CISC 102 (Fall 20)
Homework #4: Functions (20 Points)

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Solutions are due before 11:59 PM on **October 20, 2020** .

1. (2pts)
   Determine whether the mappings from $\mathbb{R}$ to $\mathbb{R}$ shown below are or are not functions, and explain your decision.

   (a) $f(x) = 1/x$
   Because there is no definition for $x = 0$, we must say that $f(x) = 1/x$ is not a function from $\mathbb{R}$ to $\mathbb{R}$

   (b) $f(x) = \sqrt{x}$
   Because $\sqrt{x}$ is not a real number for any negative number this does not work for $\mathbb{R}$ to $\mathbb{R}$, unless we define this function to only non-negative real numbers. Therefore we say: $f(x) = \sqrt{x}$ is not a function from $\mathbb{R}$ to $\mathbb{R}$.

2. (2pts)
   Determine whether each of the following functions from $\mathbb{R}$ to $\mathbb{R}$ is a bijection, and explain your decision. HINT: Plotting these functions may help you with your decision.

   (a) $f(x) = -x^2 + 2$

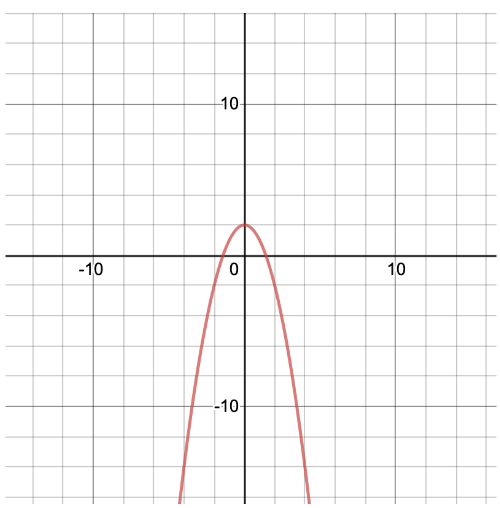

Figure 1: $f(x) = -x^2 + 2$ Graph.

As we can see from this graph, this is not a bijection because it is not unto or one-to-one. We can see this because any horizontal line put on the graph will intersect more than one, or not at all.

(b) $f(x) = x^3 - x^2$

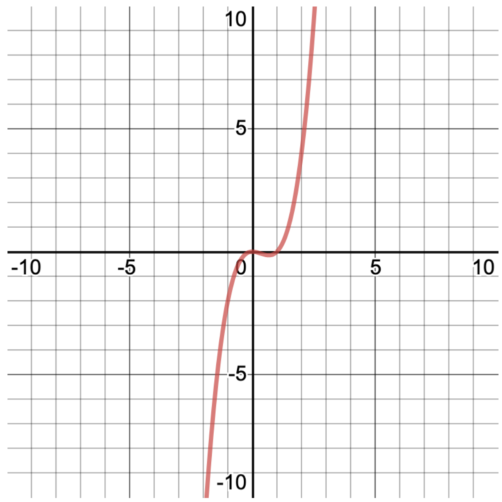

Figure 2: $f(x) = x^3 - x^2$ Graph.

As we can see from the graph, this is not a bijection for $\mathbb{R}$ to $\mathbb{R}$, because $f(x) = 0$ for both $x = 0$ and $x = 1$.

3. (2pts)
   Suppose the function $f : A \to B$ is a bijection. What can you say about the values $|A|$ and $|B|$?

   If $f : A \to B$ is a bijection, then we can say that the cardinalities of A and B are equal. In other words: $|A| = |B|$

4. (2pts)
   Let $f : \{1, 2, 3, 4, 5, 6\} \to \{red, yellow, beige, green, umber, teal\}$ be a one-to-one function. Prove, by contradiction, that f is a bijection.

   In order to prove this is a bijection by contradiction, we first must assume that this is not a bijection.
   Assign { 1,2,3,4,5,6} = A and {red,yellow,beige, green,umber,teal} = B

Assume $|A| \neq |B|$.
If $|A| = 6$
and $|B| = 6$
and $6 = 6$,
$\therefore |A| = |B|$
Marking our initial assumption wrong, and therefore proving f as a bijection.

5. (2pts) Let $A = \{1, 2, 3, 4\}$
Let $B = \{a, b\}$
Let $C = \{curling, hockey, table\text{-}tennis\}$

   (a) How many *one-to-one* functions are there from C to A?
   $|C| = x$
   $|A| = y$

   $\frac{y!}{(y-x)!} = \frac{4!}{(4-4)!}$

   $4! = 24$ and $0! = 1$
   $\therefore$ there are 24 one to one functions.

   (b) How many *onto* functions are there from C to B?
   (Hint: count the non-onto functions)
   $C(4val) \to B(2val) = C \to B$ is $2 * 2 * 2 * 2 = 16$
   However since each value in C has 2 possible images in B, there are 2 non-onto functions.
   $\therefore C \to B = 14$ onto and 2 non-onto

6. (4pts)
Decide for each of the following expressions: Is it a function? If so,
(i) what is its domain, codomain, and image? (ii) is it injective? (why or why not)
(iii) is it surjective? (why or why not) (iv) is it invertible? (why or why not)

   (a) $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^3$

   This is a function, denoted by $f(x) = x^3$
   We will refer to x as A and x^3 as B.

   (i) The domain can be defined by $\mathbb{R}$. Codomain is also defined as $\mathbb{R}$. The image is also $\mathbb{R}$.

   (ii) $f(x) = x^3$. Here we need to find if$f(a) = f(b)$:
   $f(a) = a^3$
   $f(b) = b^3$
   $b^3 = a^3 \implies \sqrt[3]{b^3} = \sqrt[3]{a^3}$

$b = a \therefore f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^3$ is injective.

(iii) $f(x) = x^3$
$f(x) = \sqrt[3]{y}^3$
$f(x) = y \therefore f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^3$ is surjective.

(iv) Because each input has a unique output, we can say that $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^3$ is invertable.

(b) $f : \mathbb{R} \times \mathbb{Z} \to \mathbb{Z}$ defined by $(r, z) \to \lceil r \rceil * z$

(i) Domain for r is $\mathbb{R}$, and the domain for z is $\mathbb{Z}$. The codomain is $\mathbb{Z}$. This image is $\mathbb{Z}$.

(ii) For this function to be injective, the cardinality of$\lceil r \rceil * z \geq (r, z)$. However we can see that this is not true, as not every r,z point will have a place to map in $\lceil r \rceil * z$, because of the ceiling function and multiplication.

(iii) This function is surjective because the cardinality of $\lceil r \rceil * z \leq (r, z)$, as so every point from the $\lceil r \rceil$ *z will have a place to map on (r,z).

(iv) This is not invertible. This is because a function must be bijective to be invertible, and because this is not injective it cannot be bijective.

7. (2pts)
Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that

(a) $f(S \cup T) = f(S) \cup f(T)$:

Let $x \in f(S \cup T)$ Then $\{\exists y \in S \cup T | f(y) = x\}$.
$y \in S \vee y \in T \therefore f(x) \in f(S) \vee f(y) \in f(T)$
Because $f(y) = x$ we can rewrite as $x \in f(S) \vee x \in x\mathcal{F}(T)$
$x \in f(S) \cup f(T) \therefore f(S \cup T) \subseteq f(S) \cup f(T)$
For the second part, we need to take the equations: $y \in S \vee y \in T$ such that $f(y) = x$
Because we are looking for the union we rewrite as $y \in S \cup T$
$f(y) \in f(S \cup T) \equiv x \in f(S \cup T)$ Then we can rewrite as $f(S) \cup f(T) \subseteq f(S \cup T)$.
From the above work we can see that:
$f(S) \cup f(T) \subseteq f(S \cup T) = f(S \cup T) \subseteq f(S) \cup f(T)$
$\therefore f(S) \cup f(T) = f(S \cup T)$

(b) $f(S \cap T) \subseteq f(S) \cap f(T)$.

Let $x \in f(S \cap T)$ Then $\{\exists y \in S \cap T | f(y) = x\}$
$y \in S \wedge t \in T \equiv f(y) \in f(S) \wedge f(y) \in f(T) \equiv x \in f(S) \wedge x \in f(T)$
To find the intersection: $x \in f(S) \cap f(T)$
$\therefore f(S \cap T) \subseteq f(S) \cap f(T)$

8. (1pts) find the inverse function of $f(x) = x^3 + 1$

   $f(x) = x^3 + 1 \equiv y = x^3 + 1$
   $x^3 = y - 1$
   $x = \sqrt[3]{y-1}$
   $\therefore f^-1(x) = \sqrt[3]{x-1}$

9. (2pts)
   Suppose that g is a function from A to B and f is a function from B to C.

   (a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

   $f(x) = f(y) \therefore x = y$. This shows that f is one to one.
   $g(x) = g(y) \therefore x = y$.. This shows that g is one to one.

   If $g : A \rightarrow B$ is one to one and $f : B \rightarrow C$ is one to one, $f \circ g$ is one to one. This is how we see this:
   Assume $(f \circ g(x)) = (f \circ g(y))$
   $f(g(x)) = f(g(y))$
   Because f and g are both one to one:
   $g(x) = g(y)$
   $x = y$

   Here we can see that $x = y \therefore f \circ g$ is a one to one function.

   (b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

   $f = \forall x \in C \exists y \in B. \therefore f(y) = x$. This shows that f is onto.

   $g = \forall y \in B \exists z \in A. \therefore g(z) = y$ This shows that g is onto.

   Using the functions found above we can see: $(f \circ g)(z) = f(g(z)) = f(y) = x$. This means that $\{\forall x \in C \exists z \in A | (f \circ g)(z) = x\}$.

   Therefore $f \circ g$ is onto.

10. (1pts)
Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from $\mathbb{R}$ to $\mathbb{R}$.

Because f and g are both from $\mathbb{R}$ to $\mathbb{R}$ then this must mean that $f \circ g$ and $g \circ f$ are also from $\mathbb{R}$. To find $f \circ g$ we do the following:

$f \circ g = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5$
$\therefore f \circ g = x^2 + 4x + 5$

To find $g \circ f$ we do the following:

$g \circ f = g(f(x)) = g(x^2 + 1) = x^2 + 1 + 2 = x^2 + 3$
$\therefore g \circ f = x^2 + 3$

Answer: $f \circ g = x^2 + 4x + 5$ and $g \circ f = x^2 + 3$