Notes on the "15" Puzzle.

I.

BY WM. WOOLSEY JOHNSON, *Annapolis, Md.*

THE puzzle described below has recently been exercising the ingenuity of many persons in Baltimore, Philadelphia and elsewhere. A ruled square of 16 compartments is numbered as in this diagram:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

the 16th square being left blank. Fifteen counters, numbered in like manner, are placed at random upon the squares so that one square is vacant. The counter occupying any adjacent square may now be moved into the vacant square—thus: If No. 7 is vacant, either of the counters occupying Nos. 3, 6, 8, 11 can be moved into it, but no diagonal move is allowed. The puzzle is to bring all the counters into their proper squares by successive moves.

It seems to be generally supposed, by those who have tried the puzzle, that this is always possible, whatever be the original random position of the counters, but this is an error, as the following demonstration will show:

When the blank or sixteenth square is the vacant one, the arrangement of the counters may be called a positive or negative one, according as the term of the 15-square determinant, which has for first and second subscripts the numbers on the squares and counters, is positive or negative. Let n

moves be made, leaving some other square vacant, and then suppose the counter which occupies the blank square to be transferred directly to the vacant square, we thus obtain a positive or a negative arrangement. Had $n+1$ moves been made before the transfer took place, the arrangement produced would have been one which can be derived from that last mentioned by a single interchange of two counters. (For example, if the $(n+1)$th move is from No. 6 to No. 7, the moved counter will be in No. 7 and the transferred counter in No. 6; whereas, had the transfer taken place after n moves, the former would have been in No. 6 and the latter in No. 7).

Now the first two moves followed by a transfer are equivalent to one interchange; therefore the displacement effected by n moves followed by a transfer is one which could have been produced by $n-1$ interchanges. Now suppose that after m moves the blank space is again left vacant, then the mth move is itself a transfer from the blank square, and therefore the displacement produced by the m moves is one which might have been produced by $m-2$ interchanges.

If the squares were coloured, as in a chess board, each move would change the colour of the vacant square, and therefore m is an even number; it follows that the displacement is one which might have been produced by an even number of interchanges, and can never change a positive to a negative arrangement or the reverse; hence the desired arrangement, which is a positive one, can never be produced if the original random arrangement happens to be a negative one. This conclusion is obviously not affected in any way by the shape of the board.

In order to make this demonstration satisfactory to non-mathematicians who may be interested in this puzzle, I add a simple demonstration of the theorem upon which the classification of the arrangements as positive and negative depends, viz: that a permutation that can be derived from a given one by an odd number of interchanges can never be produced by an even number of interchanges. Let the numbers $1, 2, \ldots n$ be written down in natural order, and under them place any other permutation of the same numbers, thus if $n=15$, as in the present case, we might have

$$\begin{array}{ccccccccccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15; \\ 12, & 10, & 8, & 15, & 9, & 6, & 3, & 4, & 11, & 1, & 7, & 2, & 14, & 5, & 13. \end{array}$$

Now group the numbers into cycles as follows: beginning with any one of the numbers, look for it in the upper row and write next to the number that which is found under it, then looking for the latter in the upper row, write next the number found under it, and so on until we find the original number in the lower row. Thus in the example above beginning with 1, we have the cycle

1, 12, 2, 10.

Then taking a number not found in this cycle, say 3, form a new cycle, and so on till the numbers are exhausted. In this case we shall find the other cycles to be

3, 8, 4, 15, 13, 14, 5, 9, 11, 7, and 6,

the last cycle happening to consist of a single number. Let m denote the number of these cycles. In the above case $m = 3$. Now let two of the lower numbers be interchanged. A little consideration will show that if these numbers belong to the same cycle, this cycle will be broken up into two cycles; but if they belong to different cycles, these cycles will be combined into a single one. In either case, the value of m will be changed from an even to an odd number, or the reverse. The same is true of the number $n - m$. Now, when the lower numbers are in natural order, there are n cycles, each composed of a single number, and $n - m = 0$. Hence, starting from this arrangement, any odd number of interchanges will produce an arrangement in which $n - m$ is odd, and any even number, one in which $n - m$ is even. The former are the negative, and the latter the positive arrangements alluded to above.

Postscript.

Since the above was written the puzzle has been published in the form of a square box containing 15 blocks, the squares *not being numbered.* The requirement is simply to "move the blocks until in regular order." It has been shown in the New York Evening Post that when it is impossible to arrange the blocks, with the block 1 in a certain corner, it is possible to obtain a regular arrangement with the block 1 in an adjacent corner.

II.

BY WILLIAM E. STORY.

IN the preceding note Mr. Johnson has proved that, with the ordinary form of the puzzle, a positive arrangement with the 16th square blank cannot be obtained from a negative arrangement with the same square blank. But, evidently, the same method of proof will shew that, on a square or rectangular board divided by any number of vertical and any number of horizontal lines into spaces or squares, if a number of counters one less than that of squares, numbered successively from 1 on, be arranged in any way, and then moved as in the "15" puzzle, a positive arrangement cannot be converted by such moves into a negative arrangement with the same square blank, nor *vice versâ*. And this result is entirely independent of the position of the blank square. Moreover we may, in forming the arrangements of the numbers of the counters, take the first number from any given square of the board, the second from any other, the third from any remaining square, and so on, without affecting the validity of the proof, provided we use the squares in the same order in all the arrangements considered. The order in which I shall suppose the squares to be employed in forming the arrangements is this: beginning at the left-hand square of the upper row, I shall take the squares in succession along the upper row from left to right, then back along the second row from right to left, and so on along the successive rows, alternately from left to right and from right to left, until all the squares on the board have been taken, omitting the vacant square. The succession of the numbers of the counters taken in this order we shall speak of simply as the *order of the arrangement*, calling the order positive or negative according as the numbers taken in this way form a positive or negative permutation. That arrangement whose order is the natural order of the numbers we will call the *standard arrangement*. We proceed now to deduce a rule for determining whether a given arrangement can be converted into the standard arrangement, and, if so, in what manner this can be effected.

1. Evidently, with any given arrangement, two squares, upon which are counters adjacent in the order of the arrangement, are either adjacent squares upon the board, or both adjacent to the blank space. Now the blank can be interchanged with any adjacent counter by simply moving the latter into the place of the former. Thus the blank can be made to pass from any one position on the board to any other, by successive interchanges with an adjacent counter, without altering the order of the arrangement. Thus the condition for the possibility of converting any given arrangement into the standard arrangement may be treated as independent of the position of the blank square, but depending only upon the order of the given arrangement, since

the arrangement itself depends only upon the position of the blank square and the order.

2. In any arrangement any counter can be made to pass over the two counters next preceding or next succeeding it in order, without otherwise altering the order; *i. e.* if a, b, c be any three successive counters in the order of the arrangement, then a may be passed over b and c. For, bring the blank space to the side of c, so that a, b, c and the blank shall occupy four successive squares, situated in two successive rows (or in one, in which case either adjacent row may be taken for the second), and these two rows, joined at both ends, form a closed circuit in which a, b, c may be moved along until they, together with the blank space, occupy the two squares nearest the same end of both rows, thus:

a	b
	c

or

	c
a	b

If then a, b, c, a be successively moved into the blank space, the arrangement in these four squares becomes

b	c
	a

or

	a
b	c

,

in the one or the other case respectively, which is the above arrangement with a, b, c respectively replaced by b, c, a. Reversing the moves by which the three counters were brought to the end of the rows, and also those by which the blank square was brought to the side of c, we have the original arrangement with a, b, c respectively replaced by b, c, a, *i. e.* a has been passed over b and c, but no other change made in the order. It is evidently not necessary to reverse the moves by which the blank was brought to the side of c, for these do not affect the *order* of the arrangement.

Whatever be the given arrangement, the counter marked 1 may be passed over the counters preceding it in order, two at a time, until it occupies either its proper position in the standard arrangement (viz: the first square on the board), or the next square; if, in this process, it comes into the square adjacent to its own, the counter which occupies its square may be passed over it and the next counter, thus leaving it in its proper place. When the 1 is in place, we may pass up the 2 and each successive number, passing it back over two counters at a time until it reaches its own square or the next; if the latter, the counter in question may be brought into its own square by causing the counter which occupies its place to pass over it and the next. This process may be continued until only the last two counters remain, when these will be either in their proper or in inverted order. Thus every arrangement may be brought into one or the other of these two *final* arrangements, differing by one interchange, and therefore of opposite characters (the first of a positive and the second of a negative order). From which it follows (since no arrangement whose order is positive can be changed into one whose order is negative, or *vice versâ*) that *every arrangement whose order is positive, and only such, can be converted into the standard arrangement.* This is the desired criterion for the possibility of a *standard solution.*

It is evident that any two arrangements, which can be converted into the same third arrangement, can be converted into each other, and that any two arrangements cannot be converted into each other, if they can be converted into two other arrangements not convertible into each other. Now every arrangement can be converted into one or the other of the two above-mentioned final arrangements. Hence any two arrangements are interchangeable if their orders are both positive or both negative, and not interchangeable otherwise. Hence, also, an arrangement whose order is positive can or cannot be converted into a given arrangement, according as the latter is convertible into the standard arrangement by an even or an odd number of interchanges; and an arrangement whose order is negative can or cannot be converted into a given arrangement, according as the latter is convertible into the standard arrangement by an odd or an even number of interchanges. Now what may be called the *natural arrangement* (*i. e.* the arrangement in which the numbers on the counters follow each other from left to right in the upper, second, third, etc. row in their natural order, and the right hand square of the bottom row is blank) can be converted into the standard arrangement by reversing the order of the counters in the second, fourth and every even row. Evidently, any row may be reversed by interchanging the counters equally distant from its two ends. Thus a row containing an even number of counters may be reversed by a number of interchanges equal to half the number of counters in the row, and a row containing an odd number of counters by half the number less one, since the position of the middle counter is not altered by reversing. Hence the number of interchanges necessary to reverse a row of an odd number of counters is the same as for a row containing a number of counters one less. The number of necessary interchanges is the same for each even row, unless the board contains an even number of rows and an even number of columns, in which case the number of interchanges for the last row will be one less than for any other even row.

Representing the number of rows on the board by r and the number of columns by c, we shall have four cases, viz:

I. r even, c even; II. r even, c odd; III. r odd, c even; IV. r odd, c odd.

The number of interchanges necessary to convert the natural arrangement into the standard arrangement in each case is

I. $\frac{1}{2}c$ in each of $\frac{1}{2}r-1$ rows and $\frac{1}{2}c-1$ in one row,
II. $\frac{1}{2}(c-1)$ in each of $\frac{1}{2}r$ rows,
III. $\frac{1}{2}c$ in each of $\frac{1}{2}(r-1)$ rows,
IV. $\frac{1}{2}(c-1)$ in each of $\frac{1}{2}(r-1)$ rows;

i. e. I. $\frac{1}{4}cr-1$, II. $\frac{1}{4}(c-1)r$, III. $\frac{1}{4}c(r-1)$, IV. $\frac{1}{4}(c-1)(r-1)$.

We may divide all possible boards into two classes, regarding as of the *first class* a board for which the number just found is even, and as of the *second class* one for which this number is odd. We have then this rule:

On a board of the first class a given arrangement can or cannot be converted into the natural arrangement, according as its order is even or odd; but on a board of the second class a given arrangement can or cannot be converted into the natural arrangement, according as its order is odd or even.

For the ordinary "15" puzzle we have $r=4$, $c=4$, which belong to Case I.; $\frac{1}{4}cr-1=3$, which being an odd number, the board is of the second kind, and the natural arrangement can be obtained from any arrangement whose order is odd, but not from one whose order is even. For a square board with five rows and five columns we have $r=5$, $c=5$, belonging to Case IV., $\frac{1}{4}(c-1)(r-1)=4$, and the board is of the first class, hence the natural arrangement can be obtained from any arrangement whose order is even, but not from one whose order is odd. We may designate as the *reversed* *natural arrangement* that which is obtained from the natural arrangement by reversing *all* the rows, leaving the left-hand square of the lower row blank. Using the notation just employed, the natural order may be reversed by $\frac{1}{2}cr - 1$ interchanges, when c is even; and by $\frac{1}{2}(c-1)r$ interchanges, when c is odd; *i. e.* the condition for the possibility of forming the reversed natural arrangement from any given arrangement will be the same or the opposite as that for the natural order, according as the number last obtained [$\frac{1}{2}cr - 1$, if c is even; and $\frac{1}{2}(c-1)r$, if c is odd] is even or odd. Thus if $r = 4$, $c = 4$; $\frac{1}{2}cr - 1 = 7$ and the reversed arrangement can always be formed when the natural arrangement cannot, and only then. If $r = 5$, $c = 5$; $\frac{1}{2}(c-1)r = 10$, and the reversed arrangement can be formed when the natural order can be, and only then. *I. e.* with the ordinary "15" puzzle, it is always possible to arrange the numbers in the natural order with the 1 in the right-hand upper square, when it is not possible to do it with the 1 in the left-hand upper square (as Mr. Johnson has remarked in the *postscript* to his note); but on a square board with five squares on a side, a solution is possible from each of the four corners, or not at all.

There are other arrangements beside the natural and reversed natural which may be regarded as solutions of the puzzle, viz: beginning at either corner of the board, the counters may be arranged in their natural order by rows or by columns, there being in all eight such solutions.

The "15" puzzle for the last few weeks has been prominently before the American public, and may safely be said to have engaged the attention of nine out of ten persons of both sexes and of all ages and conditions of the community. But this would not have weighed with the editors to induce them to insert articles upon such a subject in the American Journal of Mathematics, but for the fact that the principle of the game has its root in what all mathematicians of the present day are aware constitutes the most subtle and characteristic conception of modern algebra, viz: the law of dichotomy applicable to the separation of the terms of every complete system of permutations into two natural and indefeasible groups, a law of the inner world of thought, which may be said to prefigure the polar relation of left and right-handed screws, or of objects in space and their reflexions in a mirror. Accordingly the editors have thought that they would be doing no disservice to their science, but rather promoting its interests by exhibiting this *à priori* polar law under a concrete form, through the medium of a game which has taken so strong a hold upon the thought of the country that it may almost be said to have risen to the importance of a national institution. Whoever has made himself master of it may fairly be said to have taken his first lesson in the theory of determinants.

It may be mentioned as a parallel case that Sir William Rowan Hamilton invented, and Jacques & Co., the purveyors of toys and conjuring tricks, in London (from whom it may possibly still be procured), sold a game called the "Eikosion" game, for illustrating certain consequences of the method of quaternions.—EDS.