Assignment 7: Dynamic Decision

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Simulate data

Suppose that there is a firm and it makes decisions for $t = 1, \cdots, \infty$. We solve the model under the infinite-horizon assumption, but generate data only for $t = 1, \cdots, T$. There are $L = 5$ state $s \in \{1, 2, 3, 4, 5\}$ states for the player. The firm can choose $K + 1 = 2$ actions $a \in \{0, 1\}$.

The mean period payoff to the firm is:

$$\pi(a, s) := \alpha \ln s - \beta a,$$

where $\alpha, \beta > 0$. The period payoff is:

$$\pi(a, s) + \epsilon(a),$$

and $\epsilon(a)$ is an i.i.d. type-I extreme random variable that is independent of all the other variables.

At the beginning of each period, the state s and choice-specific shocks $\epsilon(a), a = 0, 1$ are realized, and the the firm chooses her action. Then, the game moves to the next period.

Suppose that $s > 1$ and $s < L$. If $a = 0$, the state stays at the same state with probability $1 - \kappa$ and moves down by 1 with probability κ. If $a = 1$, the state moves up by 1 with probability γ, moves down by 1 with probability κ, and stays at the same with probability $1 - \kappa - \gamma$.

Suppose that $s = 1$. If $a = 0$, the state stays at the same state with probability 1. If $a = 1$, the state moves up by 1 with probability γ and stays at the same with probability $1 - \gamma$.

Suppose that $s = L$. If $a = 0$, the state stays at the same state with probability $1 - \kappa$ and moves down by 1 with probability κ. If $a = 1$, the state moves down by 1 with probability κ, and stays at the same with probability $1 - \kappa$.

The mean period profit is summarized in Π as:

$$\Pi := \begin{pmatrix} \pi(0,1) \\ \vdots \\ \pi(K,1) \\ \vdots \\ \pi(0,L) \\ \vdots \\ \pi(K,L) \end{pmatrix}$$

The transition law is summarized in G as:

$$g(a, s, s') := \mathbb{P}\{s_{t+1} = s' | s_t = s, a_t = a\},$$

$$G := \begin{pmatrix} g(0,1,1) & \cdots & g(0,1,L) \\ \vdots & & \vdots \\ g(K,1,1) & \cdots & g(K,1,L) \\ & \vdots & \\ g(0,L,1) & \cdots & g(0,L,L) \\ \vdots & & \vdots \\ g(K,L,1) & \cdots & g(K,L,L) \end{pmatrix}.$$

The discount factor is denoted by δ. We simulate data for N firms for T periods each.

1. Set constants and parameters as follows:

```
# set seed
set.seed(1)
# set constants
L <- 5
K <- 1
T <- 100
N <- 1000
lambda <- 1e-10
# set parameters
alpha <- 0.5
beta <- 3
kappa <- 0.1
gamma <- 0.6
delta <- 0.95
```

2. Write function `compute_pi(alpha, beta, L, K)` that computes Π given parameters and compute the true Π under the true parameters. Don't use methods in `dplyr` and deal with matrix operations.

```
# compute PI
compute_PI <-
  function(alpha, beta, L, K) {
    PI <- foreach (l = 1:L,
                   .combine = "rbind") %do% {
                     PI_l <- foreach (k = 0:K,
                                      .combine = "rbind") %do% {
                                        pi_kl <- alpha * log(l) - beta * k
                                        return(pi_kl)
                                      }
                     rownames(PI_l) <- paste("k", 0:K, "_l", l, sep = "")
                     return(PI_l)
                   }
    return(PI)
  }
PI <- compute_PI(alpha, beta, L, K); PI
```

```
##             [,1]
## k0_l1  0.0000000
## k1_l1 -3.0000000
## k0_l2  0.3465736
## k1_l2 -2.6534264
## k0_l3  0.5493061
## k1_l3 -2.4506939
```

```
## k0_l4  0.6931472
## k1_l4 -2.3068528
## k0_l5  0.8047190
## k1_l5 -2.1952810
```

3. Write function `compute_G(kappa, gamma, L, K)` that computes G given parameters and compute the true G under the true parameters. Don't use methods in `dplyr` and deal with matrix operations.

```
# compute G
compute_G <-
  function(kappa, gamma, L, K) {
    G <- foreach (l = 1:L, .combine = "rbind") %do% {
      G_l <- foreach (k = 0:K, .combine = "rbind") %do% {
        g_kl <- matrix(rep(0, L), nrow = 1)
        if (l > 1) {
          g_kl[l - 1] <- kappa
          if (l < L) {
            g_kl[l + 1] <- gamma * k
            g_kl[l] <- 1 - kappa - gamma * k
          } else {
            g_kl[l] <- 1 - kappa
          }
        } else {
          g_kl[l] <- 1 - gamma * k
          g_kl[l + 1] <- gamma * k
        }
        rownames(g_kl) <- paste("k", k, "_l", l, sep = "")
        colnames(g_kl) <- paste("l", 1:L, sep = "")
        return(g_kl)
      }
      return(G_l)
    }
    return(G)
  }
G <- compute_G(kappa, gamma, L, K); G
```

```
##        l1  l2  l3  l4  l5
## k0_l1 1.0 0.0 0.0 0.0 0.0
## k1_l1 0.4 0.6 0.0 0.0 0.0
## k0_l2 0.1 0.9 0.0 0.0 0.0
## k1_l2 0.1 0.3 0.6 0.0 0.0
## k0_l3 0.0 0.1 0.9 0.0 0.0
## k1_l3 0.0 0.1 0.3 0.6 0.0
## k0_l4 0.0 0.0 0.1 0.9 0.0
## k1_l4 0.0 0.0 0.1 0.3 0.6
## k0_l5 0.0 0.0 0.0 0.1 0.9
## k1_l5 0.0 0.0 0.0 0.1 0.9
```

The exante-value function is written as a function of a conditional choice probability as follows:

$$\varphi^{(\theta_1,\theta_2)}(p) := [I - \delta\Sigma(p)G]^{-1}\Sigma(p)[\Pi + E(p)],$$

where $\theta_1 = (\alpha, \beta)$ and $\theta_2 = (\kappa, \gamma)$ and:

$$\Sigma(p) = \begin{pmatrix} p(1)' & & \\ & \ddots & \\ & & p(L)' \end{pmatrix}$$

and:

$$E(p) = \gamma - \ln p.$$

3. Write a function `compute_exante_value(p, PI, G, L, K, delta)` that returns the exante value function given a conditional choice probability. Don't use methods in `dplyr` and deal with matrix operations. When a choice probability is zero at some element, the corresponding element of $E(p)$ can be set at zero, because anyway we multiply the zero probability to the element and the corresponding element in $E(p)$ does not affect the result.

```
# compute ex-ante value function
compute_exante_value <-
  function(p, PI, G, L, K, delta) {
    # construct E_p and Sigma_p
    E_p <- -digamma(1) - log(p)
    E_p <- ifelse(is.finite(E_p), E_p, 0)
    Sigma_p <- foreach (l = 1:L) %do% {
      p_l <- p[((K + 1) * (l - 1) + 1):((K + 1) * (l - 1) + K + 1)]
      p_l <- t(matrix(p_l))
      return(p_l)
    }
    Sigma_p <-
      Matrix::bdiag(Sigma_p) %>%
      as.matrix()
    # compute exante value function
    term_1 <- diag(dim(Sigma_p)[1]) - delta * Sigma_p %*% G
    term_2 <- Sigma_p %*% (PI + E_p)
    V <-
      solve(term_1, term_2)
    # name
    rownames(V) <- paste("l", 1:L, sep = "")
    # return
    return(V)
  }

p <- matrix(rep(0.5, L * (K + 1)), ncol = 1); p
```

```
##       [,1]
##  [1,]  0.5
##  [2,]  0.5
##  [3,]  0.5
##  [4,]  0.5
##  [5,]  0.5
##  [6,]  0.5
##  [7,]  0.5
##  [8,]  0.5
##  [9,]  0.5
## [10,]  0.5
```

```
V <- compute_exante_value(p, PI, G, L, K, delta); V
```

```
##         [,1]
## l1  5.777876
## l2  7.597282
## l3  9.126304
## l4 10.115439
## l5 10.593438
```

The optimal conditional choice probability is written as a function of an exante value function as follows:

$$\Lambda^{(\theta_1,\theta_2)}(V)(a,s) := \frac{\pi(a,s) + \delta \sum_{s'} V(s') g(a,s,s')}{\sum_{a'} [\pi(a',s) + \delta \sum_{s'} V(s') g(a',s,s')]},$$

where V is an exante value function.

4. Write a function `compute_ccp(V, PI, G, L, K, delta)` that returns the optimal conditional choice probability given an exante value function. Don't use methods in `dplyr` and deal with matrix operations. To do so, write a function `compute_choice_value(V, PI, G, delta)` that returns the choice-specific value function. Use this for debugging by checking if the results are intuitive.

```
# compute choice-specific value function
compute_choice_value <-
  function(V, PI, G, delta) {
    value <- PI + delta * G %*% V
    return(value)
  }
# compute conditional choice probability
compute_ccp <-
  function(V, PI, G, L, K, delta) {
    # compute choice-specific value function
    value <- compute_choice_value(V, PI, G, delta)
    # compute the numerator
    numerator <- exp(value)
    # compute the denominator
    aggregator <- matrix(rep(1, (K + 1) * (K + 1)), ncol = K + 1)
    aggregator <- replicate(L, aggregator, simplify = FALSE)
    aggregator <- Matrix::bdiag(aggregator) %>%
      as.matrix()
    denominator <- aggregator %*% numerator
    # compute the conditional choice probability
    p <- numerator / denominator
    return(p)
  }

value <- compute_choice_value(V, PI, G, delta); value
```

```
##            [,1]
## k0_l1  5.488982
## k1_l1  3.526044
## k0_l2  7.391148
## k1_l2  5.262691
## k0_l3  9.074038
## k1_l3  6.637845
## k0_l4 10.208846
## k1_l4  7.481306
## k0_l5 10.823075
## k1_l5  7.823075
```

```
p <- compute_ccp(V, PI, G, L, K, delta); p
```

```
##             [,1]
## k0_l1 0.87685057
## k1_l1 0.12314943
## k0_l2 0.89363847
## k1_l2 0.10636153
```

```
## k0_l3 0.91954591
## k1_l3 0.08045409
## k0_l4 0.93863232
## k1_l4 0.06136768
## k0_l5 0.95257413
## k1_l5 0.04742587
```

5. Write a function that find the equilibrium conditional choice probability and ex-ante value function by iterating the update of an exante value function and an optimal conditional choice probability. The iteration should stop when $\max_s |V^{(r+1)}(s) - V^{(r)}(s)| < \lambda$ with $\lambda = 10^{-10}$.

```
# solve the dynamic decision model
solve_dynamic_decision <-
  function(PI, G, L, K, delta, lambda) {
    # initial value
    p <- matrix(rep(0.5, L * (K + 1)), ncol = 1)
    V <- compute_exante_value(p, PI, G, L, K, delta)
    distance <- 10000
    while (distance > lambda) {
      V_old <- V
      p <- compute_ccp(V, PI, G, L, K, delta)
      V <- compute_exante_value(p, PI, G, L, K, delta)
      distance <- max(abs(V - V_old))
    }
    return(list(p = p, V = V))
  }

output <- solve_dynamic_decision(PI, G, L, K, delta, lambda); output
```

```
## $p
##             [,1]
## k0_l1 0.82218962
## k1_l1 0.17781038
## k0_l2 0.80024354
## k1_l2 0.19975646
## k0_l3 0.83074516
## k1_l3 0.16925484
## k0_l4 0.87691534
## k1_l4 0.12308466
## k0_l5 0.95257413
## k1_l5 0.04742587
##
## $V
##        [,1]
## l1 15.46000
## l2 18.03675
## l3 20.86514
## l4 23.33721
## l5 25.15557
```

```
p <- output$p
V <- output$V
value <- compute_choice_value(V, PI, G, delta); value
```

```
##           [,1]
## k0_l1 14.68700
```

```
## k1_l1 13.15574
## k0_l2 17.23669
## k1_l2 15.84887
## k0_l3 20.10249
## k1_l3 18.51157
## k0_l4 22.62865
## k1_l4 20.66511
## k0_l5 24.52976
## k1_l5 21.52976
```

6. Write a function `simulate_dynamic_decision(p, s, PI, G, L, K, T, delta, seed)` that simulate the data for a single firm starting from an initial state for T periods. The function should accept a value of seed and set the seed at the beginning of the procedure inside the function, because the process is stochastic.

```
# simulate the dynamic decision model for a single player
simulate_dynamic_decision <-
  function(p, s, PI, G, L, K, T, delta, seed) {
    set.seed(seed)
    df <- data.frame(t = 1:T, s = rep(s, T), a = rep(0, T))
    for (t in 1:T) {
      # state
      s_t <- df[t, "s"]
      # draw action
      p_t <-
        p[((K + 1) * (s_t - 1) + 1):((K + 1) * (s_t - 1) + K + 1)]
      a_t <-
        rmultinom(1, 1, prob = p_t)
      a_t <- which(as.logical(a_t)) - 1
      df[t, "a"] <- a_t
      # draw next state
      if (t < T) {
        g_t <- G[(K + 1) * (s_t - 1) + a_t + 1, ]
        s_t_1 <-
          rmultinom(1, 1, prob = g_t)
        s_t_1 <- which(as.logical(s_t_1))
        df[t + 1, "s"] <- s_t_1
      }
    }
    # as tibble
    df <- tibble::as_tibble(df)
    # return
    return(df)
  }

# set initial value
s <- 1
# draw simulation for a firm
seed <- 1
df <- simulate_dynamic_decision(p, s, PI, G, L, K, T, delta, seed); df
```

```
## # A tibble: 100 x 3
##       t     s     a
##    <int> <dbl> <dbl>
##  1     1     1     0
```

```
##  2     2     1     0
##  3     3     1     0
##  4     4     1     1
##  5     5     2     1
##  6     6     1     0
##  7     7     1     0
##  8     8     1     0
##  9     9     1     0
## 10    10     1     0
## # ... with 90 more rows
```

7. Write a function `simulate_dynamic_decision_across_firms(p, s, PI, G, L, K, T, N, delta)` that returns simulation data for N firm. For firm i, set the seed at i

```
# solve the dynamic decision model for each player
simulate_dynamic_decision_across_firms <-
  function(p, s, PI, G, L, K, T, N, delta) {
    df <-
      foreach (i = 1:N, .combine = "rbind") %dopar% {
        seed <- i
        df_i <- simulate_dynamic_decision(p, s, PI, G, L, K, T, delta, seed)
        df_i <- data.frame(i = i, df_i)
        return(df_i)
      }
    # as tibble
    df <- tibble::as_tibble(df)
    # return
    return(df)
  }
df <- simulate_dynamic_decision_across_firms(p, s, PI, G, L, K, T, N, delta)
save(df, file = "data/A7_df.RData")
```

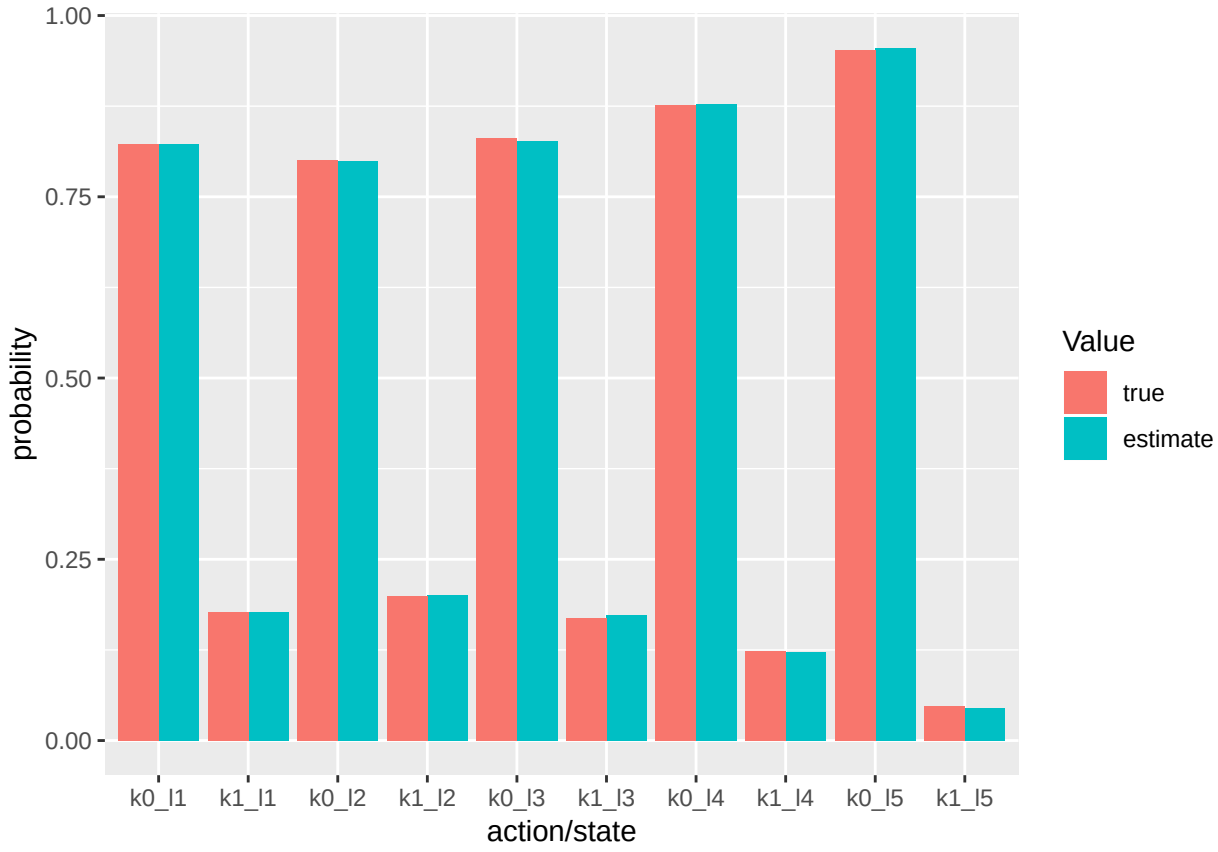
```
load(file = "data/A7_df.RData")
df
```

```
## # A tibble: 100,000 x 4
##        i     t     s     a
##    <int> <int> <dbl> <dbl>
##  1     1     1     1     0
##  2     1     2     1     0
##  3     1     3     1     0
##  4     1     4     1     1
##  5     1     5     2     1
##  6     1     6     1     0
##  7     1     7     1     0
##  8     1     8     1     0
##  9     1     9     1     0
## 10     1    10     1     0
## # ... with 99,990 more rows
```

8. Write a function `estimate_ccp(df)` that returns a non-parametric estimate of the conditional choice probability in the data. Compare the estimated conditional choice probability and the true conditional choice probability by a bar plot.

```
# non-parametrically estimate the conditional choice probability
estimate_ccp <-
  function(df) {
```

```
    p_est <- df %>%
      dplyr::group_by(s, a) %>%
      dplyr::summarise(p = length(a)) %>%
      dplyr::ungroup()
    p_est <- p_est %>%
      dplyr::group_by(s) %>%
      dplyr::mutate(p = p / sum(p)) %>%
      dplyr::ungroup() %>%
      as.matrix()
    rownames(p_est) <- paste("k", p_est[, "a"], "_l", p_est[, "s"], sep = "")
    p_est <- p_est[, "p", drop = FALSE]
    return(p_est)
  }

p_est <- estimate_ccp(df)
check_ccp <- cbind(p, p_est)
colnames(check_ccp) <- c("true", "estimate")
check_ccp <- check_ccp %>%
  reshape2::melt()
ggplot(data = check_ccp, aes(x = Var1, y = value,
                             fill = Var2)) +
  geom_bar(stat = "identity", position = "dodge") +
  labs(fill = "Value") + xlab("action/state") + ylab("probability")
```

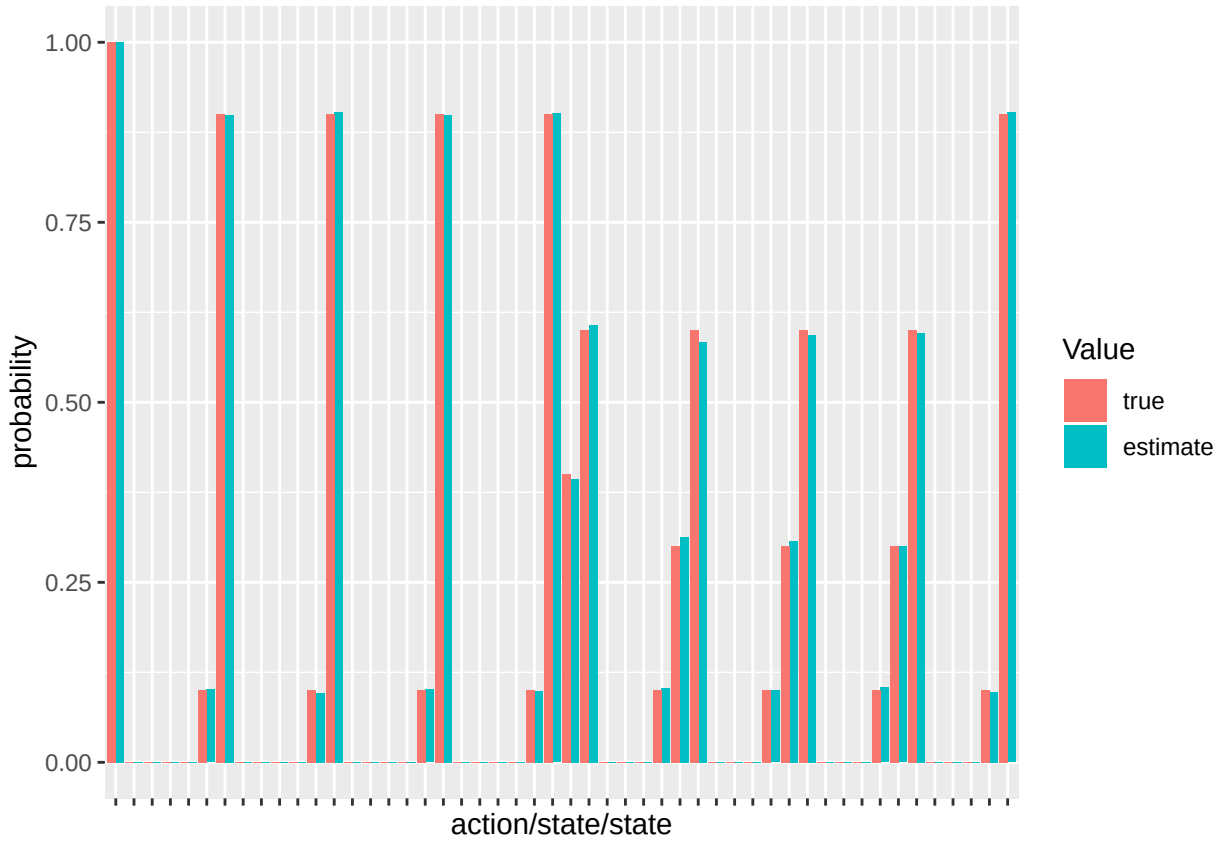


9. Write a function `estimate_G(df)` that returns a non-parametric estiamte of the transition matrix in the data. Compare the estimated transition matrix and the true transition matrix by a bar plot.

```r
# non-parametrically estimate the transition matrix
estimate_G <-
  function(df) {
    G_est <- df %>%
      dplyr::arrange(i, t) %>%
      dplyr::group_by(i) %>%
      dplyr::mutate(s_lead = dplyr::lead(s, 1)) %>%
      dplyr::filter(!is.na(s_lead)) %>%
      dplyr::ungroup() %>%
      dplyr::group_by(s, a, s_lead) %>%
      dplyr::summarise(g = length(s_lead)) %>%
      dplyr::ungroup()
    G_est <-
      G_est %>%
      dplyr::group_by(s, a) %>%
      dplyr::mutate(g = g / sum(g)) %>%
      dplyr::ungroup() %>%
      tidyr::complete(s, a, s_lead, fill = list(g = 0)) %>%
      dplyr::arrange(s, a, s_lead) %>%
      reshape2::dcast(formula = s + a ~ s_lead, value.var = "g")
    rownames(G_est) <- paste("k", G_est[, "a"], "_l", G_est[, "s"], sep = "")
    G_est <- G_est[, !colnames(G_est) %in% c("s", "a")]
    colnames(G_est) <- paste("l", 1:L, sep = "")
    G_est <- as.matrix(G_est)
    return(G_est)
  }

G_est <- estimate_G(df); G_est
```

```
##              l1         l2        l3         l4        l5
## k0_l1 1.0000000 0.00000000 0.0000000 0.00000000 0.0000000
## k1_l1 0.3930818 0.60691824 0.0000000 0.00000000 0.0000000
## k0_l2 0.1012162 0.89878384 0.0000000 0.00000000 0.0000000
## k1_l2 0.1031410 0.31276454 0.5840945 0.00000000 0.0000000
## k0_l3 0.0000000 0.09660837 0.9033916 0.00000000 0.0000000
## k1_l3 0.0000000 0.09974569 0.3071489 0.59310540 0.0000000
## k0_l4 0.0000000 0.00000000 0.1012564 0.89874358 0.0000000
## k1_l4 0.0000000 0.00000000 0.1039339 0.29966003 0.5964060
## k0_l5 0.0000000 0.00000000 0.0000000 0.09891400 0.9010860
## k1_l5 0.0000000 0.00000000 0.0000000 0.09751037 0.9024896
```

```r
check_G <- data.frame(type = "true", reshape2::melt(G))
check_G_est <- data.frame(type = "estimate", reshape2::melt(G_est))
check_G <- rbind(check_G, check_G_est)
check_G$variable = paste(check_G$Var1, check_G$Var2, sep = "_")
ggplot(data = check_G, aes(x = variable, y = value,
                           fill = type)) +
    geom_bar(stat = "identity", position = "dodge") +
  labs(fill = "Value") + xlab("action/state/state") + ylab("probability") +
  theme(axis.text.x = element_blank())
```



Estimate parameters

1. Vectorize the parameters as follows:

```
theta_1 <- c(alpha, beta)
theta_2 <- c(kappa, gamma)
theta <- c(theta_1, theta_2)
```

First, we estimate the parameters by a nested fixed-point algorithm. The loglikelihood for $\{a_{it}, s_{it}\}_{i=1,\cdots,N,t=1,\cdots,T}$ is:

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}[\log \mathbb{P}\{a_{it}|s_{it}\} + \log \mathbb{P}\{s_{i,t+1}|a_{it}, s_{it}\}],$$

with $\mathbb{P}\{s_{i,T+1}|a_{iT}, s_{iT}\} = 1$ for all i as $s_{i,T+1}$ is not observed.

2. Write a function `compute_loglikelihood_NFP(theta, df, delta, L, K)` that compute the loglikelihood.

```
# compute log likelihood function based on the NFP algorithm
compute_loglikelihood_NFP <-
  function(theta, df, delta, L, K) {
    # extract parameters
    alpha <- theta[1]
    beta <- theta[2]
    kappa <- theta[3]
    gamma <- theta[4]
    # construct PI
    PI <- compute_PI(alpha, beta, L, K)
```

```
    # construct G
    G <- compute_G(kappa, gamma, L, K)
    # solve dynamic decision
    output <- solve_dynamic_decision(PI, G, L, K, delta, lambda)
    ccp <- output$p
    # join
    ccp <- data.frame(s = gsub(".*l", "", rownames(ccp)),
                      a = gsub("_.*", "", rownames(ccp)),
                      p = as.numeric(ccp)) %>%
      dplyr::mutate(a = gsub("k", "", a),
                    s = as.integer(s),
                    a = as.integer(a))
    g <- reshape2::melt(G, value.name = "g") %>%
      dplyr::mutate(s = gsub(".*l", "", Var1),
                    a = gsub("_.*", "", Var1),
                    a = gsub("k", "", a),
                    s_lead = gsub("l", "", Var2),
                    s = as.integer(s),
                    a = as.integer(a),
                    s_lead = as.integer(s_lead)) %>%
      dplyr::select(a, s, s_lead, g)
    # likelihood
    likelihood <- df %>%
      dplyr::arrange(i, t) %>%
      dplyr::group_by(i) %>%
      dplyr::mutate(s_lead = dplyr::lead(s, 1)) %>%
      dplyr::ungroup() %>%
      dplyr::mutate(
        a = as.integer(a),
        s = as.integer(s),
        s_lead = as.integer(s_lead)) %>%
      dplyr::left_join(ccp, by = c("s", "a")) %>%
      dplyr::left_join(g, by = c("s", "s_lead", "a"))
    # compute loglikelihood
    loglikelihood <-
      sum(log(likelihood$p)) + sum(log(likelihood$g), na.rm = TRUE)
    loglikelihood <- loglikelihood / dim(df)[1]
    # return
    return(loglikelihood)
  }

loglikelihood <- compute_loglikelihood_NFP(theta, df, delta, L, K); loglikelihood
```

```
## [1] -0.7474961
```

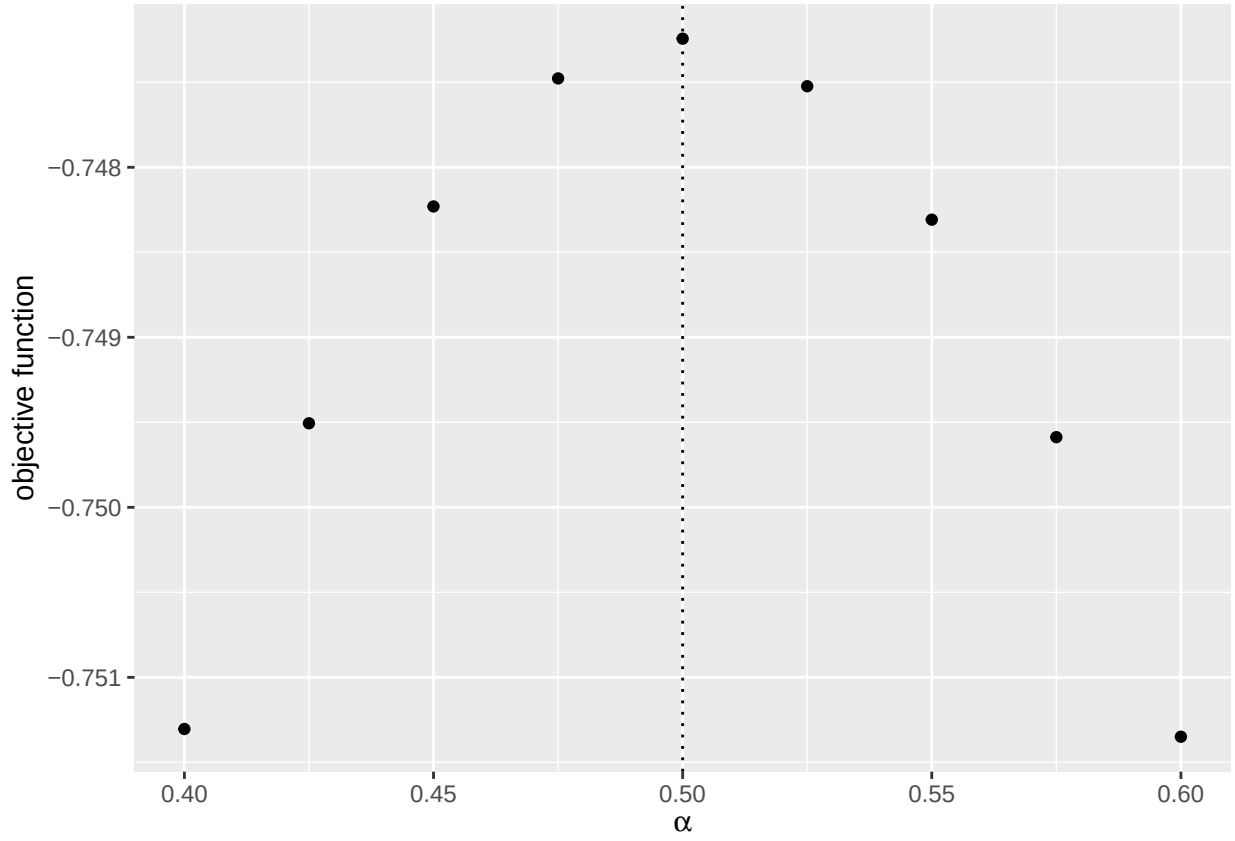
3. Check the value of the objective function around the true parameter.

```
# label
label <- c("\\alpha", "\\beta", "\\kappa", "\\gamma")
label <- paste("$", label, "$", sep = "")
# compute the graph
graph <- foreach (i = 1:length(theta)) %do% {
  theta_i <- theta[i]
  theta_i_list <- theta_i * seq(0.8, 1.2, by = 0.05)
  objective_i <-
```

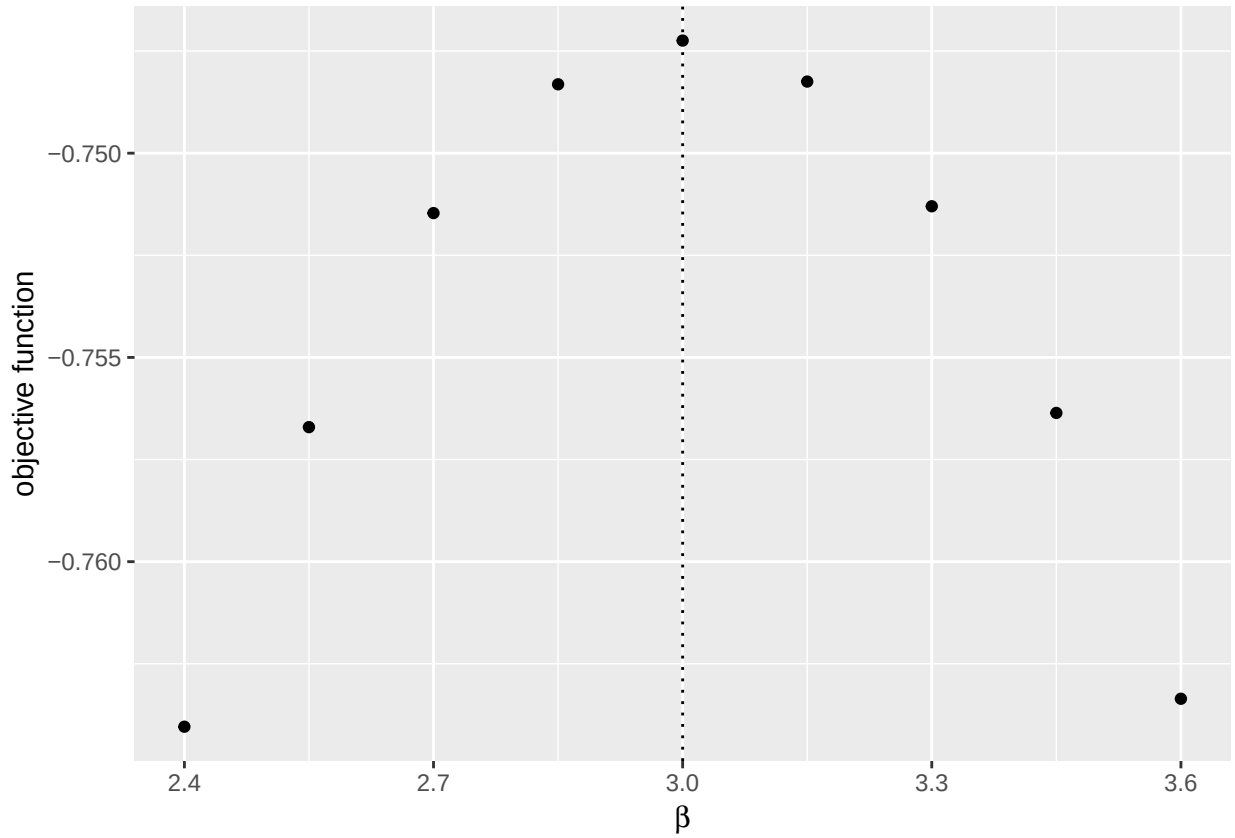
```
    foreach (j = 1:length(theta_i_list),
             .combine = "rbind") %do% {
               theta_ij <- theta_i_list[j]
               theta_j <- theta
               theta_j[i] <- theta_ij
               objective_ij <-
                 compute_loglikelihood_NFP(
                   theta_j, df, delta, L, K); loglikelihood

               return(objective_ij)
             }
  df_graph <- data.frame(x = theta_i_list, y = objective_i)
  g <- ggplot(data = df_graph, aes(x = x, y = y)) +
    geom_point() +
    geom_vline(xintercept = theta_i, linetype = "dotted") +
    ylab("objective function") + xlab(TeX(label[i]))
  return(g)
}
save(graph, file = "data/A7_NFP_graph.RData")
```

```
load(file = "data/A7_NFP_graph.RData")
graph
```

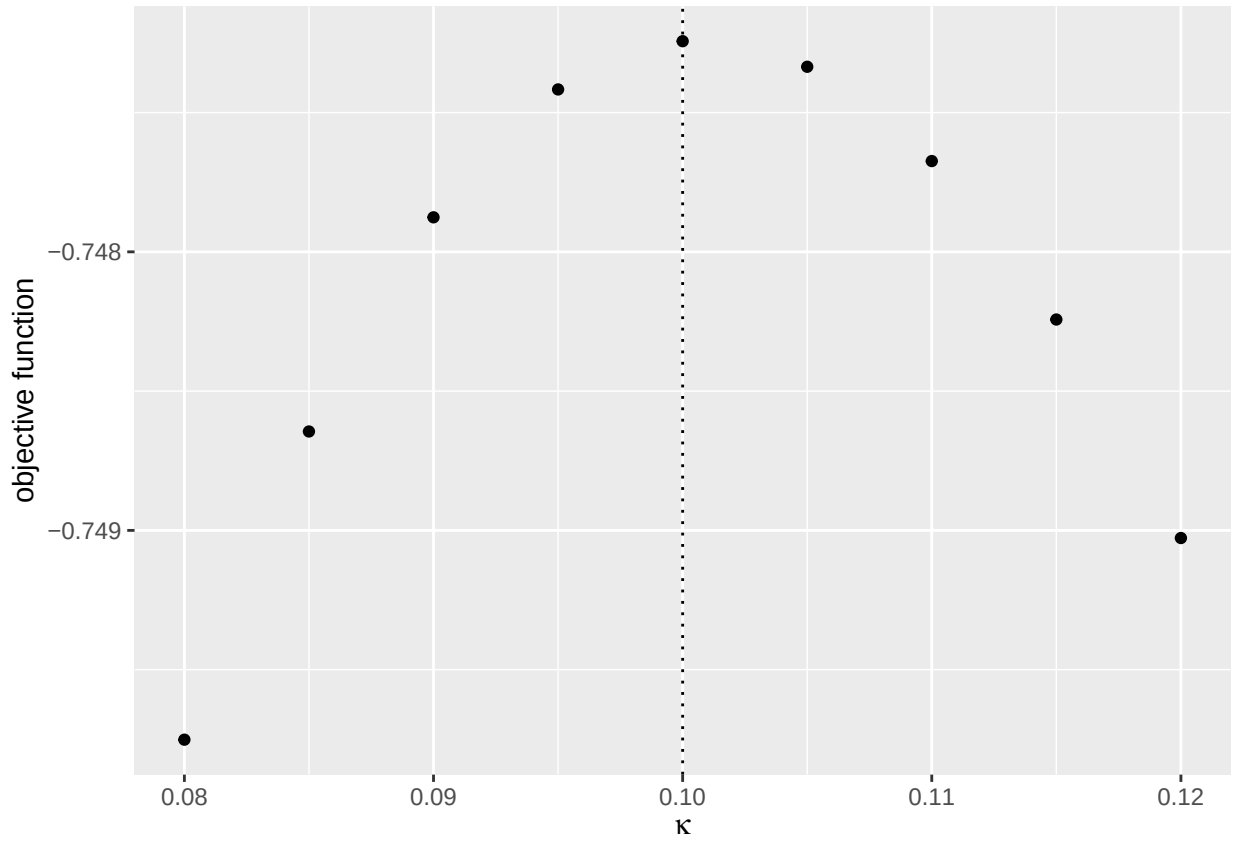
```
## [[1]]
```



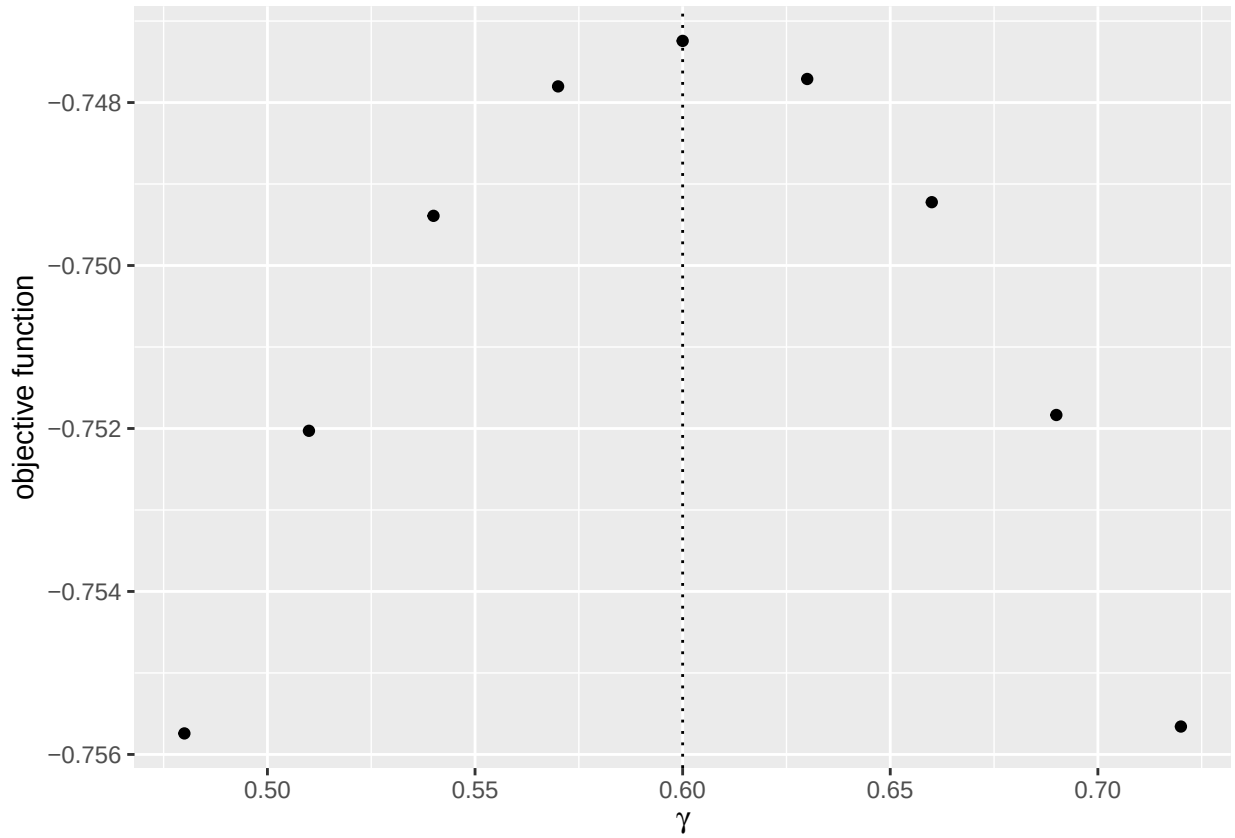

```
##
## [[2]]
```




```
##
## [[3]]
```




```
##
## [[4]]
```



4. Estiamte the parameters by maximizing the loglikelihood. To keep the model to be well-defined, impose an ad hoc lower and upper bounds such that $\alpha \in [0, 1], \beta \in [0, 5], \kappa \in [0, 0.2], \gamma \in [0, 0.7]$.

```
lower <- rep(0, length(theta))
upper <- c(1, 5, 0.2, 0.7)
NFP_result <-
  optim(par = theta,
        fn = compute_loglikelihood_NFP,
        method = "L-BFGS-B",
        lower = lower,
        upper = upper,
        control = list(fnscale = -1),
        df = df,
        delta = delta,
        L = L,
        K = K)
save(NFP_result, file = "data/A7_NFP_result.RData")
```

```
load(file = "data/A7_NFP_result.RData")
NFP_result
```

```
## $par
## [1] 0.4916153 2.9816751 0.1005993 0.6029317
##
## $value
## [1] -0.747237
##
## $counts
```

```
## function gradient
##       17       17
##
## $convergence
## [1] 0
##
## $message
## [1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"
```

```
compare <-
  data.frame(
    true = theta,
    estimate = NFP_result$par
  ); compare
```

```
##   true  estimate
## 1  0.5 0.4916153
## 2  3.0 2.9816751
## 3  0.1 0.1005993
## 4  0.6 0.6029317
```

Next, we estimate the parameters by CCP approach.

5. Write a function `estimate_theta_2(df)` that returns the estimates of κ and γ directly from data by counting relevant events.

```
# estimate theta_2
estimate_theta_2 <-
  function(df) {
    # estimate kappa
    kappa_est <- df %>%
      dplyr::arrange(i, t) %>%
      dplyr::group_by(i) %>%
      dplyr::mutate(s_lead = dplyr::lead(s, 1)) %>%
      dplyr::filter(!is.na(s_lead)) %>%
      dplyr::ungroup() %>%
      dplyr::mutate(move_down = ifelse(s_lead < s, 1, 0)) %>%
      dplyr::filter(s > 1) %>%
      dplyr::group_by(move_down) %>%
      dplyr::summarise(kappa = length(move_down)) %>%
      dplyr::ungroup() %>%
      dplyr::mutate(kappa = kappa / sum(kappa)) %>%
      dplyr::filter(move_down == 1)
    kappa_est <- kappa_est$kappa
    # estimate gamma
    gamma_est <- df %>%
      dplyr::arrange(i, t) %>%
      dplyr::group_by(i) %>%
      dplyr::mutate(s_lead = dplyr::lead(s, 1)) %>%
      dplyr::filter(!is.na(s_lead)) %>%
      dplyr::ungroup() %>%
      dplyr::mutate(move_up = ifelse(s_lead > s, 1, 0)) %>%
      dplyr::filter(s < L, a == 1) %>%
      dplyr::group_by(move_up) %>%
      dplyr::summarise(gamma = length(move_up)) %>%
      dplyr::ungroup() %>%
```

```
    dplyr::mutate(gamma = gamma / sum(gamma)) %>%
    dplyr::filter(move_up == 1)
  gamma_est <- gamma_est$gamma
  # theta_2
  theta_2_est <- c(kappa_est, gamma_est)
  # return
  return(theta_2_est)
}


theta_2_est <- estimate_theta_2(df); theta_2_est
```

```
## [1] 0.09988488 0.59551895
```

The objective function of the minimum distance estimator based on the conditional choice probability approach is:

$$\frac{1}{KL}\sum_{s=1}^{L}\sum_{a=1}^{K}\{\hat{p}(a,s) - p^{(\theta_1,\theta_2)}(a,s)\}^2,$$

where $\hat{p}$ is the non-parametric estimate of the conditional choice probability and $p^{(\theta_1,\theta_2)}$ is the optimal conditional choice probability under parameters θ_1 and θ_2.

6. Write a function `compute_CCP_objective(theta_1, theta_2, p_est, L, K, delta)` that returns the objective function of the above minimum distance estimator given a non-parametric estimate of the conditional choice probability and θ_1 and θ_2.

```
# compute the objective function of the minimum distance estimator based on the CCP approach
compute_CCP_objective <-
  function(theta_1, theta_2, p_est, L, K, delta) {
    # extract parameters
    alpha <- theta_1[1]
    beta <- theta_1[2]
    kappa <- theta_2[1]
    gamma <- theta_2[2]
    # construct PI
    PI <- compute_PI(alpha, beta, L, K)
    # construct G
    G <- compute_G(kappa, gamma, L, K)

    # solve relevant choice probability
    V <- compute_exante_value(p_est, PI, G, L, K, delta)
    ccp <- compute_ccp(V, PI, G, L, K, delta)

    # # solve dynamic decision (fixed-point)
    # output <- solve_dynamic_decision(PI, G, L, K, delta, lambda)
    # ccp <- output$p

    # minimum distance
    distance <- (ccp - p_est)^2
    distance <- distance[grepl("k1", rownames(distance)), ]
    distance <- mean(distance)
    # return
    return(distance)
  }
```

```
compute_CCP_objective(theta_1, theta_2, p_est, L, K, delta)
```

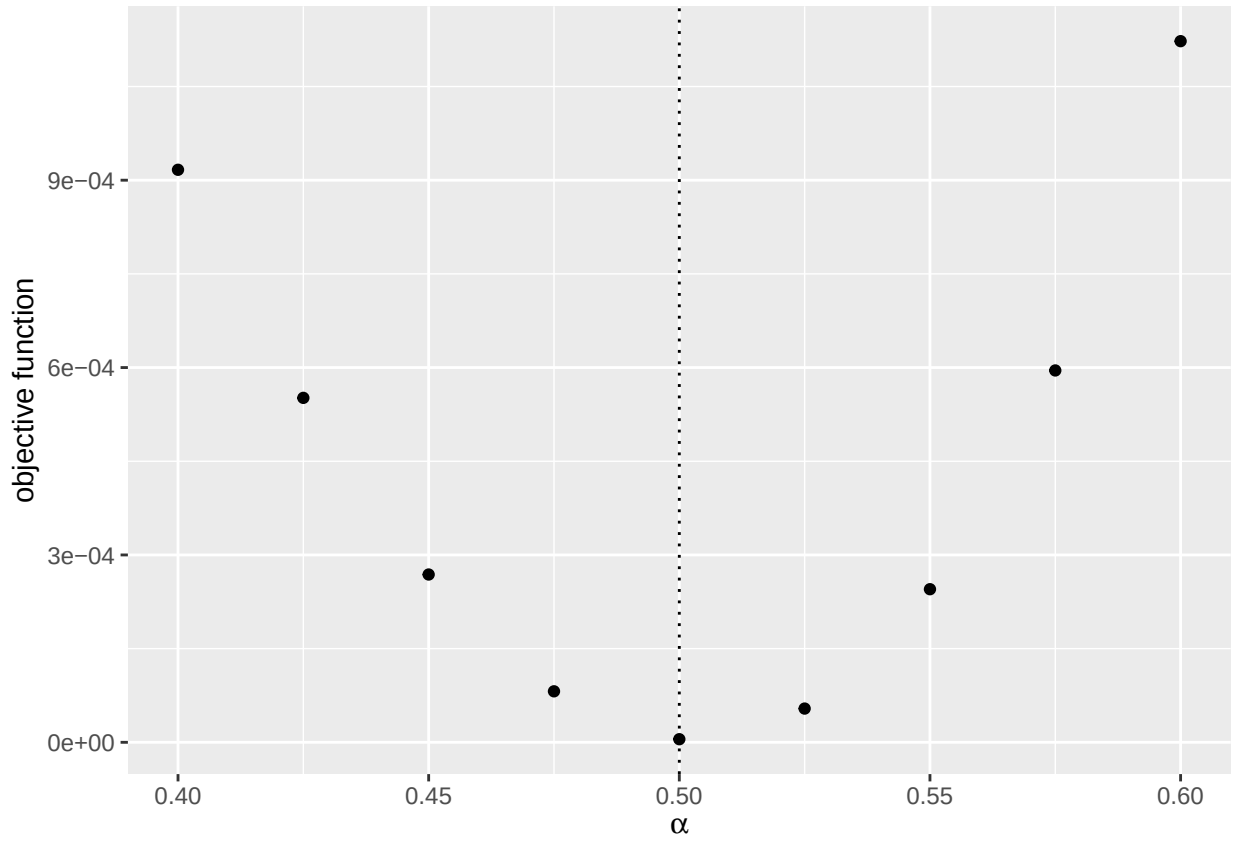
```
## [1] 5.000511e-06
```

3. Check the value of the objective function around the true parameter.

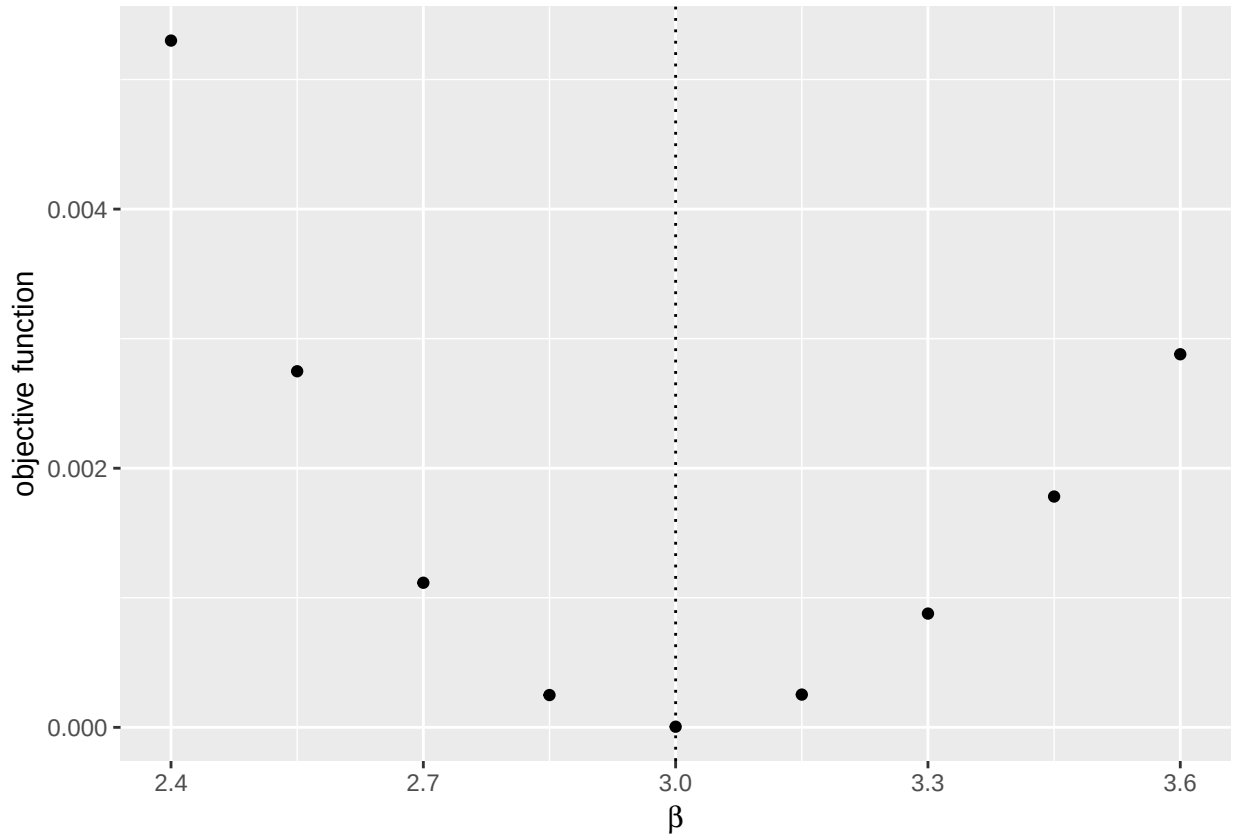
```
# label
label <- c("\\alpha", "\\beta")
label <- paste("$", label, "$", sep = "")
# compute the graph
graph <- foreach (i = 1:length(theta_1)) %do% {
  theta_i <- theta_1[i]
  theta_i_list <- theta_i * seq(0.8, 1.2, by = 0.05)
  objective_i <-
    foreach (j = 1:length(theta_i_list),
             .combine = "rbind") %do% {
               theta_ij <- theta_i_list[j]
               theta_j <- theta_1
               theta_j[i] <- theta_ij
               objective_ij <-
                 compute_CCP_objective(theta_j, theta_2, p_est, L, K, delta)
               return(objective_ij)
             }
  df_graph <- data.frame(x = theta_i_list, y = objective_i)
  g <- ggplot(data = df_graph, aes(x = x, y = y)) +
    geom_point() +
    geom_vline(xintercept = theta_i, linetype = "dotted") +
    ylab("objective function") + xlab(TeX(label[i]))
  return(g)
}
save(graph, file = "data/A7_CCP_graph.RData")
```

```
load(file = "data/A7_CCP_graph.RData")
graph
```

```
## [[1]]
```




```
##
## [[2]]
```



4. Estiamte the parameters by minimizing the objective function. To keep the model to be well-defined, impose an ad hoc lower and upper bounds such that $\alpha \in [0, 1], \beta \in [0, 5]$.

```
lower <- rep(0, length(theta_1))
upper <- c(1, 5)
CCP_result <-
  optim(par = theta_1,
        fn = compute_CCP_objective,
        method = "L-BFGS-B",
        lower = lower,
        upper = upper,
        theta_2 = theta_2_est,
        p_est = p_est,
        L = L,
        K = K,
        delta = delta)
save(CCP_result, file = "data/A7_CCP_result.RData")
```

```
load(file = "data/A7_CCP_result.RData")
CCP_result
```

```
## $par
## [1] 0.5271684 3.0644600
##
## $value
## [1] 1.790528e-06
##
## $counts
```

```
## function gradient
##       11       11
##
## $convergence
## [1] 0
##
## $message
## [1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"
```

```
compare <-
  data.frame(
    true = theta_1,
    estimate = CCP_result$par
  ); compare
```

```
##   true  estimate
## 1  0.5 0.5271684
## 2  3.0 3.0644600
```