CISC 102 (Fall 20)
Homework #5: Sequences, Recursion & Induction (25 Points)

Student Name/ID: Amy Brons / 20252295

Solutions are due before 11:59 PM on **November 1, 2020** .

1. (2pts)
   Prove by induction that
   $$\sum_{j=1}^{n} 2^j = 2^{n+1} - 2 \ ,\forall\, n \geq 1$$
   This gives us the sequence:
   $2^1 + 2^2 + 2^3 + ...2^n = 2^{n+1} - 2.$
   Here is how we get the base case:
   $\sum_{j=1}^{n} 2^j = 2^1 = 0$
   Next assume that for k $\geq$ 0 so that $\sum_{j=1}^{k} 2^j = 2^{k+1} - 2,$
   and we assume that $\sum_{j=1}^{k+1} 2^j = 2^{k+2} - 2.$
   Using recursion we can come up with the following:
   $\sum_{j=1}^{k+1} 2^j = \left(\sum_{j=1}^{k} 2^j\right) + 2^k + 1$
   $= \left(2^{k+1} - 2\right) + 2^{k+1}$
   $= 2 * 2^{k+1} - 2$
   $= 2^{k+2} - 2$
   This proves that this is true $\forall\, n \geq 1$, because even with the -2, any number will result in a positive integer.

2. (2pts)
   Prove by induction:
   $$\forall n \geq 1, (n^3 - n) \text{ is divisible by } 3$$
   For n = 1, we get 1-1 = 0, and this is divisible by 3.
   Assume that n is true
   Then we get:
   $(n + 1)^3 - (n + 1)$
   $= n^3 + 3n^2 + +3n + 1 - n - 1$
   $= (n^3 - n) + 3n^2 + 3n$
   $= 3k + 3n^2 + 3n$

$= 3(k + n^2 + n)$
$\therefore$ this is divisible by three, for all positive integers.

3. (2pts)
   Find a closed form for
   $$a_1 = 2, a_n = a_{n-1} + n + 6$$

   $a_1 = 2$
   $a_2 = a_1 + 2 + 6 = 2 + 2 + 6$
   $a_3 = a_2 + 3 + 6 = 2 + (2 + 3) + 2 * 6$
   $a_4 = a_3 + 4 + 6 = 2 + (2 + 3) + 2 * 6 + 4 + 6 = 2 + (2 + 3 + 4) + 3 * 6$

   This pattern implies that there is a pattern of :
   $a_n = 2 + \frac{(n-1)(2+n)}{2} + 6(n - 1)$
   and therefore this is the closed form.

4. (2pts)
   Find a closed form for the recurrence relation
   $$a_n = a_{n-1} + 2n, \text{ with } a_1 = 2$$
   Prove that your closed form is correct.
   $a_n = a_{n-1} + 2n$
   $= (a_{n-2} + 2(n - 1)) + 2n = a_{n-2} + 2((n - 1) + n)$
   $= (a_{n-3} + 2(n - 2)) + 2((n - 1) + n) = a_{n-3} + 2((n - 2) + (n - 1) + n)$
   $= a_1 + 2(2 + 3 + ... + (n - 2) + (n + 1) + n)$
   $= 2 + 2(2 + 3 + 4.... + (n - 2) + (n + 1) + n)$
   $= 2(1 + 2 + 3 + 4 + ... + (n - 2) + (n - 1) + n)$
   $= 2 * \frac{n(n+1)}{2} = n(n + 1)$
   $= n^2 + n$
   $\therefore$ our closed form is $n^2 + n$

   To proof that this is true, we need to look at the given.
   Let P(n) = $a_n = n^2 + n$
   $n = 1$
   $P(1) = 1^2 + 1 = 2$
   Thus P(1) is true.
   To prove that P(k+1) is true:
   $a_{k+1} = a_{(k+1)-1} + 2(k + 1)$
   $= a_k + 2(k + 1)$
   $= k^2 + k + 2(k + 1)$
   $= k^2 + k + 2k + 2$
   $= (k^2 + 2k + 1) + (k + 1)$
   $= (k + 1)^2 + (k + 1)$

$\therefore P(k+1)$ is true, and therefore P(n) is true for all positive integers.

5. (4pts)

   (a) Find a recurrence relation that defines the sequence 1, 1, 1, 1, 2, 3, 5, 9, 15, 26, ...
   (Hint: each number in the sequence is based on the four numbers just before it in the sequence.)
   This is a recurrence relation that defines this sequence:
   $P(n) = (a_{n-1} + a_{n-2} + a_{n-3}) - a_{n-4}$

   (b) Now find a different sequence that satisfies the recurrence relation you found in (a)
   Another sequence we can do is starting at 3 so:
   $3, 3, 3, 3, 6, 9, 15, 27, ...$

6. (2pts)
   Consider the following recurrence relation:

$$a_n = a_{n-1} + 2n, \text{ with } a_1 = 3$$

   Prove by induction that

$$a_n = n^2 + n + 1 \;\forall\; n \geq 1$$

   Let P(n) $= a_n = n^2 + n + 1$
   $n = 1$
   $a_n = a_1 = 3$
   $\therefore n^2 + n + 1 = 1^2 + 1 + 1 = 1 + 1 + 1 = 3$
   Therefore P(1) is true.
   We can then assume that P(k) is true. To prove this we go:
   $a_k = k^2 + k + 1$
   $a_{k+1} = a_{(k+1)-1} + 2(k+1)$
   $= a_k + 2(k+1)$
   $= k^2 + k + 1 + 2(k+1)$
   $= k^2 + k + 1 + 2k + 2$
   $= (k^2 + 2k + 1) + k + 2$
   $= (k+1)^2 + k + 2$
   $= (k+1)^2 + (k+1) + 1$

   Therefore $a_n = n^2 + n + 1 \;\forall\; n \geq 1$ because of the +1, and so this will always be greater than, or equal to 1 in the case that (k+1) is equal to zero.

7. (2pts)
   Consider the following recurrence relation:

$$a_n = 2 \cdot a_{n-1} - 3 \text{ with } a_1 = 5$$

Prove by induction that

$$(a_n = 2^n + 3 \; \forall \; n \geq 1$$

We will assign this statement as P(n):
$P(n) : a_n = 2^n + 3, n \geq 1$
$P(1) = s^1 + 3 = 5$ Therefore P(1) as our base case is true.

Assume P(k) is true:
$a_k = 2^k + 3$
$a_{k+1} = 2^{k+1} + 3$
$a_{k+1} = 2a_k - 3$
$= 2\left(2^k + 3\right) - 3$
$= 2a_k + 2 * 3 - 3$
$= 2a_k + 3$
$\therefore P(k+1)$ is true for all $n \geq 1$

8. (4pts)

(a) Find a closed-form solution for this recurrence relation:

$$a_n = 2 \cdot a_{n-1} - n + 1 \text{ with } a_1 = 2$$

$a_n = 2a_{n-1} - n + 1$
$a_1 = 2$

$a_1 = 2$
$a_2 = 2a_1 - 2 + 1 = 2(2) - 2 + 1 = 3$
$a_3 = 2a_2 - 3 + 1 = 2(3) - 3 + 1 = 4$
$a_4 = 2a_3 - 4 + 1 = 2(4) - 4 + 1 = 5$
etc...

We can say that $a_n = n + 1$

(b) Prove that your closed-form solution is correct
Let P(n) be $a_n = n + 1$
$n = 1$
$a_n = a_1 = 2$
$n + 1 = 1 + 1 = 2$
Therefore we can sat the P(1) is true.
Assume that P(k) is also true: $a_k = k + 1$
$a_k + 1 = 2a_{(k+1)-1} - (k + 1) + 1$

$= 2a_k - k$
$= 2(k + 1) - k$
$= 2k + 2 - k$
$= k + 2$
$= (k + 1) + 1$
$\therefore P(k + 1)$ is true, for all positive intgers.

9. (5 pts)
Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for all integers $n \geq 18$.

   (a) Show that the statements $P(18), P(19), P(20)$, and $P(21)$ are true, completing the basis step of a proof by strong induction that $P(n)$ is true for all integers $n \geq 18$.

   $P(18)$ is true because 18 cents can befound using two 7-cent stamps and a 4-cent stamp.
   Here we can show $2 * 7 + 4 = 18$

   $P(19)$ is true because we can use a 7-cent stamp and three 4-cent stamps to find 19.
   Here we can show that $7 + 3 * 4 = 19$

   $P(20)$ is true because we can use five 4-cent stamps to find 20.
   Here we can show that $5 * 4 = 20$

   $P(21)$ is true because we can use three 7-cent stamps to find 21.
   Here we can show that $3 * 7 = 21$

   (b) What is the inductive hypothesis of a proof by strong induction that $P(n)$ is true for all integers $n \geq 18$?

   Assume that $P(18), P(19), P(20), ...P(k)$ are true. This means that any postage from 18 to k cents can be formed using 4 and 7 cent stamps.

   (c) What do you need to prove in the inductive step of a proof that P(n) is true for all integers $n \geq 18$?

   To proof that P(n) is true for all integers $n \geq 18$ we need to show that $P(k + 1)$ is true and$k + 1$postage can be formed using 4 and 7 cent stamps.

(d) Complete the inductive step for $k \geq 21$.

Since k+1 = (k-3) +4, P(k+1) must be true because we know that P(k-3) is true through inductive hypothesis. Since this is true, the number of 7 cent stamps are the same for k+1 and k-3, while the 4 cent stamps are one more for k+1.

(e) Explain why these steps show that $P(n)$ is true for all integers $n \geq 18$.

We can determine that P(n) is true for integers $n \geq 18$ through the principle of strong induction. This is because we proved the base case P(k) as P(18), and therefore we can assume that P(k+1) is true, and therefore all integers $n \geq 18$ are true.