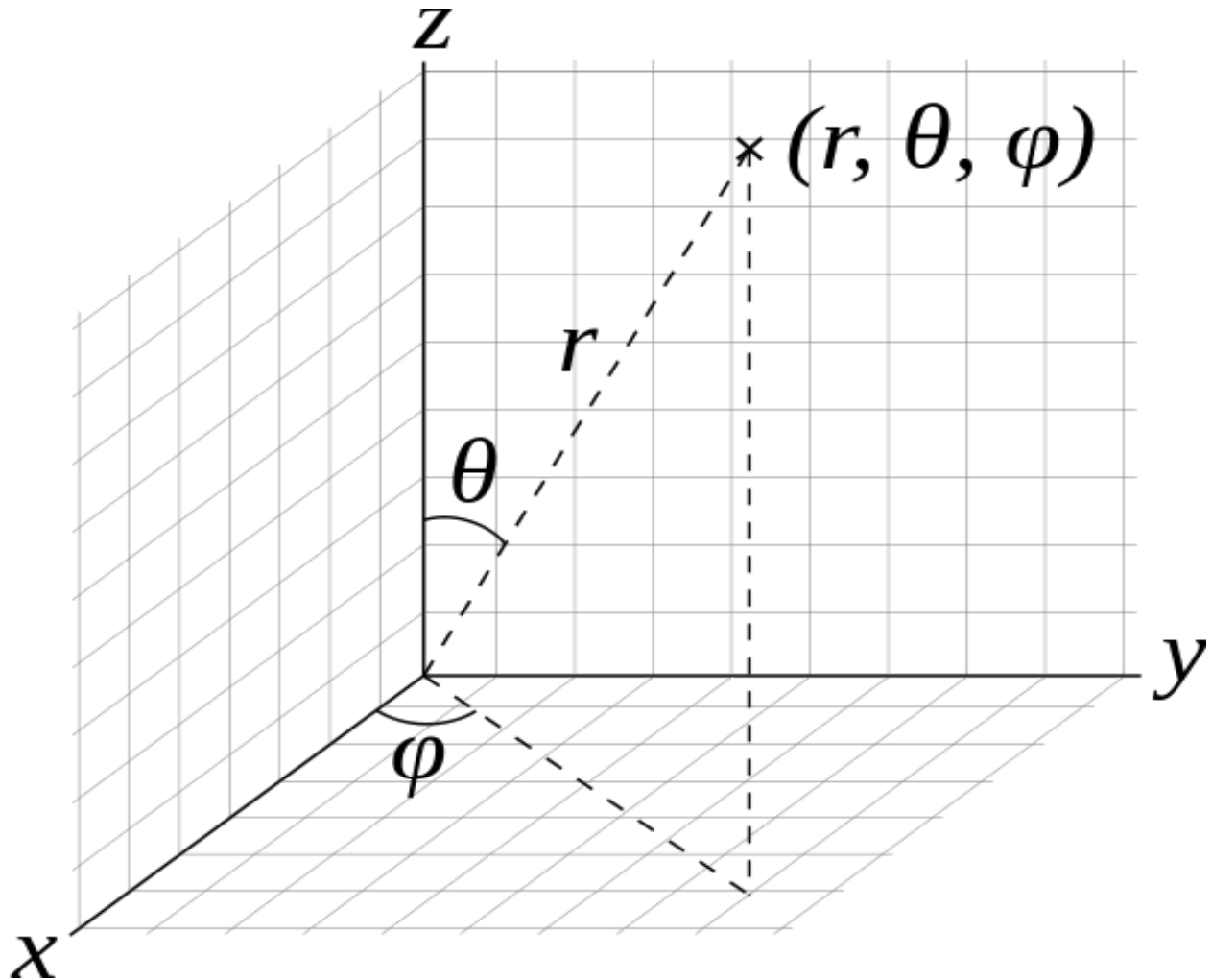


setArmPosn(r, θ, ϕ)

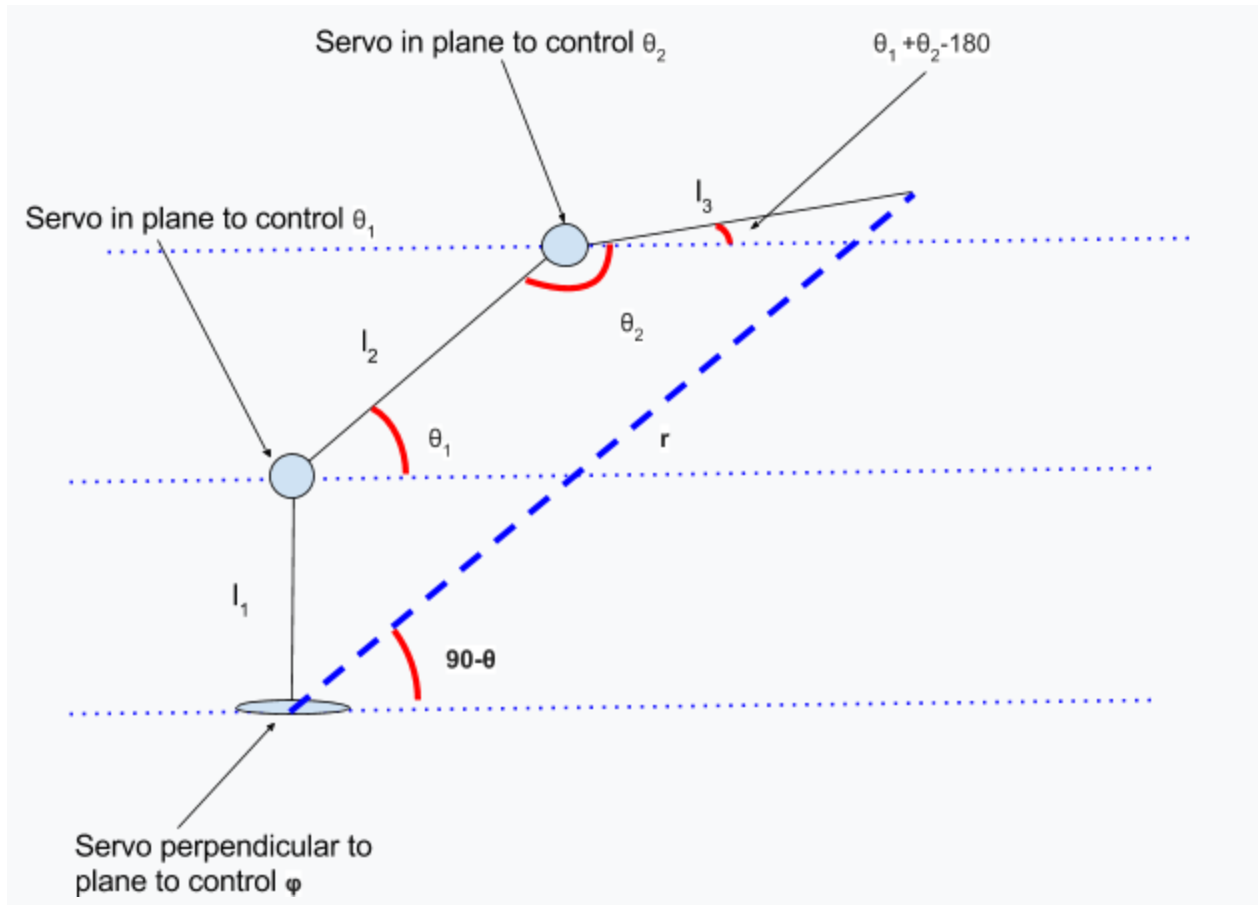
Derivation of Servo angle expressions to position robotic arm in 3D Space



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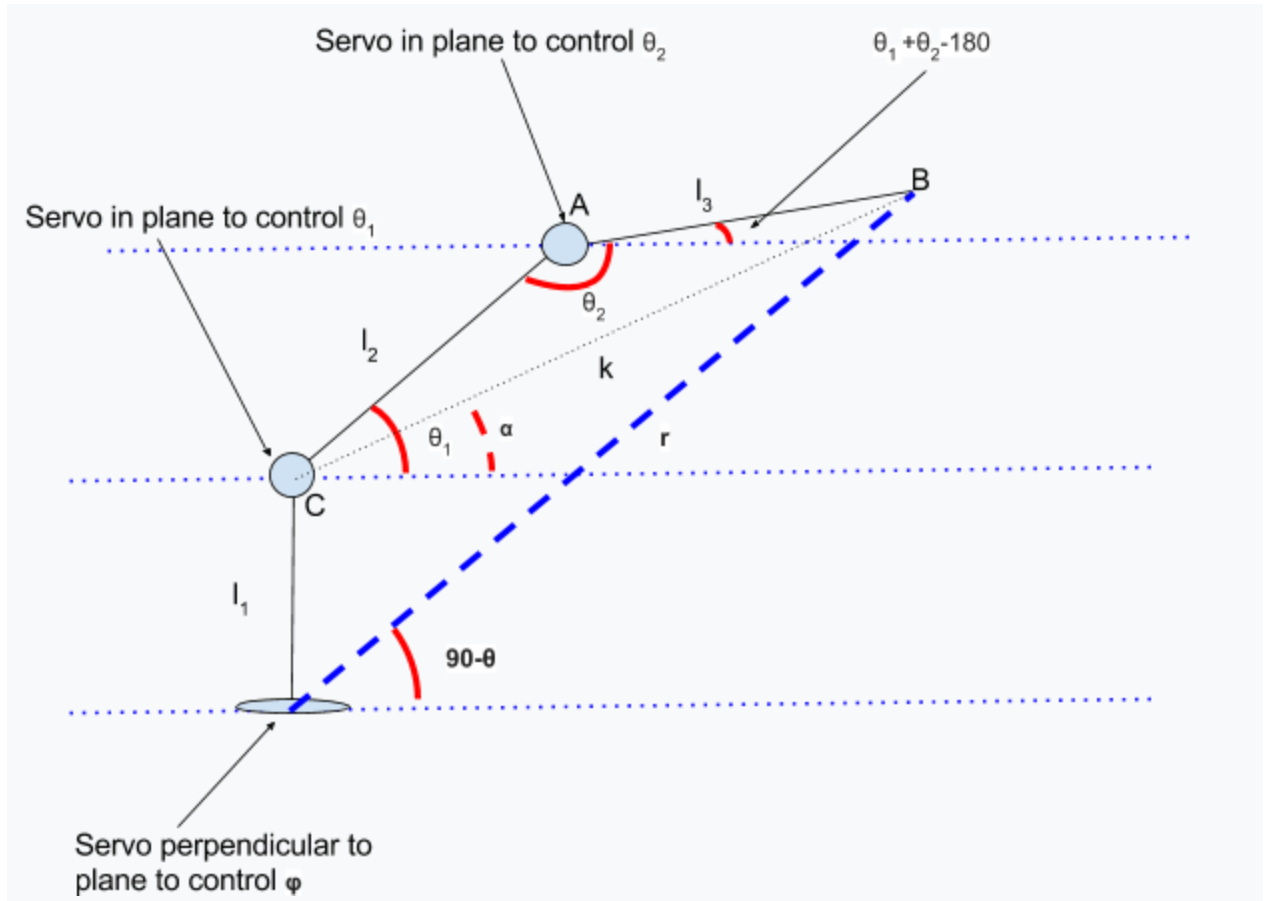
For reference , in this project I will be using the spherical coordinate system to denote points in 3D space.



2-D Representation of the Robotic Arm

Objective: The Arduino can only control the angles at which the servos are set. I need to find the servo angles as a function of the final radial distance and polar angle to be reached by the end of the robotic arm. The azimuthal angle can be set directly using a servo.

Derivation for θ_2



Assume a length k connecting B and C at angle α to the horizontal.

Applying Cosine Rule to ΔABC ,

$$\cos(\theta_2) = \frac{l_2^2 + l_3^2 - k^2}{2l_2l_3} \quad - (1)$$

By comparing vector components :

$$k \cos(\alpha) = r \cos(90 - \theta) = r \sin(\theta)$$

$$k \sin(\alpha) = r \sin(90 - \theta) - l_1 = r \cos(\theta) - l_1$$

Squaring and adding ,

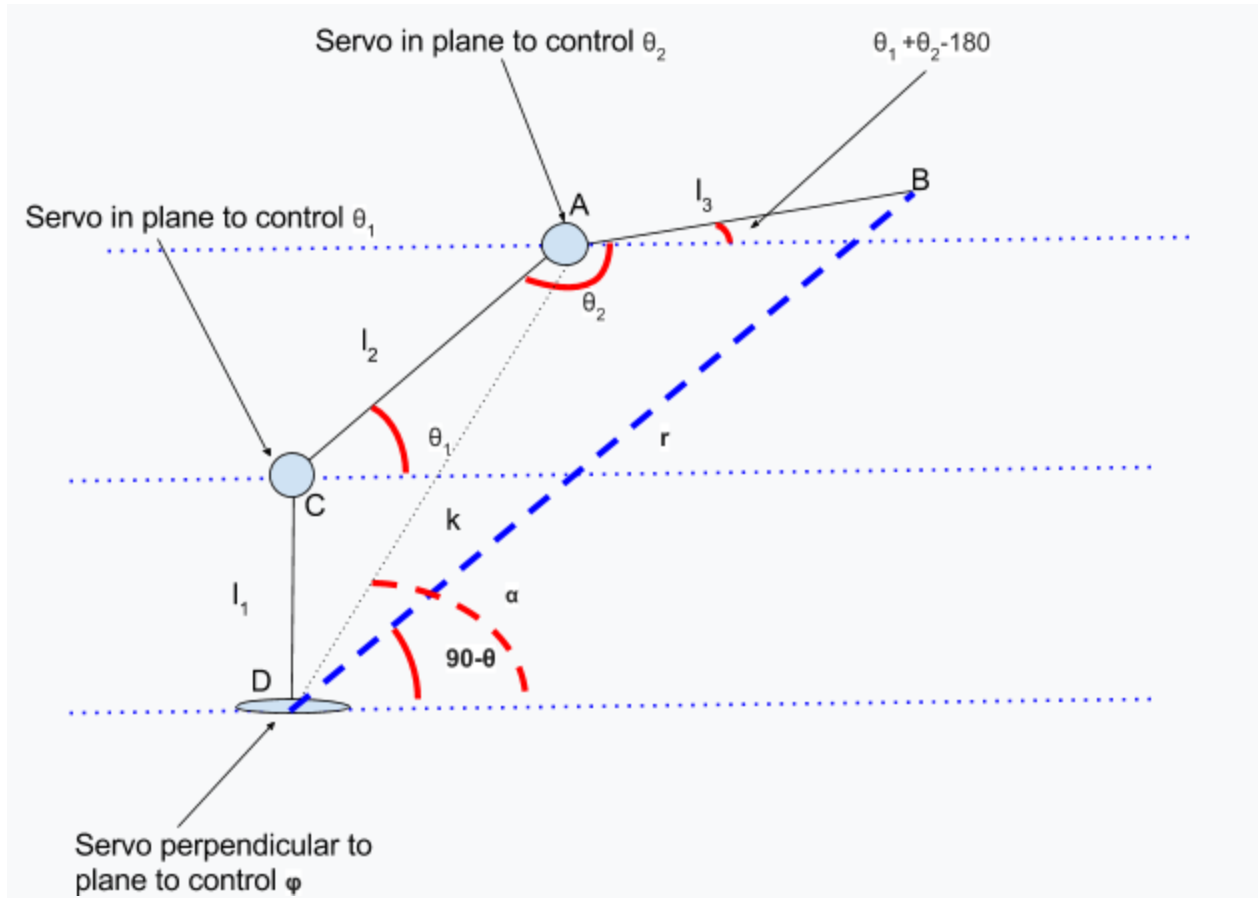
$$k^2 = r^2 + l_1^2 - 2rl_1\cos(\theta) \quad - (2)$$

Substituting back in (1),

$$\cos(\theta_2) = \frac{l_2^2 + l_3^2 - l_1^2 - r^2 + 2rl_1\cos(\theta)}{2l_2l_3}$$

$$\theta_2 = \cos^{-1}\left(\frac{l_2^2 + l_3^2 - l_1^2 - r^2 + 2rl_1\cos(\theta)}{2l_2l_3}\right)$$

Derivation for θ_1



Assume a length k connecting A and D at angle α to the horizontal.

Applying Cosine Rule to $\triangle ACD$,

$$\cos(\theta_1 + 90) = \frac{l_1^2 + l_2^2 - k^2}{2l_1l_2}$$

$$-\sin(\theta_1) = \frac{l_1^2 + l_2^2 - k^2}{2l_1l_2} \quad - (1)$$

By comparing vector components :

$$k\cos(\alpha) = r\cos(90 - \theta) - l_3\cos(\theta_1 + \theta_2 - 180) = r\sin(\theta) + l_3\cos(\theta_1 + \theta_2)$$

$$k\sin(\alpha) = r\sin(90 - \theta) - l_3\sin(\theta_1 + \theta_2 - 180) = r\cos(\theta) + l_3\sin(\theta_1 + \theta_2)$$

Squaring and adding ,

$$k^2 = r^2 + l_3^2 + 2rl_3[\sin(\theta)\cos(\theta_1 + \theta_2) + \cos(\theta)\sin(\theta_1 + \theta_2)]$$

$$k^2 = r^2 + l_3^2 + 2rl_3\sin(\theta + \theta_1 + \theta_2)$$

Substituting back in (1),

$$-\sin(\theta_1) = \frac{l_1^2 + l_2^2 - [r^2 + l_3^2 + 2rl_3\sin(\theta + \theta_1 + \theta_2)]}{2l_1l_2}$$

$$-2l_1l_2\sin(\theta_1) + [r^2 + l_3^2 + 2rl_3\sin(\theta + \theta_1 + \theta_2)] = l_1^2 + l_2^2$$

$$-2l_1l_2\sin(\theta_1) + 2rl_3\sin(\theta + \theta_1 + \theta_2) = l_1^2 + l_2^2 - l_3^2 - r^2$$

$$2\{rl_3[\sin(\theta_1)\cos(\theta + \theta_2) + \cos(\theta_1)\sin(\theta + \theta_2)] - l_1l_2\sin(\theta_1)\} = l_1^2 + l_2^2 - l_3^2 - r^2$$

$$2\{\sin(\theta_1)[rl_3\cos(\theta + \theta_2) - l_1l_2] + \cos(\theta_1)[rl_3\sin(\theta + \theta_2)]\} = l_1^2 + l_2^2 - l_3^2 - r^2$$

$$\sin(\theta_1) \frac{rl_3\cos(\theta + \theta_2) - l_1l_2}{\sqrt{[rl_3\cos(\theta + \theta_2) - l_1l_2]^2 + [rl_3\sin(\theta + \theta_2)]^2}} +$$

$$\cos(\theta_1) \frac{rl_3\sin(\theta + \theta_2)}{\sqrt{[rl_3\cos(\theta + \theta_2) - l_1l_2]^2 + [rl_3\sin(\theta + \theta_2)]^2}}$$

$$= \frac{l_1^2 + l_2^2 - l_3^2 - r^2}{2\sqrt{[rl_3\cos(\theta + \theta_2) - l_1l_2]^2 + [rl_3\sin(\theta + \theta_2)]^2}}$$

$$\begin{aligned}
& \sin(\theta_1) \frac{rl_3 \cos(\theta + \theta_2) - l_1 l_2}{\sqrt{l_1^2 l_2^2 + r^2 l_3^2 - 2rl_1 l_2 l_3 \cos(\theta + \theta_2)}} + \\
& \cos(\theta_1) \frac{rl_3 \sin(\theta + \theta_2)}{\sqrt{l_1^2 l_2^2 + r^2 l_3^2 - 2rl_1 l_2 l_3 \cos(\theta + \theta_2)}} \\
& = \frac{l_1^2 + l_2^2 - l_3^2 - r^2}{2\sqrt{l_1^2 l_2^2 + r^2 l_3^2 - 2rl_1 l_2 l_3 \cos(\theta + \theta_2)}}
\end{aligned}$$

To bring the solutions within the range of servo , multiply whole equation by -1 .

$$\begin{aligned}
& \sin(\theta_1) \frac{l_1 l_2 - rl_3 \cos(\theta + \theta_2)}{\sqrt{l_1^2 l_2^2 + r^2 l_3^2 - 2rl_1 l_2 l_3 \cos(\theta + \theta_2)}} - \\
& \cos(\theta_1) \frac{rl_3 \sin(\theta + \theta_2)}{\sqrt{l_1^2 l_2^2 + r^2 l_3^2 - 2rl_1 l_2 l_3 \cos(\theta + \theta_2)}} \\
& = \frac{l_3^2 + r^2 - l_1^2 - l_2^2}{2\sqrt{l_1^2 l_2^2 + r^2 l_3^2 - 2rl_1 l_2 l_3 \cos(\theta + \theta_2)}}
\end{aligned}$$

$$\begin{aligned}
& \sin[\theta_1 - \sin^{-1} \left(\frac{rl_3 \sin(\theta + \theta_2)}{\sqrt{l_1^2 l_2^2 + r^2 l_3^2 - 2rl_1 l_2 l_3 \cos(\theta + \theta_2)}} \right)] \\
& = \frac{l_3^2 + r^2 - l_1^2 - l_2^2}{2\sqrt{l_1^2 l_2^2 + r^2 l_3^2 - 2rl_1 l_2 l_3 \cos(\theta + \theta_2)}}
\end{aligned}$$

$$\theta_1 = \sin^{-1}\left(\frac{l_3^2 + r^2 - l_1^2 - l_2^2}{2\sqrt{l_1^2 l_2^2 + r^2 l_3^2 - 2rl_1 l_2 l_3 \cos(\theta + \theta_2)}}\right) + \sin^{-1}\left(\frac{rl_3 \sin(\theta + \theta_2)}{\sqrt{l_1^2 l_2^2 + r^2 l_3^2 - 2rl_1 l_2 l_3 \cos(\theta + \theta_2)}}\right)$$

(substitute the calculated value of θ_2 from previous expression here)

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