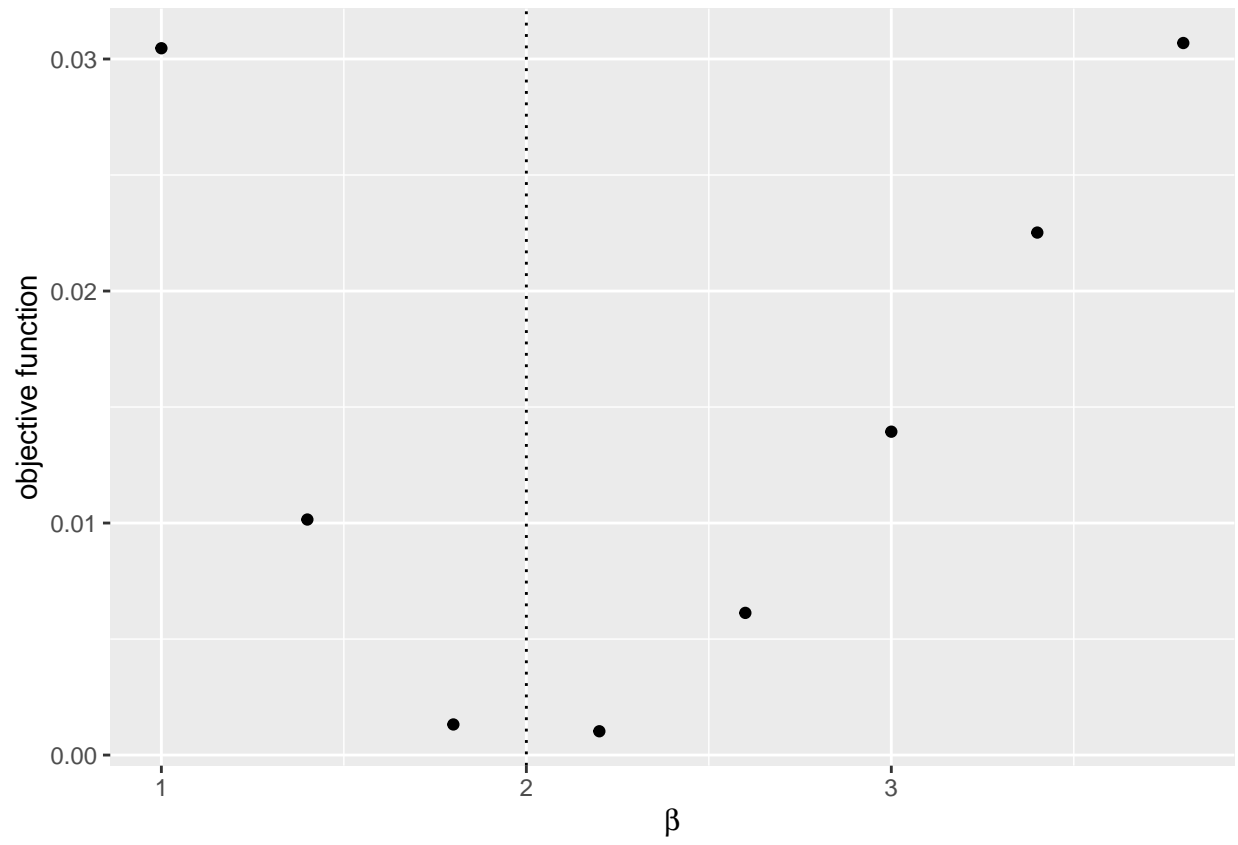
```

```
## [[1]]
```

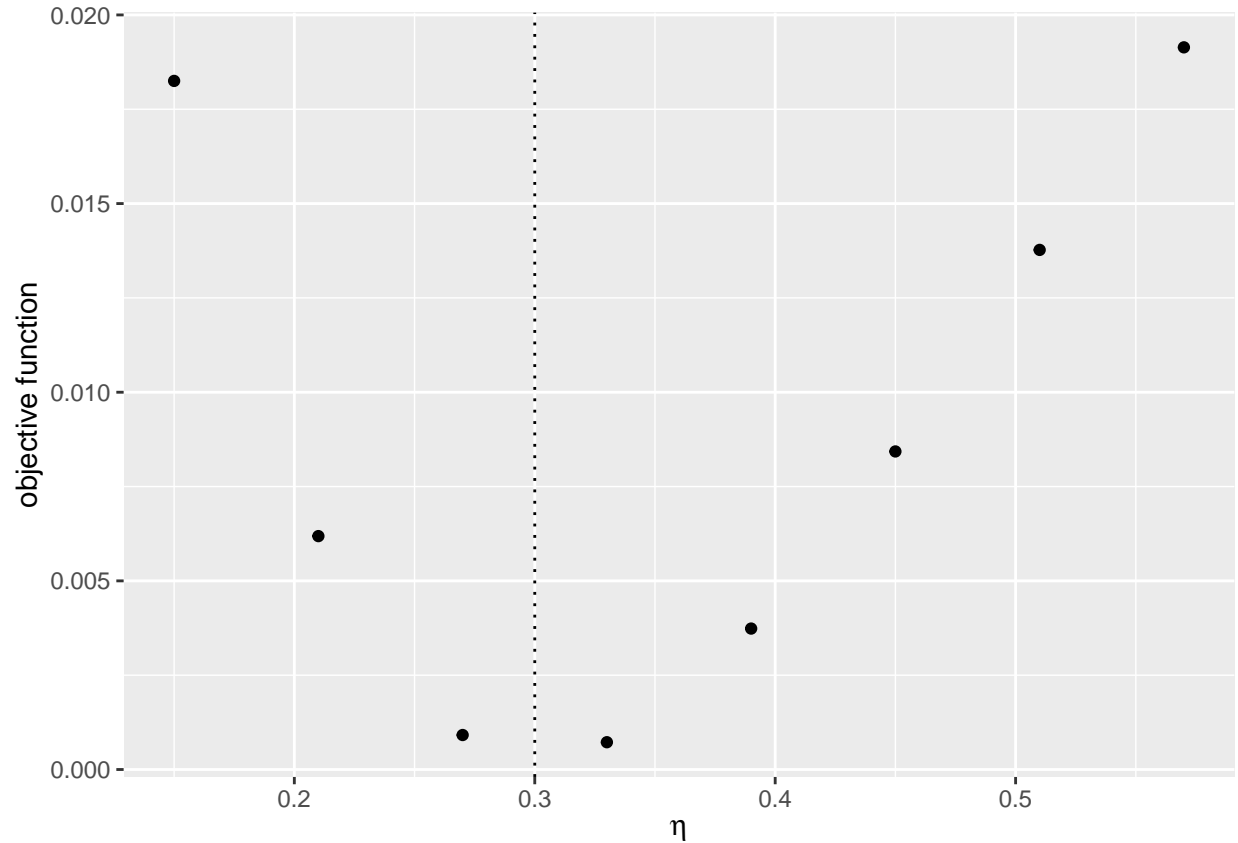


```
##
```

```
## [[2]]
```



```
##  
## [[3]]
```



4. Estimate the parameters by minimizing the objective function. To keep the model to be well-defined, impose an ad hoc lower and upper bounds such that $\alpha \in [0, 1]$, $\beta \in [0, 5]$, $\delta \in [0, 1]$.

```
lower <- rep(0, length(theta_1))
upper <- c(1, 5, 0.3)
CCP_result <-
  optim(par = theta_1,
        fn = compute_CCP_objective_game,
        method = "L-BFGS-B",
        lower = lower,
        upper = upper,
        theta_2 = theta_2_est,
        p_marginal_est = p_marginal_est,
        A = A,
        S = S,
        delta = delta,
        lambda = lambda)
save(CCP_result, file = "data/A8_CCP_result.RData")
```

```
load(file = "data/A8_CCP_result.RData")
CCP_result
```

```
## $par
## [1] 0.998164 2.000860 0.300000
##
## $value
## [1] 0.000303798
```

```
##
## $counts
## function gradient
##      5      5
##
## $convergence
## [1] 0
##
## $message
## [1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"
compare <-
  data.frame(
    true = theta_1,
    estimate = CCP_result$par
  ); compare

##   true estimate
## 1   1.0 0.998164
## 2   2.0 2.000860
## 3   0.3 0.300000
```