Assignment 7: Dynamic Decision

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Simulate data

Suppose that there is a firm and it makes decisions for $t = 1, \cdots, \infty$. We solve the model under the infinite-horizon assumption, but generate data only for $t = 1, \cdots, T$. There are $L = 5$ state $s \in \{1, 2, 3, 4, 5\}$ states for the player. The firm can choose $K + 1 = 2$ actions $a \in \{0, 1\}$.

The mean period payoff to the firm is:

$$\pi(a, s) := \alpha \ln s - \beta a,$$

where $\alpha, \beta > 0$. The period payoff is:

$$\pi(a, s) + \epsilon(a),$$

and $\epsilon(a)$ is an i.i.d. type-I extreme random variable that is independent of all the other variables.

At the beginning of each period, the state s and choice-specific shocks $\epsilon(a), a = 0, 1$ are realized, and the the firm chooses her action. Then, the game moves to the next period.

Suppose that $s > 1$ and $s < L$. If $a = 0$, the state stays at the same state with probability $1 - \kappa$ and moves down by 1 with probability κ. If $a = 1$, the state moves up by 1 with probability γ, moves down by 1 with probability κ, and stays at the same with probability $1 - \kappa - \gamma$.

Suppose that $s = 1$. If $a = 0$, the state stays at the same state with probability 1. If $a = 1$, the state moves up by 1 with probability γ and stays at the same with probability $1 - \gamma$.

Suppose that $s = L$. If $a = 0$, the state stays at the same state with probability $1 - \kappa$ and moves down by 1 with probability κ. If $a = 1$, the state moves down by 1 with probability κ, and stays at the same with probability $1 - \kappa$.

The mean period profit is summarized in Π as:

$$\Pi := \begin{pmatrix} \pi(0, 1) \\ \vdots \\ \pi(K, 1) \\ \vdots \\ \pi(0, L) \\ \vdots \\ \pi(K, L) \end{pmatrix}$$

The transition law is summarized in G as:

$$g(a, s, s') := \mathbb{P}\{s_{t+1} = s' | s_t = s, a_t = a\},$$

$$
G := \begin{pmatrix}
g(0,1,1) & \cdots & g(0,1,L) \\
\vdots & & \vdots \\
g(K,1,1) & \cdots & g(K,1,L) \\
 & \vdots & \\
g(0,L,1) & \cdots & g(0,L,L) \\
\vdots & & \vdots \\
g(K,L,1) & \cdots & g(K,L,L)
\end{pmatrix}.
$$

The discount factor is denoted by δ. We simulate data for N firms for T periods each.

1. Set constants and parameters as follows:

```
# set seed
set.seed(1)
# set constants
L <- 5
K <- 1
T <- 100
N <- 1000
lambda <- 1e-10
# set parameters
alpha <- 0.5
beta <- 3
kappa <- 0.1
gamma <- 0.6
delta <- 0.95
```

2. Write function `compute_pi(alpha, beta, L, K)` that computes Π given parameters and compute the true Π under the true parameters. Don't use methods in `dplyr` and deal with matrix operations.

```
PI <- compute_PI(alpha, beta, L, K); PI
```

```
##             [,1]
## k0_l1  0.0000000
## k1_l1 -3.0000000
## k0_l2  0.3465736
## k1_l2 -2.6534264
## k0_l3  0.5493061
## k1_l3 -2.4506939
## k0_l4  0.6931472
## k1_l4 -2.3068528
## k0_l5  0.8047190
## k1_l5 -2.1952810
```

3. Write function `compute_G(kappa, gamma, L, K)` that computes G given parameters and compute the true G under the true parameters. Don't use methods in `dplyr` and deal with matrix operations.

```
G <- compute_G(kappa, gamma, L, K); G
```

```
##        l1  l2  l3  l4  l5
## k0_l1 1.0 0.0 0.0 0.0 0.0
## k1_l1 0.4 0.6 0.0 0.0 0.0
## k0_l2 0.1 0.9 0.0 0.0 0.0
## k1_l2 0.1 0.3 0.6 0.0 0.0
## k0_l3 0.0 0.1 0.9 0.0 0.0
## k1_l3 0.0 0.1 0.3 0.6 0.0
```

```
## k0_l4 0.0 0.0 0.1 0.9 0.0
## k1_l4 0.0 0.0 0.1 0.3 0.6
## k0_l5 0.0 0.0 0.0 0.1 0.9
## k1_l5 0.0 0.0 0.0 0.1 0.9
```

The exante-value function is written as a function of a conditional choice probability as follows:

$$\varphi^{(\theta_1,\theta_2)}(p) := [I - \delta\Sigma(p)G]^{-1}\Sigma(p)[\Pi + E(p)],$$

where $\theta_1 = (\alpha, \beta)$ and $\theta_2 = (\kappa, \gamma)$ and:

$$\Sigma(p) = \begin{pmatrix} p(1)' & & \\ & \ddots & \\ & & p(L)' \end{pmatrix}$$

and:

$$E(p) = \gamma - \ln p.$$

3. Write a function `compute_exante_value(p, PI, G, L, K, delta)` that returns the exante value function given a conditional choice probability. Don't use methods in `dplyr` and deal with matrix operations. When a choice probability is zero at some element, the corresponding element of $E(p)$ can be set at zero, because anyway we multiply the zero probability to the element and the corresponding element in $E(p)$ does not affect the result.

```r
p <- matrix(rep(0.5, L * (K + 1)), ncol = 1); p
```

```
##       [,1]
##  [1,]  0.5
##  [2,]  0.5
##  [3,]  0.5
##  [4,]  0.5
##  [5,]  0.5
##  [6,]  0.5
##  [7,]  0.5
##  [8,]  0.5
##  [9,]  0.5
## [10,]  0.5
```

```r
V <- compute_exante_value(p, PI, G, L, K, delta); V
```

```
##         [,1]
## l1  5.777876
## l2  7.597282
## l3  9.126304
## l4 10.115439
## l5 10.593438
```

The optimal conditional choice probability is written as a function of an exante value function as follows:

$$\Lambda^{(\theta_1,\theta_2)}(V)(a, s) := \frac{\exp[\pi(a, s) + \delta\sum_{s'} V(s')g(a, s, s')]}{\sum_{a'} \exp[\pi(a', s) + \delta\sum_{s'} V(s')g(a', s, s')]},$$

where V is an exante value function.

4. Write a function `compute_ccp(V, PI, G, L, K, delta)` that returns the optimal conditional choice probability given an exante value function. Don't use methods in `dplyr` and deal with matrix operations. To do so, write a function `compute_choice_value(V, PI, G, delta)` that returns the choice-specific value function. Use this for debugging by checking if the results are intuitive.

```r
value <- compute_choice_value(V, PI, G, delta); value
```

```
##            [,1]
## k0_l1  5.488982
## k1_l1  3.526044
## k0_l2  7.391148
## k1_l2  5.262691
## k0_l3  9.074038
## k1_l3  6.637845
## k0_l4 10.208846
## k1_l4  7.481306
## k0_l5 10.823075
## k1_l5  7.823075
```

```r
p <- compute_ccp(V, PI, G, L, K, delta); p
```

```
##             [,1]
## k0_l1 0.87685057
## k1_l1 0.12314943
## k0_l2 0.89363847
## k1_l2 0.10636153
## k0_l3 0.91954591
## k1_l3 0.08045409
## k0_l4 0.93863232
## k1_l4 0.06136768
## k0_l5 0.95257413
## k1_l5 0.04742587
```

5. Write a function that find the equilibrium conditional choice probability and ex-ante value function by iterating the update of an exante value function and an optimal conditional choice probability. The iteration should stop when $\max_s |V^{(r+1)}(s) - V^{(r)}(s)| < \lambda$ with $\lambda = 10^{-10}$.

```r
output <- solve_dynamic_decision(PI, G, L, K, delta, lambda); output
```

```
## $p
##             [,1]
## k0_l1 0.82218962
## k1_l1 0.17781038
## k0_l2 0.80024354
## k1_l2 0.19975646
## k0_l3 0.83074516
## k1_l3 0.16925484
## k0_l4 0.87691534
## k1_l4 0.12308466
## k0_l5 0.95257413
## k1_l5 0.04742587
##
## $V
##        [,1]
## l1 15.46000
## l2 18.03675
## l3 20.86514
## l4 23.33721
## l5 25.15557
```

```r
p <- output$p
V <- output$V
value <- compute_choice_value(V, PI, G, delta); value
```

```
##            [,1]
## k0_l1 14.68700
## k1_l1 13.15574
## k0_l2 17.23669
## k1_l2 15.84887
## k0_l3 20.10249
## k1_l3 18.51157
## k0_l4 22.62865
## k1_l4 20.66511
## k0_l5 24.52976
## k1_l5 21.52976
```

6. Write a function `simulate_dynamic_decision(p, s, PI, G, L, K, T, delta, seed)` that simulate the data for a single firm starting from an initial state for T periods. The function should accept a value of seed and set the seed at the beginning of the procedure inside the function, because the process is stochastic.

```r
# set initial value
s <- 1
# draw simulation for a firm
seed <- 1
df <- simulate_dynamic_decision(p, s, PI, G, L, K, T, delta, seed); df
```

```
## # A tibble: 100 x 3
##        t     s     a
##    <int> <dbl> <dbl>
##  1     1     1     0
##  2     2     1     0
##  3     3     1     0
##  4     4     1     1
##  5     5     2     1
##  6     6     1     0
##  7     7     1     0
##  8     8     1     0
##  9     9     1     0
## 10    10     1     0
## # ... with 90 more rows
```

7. Write a function `simulate_dynamic_decision_across_firms(p, s, PI, G, L, K, T, N, delta)` that returns simulation data for N firm. For firm i, set the seed at i

```r
df <- simulate_dynamic_decision_across_firms(p, s, PI, G, L, K, T, N, delta)
save(df, file = "data/A7_df.RData")
```

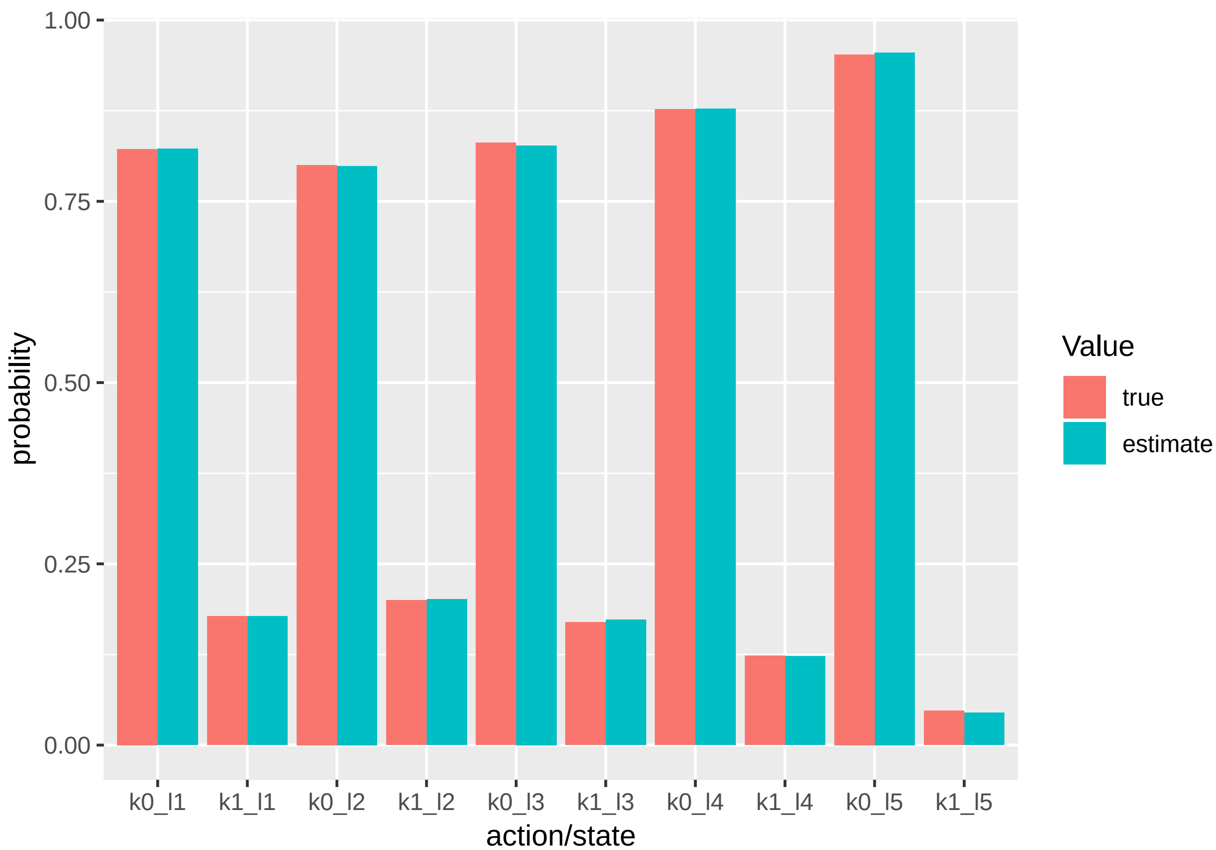
```r
load(file = "data/A7_df.RData")
df
```

```
## # A tibble: 100,000 x 4
##        i     t     s     a
##    <int> <int> <dbl> <dbl>
##  1     1     1     1     0
##  2     1     2     1     0
##  3     1     3     1     0
```

```
## 4      1      4      1      1
## 5      1      5      2      1
## 6      1      6      1      0
## 7      1      7      1      0
## 8      1      8      1      0
## 9      1      9      1      0
## 10     1     10      1      0
## # ... with 99,990 more rows
```

8. Write a function `estimate_ccp(df)` that returns a non-parametric estimate of the conditional choice probability in the data. Compare the estimated conditional choice probability and the true conditional choice probability by a bar plot.

```
p_est <- estimate_ccp(df)
check_ccp <- cbind(p, p_est)
colnames(check_ccp) <- c("true", "estimate")
check_ccp <- check_ccp %>%
  reshape2::melt()
ggplot(data = check_ccp, aes(x = Var1, y = value,
                             fill = Var2)) +
  geom_bar(stat = "identity", position = "dodge") +
  labs(fill = "Value") + xlab("action/state") + ylab("probability")
```



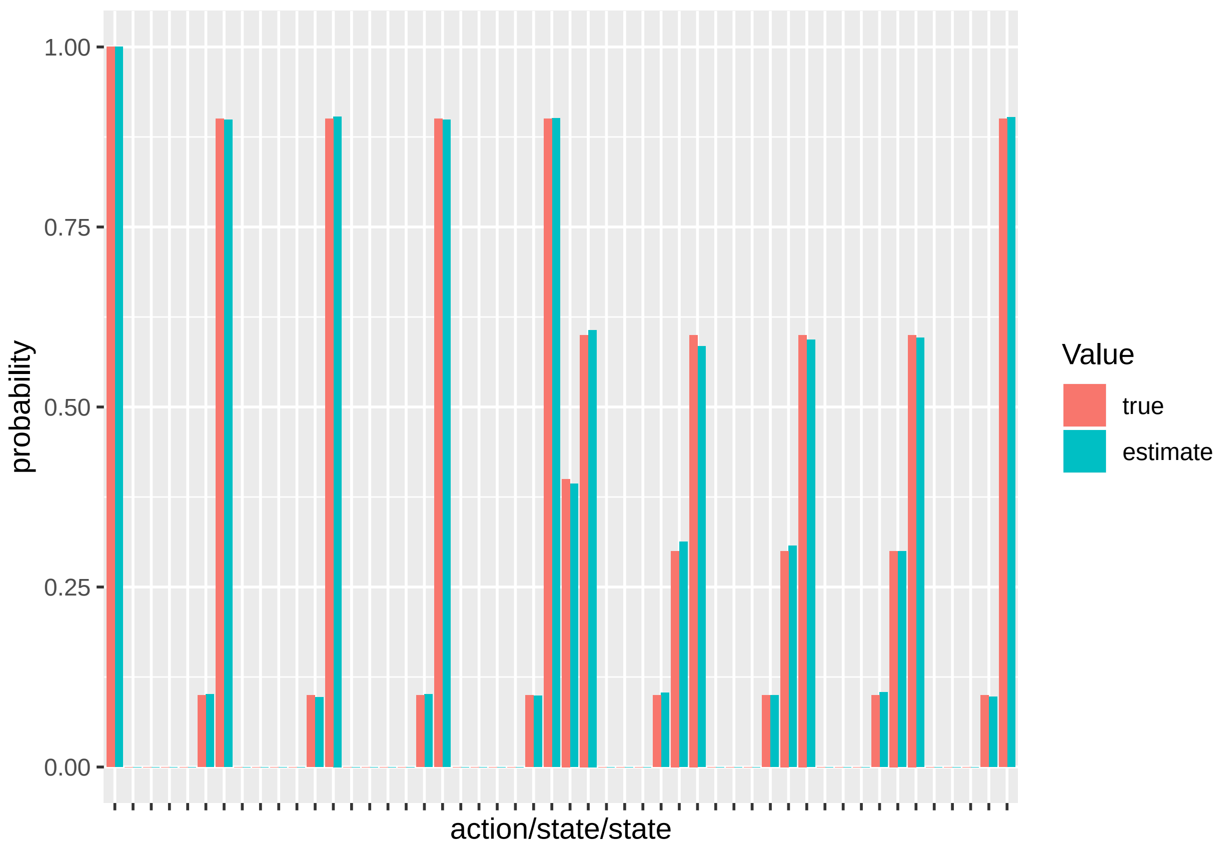
9. Write a function `estimate_G(df)` that returns a non-parametric estiamte of the transition matrix in the data. Compare the estimated transition matrix and the true transition matrix by a bar plot.

```
G_est <- estimate_G(df); G_est
```

```
##                 l1          l2         l3          l4         l5
```

```
## k0_l1 1.0000000 0.00000000 0.0000000 0.00000000 0.0000000
## k1_l1 0.3930818 0.60691824 0.0000000 0.00000000 0.0000000
## k0_l2 0.1012162 0.89878384 0.0000000 0.00000000 0.0000000
## k1_l2 0.1031410 0.31276454 0.5840945 0.00000000 0.0000000
## k0_l3 0.0000000 0.09660837 0.9033916 0.00000000 0.0000000
## k1_l3 0.0000000 0.09974569 0.3071489 0.59310540 0.0000000
## k0_l4 0.0000000 0.00000000 0.1012564 0.89874358 0.0000000
## k1_l4 0.0000000 0.00000000 0.1039339 0.29966003 0.5964060
## k0_l5 0.0000000 0.00000000 0.0000000 0.09891400 0.9010860
## k1_l5 0.0000000 0.00000000 0.0000000 0.09751037 0.9024896
```

```
check_G <- data.frame(type = "true", reshape2::melt(G))
check_G_est <- data.frame(type = "estimate", reshape2::melt(G_est))
check_G <- rbind(check_G, check_G_est)
check_G$variable = paste(check_G$Var1, check_G$Var2, sep = "_")
ggplot(data = check_G, aes(x = variable, y = value,
                           fill = type)) +
    geom_bar(stat = "identity", position = "dodge") +
  labs(fill = "Value") + xlab("action/state/state") + ylab("probability") +
  theme(axis.text.x = element_blank())
```



Estimate parameters

1. Vectorize the parameters as follows:

```
theta_1 <- c(alpha, beta)
theta_2 <- c(kappa, gamma)
```

```r
theta <- c(theta_1, theta_2)
```

First, we estimate the parameters by a nested fixed-point algorithm. The loglikelihood for $\{a_{it}, s_{it}\}_{i=1,\cdots,N,t=1,\cdots,T}$ is:

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}[\log \mathbb{P}\{a_{it}|s_{it}\} + \log \mathbb{P}\{s_{i,t+1}|a_{it}, s_{it}\}],$$

with $\mathbb{P}\{s_{i,T+1}|a_{iT}, s_{iT}\} = 1$ for all i as $s_{i,T+1}$ is not observed.

2. Write a function `compute_loglikelihood_NFP(theta, df, delta, L, K)` that compute the loglikelihood.

```r
loglikelihood <- compute_loglikelihood_NFP(theta, df, delta, L, K); loglikelihood
```

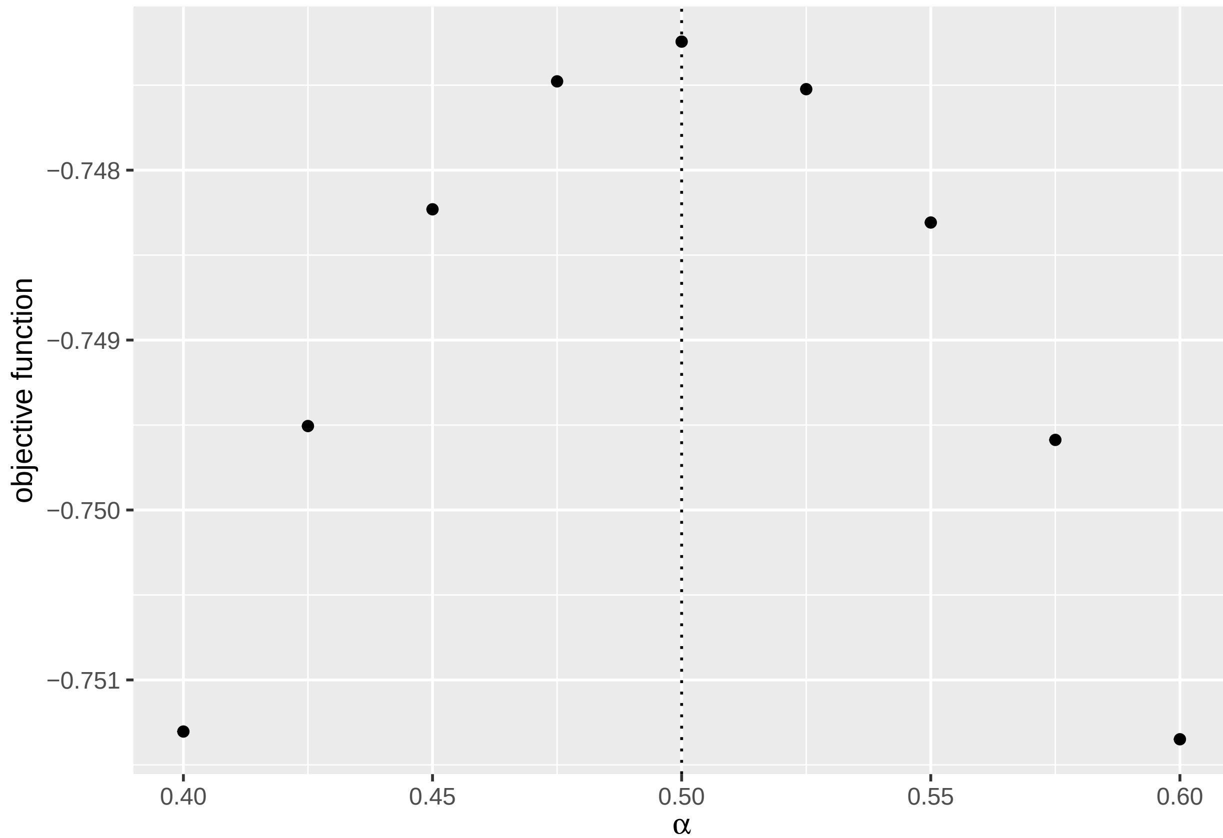
```
## [1] -0.7474961
```

3. Check the value of the objective function around the true parameter.

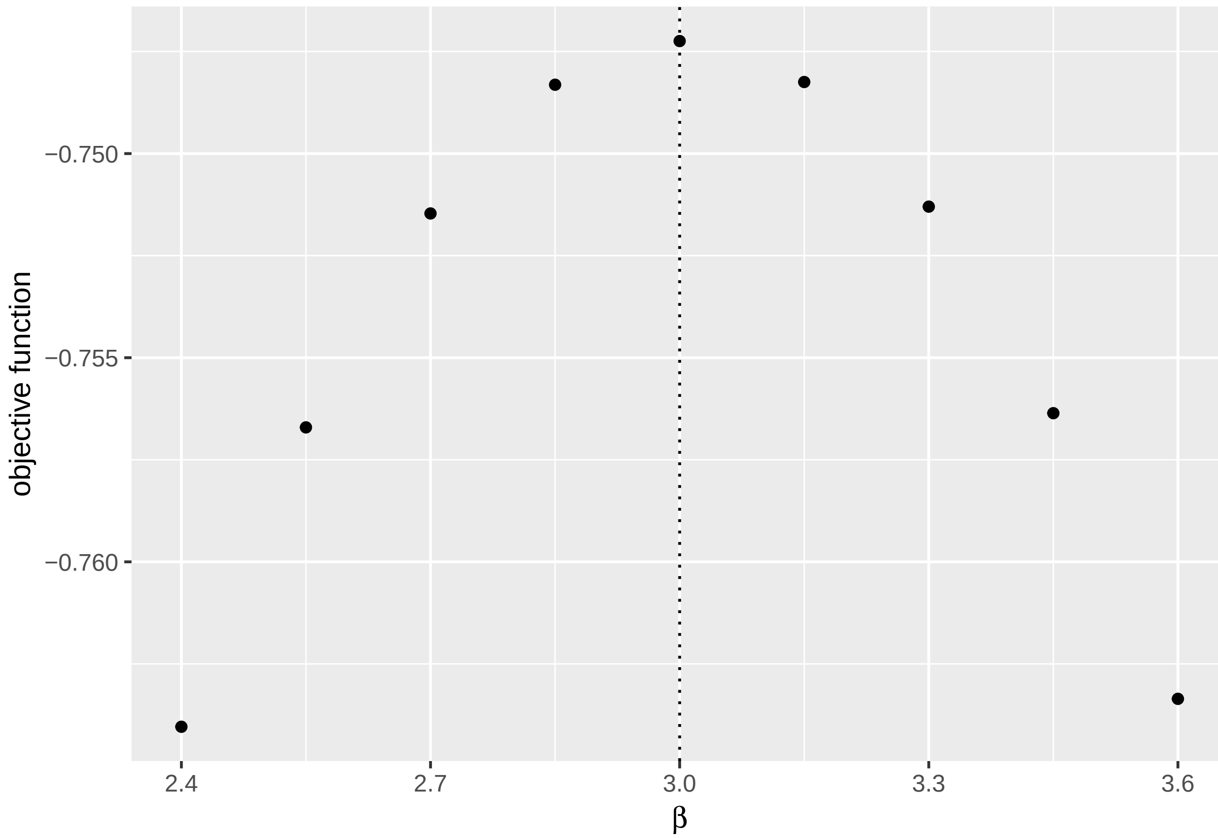
```r
# label
label <- c("\\alpha", "\\beta", "\\kappa", "\\gamma")
label <- paste("$", label, "$", sep = "")
# compute the graph
graph <- foreach (i = 1:length(theta)) %do% {
  theta_i <- theta[i]
  theta_i_list <- theta_i * seq(0.8, 1.2, by = 0.05)
  objective_i <-
    foreach (j = 1:length(theta_i_list),
             .combine = "rbind") %do% {
               theta_ij <- theta_i_list[j]
               theta_j <- theta
               theta_j[i] <- theta_ij
               objective_ij <-
                 compute_loglikelihood_NFP(
                   theta_j, df, delta, L, K); loglikelihood

               return(objective_ij)
             }
  df_graph <- data.frame(x = theta_i_list, y = objective_i)
  g <- ggplot(data = df_graph, aes(x = x, y = y)) +
    geom_point() +
    geom_vline(xintercept = theta_i, linetype = "dotted") +
    ylab("objective function") + xlab(TeX(label[i]))
  return(g)
}
save(graph, file = "data/A7_NFP_graph.RData")
```

```r
load(file = "data/A7_NFP_graph.RData")
graph
```

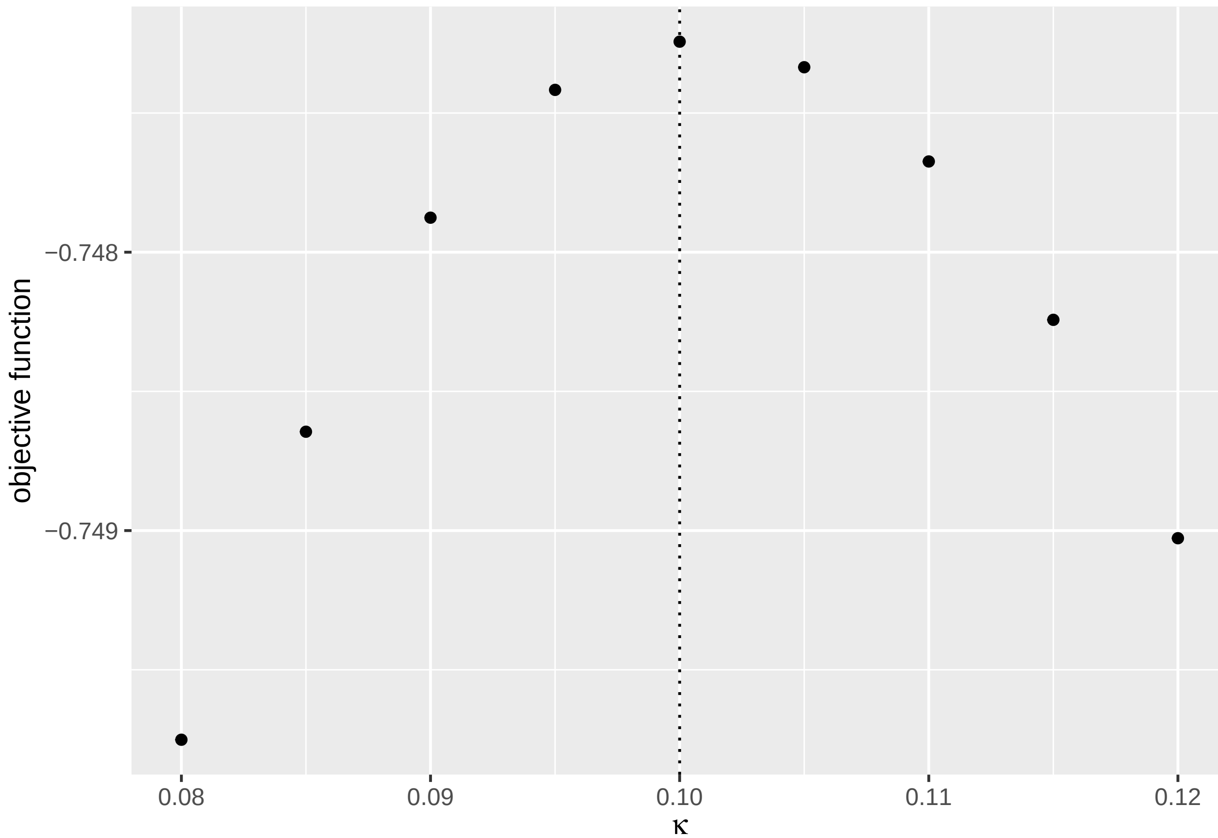
```
## [[1]]
```



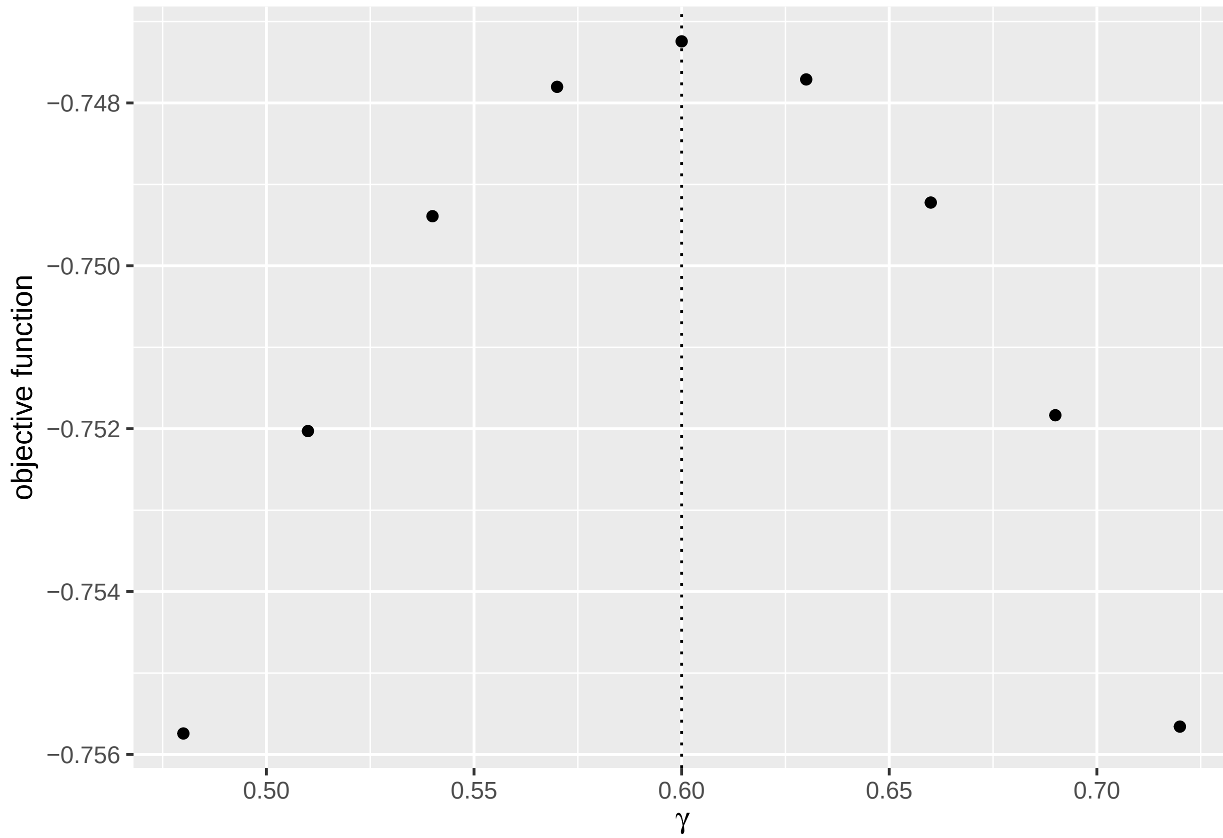

```
##
## [[2]]
```




```
##
## [[3]]
```




```
##
## [[4]]
```



4. Estiamte the parameters by maximizing the loglikelihood. To keep the model to be well-defined, impose an ad hoc lower and upper bounds such that $\alpha \in [0, 1], \beta \in [0, 5], \kappa \in [0, 0.2], \gamma \in [0, 0.7]$.

```
lower <- rep(0, length(theta))
upper <- c(1, 5, 0.2, 0.7)
NFP_result <-
  optim(par = theta,
        fn = compute_loglikelihood_NFP,
        method = "L-BFGS-B",
        lower = lower,
        upper = upper,
        control = list(fnscale = -1),
        df = df,
        delta = delta,
        L = L,
        K = K)
save(NFP_result, file = "data/A7_NFP_result.RData")
```

```
load(file = "data/A7_NFP_result.RData")
NFP_result
```

```
## $par
## [1] 0.4916153 2.9816751 0.1005993 0.6029317
##
## $value
## [1] -0.747237
##
## $counts
```

```
## function gradient
##       17       17
##
## $convergence
## [1] 0
##
## $message
## [1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"
```

```
compare <-
  data.frame(
    true = theta,
    estimate = NFP_result$par
  ); compare
```

```
##   true  estimate
## 1  0.5 0.4916153
## 2  3.0 2.9816751
## 3  0.1 0.1005993
## 4  0.6 0.6029317
```

Next, we estimate the parameters by CCP approach.

5. Write a function `estimate_theta_2(df)` that returns the estimates of κ and γ directly from data by counting relevant events.

```
theta_2_est <- estimate_theta_2(df); theta_2_est
```

```
## [1] 0.09988488 0.59551895
```

The objective function of the minimum distance estimator based on the conditional choice probability approach is:

$$\frac{1}{KL}\sum_{s=1}^{L}\sum_{a=1}^{K}\{\hat{p}(a,s) - p^{(\theta_1,\theta_2)}(a,s)\}^2,$$

where $\hat{p}$ is the non-parametric estimate of the conditional choice probability and $p^{(\theta_1,\theta_2)}$ is the optimal conditional choice probability under parameters θ_1 and θ_2.

6. Write a function `compute_CCP_objective(theta_1, theta_2, p_est, L, K, delta)` that returns the objective function of the above minimum distance estimator given a non-parametric estimate of the conditional choice probability and θ_1 and θ_2.

```
compute_CCP_objective(theta_1, theta_2, p_est, L, K, delta)
```

```
## [1] 5.000511e-06
```

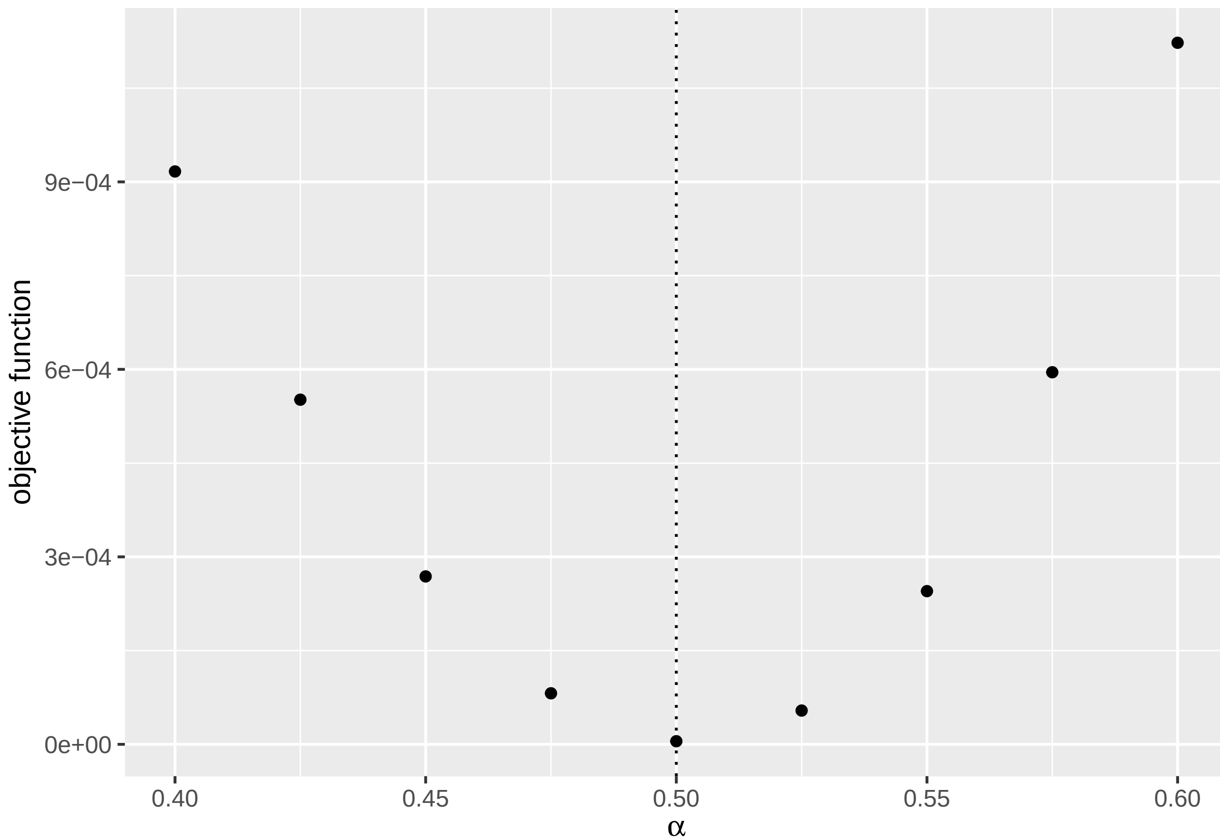
3. Check the value of the objective function around the true parameter.

```
# label
label <- c("\\alpha", "\\beta")
label <- paste("$", label, "$", sep = "")
# compute the graph
graph <- foreach (i = 1:length(theta_1)) %do% {
  theta_i <- theta_1[i]
  theta_i_list <- theta_i * seq(0.8, 1.2, by = 0.05)
  objective_i <-
    foreach (j = 1:length(theta_i_list),
             .combine = "rbind") %do% {
               theta_ij <- theta_i_list[j]
```

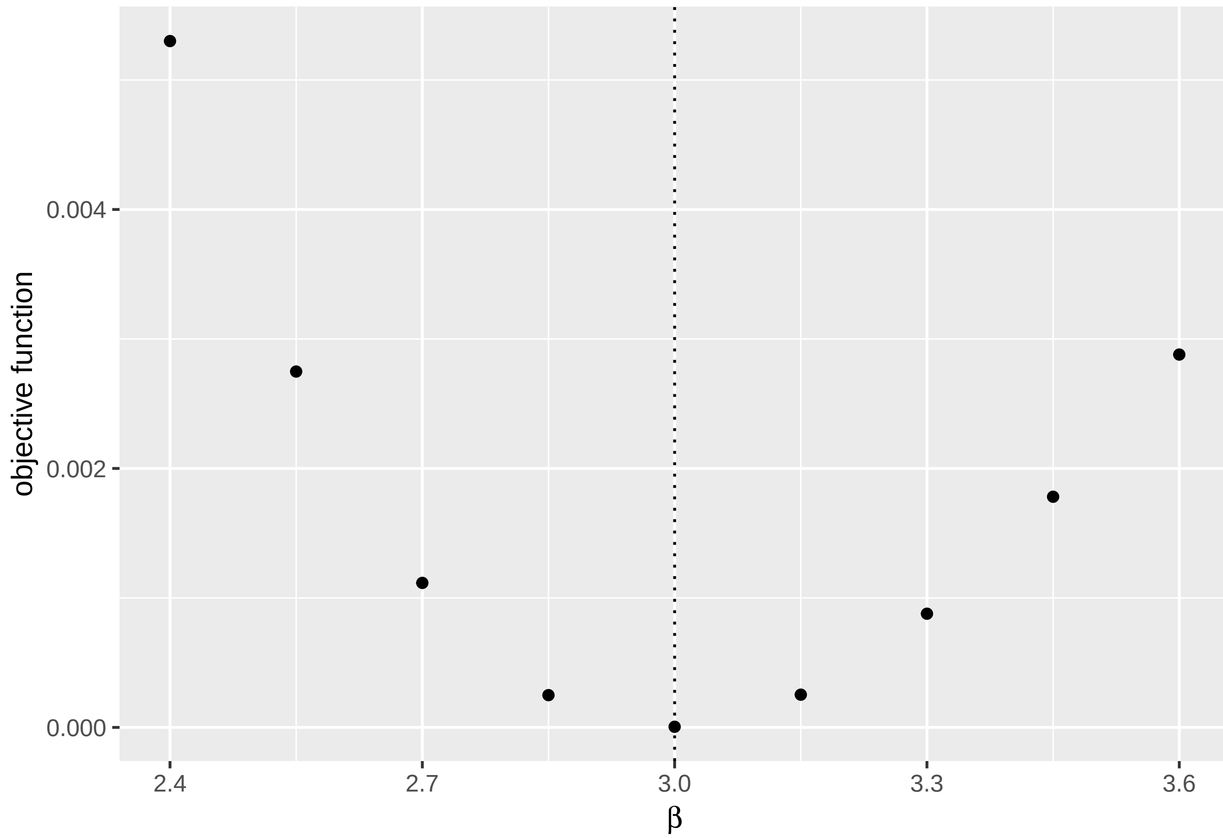
```r
                theta_j <- theta_1
                theta_j[i] <- theta_ij
                objective_ij <-
                  compute_CCP_objective(theta_j, theta_2, p_est, L, K, delta)
                return(objective_ij)
              }
  df_graph <- data.frame(x = theta_i_list, y = objective_i)
  g <- ggplot(data = df_graph, aes(x = x, y = y)) +
    geom_point() +
    geom_vline(xintercept = theta_i, linetype = "dotted") +
    ylab("objective function") + xlab(TeX(label[i]))
  return(g)
}
save(graph, file = "data/A7_CCP_graph.RData")
```

```r
load(file = "data/A7_CCP_graph.RData")
graph
```

```
## [[1]]
```




```
##
## [[2]]
```



4. Estiamte the parameters by minimizing the objective function. To keep the model to be well-defined, impose an ad hoc lower and upper bounds such that $\alpha \in [0, 1], \beta \in [0, 5]$.

```
lower <- rep(0, length(theta_1))
upper <- c(1, 5)
CCP_result <-
  optim(par = theta_1,
        fn = compute_CCP_objective,
        method = "L-BFGS-B",
        lower = lower,
        upper = upper,
        theta_2 = theta_2_est,
        p_est = p_est,
        L = L,
        K = K,
        delta = delta)
save(CCP_result, file = "data/A7_CCP_result.RData")
```

```
load(file = "data/A7_CCP_result.RData")
CCP_result
```

```
## $par
## [1] 0.5271684 3.0644600
##
## $value
## [1] 1.790528e-06
##
## $counts
```

```
## function gradient
##       11       11
##
## $convergence
## [1] 0
##
## $message
## [1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"
```

```r
compare <-
  data.frame(
    true = theta_1,
    estimate = CCP_result$par
  ); compare
```

```
##   true  estimate
## 1  0.5 0.5271684
## 2  3.0 3.0644600
```