**Question 5:**

a. Use mathematical induction to prove that for any positive integer $n$, 3 divide $n^3 + 2n$ (leaving no remainder).

proof:

1.base case:

When n = 1, $n^3 + 2n = 3$, 3 divide 3 is true.

2. Inductive steps:

Assume for k, 3 divide $k^3 + 2k$ is true. Means $k^3 + 2k = 3m$, where m is an integer.

Now we need to show for k+1,3 divide $(k+1)^3 + 2(k+1)$

$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2(k+1)$

$= 3m + 3k^2 + k+1 + 2(k+1)$

$=3m + 3k^2 + 3k+3$

$= 3(m+k^2 + k+1)$

Since m and k are integers, then $m+k^2 + k+1$ is an integer.

So $(k+1)^3 + 2(k+1) = 3t$, where t is an integer and $t = 3(m+k^2 + k+1)$

So 3 divide $(k+1)^3 + 2(k+1)$

Then for any positive integer n, 3 divide $n^3 + 2n$ ■

b. Use strong induction to prove that any positive integer $n$ ($n \geq 2$) can be written as a product of primes.

Proof:

Case 1: If n is a prime, it can be written as 1xn . the base case n = 2 is like this.

Case 2: If it is not, assume for all k <=n, it can be written as a product of primes.

Then for n + 1, we can write n+1=ab for some positive integers a,b.

Now, a,b<n+1 and so a,b⩽ k so a and b are primes or can be written as product of primes. So n+1 is a product of primes.

**Question 6:**
Solve the following questions from the Discrete Math zyBook:
**a) Exercise 7.4.1, sections a-g**
(a)Verify that P(3) is true.
When n = 3, LHS = 1 + 4 +9 =14
RHS = 3*4*7/6 = 14 = LHS
So P(3) is true.
(b)Express P(k).

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

(c)Express P(k + 1)

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

(d) In an inductive proof that for every positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the base case?

It must be proven that p(1) is true in the base case.

(e) In an inductive proof that for every positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the inductive step?

In the inductive step, need to prove p(k+1) is true.

(f)What would be the inductive hypothesis in the inductive step from your previous answer?
Assume p(k) is true.

(g) Prove by induction that for any positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

1. Base case:
When n = 1, LHS: p(1) = 1, and RHS p(1) = 1*2*3/6 = 1
So LHS = RHS, p(1) is true.
2. Inductive steps:
Assume p(k) is true, means

$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

Then prove p(k+1) is true.

The LHS of p(k+1) = $\sum_{j=1}^{k+1} j^2$ = $\sum_{j=1}^{k} j^2$+(k+1)$^2$

=$\frac{k(k+1)(2k+1)}{6}$ +(k+1)$^2$

=$\frac{(k+1)(k+2)(2k+3)}{6}$

=RHS

So p(k+1) is true.

Then for any integer n $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

## b) Exercise 7.4.3, section c

Hint: you may want to use the following fact: $\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}$

(c)

Prove that for n ≥ 1, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

1.Base case:

For n = 1, LHS = 1, RHS = 2-1 =1

So for n = 1, it is true.

2.inductive steps:

Assume that for k, p(k) is true. (p(n) is the proposition the question want to prove)

We need to prove p(k+1) is true.

LHS of p(k+1) = $\sum_{j=1}^{k+1} \frac{1}{j^2}$

= $\sum_{j=1}^{k} \frac{1}{j^2}$+$\frac{1}{k^2}$ $\leqslant$2-$\frac{1}{k}$+$\frac{1}{k^2}$

=2-$\frac{k-1}{k^2}$ $\leqslant$2-$\frac{k-1}{(k+1)k}$$\leqslant$2- $\frac{k-1}{(k+1)(k-1)}$

=2- $\frac{1}{k+1}$

= RHS of p(k+1)

So p(n) is true ■

## c) Exercise 7.5.1, section a

(a)Prove that for any positive integer n, 4 evenly divides $3^{2n}$-1.

p(n): for any positive integer n, 4 evenly divides $3^{2n}$-1

1. base case:

When n = 1, $3^{2n}$-1 = 8, is evenly divided by 4, because 8 = 2*4

2. inductive steps:

Assume p(k) is true, means $3^{2k}$-1 = 4m, where m is an integer;

Then we need to prove p(k+1) is true.

$3^{2(k+1)}-1 = 3^{2k}*3^{2}-1=9*3^{2k}-1$

$= 9*4m+8$

$=4(9m+2)$

Since m is an integer, then 9m+2 is an integer,

So 4 evenly divides $3^{2(k+1)}-1$ , p(k+1) is true.

So for any positive integer n, 4 evenly divides $3^{2n}-1$. ■