setArmPosn(r, θ, ϕ)
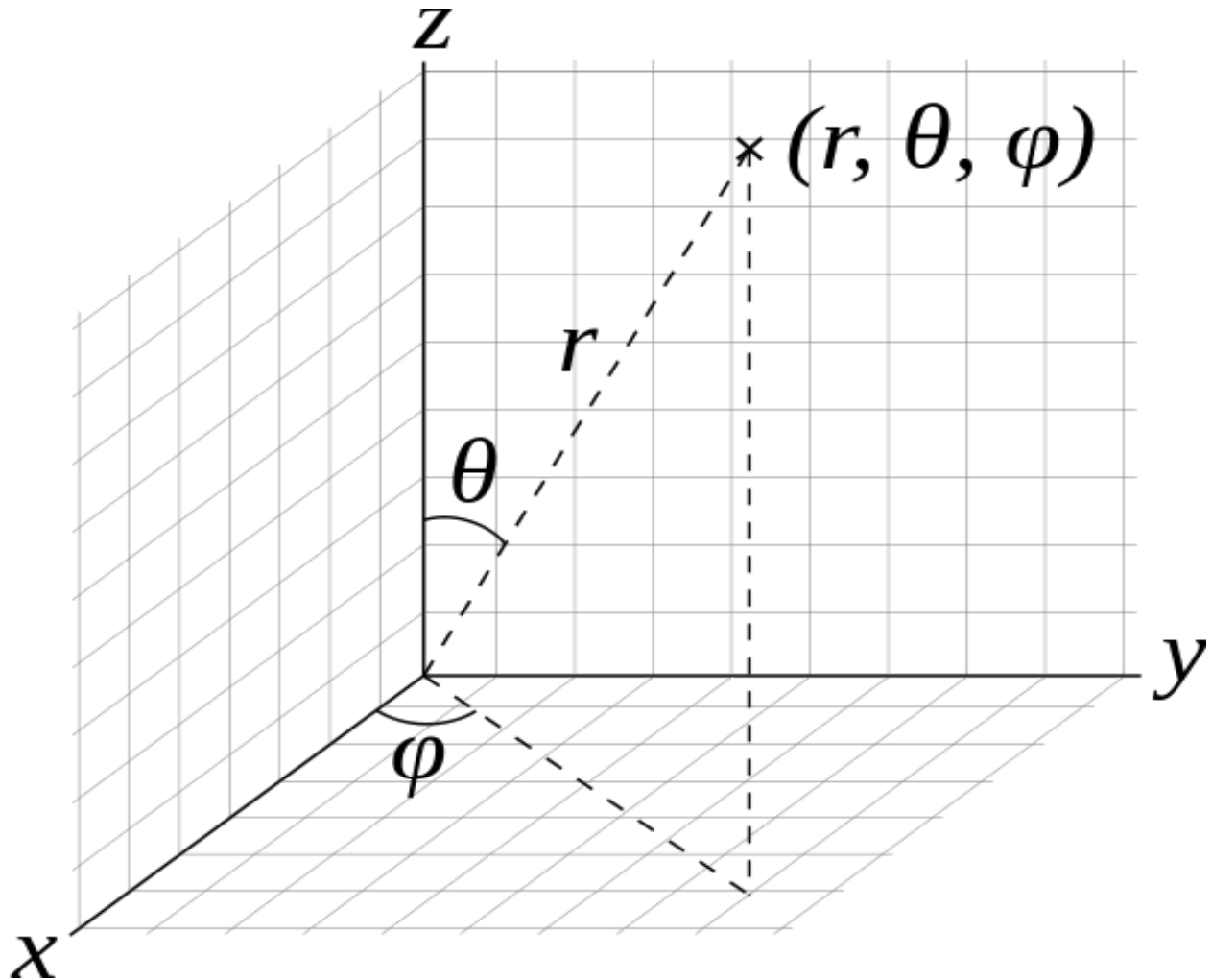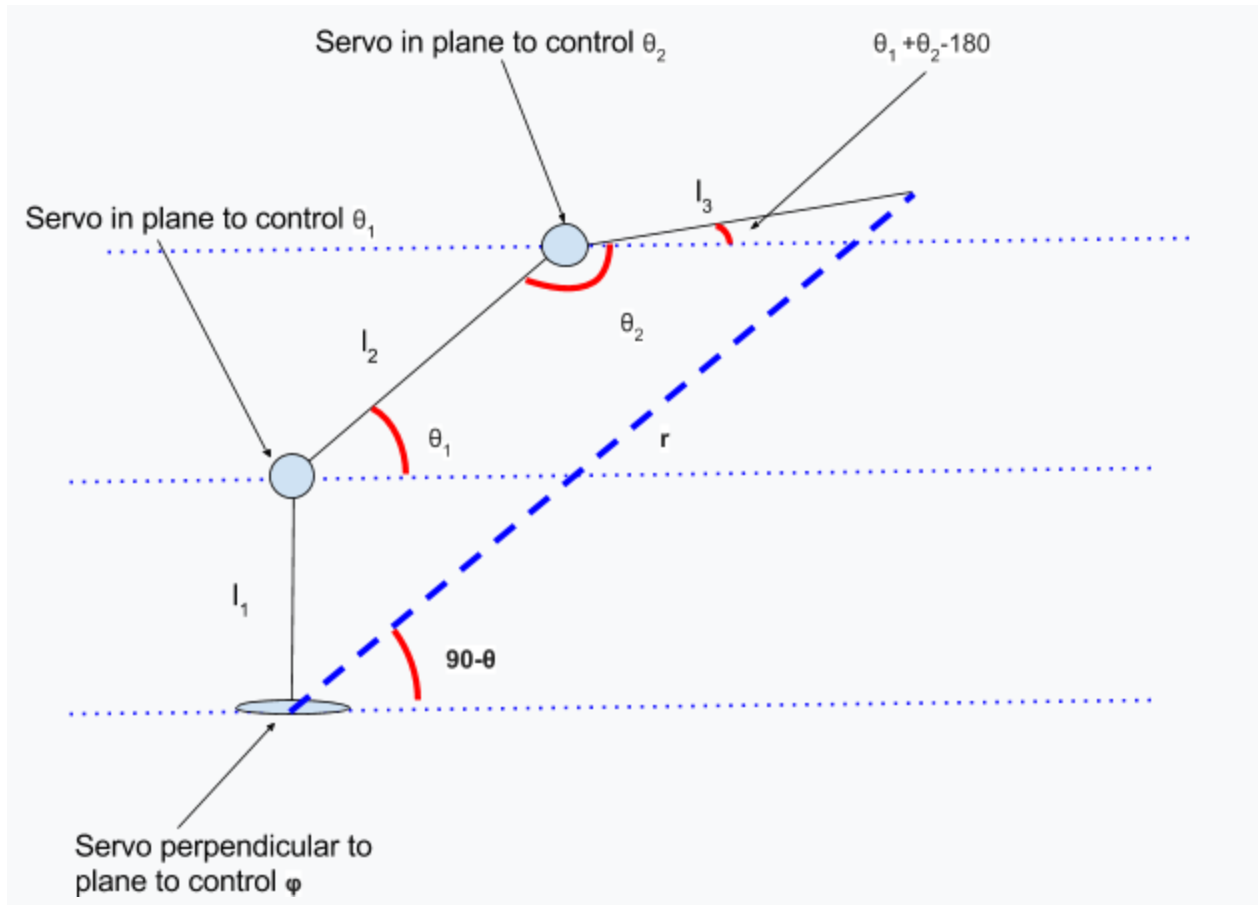
Derivation of Servo angle expressions to position robotic arm in 3D Space

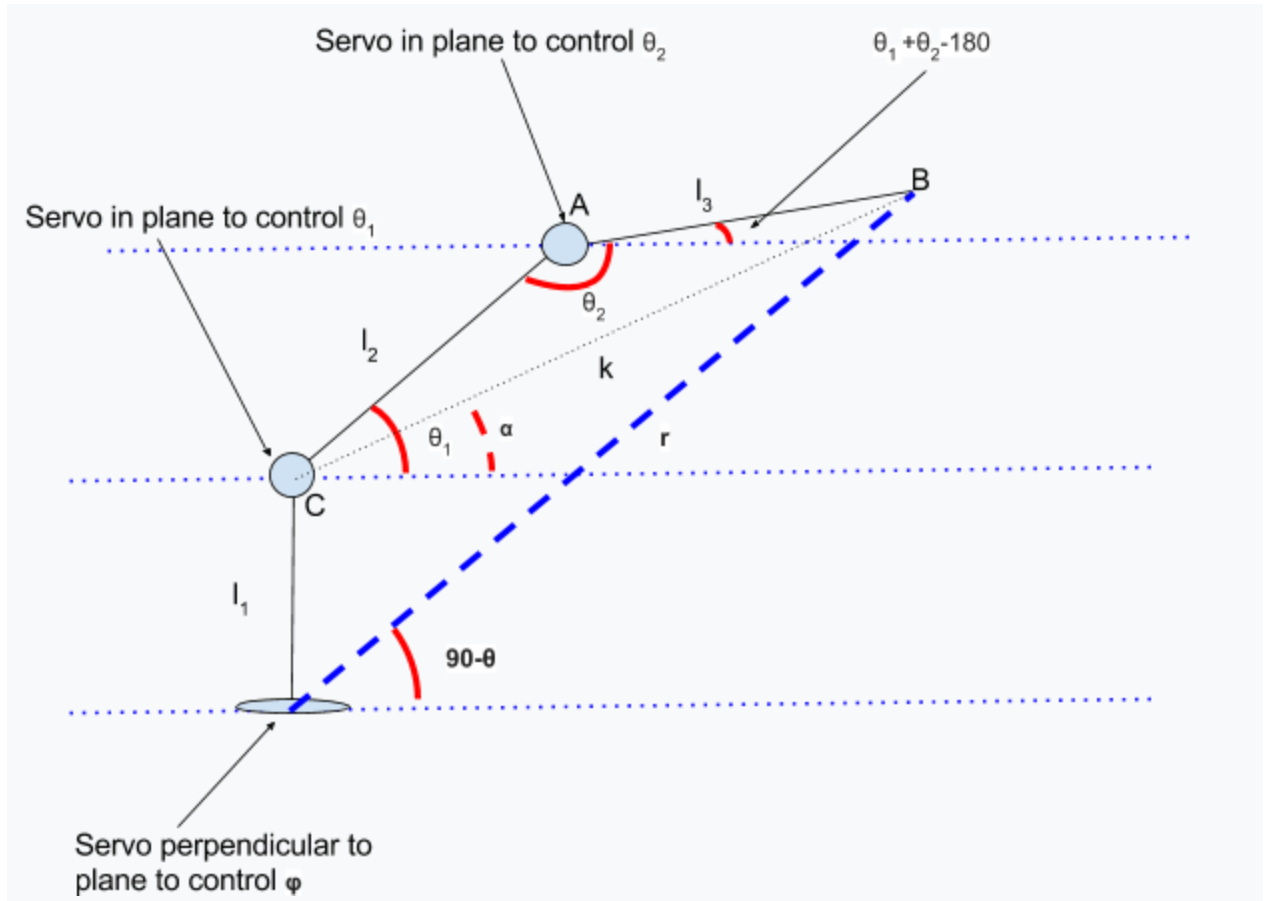**Ankur Dubey**

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For reference , in this project I will be using the spherical coordinate system to denote points in 3D space.

2-D Representation of the Robotic Arm

Objective: *The Arduino can only control the angles at which the servos are set.I need to find the servo angles as a function of the final radial distance and polar angle to be reached by the end of the robotic arm.The azimuthal angle can be set directly using a servo.*

Derivation for θ_2

Servo in plane to control θ_2

$\theta_1 + \theta_2 - 180$

Servo in plane to control θ_1

l_3

B

A

θ_2

l_2

k

θ_1 α

r

C

l_1

90-θ

Servo perpendicular to
plane to control φ

Assume a length k connecting B and C at angle α to the horizontal.
Applying Cosine Rule to ΔABC,

$$cos(\theta_2) = \frac{l_2{}^2 + l_3{}^2 - k^2}{2l_2 l_3} \quad -(1)$$

By comparing vector components :

$$kcos(\alpha) = rcos(90 - \theta) = rsin(\theta)$$

$$ksin(\alpha) = rsin(90 - \theta) - l_1 = rcos(\theta) - l_1$$

3

Squaring and adding ,
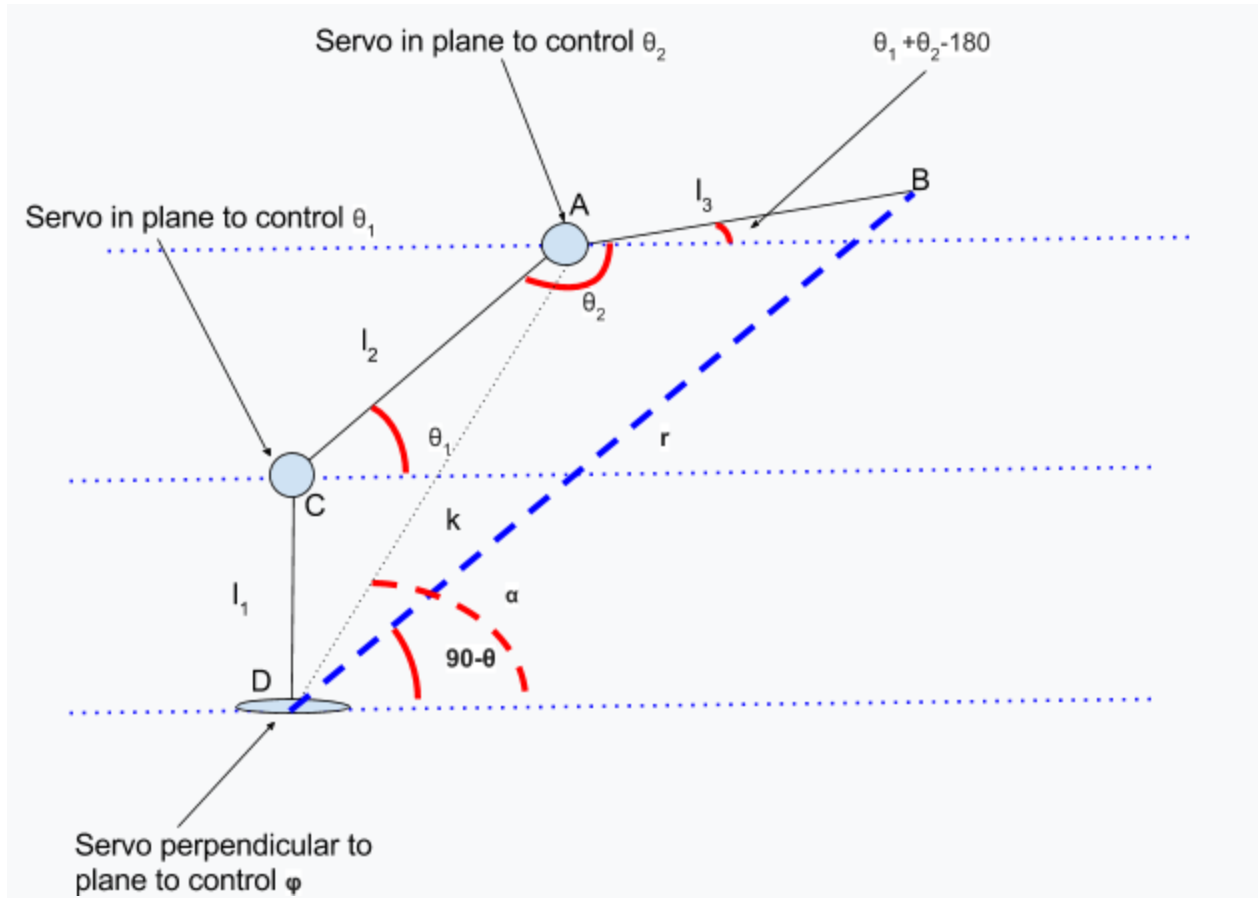
$$k^2 = r^2 + l_1{}^2 - 2rl_1\cos(\theta) \quad -(2)$$

Substituting back in (1),

$$\cos(\theta_2) = \frac{l_2{}^2 + l_3{}^2 - l_1{}^2 - r^2 + 2rl_1\cos(\theta)}{2l_2 l_3}$$

$$\theta_2 = \cos^{-1}\left(\frac{l_2{}^2 + l_3{}^2 - l_1{}^2 - r^2 + 2rl_1\cos(\theta)}{2l_2 l_3}\right)$$

Derivation for θ_1

Assume a length k connecting A and D at angle α to the horizontal.
Applying Cosine Rule to ΔACD,

$$cos(\theta_1 + 90) = \frac{l_1{}^2 + l_2{}^2 - k^2}{2l_1 l_2}$$

$$-sin(\theta_1) = \frac{l_1{}^2 + l_2{}^2 - k^2}{2l_1 l_2} \qquad -(1)$$

By comparing vector components :

$$k\cos(\alpha) = r\cos(90 - \theta) - l_3\cos(\theta_1 + \theta_2 - 180) = r\sin(\theta) + l_3\cos(\theta_1 + \theta_2)$$
$$k\sin(\alpha) = r\sin(90 - \theta) - l_3\sin(\theta_1 + \theta_2 - 180) = r\cos(\theta) + l_3\sin(\theta_1 + \theta_2)$$

Squaring and adding ,

$$k^2 = r^2 + l_3{}^2 + 2rl_3[\sin(\theta)\cos(\theta_1 + \theta_2) + \cos(\theta)\sin(\theta_1 + \theta_2)]$$
$$k^2 = r^2 + l_3{}^2 + 2rl_3\sin(\theta + \theta_1 + \theta_2)$$

Substituting back in (1),

$$-\sin(\theta_1) = \frac{l_1{}^2 + l_2{}^2 - [r^2 + l_3{}^2 + 2rl_3\sin(\theta+\theta_1+\theta_2)]}{2l_1 l_2}$$

$$-2l_1 l_2\sin(\theta_1) + [\ r^2 + l_3{}^2 + 2rl_3\sin(\theta + \theta_1 + \theta_2)] = l_1{}^2 + l_2{}^2$$

$$-2l_1 l_2\sin(\theta_1) + 2rl_3\sin(\theta + \theta_1 + \theta_2) = l_1{}^2 + l_2{}^2 - l_3{}^2 - r^2$$

$$2\{rl_3[\sin(\theta_1)\cos(\theta + \theta_2) + \cos(\theta_1)\sin(\theta + \theta_2)] - l_1 l_2\sin(\theta_1)\} = l_1{}^2 + l_2{}^2 - l_3{}^2 - r^2$$

$$2\{\sin(\theta_1)[rl_3\cos(\theta + \theta_2) - l_1 l_2] + \cos(\theta_1)[rl_3\sin(\theta + \theta_2)]\} = l_1{}^2 + l_2{}^2 - l_3{}^2 - r^2$$

$$\sin(\theta_1)\ \frac{rl_3\cos(\theta +\theta_2) - l_1 l_2}{\sqrt{[rl_3\cos(\theta +\theta_2) - l_1 l_2]^2 + [rl_3\sin(\theta+\theta_2)]^2}}\ +$$

$$\cos(\theta_1)\ \frac{rl_3\sin(\theta+\theta_2)}{\sqrt{[rl_3\cos(\theta +\theta_2) - l_1 l_2]^2 + [rl_3\sin(\theta+\theta_2)]^2}}$$

$$= \frac{l_1{}^2 + l_2{}^2 - l_3{}^2 - r^2}{2\sqrt{[rl_3\cos(\theta +\theta_2) - l_1 l_2]^2 + [rl_3\sin(\theta+\theta_2)]^2}}$$

$$sin(\theta_1)\ \frac{rl_3cos(\theta+\theta_2)-l_1l_2}{\sqrt{l_1^2l_2^2+r^2l_3^2-2rl_1l_2l_3cos(\theta+\theta_2)}}\ +$$

$$cos(\theta_1)\ \frac{rl_3sin(\theta+\theta_2)}{\sqrt{l_1^2l_2^2+r^2l_3^2-2rl_1l_2l_3cos(\theta+\theta_2)}}$$

$$=\ \frac{l_1^2+l_2^2-l_3^2-r^2}{2\sqrt{l_1^2l_2^2+r^2l_3^2-2rl_1l_2l_3cos(\theta+\theta_2)}}$$

To bring the solutions within the range of servo , multiply whole equation by -1 .

$$sin(\theta_1)\ \frac{l_1l_2-rl_3cos(\theta+\theta_2)}{\sqrt{l_1^2l_2^2+r^2l_3^2-2rl_1l_2l_3cos(\theta+\theta_2)}}\ -$$

$$cos(\theta_1)\ \frac{rl_3sin(\theta+\theta_2)}{\sqrt{l_1^2l_2^2+r^2l_3^2-2rl_1l_2l_3cos(\theta+\theta_2)}}$$

$$=\ \frac{l_3^2+r^2-l_1^2-l_2^2}{2\sqrt{l_1^2l_2^2+r^2l_3^2-2rl_1l_2l_3cos(\theta+\theta_2)}}$$

$$sin[\theta_1-sin^{-1}\ (\frac{rl_3sin(\theta+\theta_2)}{\sqrt{l_1^2l_2^2+r^2l_3^2-2rl_1l_2l_3cos(\theta+\theta_2)}})\ ]$$

$$=\ \frac{l_3^2+r^2-l_1^2-l_2^2}{2\sqrt{l_1^2l_2^2+r^2l_3^2-2rl_1l_2l_3cos(\theta+\theta_2)}}$$

$$\theta_1 =$$

$$sin^{-1}\left(\frac{l_3^2+r^2-l_1^2-l_2^2}{2\sqrt{l_1^2l_2^2+r^2l_3^2-2rl_1l_2l_3cos(\theta+\theta_2)}}\right) + sin^{-1}\left(\frac{rl_3sin(\theta+\theta_2)}{\sqrt{l_1^2l_2^2+r^2l_3^2-2rl_1l_2l_3cos(\theta+\theta_2)}}\right)$$

(substitute the calculated value of θ_2 from previous exression here)

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