CISC 102 (Fall 20)
Homework #9: Relations (31 Points)

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1. (6 pts) Let R be the relation on the natural numbers defined by

$$R = \{(x, y) : x, y \in \mathbb{N}, x^2 + y^2 \leq 8\}.$$

   (a) Write out the elements of R as a set of ordered pairs

   Considering natural numbers include 0, and $x^2 + y^2 \leq 8$, here are the orded pairs:
   $R = (0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)$

   (b) Is R an equivalence relation, and explain why or why not?

   To be an equivalence relation, the relation has to be reflexive, symmetric and transitive. Here are the ordered pairs that prove each type of relation;
   Reflexive: $(0,0), (1,1), (2,2)$
   Symmetric: $(0,1), (0,2), (1,0), (2,0), (1,2), (2,1)$
   Transitive: $(1,2), (2,4), (1,4)$
   $\therefore$ R is an equivalence relation.

   (c) Is R a partial order and explain why or why not?

   No R is not a partial order because it is not asymmetric.
   There are not ordered pairs that do not fall into the reflexive or symmetric form.

2. (2 pts)
   Let A be a set with n elements, and let B be a set with m elements ($n \geq 0, m \geq 0$)
   How many relations are there from A to B?

   Given both elements are positive we can use the rule of Cartesian product. To be able to tell how many relations there are, we can use the formula 2^{mn}.

3. (4 pts) Let A be a set and let R and S be relations on A.

(a) Suppose R is anti-symmetric. Prove that $R \cap S$ is also anti-symmetric.

If R is anti-symmetric it follows that $R \cap S$ is anti-symmetric as any subset of R is.
If $(x, y), (y, x) \in R \cap S$
$(x, y), (y, x) \in R$
$\therefore x = y$ and therefore $R \cap S$ is anti-symmetric.

(b) Suppose R and S are both transitive. Prove that $R \cap S$ is also transitive.

If R and S are transitive, then:
$(x, y) \in R \cap S$ and $(y, z) \in R \cap S$
$(x, y) \in R$ and $(x, y) \in S$
$(y, z) \in R$ and $(y, z) \in S$
Because R and S are transitive then $(x, z) \in R$ and $(x, z) \in S$
$(x, z) \in R \cap S$ and therefore $R \cap S$ is transitive.

4. (2 pts) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$
Let R be the relation on A defined by
$(a, b) \in R$ if and only if $b = a + 3m$ for some non-negative integer m
For example, $(5, 11) \in R$ because $11 = 5 + 3 \cdot 2$
Is R a partial ordering on A? Prove your answer.

This would not be partial ordering. This is because there would be no ordered pairs that would be reflexive, as b will always be greater than a, as the value inputs must be non negative.

5. (2 pts) Let R be a relation defined on the set $\{0, 1, 2, 3, \dots\}$ as follows:
$(a, b) \in R$ if and only if $(a + b)$ is a multiple of 2
(a) is R reflexive?

Let $a \in Z$ and $a + a = 2a$, a multiple of 2.
$\therefore aRa, \forall a \in Z$ and R is reflexive.

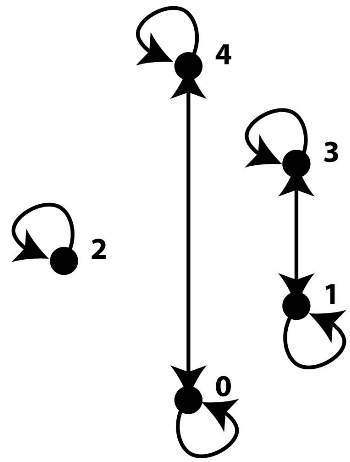
(b) is R symmetric?

R is symmetric because; if $a + b = 2k$ then $(a, b) = (b, a)$, and
$\forall a, b \in R(aRb \leftrightarrow bRa)$
We can also tell this because the relation is not antisymmetric.

6. (6 pts) Let $A = \{0, 1, 2, 3, 4\}$ and define a relation R on A as follows:

$$R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}.$$

(a) Draw the directed graph of R



(b) Find the equivalence class of every element of A.

Equivalence classes are any elements in A related to R so :
$0 = [0] = \{0, 4\}$
$1 = [1] = \{1, 3\}$
$2 = [2] = \{2\}$
$3 = [3] = \{1, 3\} = [1]$
$4 = [4] = \{0, 4\} = [0]$

(c) Find the distinct equivalence classes of the relation (Usually several of the classes will contain exactly the same elements, so you must take a careful look at the classes to determine which are the same. You then indicate the distinct equivalence classes by describing them without duplication.)

Because we have determined that 0 and 4, and 1 and 3 share equivalence classes, we can see that only 0,1 and 2 are the distinct, and therefore:
$\{\{0, 4\}, \{1, 3\}, \{2\}\}$ are the set of distinct equivalence classes of A under R.

7. (5 pts)
Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? For those that are not, what properties do they lack?

(i) $\{0 \sim 0, 1 \sim 1, 2 \sim 2, 3 \sim 3\}$

Reflexive: $(0, 0), (1, 1), (2, 2), (3, 3)$, yes!
Symmetric: No.
Transitive: No.
$\therefore R$ is not an equivalence relation.

(ii) $\{0 \sim 0, 0 \sim 2, 2 \sim 0, 2 \sim 2, 2 \sim 3, 3 \sim 2, 3 \sim 3\}$

Reflexive: $(0,0),(2,2),(3,3)$, yes!
Symmetric: $(0,2),(2,0),(2,3),(3,2)$, yes!
Transitive: $(0,2),(2,0),(2,2)$, yes!
$\therefore R$ is an equivalence relation.

(iii) $\{0 \sim 0, 1 \sim 1, 1 \sim 2, 2 \sim 1, 2 \sim 2, 3 \sim 3\}$

Reflexive: $(0,0),(1,1),(2,2),(3,3)$, yes!
Symmetric: $(1,2),(2,1)$, yes!
Transitive: $(1,2),(2,1),(2,2)$, yes!
$\therefore R$ is an equivalence relation.

(iv) $\{0 \sim 0, 1 \sim 1, 1 \sim 3, 2 \sim 2, 2 \sim 3, 3 \sim 1, 3 \sim 2, 3 \sim 3\}$

Reflexive: $(0,0),(1,1),(2,2),(3,3)$,yes!
Symmetric: $(1,3),(3,1),(2,3),(3,2)$, yes!
Transitive: $(1,3),(3,1),(3,3)$, yes!
$\therefore R$ is an equivalence relation.

(v) $\{0 \sim 0, 0 \sim 1, 0 \sim 2, 1 \sim 0, 1 \sim 1, 1 \sim 2, 2 \sim 0, 2 \sim 2, 3 \sim 3\}$

Reflexive: $(0,0),(1,1),(2,2),(3,3)$, yes!
Symmetric: $(0,1),(0,2),(1,0),(2,0)$, yes!
Transitive: $(0,2),(2,0),(2,2)$, yes!
$\therefore R$ is an equivalence relation.

8. (4 pts)
For the following relations on A determine whether they are reflexive, symmetric, and/or transitive. State whether they are equivalence relations or not, and if they are describe their equivalence classes.

(a) Let $A = \mathbb{Z}$ and define $\sim$ by $a \sim b$ whenever $a - b$ is odd.
Considering a-b must be odd, this means that a $\neq$ b, therefore this is not reflexive. Therefore there is not a description for the equivalence classes.

(b) Let $A = \mathbb{R}$ and define $\sim$ by $a \sim b$ whenever $ab \neq 0$.
Considering ab must not equal zero, this means that any a or b that is zero is not included, therefore this is not reflexive, or symmetrical. Therefore there is not a description for the equivalence classes.