# CISC 102 (Fall 20)
# Homework #8: Counting (24 Points)

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1. (2 pts)
   How many ways are there to select a 5 card poker hand from a standard deck of 52 cards, such that none of the cards are clubs?
   To determine this we can take the number of clubs and take them out of the deck. Then we get 39.

   $\therefore$ we have $\binom{39}{5}$ ways to make the selection. To further calculate this :

   $\frac{n!}{k!(n-k)!} = \$\frac{39!}{5!(39-5)!} = 575757$

2. (2pts)
   How many ways are there to select a 5 card poker hand from a standard deck of 52 cards, such that at least one of the cards is a club?
   To determine this, we need to subtract the selections with clubs out of all the cards:

   $\therefore$ we have $\binom{52}{5} - \binom{39}{5}$ ways to make the selection. To further calculate:

   $\frac{n!}{k!(n-k)!} = \frac{52!}{5!(52-5)!} - \frac{39!}{5!(39-5)!} = 2598960 - 575757 = 2023203.$

3. (6 pts)
   You are planning a dinner party and want to choose 5 people to attend from a list of 11 close personal friends.

   (a) In how many ways can you select the 5 people to invite.

   $\therefore$ we have $\binom{11}{5}$ ways to make the selection.

   To further calculate this :

   $\frac{n!}{k!(n-k)!} = \quad \frac{11!}{5!(11-5)!} = 462$

(b) Suppose two of your friends are a couple and will not attend unless the other is invited. How many different ways can you invite 5 people under these constraints? We would have to add the count with the couple and without the couple in the group.

$\therefore$ we have $\binom{9}{3} + \binom{9}{5}$

To further calculate this :

$\frac{n!}{k!(n-k)!} = \frac{9!}{3!(9-3)!} + \frac{9!}{5!(9-5)!} = 84 + 126 = 210$

(c) Suppose two of your friends are enemies, and will not attend unless the other is not invited. How many different ways can you invite 5 people under these constraints? To find this we add the binomial of the group without the enemies, and add it to the enemies( for two enemies).

$\therefore$ we have $\binom{9}{5} + 2\binom{9}{4}$

$\frac{n!}{k!(n-k)!} = \frac{9!}{5!(9-5)!} + 2\frac{9!}{4!(9-4)!} = 126 + 2(126) = 372$

4. (2 pts)
Prove that for $n \geq 2,\ 2 \cdot \binom{n}{2} + \binom{n}{1} = n^2$

To prove this we must expand to:
$2\binom{n}{2} + \binom{n}{1}$

from here we know that $\binom{n}{1} = n$

we can also tell that $\binom{n}{2} = n(n-1)/2.)$

$2 \cdot \binom{n}{2} + \binom{n}{1} =$

$2 \cdot (n(n-1)/2. + n =$

$n(n-1) + n =$

$n^2 - n + n =$

$n^2$

To determine the $n \geq 2$, we can see this because if $\binom{n}{2} = n(n-1)/2)$ we need to have a number 2 or greater, considering the n-1 portion of this equation, and if the number is 1 or less, this will be negative, and therefore impossible.

5. (2 pts)
   Let $S = \{1, 2, 3, ...n\}$ where $n$ is even. What is the relationship between the number of subsets $S$ of with odd cardinality, and the number of subsets of $S$ with even cardinality. Explain your answer.
   Hint: Each subset either contains the element 1 or it doesn't. Consider all the subsets that do not contain the element 1. Some of these have odd cardinality and some have even cardinality. What happens when you include the element 1 in each of these subsets?

   By using the binomial theorem we can say:

   There are $\binom{n}{0}$ subsets with 0 elements, $\binom{n}{1}$ with one elements etc.

   Therefore:

   $\sum_{k=0}^{n} \binom{n}{k}$ can count the total amount of subsets with n elements.

   Therefore we can equate and see that the total amount of elements in set S is $2^n$.
   If $n$ is even then, if we assume there is an empty set then there will be an even number Considering the subset must either have 1 in it or not, we can define the amount of elements as follows:

   There are $2^{n-1}$ subsets with and even number of elements.
   There are a $2^{n-1}$ subsets with an odd number of elements.

6. (2 pts) Lotto 6/49 lets you pay \$3 for the thrill of choosing six different numbers from the set $\{1, 2, 3, \ldots 49\}$. Rumour has it that there may (on extremely rare occasions) be other benefits to buying a lottery ticket.
   How many ways are there to choose the six numbers? Work out the exact value.

   If the order doesn't matters: To get this we need to compute the total number of ways six numbers can be drawn. For this we use the expression ${}_{49}C_6$ or $C(49, 6)$
   Therefore we can rewrite as we have $\binom{49}{1}$ ways to make the selection. To further calculate this :

   $\frac{n!}{k!(n-k)!} = \$\frac{49!}{6!(49-6)!} = 13,983,816$ ways to choose these six numbers.

   If the order matters: To get this we need to compute the total number of ways six numbers can be drawn. For this we use the expression ${}_{49}P_6$ or $P(49, 6)$
   Therefore we can rewrite as we have $\binom{49}{1}$ ways to make the selection. To further calculate this :

$\frac{n!}{k!(n-k)!} = \$\frac{49!}{(49-6)!} = 10068347520$ ways to choose these six numbers.

7. (2 pts)
   Prove that
   $$\sum_{i=1}^{n} \binom{i}{2} = \binom{n+1}{3} \quad \forall\, n \geq 2$$
   We can rewrite this as:
   $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} ... + \binom{n}{2}$

   To consider $i = 0$ we can rewrite this as:

   $\binom{n+i}{2+i} = \binom{n+1}{2+1} = \binom{n+1}{3}$

   $\therefore \binom{n+1}{3} = \sum_{i=1}^{n} \binom{i}{2}$

8. (2 pts)
   What is the number of ways to colour $n$ different objects, one colour per object with 2 colours? What is the number of ways to colour $n$ different objects with 2 colours, so that each colour is used at least once.
   To find the amount of ways to colour $n$ objects using 2 colours, we can use the format $2^n$. Because we know that their must be 2 ways where one colour is used, we subtract 2.
   $\therefore 2^n - 2$ is the amount of ways to do this.

9. (2pts)
   Prove that
   $$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$$
   Note that this equation can also be written as follows:
   $$\sum_{i=0}^{n} \binom{n}{i} (-1^i) = 0$$
   HINT: This can be viewed as a special case of the binomial theorem.

   We can rewrite like this:
   $\sum_{i=0}^{n} \binom{n}{i} (-1^i)$

   $\sum_{i=0}^{n} \binom{n}{i} (-1)^i 1^{n-i}$

We know that:
$\binom{n}{0} = 1$ and $(-1)^0 = 1$

$\therefore = ((-1) + 1)^n$
$= 0^n$
$= 0$

10. (2 pts)
A class of 100 students must select 2 safety officers per week for 7 weeks. There are no constraints on how the selection is made, and the same student can be selected more that once. 950n how many ways can this selection be made?

We have $\binom{100}{2} = \frac{100x99}{2x1} = \frac{9900}{2} = 4950$ for one week, however for seven weeks:

$\binom{100}{2}^7 = \frac{100x99}{2x1}^7 = \frac{9900}{2}^7 = 4950^7$ is the probability for seven weeks, assuming that

students can be selected more than once.