CISC 203 Problem Set 1

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1. (a) Prove that$A \times (B) = (A \times B) \cap (A \times C)$

   Suppose : $(x, y) \in A \times (B \cap C)$

   $x \in A, y \in B \cap C = x \in A, y \in (B \cap C)$
   $x \in A, y \in B, y \in C$
   $\implies \{x \in A, y \in B\}$ and $\{x \in A, y \in C\}$
   $(x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$
   $(x, y) \in (A \times B) \cap (A \times C)$
   $(x, y) \in A \times (B \cap C) \implies (x, y) \in (A \times B) \cap (A \times C)$

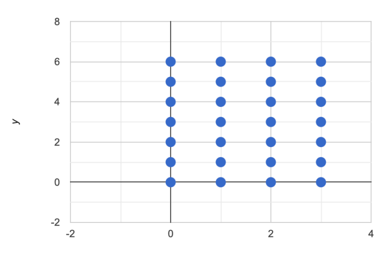
   Therefore : $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

   (b) For : $A = \{a \in \mathbb{R} : |a - 1| \leq 2\}$
   This can be described as the following:
   All numbers in the set A are Real Numbers, and the size of each element minus 1, is less than or equal to 2.
   This can be summarized as $A = \{(|a - 1|), ..., (2)\}$
   And re-written as A $= \{a \in \mathbb{R} : a \leq 3\}$

   For$B = \{b \in \mathbb{R} : |b - 4| \leq 2\}$
   This can be described as the following:
   All numbers in the set B are Real Numbers, and the size of each element minus 4, is less than or equal to 2.
   This can be summarized as $B = \{(|b - 4|), ...(2)\}$
   And re-written as $B = \{b \in R : b \leq 6\}$

   $A = 3, 2, 1, 0$
   $B = \{6, 5, 4, 3, 2, 1, 0\}$

   $\therefore$A$\times B = \{(3, 6), (3, 5), (3, 4), (3, 3), (3, 2), (3, 1), (3, 0), (2, 6), (2, 5), (2, 4), (2, 3), (2, 2), (2, 1), (2, 0), (1, 6), (1, 5), (1, 4), (1, 3), (1, 2), (1, 1), (0, 6), (0, 5), (0, 4), (0, 3), (0, 2), (0, 1), (0, 0)\}$



(c) Let $n \in \mathbb{N}$ and $n > 0$, and let the set $A_n = \{x \in \mathbb{R} : -\frac{1}{n} \leq x \leq \frac{1}{n}\}$

(i): $\cup_{n=1}^{\infty} A_n$

Let $x \in \cup_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$

Then $x \in (-\frac{1}{n}, \frac{1}{n})$ for some $n > 0$

$n > 0 \implies \frac{1}{n} < 0, \frac{-1}{n} < x < 0$

$\therefore x \in (\frac{-1}{n}, 0)$

Since all $n \in \mathbb{R}$ and n=1

$\therefore \cup_{n=1}^{\infty} A_n = (1, 0)$

(ii): $\cap_{n=1}^{\infty} A_n$

To start, we can determine that the empty set is a subset

Therefore: $\emptyset \subset \cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$

To find the rest of the sets:
$= A_1 \cap A_2 \cap ....$

Because A_n will equal different outcomes on each n value, the only intersection is the empty set.

$\therefore \cap_{n=1}^{\infty} A_n = \emptyset$

2. (a) Let $A = \{a, b, c, d, e, f, g, h\}$ and let R be an equivalence relation on A such that aRc, cRd, dRg, and bRh.

To determine the equivalence classes, we need to look at the related:

$aRc, cRd,$ and dRg are related so we can sort into the first equivalence class:
$[a] = \{a, c, d, g\}$

bRh can be the second equivalence class:
$[b] = \{b, h\}$

Now is we take these elements out of A, we are left with two equivalence classes:
$[e] = \{e\}$ and $[f] = \{f\}$

Therefore the four equivalence classes are $[a][b][e][f]$ as follows:
$[a] = \{a, c, d, g\}$
$[b] = \{b, h\}$
$[e] = \{e\}$
$[f] = \{f\}$

Therefore the elements of R are:

$R = \{(a, a,), (a, c), (a, d), (a, g), (c, c), (c, a), (c, d), (c, g), (d, d), (d, a), (d, c), (d, g), (g, g), (g, a), (g, c), (g, d), (b, b), (b, h), (h, b), (h, h), (e, e), (f, f)\}$

(b) Let $B = \{2^n : n \in \mathbb{Z}\}$ and R is a positive real number, such that xRy if $\frac{x}{y} \in B$.

To determine if R is reflexive, we must determine if each element is related to itself.

$\forall x \in B, xRx$

The existence of $\frac{x}{y}$ implies that this that xy exists, and therefore so does yx. This relation is reflexive.

To determine if R is symmetric, we must determine id each element is related to each other in the form x,y $\rightarrow y, x$

$\forall x, y \in B, xRy, yRx$

The existence of $\frac{x}{y}$ implies that the expression is not symmetric, because x cannot map on $\frac{x}{y}$ in the same way that y does.
Therefore this is not symmetric.

To determine if R is transitive, we must determine if each element can be mapped onto one, the next and then each other. In other words:

$\forall x, y, z \in B, xRy, yRz, xRz$

The existence of the fraction implies that this is not transitive because:
$\frac{x}{y} \implies y, y \implies z$ is true.

However $z \implies \frac{x}{y}$ is not true.

Therefore this is not transitive.

Because this equation is only reflexive, but not symmetric and not transitive, it is not an equivalence relation.

3. (a) To determine these questions, first we can state what is known:

Give DOTA2 the set A, CS:GO the set B and TF2 the set C

$|A| = 46$
$|B| = 52$
$|C| = 50$
$|A \cap B| = 25$
$|A \cap C| = 21$
$|B \cap C| = 23$
$|A \cap B \cap C| = 16$

(i) To determine the amount of students who play at least one of the games, we will include the total amount of game + game + game. Then exclude the double counted values, where the games intersect. Then we will include the values that intersect all of the games, as they would be excluded at this point. This gives us this exclusion-inclusion formula:

$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |C \cap B| + |A \cap B \cap C|$

$46 + 52 + 50 - 25 - 21 - 23 + 16$

$\therefore$ The amount of students who play at least one game is :95

(ii) To determine the amount of students who only play one game, we will include the games separately, then exclude each games intersect with other games, and include the over-excluded values.

DOTA2 : $|A| - (|A \cap B| + |A \cap C|) + |A \cap B \cap C|$

$= 46 - (25 + 21) + 16$

$= 16$

CS:GO : $|B| - (|A \cap B| + |B \cap C|) + |A \cap B \cap C|$

$= 52 - (25 + 23) + 16$

$= 20$

TF2 : $|C| - (|C \cap B| + |A \cap C|) + |A \cap B \cap C|$

$= 50 - (23 + 21) + 16$

$= 22$

$16 + 20 + 22 = 58$

$\therefore$ The players of only one game : 58

(iii) To determine how many students play none of the games, we will include the total students. Then we will exclude the game + game + game value. Next we will include the intersections, and finally exclude the over included values.

$= 120 - (|A| + |B| + |C|) + (|A \cap B| + |A \cap C| + |B \cap C|) - |A \cap B \cap C|$

$= 120 - (46 + 52 + 50) + (25 + 21 + 23) - 16$

$= 120 - 148 + 69 - 16$

$= 25$

$\therefore$ The students that play none of the games :25

(iv) To determine how many students play only DOTA2, but not CS:GO or TF2, we will include the total amount of DOTA2 players, then exclude those who play DOTA2 and CS:GO, and those who play DOTA2 and TF2. Finally we will include the over-excluded intersection of all three games.

DOTA2 : $|A| - (|A \cap B| + |A \cap C|) + |A \cap B \cap C|$

$= 46 - (25 + 21) + 16$

$= 16$

$\therefore$ The amount of students that only plat DOTA2 $= 16$

(v) To determine the players of DOTA2 and CS:GO, who do not play TF2, first we must include all players of DOTA2 and CS:GO. We can do this by adding the total players of CS:GO to the students who only play DOTA2, as determined in section (iv). Then we will exclude the intersection of CS:GO and TF2.

$16 + |B| - |B \cap C|$

$= 16 + 52 - 23$

$= 45$

$\therefore$ The amount of student who play DOTA2 and CS:GO, but TF2 $= 45$

(b) (i) The number of elements in the set $A = 1, 2, ...9999$ that are divisible by at least one of 2,3,5, and 7 can be found using inclusion-exclusion.
Basically the question is asking us to determine:

$|A \cup B \cup C \cup D|$

Include the numbers divisible by $2 = |A| = [\frac{9999}{2}] = 4999$

Include the numbers divisible by $3 = |B| = [\frac{9999}{3}] = 3333$

Include the numbers divisible by $5 = |C| = [\frac{9999}{5}] = 1999$

Include the numbers divisible by $7 = |D| = [\frac{9999}{7}] = 1428$

Exclude the intersect of the numbers divisible by 2 and $3 = |A \cap |B = [\frac{9999}{2 \cdot 3} = 1666$

Exclude the intersect of the numbers divisible by 2 and $5 = |A \cap C| = [\frac{9999}{2 \cdot 5}] = 999$

Exclude the intersect of the numbers divisible by 2 and $7 = |A \cap D| = [\frac{9999}{2 \cdot 7}] = 714$

Exclude the intersect of the numbers divisible by 3 and $5 = |B \cap C| = [\frac{9999}{3 \cdot 5}] = 666$

Exclude the intersect of the numbers divisible by 3 and $7 = |B \cap D| = [\frac{9999}{3 \cdot 7}] = 476$

Exclude the intersect of the numbers divisible by 5 and $7 = |C \cap D| = [\frac{9999}{5 \cdot 7}] = 285$

Include the intersect of the numbers divisible by 2,3,5 $= |A \cap B \cap C| = [\frac{9999}{2 \cdot 3 \cdot 5}] = 333$

Include the intersect of the numbers divisible 2,5,7 $= |A \cap B \cap D| = [\frac{9999}{2 \cdot 3 \cdot 7}] = 238$

Include the intersect of the numbers divisible by 3,5,7 $= |B \cap C \cap D| = [\frac{9999}{3 \cdot 5 \cdot 7}] = 95$

Exclude the intersect of the numbers divisible by all 2,3,5,7 $= |A \cap B \cap C \cap D| = [\frac{9999}{2 \cdot 3 \cdot 5 \cdot 7}] = 47$

$(A \cup B \cup C \cup D) = (|A| + |B| + |C| + |D|) - (|A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|) + (|A \cap B \cap C| + |A \cap B \cap D| + |B \cap C \cap D|) - (|A \cap B \cap C \cap D|)$

$= (4999 + 3333 + 1999 + 1428) - (1666 + 999 + 714 + 666 + 476 + 285) + (333 + 238 + 95) - 47$

$= 11759 - 4806 + 666 - 47$

$= 7572$

Therefore, using the inclusion-exclusion principle, we can see that 7572 elements are divisible by at least one of 2,3,5,7.

(ii) The number of elements in the set $A = 1, 2, ...9999$ that are not divisible by any of 2,3,5, and 7 can be found using inclusion-exclusion. This question is basically asking:

$9999 - |A \cup B \cup C \cup D|$

Exclude the numbers divisible by 2 $= |A| = [\frac{9999}{2}] = 4999$

Exclude the numbers divisible by 3 $= |B| = [\frac{9999}{3}] = 3333$

Exclude the numbers divisible by 5 $= |C| = [\frac{9999}{5}] = 1999$

Exclude the numbers divisible by 7 $= |D| = [\frac{9999}{7}] = 1428$

Include the intersect of the numbers divisible by 2 and 3 $= |A \cap |B = [\frac{9999}{2 \cdot 3} = 1666$

Include the intersect of the numbers divisible by 2 and 5 $= |A \cap C| = [\frac{9999}{2 \cdot 5}] = 999$

Include the intersect of the numbers divisible by 2 and 7 $= |A \cap D| = [\frac{9999}{2 \cdot 7}] = 714$

Include the intersect of the numbers divisible by 3 and 5 $= |B \cap C| = [\frac{9999}{3 \cdot 5}] = 666$

Include the intersect of the numbers divisible by 3 and 7 $= |B \cap D| = [\frac{9999}{3 \cdot 7}] = 476$

Include the intersect of the numbers divisible by 5 and 7 $= |C \cap D| = [\frac{9999}{5 \cdot 7}] = 285$

Exclude the intersect of the numbers divisible by 2,3,5 $= |A \cap B \cap C| = [\frac{9999}{2 \cdot 3 \cdot 5}] = 333$

Exclude the intersect of the numbers divisible 2,5,7 $= |A \cap B \cap D| = [\frac{9999}{2 \cdot 3 \cdot 7}] = 238$

Exclude the intersect of the numbers divisible by 3,5,7 $= |B \cap C \cap D| = [\frac{9999}{3 \cdot 5 \cdot 7}] = 95$

Include the intersect of the numbers divisible by all 2,3,5,7 = $|A \cap B \cap C \cap D| = [\frac{9999}{2 \cdot 3 \cdot 5 \cdot 7}] = 47$

$= 9999 - (A \cup B \cup C \cup D)$

$= 9999 - (|A| + |B| + |C| + |D|) + (|A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|) - (|A \cap B \cap C| + |A \cap B \cap D| + |B \cap C \cap D|) + (|A \cap B \cap C \cap D|)$

$= 9999 - (4999 + 3333 + 1999 + 1428) + (1666 + 999 + 714 + 666 + 476 + 285) - (333 + 238 + 95) + 47$

$= 9999 - 11759 + 4806 - 666 + 47$

$= 2427$

Therefore, using the inclusion-exclusion principle, we can see that 2427 elements are divisible by none of 2,3,5,7.

4. $2 \cdot \binom{n}{1}\binom{n}{2} = \binom{2n}{3} - 2 \cdot \binom{n}{3}$

   Suppose we have a 2n set. We can call it S. Then $\binom{2n}{3}$ is the number of 3 element subsets of S. Next, we will rearrange the equation, to make it easier to proof.

   $\binom{2n}{3} = 2 \cdot \binom{n}{3} + 2 \cdot \binom{n}{1}\binom{n}{2}$
   .
   Here we can see that the choices of 3 element, 2, element and one element subsets of n is written:

   $\binom{n}{3} + \binom{n}{1}\binom{n}{2}$

   To re adjust this to be a set of 2n:

   $2 \cdot \binom{n}{3} + 2 \cdot \binom{n}{1}\binom{n}{2}$

   Therefore:

   $2 \cdot \binom{n}{1}\binom{n}{2} = \binom{2n}{3} - 2 \cdot \binom{n}{3}$

5. A website requires a password to be chosen containing characters. They can be upper case letters, lower case letters or digits. How many disallowed passwords are there, given a character cannot repeat?
   First, we must determine how many possible characters can fit in each character of the password.
   The total amount of characters to choose from is uppercase + lowercase + numeric digit.
   This can be shown as:

   $|L| + |U| + |N| = 26 + 26 + 10 = 62$

   There are 62 total characters, and they can repeat but not next to each other.
   This means the first digit can be a possibility of 62 characters, and and other characters after can be a total of 61 possible characters, as the previous character needs to be taken out of possibilities.
   To simplify this for making an equation, we can determine that $62 = a$.

   Therefore we need to find the disallowed characters using the following:
   $a \times a \times a ... a_n$

This will determine the illicit passwords up until the length of password, marked by n.
To re write this:

a^n would denote the total passwords is numbers could repeat. Given numbers cannot repeat right away:

$= a^n - (a \times a - 1 \times a - 1....a1_n)$

For example, if the length of password was 4, given the amount of characters being 62:

$= 62^4 - (62 \times 61 \times 61 \times 61)$

$= 16777216 - 14072822$

$= 2704394$ passwords would be disallowed, in this example.

This proves that our amount of disallowed passwords–denoted with a D–can be determined with the following equation:

$D = a^n - (a \times (n(a - 1)))$