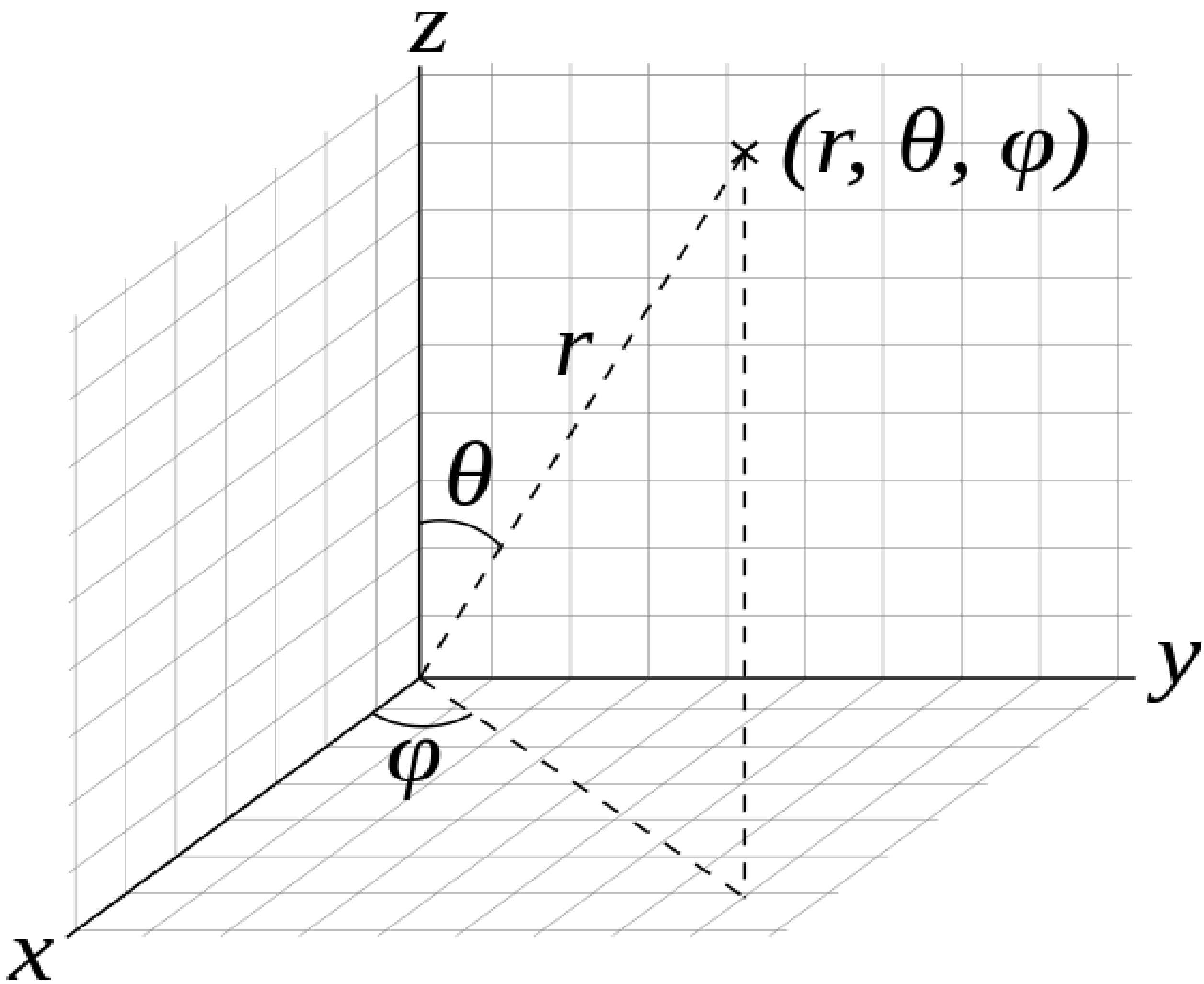
setArmPosn(r, θ , ϕ)

Derivation of Servo angle expressions to position robotic arm in 3D Space



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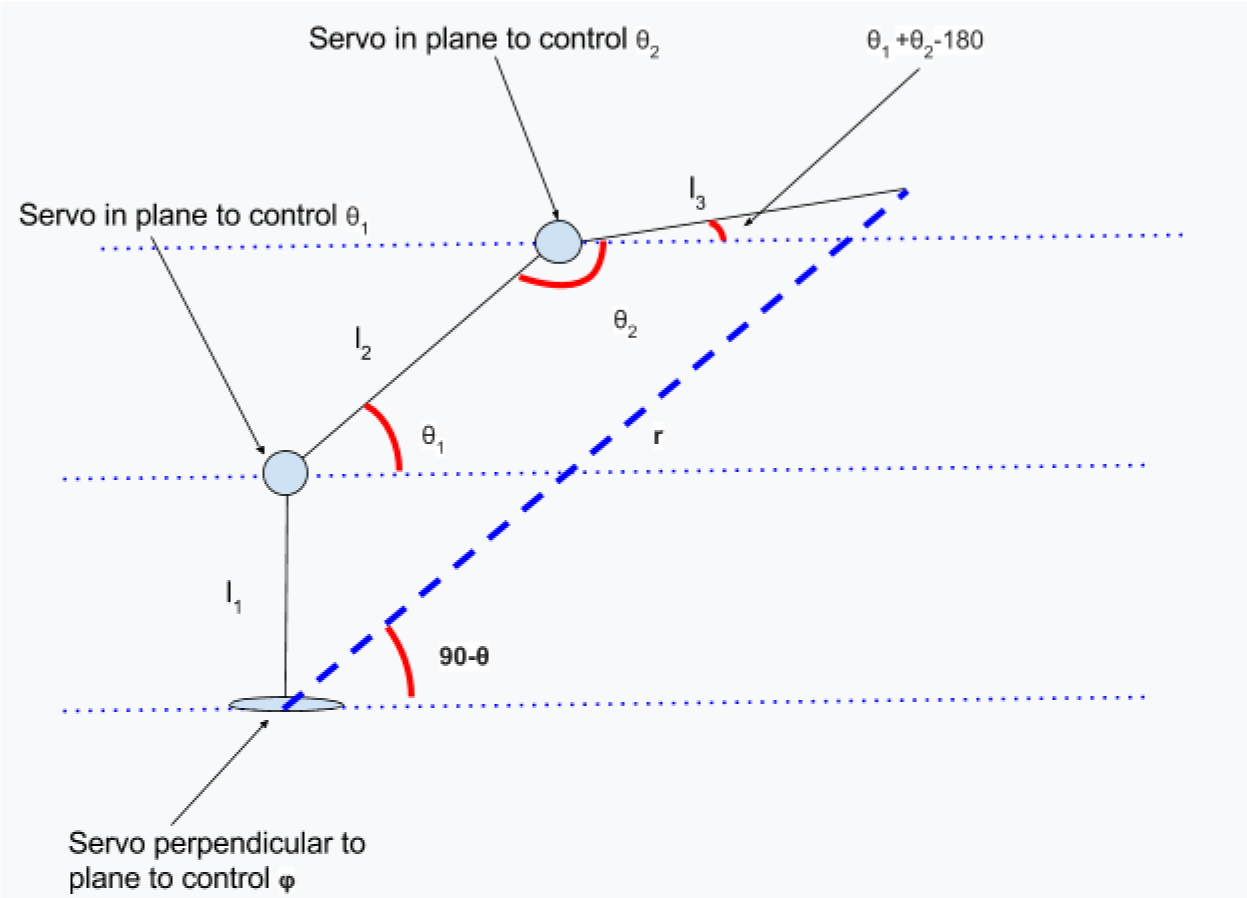
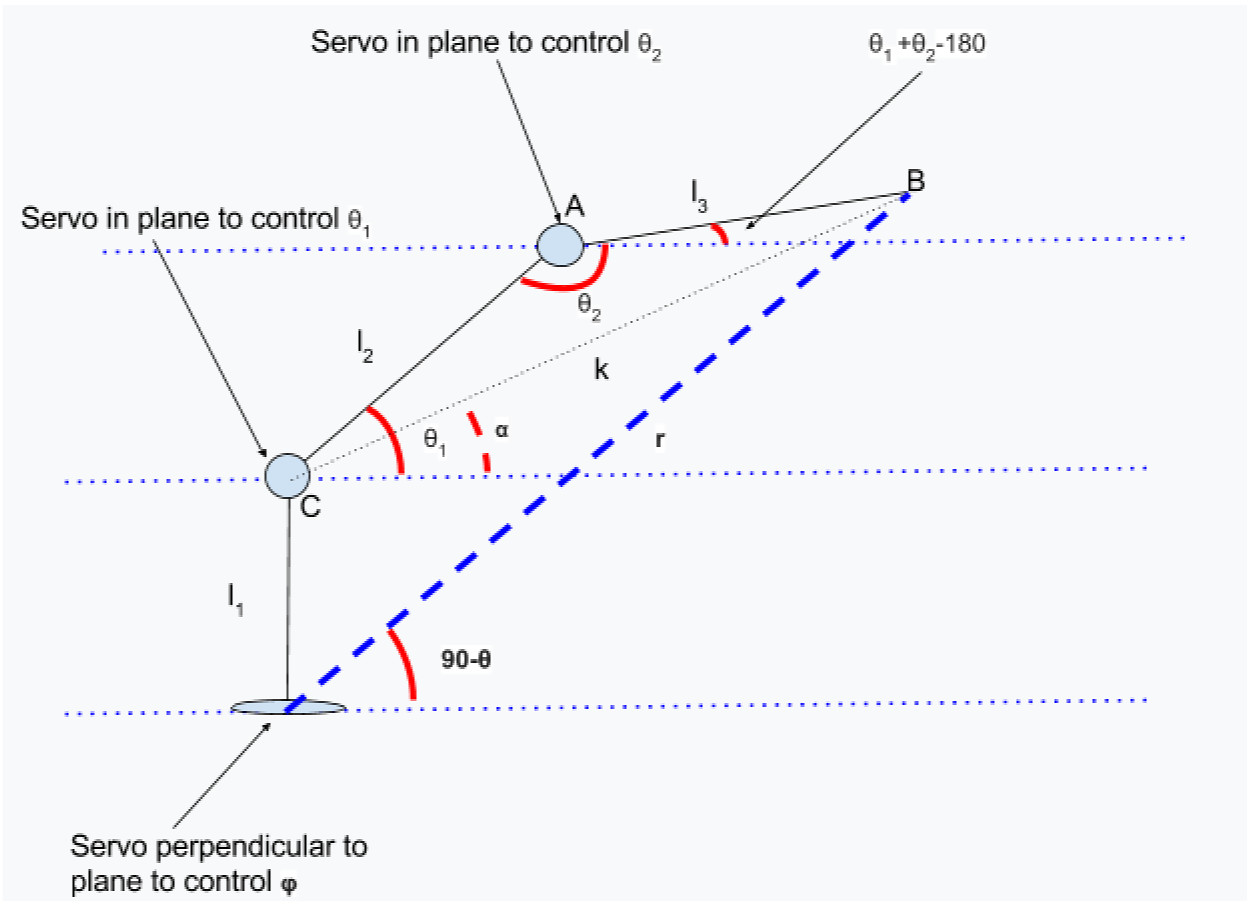
For reference , in this project I will be using the spherical coordinate system to denote points in 3D space.



2-D Representation of the Robotic Arm

Objective: *The Arduino can only control the angles at which the servos are set.I need to find the servo angles as a function of the final radial distance and polar angle to be reached by the end of the robotic arm.The azimuthal angle can be set directly using a servo.*

Derivation for θ_2



Assume a length k connecting B and C at angle α *to the horizontal.*
Applying Cosine Rule to ΔABC,

$$cos(\theta_2) = \frac{l_2^{\,2}+l_3^{\,2}-k^2}{2l_2l_3} \qquad -(1)$$

By comparing vector components :

$$kcos(\alpha) = rcos(90-\theta) = rsin(\theta)$$

$$ksin(\alpha) = rsin(90-\theta) - l_1 = rcos(\theta) - l_1$$

Squaring and adding ,

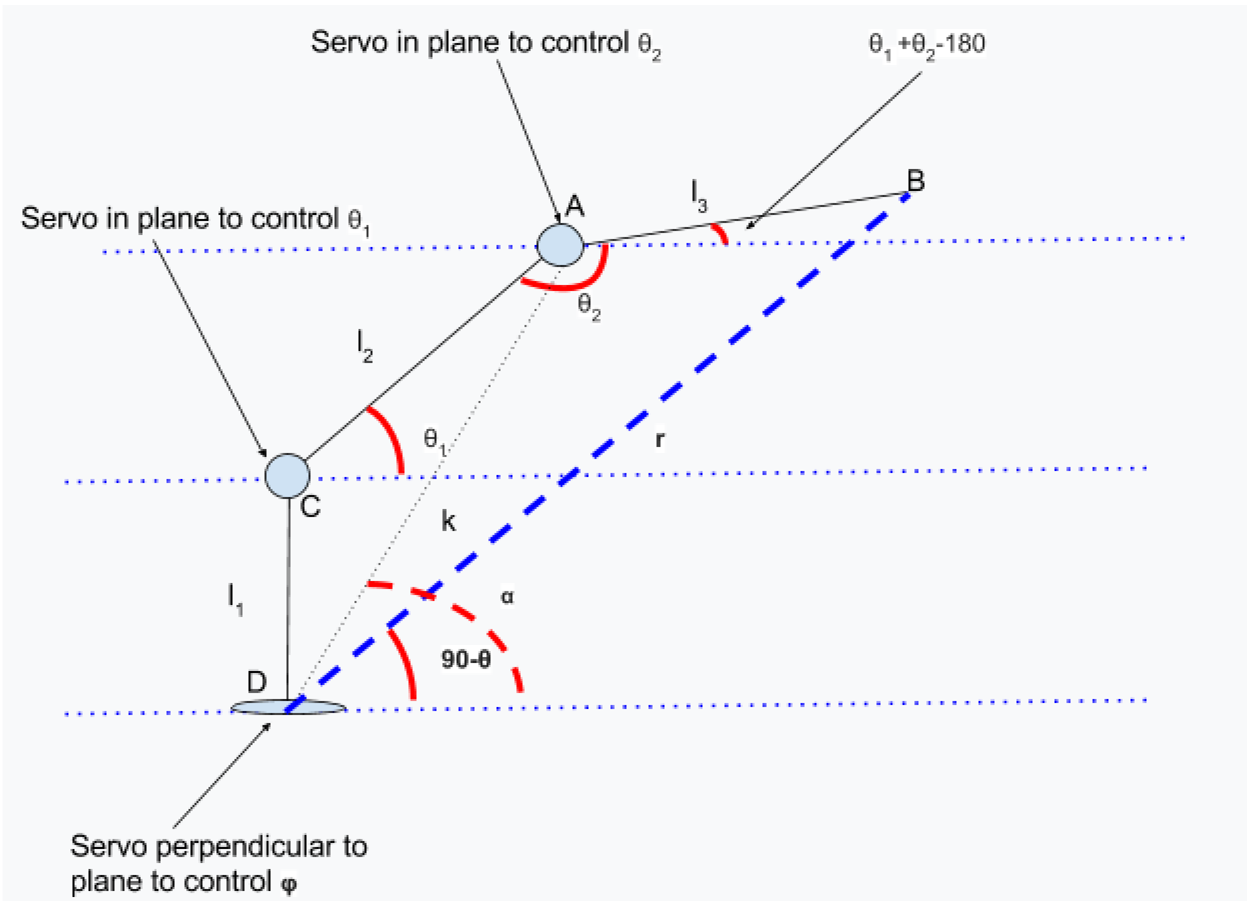
$$k^2 = r^2 + {l_1}^2 - 2rl_1 cos(\theta) \quad -(2)$$

Substituting back in (1),

$$cos(\theta_2) = \frac{{l_2}^2+{l_3}^2-{l_1}^2-r^2+2rl_1 cos(\theta)}{2l_2 l_3}$$

$$\theta_2 = cos^{-1}\left(\frac{{l_2}^2+{l_3}^2-{l_1}^2-r^2+2rl_1 cos(\theta)}{2l_2 l_3}\right)$$

Derivation for θ_1



Assume a length k connecting A and D at angle α *to the horizontal.*
Applying Cosine Rule to ΔACD,

$$cos(\theta_1 + 90) = \frac{{l_1}^2 + {l_2}^2 - k^2}{2 l_1 l_2}$$

$$-\,sin(\theta_1) = \frac{{l_1}^2 + {l_2}^2 - k^2}{2 l_1 l_2} \qquad -\,(1)$$

By comparing vector components :

$$kcos(\alpha) = rcos(90 - \theta) - l_3cos(\theta_1 + \theta_2 - 180) = rsin(\theta) + l_3cos(\theta_1 + \theta_2)$$
$$ksin(\alpha) = rsin(90 - \theta) - l_3sin(\theta_1 + \theta_2 - 180) = rcos(\theta) + l_3sin(\theta_1 + \theta_2)$$

Squaring and adding ,

$$k^2 = r^2 + {l_3}^2 + 2rl_3[sin(\theta)cos(\theta_1 + \theta_2) + cos(\theta)sin(\theta_1 + \theta_2)]$$
$$k^2 = r^2 + {l_3}^2 + 2rl_3sin(\theta + \theta_1 + \theta_2)$$

Substituting back in (1),

$$- sin(\theta_1) = \frac{{l_1}^2+{l_2}^2-[r^2+{l_3}^2+2rl_3sin(\theta+\theta_1+\theta_2)]}{2l_1l_2}$$
$$- 2l_1l_2sin(\theta_1) + [\ r^2 + {l_3}^2 + 2rl_3sin(\theta + \theta_1 + \theta_2)] = {l_1}^2 + {l_2}^2$$
$$- 2l_1l_2sin(\theta_1) + 2rl_3sin(\theta + \theta_1 + \theta_2) = {l_1}^2 + {l_2}^2 - {l_3}^2 - r^2$$
$$2\{rl_3[sin(\theta_1)cos(\theta + \theta_2) + cos(\theta_1)sin(\theta + \theta_2)] - l_1l_2sin(\theta_1)\} = {l_1}^2 + {l_2}^2 - {l_3}^2 - r^2$$
$$2\{sin(\theta_1)[rl_3cos(\theta\ + \theta_2) - l_1l_2] + cos(\theta_1)[rl_3sin(\theta + \theta_2)]\} = {l_1}^2 + {l_2}^2 - {l_3}^2 - r^2$$

$$sin(\theta_1)\ \frac{rl_3cos(\theta +\theta_2)-l_1l_2}{\sqrt{[rl_3cos(\theta +\theta_2)-l_1l_2]^2+[rl_3sin(\theta+\theta_2)]^2}}\ +$$
$$cos(\theta_1)\ \frac{rl_3sin(\theta+\theta_2)}{\sqrt{[rl_3cos(\theta +\theta_2)-l_1l_2]^2+[rl_3sin(\theta+\theta_2)]^2}}$$
$$= \frac{{l_1}^2+{l_2}^2-{l_3}^2-r^2}{2\sqrt{[rl_3cos(\theta +\theta_2)-l_1l_2]^2+[rl_3sin(\theta+\theta_2)]^2}}$$

$$sin(\theta_1)\ \frac{rl_3cos(\theta+\theta_2)-l_1l_2}{\sqrt{l_1^{\ 2}l_2^{\ 2}+r^2l_3^{\ 2}-2rl_1l_2l_3cos(\theta+\theta_2)}}+$$
$$cos(\theta_1)\ \frac{rl_3sin(\theta+\theta_2)}{\sqrt{l_1^{\ 2}l_2^{\ 2}+r^2l_3^{\ 2}-2rl_1l_2l_3cos(\theta+\theta_2)}}$$
$$=\frac{l_1^{\ 2}+l_2^{\ 2}-l_3^{\ 2}-r^2}{2\sqrt{l_1^{\ 2}l_2^{\ 2}+r^2l_3^{\ 2}-2rl_1l_2l_3cos(\theta+\theta_2)}}$$

To bring the solutions within the range of servo , multiply whole equation by -1 .

$$sin(\theta_1)\ \frac{l_1l_2-rl_3cos(\theta+\theta_2)}{\sqrt{l_1^{\ 2}l_2^{\ 2}+r^2l_3^{\ 2}-2rl_1l_2l_3cos(\theta+\theta_2)}}-$$
$$cos(\theta_1)\ \frac{rl_3sin(\theta+\theta_2)}{\sqrt{l_1^{\ 2}l_2^{\ 2}+r^2l_3^{\ 2}-2rl_1l_2l_3cos(\theta+\theta_2)}}$$
$$=\frac{l_3^{\ 2}+r^2-l_1^{\ 2}-l_2^{\ 2}}{2\sqrt{l_1^{\ 2}l_2^{\ 2}+r^2l_3^{\ 2}-2rl_1l_2l_3cos(\theta+\theta_2)}}$$

$$sin[\theta_1-sin^{-1}\left(\frac{rl_3sin(\theta+\theta_2)}{\sqrt{l_1^{\ 2}l_2^{\ 2}+r^2l_3^{\ 2}-2rl_1l_2l_3cos(\theta+\theta_2)}}\right)]$$
$$=\frac{l_3^{\ 2}+r^2-l_1^{\ 2}-l_2^{\ 2}}{2\sqrt{l_1^{\ 2}l_2^{\ 2}+r^2l_3^{\ 2}-2rl_1l_2l_3cos(\theta+\theta_2)}}$$

$$\theta_1 =$$

$$sin^{-1}\left(\frac{{l_3}^2+r^2-{l_1}^2-{l_2}^2}{2\sqrt{{l_1}^2{l_2}^2+r^2{l_3}^2-2rl_1l_2l_3cos(\theta+\theta_2)}}\right)+sin^{-1}\left(\frac{rl_3sin(\theta+\theta_2)}{\sqrt{{l_1}^2{l_2}^2+r^2{l_3}^2-2rl_1l_2l_3cos(\theta+\theta_2)}}\right)$$

(substitute the calculated value of θ_2 from previous expression here)

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