Breit-Rabi formula

Philippe W. Courteille, 27/01/2021

Solution: *The purpose is to diagonalize the following matrix,*

$$\begin{aligned}\langle m_I'm_J'|V_{hfs}+V_{zee}(\mathcal{B})|m_Im_J\rangle &= \langle m_I'm_J'|\tfrac{A_J}{\hbar^2}\mathbf{I}\cdot\mathbf{J}+g_J\mu_B\vec{\mathcal{B}}\cdot\mathbf{J}-g_I\mu_N\vec{\mathcal{B}}\cdot\mathbf{I}|m_Im_J\rangle \\ &= \langle m_I'm_J'|A_J(I_zJ_z+\left(\tfrac{1}{2}I_+J_-+\tfrac{1}{2}I_-J_+\right)+g_J\mu_B\mathcal{B}J_z-g_I\mu_N\mathcal{B}I_z|m_Im_J\rangle\ .\end{aligned}$$

Now, since $J=\frac{1}{2}$ we know that there are only two hyperfine states $F=I\pm J=I\pm\frac{1}{2}$. We also know $m_F=m_I+m_J$ and $m_J\pm\frac{1}{2}$. Using the formula,

$$\hat{J}_\pm|m_I,m_J\rangle=\hbar\sqrt{J(J+1)-m_J(m_J\pm1)}|m_I,m_J\pm1\rangle\ ,$$

we find,

$$\hat{J}_\pm|m_I,\pm\tfrac{1}{2}=0\qquad\textit{and}\qquad\hat{J}_\pm|m_I,\mp\tfrac{1}{2}\rangle=|m_I,\mp\tfrac{1}{2}\rangle\ .$$

With this, we can calculate the components of the matrix:

$$\begin{aligned}\langle m_I',\tfrac{1}{2}|V_{hfs}+V_{zee}(B)|m_I,\tfrac{1}{2}\rangle &= (A_Jm_I\tfrac{1}{2}+\hbar g_J\mu_B\mathcal{B}\tfrac{1}{2}-\hbar g_I\mu_N\mathcal{B}m_I)\delta_{m_I'm_I} \\ &= \left[\tfrac{A_J}{2}(m_F-\tfrac{1}{2})+\tfrac{\hbar}{2}(g_J\mu_B-g_I\mu_N)\mathcal{B}-\hbar g_I\mu_N\mathcal{B}m_F\right]\delta_{m_I'm_I} \\ \langle m_I',-\tfrac{1}{2}|V_{hfs}+V_{zee}(B)|m_I,-\tfrac{1}{2}\rangle &= (-A_Jm_I\tfrac{1}{2}-\hbar g_J\mu_B\mathcal{B}\tfrac{1}{2}-\hbar g_I\mu_N\mathcal{B}m_I)\delta_{m_I'm_I} \\ &= \left[-\tfrac{A_J^2}{2}(m_F+\tfrac{1}{2})-\tfrac{\hbar}{2}(g_J\mu_B-g_I\mu_N)\mathcal{B}-\hbar g_I\mu_N\mathcal{B}m_F\right]\delta_{m_I'm_I} \\ \langle m_I',\tfrac{1}{2}|V_{hfs}+V_{zee}(\mathcal{B})|m_I,-\tfrac{1}{2}\rangle &= \tfrac{A_J}{2}\sqrt{I(I+1)+m_I(m_I-1)}\delta_{m_I'm_I-1} \\ &= \tfrac{A_J}{2}\sqrt{I(I+1)+(m_F^2-\tfrac{1}{4})}\delta_{m_I'm_I-1} \\ \langle m_I',-\tfrac{1}{2}|V_{hfs}+V_{zee}(\mathcal{B})|m_I,\tfrac{1}{2}\rangle &= \tfrac{A_J}{2}\sqrt{I(I+1)+m_I(m_I+1)}\delta_{m_I'm_I+1} \\ &= \tfrac{A_J}{2}\sqrt{I(I+1)+(m_F^2-\tfrac{1}{4})}\delta_{m_I'm_I+1}\ .\end{aligned}$$

The eigenvalues of this 2×2 matrix are,

$$\Delta E_{hfs}+\Delta E_{zee}(\mathcal{B})=-\frac{A_J}{4}-g_I\mu_N\mathcal{B}m_F\pm\frac{A_J}{4}\sqrt{-1+8m_F^2+4A_J^2I(I+1)+4m_Fx+x^2}\ ,$$

with the abbreviation $x\equiv\frac{2(\mu_Bg_J-\mu_Ng_I)\mathcal{B}}{A_J}$. This is the Breit-Rabi formula. Note that the derivation can also be made in the base $|(I,J),F,m_F\rangle$ using the formula $\mathbf{I}\cdot\mathbf{J}=\frac{1}{2}(\mathbf{F}^2-\mathbf{J}^2-\mathbf{I}^2)$.