CISC 102 (Fall 20)
Homework #3: Proofs (24 Points)

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Solutions are due before 11:59 PM on **Friday Midnight October 10, 2020** .

1. (2 pts) Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?
   If George does not have eight legs, then he is not a spider.
   George is a spider.

   ∴ George has eight legs.

   p = George has eight legs
   q= George is a spider

   $\neg p \rightarrow \neg q$
   q

   $\therefore p$

   This argument form is Modus Tollens, and it is valid. This is true if all premises are true.

2. (2 pts) What rules of inference are used in this famous argument? "All men are mortal. Socrates is a man. Therefore, Socrates is mortal."

   $A(x)$ =x are mortal
   $B(x)$ = x is a man

   $\forall A(men) \rightarrow B(socrates)$

   $\therefore A(socrates)$

   The rules of inference used in this famous argument is Universal Instantiation and Modus Ponens.

3. (2 pts) Use rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go

on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."

p= It rained
q = it is foggy
r = the sailing race will be held
s = lifesaving demonstration will be held
t = the trophy will be awarded

$[\neg p \vee \neg q] \rightarrow [r \wedge s]$
$r \rightarrow t$

$\neg t$

Taking $r \rightarrow t$ and $\neg t$ into the inference of Modus Tollens:
$\neg r \rightarrow \neg t$
$\neg t$

$\therefore \neg r$

Using De Morgan's Law we can see:
$[\neg p \vee \neg q] \rightarrow [r \wedge s] =$
$\neg[r \wedge s] \rightarrow \neg[\neg p \vee \neg q] =$
$[\neg r \vee \neg s] \rightarrow [p \wedge q]$

Addition:
$\neg r$

$\therefore [\neg r \vee \neg s]$

Modus Ponens:
$[\neg r \vee \neg s] \rightarrow [p \wedge q]$
$[\neg r \vee \neg s]$

$\therefore p \wedge q$

Simplification:
$p \wedge q$

$\therefore p$

4. (2 pts) Prove, or find a counterexample: the sum of the squares of two consecutive positive integers is odd.
$n^2 + (n+1)^2$
$2n^2 + 2n + 1$

$2(n^2+n)+1$
This is proof that $n^2+(n+1)^2$ is always odd because if n is an even number, it will be added to one which will make it odd. If n is odd, it will be multiplied by two, making it even and then added to one. The multiplication of any number makes it even, and the addition of one makes that number odd.
For example:
$n=1$
$2(n^2+n)+1$
$2(1^2+1)+1=5$

$n=2$
$2(2^2+1)+1=11$

5. (2 pts) Prove that if n is any integer then n^3+2n^2+n+4 is even
Hint: do two cases: one for when n is even, and one for when n is odd.
n^3+2n^2+n+4
$n(n^2+2n+1)+4$

For any even or odd number the out come will also be even because any number times itself will be even, plus 4 will still be even. For example:
Odd: $3(3^2+6+1)+4=48+4=52$
Even: $2(2^2+4+1)+4=18+4=22$

6. (4 pts) Prove by contradiction the following. For all rational number x and irrational number y, the sum of x and y is irrational.

Assume that $x \in \mathbb{Q}'$
and $y \in \mathbb{Q}$.
Our proposition is then that $x+y \in \mathbb{Q}'$

$y=\frac{a}{b}$
$\frac{a}{b}+x=\frac{c}{d}$ rewritten as $x=\frac{c}{d}-\frac{a}{b}$
$x=\frac{bc-ad}{bd}$

$x \in \mathbb{Q}$ an is rational number, as it can be written as a faction, contradicting our proposition that $x \in \mathbb{Q}'$.
$\therefore$ if $x \in \mathbb{Q}$ and $y \in \mathbb{Q}'$, $(x+y) \in \mathbb{Q}'$

7. (2 pts) Prove the proposition $P(1)$, where $P(n)$ is the proposition "If n is a positive integer, then $n^2 \geq n$." What kind of proof did you use?

$n \in \mathbb{Z}^+, n^2 \geq n$ is true. We can see that no matter what the number chosen as n, because $n \in \mathbb{Z}^+$, and therefore will always be less or equal to itself squared. For example:
$P(1)$
$n^2 = 1^2 = 1$
$\therefore n^2 \geq n = 1^2 \geq 1 = 1 \geq 1$ is true.
This is a trivial proof.

8. (2 pts) Prove, by contradiction, that at least three of any 25 days chosen must fall in the same month of the year.
$p =$ "at least three of any 25 days chosen must fall in the same month of the year"
Assume: $\neg p$ is true. This would mean that at most two of any 25 days chosen must fall in the same month.
There are 12 months in the year. This means that at most 24 days will have been chosen, as 12 x 2 days each = 24.
This is not possible because there are 25 days, and so more than 2 days per month.
$r =$ "24 days chosen"
Because 12 months x 2 days chosen each = 24, we can assume the following.
$\neg p \to (r \wedge \neg r)$
From this we can see that p must be true.

9. (3 pts) Use proof by contraposition to show that these statements about the integer x are equivalent: (i) $3x + 2$ is even, (ii) $x + 5$ is odd, (iii) x^2 is even.

$3x + 2$
Because we can see that x must be even, we will write x as 2y, therefore $3x + 2 = 2(2y + 1) + 2 = 6y + 3 + 2 = 6y + 5 = 2(3y + 2) + 1$
$3y + 2 = k, \therefore 3x + 2 = 2k + 1$
$3x + 2 = 3(2y) + 2 = 2(3y + 1)$
If integer $3y + 1 = k$ then $3x + 2 = 2k$
Meaning that $2k$ must be even.

Assume $x + 5$ is even. Therefore x must be odd and can be written as $2y + 1$
$x + 5 = 2y + 1 + 5 = 2y + 6 = 2(y + 3)$
$k = y + 3$ and therefore $x + 5 = 2k$
$x + 5 = 2y + 5 = 2(y + 2) + 1$
$k = y + 2$ if $x + 5 = 2k + 1$.
$\therefore x + 5$ is odd.

Assume x^2 is odd.
$x^2 = (2y + 1)^2 = 4y^2 + 4y + 1 = 2(2y^2 + 2y) + 1$
$k = 2y^2 + 2y$ so $x^2 = 2k + 1$
$x^2 = 4y^2 = 2(2y^2)$

$k = 2y^2$ such that $x^2 = 2k$
Therefore x^2 must be even.

This is proof that$(i) \equiv (ii) \equiv (iii)$

10. (3 pts) Show that the propositions $p_1, p_2, p_3,$ and p_4 can be shown to be equivalent by showing that $p_1 \leftrightarrow p_4, p_2 \leftrightarrow p_3,$ and $p_1 \leftrightarrow p_3$.

    Equivalencies: $p \iff q \equiv (p \to q) \wedge (q \to p)$
    1.$p_1 \iff p_4 \equiv (p_1 \to p_4) \wedge (p_4 \to p_1)$
    2.$p_2 \iff p_3 \equiv (p_2 \to p_3) \wedge (p_3 \to p_2)$
    3.$p_1 \iff p_3 \equiv (p_1 \to p_3) \wedge (p_3 \to p_1)$

    Then simplification we can see:
    1.$(p_1 \to p_4) \wedge (p_4 \to p_1) \therefore (p_1 \to p_4)$
    1.$(p_1 \to p_4) \wedge (p_4 \to p_1) \therefore (p_4 \to p_1)$
    2.$(p_2 \to p_3) \wedge (p_3 \to p_2) \therefore (p_2 \to p_3)$
    2.$(p_2 \to p_3) \wedge (p_3 \to p_2) \therefore (p_3 \to p_2)$
    3.$(p_1 \to p_3) \wedge (p_3 \to p_1) \therefore (p_1 \to p_3)$
    3.$(p_1 \to p_3) \wedge (p_3 \to p_1) \therefore (p_3 \to p_1)$

    Using hypothetical syllogism:
    $(p_4 \to p_1)$ and $(p_1 \to p_3) \therefore p_4 \to p_3$
    then $(p_3 \to p_2)$ and $(p_4 \to p_3) \therefore p_4 \to p_2$
    then $(p_2 \to p_3)$ and $(p_3 \to p_1) \therefore p_2 \to p_1$
    then $(p_3 \to p_1)$ and $(p_1 \to p_4) \therefore p_3 \to p_4$
    then $(p_2 \to p_3)$ and $(p_3 \to p_4) \therefore p_2 \to p_4$
    then $(p_3 \to p_2)$ and $(p_1 \to p_3) \therefore p_1 \to p_2$
    then
    Using conjunction:
    $(p_1 \to p_2) \wedge (p_2 \to p_1)$
    $(p_3 \to p_4) \wedge (p_4 \to p_3)$
    $(p_2 \to p_4) \wedge (p_4 \to p_2)$

    Logically we can see then that $p \iff q \equiv (p \to q) \wedge (q \to p)$ means:
    $(p_1 \to p_2) \wedge (p_2 \to p_1) \equiv p_1 \iff p_2$
    $(p_3 \to p_4) \wedge (p_4 \to p_3) \equiv p_3 \iff p_4$
    $(p_2 \to p_4) \wedge (p_4 \to p_2) \equiv p_2 \iff p_4$

    Therefore we can see that $p_1, p_2, p_3,$ and p_4 are equivalent.