CISC 203 Problem Set 3

Amy Brons

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1. (a) Let R be a relation on R and let $x, y \in R$. Then, xRy if and only if $x^2 + y^2 = 1$. Determine in R is a function.

   Given the formula, we can determine that this relation is symmetric because if $x^2 + y^2 = 1$ therefore $x\ y$.

   Another way to think about it is as such:

   $x^2 = 1 - y^2$

   $1 - y^2 + y^2 = 1$

   $1 = 1$

   Evidently, these y will only have one x, when looking to solve.
   This is by definition symmetric as a relation. Although this is not reflexive or transitive, it can be determined that this statement is in fact a function. This is because each x will map exactly one output to y.

   Therefore given that $x, y \in R$ and $x^2 + y^2 = 1$, the relation $R \longrightarrow R$ is a function.

   (b) Let $x \in R$ and let $f(x) = \frac{x^2}{(x-2)^2}$. Determine dom f and im f

   To find the domain of the function $f(x)$, we need to find the set of all of the first elements of the ordered pairs in f:

   $domf = a \in A : \exists b \in B, (a, b) \in f$

   This means that the domain will be the value of (x,f(x)), respectively.

   $f(x) = \frac{x^2}{(x-2)^2}$

   Because of the value of the denominator:
   $(x - 2)^2$, the dom will be 2.

   Therefore the value of $domf = 2$

   To find the image of the function, we need to look at all the second elements in the ordered pairs, ie:

   $imf = b \in B : \exists a \in A, (a, b) \in f$
   So for this, we fill the domain in as is:

$2 = \frac{x^2}{(x-2)^2}$

$2((x-2)(x-2)) = 4$

$(x-2)(x-2) = 2$
$x^2 - 4x = 0$
$x(x-4) =$

Therefore :
$imf = 4$

(c) Consider the function $f : R \to R$ defined by:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if x is even} \\ x+2 & \text{if x is odd.} \end{cases}$$

1. Determine whether f is one-to-one and/or onto.
To determine if f is one-to-one we need to asses the two cases.

$\frac{x}{2}$, when x is even must mean that $f(x)$ is equal to a whole number, even or odd.

$x + 2$ if the number is odd determines that $f(x) =$ an even whole number.

This means that the function is one-to-one, given the fact that the number determined only has one possible outcome per value of x. The function however is not one given that it is onto.

2. Does an inverse function f^{-1} of f exist? If so, determine f^1.
Yes, it does exist and can be determined by denoting $f(x)$ as y, and replacing x and y with each other.

$$x = \begin{cases} \frac{y}{2} & \text{if y is even} \\ y+2 & \text{if y is odd.} \end{cases}$$

This is the denotation of $f(x)^{-1}$, on $f : R \longrightarrow R$

2. (a) i. If $g \circ f = h$, then $g = h$;

This can be determined through equality of functions:
Let $a \in A, b \in B, c \in B$
Given $\circ f = h \circ f$
$g(f(a)) = h(f(a))$ for $a \in A$
from this we can determine that g = b, only in f(a) = b
Therefore there is an element b in B for each element a.

Therefore this is disproved because $g \circ f \neq h \circ f$ when f(a) is not equal to b.

ii. If f is one-to-one and $g \circ f = h \circ f$, then $g = h$.
If we look at the function $g = h$, as determined in (i), we can see that $g(b) = h(b)$, so therefore if we assume that $g \neq h, g(b) \neq h(b)$.

Considering that f is one-to-one, there must be an $a \in A \forall b \in B$, such that $f(a) = b$
Therefore it must be determined if:

$g \circ f(a) \neq h \circ f(a)$
Therefore this is a contradiction that exists, meaning that this is disproved and if f is one-to-one then $g \circ f \neq h \circ f$

(b) To figure this out, we can look at the statement:
$\sigma \circ \pi = \tau$,
We need to determine σ, given the sets of τ and π.

First σ must be isolated, by multiplying both sides by π^{-1}

$\sigma \circ \pi \circ \pi^{-1} = \tau\pi^{-1}$

$\sigma = \tau \circ \pi^{-1}$

Next lets put this in cycle notation to make it easier to use:
$\pi = (1, 2, 6, 3, 4)(5)$
$\pi^{-1} = (4, 3, 6, 2, 1)(5)$
$\tau = (1, 5)(2, 3, 6, 4)$

Given these cycle notations, we can now find the function of $\tau \circ \pi^{-1}$:

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 2 & 1 & 4 & 3 \end{bmatrix}$$

In cycle notation can be written as:

$\sigma = (1, 5, 4)(2, 6, 3)$

3. To find the possibility of Alice hitting at least one bulls eye in three darts, we need to find the possible throw outcomes, and then subtract the outcomes where the bulls eye was not hit to make at least one outcome a bulls eye. If we denote the outcome of no-bulls eye by $P(N)$, this must be found:

$P(X) = 1 - P(N_1 \cap N_2 \cap N_3)$

To find $P(N)$ lets subtract the bulls eye outcome:

$P(B) = \frac{1}{3}$

$P(N) = 1 - P(B) = 1 - \frac{1}{3}$

$P(N) = \frac{2}{3}$

Therefore:

$P(X) = 1 - P(\frac{2}{3} \cap \frac{2}{3} \cap \frac{2}{3})$

$P(X) = 1 - (\frac{2}{3})^3$

$P(X) = 1 - \frac{8}{27} = \frac{27}{27} - \frac{8}{27}$

$P(X) = \frac{19}{27}$

Given that the outcome of a throw does not affect the outcome of any other throw, the odds that Alice throw at least one bulls eye when throwing three darts in succession is $\frac{19}{27}$

4. To figure out this probability, the formula solved must figure out the odds of Billy playing Among Us, and divide it by the odds of Billy playing PLUS Bobby playing Among Us.
   Basically the formula needs to be"

   $P(X) = \frac{\text{Billy being logged on x Billy playing Among Us}}{\text{(Billy being logged on x Billy playing Among Us)+(Bobby being logged on x Bobby playing Among Us)}}$

   To do this, we need to determine the values of Billy being logged on as $P(I)$ and Bobby being logged on as $P(B)$. These are equal odds, and therefore can be defined as:

   $P(I) = \frac{1}{2}$
   $P(B) = \frac{1}{2}$

   Next, to define the odds of Among Us being played by each player. Billy playing Among Us is defined by $P(IA)$ and Bobby playing Among Us can be defined as $P(BA)$. This values are stated in the question:

   $P(IA) = 1$
   $P(BA) = \frac{1}{4}$

   Therefore the formula can be written as:

   $P(X) = \frac{P(I) \cap P(IA)}{(P(I) \cap P(IA)) + (P(B) \cap P(BA))}$

   $P(X) = \frac{\frac{1}{2} \times 1}{(\frac{1}{2} \times 1) + (\frac{1}{2} \times \frac{1}{4})}$

   $P(X) = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{8}} = \frac{\frac{1}{2}}{\frac{4}{8} + \frac{1}{8}}$

   $P(X) = \frac{\frac{1}{2}}{\frac{5}{8}}$

   $P(X) = \frac{4}{5}$

   Therefore the odds of Billy playing Among Us, when the account is playing this game is $\frac{4}{5}$

5. (a) Show that $P(X = x)$ is a valid probability function.
   Given that:

   $P(X = x) = \frac{2}{(x+2)(x+3)}, x \geq 0$

   Using partial fraction decomposition we can determine:

   $\frac{2}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$

   $= \frac{A(x+3)}{(x+2)(x+3)} + \frac{B(x+2)}{(x+3)(x+2)}$

   $= \frac{Ax+3A+Bx+2B}{(x+2)(x+3)}$

$= \frac{(A+B)_x+3A+2B}{(x+2)(x+3)} = \frac{2+0_x}{(x+2)(x+3)}$

$(A+B)_x = 0, 3A+2B = 2$

$A+B=0$
$3A+2B=2$
$-A-B=0$
$=$
$2A+B=2$
$-A-B=0$
$=$
$A=2$

$3(2)+2B = 6+2B = 2$
$2B=-4$
$B=-2$
Checking:
$A+B=0$
$2-2=0$

Therefore, lets determine the validity using:

$\frac{A}{x+2} + \frac{B}{x+3} = \frac{2}{x+2} + \frac{-2}{x+3}$

$P(X=x) = \frac{2}{x+2} - \frac{2}{x+3}$

$P(0) = 1 - \frac{2}{3}$

$P(1) = \frac{2}{3} - \frac{2}{4}$

$P(2) = \frac{2}{4} - \frac{2}{5}$

. . . etc.

Check that

$\sum_{x=0}^{\infty} P(X=x)^2 = 1$
$\sum_{x=0}^{\infty} \frac{2}{x+2} - \frac{2}{x+3}$
$= (1-\frac{2}{3}) + (\frac{2}{3}-\frac{1}{2}) + (\frac{1}{2}-\frac{2}{5}) + ...$

$= 1 + (\frac{2}{3}) + (\frac{2}{3}-\frac{1}{2}) + (\frac{1}{2}-\frac{2}{5}) + ...$

$= 1$

Therefore this is satisfied.

Finally, we need to check that

$P(X=x) \geq 0 \forall x \geq 0$

This is easy to check, because the numerator is 2, therefore if $x \geq 0, 2 > 0$. And therefore it must be true that $P(X = x) > 0$.

Therefore this probability function is valid, because it satisfies all the requirements of a probability function.

(b) The probability that there was at least one meme shared on a day, given that at most 5 memes are shared a day, we need to use the conditional probability formula:

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

Given the formula, we can denote that:
$A = X \geq 1$
$B = X \leq 5$

Therefore:

$P(X \geq 1|X \leq 5) = \frac{P(1 \leq X \leq 5}{P(X \leq 5)}$

Using the formula in part (a), lets add the values determined:

$$\sum_{1 \geq X \leq 5}^{\infty} = \frac{2}{x+2} - \frac{2}{x+3}$$

$$= \frac{(\frac{2}{3} - \frac{1}{2}) + (\frac{1}{2} - \frac{2}{5}) + (\frac{2}{5} - \frac{1}{3}) + (\frac{1}{3} - \frac{2}{7}) + (\frac{2}{7} - \frac{1}{4})}{(1 - \frac{2}{3})(\frac{2}{3} - \frac{1}{2}) + (\frac{1}{2} - \frac{2}{5}) + (\frac{2}{5} - \frac{1}{3}) + (\frac{1}{3} - \frac{2}{7}) + (\frac{2}{7} - \frac{1}{4})}$$
$$= \frac{\frac{2}{3} - \frac{1}{4}}{1 - \frac{1}{4}}$$
$$= \frac{\frac{5}{12}}{\frac{3}{4}}$$
$$= \frac{\frac{5}{12}}{\frac{9}{12}}$$
$$= \frac{5}{9}$$

Therefore the odds that at least one meme is shared on a day, given that at most 5 memes are shared a day is $\frac{5}{9}$.

6. (i) No they are not mutally exclusive.

A and B are not mutually exclusive events, because both events can occur simultaneously. There are 4 cards with the value of 7 in a 52 card deck, meaning these two draws can be 7 at the same time.

(ii) No they are not independent.

They are not independent, because the result of A effects the probability of event B. Intuitively, if we think about the amount of cards in a deck, only 4 are of the value 7. So if a seven is taken out of the deck on the first draw, the odds of a seven being drawn second goes down, as there are less 7 cards in the deck.

To think about this mathematically, let's denote $A =$ one 7 is drawn. and $B =$ a second 7 is drawn.

Therefore we can see that:

$P(A) = \frac{4}{52} = \frac{1}{13}$

$P(B) = \frac{3}{51} = \frac{1}{17}$

Therefore we can see that is only one 7 is drawn the odds are higher, but if both the first and second cards are drawn the odds go down.

7. (i) To determine the value of X, lets first determine the possible flips:

H, TH, TTH, TTTH, TTTTH, TTTTTH, TTTTTT

If we determine each game by it's number of flips so:
H = 1
TH = 2
TTH = 3
TTTH = 4
TTTTH = 5
TTTTTH = 6
TTTTTT = 6

and to find the odds, the formula is

$\frac{1}{2^x}$

To determine the expected value, lets multiply the flips by the probability:

$P(x) = (1 \times \frac{1}{2^1}) + (2 \times \frac{1}{2^2}) + (3 \times \frac{1}{2^3}) + (4 \times \frac{1}{2^4}) + (5 \times \frac{1}{2^5}) + (6 \times \frac{1}{2^6})$

$= (\frac{1}{2}) + (\frac{1}{2}) + (\frac{3}{8}) + (\frac{1}{4}) + (\frac{5}{32}) + (\frac{3}{32})$

$= 1 + \frac{5}{8} + \frac{1}{4}$

$= 1 + \frac{7}{8}$

$= \frac{15}{8}$

(ii) To determine the variance of X we need to square each number of flips, and multiply them by the probabilities, and then subtract the expected value:

$Var(x) = (1^2 \times \frac{1}{2^1}) + (2^2 \times \frac{1}{2^2}) + (3^2 \times \frac{1}{2^3}) + (4^2 \times \frac{1}{2^4}) + (5^2 \times \frac{1}{2^5}) + (6^2 \times \frac{1}{2^6}) - (\frac{15}{8})^2$

$= (\frac{1}{2}) + (4 \times \frac{1}{4}) + (9 \times \frac{1}{8}) + (16 \times \frac{1}{16}) + (25 \times \frac{1}{32}) + (36 \times \frac{1}{64}) - (\frac{15}{8})^2$

$= \frac{1}{2} + 1 + \frac{9}{8} + 1 + \frac{25}{32} + \frac{9}{16} - (\frac{15}{8})^2$

$= \frac{1}{2} + 2 + \frac{79}{32} - \frac{225}{64}$

$= \frac{1}{2} + 2 - \frac{67}{64}$

$= 2 - \frac{35}{64}$

$= 2 - 0.546875 = 1.453125$

The variance of X is 1.453125.

(iii) To determine this, lets re state the possible cases:

H, TH, TTH, TTTH, TTTT
If we re-work the formula from part (i), it would look like:

$P(x) = (1 \times \frac{1}{2^1}) + (2 \times \frac{1}{2^2}) + (3 \times \frac{1}{2^3}) + (4 \times \frac{1}{2^4})$

$P(x) = (\frac{1}{2}) + (\frac{1}{2}) + (\frac{3}{8}) + (\frac{1}{4})$

$= 1 + \frac{5}{8}$
$= \frac{13}{8}$

Next to find the variance, lets rework the formula from part (ii):

$Var(x) = (1^2 \times \frac{1}{2^1}) + (2^2 \times \frac{1}{2^2}) + (3^2 \times \frac{1}{2^3}) + (4^2 \times \frac{1}{2^4}) - (\frac{13}{8})^2$

$= \frac{1}{2} + 1 + \frac{9}{8} + 1 - \frac{169}{64}$

$= 2 - \frac{65}{64}$

$= 2 - 1.015625$

$= 0.984375$

Therefore the new variance is $Var(x) = 0.984375$

To explain this to a layperson:
The reason for the drop in variance can be attributed to the fact that less possible outcomes need to be accounted for. So in the step (ii), the variance is greater because more steps can have difference between excepted outcome, and actual outcome. It only makes sense that is there is less outcomes to worry about, there is less variance to worry about, because less actual outcomes have less expected outcomes. If you roll a die 10 times you and guess the roll value for each roll, there is a chance of being wrong each of those 10 times. If you roll a die 3 times and guess the value each time, you may be wrong those three times. Because being wrong 3 times is less than being wrong 10 times, the variance in lower.