

# Delta Method, Bootstrap, and Cross Validation

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Applied Econometrics II

# Bootstrap and Delta Method

- We know how to construct confidence intervals for parameter estimates:  
 $\hat{\theta}_k \pm 1.96SE(\hat{\theta}_k)$
- Often we are asked to construct standard errors or confidence intervals around model outputs that are not just parameter estimates: ie:  $g(x_i, \hat{\theta})$ .
- Sometimes we can't even write  $g(x_i, \theta)$  as an explicit function of  $\theta$  ie:  
 $\Psi(g(x_i, \theta), \theta) = 0$ .
- Two options:
  1. Delta Method
  2. Bootstrap

## Delta Method

Delta method works by considering a **Taylor Expansion** of  $g(x_i, \theta)$ .

$$g(z) \approx g(z_0) + g'(z_0)(z - z_0) + o(\|z - z_0\|)$$

Assume that  $\theta_n$  is asymptotically normally distributed so that:

$$\sqrt{n}(\theta_n - \theta_0) \sim N(0, \Sigma)$$

(How do we get this: OLS? GMM? MLE?).

Then we have that

$$\sqrt{n}(g(\theta_n) - g(\theta_0)) \sim N(0, D(\theta)' \Sigma D(\theta))$$

Where  $D(\theta) = \frac{\partial g(x_i, \theta)}{\partial \theta}$  is the Jacobian of  $g$  with respect to  $\theta$  evaluated at  $\theta$ .

We need  $g$  to be continuously differentiable around the center of our expansion  $\theta$ .

## Delta Method: Examples

Start with something simple:  $Y = \bar{X}_1 \cdot \bar{X}_2$  with  $(X_{1i}, X_{2i}) \sim IID$ . We know the CLT applies so that:

$$\sqrt{n} \begin{pmatrix} \bar{X}_1 - \mu_1 \\ \bar{X}_2 - \mu_2 \end{pmatrix} \sim N \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma \right]$$

The Jacobian is just  $D(\theta) = \begin{pmatrix} \frac{\partial g(\theta)}{\partial \theta_1} \\ \frac{\partial g(\theta)}{\partial \theta_2} \end{pmatrix} = \begin{pmatrix} s_2 \\ s_1 \end{pmatrix}$

So,

$$V(Y) = D(\theta)' \Sigma D(\theta) = \begin{pmatrix} \mu_2 & \mu_1 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \mu_2 \\ \mu_1 \end{pmatrix}$$
$$\sqrt{n}(\bar{X}_1 \bar{X}_2 - \mu_1 \mu_2) \sim N(0, \mu_2^2 \sigma_{11}^2 + 2\mu_1 \mu_2 \sigma_{12} + \mu_1^2 \sigma_{22}^2)$$

## Delta Method: Examples

Think about a simple logit:

$$P(Y_i = 1|X_i) = \frac{\exp^{\beta_0 + \beta_1 X_i}}{1 + \exp^{\beta_0 + \beta_1 X_i}} \quad P(Y_i = 0|X_i) = \frac{1}{1 + \exp^{\beta_0 + \beta_1 X_i}}$$

Remember the “trick” to use GLM (log-odds):

$$\log P(Y_i = 1|X_i) - \log P(Y_i = 0|X_i) = \beta_0 + \beta_1 X_i$$

- Suppose that we have estimated  $\hat{\beta}_0, \hat{\beta}_1$  via GLM/MLE but we want to know the confidence interval for the probability:  $P(Y_i = 1|X_i, \hat{\theta})$
- The derivatives are a little bit tricky, but the idea is the same.
- This is what STATA should be doing when you type: `mfxx, compute`

## Delta Method: Other Examples

Often we have a regression like:

$$\log Y_i = \beta_0 + \beta_1 X_i + \gamma \text{Income}_i + \epsilon_i$$

And we are interested in  $\beta_1/\gamma$  so that we have  $\beta_i$  in units of “dollars”. Again Delta Method Works fine here.

## Delta Method: Some Failures

But we need to be careful. Suppose that  $\theta \approx 0$  and

- $g(x) = |X|$
- $g(x) = 1/X$
- $g(x) = \sqrt{X}$

These situations can arise in practice when we have weak instruments or other problems.

# Bootstrap

- Bootstrap takes a different approach.
  - Instead of estimating  $\hat{\theta}$  and then using a first-order Taylor Approximation...
  - What if we directly tried to construct the **sampling distribution** of  $\hat{\theta}$ ?
- Our data  $(X_1, \dots, X_n) \sim P$  are drawn from some measure  $P$ 
  - We can form a **nonparametric estimate**  $\hat{P}$  by just assuming that each  $X_i$  has weight  $\frac{1}{n}$ .
  - We can then simulate a new sample  $X^* = (X_1^*, \dots, X_n^*) \sim \hat{P}$ .
    - Easy: we take our data and construct  $n$  observations by **sampling with replacement**
  - Compute whatever statistic of  $X^*$ ,  $S(X^*)$  we would like.
    - Could be the OLS coefficients  $\beta_1^*, \dots, \beta_k^*$ .
    - Or some function  $\beta_1^* / \beta_2^*$ .
    - Or something really complicated: estimate parameters of a game  $\hat{\theta}^*$  and now find Nash Equilibrium of the game  $S(X^*, \hat{\theta}^*)$  changes.
  - Do this  $B$  times and calculate at  $Var(S_b)$  or  $CI(S_1, \dots, S_b)$ .



## Bootstrap: Bias Correction

The main idea is that  $\hat{\theta}^{1*}, \dots, \hat{\theta}^{B*}$  approximates the **sampling distribution** of  $\hat{\theta}$ . There are lots of things we can do now:

- We already saw how to calculate  $\text{Var}(\hat{\theta}^{1*}, \dots, \hat{\theta}^{B*})$ .

$$\frac{1}{B-1} \sum_{b=1}^B (\hat{\theta}_{(b)}^* - \bar{\theta}^*)^2$$

- Calculate  $E(\hat{\theta}_{(1)}^*, \dots, \hat{\theta}_{(B)}^*) = \bar{\theta}^* = \frac{1}{B} \sum_{b=1}^B \hat{\theta}_{(b)}^*$ .
  - We can use the estimated bias to **bias correct** our estimates

$$\begin{aligned} \text{Bias}(\hat{\theta}) &= E[\hat{\theta}] - \theta \\ \text{Bias}_{bs}(\hat{\theta}) &= \bar{\theta}^* - \hat{\theta} \end{aligned}$$

Recall  $\theta = E[\hat{\theta}] - \text{Bias}[\hat{\theta}]$ :

# Bootstrap: Confidence Intervals

There are actually three ways to construct bootstrap CI's:

1. Obvious way: sort  $\hat{\theta}^*$  then take  $CI : [\hat{\theta}_{\alpha/2}^*, \hat{\theta}_{1-\alpha/2}^*]$ .
2. Asymptotic Normal:  $CI : \hat{\theta} \pm 1.96 \sqrt{V(\hat{\theta}^*)}$ . (CLT).
3. Better Way: let  $W = \hat{\theta} - \theta$ . If we knew the distribution of  $W$  then:  
 $Pr(w_{1-\alpha/2} \leq W \leq w_{\alpha/2})$ :

$$CI : [\hat{\theta} - w_{1-\alpha/2}, \hat{\theta} - w_{\alpha/2}]$$

We can estimate with  $W^* = \hat{\theta}^* - \hat{\theta}$ .

$$CI : [\hat{\theta} - w_{1-\alpha/2}^*, \hat{\theta} - w_{\alpha/2}^*] = [2\hat{\theta} - \theta_{1-\alpha/2}^*, 2\hat{\theta} - \theta_{\alpha/2}^*]$$

Why is this preferred? Bias Correction!

## Bootstrap: Why do people like it?

- Econometricians like the bootstrap because under certain conditions it is **higher order efficient** for the confidence interval construction (but not the standard errors).
  - Intuition: because it is non-parametric it is able to deal with more than just the first term in the Taylor Expansion (actually an **Edgeworth Expansion**).
  - Higher-order asymptotic theory is best left for real econometricians!
- Practitioner's like the bootstrap because it is easy.
  - If you can estimate your model once in a reasonable amount of time, then you can construct confidence intervals for most parameters and model predictions.

## Bootstrap: When Does It Fail?

- Bootstrap isn't magic. If you are constructing standard errors for something that isn't asymptotically normal, don't expect it to work!
- The Bootstrap exploits the notion that your sample is IID (by sampling with replacement). If IID does not hold, the bootstrap may fail (but we can sometimes fix it!).
- Bootstrap depends on asymptotic theory. In small samples weird things can happen. We need  $\hat{P}$  to be a good approximation to the true  $P$  (nothing missing).

## Bootstrap: Variants

The bootstrap I have presented is sometimes known as the **nonparametric bootstrap** and is the most common one.

**Parametric Bootstrap** ex: if  $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$  then we can estimate  $(\hat{\beta}_0, \hat{\beta}_1)$  via OLS.

Now we can generate a bootstrap sample by drawing an  $x_i$  at random with replacement  $\hat{\beta}_0 + \hat{\beta}_1$  and then drawing **independently** from the distribution of estimated residuals  $\hat{\epsilon}_i$ .

**Wild Bootstrap** Similar to parametric bootstrap but we rescale  $\epsilon_i$  to allow for **heteroskedasticity**

**Block Bootstrap** For correlated data (e.g.: time series). Blocks can be overlapping or not.

# Bootstrap vs Delta Method

- Delta Method works best when working out Jacobian  $D(\theta)$  is easy and statistic is well approximated with a linear function (not too curvy).
- I would almost always advise Bootstrap unless:
  - Delta method is trivial e.g.:  $\beta_1/\beta_2$  in linear regression.
  - Computing model takes many days so that 10,000 repetitions would be impossible.
- Worst case scenario: rent time on Amazon EC2!
  - I “bought” over \$1,000 of standard errors recently.
- But neither is magic and both can fail!

# Cross Validation

Cross Validation appears superficially similar to bootstrap but asks a different question.

- Bootstrap tries to construct an empirical analogue to the sampling distribution of  $\hat{\theta}$ .
- CV tries to measure what the expected out of sample (OOS or EPE) prediction error of a new never seen before dataset.
- The main consideration is to prevent **overfitting**.
  - In sample fit is always going to be maximized by the most complicated model.
  - OOS fit might be a different story.
  - 1-NN might do really well in-sample, but with a new sample might perform badly.

## Sample Splitting/Holdout Method and CV

Cross Validation is actually a more complicated version of **sample splitting** that is one of the organizing principles in machine learning literature.

**Training Set** This is where you estimate parameter values.

**Validation Set** This is where you choose a model- a bandwidth  $h$  or tuning parameter  $\lambda$  by computing the error.

**Test Set** You are only allowed to look at this after you have chosen a model.

**Only Test Once:** compute the error again on fresh data.

- Conventional approach is to allocate 50-80% to training and 10-20% to Validation and Test.
- Sometimes we don't have enough data to do this reliably.

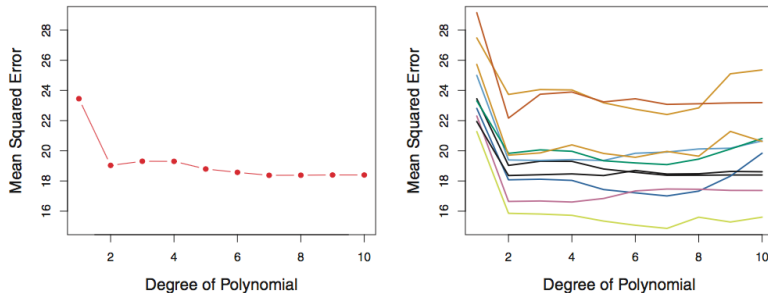


# Sample Splitting/Holdout Method



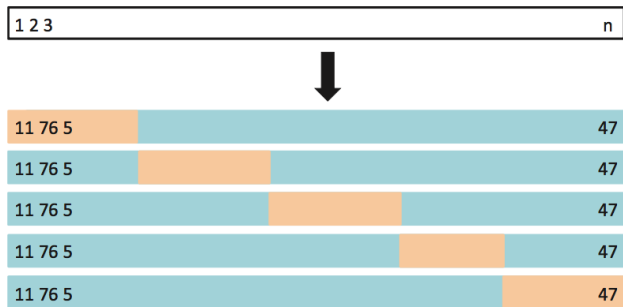
**FIGURE 5.1.** A schematic display of the validation set approach. A set of  $n$  observations are randomly split into a training set (shown in blue, containing observations 7, 22, and 13, among others) and a validation set (shown in beige, and containing observation 91, among others). The statistical learning method is fit on the training set, and its performance is evaluated on the validation set.

# Challenge with Sample Splitting



**FIGURE 5.2.** The validation set approach was used on the **Auto** data set in order to estimate the test error that results from predicting **mpg** using polynomial functions of **horsepower**. Left: Validation error estimates for a single split into training and validation data sets. Right: The validation method was repeated ten times, each time using a different random split of the observations into a training set and a validation set. This illustrates the variability in the estimated test MSE that results from this approach.

# Cross Validation



**FIGURE 5.5.** A schematic display of 5-fold CV. A set of  $n$  observations is randomly split into five non-overlapping groups. Each of these fifths acts as a validation set (shown in beige), and the remainder as a training set (shown in blue). The test error is estimated by averaging the five resulting MSE estimates.

## $k$ -fold Cross Validation

- Break the dataset into  $k$  equally sized “folds” (at random).
- Withhold  $i = 1$  fold
  - Estimate the model parameters  $\hat{\theta}^{(-i)}$  on the remaining  $k - 1$  folds
  - Predict  $\hat{y}^{(-i)}$  using  $\hat{\theta}^{(-i)}$  estimates for the  $i$ th fold (withheld data).
  - Compute  $MSE_i = \frac{1}{k \cdot N} \sum_j (y_j^{(-i)} - \hat{y}_j^{(-i)})^2$ .
  - Repeat for  $i = 1, \dots, k$ .
- Construct  $\widehat{MSE}_{k,CV} = \frac{1}{k} \sum_i MSE_i$

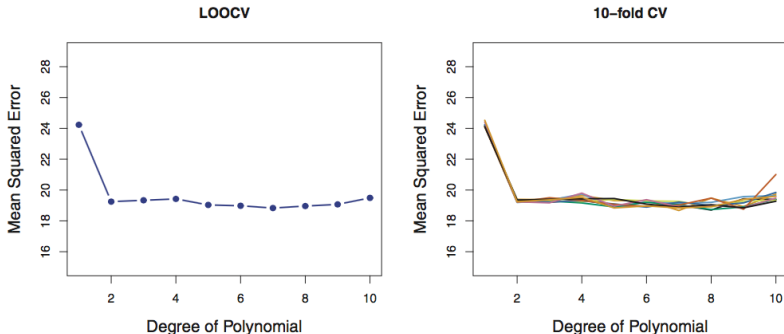
## Leave One Out Cross Validation (LOOCV)

Same as  $k$ -fold but with  $k = N$ .

- Withhold a single observation  $i$
- Estimate  $\hat{\theta}_{(-i)}$ .
- Predict  $\hat{y}_i$  using  $\hat{\theta}^{(-i)}$  estimates
- Compute  $MSE_i = \frac{1}{N} \sum_j (y_i - \hat{y}_i(\hat{\theta}^{(-i)}))^2$ .

Note: this requires estimating the model  $N$  times which can be costly.

# Cross Validation



**FIGURE 5.4.** Cross-validation was used on the **Auto** data set in order to estimate the test error that results from predicting **mpg** using polynomial functions of **horsepower**. Left: The LOOCV error curve. Right: 10-fold CV was run nine separate times, each with a different random split of the data into ten parts. The figure shows the nine slightly different CV error curves.

# Cross Validation

- Main advantage of cross validation is that we use all of the data in both **estimation** and in **validation**.
  - For our purposes validation is mostly about choosing the right bandwidth or tuning parameter.
- We have much lower variance in our estimate of the OOS mean squared error.
  - Hopefully our bandwidth choice doesn't depend on randomness of splitting sample.

- In Statistics/Machine learning there is a tradition to withhold 10% of the data as **Test Data**.
- This is **completely new data** that was not used in the CV procedure.
- The idea is to report the results using this test data because it most accurately simulates true OOS performance.
- We don't do much of this in economics.  
(Should we do more?)