Geometric Transformation II

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3D 기하 변환
(3D Geometric Transformation)

- 모든 2D 기하 변환은 3D에도 적용 가능
- Homogeneous coordinates in 3D:

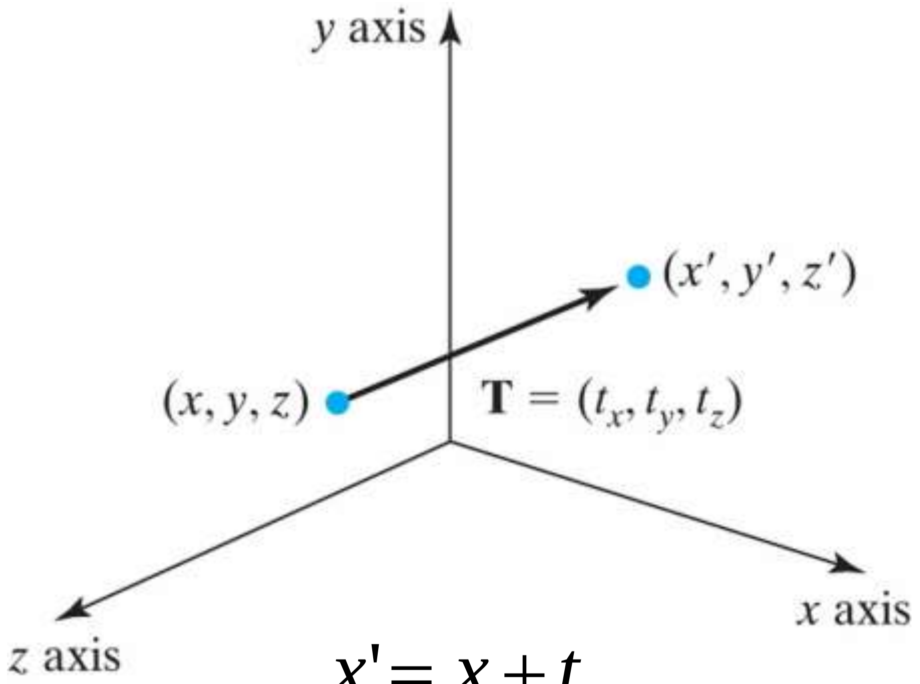
$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \text{(for a 3D point)}, \qquad \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} \text{(for a 3D vector)}$$

- Homogeneous coordinate → Cartesian coordinate:

$$(x, y, z, h) \rightarrow \left(\frac{x}{h}, \frac{y}{h}, \frac{z}{h}\right)$$

ex) (2,0,0,1) = (4,0,0,2) = (6,0,0,3) = (2,0,0)

3D 평행이동 (Translation)



$$x' = x + t_x$$

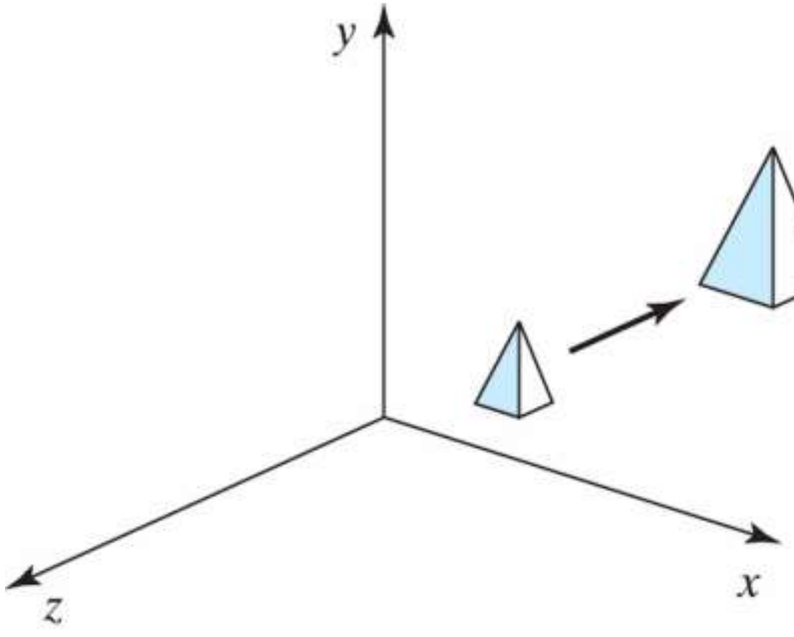
$$y' = y + t_y$$

$$z' = z + t_z$$

$$\mathbf{P}' = \mathbf{T} \cdot \mathbf{P}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D 축소 변환 (Scaling)



$$x' = s_x \cdot x$$

$$y' = s_y \cdot y$$

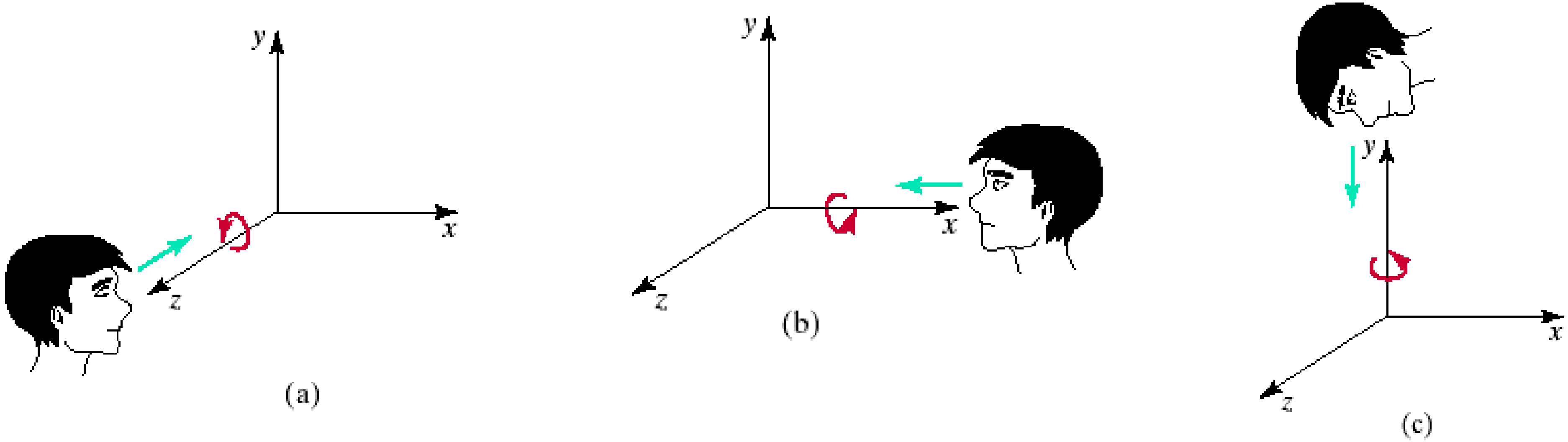
$$z' = s_z \cdot z$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

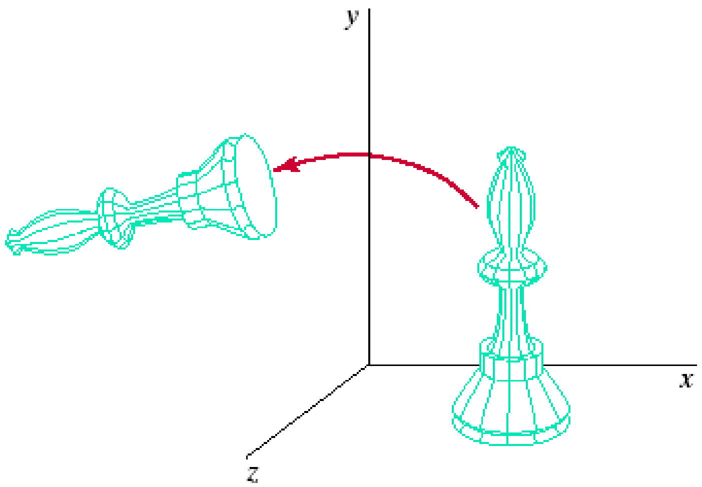
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D 회전 변환 (Rotation)

- 3D 회전변환은 회전 축(axis)를 중심으로 θ 각도만큼 회전
  - 축의 반대 방향으로 바라보며 반시계방향으로 회전
- 2D 회전변환은 z축을 중심으로 θ 만큼 회전한 것으로 해석



Z-축을 중심으로 회전

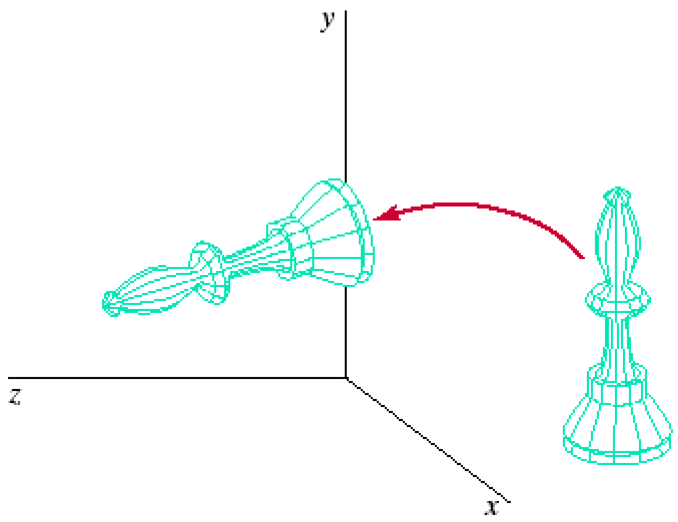


$$\mathbf{P}' = \mathbf{R_z}(\theta) \cdot \mathbf{P}$$

$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$
$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

X-축을 중심으로 회전

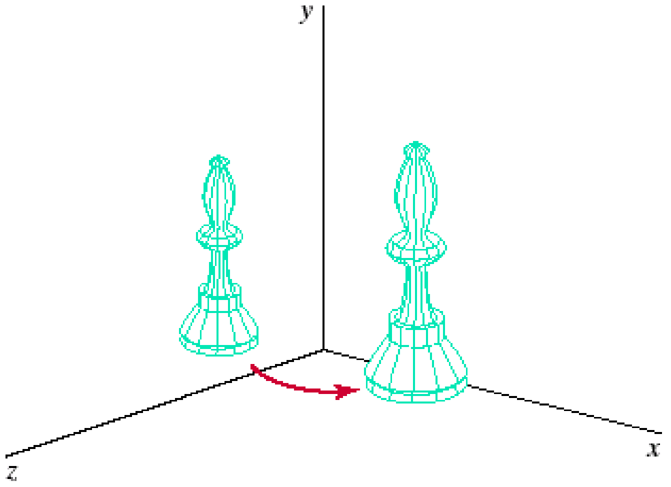


$$\mathbf{P'} = \mathbf{R_x}(\theta) \cdot \mathbf{P}$$

$$y' = y \cdot \cos\theta - z \cdot \sin\theta$$
$$z' = y \cdot \sin\theta + z \cdot \cos\theta$$
$$x' = x$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Y-축을 중심으로 회전



$$\mathbf{P}' = \mathbf{R_y}(\theta) \cdot \mathbf{P}$$

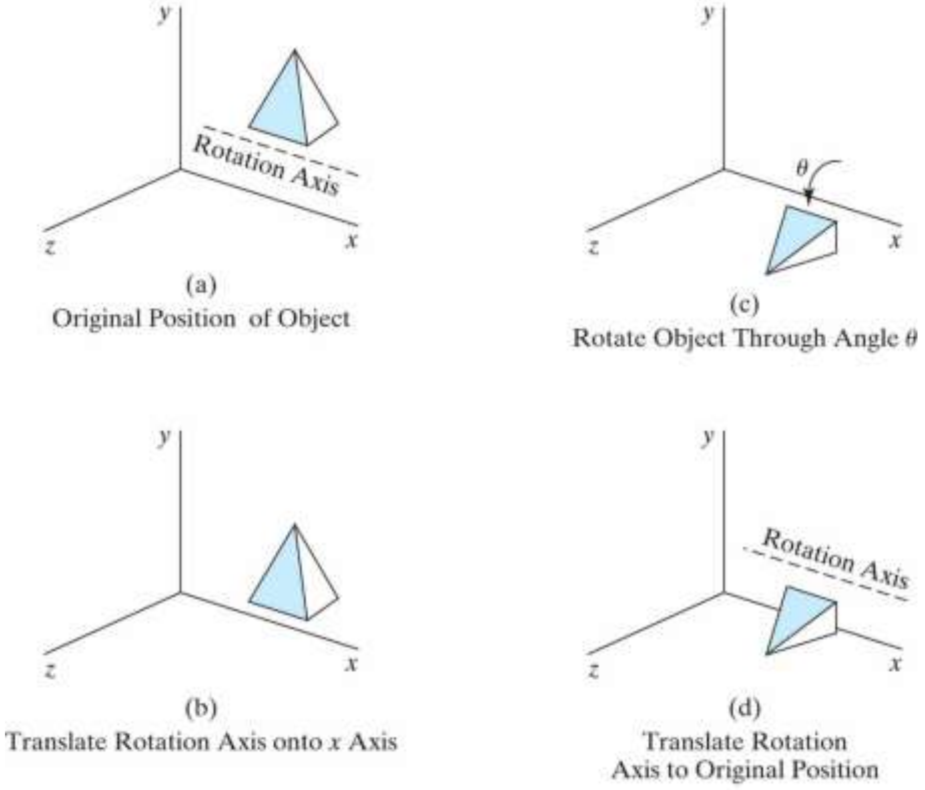
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$x' = z \bullet \sin\theta + x \bullet \cos\theta$$
$$z' = z \bullet \cos\theta - x \bullet \sin\theta$$
$$y' = y$$

일반적인 3D 회전 변환

(방법1) 임의의 축을 중심으로 하는 3D 회전 변환

- 1-1. Special Case: 회전축이 x,y 또는 z축에 평행함

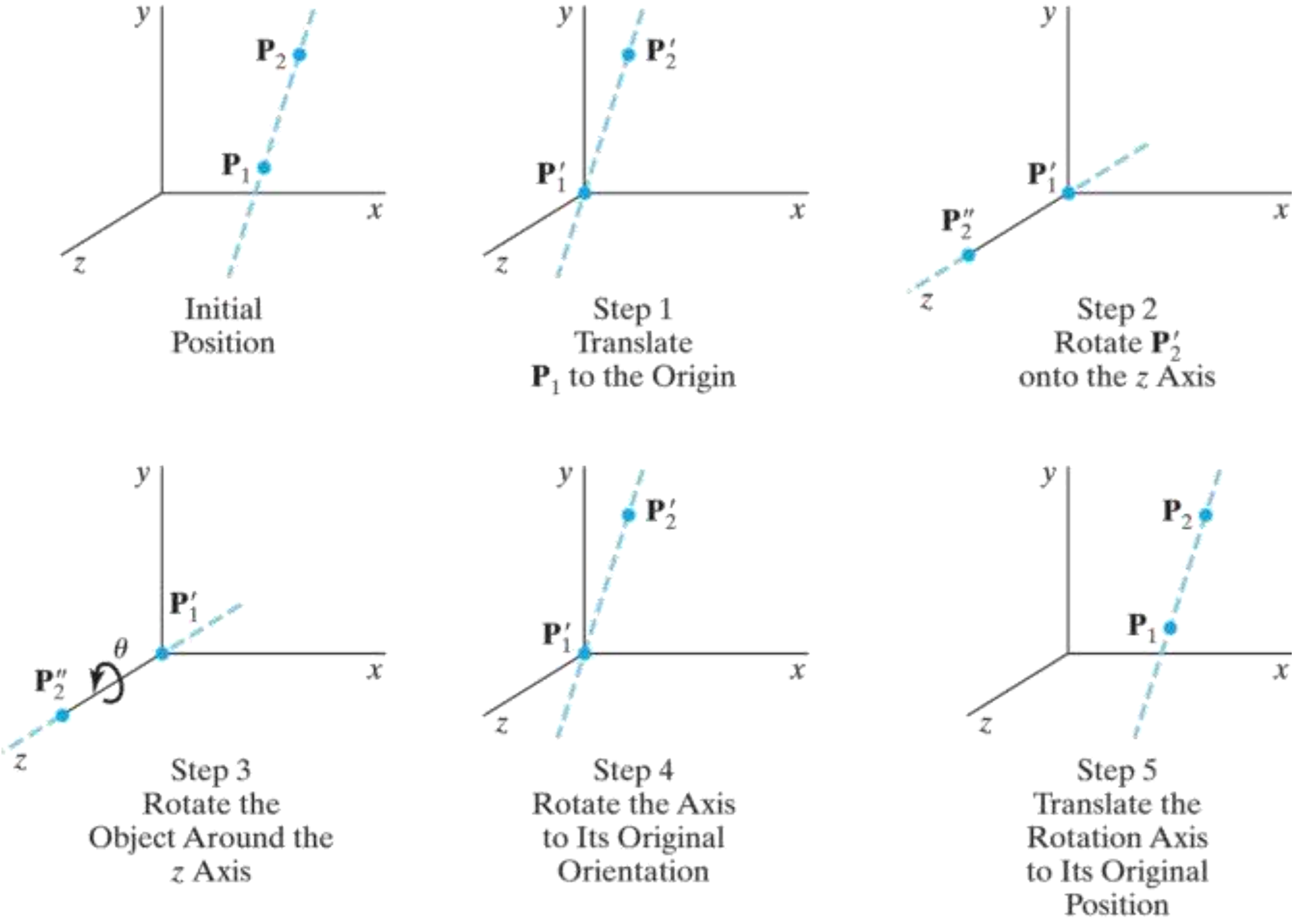


$$\mathbf{R_{sc}}(\theta) = \mathbf{T}^{-1} \cdot \mathbf{R_x}(\theta) \cdot \mathbf{T}$$

$$\mathbf{P}' = \mathbf{R_{sc}}(\theta) \cdot \mathbf{P} = \mathbf{T}^{-1} \cdot \mathbf{R_x}(\theta) \cdot \mathbf{T} \cdot \mathbf{P}$$

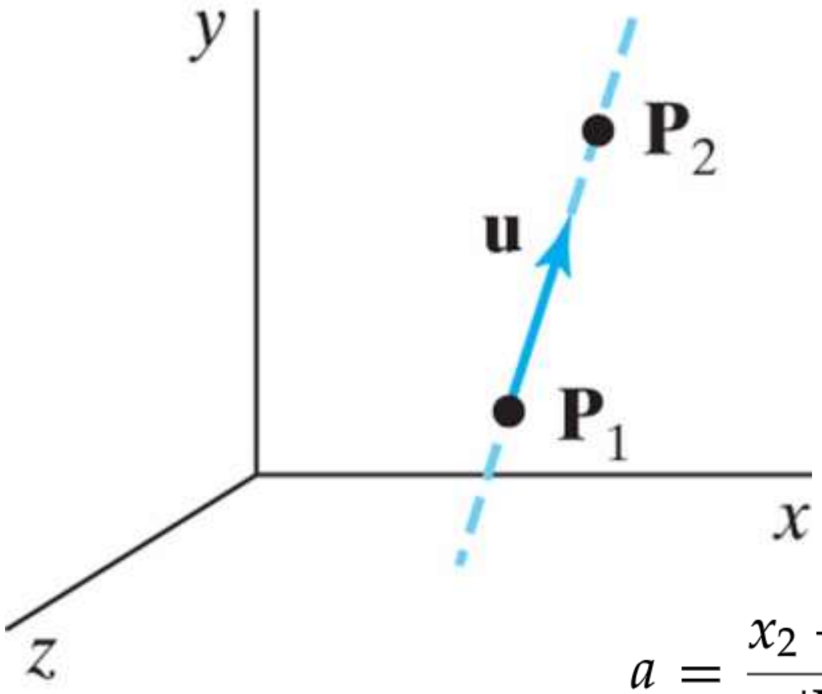
일반적인 3D 회전 변환

- 1-2. 회전축이 x,y,z축에 평행하지 않음



임의의 회전축에 대한 회전

Step 0.



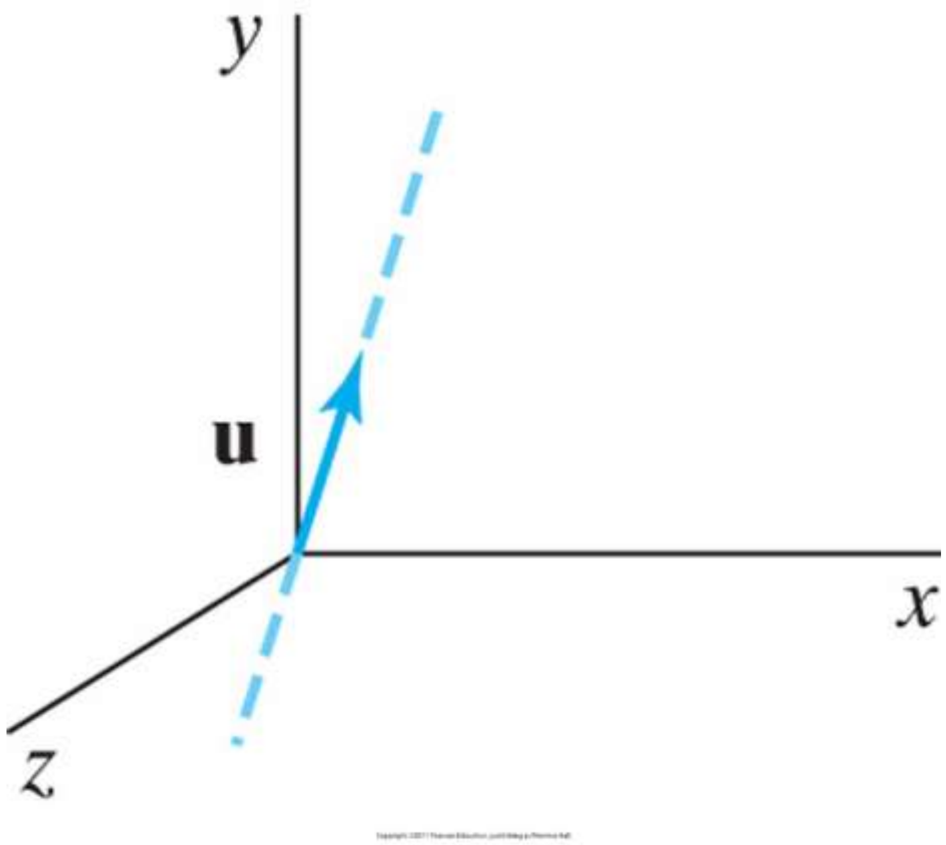
$$\mathbf{V} = \mathbf{P}_2 - \mathbf{P}_1$$
$$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

$$\mathbf{u} = \frac{\mathbf{V}}{|\mathbf{V}|} = (a, b, c)$$

$$a = \frac{x_2 - x_1}{|\mathbf{V}|}, \qquad b = \frac{y_2 - y_1}{|\mathbf{V}|}, \qquad c = \frac{z_2 - z_1}{|\mathbf{V}|}$$

임의의 회전축에 대한 회전

Step 1. 축을 원점으로 평행이동

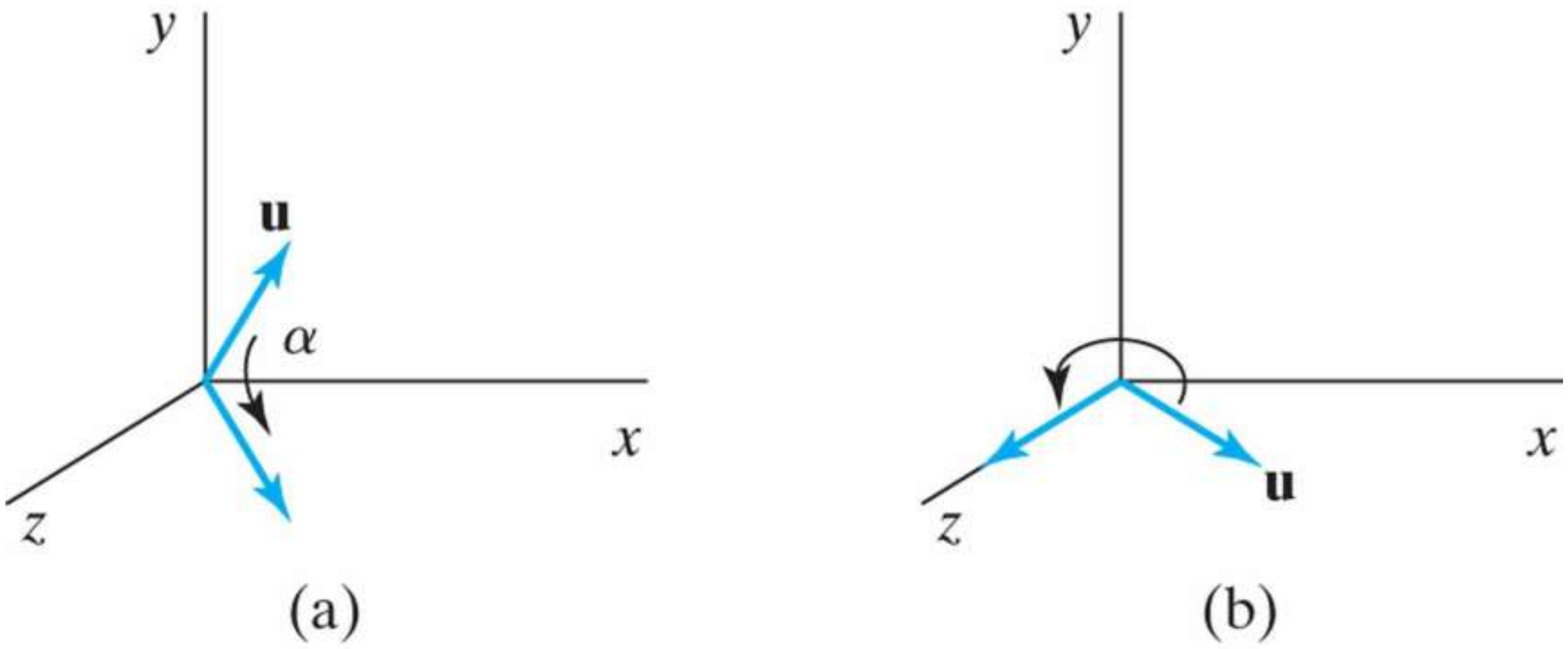


$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

임의의 회전축에 대한 회전

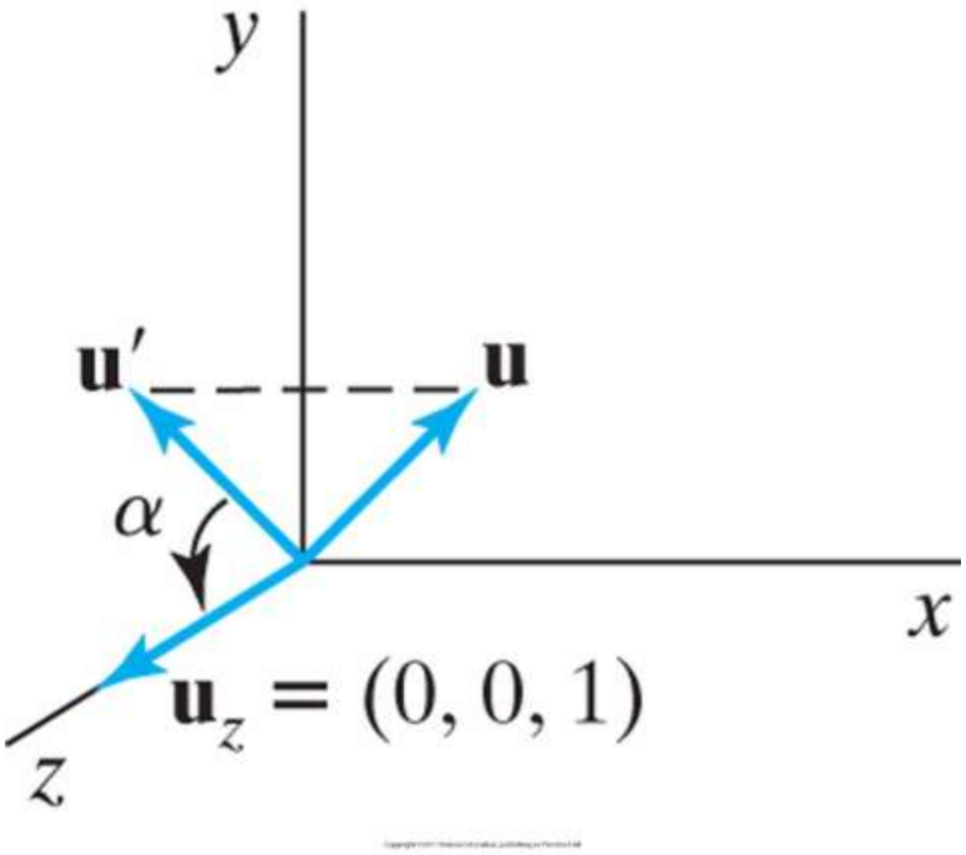
Step 2. Unit vector **u**를 x축 중심으로 회전하여 xz 평면에 내림

Step 3. xz 평면상의 **u**를 y축 중심으로 회전



임의의 회전축에 대한 회전

Step 2. x축 회전의 각도는?



- u를 yz 평면에 투영

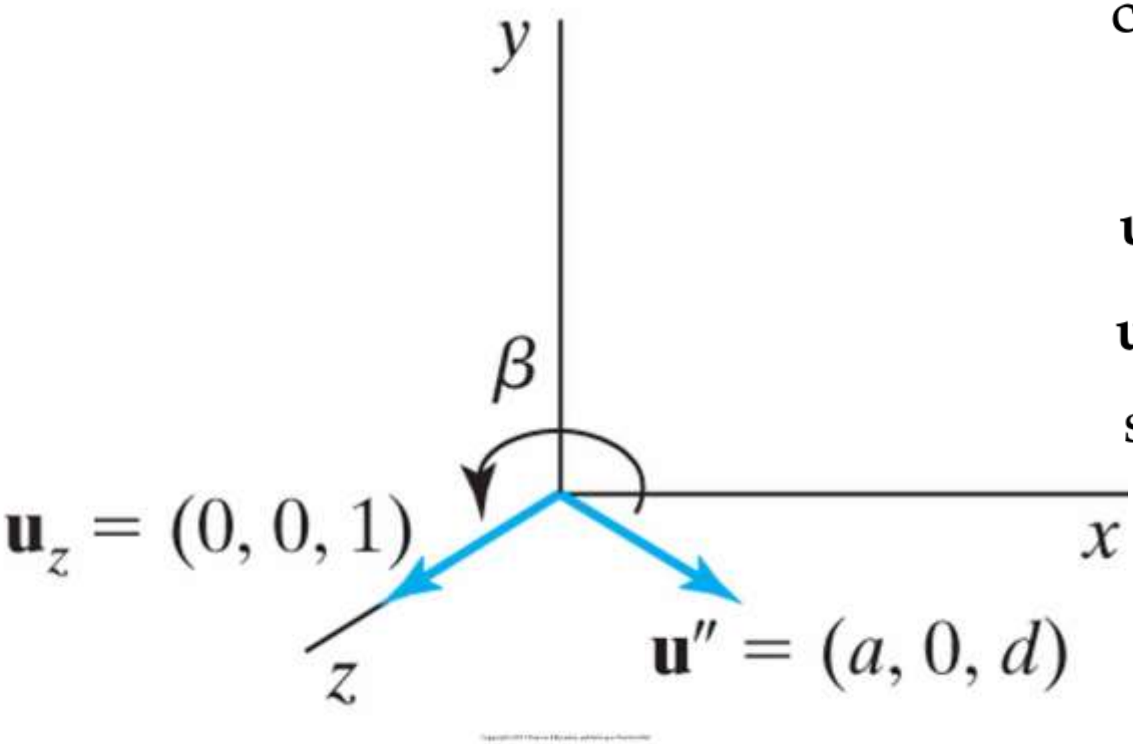
$$\cos\alpha = \frac{\mathbf{u}' \cdot \mathbf{u}_z}{|\mathbf{u}'|\,|\mathbf{u}_z|} = \frac{c}{d}$$

$$\sin\alpha = \frac{b}{d} \qquad d = \sqrt{b^2 + c^2}$$

$$\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{d} & -\frac{b}{d} & 0 \\ 0 & \frac{b}{d} & \frac{c}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

임의의 회전축에 대한 회전

Step 3. y축 회전의 각도는?



$$\cos\beta = \frac{\mathbf{u}'' \cdot \mathbf{u}_z}{|\mathbf{u}''|\,|\mathbf{u}_z|} = d \qquad \text{because } |\mathbf{u}_z| = |\mathbf{u}''| = 1$$

$$\mathbf{u}'' \times \mathbf{u}_z = \mathbf{u}_y\,|\mathbf{u}''|\,|\mathbf{u}_z|\sin\beta$$

$$\mathbf{u}'' \times \mathbf{u}_z = \mathbf{u}_y \cdot (-a)$$

$$\sin\beta = -a$$

$$\mathbf{R}_y(\beta) = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

임의의 회전축에 대한 회전

Step 4. z축을 중심으로 θ 만큼 회전

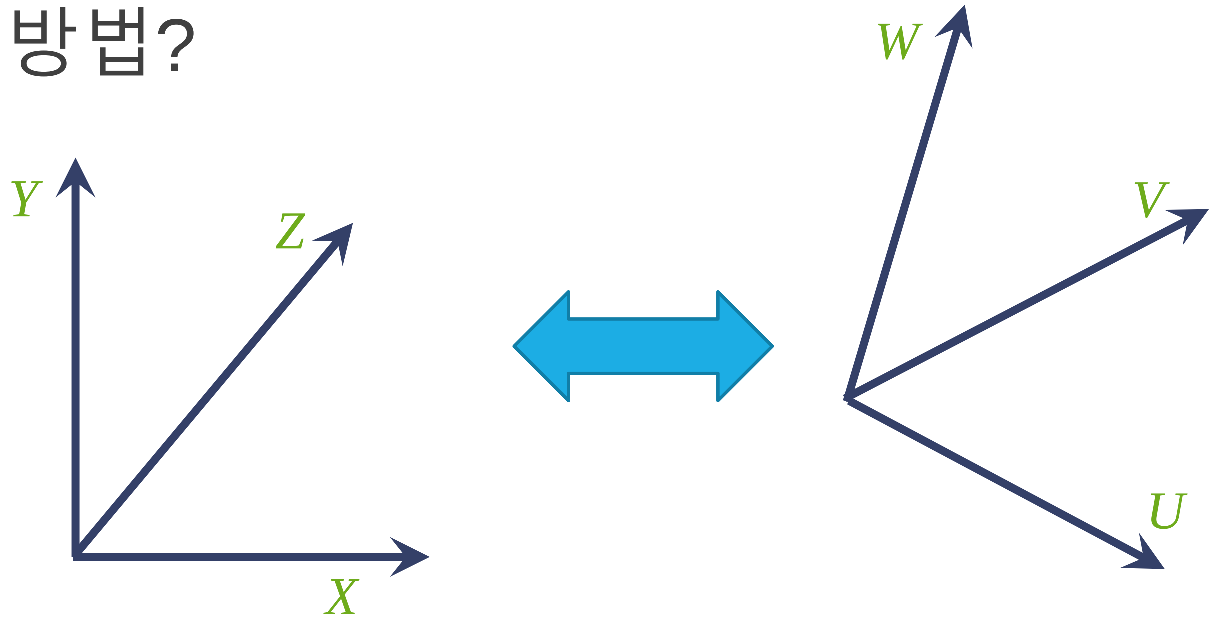
$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5~7. Step 3~1의 역행렬변환 적용

$$\mathbf{R}(\theta) = \mathbf{T}^{-1} \cdot \mathbf{R}_x^{-1}(\alpha) \cdot \mathbf{R}_y^{-1}(\beta) \cdot \mathbf{R}_z(\theta) \cdot \mathbf{R}_y(\beta) \cdot \mathbf{R}_x(\alpha) \cdot \mathbf{T}$$

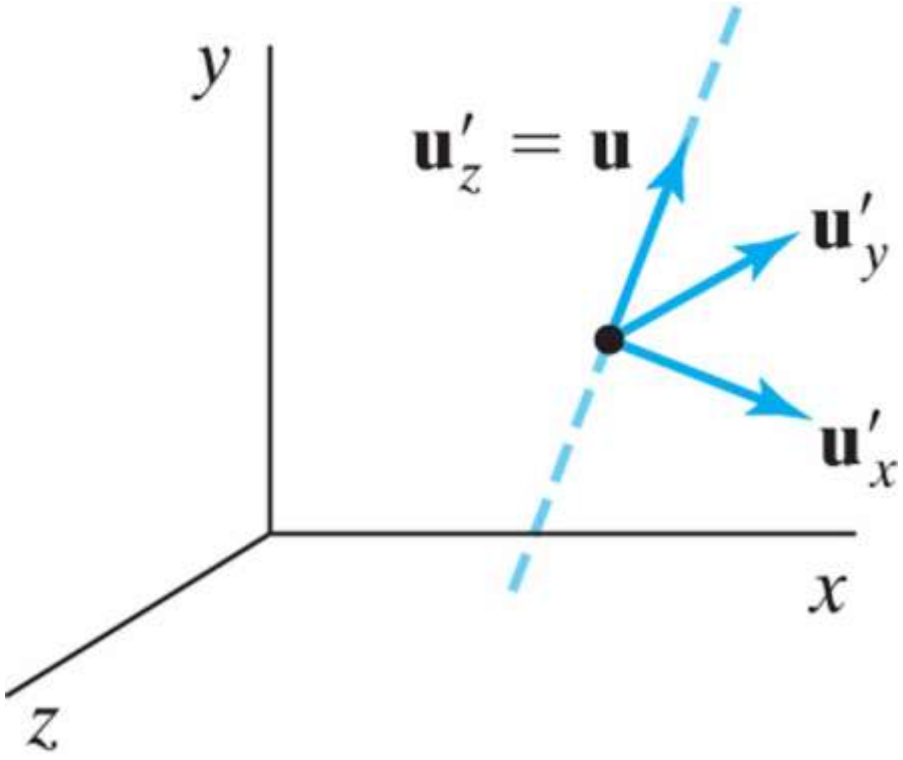
임의의 3D 회전

좀더 간단한 방법?



- $R = R_y(\beta)R_x(\alpha)$의 회전 변환은 회전축을 W축으로 하는 직교좌표계 (orthonormal coordinate system) UVW를 표준 직교좌표계 XYZ로변환하는 것과 같음

임의의 3D 회전



$$\mathbf{u}'_z = \mathbf{u}$$

$$\mathbf{u}'_y = \frac{\mathbf{u} \times \mathbf{u}_x}{|\mathbf{u} \times \mathbf{u}_x|}$$

$$\mathbf{u}'_x = \mathbf{u}'_y \times \mathbf{u}'_z$$

임의의 3D 회전

- UVW coordinate system의 세 축 (**u,v,w**)를 계산한 후,

$$\mathbf{u} = x_u\mathbf{x} + y_u\mathbf{y} + z_u\mathbf{z}$$
$$\mathbf{v} = x_v\mathbf{x} + y_v\mathbf{y} + z_v\mathbf{z}$$
$$\mathbf{w} = x_w\mathbf{x} + y_w\mathbf{y} + z_w\mathbf{z}$$

$$\mathbf{R}_{uvw}\mathbf{u} = \begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{bmatrix} \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix} = \begin{bmatrix} x_u x_u + y_u y_u + z_u z_u \\ x_v x_u + y_v y_u + z_v z_u \\ x_w x_u + y_w y_u + z_w z_u \end{bmatrix}$$

UVW 좌표계에서 XYZ 좌표계로

$$\mathbf{R}_{uvw}^{\mathrm{T}}\mathbf{y} = \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} = \mathbf{v}.$$

XYZ 좌표계에서 UVW 좌표계로

임의의 3D 회전

$$\mathbf{R}(\theta) = \mathbf{T}^{-1} \cdot \boxed{\mathbf{R}_x^{-1}(\alpha) \cdot \mathbf{R}_y^{-1}(\beta) \cdot \mathbf{R}_z(\theta) \cdot \mathbf{R}_y(\beta) \cdot \mathbf{R}_x(\alpha)} \cdot \mathbf{T}$$

$$\boxed{\begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{bmatrix}}$$

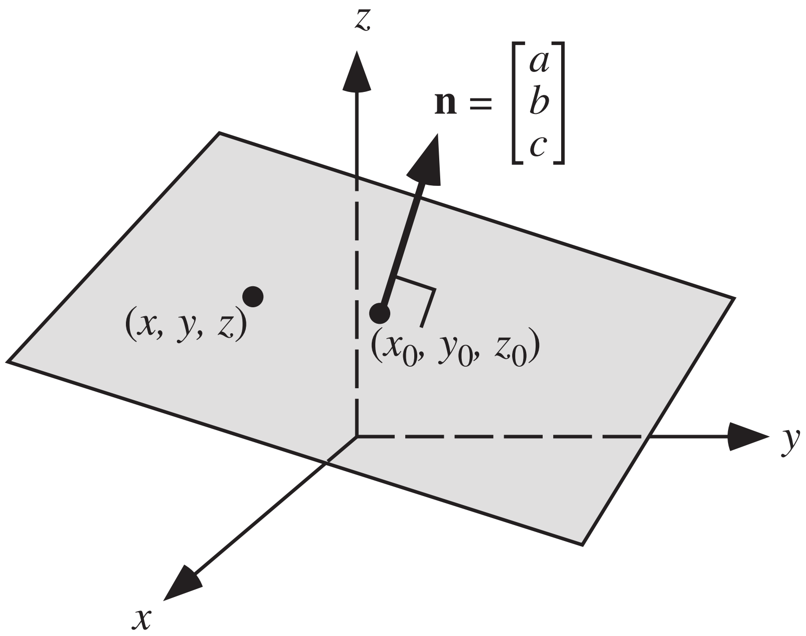
법선 벡터의 변환 (Transformation of Normal Vectors)

- 3D에서 평면을 정의하는 방법 – 평면의 방정식

$$ax + by + cz + d = 0$$

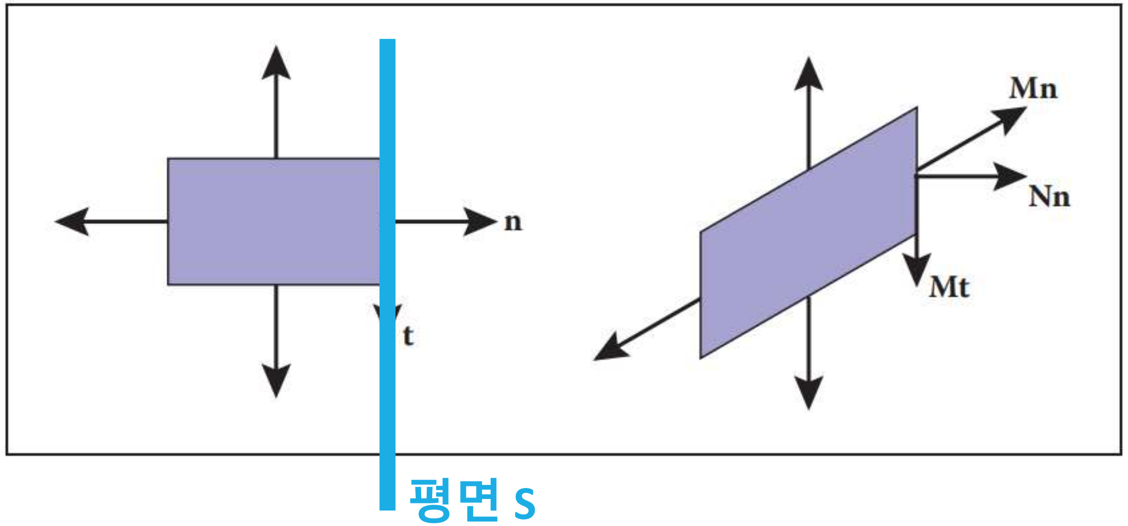
- 평면의 방정식은 다음과 같이 풀이됨

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$$



n = (a,b,c)
Normal vector (법선 벡터)

법선 벡터의 변환 (Transformation of Normal Vectors)



n:

- 평면 s
벡터 t = 가 평면 s 상에 있는 임의의 벡터이면 t와 s의
법선 벡터 n은 항상 다음의 식을 만족함

$$n \cdot t = 0$$

법선 벡터의 변환 (Transformation of Normal Vectors)

Q) 평면 위의 벡터 $\mathbf{x}-\mathbf{x}_0$는 임의의 기하 변환 후에도 보전되는가?

→ 그렇다, (왜냐하면 $\mathbf{x}'-\mathbf{x}_0'=\mathbf{M}\mathbf{x}-\mathbf{M}\mathbf{x}_0=\mathbf{M}(\mathbf{x}-\mathbf{x}_0)$ 이기 때문)

Q) 평면의 법선 벡터 $\mathbf{n}$은 임의의 기하 변환 $\mathbf{M}$ 적용 후에도 여전히 법선인가?

→ 아니다, (왜냐하면, $\mathbf{n}'^{\mathrm{T}}(\mathbf{x}'-\mathbf{x}_0')=(\mathbf{M}\mathbf{n})^{\mathrm{T}}(\mathbf{M}\mathbf{x}-\mathbf{M}\mathbf{x}_0)\neq 0$ if $\mathbf{n}'=\mathbf{M}\mathbf{n}$)

법선 벡터의 변환
(Transformation of Normal Vectors)

- 임의의 기하 변환 M을 적용할 때, 법선 벡터에 적용되는 변환 N은?

$\Rightarrow$ Find $\mathbf{N}$ s.t. $\mathbf{n}' = \mathbf{N}\mathbf{n}$ and $\mathbf{n}'^{\mathbf{T}}(\mathbf{x}' - \mathbf{x}'_{\mathbf{0}}) = 0$

$\Rightarrow \mathbf{0} = \mathbf{n}'^{\mathbf{T}}(\mathbf{x}' - \mathbf{x}'_{\mathbf{0}}) = \mathbf{n}'^{\mathbf{T}}(\mathbf{M}\mathbf{x} - \mathbf{M}\mathbf{x}_{\mathbf{0}}) = \mathbf{n}'^{\mathbf{T}}\mathbf{M}(\mathbf{x} - \mathbf{x}_{\mathbf{0}})$

$\mathbf{n}'^{\mathbf{T}} = (\mathbf{N}\mathbf{n})^{\mathbf{T}} = \mathbf{n}^{\mathbf{T}}\mathbf{N}^{\mathbf{T}}$이므로

$\Rightarrow \mathbf{0} = \mathbf{n}^{\mathbf{T}}\mathbf{N}^{\mathbf{T}}\mathbf{M}(\mathbf{x} - \mathbf{x}_{\mathbf{0}})$.

$\mathbf{N}^{\mathbf{T}} = \mathbf{M}^{-\mathbf{1}}$로 정의하면,

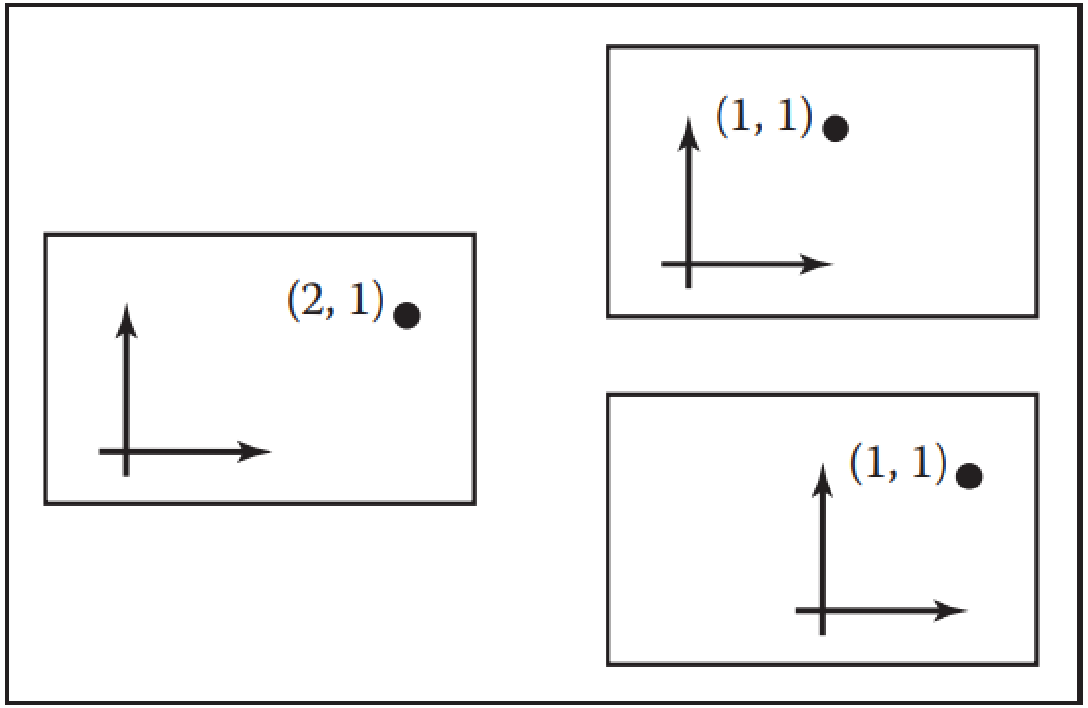
$\Rightarrow \mathbf{n}^{\mathbf{T}}\mathbf{N}^{\mathbf{T}}\mathbf{M}(\mathbf{x} - \mathbf{x}_{\mathbf{0}}) = \mathbf{n}^{\mathbf{T}}\mathbf{I}(\mathbf{x} - \mathbf{x}_{\mathbf{0}}) = \mathbf{0}$

<u>**따라서** $\mathbf{N} = \left(\mathbf{M}^{-\mathbf{1}}\right)^{\mathbf{T}}$</u>.

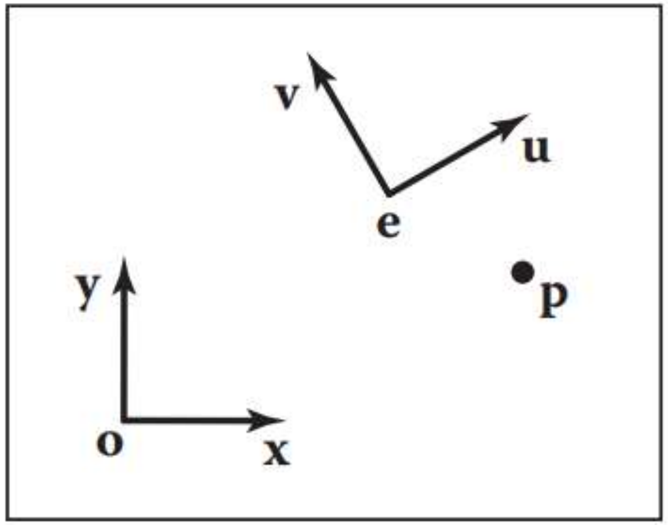
좌표계 변환 (Coordinate Transformation)

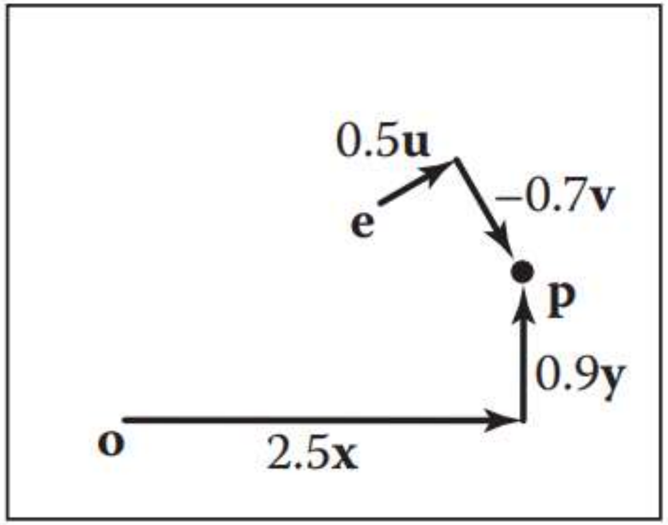
다음의 변환을 어떻게 해석하는가?

- 점 p의 좌표가 (2,1)에서 (1,1)로 평행이동함



좌표계 변환
(Coordinate Transformation)





- Frame-to-canonical matrix:

$$\mathbf{p}_{xy} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{e} \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}_{uv}$$

- Canonical-to-frame matrix:

$$\mathbf{p}_{uv} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{e} \\ 0 & 0 & 1 \end{bmatrix}^{-1} \mathbf{p}_{xy}.$$

$$\mathbf{p}_{uv} = \begin{bmatrix} \mathbf{x}_{uv} & \mathbf{y}_{uv} & \mathbf{o}_{uv} \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}_{xy}.$$

Viewing Transformation

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뷰잉 변환 (Viewing Transformation)

Q) 3D물체를 어떻게 2D 화면에 그리는가?

"거울 앞 소녀", 파블로 피카소, 1932

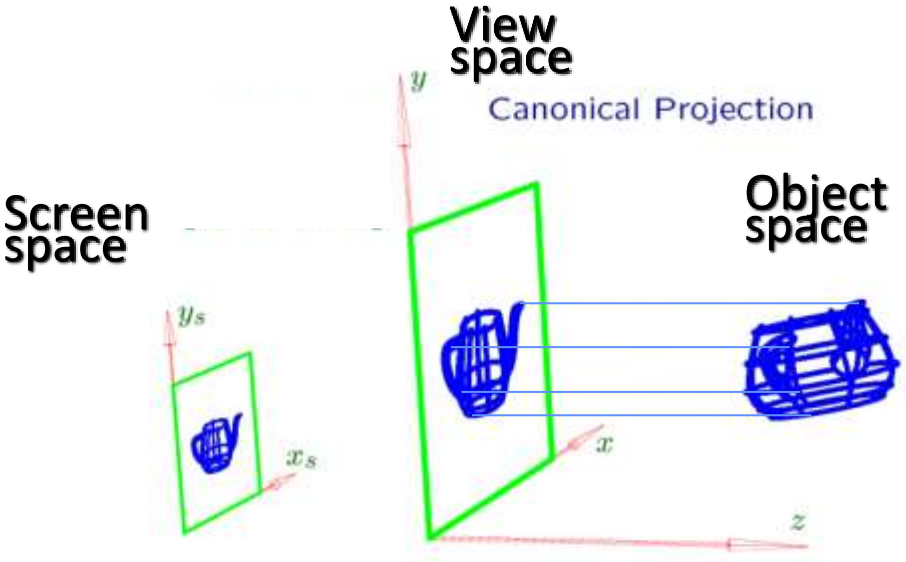
어안렌즈 이미지

뷰잉 변환(Viewing Transformation)

Q) 3D 물체를 2D 화면에 어떻게 그리는가?

- 3D 물체는 image plane에 투영(projection)됨

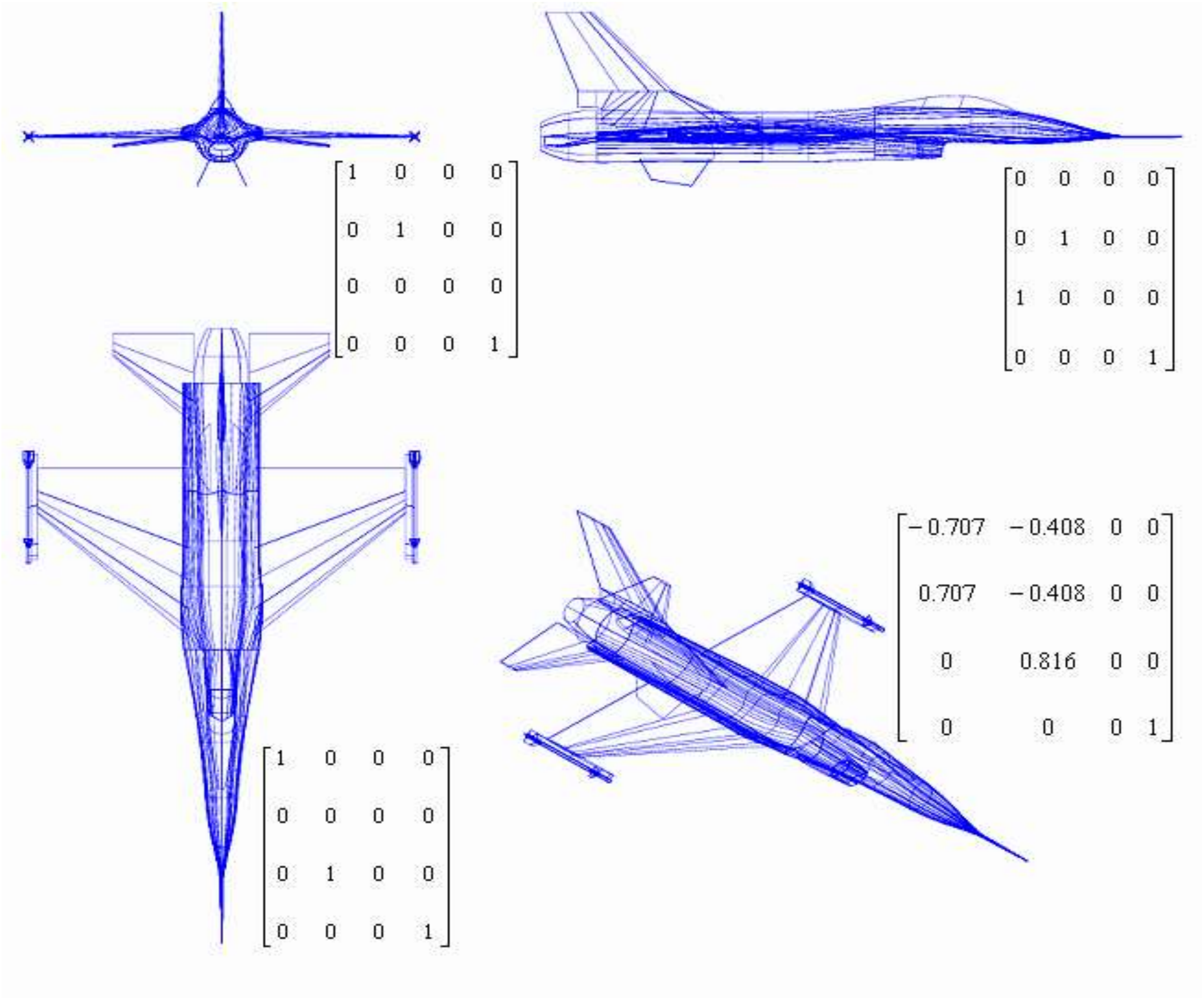
  (직선이 직선으로 투영됨)



투영의 한 예시: 3D 물체를 xy-plane에 투영
(이때, z값은 무시됨)

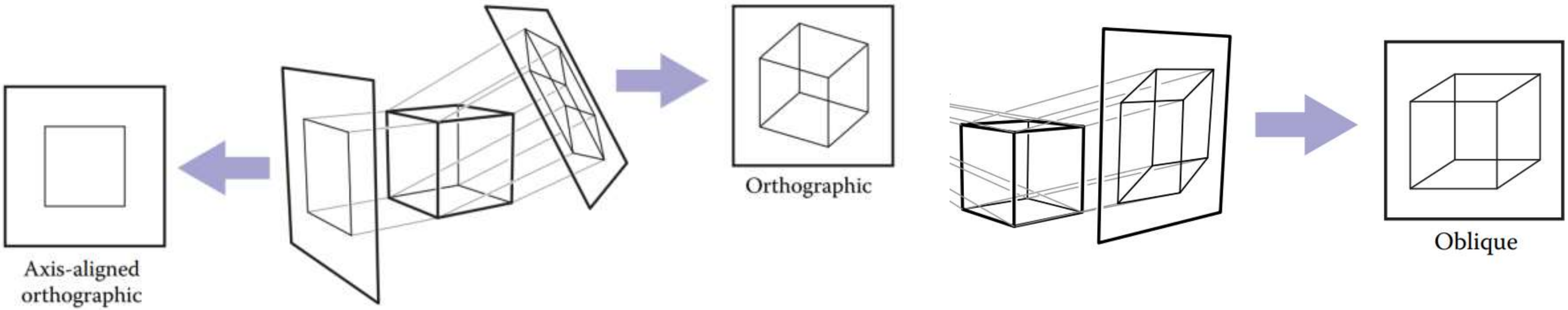
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

뷰잉 변환(Viewing Transformation)



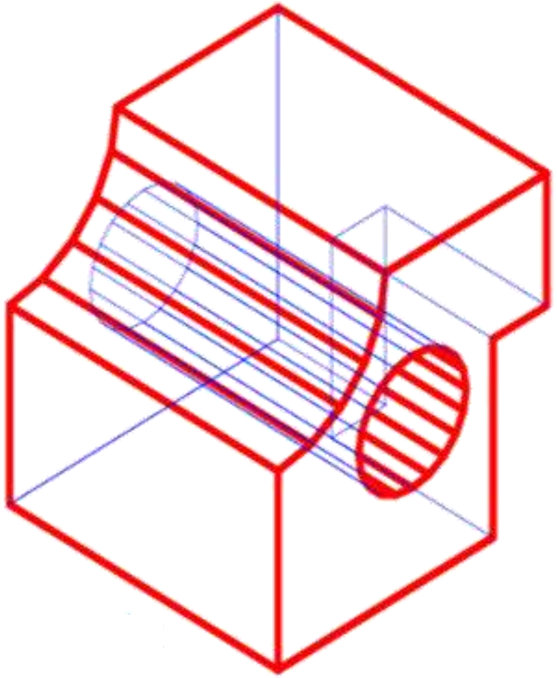
Projection의 종류

- Parallel projection (평행 투영)
  - 투영 직선 (projection line)들이 모두 평행임
  - Orthographic projection (정사 투영): 투영 직선들이 이미지에 수직
  - Oblique projection (경사 투영): 투영 직선들이 이미지에 비스듬히 만남



Projection의 종류

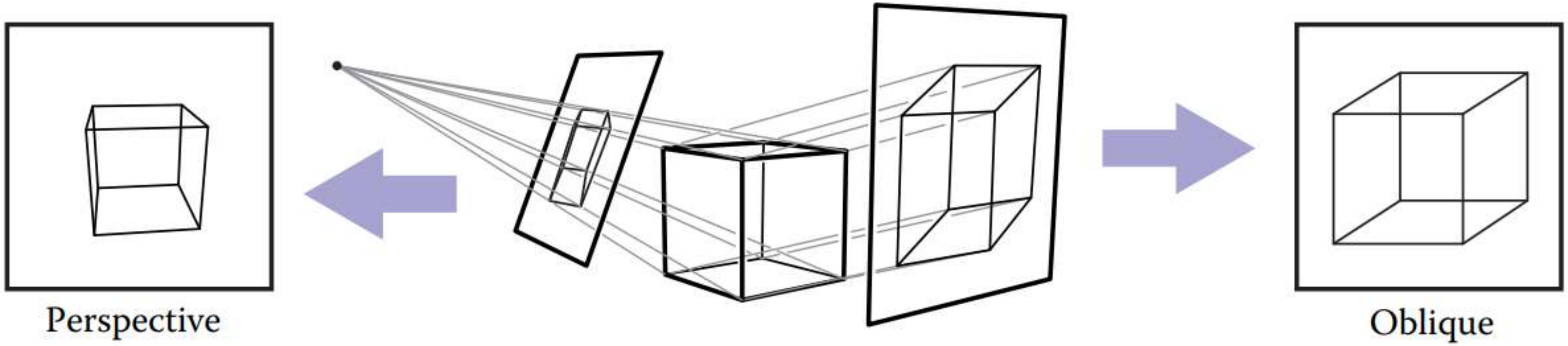
- Parallel projection
  - 물체의 길이와 크기 보전, 평행선은 평행선으로 투영
  - 기계 및 건축 설계에서 자주 쓰임



But….

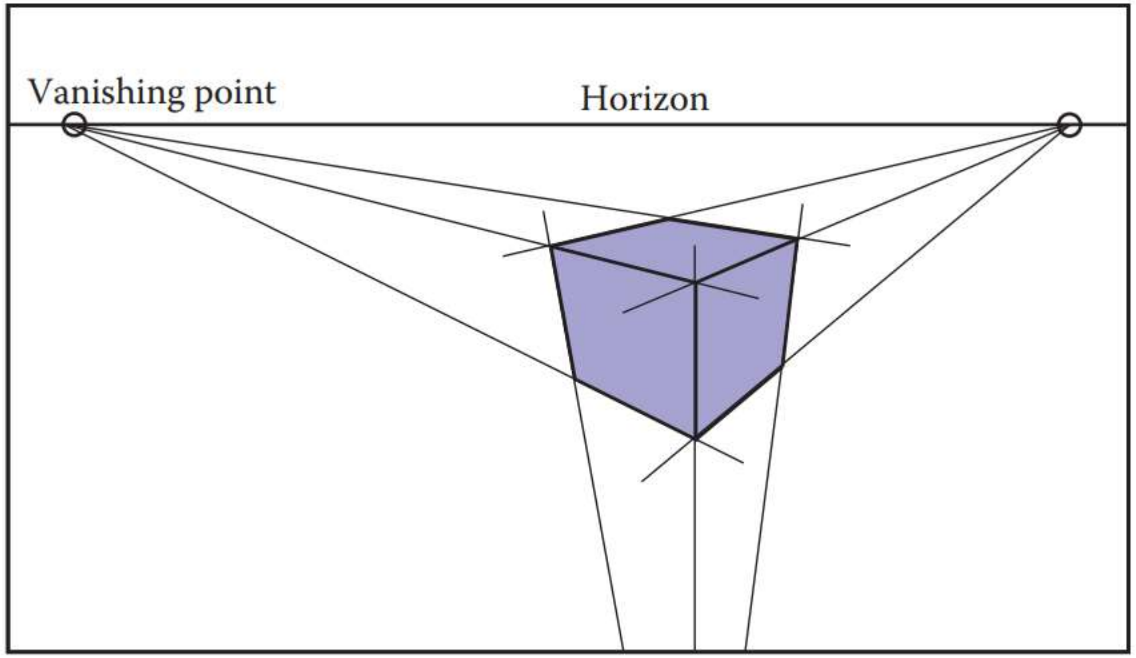
Projection의 종류

- Perspective projection (원근 투영)
  - 투영 직선이 한점(viewpoint)를 지남
  - 3D 물체에서 viewpoint를 잇는 투영 직선들과 image plane의 교점들로 2D 이미지 생성됨



Projection의 종류

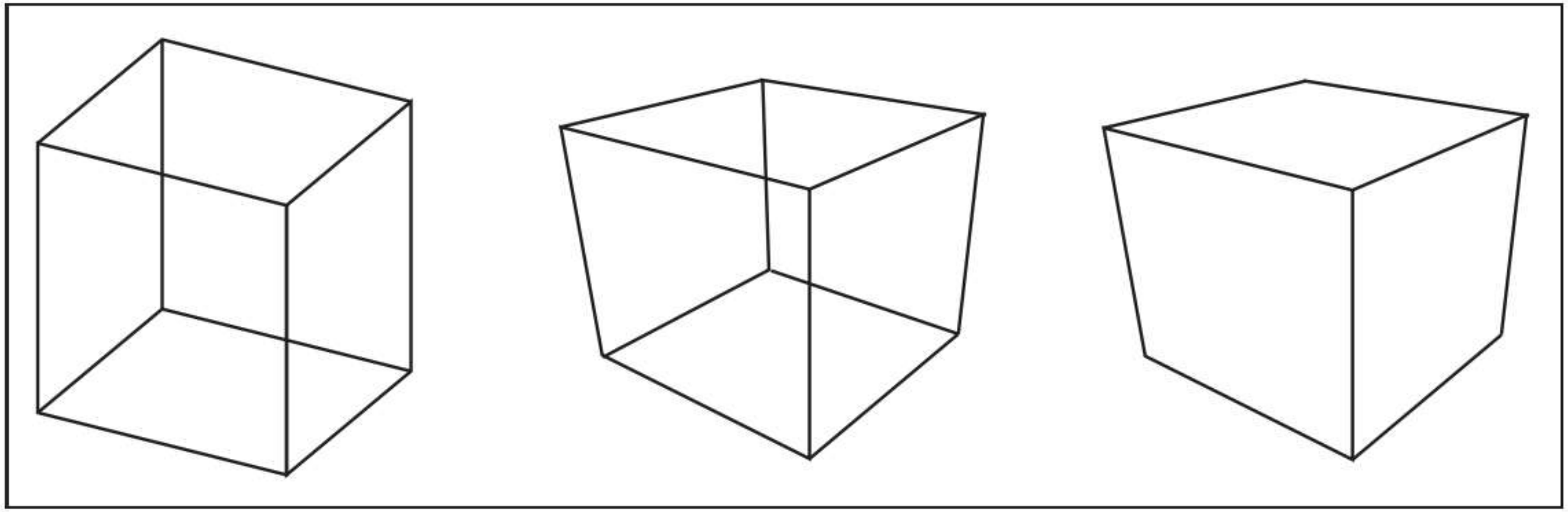
- Perspective projection



삼점투시 (Three-point perspective projection)

Projection의 종류

정사 투영(Orthographic)과 원근 투영(Perspective)의 비교



Orthographic projection

Perspective projection

Perspective projection
(with hidden edges)