CISC 203 Problem Set 2

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1. Prove that $3|(2n^2 + 4)$ if and only if $3 \not| n$, for all $n \in \mathbb{Z}$.

   This proof will be conducted with mathematical induction.

   First, base cases are proven:

   Show that $P(1) = T$
   $(2n^2 + 4) = 2(1)^2 + 4$
   $7 = 3|1$

   Therefore since 7 is not divisible by 3, and 3 is divisible by 1,
   P(1) is true.

   Show that $P(2) = T$
   $(2n^2 + 4) = 2(2)^2 + 4$
   $12 = 3 \not| 2$

   Therefore since 12 is divisible by 3, and 3 is not divisible by 2,
   P(2) is true.

   Show that $P(4) = T$
   $(2n^2 + 4) = 2(4)^2 + 4$
   $36 = 3 \not| 4$

   Therefore since 36 is divisible by 3, and 3 is not divisible by 4,
   P(2) is true.

   As we can see in all three base cases, when 3 is not divisible by n, to make an integer, the expression $(2n^2 + 4)$ is divisible by 3.

   Hypothesis: $P(k)$ is true when $3 \not| k$

   Induction: Show that $P(k + 1)$ is true when $3 \not| (k + 1)$

   Therefore $3|2(k + 1)^2 + 4 = \mathbb{Z}$

   $2((k + 1)(k + 1)) + 4 = 2(k^2 + 2k + 1) + 4$

   $2k^2 + 4k + 2 + 4 = 2(k + 1)^2 + 4$

$2(k+1)^2 + 4$
$k+1 \not| 3 \implies = \frac{2k^2+4k+6}{3} \neq \mathbb{Z}$

$\therefore \implies 3|(2n^2+4)$ if and only if $3 \not| n$, for all $n \in \mathbb{Z}$

Therefore this statment is proven.

2. Does there exist at least one pair of odd integers a and b such that $3a^2 + 7b^2 \equiv 0 \ mod\ 4$? Make a conjecture. Then, use an appropriate technique covered in class to prove your conjecture.

   Because 0 mod any number is 0, we need to find if $3a^2 + 7b^2$ could be equivalent to this.

   Because the two values of $3a^2$ and $7b^2$ need to add together to be zero, the values must be a negative, plus a positive; or a postive plus a negative. However, when examining this statement, we can see how this is impossible, given that any number squared is a positive. Therefore any number that is put in the place of a or b, no matter the sign will not add to 0.

   Lets make a conjecture, with examining some examples:

   $3(-5)^2 + 7(3)^2 = 75 + 63 \not\equiv 0$
   $3(-3)^2 + 7(-1)^2 = 27 + 7 \not\equiv 0$
   $3(-1)^2 + 7(1)^2 = 3 + 7 \not\equiv 0$
   $3(1)^2 + 7(3)^2 = 3 + 27$
   etc...

   Except:
   $3(0)^2 + 7(0)^2 = 0 + 0 \equiv 0 \ mod\ 4$
   However, this is not an odd integer, so it does not count.

   Conjecture: Given the above statement, there is no pair of odd integers a and b such that $3a^2 + 7b^2 \equiv 0 \ mod\ 4$. This will always be the case as any number squared is positive, and any postive values will add up to greater than $0 \ mod\ 4$.

   Therefore we can look at this like a direct if-then(conditional) proof:

   $3a^2 + 7b^2 = 0$
   $3a^2 = -7b^2$

   $a^2 = \frac{7b^2}{3}$

   $a = \frac{\sqrt{7b^2}}{3}$

   $a = \frac{\sqrt{7} \cdot b}{3}$

   $a = \frac{\sqrt{7}}{3} \cdot \frac{b}{3}$

   $3a = \sqrt{7} \cdot b$

   Because any number times $\sqrt{7}$ will equal to a non integer, the value of a and b can not exist as integers. Therefore we can say, that if a and b are integers, then there is no such way that $3a^2 + 7b^2 \equiv 0 \ mod\ 4$.

   Therefore it is proven through direct proofs that there is no two odd integers a and b such that $3a^2 + 7b^2 \equiv 0 \ mod\ 4$. In other words if $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ then, $3a^2 + 7b^2 \not\equiv 0 \ mod\ 4$.

3. Let $n \in \mathbb{N}$ and $n \geq 1$. Give proof by smallest counterexample that $3|(2^n + 2^{n-1})$.

   To complete a proof by the smallest counterexample, we must first claim the statement is false. Therefore:

   $3 \not| (2^n + 2^{n-1})$ for $n \geq 1$

   Next we need to show that this holds true for the smallest example. Since 1 is the smallest Natural number, and n must be an element of the set of Natural numbers, we plug in 1:

   $3|(2^1 + 2^{1-1})$
   $3|(2+1) = 3|3 = 1$

   This proves the basis step true.

   Next lets show that this is true for (n-1):

   $3|(2^{1-1} + 2^{1-1-1})$

   $3|(1 + .5) = 3|1.5 = 0.5 \notin \mathbb{N}$

   Because this is not in the natural numbers, and also not divisible by 3, the statement still holds true.

   Next we need to find a contradiction to the statement being false:
   For this, we can use any natural number, greater than our base case. Given we are required to use the smallest counter example, lets chose n+1.

   $3|(2^{1+1} + 2^{1+1-1})$

   $3|(4 + 2) = 3|6 = 2 \in \mathbb{N}$

   Given that this value is divisible by 3, and a natural number, this statement is proven true, therefore disproving our assertion that $3|(2^n + 2^{n-1})$ is false.

   Therefore proof is completed by smallest counterexample.

4. Sequence a_n is defined by:

   $a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n - 3)$

   For $n \geq 4$ and with initial terms $a_1 = 1, a_2 = 4$, and $a_3 = 9$

   (a) Calculate a_4, a_5, a_6, and a_7 by hand.

   $a_4 = a_{4-1} - a_{4_2} + a_{4-3} + 2(2(4) - 3)$
   $a_4 = 9 - 4 + 1 + 10$
   $a_4 = 16$

   $a_5 = a_{5-1} - a_{5_2} + a_{5-3} + 2(2(5) - 3)$
   $a_5 = 16 - 9 + 4 + 14$
   $a_5 = 25$

$a_6 = a_{6-1} - a_{6_2} + a_{6-3} + 2(2(6) - 3)$
$a_6 = 25 - 16 + 9 + 18$
$a_6 = 36$

$a_7 = a_{7-1} - a_{7_2} + a_{7-3} + 2(2(7) - 3)$
$a_7 = 36 - 25 + 16 + 22$
$a_7 = 49$

(b) Using a script or spreadsheet, calculate 100 terms and plot the values on an xy-plane.

I created a recursive python script to solve this, and placed the values on an xy- plane. The full script can be seen in part (d) of this question. Here is the xy-plane:

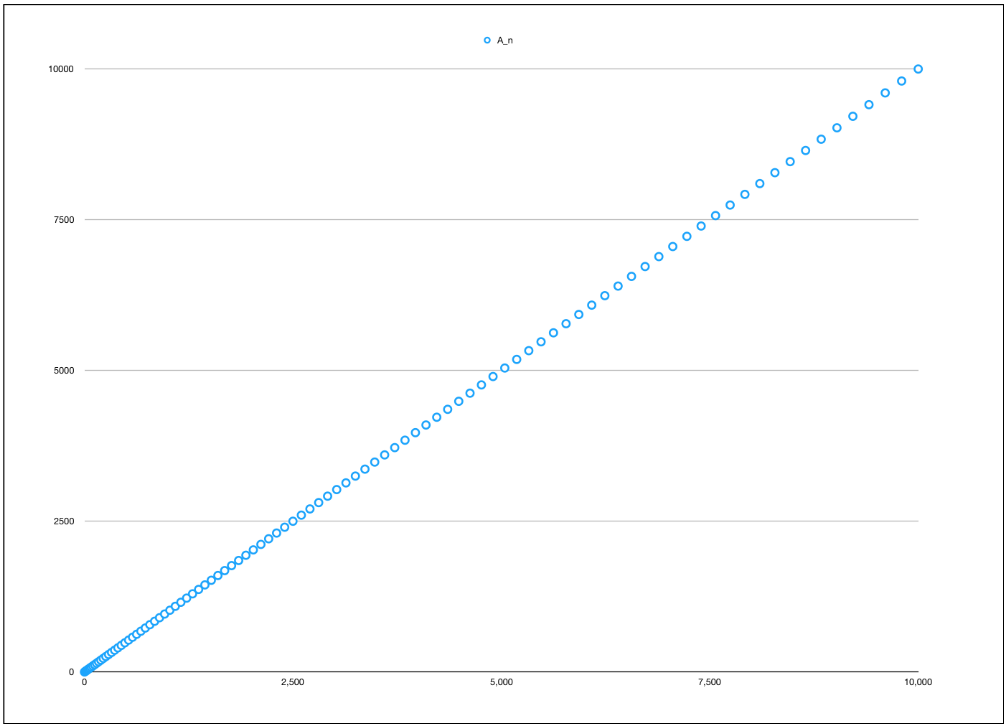

Figure 1: xy-plane of the first 100 terms of a$_n$

(c) My closed form conjecture is as follows:

$a_n = n^2$

I arrived at this conjecture after calculating the recursion, though python in part b. I put the values of n and a_n next to each other as the xy values, and noticed that in all cases this expression was true.
Addtionally, when looking at the xy-plane, the linear progression is familiar, and is in fact the same as a $f(x) = x^2$, which leads to the logical conclusion that a must have a relation to n^2.

(d) Given that we have all the cases of the original sequence for any value n up to 100, let's find a base case for our new closed sequence.

$a_n = n^2$

$a_{n-1} - a_{n-2} + a_{n-3} + 2(2n-3) = n^2$

Given the calculated values in part b of the question:

$a_{4-1} - a_{4-2} + a_{4-3} + 2(2(4)-3) = 4^2$
$9 - 4 + 1 + 10 = 4 \cdot 4$
$16 = 16$
Therefore this is true for the base case a_4

$a_{5-1} - a_{5-2} + a_{5-3} + 2(2(5)-3) = 5^2$
$16 - 9 + 4 + 14 = 5 \cdot 5$
$25 = 25$
Therefore this is true for the base case a_5

$a_{6-1} - a_{6-2} + a_{6-3} + 2(2(6)-3) = 6^2$
$25 - 16 + 9 + 18 = 6 \cdot 6$
$36 = 36$
Therefore this is true for the base case a_6

$a_{7-1} - a_{7-2} + a_{7-3} + 2(2(7)-3) = 7^2$
$36 - 25 + 16 + 22 = 7 \cdot 7$
$49 = 49$
Therefore this is true for the base case a_7

To continue this, we must prove this for a_{k+1} :

$a_{n-1} - a_{n-2} + a_{n-3} + 2(2n-3) = n^2$

$a_{(k+1)-1} - a_{(k+1)-2} + a_{(k+1)-3} + 2(2(k+1)-3) = (k+1)^2$

$a_{(k+1)-1} - a_{(k+1)-2} + a_{(k+1)-3} + 2(2(k+1)-3) = (k+1)^2$

$a_k - a_{k-1} + a + k - 2 + 2(2(k+1)-3) = n^2 - (n-1)^2 + (n-2)^2 + 2(2(k+1)-3)$

$n^2 - n^2 + 2n + 1 + n^2 - 4n + 4 + 2(2(k+1)-3) = 2n + 1 + n^2 + 4 + 2(2(k+1)-3)$

$(n+1)^2 + 4 + 2(2(k+1)-3) = (k+1)^2$

$(n+1)^2 = (k+1)^2 - 4 - 2(2(k+1)-3)$

$(n+1)^2 = k^2 + 2k + 1 - 4 - 4k - 2 + 3$

$(n+1)^2 = k^2 - 2k - 2$

$\implies a_n = n^2$ For all integers, this is true. This can be seen in my modeling and experimentation. Proving $a_n = n^2$ is easy, when looking at the outcomes from the experiments from part b of this question.

Here are the first 100 steps of the sequence, according to my python program (Please note that this output has been reformatted to better fit this page. Therefore read any value in a grey column as the value of n, and any value in the corresponding white column as the value of a_n):

n	A_n
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
22	484
23	529
24	576
25	625
26	676
27	729
28	784
29	841
30	900
31	961
32	1024
33	1089
34	1156
35	1225
36	1296
37	1369
38	1444
39	1521
40	1600
41	1681
42	1764
43	1849
44	1936
45	2025
46	2116
47	2209
48	2304
49	2401
50	2500
51	2601
52	2704
53	2809
54	2916
55	3025
56	3136
57	3249
58	3364
59	3481
60	3600
61	3721
62	3844
63	3969
64	4096
65	4225
66	4356
67	4489
68	4624
69	4761
70	4900
71	5041
72	5184
73	5329
74	5476
75	5625
75	5625
76	5776
77	5929
78	6084
79	6241
80	6400
81	6561
82	6724
83	6889
84	7056
85	7225
86	7396
87	7569
88	7744
89	7921
90	8100
91	8281
92	8464
93	8649
94	8836
95	9025
96	9216
97	9409
98	9604
99	9801
100	10000

This was the python program that caused this output. Here we can see that this is the output when considering the original sequence recursively, and taking into account the predetermined base cases:

```python
# Defining a recursive function to determine the first 100  terms of
# the sequence.
def recur(n):

    # Base Case 1
    if n == 1:
        return 1

    # Base Case 2
    if n == 2:
        return 4

    # Base Case 3
    if n == 3:
        return 9

    # Defining if the input is less than 1, therefore error catching
    if n <= 1:
        return n

    # Recursive function to find the remaining terms
    else:
        return(recur(n-1) - recur(n-2) + recur(n-3) + 2*((2*n)-3))

# Number of terms to count to
nterms = 100

# If the number of terms is less than or equal to one, print message.
if nterms <= 0:
    print("There are no more terms.")

# For each value in the nterms, the recursive fuction goes through, and
# returns a value.
else:
    print("Sequence:")
    for i in range(nterms):
        print(recur(i));
```

As proven by mathematical induction, programming, graphing and experimentation, the closed form $a_n = n^2$ is correct, as it causes equivalent outcomes to the original sequence $a_n = a_{n-1} - a_{n-2} + a_{a-3} + 2(2n-3)$.

5. Your friend shows you a strange proof written by an unknown "Expert" that they found online. The proof claims that $a^n = 1$ for all nonzero $a \in \mathbb{R}$ and all natural numbers $n \geq 0$. Their proof uses strong induction, as follows:
Base case: $a^0 = 1$, because any number to the power of zero is 1 by definition.
Inductive case: Assume that $a^j = 1$ for all natural numbers $0 \leq j \leq k$. Then

   $a^{k+1} = \frac{a^k \cdot a^k}{a^{k1}} = \frac{1 \cdot 1}{1} = 1$

   However, since you are a sophisticated CISC 203 student, you immediately recognize that this proof is incorrect. What is the flaw in the proof? Identify the problem and explain what the "Expert" did wrong.

   In this case the expert attempts to use strong induction. This should mean that the expert should solve the basis step. This would require the expert to solve the sequence for a^k for a finite amount of steps. Assuming $a^0 = 1$ is not good enough to form a strong induction. Only after forming a solid basis, can the inductive case be introduced. Because the basis step does not take into account that natural numbers do not include 0, a^0 is not a good enough base case. This means the entire proof can not be based on a sound inductive case, and this is why the proof does not work.