Average-case matrix discrepancy: satisfiability bounds

TODO:

1. Fix the anonymity, I need to re-read to make sure I don't disclose my identity. **DONE**
2. Finish writing the proof for the failure of second moment.
3. Add a sentence that in upcoming work we will characterize the second moment sharply using methods from statistical physics, as well as the typical number of solutions.
4. Re-read all new passages, and the new Section and proof.
5. Check that there is no missing mention of the new results in the rest.

Abstract

Given a sequence of $d \times d$ symmetric matrices $\{\mathbf{W}_i\}_{i=1}^n$, and a margin $\Delta > 0$, we investigate whether it is possible to find signs $(\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n$ such that the operator norm of the signed sum satisfies $\|\sum_{i=1}^n \varepsilon_i \mathbf{W}_i\|_{\mathrm{op}} \le \Delta$. Kunisky and Zhang (2023) recently introduced a random version of this problem, where the matrices $\{\mathbf{W}_i\}_{i=1}^n$ are drawn from the Gaussian orthogonal ensemble. This model can be seen as a random variant of the celebrated Matrix Spencer conjecture, and as a matrix-valued analog of the symmetric binary perceptron in statistical physics. In this work, we establish a satisfiability transition in this problem as $n, d \to \infty$ with $n/d^2 \to \tau > 0$. First, we prove that the expected number of solutions with margin $\Delta = \kappa\sqrt{n}$ has a sharp threshold at a critical $\tau_1(\kappa)$: for $\tau < \tau_1(\kappa)$ the problem is typically unsatisfiable, while for $\tau > \tau_1(\kappa)$ the average number of solutions becomes exponentially large. Second, combining a second-moment method with recent results from Altschuler (2023) on margin concentration in perceptron-type problems, we identify a second threshold $\tau_2(\kappa)$, such that for $\tau > \tau_2(\kappa)$ the problem admits solutions with high probability. In particular, we establish for the first time that a system of $n = \Theta(d^2)$ Gaussian random matrices can be balanced so that the spectrum of the resulting matrix macroscopically shrinks compared to the typical semicircle law. Finally, we prove that there exists values of (τ, κ) for which the average number of solutions is both exponentially large and does not concentrate around its average, implying the failure of the second moment method, and uncovering a richer picture than in the symmetric binary perceptron. Our proofs rely on deriving concentration inequalities and large deviation properties for the law of correlated Gaussian matrices under spectral norm constraints, extending results in the literature and offering insights of independent interest.

Contents

1 Introduction and main results

subsectionSetting and related literature

We start by introducing in Sections 1.0.1 and 1.0.2 two sets of independent definitions and models, both being important motivations behind the problem we study, which is discussed in Section 1.0.3.

1.0.1 Discrepancy theory

Computing the *discrepancy* (i.e. the best possible balancing) of a collection of sets or vectors is a classical question in mathematics and theoretical computer science. It has wide-ranging applications in combinatorics, computational geometry, experimental design, and the theory of approximation algorithms, to name a few. The reader will find in Spencer (1994), Matousek (2009), and Chen, Srivastav, and Travaglini (2014) a detailed account of the history and applications of discrepancy theory. Arguably one of the most celebrated results in this area is the following theorem, known as Spencer's "six deviations suffice" (Spencer, 1985).

Theorem 1.1 (Spencer (1985))**.** *There exists $C > 0$ such that for all $n, d \geq 1$, and all $\mathbf{u}_1, \cdots, \mathbf{u}_n \in \mathbb{R}^d$ with $\|\mathbf{u}_i\|_\infty \leq 1$ for all $i \in [n]$:*

$$\mathrm{disc}(\mathbf{u}_1, \cdots, \mathbf{u}_n) \coloneqq \min_{\varepsilon \in \{\pm 1\}^n} \left\| \sum_{i=1}^n \varepsilon_i \mathbf{u}_i \right\|_\infty \leq C \sqrt{n \max\left(1, \log \frac{d}{n}\right)}.$$

Interestingly, Spencer's theorem shows that one can drastically improve over a naive pick of random signings $\varepsilon_1, \cdots, \varepsilon_n \overset{\text{i.i.d.}}{\sim} \mathrm{Unif}(\{\pm 1\})$. Indeed, it is not hard to show that (in the worst case over $(\mathbf{u}_i)_{i=1}^n$), such random signings achieve $\|\sum_{i=1}^n \varepsilon_i \mathbf{u}_i\|_\infty = \Theta(\sqrt{n \log d})$, with high probability as $n, d \to \infty$. Spencer's theorem implies the existence of a signing $\varepsilon \in \{\pm 1\}^n$, which depends on the value of the $\mathbf{u}_i$'s, and whose discrepancy improves over the one of random signs by a logarithmic factor. While the original proof of Spencer (1985) is not constructive, there has been recently a great number of results regarding efficient algorithmic constructions of these signings. We will discuss this point further in Section 1.2.

Matrix discrepancy – In the present work we consider the problem of *matrix discrepancy*. Given a set of n symmetric $d \times d$ matrices $\mathbf{A}_1, \cdots, \mathbf{A}_n$, we aim to characterize the following discrepancy objective ($\|\cdot\|_{\mathrm{op}}$ is the spectral, or operator, norm):

$$\mathrm{disc}(\mathbf{A}_1, \cdots, \mathbf{A}_n) \coloneqq \min_{\varepsilon \in \{\pm 1\}^n} \left\| \sum_{i=1}^n \varepsilon_i \mathbf{A}_i \right\|_{\mathrm{op}}. \tag{1}$$

As already mentioned, a foundational question in discrepancy is whether the objective can be made significantly smaller than the result given by a *random* choice of the signings $\varepsilon_i \overset{\text{i.i.d.}}{\sim} \mathrm{Unif}(\{\pm 1\})$. As such, matrix discrepancy is intimately connected to the study of large random matrices (Bandeira, Boedihardjo, and Handel, 2023). Since many problems related to the spectra of large matrices can be viewed as questions on matrix discrepancy, it has also been shown to have implications in the theory of quantum random access codes (Hopkins, Raghavendra, and Shetty, 2022; Bansal, Jiang, and Meka, 2023), in generalizations of the Kadison-Singer problem (Marcus, Spielman, and Srivastava, 2015; Kyng, Luh, and Song, 2020), as well as graph sparsification (Batson, Spielman, and Srivastava, 2014) to name a few.

The celebrated "Matrix Spencer" conjecture (Zouzias, 2012; Meka, 2014) is likely the most important open problem in matrix discrepancy. It asserts that Spencer's Theorem 1.1 can be generalized to the matrix setting, as follows.

Conjecture 1.2 (Matrix Spencer (Zouzias, 2012; Meka, 2014))**.** *There exists $C > 0$ such that for all $n, d \geq 1$, and all $\mathbf{A}_1, \cdots, \mathbf{A}_n$ symmetric $d \times d$ matrices with $\|\mathbf{A}_i\|_{\mathrm{op}} \leq 1$ for all $i \in [n]$:*

$$\mathrm{disc}(\mathbf{A}_1, \cdots, \mathbf{A}_n) \leq C\sqrt{n \max\left(1, \log \frac{d}{n}\right)}.$$

We stress that Spencer's theorem can be seen as the special case of Conjecture 1.2 in which all $\mathbf{A}_i$ commute with each other (and are thus diagonalizable in the same basis). Moreover, a weaker form of Conjecture 1.2, with a bound $\mathcal{O}(\sqrt{n \log d})$ on the right-hand side, can easily be shown to be achievable using a random choice of signs $\varepsilon_i \overset{\text{i.i.d.}}{\sim} \mathrm{Unif}(\{\pm 1\})$, using the non-commutative Khintchine inequality of Lust-Piquard and Pisier (1991). Despite a recent surge in efforts (Hopkins, Raghavendra, and Shetty, 2022; Dadush, Jiang, and Reis, 2022), Conjecture 1.2 remains open at the time of this writing. The best-known result is a proof of Matrix Spencer when $\mathrm{rk}(\mathbf{A}_i) \lesssim n/\log^3 n$ (Bansal, Jiang, and Meka, 2023), and is based on the recent improvements over the non-commutative Khintchine inequality of Bandeira, Boedihardjo, and Handel (2023). In this work, we consider an average-case version of Conjecture 1.2, introduced in Section 1.0.3.

1.0.2 The symmetric binary perceptron

The *symmetric binary perceptron* (SBP) is a random constraint satisfaction problem introduced by Aubin, Perkins, and Zdeborová (2019). It emerged as a variant to the classical asymmetric binary perceptron, a simple model of a neural network storing random patterns, which has a long history of study in computer science, statistical physics, and probability theory (Cover, 1965; Gardner, 1988; Gardner and Derrida, 1988; Krauth and Mézard, 1989; Sompolinsky, Tishby, and Seung, 1990; Talagrand, 1999a; Talagrand, 2010). Given a *margin* $K > 0$, the question of satisfiability in the SBP can be laid out as follows:

Given $\mathbf{g}_1, \cdots, \mathbf{g}_d \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \mathrm{I}_n)$, can we find $\varepsilon \in \{\pm 1\}^n$ such that $\max_{i \in [d]} |\langle \mathbf{g}_i, \varepsilon \rangle| \leq K\sqrt{n}$?

The SBP has received significant attention in the recent literature: its relative simplicity allows studying in detail the relation between its structural properties and the performance of solving algorithms, which can then guide the theory of more complex statistical models, see e.g. Aubin, Perkins, and Zdeborová (2019), Perkins and Xu (2021), Abbe, Li, and Sly (2022), Gamarnik et al. (2022), Kizildag and Wakhare (2023), Barbier et al. (2024), El Alaoui and Gamarnik (2024), and Barbier (2024). Letting $(\mathbf{u}_i)_k \coloneqq (\mathbf{g}_k)_i$ so that $(\mathbf{u}_i)_{i=1}^n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \mathrm{I}_d)$, the question above can be reformulated as whether the set

$$S_K \coloneqq \left\{ \varepsilon \in \{\pm 1\}^n : \left\| \sum_{i=1}^n \varepsilon_i \mathbf{u}_i \right\|_\infty \leq K\sqrt{n} \right\} \tag{2}$$

is non-empty. In this form, it is clear that the SBP can be thought as an average-case version of the discrepancy question solved by Spencer's Theorem 1.1. While Spencer's Theorem 1.1 does not apply to such random vectors (as one can easily show that $\|\mathbf{u}_i\|_\infty \sim \sqrt{2\log n} \gg 1$) the randomness makes the problem amenable to a detailed mathematical analysis with different tools. Concretely, it is shown in Aubin, Perkins, and Zdeborová (2019), Abbe, Li, and Sly (2022), and Gamarnik et al. (2022) (among other results) that the SBP undergoes the following sharp satisfiability/unsatisfiability transition.

Theorem 1.3 (Sharp threshold for the SBP (Aubin, Perkins, and Zdeborová, 2019; Abbe, Li, and Sly, 2022))**.** *Let*

$$\beta_1(K) \coloneqq -\frac{\log \mathbb{P}_{z\sim\mathcal{N}(0,1)}[|z| \leq K]}{\log 2}.$$

Let $Z_K \coloneqq |S_K|$*, see eq.* (2)*, where* $\mathbf{u}_1, \cdots, \mathbf{u}_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \mathrm{I}_d)$*. Then in the limit* $n, d \to \infty$ *with* $n/d \to \beta > 0$*:*

(i) If $\beta < \beta_1(K)$*,* $\lim_{d\to\infty} \mathbb{P}[Z_K \geq 1] = 0$*.*

(ii) If $\beta > \beta_1(K)$*,* $\lim_{d\to\infty} \mathbb{P}[Z_K \geq 1] = 1$*.*

The analysis of Aubin, Perkins, and Zdeborová (2019) is based on the second moment method, and has been refined in subsequent works (Abbe, Li, and Sly, 2022; Gamarnik et al., 2022). In particular, $\beta_1(K)$ can be derived as the critical value of β where the value of $\mathbb{E}[Z_K]$ transitions from being exponentially small (in n, d) to exponentially large.

1.0.3 Average-case matrix discrepancy

In this work, alongside Kunisky and Zhang (2023), we initiate the study of the average-case matrix discrepancy problem. Concretely, given $n, d \geq 1$ a *margin* $\kappa > 0$, and $\mathbf{W}_1, \cdots, \mathbf{W}_n$ random independent matrices, drawn with centered Gaussian i.i.d. elements (up to symmetry) with variance $1/d$, we seek to answer the question:

$$\textbf{(P)} : \textit{Can we find signs } (\varepsilon_1, \cdots, \varepsilon_n) \in \{\pm 1\}^n \textit{ such that } \|\textstyle\sum_{i=1}^n \varepsilon_i \mathbf{W}_i\|_{\mathrm{op}} \leq \kappa\sqrt{n}?$$

This problem can be seen as an average-case analog of Conjecture 1.2, with the simple Gaussian random matrix model serving as a natural starting point. By investigating this simplified case in great detail, we aim to gain insight into Conjecture 1.2, to then probe it for increasingly complex random matrix models. This would form an alternative direction to other recent works (Hopkins, Raghavendra, and Shetty, 2022; Dadush, Jiang, and Reis, 2022; Bansal, Jiang, and Meka, 2023) towards the Matrix Spencer conjecture.

Additionally, **(P)** can naturally be thought of as the matrix discrepancy analog to the symmetric binary perceptron described above: in the SBP one would consider diagonal matrices $\mathbf{W}_i$ with i.i.d. elements drawn from $\mathcal{N}(0,1)$ on the diagonal, and $(\mathbf{g}_i)_k = (\mathbf{W}_i)_{kk}$. Our approach to tackle **(P)** further builds on the methods introduced by Aubin, Perkins, and Zdeborová (2019) for the SBP model. Ultimately, the main goal we pursue is to obtain a detailed understanding of the satisfiability properties of **(P)**, and to reach a counterpart to Theorem 1.3 in the context of average-case matrix discrepancy.

1.1 Main results

Notations and background

We denote $[d] \coloneqq \{1, \cdots, d\}$ the set of integers from 1 to d, and $\mathcal{S}_d$ the set of $d \times d$ real symmetric matrices. For a function $V : \mathbb{R} \to \mathbb{R}$, and $\mathbf{S} \in \mathcal{S}_d$ with eigenvalues $(\lambda_i)_{i=1}^d$, we define $V(\mathbf{S})$ as the matrix with the same eigenvectors as $\mathbf{S}$, and eigenvalues $(V(\lambda_i))_{i=1}^d$. We denote by $\|\mathbf{S}\|_{\mathrm{op}} \coloneqq \max_{i\in[d]} |\lambda_i|$ the operator norm. For any $B \subseteq \mathbb{R}$ we denote by $\mathcal{M}_1^+(B)$ the set of real probability distributions on B. Finally, for a probability measure $\mu \in \mathcal{M}_1^+(\mathbb{R})$ we denote $\Sigma(\mu) \coloneqq \int \mu(\mathrm{d}x)\mu(\mathrm{d}y) \log|x-y|$ its non-commutative entropy. We say that a random matrix $\mathbf{Y} \in \mathcal{S}_d$ is generated from the *Gaussian Orthogonal Ensemble* $\mathrm{GOE}(d)$ if

$$Y_{ij} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, (1+\delta_{ij})/d) \text{ for } i \leq j.$$

The following celebrated result dates back to Wigner (1955), and is one of the foundational results of random matrix theory. For a modern proof and further results, see Anderson, Guionnet, and Zeitouni (2010).

Theorem 1.4 (Semicircle law)**.** *Let $\mathbf{Y} \sim \mathrm{GOE}(d)$, with eigenvalues $(\lambda_i)_{i=1}^d$, and denote $\mu_{\mathbf{Y}} \coloneqq (1/d)\sum_{i=1}^d \delta_{\lambda_i}$ its empirical eigenvalue distribution. Then, almost surely,*

$$\begin{cases} \mu_{\mathbf{Y}} & \xrightarrow[d\to\infty]{\text{weakly}} \rho_{\text{s.c.}}(\mathrm{d}x) = \dfrac{\sqrt{4-x^2}}{2\pi} \mathbb{1}\{|x| \leq 2\}, \\ \|\mathbf{Y}\|_{\mathrm{op}} = \max\limits_{i\in[d]} |\lambda_i| & \xrightarrow[d\to\infty]{} 2. \end{cases}$$

$\rho_{\text{s.c.}}$ in Theorem 1.4 is often called the *semicircle law.* Finally, we generically denote constants as $C > 0$ (or $C_1 > 0, C_2 > 0, \cdots$), whose value may vary from line to line. We will specify their possible dependency on parameters of the problem when relevant.

1.1.1 Setting of the problem

Let $n, d \geq 1$, and $\mathbf{W}_1, \cdots, \mathbf{W}_n \overset{\text{i.i.d.}}{\sim} \mathrm{GOE}(d)$. For any $\kappa > 0$, we define:

$$Z_\kappa \coloneqq \# \left\{ \varepsilon \in \{\pm 1\}^n \text{ s.t. } \left\| \sum_{i=1}^n \varepsilon_i \mathbf{W}_i \right\|_{\mathrm{op}} \leq \kappa\sqrt{n} \right\}. \tag{3}$$

We refer to $n^{-1/2}\|\sum_{i=1}^n \varepsilon_i \mathbf{W}_i\|_{\mathrm{op}}$ as the *margin* of $\varepsilon \in \{\pm 1\}^n$. Z_κ thus counts the number of signings $\varepsilon \in \{\pm 1\}^n$ with margin at most κ.

The case $\kappa > 2$ – Clearly $(1/\sqrt{n})\sum_{i=1}^n \mathbf{W}_i \sim \mathrm{GOE}(d)$, so that (with $\varepsilon_i = 1$ for all $i \in [n]$), for any $\kappa > 2$,

$$\mathbb{P}[Z_\kappa \geq 1] \geq \mathbb{P}_{\mathbf{W}\sim\mathrm{GOE}(d)}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa] \overset{\text{(a)}}{=} 1 - o_d(1),$$

using Theorem 1.4 in (a). Notice that this bound is also the one given by a random choice of $\varepsilon_i \overset{\text{i.i.d.}}{\sim} \text{Unif}(\{\pm 1\})$ (independently of $\{\mathbf{W}_i\}_{i=1}^n$), as again in this case $n^{-1/2}\sum_i \varepsilon_i \mathbf{W}_i \sim \text{GOE}(d)$. In what follows, we therefore focus on the (interesting) regime $\kappa \in (0, 2]$, in which a choice of signings *dependent on the* $\mathbf{W}_i$ must be made in order to get a solution with margin at most κ.

1.1.2 Asymptotics of the first moment

Define, for $\kappa \in (0, 2]$:

$$\tau_1(\kappa) := \frac{1}{\log 2}\left[-\frac{\kappa^4}{128} + \frac{\kappa^2}{8} - \frac{1}{2}\log\frac{\kappa}{2} - \frac{3}{8}\right]. \tag{4}$$

Notice that $\tau_1(2) = 0$, and $\tau_1(\kappa) \to +\infty$ as $\kappa \downarrow 0$. Our first main result is the following sharp asymptotics for the expectation of Z_κ.

Theorem 1.5 (Asymptotics of the first moment)**.** *Let* $\kappa \in (0, 2]$*, and* $\mathbf{W}_1, \cdots, \mathbf{W}_n \overset{\text{i.i.d.}}{\sim} \text{GOE}(d)$*. Assume* $n/d^2 \to \tau \in [0, \infty)$ *as* $n, d \to \infty$*. Then*

$$\lim_{d\to\infty} \frac{1}{d^2}\log \mathbb{E}Z_\kappa = (\tau - \tau_1(\kappa))\log 2, \tag{5}$$

where Z_κ *is defined in eq. (3). In particular, if* $\tau < \tau_1(\kappa)$*, then*

$$\lim_{d\to\infty} \mathbb{P}[Z_\kappa = 0] = \lim_{d\to\infty} \mathbb{P}\left[\min_{\varepsilon\in\{\pm 1\}^n}\left\|\sum_{i=1}^n \varepsilon_i \mathbf{W}_i\right\|_{\text{op}} > \kappa\sqrt{n}\right] = 1.$$

In Section 2 we carry out the proof of Theorem 1.5. It relies on Proposition 2.1, which establishes the large deviations of the operator norm of a $\text{GOE}(d)$ matrix, in the scale d^2. The existence of such a large deviation principle is not surprising given existing results on the large deviations of the empirical measure of $\text{GOE}(d)$ matrices (Ben Arous and Guionnet, 1997), and we partially build upon these results in the first part of our proof. Further, using results of logarithmic potential theory (Edward B Saff and Totik, 2013) and the Tricomi theorem (Tricomi, 1985), we are able to then solve the variational principle arising from this large deviations result, and its outcome gives eq. (4). Our proof also implies a limiting theorem for the spectral measure of a matrix $\mathbf{W} \sim \text{GOE}(d)$ constrained on the event $\|\mathbf{W}\|_{\text{op}} \leq \kappa$, see Theorem 2.2.

Relation to previous results – Theorem 1.5 is a refinement of Theorem 1.13 of Kunisky and Zhang (2023), which show that $Z_\kappa = 0$ with high probability if $\kappa \leq \delta \cdot 4^{-\tau}$, for an (unspecified) absolute constant $\delta > 0$. For instance, the bound of Kunisky and Zhang (2023) correctly predicts $\tau_1(\kappa) \sim -\log\kappa/\log 4$ for $\kappa \to 0$, but fails to capture that $\tau_1(\kappa) \to 0$ continuously as $\kappa \to 2$.

1.1.3 The satisfiability region

We start by introducing a function $\tau_2(\kappa)$, which will serve as a threshold for the validity of our satisfiability analysis.

Proposition 1.6. *For any* $\eta > 0$*, let* $\delta_\eta \in (0, 1)$ *to be the unique solution to:*

$$-\frac{1+\delta}{2}\log\frac{1+\delta}{2} - \frac{1-\delta}{2}\log\frac{1-\delta}{2} = \frac{\eta}{1+\eta}\log 2. \tag{6}$$

For any $\eta > 0$, let

$$\widetilde{\tau}(\eta, \kappa) := \max\left\{(1+\eta)\tau_1(\kappa), \frac{1+\delta_\eta^2}{2(1-\delta_\eta^2)^2} + \left[\frac{2\delta_\eta(3+\delta_\eta^2) - (1+3\delta_\eta^2)}{4(1-\delta_\eta^2)^3} + \frac{2(1+\delta_\eta)^5}{(1-\delta_\eta^2)^4}\right]\kappa^2\right\}. \tag{7}$$

We define

$$\tau_2(\kappa) := \min_{u \in [0,\kappa]} \min_{\eta > 0} \widetilde{\tau}(\eta, u). \tag{8}$$

Then $\kappa \mapsto \tau_2(\kappa)$ is a continuous and non-increasing function of κ.

Proposition 1.6 is elementary, we prove it in Section 3.1 for completeness, along with a straightforward way to evaluate numerically $\tau_2(\kappa)$, see eq. (31). We are now ready to state our main result on the existence of solutions with a given required margin. It is based on the second moment method, building upon similar techniques to the ones used in the symmetric binary perceptron (Aubin, Perkins, and Zdeborová, 2019).

Theorem 1.7 (Satisfiability region). *Let $\kappa \in (0, 2]$. Let $n, d \geq 1$, such that, as $d \to \infty$, $n/d^2 \to \tau > \tau_2(\kappa)$ defined in eq. (8). For $\mathbf{W}_1, \cdots, \mathbf{W}_n \overset{\text{i.i.d.}}{\sim} \mathrm{GOE}(d)$, we have (recall the definition of Z_κ in eq. (3)):*

$$\lim_{d\to\infty} \mathbb{P}[Z_\kappa \geq 1] = \lim_{d\to\infty} \mathbb{P}\left[\min_{\varepsilon \in \{\pm 1\}^n} \left\|\sum_{i=1}^n \varepsilon_i \mathbf{W}_i\right\|_{\mathrm{op}} \leq \kappa\sqrt{n}\right] = 1.$$

In Section 3, we prove Theorem 1.7. We establish concentration of Z_κ in Proposition 3.2, upper bounding $\mathbb{E}[Z_\kappa^2]/\mathbb{E}[Z_\kappa]^2$ by some (large) constant as $d \to \infty$. Our proof relies on a discrete analog of Laplace's method, combined with showing concentration results (via a log-Sobolev inequality) for the distribution of correlated Gaussian matrices under spectral norm constraints. We finally strengthen the result of Proposition 3.2 thanks to the general techniques on sharp transitions for integer feasibility problems developed in Altschuler (2023), and deduce Theorem 1.7. Let us emphasize that having access to two-sided bounds on the first moment asymptotics (which are here sharp, and given in Theorem 1.5) is crucial in order to develop our second moment analysis.

1.1.4 Failure of the second moment method

Finally, we show that in average-case matrix discrepancy, an equivalent to Theorem 1.3 can not hold, in the sense that the satisfiability threshold is not always given by the asymptotics of the first moment of Z_κ.

Theorem 1.8 (Failure of the second moment method in part of the phase diagram). *For $\kappa \in (0, 2]$, let*

$$\tau_{\text{fail.}}(\kappa) := \frac{1}{2}\left(\frac{\kappa^2}{4} - 1\right)^4. \tag{9}$$

Then $n, d \to \infty$ with $n/d^2 \to \tau$, if $\tau < \tau_{\text{fail.}}(\kappa)$

$$\liminf_{d\to\infty} \frac{1}{d^2} \log \frac{\mathbb{E}[Z_\kappa^2]}{\mathbb{E}[Z_\kappa]^2} > 0. \tag{10}$$

Theorem 1.8 is based on a local analysis of a so-called second moment potential. This analysis leverages the concentration results developed for the proof of Theorem 1.7, and we carry out the proof in Section 4. We will discuss further in Section 1.2 the consequences of Theorem 1.8 in relation to our other main results.

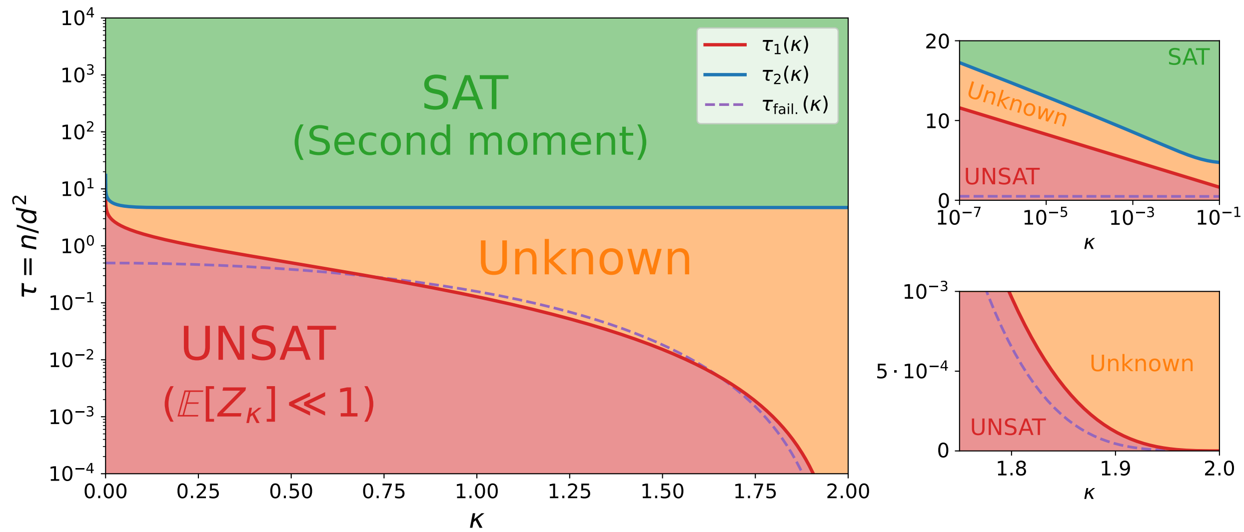

Figure 1: Sketch of the satisfiable (SAT) and unsatisfiable (UNSAT) regimes in average-case matrix discrepancy, as proven by Theorems 1.5 and 1.7. The border of the SAT region is given by $\tau_2(\kappa)$, see Proposition 1.6. Numerically, we find $\tau_2(\kappa \to 2) \simeq 4.71$. The orange region is not characterized by our results, and remains open. The dotted purple line shows $\tau_{\text{fail.}}(\kappa)$: according to Theorem 1.8, for $\tau < \tau_{\text{fail.}}(\kappa)$ the number of solutions does not concentrate around its expectation (see also Fig. 2). The right plots show the limits $\kappa \downarrow 0$ (top) and $\kappa \uparrow 2$ (bottom). We emphasize that $\tau_1(\kappa), \tau_2(\kappa) \to +\infty$ as $\kappa \downarrow 0$.

1.2 Discussion and consequences

In Figure 1, we plot a sketch of the phase diagram of the problem, as established by Theorems 1.5,1.7 and 1.8. Our results characterize a large part of the (κ, τ) phase diagram: we discuss in the following some important consequences, as well as open directions stemming from our analysis.

Balancing $\Theta(d^2)$ random matrices – An immediate consequence of Theorem 1.7 is the following corollary.

Corollary 1.9. *For any $\tau > 0$ large enough*[1]*, if $n, d \to \infty$ with $n/d^2 \to \tau$, and letting $\mathbf{W}_1, \cdots, \mathbf{W}_n \overset{\text{i.i.d.}}{\sim}$ $\mathrm{GOE}(d)$, then (with high probability) there exists $\varepsilon \in \{\pm 1\}^n$ with margin*

$$\frac{1}{\sqrt{n}} \left\| \sum_{i=1}^{n} \varepsilon_i \mathbf{W}_i \right\|_{\text{op}} \leq \kappa_c(\tau) < 2.$$

Furthermore, $\kappa_c(\tau) \to 0$ as $\tau \to \infty$.

Corollary 1.9 establishes that for n large enough but still in the scale $n = \Theta(d^2)$, one can find solutions with margin arbitrarily close to 0. As far as we know, our result is the first proof that a number $n = \Theta(d^2)$ of Gaussian random matrices can be balanced in a way to make their spectrum *macroscopically shrink* compared to the typical spectrum of a Gaussian matrix. This solves an open problem posed by Kunisky and Zhang (2023) (see Remark 2.6 there).

Tightness of the second moment analysis – Similarly to Aubin, Perkins, and Zdeborová (2019) (and subsequent works) for the symmetric binary perceptron (SBP), we use a second moment

[1] Numerically, we find $\tau \geq 5$ is enough, see Fig. 1.

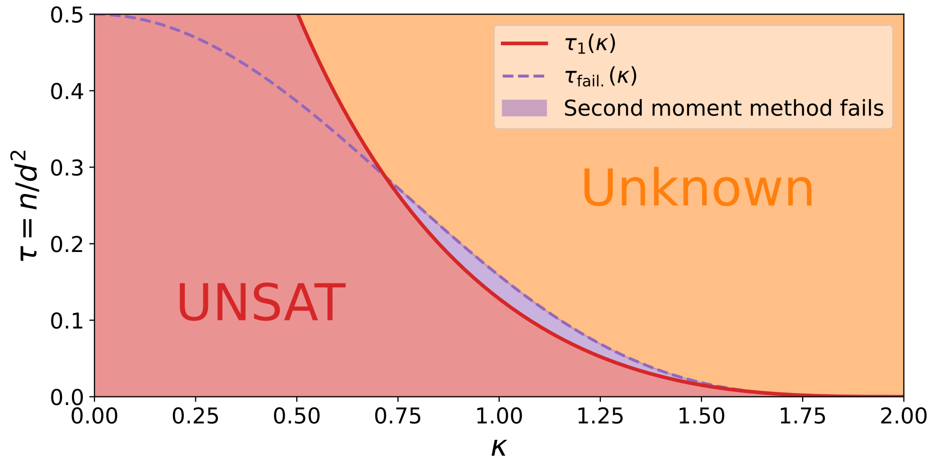

Figure 2: Illustration of Theorem 1.8. There exists a region of parameters (in purple) $\tau \in (\tau_1(\kappa), \tau_c(\kappa))$ for which the number of solutions to average-case matrix discrepancy does not concentrate, and the second moment method fails at characterizing the feasibility of the problem.

approach to characterize the feasibility of this problem. However, while this method gives a tight threshold for the SBP in Theorem 1.3 (and thus leaves no "Unknown" region in the SBP counterpart to Figure 1), the situation in average-case matrix discrepancy is quite different.

First, the threshold $\tau_2(\kappa)$ is likely not of eq. (8) is optimal, as the upper bounds on $\mathbb{E}[Z_\kappa^2]$ proven in Section 3 are not expected to be tight. For this reason, the second moment ratio $\mathbb{E}[Z_\kappa^2]/\mathbb{E}[Z_\kappa]^2$ might still be bounded by a constant (cf. Proposition 3.2) even for some values $\tau \leq \tau_2(\kappa)$. For instance, one might conjecture that this includes values of τ arbitrarily close to 0 when κ approaches 2, which is not captured by our current bounds. Improving the estimates we obtain in Section 3 to obtain a sharp study of the range of parameters (τ, κ) with bounded second moment ratio is significantly more complex than in the SBP setting: it requires a precise understanding of the large deviation properties of the law of $(\|\mathbf{W}_1\|_{\mathrm{op}}, \|\mathbf{W}_2\|_{\mathrm{op}})$, where $\mathbf{W}_1, \mathbf{W}_2$ are two correlated $\mathrm{GOE}(d)$ matrices, both conditioned on having small spectral norm (see eq. (36)). This is an involved problem of large deviations theory applied to random matrices (Guionnet, 2022), and we leave it as a future direction.

Furthermore, and very interestingly, we show that *the second moment method does not yield a sharp satisfiability threshold* in average-case matrix discrepancy. This is implies by Theorem 1.8: indeed, there exists a range of $\kappa \in (0, 2)$ such that $\tau_1(\kappa) < \tau_c(\kappa)$, so that if $\tau \in (\tau_1(\kappa), \tau_c(\kappa))$, then both $\lim(1/d^2) \log \mathbb{E}[Z_\kappa] > 0$ and $\liminf(1/d^2) \log \mathbb{E}[Z_\kappa^2]/\mathbb{E}[Z_\kappa]^2 > 0$. For such values of (τ, κ), the variance of Z_κ is thus exponentially large (in d^2), and an approach based solely on the second moment method will fail at characterizing the feasibility of the problem. Numerically, we find that $\tau_1(\kappa) < \tau_c(\kappa)$ in the range $0.718 \lesssim \kappa \lesssim 1.652$. We summarize these findings in Fig. 2.

We stress that the result of Theorem 1.8 is in stark contrast with the symmetric binary perceptron, in which the second moment method succeeds in the entirety of the phase diagram (Theorem 1.3). In particular, this unveils a potentially richer picture for the geometry of the solution space than in the SBP. In this regard, we emphasize that the failure of the second moment method shown in Theorem 1.8 might extend to other parts of the "Unknown" region of the phase diagram (beyond the purple region of Fig. 2) which are not captured by the local analysis we use in our proof: we refer to Section 4 for details.

Sharp threshold sequence – Theorem 7 of Altschuler (2023) (see also Lemma 3.3) directly implies the existence of a sharp threshold sequence for average-case matrix discrepancy. Using Theorems 1.5 and 1.7, we can locate it in the closure of the "Unknown region" of Fig. 1.

Corollary 1.10 (Sharp threshold sequence)**.** *Let $n, d \geq 1$, and assume $n/d^2 \to \tau > 0$ as $d \to \infty$. There exists a sequence $\kappa_c(d, \tau)$ such that:*

(i) For any $\varepsilon > 0$, $\mathbb{P}[Z_{\kappa_c - \varepsilon} \geq 1] = o_d(1)$, while $\mathbb{P}[Z_{\kappa_c + \varepsilon} \geq 1] = 1 - o_d(1)$.

(ii) $\tau_1(\kappa_c) \leq \tau \leq \tau_2(\kappa_c)$, i.e. (κ_c, τ) is in the closure of the "Unknown" region of Fig. 1.

The existence of a sharp threshold sequence has been established recently in a variety of other perceptron-like problems, see for instance Talagrand (1999b), Talagrand (2011), Xu (2021), Nakajima and Sun (2023), and Altschuler (2023). Corollary 1.10 shows the existence of a sharp threshold κ_c depending on d, and bounds it in an interval of size $\Theta(1)$. Note that we further conjecture $\kappa_c(d, \tau)$ to converge to a well-defined limit, as we plan to discuss in a future work.

Structure of the solution space – Beyond satisfiability, a natural question concerns the geometric structure of the space of solutions $\{\varepsilon \in \{\pm 1\}^n : \|\sum_i \varepsilon_i \mathbf{W}_i\|_{\mathrm{op}} \leq \kappa\sqrt{n}\}$. In the symmetric binary perceptron, the geometric structure of the solution space was investigated by Aubin, Perkins, and Zdeborová (2019), Perkins and Xu (2021), and Abbe, Li, and Sly (2022), and was shown to exhibit a "frozen-1RSB" structure: that is, typically, solutions are isolated and far apart. Whether the solution space to average-case matrix discrepancy presents similar geometric features is a stimulating question, especially as the failure of the second moment method in parts of the phase diagram might indicate a richer picture than in the SBP. We leave this question open for future investigation.

Algorithmic discrepancy – The past decade has seen a surge of interest in *algorithmic discrepancy*, that is the design of efficient algorithms to produce signings $\varepsilon \in \{\pm 1\}^n$ minimizing a discrepancy objective such as eq. (1). In the classical context of vector discrepancy, this line of work was initiated by Bansal (2010), and we refer to Bansal, Jiang, and Meka (2023) and Kunisky and Zhang (2023) for a more detailed description of the literature that followed. For the problem of average-case matrix discrepancy, Kunisky and Zhang (2023) propose an online algorithm that is able to achieve a discrepancy $\|\sum_i \varepsilon_i \mathbf{W}_i\|_{\mathrm{op}} \lesssim d \log(n + d)$ for a large class of random matrices $\mathbf{W}_i$, including the GOE distribution. In the regime $n = \Theta(d^2)$, this unfortunately falls short of obtaining a discrepancy lower than $2\sqrt{n}$. Whether one can design a polynomial-time algorithm that can output (with large probability) a low-discrepancy $\varepsilon \in \{\pm 1\}^n$ in parts of the SAT phase described in Fig. 1 remains an open question. For the symmetric binary perceptron, which is the vector analog to our problem, it was recently shown that the phase diagram presents a large computational-to-statistical gap in which low-discrepancy solutions exist, while large classes of polynomial-time algorithms can not find them, and that this was related to the geometric structure of the solution space (Bansal and Spencer, 2020; Gamarnik et al., 2022).

2 Sharp asymptotics of the first moment

We carry out in this section the proof of Theorem 1.5. A direct computation from eq. (3) using linearity of expectation yields:

$$\mathbb{E} Z_\kappa = \sum_{\varepsilon \in \{\pm 1\}^n} \mathbb{P}\left[\left\|\sum_{i=1}^n \varepsilon_i \mathbf{W}_i\right\|_{\mathrm{op}} \leq \kappa\sqrt{n}\right] = 2^n \, \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa], \tag{11}$$

where $\mathbf{W} \sim \mathrm{GOE}(d)$. The main result we prove is the following.

Proposition 2.1 (Left large deviations for the operator norm of a GOE(d) matrix)**.** *Let* $\mathbf{W} \sim \mathrm{GOE}(d)$*. For any* $\kappa > 0$*:*

$$\lim_{d\to\infty} \frac{1}{d^2} \log \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa] = \begin{cases} \frac{\kappa^4}{128} - \frac{\kappa^2}{8} + \frac{1}{2}\log\frac{\kappa}{2} + \frac{3}{8} & \text{if } \kappa \leq 2, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Using Proposition 2.1 in eq. (11) (recall that $n/d^2 \to \tau$) yields eq. (5). The second result of Theorem 1.5 is a direct consequence of Markov's inequality combined with eq. (5). Notice that Markov's inequality even shows that $\mathbb{P}[Z_\kappa > 0]$ goes to zero exponentially fast in d^2 for $\tau < \tau_1(\kappa)$.

In the remainder of Section 2, we focus on proving Proposition 2.1. We note first that for $\kappa > 2$ we have $\log \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa] = o_d(1)$ as a consequence of Theorem 1.4. We thus focus on the case $\kappa \in (0, 2]$ in what follows.

Sketch of proof and important related work – Our proof of Proposition 2.1 builds upon the seminal results and proof techniques of Ben Arous and Guionnet (1997), which established the large deviations of the empirical spectral measure of $\mathbf{W} \sim \mathrm{GOE}(d)$, with respect to the weak topology. Our approach yields the value of the limit in eq. (12) as a variational principle over probability measures supported in $[-\kappa, \kappa]$, which we can solve using the theory of logarithmic potentials (Edward B Saff and Totik, 2013) and Tricomi's theorem (Tricomi, 1985), similarly to the alternative proof of Wigner's semicircle law obtained by Ben Arous and Guionnet (1997) from their large deviations result. Similar arguments also appeared in the theoretical physics literature (Dean and Majumdar, 2006; Vivo, Majumdar, and Bohigas, 2007; Dean and Majumdar, 2008; Majumdar and Schehr, 2014). We refer to Anderson, Guionnet, and Zeitouni (2010) and Guionnet (2022) for more background and open problems in the theory of large deviations for random matrices.

Remark – The following result is a byproduct of our analysis.

Theorem 2.2 (Limiting spectral density of a constrained GOE(d) matrix)**.** *For* $\kappa \in (0, 2]$*, denote* $\mathbb{P}_\kappa$ *the law of* $\mathbf{W} \sim \mathrm{GOE}(d)$ *conditioned on* $\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa$*. If* $\mathbf{W} \sim \mathbb{P}_\kappa$*, then its empirical spectral density* $\mu_{\mathbf{W}}$ *converges weakly (as* $d \to \infty$*, and a.s.) to* $\mu_\kappa(\mathrm{d}x) := \rho_\kappa(x)\mathrm{d}x$ *given, for* $x \in (-\kappa, \kappa)$*, by:*

$$\rho_\kappa(x) := \frac{4 + \kappa^2 - 2x^2}{4\pi\sqrt{\kappa^2 - x^2}}. \tag{13}$$

Notice that $\rho_2(x) = \sqrt{4 - x^2}/(2\pi)$ *is Wigner's semicircle law, see Theorem 1.4.*

We illustrate the form of $\rho_\kappa(x)$ in Figure 3. Theorem 2.2 is proven in Section 2.4.

2.1 Background on large deviations

Our proof leverages the main result of Ben Arous and Guionnet (1997), which is a large deviation principle for the empirical eigenvalue distribution of GOE(d) matrices[1]. The space $\mathcal{M}_1^+(\mathbb{R})$ of probability measures on $\mathbb{R}$ is endowed with its usual weak topology. It is metrizable by the Dudley metric (Bogachev and Ruas, 2007):

$$d(\mu, \nu) := \left\{ \left| \int f \mathrm{d}\mu - \int f \mathrm{d}v \right| \, : \, |f(x)| \leq 1 \text{ and } |f(x) - f(y)| \leq |x - y|, \forall (x, y) \in \mathbb{R}^2 \right\}. \tag{14}$$

[1]Notice that Ben Arous and Guionnet (1997) use a convention where GOE(d) matrices have off-diagonal entries with variance $1/(2d)$. We state their result adapted to our conventions.

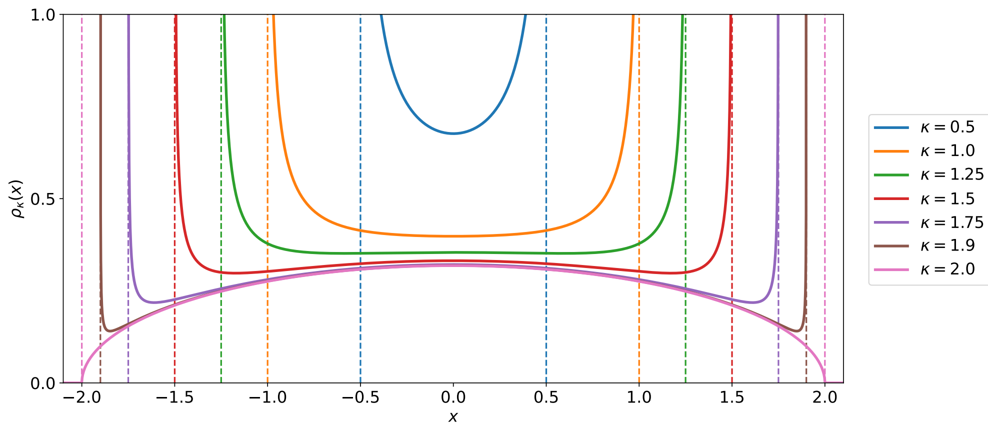

Figure 3: $\rho_\kappa(x)$ of eq. (13) for different values of κ. For $\kappa = 2$ one recovers the semicircle law.

Recall that $\Sigma(\mu) := \int \mu(\mathrm{d}x)\mu(\mathrm{d}y) \log|x-y|$.

Proposition 2.3 (Ben Arous and Guionnet (1997))**.** *Let* $\mathbf{W} \sim \mathrm{GOE}(d)$*. We denote* $\mu_{\mathbf{W}} := (1/d)\sum_{i=1}^d \delta_{\lambda_i(\mathbf{W})}$ *its empirical spectral distribution. For* $\mu \in \mathcal{M}_1^+(\mathbb{R})$*, we define:*

$$I(\mu) := -\frac{1}{2}\Sigma(\mu) + \frac{1}{4}\int \mu(\mathrm{d}x)\, x^2 - \frac{3}{8}. \tag{15}$$

Then:

(i) $I : \mathcal{M}_1^+(\mathbb{R}) \to [0, \infty]$ *is a strictly convex function and a good rate function, i.e.* $\{I_1 \le M\}$ *is a compact subset of* $\mathcal{M}_1^+(\mathbb{R})$ *for any* $M > 0$*.*

(ii) The law of $\mu_{\mathbf{W}}$ *satisfies a large deviation principle, in the scale* d^2*, with rate function* I*, that is for any open (respectively closed) subset* $O \subseteq \mathcal{M}_1^+(\mathbb{R})$ *(respectively* $F \subseteq \mathcal{M}_1^+(\mathbb{R})$*):*

$$\begin{cases} \liminf\limits_{d\to\infty} \dfrac{1}{d^2}\log \mathbb{P}[\mu_{\mathbf{W}} \in O] \ge -\inf\limits_{\mu\in O} I(\mu), \\ \limsup\limits_{d\to\infty} \dfrac{1}{d^2}\log \mathbb{P}[\mu_{\mathbf{W}} \in F] \le -\inf\limits_{\mu\in F} I(\mu). \end{cases}$$

More general large deviations results – Ben Arous and Guionnet (1997) also prove large deviations results for the empirical measure of $\mathbf{W}$ sampled under more general distributions, with density proportional to $\exp\{-\mathrm{Tr}[V(\mathbf{W})]\}$, for a continuous potential V growing fast enough at infinity, see also Section 2.6 of Anderson, Guionnet, and Zeitouni (2010). One could approach Proposition 2.1 by noticing that the law of $\mathbf{W} \sim \mathrm{GOE}(d)$ constrained to $\|\mathbf{W}\|_{\mathrm{op}} \le \kappa$ can be written in this form, with a discontinuous potential $V(x) = x^2 + \mathbb{1}\{|x| > \kappa\} \times \infty$. However, we will see that the large deviations of the empirical measure is only used to prove the upper bound in Proposition 2.1, and the particular form of the operator norm constraint allows us to rely solely on Proposition 2.3, rather than adapting the general results of Ben Arous and Guionnet (1997) and Anderson, Guionnet, and Zeitouni (2010) to this discontinuous potential.

2.2 Large deviation upper bound for the operator norm

We prove here the upper bound for $(1/d^2)\log \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \le \kappa]$ in Proposition 2.1.

Let $Q \coloneqq \{\mu \in \mathcal{M}_1^+(\mathbb{R}) \,:\, \mu([-\kappa,\kappa]) = 1\}$. By the Portmanteau theorem, Q is sequentially closed under weak convergence, and thus closed since the weak topology on $\mathcal{M}_1^+(\mathbb{R})$ is metrizable. We apply Proposition 2.1 to get:

$$\begin{aligned}
\limsup_{d\to\infty} \frac{1}{d^2} \log \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \le \kappa] &= \limsup_{d\to\infty} \frac{1}{d^2} \log \mathbb{P}[\mu_{\mathbf{W}} \in Q], \\
&\le - \inf_{\mu\in Q} I(\mu), \\
&= - \inf_{\mu \in \mathcal{M}_1^+([-\kappa,\kappa])} I(\mu).
\end{aligned} \tag{16}$$

The last equality follows since $I(\mu_{|[-\kappa,\kappa]}) = I(\mu)$ for all $\mu \in Q$, see eq. (15). In order to characterize the minimizer of $I(\mu)$ for $\mu \in \mathcal{M}_1^+([-\kappa,\kappa])$, we rely on classical results of logarithmic potential theory, such as Theorem 1.3 of Chapter I of Edward B Saff and Totik (2013), see also Mhaskar and E. Saff (1985) and Theorem 2.4 of Ben Arous and Guionnet (1997). In our context, the "admissible weight function" of Edward B Saff and Totik (2013) reads

$$w(x) = \exp\{-x^2/4\}\mathbb{1}\{|x| \le \kappa\}.$$

Adapting the results of the aforementioned literature to our notations, we obtain the following theorem.

Theorem 2.4 (Edward B Saff and Totik (2013))**.**
Let $\kappa > 0$ and $E_\kappa \coloneqq \inf_{\mu\in\mathcal{M}_1^+([-\kappa,\kappa])} I(\mu)$. Then

(i) $E_\kappa < \infty$.

(ii) There exists a unique $\mu_\kappa^\star \in \mathcal{M}_1^+([-\kappa,\kappa])$ such that $I(\mu_\kappa^\star) = E_\kappa$.

(iii) $\mu_\kappa^\star$ is the unique measure in $\mathcal{M}_1^+([-\kappa,\kappa])$ such that for $\mu_\kappa^\star$-almost all x:

$$\int \mu_\kappa^\star(\mathrm{d}y)\, \log|x-y| = \frac{x^2}{4} + \frac{1}{4}\int \mu_\kappa^\star(\mathrm{d}y)\, y^2 - \frac{3}{4} - 2E_\kappa.$$

One can further show that this theorem allows to prove that a candidate measure is the optimal one without computing E_κ.

Lemma 2.5. *For $\kappa \in (0,2]$, assume that $\mu \in \mathcal{M}_1^+([-\kappa,\kappa])$ and $C \in \mathbb{R}$ are such that for all $x \in (-\kappa,\kappa)$:*

$$\int \mu(\mathrm{d}y)\, \log|x-y| = \frac{x^2}{4} + C. \tag{17}$$

Then $\mu = \mu_\kappa^\star$.

Note that Lemma 2.5 can also be seen as a consequence of Theorem 3.3 of Chapter I of Edward B Saff and Totik (2013). We give here a short proof for the sake of completeness.

Proof of Lemma 2.5 –. For any σ, ν real signed measures, we have (recall eq. (15))

$$I(\sigma+\nu) - I(\sigma) = -\frac{1}{2}\Sigma(\nu) + \int \nu(\mathrm{d}x)\left[\frac{x^2}{4} - \int \sigma(\mathrm{d}y)\log|x-y|\right].$$

Applying this formula to $\sigma = \mu$ and $\nu = \mu_\kappa^\star - \mu$, we reach:

$$
\begin{aligned}
I(\mu_\kappa^\star) - I(\mu) &= -\frac{1}{2}\Sigma(\mu_\kappa^\star - \mu) + \int (\mu_\kappa^\star - \mu)(\mathrm{d}x)\left[\frac{x^2}{4} - \int \mu(\mathrm{d}y)\log|x-y|\right], \\
&\overset{\text{(a)}}{=} -\frac{1}{2}\Sigma(\mu_\kappa^\star - \mu), \\
&\overset{\text{(b)}}{\geq} 0.
\end{aligned}
$$

We used eq. (17) in (a) and the fact that $\mu_\kappa^\star$ has no atom since $I(\mu_\kappa^\star) < \infty$ (Ben Arous and Guionnet, 1997), so we can restrict the integral to $x \in (-\kappa, \kappa)$. In (b) we used the following classical property of the non-commutative entropy $\Sigma(\mu)$, which can be found e.g. as Proposition II.2.2 in Faraut (2014).

Lemma 2.6. *Let μ be a signed measure on $\mathbb{R}$ with compact support, and such that $\int_{\mathbb{R}} \mu(\mathrm{d}x) = 0$. Let $\hat{\mu}(t) := \int \mu(\mathrm{d}x)\, e^{itx}$ be the Fourier transform of μ. Then*

$$
\Sigma(\mu) := \int \mu(\mathrm{d}x)\mu(\mathrm{d}y)\log|x-y| = -\int_0^\infty \frac{|\hat{\mu}(t)|^2}{t}\mathrm{d}t.
$$

In particular, $\Sigma(\mu) \leq 0$.

This shows that $I(\mu) \leq I(\mu_\kappa^\star) = \inf_{\mu \in \mathcal{M}_1^+([-\kappa,\kappa])} I(\mu)$. By (*ii*) of Theorem 2.4, $\mu = \mu_\kappa^\star$. $\square$

Thanks to Lemma 2.5, we can give an exact formula for $\mu_\kappa^\star$ by exhibiting a candidate measure satisfying eq. (17), which in turns implies an exact formula for E_κ.

Proposition 2.7. *Let $\kappa \in (0, 2]$. Recall that $E_\kappa = \inf_{\mu \in \mathcal{M}_1^+([-\kappa,\kappa])} I(\mu)$ is reached in a unique measure $\mu_\kappa^\star$. Let $\mu_\kappa(\mathrm{d}x) := \rho_\kappa(x)\mathrm{d}x$ be given, for $x \in (-\kappa, \kappa)$, by:*

$$
\rho_\kappa(x) := \frac{4 + \kappa^2 - 2x^2}{4\pi\sqrt{\kappa^2 - x^2}}. \tag{18}
$$

Then $\mu_\kappa = \mu_\kappa^\star$, and

$$
E_\kappa = I(\mu_\kappa^\star) = -\frac{\kappa^4}{128} + \frac{\kappa^2}{8} - \frac{1}{2}\log\frac{\kappa}{2} - \frac{3}{8}.
$$

Combining Proposition 2.7 with eq. (16), we obtain the upper bound, for $\kappa \in (0, 2]$:

$$
\limsup_{d\to\infty} \frac{1}{d^2}\log \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa] \leq \frac{\kappa^4}{128} - \frac{\kappa^2}{8} + \frac{1}{2}\log\frac{\kappa}{2} + \frac{3}{8}. \tag{19}
$$

It therefore just remains to prove Proposition 2.7.

Proof of Proposition 2.7. By Lemma 2.5, to show that $\mu_\kappa = \mu_\kappa^\star$ it is enough to show that eq. (17) holds for μ_κ. Since $U(x) := \int \mu_\kappa(\mathrm{d}y)\log|x-y|$ satisfies (in the sense of distributions)

$$
U'(x) = \mathrm{P.V.}\int \frac{\rho_\kappa(y)}{x-y}\mathrm{d}y,
$$

it is enough to check for any $x \in (-\kappa, \kappa)$:

$$
\mathrm{P.V.}\int_{-r}^{r} \frac{\rho_\kappa(y)}{x-y}\mathrm{d}y = \frac{x}{2}, \tag{20}
$$

where P.V. refers to the principal value. Notice that

$$\text{P.V.} \int_{-r}^{r} \frac{\rho_\kappa(y)}{x-y} \mathrm{d}y = \lim_{\varepsilon \downarrow 0} \mathrm{Re} \left\{ \int_{-r}^{r} \frac{\rho_\kappa(y)}{x + i\varepsilon - y} \mathrm{d}y \right\}.$$

We compute $G_\kappa(z) \coloneqq \int_{-r}^{r} \rho_\kappa(y)/(z-y)\mathrm{d}y$ for all z such that $\mathrm{Im}(z) > 0$. Changing variables to $y = \kappa \cos\theta$, and then to $\zeta = e^{i\theta}$, we have:

$$\begin{aligned} G_\kappa(z) &= \frac{1}{4\pi} \int_0^\pi \frac{4 + \kappa^2 - 2\kappa^2 \cos^2\theta}{z - \kappa\cos\theta} \mathrm{d}\theta, \\ &= \frac{1}{8\pi} \int_{-\pi}^{\pi} \frac{4 + \kappa^2 - 2\kappa^2 \cos^2\theta}{z - \kappa\cos\theta} \mathrm{d}\theta, \\ &= \frac{1}{8i\pi} \oint_{|\zeta|=1} \frac{\kappa^2\zeta^4 - 8\zeta^2 + \kappa^2}{\zeta^2(\kappa\zeta^2 - 2\zeta z + \kappa)} \mathrm{d}\zeta. \end{aligned}$$

The denominator has three poles: $\{0, \zeta_-, \zeta_+\}$, where $\zeta_\pm \coloneqq (z \pm \sqrt{z^2 - \kappa^2})/\kappa$ (we choose the branch of the square root such that $\mathrm{Im}[\sqrt{w}] \geq 0$ for all $w \in \mathbb{C}$). Since $\mathrm{Im}(z) > 0$, it is easy to show that $|\zeta_-| < 1 < |\zeta_+|$. We then apply the residue theorem and find:

$$G_\kappa(z) = \frac{z}{2} + \frac{4 + \kappa^2 - 2z^2}{4\sqrt{z^2 - \kappa^2}}. \tag{21}$$

Taking $\lim_{\varepsilon \to 0} \mathrm{Re}[G_\kappa(x + i\varepsilon)]$ for $|x| < \kappa$ yields eq. (20), and thus $\mu_\kappa = \mu_\kappa^\star$. It remains to compute $E_\kappa = I(\mu_\kappa^\star)$. One can use the same arguments as above (based on the residue theorem) to show

$$\int \mu_\kappa^\star(\mathrm{d}x)\, x^2 = \frac{\kappa^2(8 - \kappa^2)}{16}. \tag{22}$$

By (iii) of Theorem 2.4 and Lemma 2.5, we then have for all $x \in (-\kappa, \kappa)$:

$$E_\kappa = \frac{\kappa^2(8-\kappa^2)}{128} - \frac{3}{8} + \frac{x^2}{8} - \frac{1}{2}\int_{-\kappa}^{\kappa} \rho_\kappa(y) \log|x - y| \,\mathrm{d}y. \tag{23}$$

Notice that eq. (21) is valid for all z with $\mathrm{Im}(z) > 0$. In particular, we reach from it, that for all $x \geq 0$:

$$\text{P.V.} \int_{-r}^{r} \frac{\rho_\kappa(y)}{x - y} \mathrm{d}y = \begin{cases} \dfrac{x}{2} & \text{if } x < \kappa, \\ \dfrac{x}{2} + \dfrac{4 + \kappa^2 - 2x^2}{4\sqrt{x^2 - \kappa^2}} & \text{if } x > \kappa. \end{cases}$$

Since this is an integrable function, we have

$$\int_{-\kappa}^{\kappa} \rho_\kappa(y) \log|x-y| \,\mathrm{d}y = \begin{cases} \dfrac{x^2}{4} + C & \text{if } x \leq \kappa, \\ \dfrac{x^2}{4} + C - \dfrac{x\sqrt{x^2-\kappa^2}}{4} - \log\left(\dfrac{x - \sqrt{x^2 - \kappa^2}}{\kappa}\right) & \text{if } x > \kappa. \end{cases}$$

Using that $\int_{-\kappa}^{\kappa} \rho_\kappa(y) \log|x-y|\mathrm{d}y - \log x \to 0$ as $x \to \infty$ yields $C = \log(\kappa/2) - \kappa^2/8$. Eq. (23) becomes:

$$\begin{aligned} E_\kappa = I(\mu_\kappa^\star) &= \frac{\kappa^2(8-\kappa^2)}{128} - \frac{3}{8} - \frac{1}{2}\left[\log\frac{\kappa}{2} - \frac{\kappa^2}{8}\right], \\ &= -\frac{\kappa^4}{128} + \frac{\kappa^2}{8} - \frac{1}{2}\log\frac{\kappa}{2} - \frac{3}{8}, \end{aligned}$$

which ends the proof. □

Remark: predicting the form of ρ_κ – In order to predict the density ρ_κ given by eq. (18), we used an argument based on a heuristic application of Tricomi's theorem (Tricomi, 1985), which states that if eq. (20) is satisfied and ρ_κ is supported on $[-\kappa, \kappa]$, then

$$\rho_\kappa(x) = \frac{1}{\pi\sqrt{\kappa^2 - x^2}} \left[C - \frac{1}{\pi} \mathrm{P.V.} \int_{-\kappa}^{\kappa} \frac{\sqrt{\kappa^2 - y^2}}{x - y} \times \left(\frac{y}{2}\right) \mathrm{d}y \right], \tag{24}$$

for some constant C chosen to ensure $\int_{-\kappa}^{\kappa} \rho_\kappa(x)\mathrm{d}x = 1$. A careful evaluation of eq. (24) based on the residue theorem yields eq. (18).

2.3 Large deviation lower bound for the operator norm

We focus now on the lower bound for $(1/d^2) \log \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa]$ in Proposition 2.1.

Unfortunately the large deviation statement of Proposition 2.3 is not enough to obtain the corresponding lower bound to eq. (16), because the set of probability measures supported in $[-\kappa, \kappa]$ has empty interior under the weak topology[1]. Instead, we come back to the joint law of the eigenvalues of a GOE(d) matrix, and restrict the integration domain to a small neighborhood of the quantiles of $\mu_\kappa^\star$. This strategy is similar to the one used in the proof of the large deviation lower bound in Ben Arous and Guionnet (1997).

The joint law of the eigenvalues $(\lambda_1, \cdots, \lambda_d)$ of $\mathbf{W}$ is well-known thanks to the rotation invariance of the law of $\mathbf{W}$. We have (see e.g. Theorem 2.5.2 of Anderson, Guionnet, and Zeitouni (2010)):

$$\mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa] = \frac{\int_{[-\kappa,\kappa]^d} \prod_{i<j} |\lambda_i - \lambda_j| e^{-\frac{d}{4}\sum_{i=1}^d \lambda_i^2} \prod_{i=1}^d \mathrm{d}\lambda_i}{\int_{\mathbb{R}^d} \prod_{i<j} |\lambda_i - \lambda_j| e^{-\frac{d}{4}\sum_{i=1}^d \lambda_i^2} \prod_{i=1}^d \mathrm{d}\lambda_i}. \tag{25}$$

The denominator (or partition function) can be computed from Selberg's integrals (Mehta, 2014). Its limit is given by

$$\lim_{d\to\infty} \frac{1}{d^2} \log \int_{\mathbb{R}^d} \prod_{i<j} |\lambda_i - \lambda_j| e^{-\frac{d}{4}\sum_{i=1}^d \lambda_i^2} \prod_{i=1}^d \mathrm{d}\lambda_i = -\frac{3}{8}. \tag{26}$$

Let $\delta \in (0, \kappa)$, and recall the definition of ρ_κ in eq. (18). In what follows, we let $\nu_\delta \coloneqq \rho_{\kappa-\delta}$. We define the quantiles of ν_δ as

$$-(\kappa - \delta) = x_0^{(d)} < x_1^{(d)} < \cdots < x_d^{(d)} < x_{d+1}^{(d)} = \kappa - \delta,$$

with, for all $i \in \{0, \cdots, d\}$:

$$\int_{x_i^{(d)}}^{x_{i+1}^{(d)}} \nu_\delta(u)\,\mathrm{d}u = \frac{1}{d+1}.$$

We will drop the subscript and write x_i for $x_i^{(d)}$ to lighten notations.

Clearly, the empirical measure $(1/d)\sum_{i=1}^d \delta_{x_i}$ weakly converges to ν_δ as $d \to \infty$. Notice that $\{(\lambda_i)_{i=1}^d : |\lambda_i - x_i| \leq \delta, \ \forall i \in [1, d]\} \subseteq [-\kappa, \kappa]^d$, so that from eqs. (25) and (26):

$$e^{-\frac{3d^2}{8} + o(d^2)} \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa] \tag{27}$$

[1] For any $\mu \in \mathcal{M}_1^+(\mathbb{R})$, there is a sequence μ_n weakly converging to μ while $\mathrm{supp}(\mu_n) \not\subseteq [-\kappa, \kappa]$.

$$
\begin{aligned}
&\geq \int_{[-\delta,\delta]^d} \prod_{i<j} |u_i - u_j + x_i - x_j| e^{-\frac{d}{4}\sum_{i=1}^d (u_i+x_i)^2} \prod_{i=1}^d \mathrm{d}u_i, \\
&\overset{\text{(a)}}{\geq} \int_{[-\delta,\delta]^d \cap \Delta_d} \prod_{i<j} (u_j - u_i + x_j - x_i) e^{-\frac{d}{4}\sum_{i=1}^d (u_i+x_i)^2} \prod_{i=1}^d \mathrm{d}u_i, \\
&\overset{\text{(b)}}{\geq} \int_{[-\delta,\delta]^d \cap \Delta_d} \prod_{i+1<j} (x_j - x_i) \prod_{i=1}^{d-1} [x_{i+1} - x_i]^{1/2} [u_{i+1} - u_i]^{1/2} e^{-\frac{d}{4}\sum_{i=1}^d (\delta + |x_i|)^2} \prod_{i=1}^d \mathrm{d}u_i,
\end{aligned}
$$

where we defined in (a) the set $\Delta_d \coloneqq \{u_1 < \cdots < u_d\}$, and used in (b) that $u_i \leq u_j$ and $x_i \leq x_j$, as well as the inequality $A + B \geq \sqrt{AB}$ for $A, B \geq 0$. The integral on the variables u_i can be lower-bounded as follows:

$$
\begin{aligned}
\int_{[-\delta,\delta]^d \cap \Delta_d} \prod_{i=1}^{d-1} \sqrt{u_{i+1} - u_i} \prod_{i=1}^d \mathrm{d}u_i &= \delta^{(3d-1)/2} \int_{[-1,1]^d \cap \Delta_d} \prod_{i=1}^{d-1} \sqrt{u_{i+1} - u_i} \prod_{i=1}^d \mathrm{d}u_i, \\
&\geq \delta^{(3d-1)/2} \prod_{i=1}^d \int_{-1+\frac{2(i-1)}{d}}^{-1+\frac{2i-1}{d}} \mathrm{d}u_i \left(\prod_{i=1}^{d-1} \sqrt{u_{i+1} - u_i}\right), \\
&\geq \left(\frac{\delta}{d}\right)^{(3d-1)/2}.
\end{aligned} \tag{28}
$$

Combining eqs. (27) and (28):

$$
\begin{aligned}
\liminf_{d\to\infty} \frac{1}{d^2} \log \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa] \geq \frac{3}{8} - \frac{\delta^2}{4} + \liminf_{d\to\infty} \Bigg[&-\frac{\delta}{2d} \sum_{i=1}^d |x_i| - \frac{1}{4d} \sum_{i=1}^d x_i^2 \\
&+ \frac{1}{2d^2} \sum_{i=1}^{d-1} \log(x_{i+1} - x_i) + \frac{1}{d^2} \sum_{i,j=1}^d \log(x_j - x_i) \mathbb{1}\{j > i+1\} \Bigg].
\end{aligned}
$$

By the weak convergence described above, and since $\sum_{i=1}^d x_i^2 = \sum_{i=1}^d x_i^2 \mathbb{1}\{|x_i| \leq \kappa\}$, we get by the Portmanteau theorem:

$$
\begin{aligned}
\liminf_{d\to\infty} \frac{1}{d^2} \log \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa] \geq \frac{3}{8} - \frac{\delta^2}{4} - \frac{\delta}{2} \int |x| \nu_\delta(\mathrm{d}x) - \frac{1}{4} \int x^2 \nu_\delta(\mathrm{d}x) \\
+ \liminf_{d\to\infty} \left[\frac{1}{2d^2} \sum_{i=1}^{d-1} \log(x_{i+1} - x_i) + \frac{1}{d^2} \sum_{i,j=1}^d \log(x_j - x_i) \mathbb{1}\{j > i+1\} \right].
\end{aligned} \tag{29}
$$

Finally, notice that

$$
\begin{aligned}
\Sigma(\nu_\delta) &= 2 \sum_{0 \leq i,j \leq d} \int_{x_i}^{x_{i+1}} \nu_\delta(\mathrm{d}x) \int_{x_j}^{x_{j+1}} \nu_\delta(\mathrm{d}y) \log(y - x) \mathbb{1}\{x < y\}, \\
&\leq \sum_{i=0}^d \int_{x_i}^{x_{i+1}} \nu_\delta(\mathrm{d}x) \int_{x_i}^{x_{i+1}} \nu_\delta(\mathrm{d}y) \log|y - x| + 2 \sum_{0 \leq i < j \leq d} \frac{\log(x_{j+1} - x_i)}{(d+1)^2}, \\
&\leq \frac{1}{(d+1)^2} \left[\sum_{i=0}^d \log(x_{i+1} - x_i) + 2 \sum_{0 \leq i < j \leq d} \log(x_{j+1} - x_i) \right], \\
&\overset{\text{(a)}}{\leq} \frac{1}{(d+1)^2} \left[\sum_{i=1}^{d-1} \log(x_{i+1} - x_i) + 2 \sum_{i,j=1}^d \log(x_j - x_i) \mathbb{1}\{j > i+1\} \right] + \frac{2d+1}{(d+1)^2} \log 2\kappa.
\end{aligned}
$$

We used in (a) that $|x_i - x_j| \leq 2(\kappa - \delta) \leq 2\kappa$ for all i, j. Using this in eq. (29) gives:

$$\liminf_{d\to\infty} \frac{1}{d^2} \log \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa] \geq \frac{3}{8} - \frac{\delta^2}{4} - \frac{\delta}{2}\int |x|\nu_\delta(\mathrm{d}x) - \frac{1}{4}\int x^2 \nu_\delta(\mathrm{d}x) + \frac{1}{2}\Sigma(\nu_\delta),$$
$$\geq -\frac{\delta}{2}\int |x|\nu_\delta(\mathrm{d}x) - \frac{\delta^2}{4} - I(\nu_\delta).$$

Recall that $\nu_\delta = \rho_{\kappa-\delta}$, so that taking the limit $\delta \to 0$, we get:

$$\liminf_{d\to\infty} \frac{1}{d^2} \log \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa] \geq -I(\rho_\kappa),$$

Proposition 2.7 gives the value of $I(\rho_\kappa)$, which implies the lower bound

$$\liminf_{d\to\infty} \frac{1}{d^2} \log \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa] \geq \frac{\kappa^4}{128} - \frac{\kappa^2}{8} + \frac{1}{2}\log\frac{\kappa}{2} + \frac{3}{8}. \tag{30}$$

Combining eqs. (16) and (30) ends the proof of Proposition 2.1. $\square$

2.4 Limiting spectral distribution of a norm-constrained Gaussian matrix

We prove here Theorem 2.2. A large deviations upper bound for the law of $\mu_{\mathbf{W}}$ can be obtained by combining Proposition 2.3 and Proposition 2.1, as we state in the following corollary.

Corollary 2.8. *Let $\kappa \in (0, 2]$ and $\mathbf{W} \sim \mathbb{P}_\kappa$. We denote $\mu_{\mathbf{W}} := (1/d)\sum_{i=1}^d \delta_{\lambda_i(\mathbf{W})}$ its empirical spectral distribution. Recall the definition of $I(\mu)$ in eq. (15), and of $\mu_\kappa^\star(\mathrm{d}x) = \rho_\kappa(x)\mathrm{d}x$ in eq. (18). Then the law of $\mu_{\mathbf{W}}$ satisfies a large deviation upper bound, in the scale d^2, with rate function $J_\kappa(\mu) := I(\mu) - I(\mu_\kappa^\star)$, that is for any closed subset $F \subseteq \mathcal{M}_1^+([-\kappa, \kappa])$:*

$$\limsup_{d\to\infty} \frac{1}{d^2} \log \mathbb{P}_\kappa[\mu_{\mathbf{W}} \in F] \leq -\inf_{\mu\in F} J_\kappa(\mu).$$

Proof of Corollary 2.8. Because $\mu_{\mathbf{W}} \in F \Rightarrow \|\mathbf{W}\|_{\mathrm{op}} \leq \kappa$, we have

$$\mathbb{P}_\kappa[\mu_{\mathbf{W}} \in F] = \mathbb{P}_{\mathbf{W}\sim\mathrm{GOE}(d)}[\mu_{\mathbf{W}} \in F \,|\, \|\mathbf{W}\|_{\mathrm{op}} \leq \kappa],$$
$$= \frac{\mathbb{P}_{\mathbf{W}\sim\mathrm{GOE}(d)}[\mu_{\mathbf{W}} \in F]}{\mathbb{P}_{\mathbf{W}\sim\mathrm{GOE}(d)}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa]}.$$

Since F is closed in $\mathcal{M}_1^+([-\kappa, \kappa])$ under the weak topology, it is closed as well in $\mathcal{M}_1^+(\mathbb{R})$. Thus:

$$\limsup_{d\to\infty} \frac{1}{d^2} \log \mathbb{P}_\kappa[\mu_{\mathbf{W}} \in F] \overset{\text{(a)}}{=} \limsup_{d\to\infty} \frac{1}{d^2} \log \mathbb{P}_{\mathbf{W}\sim\mathrm{GOE}(d)}[\mu_{\mathbf{W}} \in F] + I(\mu_\kappa^\star),$$
$$\overset{\text{(b)}}{\leq} -\inf_{\mu\in F} J_\kappa(\mu),$$

using Proposition 2.1 in (a), and Proposition 2.3 in (b). $\square$

While we surely anticipate that one can obtain a corresponding large deviation lower bound for $\mu_{\mathbf{W}}$ (and thus a full large deviation principle), Corollary 2.8 is enough to imply Theorem 2.2. Indeed, for any $\delta > 0$, if $B(\mu_\kappa^\star, \delta) \subseteq \mathcal{M}_1^+([-\kappa, \kappa])$ is the open ball of radius δ centered in $\mu_\kappa^\star$ for the distance of eq. (14), then

$$\limsup_{d\to\infty} \frac{1}{d^2} \log \mathbb{P}_\kappa[\mu_{\mathbf{W}} \notin B(\mu_\kappa^\star, \delta)] \leq -\inf_{\mu\in B(\mu_\kappa^\star,\delta)^c} [J_\kappa(\mu)].$$

Since J_κ is a good rate function (see Proposition 2.3) and has a unique minimizer (cf. (ii) of Theorem 2.4), $\inf_{\mu \in B(\mu^\star_\kappa, \delta)^c}[J_\kappa(\mu)] > 0$. Therefore, by the Borel-Cantelli lemma,

$$\mathbb{P}[\limsup_{d\to\infty} d(\mu_{\mathbf{W}}, \mu^\star_\kappa) \leq \delta] = 1,$$

which ends the proof by taking the limit $\delta \to 0$. $\square$

3 The satisfiability region

Section 3.1 is dedicated to studying the properties of the threshold $\tau_2(\kappa)$. The proof of Theorem 1.7, which is the main goal of this section, is outlined in Section 3.2, and details are given in the remainder of Section 3.

3.1 Properties of the satisfiability bound

We prove here Proposition 1.6, which follows from the following lemma.

Lemma 3.1. *Define, for any $\kappa > 0$, $\bar{\tau}(\kappa) \coloneqq \min_{\eta>0} \widetilde{\tau}(\eta, \kappa)$. Then $\bar{\tau}(\kappa) = \widetilde{\tau}(\eta^\star(\kappa), \kappa)$, where $\eta^\star(\kappa)$ is the unique value of $\eta > 0$ such that:*

$$(1+\eta)\tau_1(\kappa) = \frac{1+\delta_\eta^2}{2(1-\delta_\eta^2)^2} + \left[\frac{2\delta_\eta(3+\delta_\eta^2) - (1+3\delta_\eta^2)}{4(1-\delta_\eta^2)^3} + \frac{2(1+\delta_\eta)^5}{(1-\delta_\eta^2)^4}\right]\kappa^2. \tag{31}$$

Moreover, $\kappa \mapsto \bar{\tau}(\kappa)$ is a continuous function of κ.

Recall that $\tau_2(\kappa) = \min_{u\in[0,\kappa]} \bar{\tau}(u)$, as defined in Lemma 3.1, so that $\kappa \mapsto \tau_2(\kappa)$ is clearly continuous and non-increasing. Moreover, solving eq. (31) gives a simple way to numerically evaluate $\kappa \in [0, 2] \mapsto \bar{\tau}(\kappa)$, which then yields the values of τ_2.

Proof of Lemma 3.1. Let

$$\begin{cases} f_\kappa(\eta) & \coloneqq (1+\eta)\tau_1(\kappa), \\ g_\kappa(\eta) & \coloneqq \dfrac{1+\delta_\eta^2}{2(1-\delta_\eta^2)^2} + \left[\dfrac{2\delta_\eta(3+\delta_\eta^2) - (1+3\delta_\eta^2)}{4(1-\delta_\eta^2)^3} + \dfrac{2(1+\delta_\eta)^5}{(1-\delta_\eta^2)^4}\right]\kappa^2. \end{cases}$$

Recall that δ_η is defined as the unique solution to $H[(1+\delta)/2]/\log 2 = \eta/(1+\eta)$, with $H(p) = -p\log p - (1-p)\log(1-p)$. If $G(\delta) \coloneqq H[(1+\delta)/2]/\log 2$, then G is smooth and strictly decreasing on $[0, 1]$. So $\delta_\eta = G^{-1}[\eta/(1+\eta)]$ is a smooth and strictly decreasing function of $\eta > 0$. It is then immediate by elementary arguments (as done e.g. in eq. (44)) that $g_\kappa(\eta)$ is also a smooth and strictly decreasing function of η.

Moreover, we have $g_\kappa(0^+) = \infty$, $f_\kappa(0^+) = \tau_1(\kappa) < \infty$, and $g_\kappa(\infty) < \infty$, $f_\kappa(\infty) = \infty$. It is then elementary to show that for any $\kappa > 0$, $\min_{\eta>0} \widetilde{\tau}(\eta, \kappa) = \min_{\eta>0} \max\{f_\kappa(\eta), g_\kappa(\eta)\}$ is reached in a unique $\eta^\star(\kappa)$, such that $f_\kappa(\eta^\star(\kappa)) = g_\kappa(\eta^\star(\kappa))$, and that $\eta^\star(\kappa)$ is a continuous function of κ, which implies that $\bar{\tau}(\kappa) = f_\kappa(\eta^\star(\kappa))$ is also continuous. $\square$

3.2 Reduction to a second moment upper bound

The main element of our analysis is the following upper bound.

Proposition 3.2. *Let $\kappa \in (0, 2]$. Recall the definition of Z_κ in eq. (3), and of $\bar{\tau}(\kappa)$ in Lemma 3.1. Assume that $\tau > \bar{\tau}(\kappa)$. Then, for $n, d \to \infty$ with $n/d^2 \to \tau$:*

$$\limsup_{d\to\infty} \frac{\mathbb{E}[Z_\kappa^2]}{\mathbb{E}[Z_\kappa]^2} \leq L \cdot \left[1 - \frac{\bar{\tau}(\kappa)}{\tau}\right]^{-1/2},$$

for an absolute constant $L > 0$.

We will detail the proof of Proposition 3.2 in Section 3.3, deferring some intermediate results to Section 3.4 and 3.5. We first show how to deduce Theorem 1.7.

Proof of Theorem 1.7. Since $\tau > \tau_2(\kappa) = \min_{u\in[0,\kappa]} \bar{\tau}(u)$ (see Lemma 3.1), and $Z_{\kappa'} \leq Z_\kappa$ for any $\kappa' \leq \kappa$, we can assume without loss of generality that $\tau > \bar{\tau}(\kappa)$ in order to prove Theorem 1.7.

Because the bound on the right-hand side is strictly higher than 1, Proposition 3.2 is not strong enough to directly guarantee the existence of solutions with high probability. This is a recurring challenge in many random constraint satisfaction problems where $\mathbb{E}[Z_\kappa^2]/\mathbb{E}[Z_\kappa]^2 \to C > 1$. This occurs e.g. in the symmetric binary perceptron, the vector analog of our matrix discrepancy task, see Abbe, Li, and Sly (2022), and prevents from applying the classical second moment method to get high-probability bounds. Fortunately, in Altschuler (2023) the author develops general techniques on sharp transitions for integer feasibility problems, and applies them to show the concentration of the discrepancy $\min_{\varepsilon\in\{\pm1\}^n} \|\sum_{i=1}^n \varepsilon_i \mathbf{W}_i\|_{\mathrm{op}}$.

Lemma 3.3 (Theorem 7 of Altschuler (2023))**.** *Let $d \geq 1$, and $\mathbf{W}_1, \cdots, \mathbf{W}_n \overset{\text{i.i.d.}}{\sim} \mathrm{GOE}(d)$. Let $\mathrm{disc}(\mathbf{W}_1, \cdots, \mathbf{W}_n) \coloneqq \min_{\varepsilon\in\{\pm1\}^n} \|\sum_{i=1}^n \varepsilon_i \mathbf{W}_i\|_{\mathrm{op}}$. Assume that $n/d^2 \to \tau$ as $d \to \infty$. Then there exists $c(\tau) > 0$ such that*

$$\frac{\mathbb{E}[\mathrm{disc}(\mathbf{W}_1, \cdots, \mathbf{W}_n)]}{\sqrt{\mathrm{Var}[\mathrm{disc}(\mathbf{W}_1, \cdots, \mathbf{W}_n)]}} \geq c(\tau)\sqrt{d}.$$

Let us see how the combination of Lemma 3.3 with our second moment estimates (Proposition 3.2) ends the proof of Theorem 1.7.

Since $\bar{\tau}(\kappa)$ is a continuous function of κ by Lemma 3.1, we choose $\delta > 0$ small enough such that $\tau > \bar{\tau}(\kappa - \delta)$. Let $X \coloneqq \mathrm{disc}(\mathbf{W}_1, \cdots, \mathbf{W}_n)$. Notice that $X \leq \|\sum_{i=1}^n \mathbf{W}_i\|_{\mathrm{op}}$, so that $\mathbb{E}X \leq 2\sqrt{n}$ since $(1/\sqrt{n})\sum_{i=1}^n \mathbf{W}_i \sim \mathrm{GOE}(d)$, and $\mathbb{E}\|\mathbf{Y}\|_{\mathrm{op}} \leq 2$ for $\mathbf{Y} \sim \mathrm{GOE}(d)$ (see e.g. Exercise 7.3.5 of Vershynin (2018)). From Lemma 3.3 we thus get

$$\mathrm{Var}(X) \leq \frac{4n}{c(\tau)^2 d}. \tag{32}$$

Recall that the Paley-Zygmund inequality states that for any random variable $X \geq 0$:

$$\mathbb{P}[X > 0] \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$

Applying it to $Z_{\kappa-\delta}$ and using Proposition 3.2, we get that

$$\mathbb{P}[X \leq (\kappa - \delta)\sqrt{n}] \geq L^{-1} \cdot \left[1 - \frac{\bar{\tau}(\kappa - \delta)}{\tau}\right]^{1/2} + o_d(1). \tag{33}$$

Let us denote $C(\tau, \kappa, \delta) := L \cdot [1 - \bar{\tau}(\kappa - \delta)/\tau]^{-1/2}$. By Chebyshev's inequality and eq. (32), we further have for all $t > 0$:

$$\mathbb{P}[X \geq \mathbb{E}X - t] \geq 1 - \frac{4n}{c(\tau)^2 dt^2}.$$

In particular, if $t = c(\tau)^{-1}\sqrt{8C(\tau, \kappa, \delta)n/d}$, we have $\mathbb{P}[X \geq \mathbb{E}X - t] \geq 1 - (2C(\tau, \kappa, \delta))^{-1}$, which combined with eq. (33) implies that

$$\mathbb{E}X \leq (\kappa - \delta)\sqrt{n} + c_2(\tau, \kappa, \delta)\sqrt{\frac{n}{d}},$$

where we redefined the constant $c_2(\tau, \kappa, \delta) > 0$. Again by Chebyshev's inequality and eq. (32), this implies that for all $u > 0$:

$$\mathbb{P}[X \leq (\kappa - \delta)\sqrt{n} + c_2\sqrt{\frac{n}{d}} + u] \geq 1 - \frac{c_1(\tau)n}{du^2}.$$

Picking $u = \delta\sqrt{n} - c_2\sqrt{n/d}$, we have $u \geq (\delta/2)\sqrt{n}$ for n, d large enough, and this yields:

$$\mathbb{P}[X \leq \kappa\sqrt{n}] \geq 1 - \frac{4c_1(\tau)}{\delta^2 d} \to_{d\to\infty} 1,$$

which ends the proof. □

3.3 Proof of the second moment upper bound

We prove here Proposition 3.2. We compute the second moment as:

$$\begin{aligned}
\mathbb{E}[Z_\kappa^2] &= \sum_{\varepsilon, \varepsilon' \in \{\pm 1\}^n} \mathbb{P}\left[\left\|\sum_{i=1}^n \varepsilon_i \mathbf{W}_i\right\|_{\mathrm{op}} \leq \kappa\sqrt{n} \text{ and } \left\|\sum_{i=1}^n \varepsilon'_i \mathbf{W}_i\right\|_{\mathrm{op}} \leq \kappa\sqrt{n}\right], \\
&\overset{\text{(a)}}{=} 2^n \sum_{\varepsilon \in \{\pm 1\}^n} \mathbb{P}\left[\left\|\sum_{i=1}^n \mathbf{W}_i\right\|_{\mathrm{op}} \leq \kappa\sqrt{n} \text{ and } \left\|\sum_{i=1}^n \varepsilon_i \mathbf{W}_i\right\|_{\mathrm{op}} \leq \kappa\sqrt{n}\right], \\
&\overset{\text{(b)}}{=} 2^n \sum_{l=0}^n \binom{n}{l} \mathbb{P}\left[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa \text{ and } \|q_l \mathbf{W} + \sqrt{1 - q_l^2}\mathbf{Z}\|_{\mathrm{op}} \leq \kappa\right].
\end{aligned} \tag{34}$$

In (a) and (b) we used the rotation invariance of the $\mathrm{GOE}(d)$ distribution. In eq. (34), we changed variables to $l := (\langle \varepsilon, \mathbf{1}_n \rangle + n)/2$ and defined the "overlap" $q_l := (1/n)\langle \varepsilon, \mathbf{1}_n \rangle = 2(l/n) - 1$. Furthermore, $\mathbf{W}, \mathbf{Z} \sim \mathrm{GOE}(d)$ independently. We get from eq. (34) that (recall as well eq. (11)):

$$\frac{\mathbb{E}[Z_\kappa^2]}{\mathbb{E}[Z_\kappa]^2} = \frac{1}{2^n} \sum_{l=0}^n \binom{n}{l} \exp\{nG_d(q_l)\}, \tag{35}$$

where for $q \in [-1, 1]$:

$$G_d(q) := \frac{1}{n} \log \frac{\mathbb{P}\left[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa \text{ and } \|q\mathbf{W} + \sqrt{1 - q^2}\mathbf{Z}\|_{\mathrm{op}} \leq \kappa\right]}{\mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa]^2}. \tag{36}$$

Let $H(p) := -p \log p - (1-p)\log(1-p)$ denote the "binary entropy" function. We will leverage the following lemma, which is based on standard asymptotic techniques, and whose proof is deferred to Section 3.4.

Lemma 3.4. *Let $n \geq 1$, and $F_n : [-1,1] \to \mathbb{R}$ such that $F_n(0) = 0$ and $F_n'(0) = 0$. Assume that there exists $(\gamma, \delta) > 0$ such that:*

(i) $\limsup_{n\to\infty} \sup_{|q|\leq \delta} F_n''(q) \leq 1-\gamma$.

(ii) $\limsup_{n\to\infty} \sup_{|q|\geq \delta} \left[F_n(q) + H\left(\frac{1+q}{2}\right)\right] < \log 2$.

Then (with $q_l \coloneqq 2l/n - 1 \in [-1,1]$ for $l \in \{0, \cdots, n\}$):

$$\limsup_{n\to\infty} \frac{1}{2^n} \sum_{l=0}^{n} \binom{n}{l} \exp\{nF_n(q_l)\} \leq \frac{C}{\sqrt{\gamma}},$$

for a global constant $C > 0$.

From eq. (35), in order to finish the proof of Proposition 3.2, it suffices to check conditions (i) and (ii) of Lemma 3.4 for G_d defined in eq. (36), for $\tau > \bar{\tau}(\kappa)$, $\gamma = 1 - \bar{\tau}(\kappa)/\tau$, and some $\delta > 0$.

Recall the definition of $\eta^\star(\kappa)$ in Lemma 3.1. We let $\delta \coloneqq \delta_{\eta^\star(\kappa)}$ as defined by eq. (6).

Condition (ii) – Notice that $G_d(0) = 0$ and that G_d is clearly an even function of q, so $G_d'(0) = 0$ (the smoothness of G_d can be shown by direct computation, as we will see in eq. (38)). Furthermore, we have the trivial bound:

$$G_d(q) + H\left(\frac{1+q}{2}\right) \leq H\left(\frac{1+q}{2}\right) - \frac{1}{n}\log \mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa].$$

Recall that $q \mapsto H[(1+q)/2]$ is even, and strictly decreasing on $[0,1]$. Using Proposition 2.1, and the definition of $\tau_1(\kappa)$ in eq. (4), we get

$$\begin{aligned}
\limsup_{d\to\infty} \sup_{|q|\geq\delta} \left[G_d(q) + H\left(\frac{1+q}{2}\right)\right] &\leq H\left(\frac{1+\delta}{2}\right) + \frac{\tau_1(\kappa)}{\tau}\log 2, \\
&\overset{\text{(a)}}{<} H\left(\frac{1+\delta}{2}\right) + \left[\frac{1}{1+\eta^\star(\kappa)}\right]\log 2, \\
&\overset{\text{(b)}}{=} \left[\frac{\eta^\star(\kappa)}{1+\eta^\star(\kappa)} + \frac{1}{1+\eta^\star(\kappa)}\right]\log 2, \\
&= \log 2,
\end{aligned}$$

using $\tau > \bar{\tau}(\kappa) = (1+\eta^\star(\kappa))\tau_1(\kappa)$ in (a), and the definition of $\delta = \delta_{\eta^\star(\kappa)}$ in (b), cf. eq. (6). We have thus checked condition (ii) of Lemma 3.4.

Condition (i) – We will show that for any $\delta \in (0,1)$:

$$\limsup_{d\to\infty} \sup_{|q|\leq\delta} G_d''(q) \leq \frac{1}{\tau}\left\{\frac{1+\delta^2}{2(1-\delta^2)^2} + \left[\frac{2\delta(3+\delta^2) - (1+3\delta^2)}{4(1-\delta^2)^3} + \frac{2(1+\delta)^5}{(1-\delta^2)^4}\right]\kappa^2\right\}. \tag{37}$$

Let us first show how eq. (37) finishes the proof of Proposition 3.2. We apply if to $\delta = \delta_{\eta^\star(\kappa)}$. By Lemma 3.1, eq. (37) can be rewritten for this value of δ as

$$\limsup_{d\to\infty} \sup_{|q|\leq\delta} G_d''(q) \leq \frac{\bar{\tau}(\kappa)}{\tau} = 1-\gamma,$$

with $\gamma := (1 - \bar{\tau}(\kappa)/\tau)$. This implies that condition (i) of Lemma 3.4 holds with this value of γ, and thus ends the proof of Proposition 3.2, as described above.

Proof of eq. (37) – There remains to show eq. (37). Let $q \in [0, 1)$. We have ($\mathrm{d}\mathbf{W} = \prod_{i \le j} \mathrm{d}W_{ij}$ is the Lebesgue measure over the space $\mathcal{S}_d$ of symmetric matrices):

$$
\begin{aligned}
&\mathbb{P}\left[\|\mathbf{W}\|_{\mathrm{op}} \le \kappa \text{ and } \|q\mathbf{W} + \sqrt{1-q^2}\mathbf{Z}\|_{\mathrm{op}} \le \kappa\right] \\
&= \frac{\int \mathbb{1}\{\|\mathbf{W}\|_{\mathrm{op}} \le \kappa\} e^{-\frac{d}{4}\mathrm{Tr}[\mathbf{W}^2]} \mathbb{P}\left[\|q\mathbf{W} + \sqrt{1-q^2}\mathbf{Z}\|_{\mathrm{op}} \le \kappa\right] \mathrm{d}\mathbf{W}}{\int e^{-\frac{d}{4}\mathrm{Tr}[\mathbf{W}^2]} \mathrm{d}\mathbf{W}}, \\
&= \frac{\int \mathbb{1}\{\|\mathbf{W}\|_{\mathrm{op}} \le \kappa\} e^{-\frac{d}{4}\mathrm{Tr}[\mathbf{W}^2]} \left(\int \mathrm{d}\mathbf{Y} e^{-\frac{d}{4(1-q^2)}\mathrm{Tr}[(\mathbf{Y}-q\mathbf{W})^2]} \mathbb{1}\{\|\mathbf{Y}\|_{\mathrm{op}} \le \kappa\}\right) \mathrm{d}\mathbf{W}}{\left(\int e^{-\frac{d}{4}\mathrm{Tr}[\mathbf{W}^2]} \mathrm{d}\mathbf{W}\right) \left(\int \mathrm{d}\mathbf{Y} e^{-\frac{d}{4(1-q^2)}\mathrm{Tr}[\mathbf{Y}^2]}\right)}, \\
&= \frac{\int \mathbb{1}\{\|\mathbf{W}\|_{\mathrm{op}}, \|\mathbf{Y}\|_{\mathrm{op}} \le \kappa\} e^{-\frac{d}{4(1-q^2)}(\mathrm{Tr}[\mathbf{W}^2] + \mathrm{Tr}[\mathbf{Y}^2]) + \frac{dq}{2(1-q^2)}\mathrm{Tr}[\mathbf{Y}\mathbf{W}]} \mathrm{d}\mathbf{Y}\mathrm{d}\mathbf{W}}{\left(\int \mathrm{d}\mathbf{W} e^{-\frac{d}{4}\mathrm{Tr}[\mathbf{W}^2]}\right)^2 (1-q^2)^{d(d+1)/4}}.
\end{aligned} \tag{38}
$$

Starting from eq. (38), we can compute the derivatives of $G_d(q)$. We will use the shorthand notation

$$
\langle \cdot \rangle_{q,\kappa} := \frac{\int (\cdot) \mathbb{1}\{\|\mathbf{W}\|_{\mathrm{op}}, \|\mathbf{Y}\|_{\mathrm{op}} \le \kappa\} e^{-\frac{d}{4(1-q^2)}(\mathrm{Tr}[\mathbf{W}^2] + \mathrm{Tr}[\mathbf{Y}^2]) + \frac{dq}{2(1-q^2)}\mathrm{Tr}[\mathbf{Y}\mathbf{W}]} \mathrm{d}\mathbf{Y}\mathrm{d}\mathbf{W}}{\int \mathbb{1}\{\|\mathbf{W}\|_{\mathrm{op}}, \|\mathbf{Y}\|_{\mathrm{op}} \le \kappa\} e^{-\frac{d}{4(1-q^2)}(\mathrm{Tr}[\mathbf{W}^2] + \mathrm{Tr}[\mathbf{Y}^2]) + \frac{dq}{2(1-q^2)}\mathrm{Tr}[\mathbf{Y}\mathbf{W}]} \mathrm{d}\mathbf{Y}\mathrm{d}\mathbf{W}}, \tag{39}
$$

i.e. $\langle \cdot \rangle_{q,\kappa}$ is the law of $(\mathbf{W}, \mathbf{Y})$ two correlated GOE(d) matrices (with correlation q), conditioned on the event $\|\mathbf{W}\|_{\mathrm{op}}, \|\mathbf{Y}\|_{\mathrm{op}} \le \kappa$. We get

$$
G_d'(q) = \frac{d(d+1)q}{2n(1-q^2)} + \frac{1}{2n}\left\langle -\frac{dq}{(1-q^2)^2}\mathrm{Tr}[\mathbf{W}^2 + \mathbf{Y}^2] + \frac{d(1+q^2)}{(1-q^2)^2}\mathrm{Tr}[\mathbf{W}\mathbf{Y}] \right\rangle_{q,\kappa}.
$$

Differentiating further, we obtain:

$$
\begin{aligned}
G_d''(q) = &\underbrace{\frac{d(d+1)(1+q^2)}{2n(1-q^2)^2}}_{=:I_1(q)} + \underbrace{\frac{1}{2n}\left\langle -\frac{d(1+3q^2)}{(1-q^2)^3}\mathrm{Tr}[\mathbf{W}^2 + \mathbf{Y}^2] + \frac{2dq(3+q^2)}{(1-q^2)^3}\mathrm{Tr}[\mathbf{W}\mathbf{Y}] \right\rangle_{q,\kappa}}_{=:I_2(q)} \\
&+ \underbrace{\frac{1}{4n}\mathrm{Var}_{\langle \cdot \rangle_{q,\kappa}}\left(-\frac{dq}{(1-q^2)^2}\mathrm{Tr}[\mathbf{W}^2 + \mathbf{Y}^2] + \frac{d(1+q^2)}{(1-q^2)^2}\mathrm{Tr}[\mathbf{W}\mathbf{Y}]\right)}_{=:I_3(q)}.
\end{aligned} \tag{40}
$$

We bound successively the different terms $\{I_a\}_{a=1}^3$ in eq. (40). Since $n/d^2 \to \tau$, we have:

$$
\limsup_{d \to \infty} \sup_{|q| \le \delta} I_1(q) = \frac{1}{\tau} \sup_{|q| \le \delta} \frac{(1+q^2)}{2(1-q^2)^2} = \frac{(1+\delta^2)}{2\tau(1-\delta^2)^2}. \tag{41}
$$

Note that under the law of $\langle \cdot \rangle_{q,\kappa}$ of eq. (39), $\mathbf{W}$ is distributed as a GOE(d) matrix, conditioned to satisfy $\|\mathbf{W}\|_{\mathrm{op}} \le \kappa$. By Theorem 2.2, we know that $\mu_{\mathbf{W}}$ weakly converges (a.s.) to $\mu_\kappa^\star$. Since $\int \mu_{\mathbf{W}}(\mathrm{d}x) x^2 = \int \mu_{\mathbf{W}}(\mathrm{d}x) x^2 \mathbb{1}\{|x| \le \kappa\}$, we have by the Portmanteau theorem and dominated convergence:

$$
\lim_{d \to \infty} \frac{1}{d}\mathbb{E}[\mathrm{Tr}\mathbf{W}^2] = \int \mu_\kappa^\star(\mathrm{d}x)\, x^2\, \mathbb{1}\{|x| \le \kappa\}
$$

$$= \int \mu_\kappa^\star(\mathrm{d}x)\, x^2,$$
$$\stackrel{\text{(a)}}{=} \frac{\kappa^2(8-\kappa^2)}{16},$$

using eq. (22) in (a). In particular this easily implies:

$$\frac{\kappa^2}{4} \le \lim_{d\to\infty} \frac{1}{d}\mathbb{E}[\mathrm{Tr}\mathbf{W}^2] \le \frac{\kappa^2}{2}. \tag{42}$$

By the Cauchy-Schwarz inequality

$$\langle \mathrm{Tr}[\mathbf{W}\mathbf{Y}]\rangle_{q,\kappa} \le \langle (\mathrm{Tr}[\mathbf{W}^2]\mathrm{Tr}[\mathbf{Y}^2])^{1/2}\rangle_{q,\kappa} \le \sqrt{\langle (\mathrm{Tr}\mathbf{W}^2)(\mathrm{Tr}\mathbf{Y}^2)\rangle_{q,\kappa}} \le [\mathbb{E}[(\mathrm{Tr}[\mathbf{W}^2])^2]]^{1/2}.$$

Using the same arguments as above, one gets from this:

$$\limsup_{d\to\infty} \sup_{|q|\le 1} \frac{1}{d}\langle \mathrm{Tr}[\mathbf{W}\mathbf{Y}]\rangle_{q,\kappa} \le \sqrt{\lim_{d\to\infty}\frac{1}{d^2}\mathbb{E}[(\mathrm{Tr}[\mathbf{W}^2])^2]} = \int \mu_\kappa^\star(\mathrm{d}x)\, x^2 \le \frac{\kappa^2}{2}. \tag{43}$$

For a small $\sigma > 0$, we let n, d large enough (by eqs. (42),(43)) such that $\mathbb{E}[\mathrm{Tr}\mathbf{W}^2] \ge d\kappa^2(1-\sigma)/4$ and $\langle \mathrm{Tr}[\mathbf{W}\mathbf{Y}]\rangle_{q,\kappa} \le d\kappa^2(1+\sigma)/2$ for all $|q| \le 1$. We get:

$$\sup_{|q|\le\delta} I_2(q) \le \frac{d^2\kappa^2}{4n} \max_{|q|\le\delta}\left[\frac{2q(3+q^2)(1+\sigma)-(1+3q^2)(1-\sigma)}{(1-q^2)^3}\right].$$

If $f(q) \coloneqq 2q(3+q^2)(1+\sigma)-(1+3q^2)(1-\sigma)$, then for all $q \in [-1,1]$:

$$f'(q) = 6[1+\sigma-q(1-q)+\sigma q(1+q)] \ge 6\left[\frac{3}{4}+\sigma\right] > 0, \tag{44}$$

so that

$$\sup_{|q|\le\delta} I_2(q) \le \frac{d^2\kappa^2}{4n}\left[\frac{2\delta(3+\delta^2)(1+\sigma)-(1+3\delta^2)(1-\sigma)}{(1-\delta^2)^3}\right].$$

Finally, taking $d\to\infty$ and $\sigma\to 0$:

$$\limsup_{d\to\infty}\sup_{|q|\le\delta} I_2(q) \le \frac{\kappa^2}{4\tau}\cdot\frac{2\delta(3+\delta^2)-(1+3\delta^2)}{(1-\delta^2)^3}. \tag{45}$$

Recall that for a real random variable X, we define the sub-Gaussian norm $\|X\|_{\psi_2}$ of X as (Vershynin, 2018):

$$\|X\|_{\psi_2} \coloneqq \inf\{t>0 \,:\, \mathbb{E}[\exp(X^2/t^2)] \le 2\}.$$

To bound I_3, we rely on the following crucial result, which we prove in Section 3.5.

Lemma 3.5 (Concentration of moments under $\langle\cdot\rangle_{q,\kappa}$)**.** *Let $q \in (-1,1)$, $\kappa \in (0,2]$, and*

$$P(X_1,X_2) \coloneqq \sum_{p\ge 0}\sum_{i_1,\cdots,i_p\in\{1,2\}} a_{i_1\cdots i_p} X_{i_1}\cdots X_{i_p}$$

be a polynomial in two non-commutative random variables (X_1, X_2)*. Let* $(\mathbf{W}, \mathbf{Y}) \sim \langle\cdot\rangle_{q,\kappa}$ *given by eq.* (39)*. Then:*

$$\|\mathrm{Tr}\, P(\mathbf{W}, \mathbf{Y}) - \langle \mathrm{Tr}\, P(\mathbf{W}, \mathbf{Y})\rangle_{q,\kappa}\|_{\psi_2} \leq C\sqrt{1+q}\sum_{p\geq 0} p \cdot \kappa^{p-1} \sum_{i_1,\cdots,i_p \in \{1,2\}} |a_{i_1\cdots i_p}|,$$

where $C > 0$ *is an absolute constant. Furthermore, we have the fully explicit bound:*

$$\mathrm{Var}_{\langle\cdot\rangle_{q,\kappa}}[\mathrm{Tr}\, P(\mathbf{W}, \mathbf{Y})] \leq 2(1+q)\left(\sum_{p\geq 0} p \cdot \kappa^{p-1} \sum_{i_1,\cdots,i_p \in \{1,2\}} |a_{i_1\cdots i_p}|\right)^2. \tag{46}$$

Lemma 3.5 is a consequence of a log-Sobolev inequality we prove for $\langle\cdot\rangle_{q,\kappa}$. We apply eq. (46) to $P(X, Y) = -q(X^2 + Y^2) + (1 + q^2)XY$, which yields:

$$\begin{aligned} I_3(q) &\leq \frac{d^2}{4n(1-q^2)^4} \cdot 2(1+q)\left(2\kappa[2q+1+q^2]\right)^2, \\ &= \frac{2d^2(1+q)^5\kappa^2}{n(1-q^2)^4}. \end{aligned}$$

So finally we get:

$$\limsup_{d\to\infty} \sup_{|q|\leq\delta} I_3(q) \leq \frac{2(1+\delta)^5\kappa^2}{\tau(1-\delta^2)^4}. \tag{47}$$

Combining eqs. (41),(45),(47) finishes the proof of eq. (37). As we discussed above, this ends the proof of Proposition 3.2. $\square$

3.4 Discrete Laplace's method for a dimension-dependent exponent

We prove here Lemma 3.4. By hypothesis (ii), we fix $\varepsilon > 0$ such that, for n large enough:

$$\sup_{|q|\geq\delta}\left[F_n(q) + H\left(\frac{1+q}{2}\right)\right] \leq \log 2 - \varepsilon. \tag{48}$$

Recall the classical inequality:

$$\binom{n}{l} \leq e^{nH(l/n)}, \qquad \text{for } l \in \{0, \cdots, n\}. \tag{49}$$

Combining eqs. (48) and (49), we have

$$\begin{aligned} \frac{1}{2^n}\sum_{l=0}^{n} \mathbb{1}\left\{\left|l - \frac{n}{2}\right| > \frac{n\delta}{2}\right\}\binom{n}{l}\exp\{nF_n(q_l)\} &\leq \frac{1}{2^n}\sum_{l=0}^{n} \mathbb{1}\left\{\left|l - \frac{n}{2}\right| > \frac{n\delta}{2}\right\}\exp\{n(\log 2 - \varepsilon)\}, \\ &\leq n\exp\{-n\varepsilon\}. \end{aligned} \tag{50}$$

Let $\sigma \in (0, \gamma)$. By hypothesis (i), we get that for n large enough $F_n''(q) \leq (1 - \gamma + \sigma)$ for all $|q| \leq \delta$. Since $F_n(0) = 0$ and $F_n'(0) = 0$, this implies $F_n(q) \leq (1 - \gamma + \sigma)q^2/2$ for all $|q| \leq \delta$. Therefore,

$$\frac{1}{2^n}\sum_{l=0}^{n} \mathbb{1}\left\{\left|l - \frac{n}{2}\right| \leq \frac{n\delta}{2}\right\}\binom{n}{l}\exp\{nF_n(q_l)\} \leq \frac{1}{2^n}\sum_{l=0}^{n}\binom{n}{l}e^{\frac{n(1-\gamma+\sigma)}{2}q_l^2}. \tag{51}$$

Recall that $q_l = 2(l/n) - 1$. The right-hand side of eq. (51) can now be analyzed with standard extensions of Laplace's method. We use here the following statement, which is a consequence of the proof of Lemma 2 of Achlioptas and Moore (2002).

Lemma 3.6 (Achlioptas and Moore (2002))**.** *There exists $B, C > 0$ such that the following holds. Let G a real analytic positive function on $[0,1]$, and define for $\alpha \in [0,1]$:*

$$g(\alpha) := \frac{G(\alpha)}{\alpha^\alpha (1-\alpha)^{1-\alpha}}.$$

If there exists $\alpha_{\max} \in (0,1)$ a strict global maximum of g in $[0,1]$ such that $g''(\alpha_{\max}) < 0$, then for sufficiently large n:

$$B \cdot \frac{g(\alpha_{\max})^{n+1/2}}{\sqrt{-g''(\alpha_{\max})}} \leq \sum_{l=0}^{n} \binom{n}{l} G(l/n)^n \leq C \cdot \frac{g(\alpha_{\max})^{n+1/2}}{\sqrt{\alpha_{\max}(1-\alpha_{\max})(-g''(\alpha_{\max}))}}.$$

Remark – Lemma 3.6 is stated in Achlioptas and Moore (2002) as

$$C_1 \cdot g(\alpha_{\max})^n \leq \sum_{l=0}^{n} \binom{n}{l} G(l/n)^n \leq C_2 \cdot g(\alpha_{\max})^n,$$

where the constants C_1, C_2 might depend on $\alpha_{\max}$ and $g(\alpha_{\max})$. Their proof (see Appendix A of Achlioptas and Moore (2002)) reveals the dependency of C_1, C_2 on $\alpha_{\max}$ and $g''(\alpha_{\max})$, which we make explicit here.

We apply Lemma 3.6 in eq. (51), with

$$G(x) := \frac{1}{2} e^{\frac{(1-\gamma+\sigma)(2x-1)^2}{2}}$$

Let

$$g(x) := \frac{G(x)}{x^x (1-x)^{1-x}} = \frac{e^{\frac{(1-\gamma+\sigma)(2x-1)^2}{2}}}{2x^x(1-x)^{1-x}}.$$

It is clear that $g(x) = g(1-x)$ for all $x \in [0,1]$, and moreover

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(\log g)(x) = -(1-\gamma+\sigma)(1-2x) + \operatorname{arctanh}(1-2x).$$

Since $\operatorname{arctanh}(u) \geq u$ for all $u \in [0,1)$, we get that for all $x \in (0, 1/2]$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log g)(x) \geq 2(\gamma - \sigma)(1-2x).$$

Combining this with the symmetry $g(x) = g(1-x)$, we obtain that g has a strict global maximum in $x = 1/2$, and we compute $g(1/2) = 1$. Moreover, we get by direct computation that $g''(1/2) = -4(\gamma - \sigma) < 0$. All in all, we reach that for n large enough:

$$\frac{1}{2^n}\sum_{l=0}^{n} \binom{n}{l} e^{-\frac{n(1-\gamma)}{2} q_l^2} \leq \frac{C}{\sqrt{\gamma - \sigma}}. \tag{52}$$

Combining eqs. (50),(51) and (52), we get:

$$\limsup_{n\to\infty} \frac{1}{2^n}\sum_{l=0}^{n} \binom{n}{l} \exp\{nF_n(q_l)\} \leq \limsup_{n\to\infty} [ne^{-n\varepsilon} + C(\gamma-\sigma)^{-1/2}] = C(\gamma-\sigma)^{-1/2}.$$

Letting $\sigma \downarrow 0$ ends the proof of Lemma 3.4. $\square$

3.5 Log-Sobolev inequality for the conditioned law of two correlated $\mathrm{GOE}(d)$

In this section, we start by reminders on log-Sobolev inequalities, before proving such a property for the law of eq. (39), and finally proving Lemma 3.5.

3.5.1 Log-Sobolev inequalities and concentration of measure

Definition 3.1. Let $d \geq 1$. A probability measure $\mu \in \mathcal{M}_1^+(\mathbb{R}^d)$ is said to satisfy the *Logarithmic Sobolev Inequality* (LSI) with constant $c > 0$ if, for any differentiable function f in $L^2(\mu)$, we have

$$\int f^2 \log \frac{f^2}{\int f^2 \,\mathrm{d}\mu} \mathrm{d}\mu \leq 2c \int \|\nabla f\|_2^2 \,\mathrm{d}\mu. \tag{53}$$

We refer the reader to Guionnet (2009) and Anderson, Guionnet, and Zeitouni (2010) for more on the theory of log-Sobolev inequalities and their applications to concentration results in random matrix theory. A particularly useful consequence of the LSI is the following.

Lemma 3.7 (Herbst)**.** *Assume that $\mu \in \mathcal{M}_1^+(\mathbb{R}^d)$ satisfies the LSI with constant c. Let $G : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function, with Lipschitz constant $\|G\|_L$. Then for all $\lambda \in \mathbb{R}$:*

$$\mathbb{E}_\mu \left[e^{\lambda[G - \mathbb{E}G]}\right] \leq \exp\left\{\frac{c\|G\|_L^2 \lambda^2}{2}\right\}.$$

Therefore, for all $\delta > 0$:

$$\mu(|G - \mathbb{E}G| \geq \delta) \leq 2\exp\left\{-\frac{\delta^2}{2c\|G\|_L^2}\right\}.$$

Finally, we will use that a necessary condition for a measure to satisfy the LSI is the so-called *Bakry-Emery (BE)* condition.

Theorem 3.8 (Theorem 4.4.17 of Anderson, Guionnet, and Zeitouni (2010))**.** *Let $d \geq 1$ and $\Phi : \mathbb{R}^d \to \mathbb{R}$ a $\mathcal{C}^2$ function. Assume that Φ satisfies the Bakry-Emery condition:*

$$\mathrm{Hess}\,\Phi(x) \succeq \frac{1}{c}\mathrm{I}_d,$$

for all $x \in \mathbb{R}^d$, for some $c > 0$. Then the measure

$$\mu_\Phi(\mathrm{d}x) := \frac{1}{\mathcal{Z}} e^{-\Phi(x)} \mathrm{d}x$$

satisfies the LSI with constant c.

3.5.2 A log-Sobolev inequality for the law of eq. (39)

We show the following lemma.

Lemma 3.9. *For any $q \in (-1, 1)$ and $\kappa > 0$, the law $\langle\cdot\rangle_{q,\kappa}$ of eq. (39) satisfies the LSI with constant $2(1+q)/d$.*

Proof of Lemma 3.9. Let $\varepsilon > 0$. We denote $\phi_\varepsilon(x) := e^{-x^2/(2\varepsilon)}/\sqrt{2\pi\varepsilon}$. We define

$$V_\varepsilon(x) := -\log\left(\int_{-\kappa}^{\kappa} \mathrm{d}y\, e^{-y^2/2}\, \phi_\varepsilon(x - y)\right).$$

Reminders on log-concavity – A real positive integrable function p is said to be *strongly log-concave* with variance parameter σ^2 (denoted $\mathrm{SLC}(\sigma^2)$) if $p(x) = \phi_{\sigma^2}(x) \cdot e^{\varphi(x)}$, for some concave function $\varphi : \mathbb{R} \to [-\infty, \infty)$. We refer the reader to Saumard and Wellner (2014) for properties of log-concave and strongly log-concave functions and probability distributions. It is clear that $x \mapsto e^{-x^2/2}\mathbb{1}\{|x| \le \kappa\}$ is SLC(1), and that ϕ_ε is $\mathrm{SLC}(\varepsilon)$. By Theorem 3.7 of Saumard and Wellner (2014), if f is $\mathrm{SLC}(\sigma_1^2)$ and g is $\mathrm{SLC}(\sigma_2^2)$, their convolution $f \star g$ is $\mathrm{SLC}(\sigma_1^2 + \sigma_2^2)$. Therefore, e^{-V_ε} is $\mathrm{SLC}(1+\varepsilon)$, which implies (since V_ε is smooth) that $V_\varepsilon''(x) \ge (1+\varepsilon)^{-1}$ for all $x \in \mathbb{R}$. Since V_ε is even, and

$$V_\varepsilon(0) \ge -\log \int_{-\kappa}^{\kappa} \mathrm{d}y \, \phi_\varepsilon(y) \ge 0,$$

we get that $V_\varepsilon(x) \ge x^2/[2(1+\varepsilon)]$ for all $x \in \mathbb{R}$. We define μ_ε as:

$$\mu_\varepsilon(\mathrm{d}\mathbf{Y}, \mathrm{d}\mathbf{W}) \coloneqq \frac{e^{-\frac{d}{2(1-q^2)}(\mathrm{Tr}V_\varepsilon(\mathbf{W}) + \mathrm{Tr}V_\varepsilon(\mathbf{Y})) + \frac{dq}{2(1-q^2)}\mathrm{Tr}[\mathbf{Y}\mathbf{W}]}\mathrm{d}\mathbf{Y}\mathrm{d}\mathbf{W}}{\int e^{-\frac{d}{2(1-q^2)}(\mathrm{Tr}V_\varepsilon(\mathbf{W}) + \mathrm{Tr}V_\varepsilon(\mathbf{Y})) + \frac{dq}{2(1-q^2)}\mathrm{Tr}[\mathbf{Y}\mathbf{W}]}\mathrm{d}\mathbf{Y}\mathrm{d}\mathbf{W}}. \tag{54}$$

Recall $\mathcal{S}_d$ is the set of symmetric $d \times d$ matrices. Since $x \mapsto V_\varepsilon(x) - x^2/(2[1+\varepsilon])$ is convex, by Klein's lemma (cf. Lemma 4.4.12 of Anderson, Guionnet, and Zeitouni (2010) or Lemma 6.4 of Guionnet (2009)), the function $\mathbf{W} \mapsto \mathrm{Tr}V_\varepsilon(\mathbf{W}) - \mathrm{Tr}[\mathbf{W}^2]/(2[1+\varepsilon])$ is also convex. Thus, for all $\mathbf{W} \in \mathcal{S}_d$:

$$\mathrm{Hess}\, V_\varepsilon(\mathbf{W}) \succeq \frac{1}{1+\varepsilon}\mathrm{I}_{\mathcal{S}_d}.$$

All in all we get for any $\mathbf{W}$ and $\mathbf{Y}$:

$$\mathrm{Hess}\left[\frac{d}{2(1-q^2)}(\mathrm{Tr}V_\varepsilon(\mathbf{W}) + \mathrm{Tr}V_\varepsilon(\mathbf{Y})) - \frac{dq}{2(1-q^2)}\mathrm{Tr}[\mathbf{Y}\mathbf{W}]\right] \succeq \frac{d}{2(1-q^2)}\begin{pmatrix} \frac{\mathrm{I}_{\mathcal{S}_d}}{1+\varepsilon} & -q\mathrm{I}_{\mathcal{S}_d} \\ -q\mathrm{I}_{\mathcal{S}_d} & \frac{\mathrm{I}_{\mathcal{S}_d}}{1+\varepsilon} \end{pmatrix},$$

which means

$$\lambda_{\min}\left(\mathrm{Hess}\left[\frac{d}{2(1-q^2)}(\mathrm{Tr}V_\varepsilon(\mathbf{W}) + \mathrm{Tr}V_\varepsilon(\mathbf{Y})) - \frac{dq}{2(1-q^2)}\mathrm{Tr}[\mathbf{Y}\mathbf{W}]\right]\right) \ge \frac{d(1-q-\varepsilon q)}{2(1+\varepsilon)(1-q^2)}.$$

Therefore, by Theorem 3.8, μ_ε satisfies the LSI with constant

$$\frac{2(1+\varepsilon)(1-q^2)}{d(1-q-\varepsilon q)} = \frac{2(1+q)}{d} + o_{\varepsilon\to 0}(1).$$

Finally, notice that we have $V_\varepsilon(x) \to_{\varepsilon \to 0} V(x)$ pointwise, with $V(x)$ defined as:

$$V(x) \coloneqq \begin{cases} \frac{x^2}{2} & \text{if } |x| < \kappa, \\ \frac{\kappa^2}{2} + \log 2 & \text{if } |x| = \kappa, \\ +\infty & \text{if } |x| > \kappa. \end{cases}$$

Since $V_\varepsilon(x) \ge x^2/4$ for $\varepsilon \le 1/2$, we get by dominated convergence and the Portmanteau theorem that $\mu_\varepsilon \to_{\varepsilon \to 0} \mu_0$ weakly, where μ_0 is defined as in eq. (54), replacing V_ε by V. Because the set $\{\|\mathbf{W}\|_{\mathrm{op}} = \kappa\}$ has Lebesgue measure zero, we further have that $\mu_0 = \langle \cdot \rangle_{q,\kappa}$. Since μ_ε satisfies the LSI with constant $2(1+q)/d + o_{\varepsilon\to 0}(1)$, and weakly converges to $\langle \cdot \rangle_{q,\kappa}$ as $\varepsilon \downarrow 0$, we deduce that $\langle \cdot \rangle_{q,\kappa}$ satisfies the LSI[1] with constant $2(1+q)/d$. □

[1] By taking the limit of eq. (53) for well-behaved functions f using weak convergence, and extending to all differentiable and square integrable functions by density. See e.g. the proof of Theorem 4.4.17 in Anderson, Guionnet, and Zeitouni (2010) for details.

3.5.3 Proof of Lemma 3.5

Let $P(X_1, X_2) = \sum_{p\geq 0} \sum_{i_1,\cdots,i_p\in\{1,2\}} a_{i_1\cdots i_p} X_{i_1} \cdots X_{i_p}$. We make use of the following elementary result.

Lemma 3.10 (Lemma 6.2 of Guionnet (2009))**.** *Let Q be a polynomial in two non-commutative variables. Then, for any $\kappa > 0$, the function*

$$(\mathbf{W}, \mathbf{Y}) \in B_{\mathrm{op}}(\kappa) \times B_{\mathrm{op}}(\kappa) \mapsto \mathrm{Tr}[Q(\mathbf{W}, \mathbf{Y})]$$

is Lipschitz with respect to the Euclidean norm, with Lipschitz norm bounded by $\sqrt{d}C(Q, \kappa)$ for some constant $C(Q, \kappa) > 0$. If Q is a monomial of degree p, one can take $C(Q, \kappa) = p\kappa^{p-1}$.

Notice that $\mathrm{supp}(\langle\cdot\rangle_{q,\kappa}) \subseteq B_{\mathrm{op}}(\kappa) \times B_{\mathrm{op}}(\kappa)$. By Lemma 3.10, $f : (\mathbf{W}, \mathbf{Y}) \mapsto \mathrm{Tr}P(\mathbf{W}, \mathbf{Y})$ is thus Lipschitz on the support of $\langle\cdot\rangle_{q,\kappa}$, with Lipschitz constant

$$\|f\|_L \leq \sqrt{d} \sum_{p\geq 0} p \cdot \kappa^{p-1} \sum_{i_1,\cdots,i_p\in\{1,2\}} |a_{i_1\cdots i_p}|.$$

Combining Lemmas 3.9 and 3.7 finishes the proof of Lemma 3.5. Indeed, notice that if X is a random variable such that $\mathbb{E}[X] = 0$ and $\mathbb{E}[e^{\lambda X}] \leq e^{\lambda^2 K^2/2}$ for some $K > 0$ and all $\lambda \in \mathbb{R}$, then $\|X\|_{\psi_2} \leq CK$, and moreover by Taylor expansion close to $\lambda = 0$, we get $\mathrm{Var}(X) \leq K^2$. $\square$

4 Limitations of the second moment method

We prove here Theorem 1.8. Let $\kappa \in (0, 2]$ and $\tau > 0$. We start again from the second moment computation detailed in Section 3.3, and more precisely from eq. (35), which we recall here:

$$\frac{\mathbb{E}[Z_\kappa^2]}{\mathbb{E}[Z_\kappa]^2} = \frac{1}{2^n} \sum_{l=0}^{n} \binom{n}{l} \exp\{nG_d(q_l)\}, \tag{55}$$

where for $q \in [-1, 1]$:

$$G_d(q) := \frac{1}{n} \log \frac{\mathbb{P}\left[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa \text{ and } \|q\mathbf{W} + \sqrt{1-q^2}\mathbf{Z}\|_{\mathrm{op}} \leq \kappa\right]}{\mathbb{P}[\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa]^2}.$$

Using eq. (38) we further have, for any $q \in (-1, 1)$:

$$G_d(q) = \frac{1}{n} \log \frac{\int \mathbb{1}\{\|\mathbf{W}_1\|_{\mathrm{op}}, \|\mathbf{W}_2\|_{\mathrm{op}} \leq \kappa\} e^{-\frac{d}{4(1-q^2)}(\mathrm{Tr}[\mathbf{W}_1^2]+\mathrm{Tr}[\mathbf{W}_2^2])+\frac{dq}{2(1-q^2)}\mathrm{Tr}[\mathbf{W}_1\mathbf{W}_2]} \mathrm{d}\mathbf{W}_1 \mathrm{d}\mathbf{W}_2}{\left(\int \mathrm{d}\mathbf{W}\, \mathbb{1}\{\|\mathbf{W}\|_{\mathrm{op}} \leq \kappa\}\, e^{-\frac{d}{4}\mathrm{Tr}[\mathbf{W}^2]}\right)^2 (1-q^2)^{d(d+1)/4}}. \tag{56}$$

Our proof of Theorem 1.8 uses then the following two technical lemmas, which will allow to control $G_d(q)$ close to $q = 0$.

Lemma 4.1. *Recall the definition of $\tau_{\text{fail.}}(\kappa)$ in eq. (9). We have*

$$\lim_{d\to\infty} G_d''(0) = \frac{\tau_{\text{fail.}}(\kappa)}{\tau}.$$

FIXME: I have an issue: with my current results I am not sure that I can show the following lemma. Worst case scenario I can always assume it (or assume a weaker version where I just assume the second derivative is continuous uniformly in d – at fixed d it is trivial), and mention that I prove the last result under a technical assumption.

Lemma 4.2. *For any $\varepsilon \in (0,1)$, we have:*

$$\limsup_{d\to\infty} \sup_{|q|\le\varepsilon} |G_d^{(3)}(q)| \le \frac{C(\varepsilon,\kappa)}{\tau}.$$

Lemmas 4.1 and 4.2 are proven respectively in Sections 4.1 and 4.2. Their proofs crucially rely on the limiting theorem for spectral distributions we established in Theorem 2.2, as well as on the concentration properties we established for the moments of correlated GOE(d) matrices under spectral norm constraints, see Lemma 3.5.

If we assume that $\tau < \tau_{\text{fail.}}(\kappa)$, by Lemmas 4.1 and 4.2, there exists $\delta > 0$ and $\varepsilon > 0$ (depending on τ, κ) such that, for d large enough:

$$\inf_{|q|\le\varepsilon} G_d''(q) \ge (1+\delta). \tag{57}$$

Recall that $H(p) := -p\log p - (1-p)\log(1-p)$ is the "binary entropy" function. We define $S(q) := H[(1+q)/2] - \log 2$, and

$$\Phi_d(q) := G_d(q) + S(q). \tag{58}$$

Since S is clearly a smooth function of q, and $S''(0) = -1$, from eq. (57) there exists new constants $(\varepsilon,\delta) > 0$ such that

$$\inf_{|q|\le\varepsilon} \Phi_d''(q) \ge \delta. \tag{59}$$

Notice that $\Phi_d(0) = 0$ and that $\Phi_d'(0) = 0$ (since Φ_d is clearly an even function of q), so eq. (59) directly implies that for d large enough:

$$\inf_{|q|\le\varepsilon} \left[\Phi_d(q) - \frac{\delta q^2}{2}\right] \ge 0. \tag{60}$$

Using the classical inequality that for any $l \in \{0,\cdots,n\}$:

$$\binom{n}{l} \ge \frac{1}{n+1} 2^{nH(l/n)},$$

we obtain from eq. (55):

$$\frac{\mathbb{E}[Z_\kappa^2]}{\mathbb{E}[Z_\kappa]^2} \ge \frac{1}{n+1}\sum_{l=0}^{n} \exp\{n\Phi_d(q_l)\} \overset{\text{(a)}}{\ge} \frac{1}{n+1} \sum_{\substack{0\le l\le n \\ |q_l|\le\varepsilon}} \exp\left\{\frac{n\delta q_l^2}{2}\right\},$$

where recall $q_l = 2(l/n) - 1$, and we used eq. (60) in (a). Choosing $l \in \{0,\cdots,n\}$ such that $\varepsilon/2 \le |q_l| \le \varepsilon$, we reach (recall that $n/d^2 \to \tau$):

$$\liminf_{d\to\infty} \frac{1}{d^2} \log \frac{\mathbb{E}[Z_\kappa^2]}{\mathbb{E}[Z_\kappa]^2} \ge \frac{\delta\tau\varepsilon^2}{8} > 0,$$

which ends the proof of Theorem 1.8. $\square$

Remark – Notice that a statement akin to Theorem 1.8 might still hold even if $\Phi_d''(0) < 0$ for large d, as long as Φ_d reaches its global maximum in a value q which is far from 0 as $d \to \infty$, as our proof can then be straightforwardly applied under this assumption. As such, we do not know if $\tau_{\text{fail.}}(\kappa)$ (which comes out of our local analysis around $q = 0$) is a sharp threshold for the concentration of the random variable Z_κ around its expectation.

4.1 Proof of Lemma 4.1

We start from eq. (40), which for $q = 0$ gives:

$$G_d''(0) = \frac{d(d+1)}{2n} - \frac{d}{n}\mathbb{E}[\mathrm{Tr}\mathbf{W}^2] + \frac{d^2}{4n}\mathrm{Var}[\mathrm{Tr}[\mathbf{W}\mathbf{W}']]. \tag{61}$$

In eq. (61), $\mathbf{W}$ and $\mathbf{W}'$ are sampled independently according to the law $\mathbb{P}_\kappa$ of $\mathbf{Z} \sim \mathrm{GOE}(d)$ conditioned on $\|\mathbf{Z}\|_{\mathrm{op}} \leq \kappa$, i.e. for any test function φ:

$$\mathbb{E}_{\mathbb{P}_\kappa}[\varphi(\mathbf{Z})] = \frac{\int \varphi(\mathbf{Z})\, \mathbb{1}\{\|\mathbf{Z}\|_{\mathrm{op}} \leq \kappa\} e^{-\frac{d}{4}\mathrm{Tr}[\mathbf{Z}^2]}\mathrm{d}\mathbf{Z}}{\int \mathbb{1}\{\|\mathbf{Z}\|_{\mathrm{op}} \leq \kappa\} e^{-\frac{d}{4}\mathrm{Tr}[\mathbf{Z}^2]}\mathrm{d}\mathbf{Z}}. \tag{62}$$

We know that for $\mathbf{W} \sim \mathbb{P}_\kappa$, $\mu_{\mathbf{W}}$ weakly converges (a.s.) to $\mu_\kappa^\star$ given by Theorem 2.2. Since $\int \mu_{\mathbf{W}}(\mathrm{d}x)x^2 = \int \mu_{\mathbf{W}}(\mathrm{d}x)x^2 \mathbb{1}\{|x| \leq \kappa\}$, we have by the Portmanteau theorem and dominated convergence:

$$\lim_{d\to\infty} \frac{1}{d}\mathbb{E}[\mathrm{Tr}\mathbf{W}^2] = \int \mu_\kappa^\star(\mathrm{d}x)\, x^2\, \mathbb{1}\{|x| \leq \kappa\} = \int \mu_\kappa^\star(\mathrm{d}x)\, x^2. \tag{63}$$

We now focus on the last term of eq. (61). Notice that $\mathbb{E}[\mathrm{Tr}[\mathbf{W}\mathbf{W}']] = \mathrm{Tr}[(\mathbb{E}\mathbf{W})^2] = 0$, since $\mathbb{E}\mathbf{W}$ because $\mathbb{P}_\kappa$ is symmetric under $\mathbf{W} \leftrightarrow -\mathbf{W}$. Using that, for any orthogonal matrix $\mathbf{O} \in \mathcal{O}(d)$, $\mathbf{W} \overset{\mathrm{d}}{=} \mathbf{O}\mathbf{W}\mathbf{O}^\top$ (as is directly seen from eq. (62)), we further have:

$$\mathrm{Var}[\mathrm{Tr}[\mathbf{W}\mathbf{W}']] = \mathbb{E}[\mathrm{Tr}[\mathbf{W}\mathbf{W}']^2] = \mathbb{E}_{\mathbf{O},\mathbf{\Lambda},\mathbf{\Lambda}'}[\mathrm{Tr}[\mathbf{O}\mathbf{\Lambda}\mathbf{O}^\top\mathbf{\Lambda}']^2]. \tag{64}$$

In eq. (64), $\mathbf{\Lambda} = \mathrm{Diag}(\{\lambda_i\})$ is a diagonal matrix containing the eigenvalues of $\mathbf{W}$ (and similarly for $\mathbf{\Lambda}'$). and $\mathbf{O}$ is an orthogonal matrix sampled from the Haar measure on $\mathcal{O}(d)$, independently of $\mathbf{W}, \mathbf{W}'$. Thus:

$$\mathrm{Var}[\mathrm{Tr}[\mathbf{W}\mathbf{W}']] = \sum_{i,j,k,l} \mathbb{E}[\lambda_i\lambda_k]\mathbb{E}[\lambda_j\lambda_l]\mathbb{E}[O_{ij}^2 O_{kl}^2]. \tag{65}$$

The terms involving λ_i eq. (65) can be computed using the permutation invariance of the law of $\{\lambda_i\}$ as well as the invariance under $\mathbf{\Lambda} \leftrightarrow -\mathbf{\Lambda}$, so that for all $i \in [d]$:

$$\mathbb{E}[\lambda_i^2] = \mathbb{E}[\lambda_1^2] = \frac{1}{d}\sum_{j=1}^{d}\mathbb{E}[\lambda_i^2] = \frac{1}{d}\mathbb{E}[\mathrm{Tr}\mathbf{W}^2], \tag{66}$$

and for $i \neq j$:

$$\mathbb{E}[\lambda_i\lambda_j] = \mathbb{E}[\lambda_1\lambda_2] = \frac{1}{d-1}\mathbb{E}\left[\lambda_1 \sum_{k\geq 2}\lambda_k\right] = \frac{1}{d(d-1)}\mathbb{E}[(\mathrm{Tr}\mathbf{W})^2 - \mathrm{Tr}(\mathbf{W}^2)]. \tag{67}$$

The first moments of the matrix elements of a Haar-sampled orthogonal matrix are elementary, see e.g. Banica, Collins, and Schlenker (2011) for general results, which prove:

$$\mathbb{E}[O_{ij}^2 O_{kl}^2] = \begin{cases} \dfrac{3}{d(d+2)} & (i = k \text{ and } j = l), \\ \dfrac{1}{d(d+2)} & (i = k \text{ and } j \neq l, \text{ or } i \neq k \text{ and } j = l), \\ \dfrac{d+1}{d(d-1)(d+2)} & (i \neq k \text{ and } j \neq l). \end{cases} \tag{68}$$

Using eq. (68) in eq. (65), separating cases in the sum, we get:

$$\begin{aligned} \mathrm{Var}[\mathrm{Tr}[\mathbf{W}\mathbf{W}']] &= \frac{3}{d(d+2)} \cdot d^2 \cdot \mathbb{E}[\lambda_1^2]^2 + \frac{1}{d(d+1)} \cdot 2d^2(d-1) \cdot \mathbb{E}[\lambda_1^2]\mathbb{E}[\lambda_1\lambda_2] \\ &+ \frac{d+1}{d(d-1)(d+2)} \cdot d^2(d-1)^2 \cdot \mathbb{E}[\lambda_1\lambda_2]^2, \\ &= [1 + o_d(1)] \left(3\mathbb{E}[\lambda_1^2]^2 + 2d\mathbb{E}[\lambda_1\lambda_2]\mathbb{E}[\lambda_1^2] + d^2\mathbb{E}[\lambda_1\lambda_2]^2\right). \end{aligned} \tag{69}$$

From eqs. (63) and (66), we have $\mathbb{E}[\lambda_1^2] \to \mathbb{E}_{\mu_\kappa^\star}[X^2]$ as $d \to \infty$. Furthermore, by Lemma 3.5, $\mathbb{E}[(\mathrm{Tr}\mathbf{W})^2] = \mathrm{Var}[\mathrm{Tr}\mathbf{W}] = \mathcal{O}(1)$ as $d \to \infty$, so eq. (67) gives that $d\mathbb{E}[\lambda_1\lambda_2] \to -\mathbb{E}_{\mu_\kappa^\star}[X^2]$ as $d \to \infty$. Plugging these limits in eq. (69) we get:

$$\mathrm{Var}[\mathrm{Tr}[\mathbf{W}\mathbf{W}']] = 2\left(\int \mu_\kappa^\star(\mathrm{d}x)\, x^2\right)^2 + o_{d\to\infty}(1). \tag{70}$$

Finally, combining eqs. (61), (63) and (70) we obtain (recall $n/d^2 \to \tau$):

$$\lim_{d\to\infty} G_d''(0) = \frac{1}{\tau}\left[\frac{1}{2} - \int \mu_\kappa^\star(\mathrm{d}x)\, x^2 + \frac{1}{2}\left(-\frac{1}{2}\int \mu_\kappa^\star(\mathrm{d}x)\, x^2\right)^2\right]. \tag{71}$$

The integral in eq. (71) was already computed in eq. (22): plugging its value in eq. (71) shows that $\lim_{d\to\infty} G_d''(0) = \tau_{\text{fail.}}(\kappa)/\tau$, which ends the proof of Lemma 4.1.

4.2 Proof of Lemma 4.2

We start from eq. (56), which we rewrite as:

$$G_d(q) = \frac{1}{n}\log \mathcal{Z}_d(q) - \frac{d(d+1)}{4n}\log(1-q^2) - \frac{2}{n}\log \int \mathrm{d}\mathbf{W}\, \mathbb{1}\{\|\mathbf{W}\|_{\mathrm{op}} \le \kappa\}\, e^{-\frac{d}{4}\mathrm{Tr}[\mathbf{W}^2]}, \tag{72}$$

where

$$\mathcal{Z}_d(q) := \int \mathbb{1}\{\|\mathbf{W}_1\|_{\mathrm{op}}, \|\mathbf{W}_2\|_{\mathrm{op}} \le \kappa\} e^{-\frac{d}{4(1-q^2)}(\mathrm{Tr}[\mathbf{W}_1^2] + \mathrm{Tr}[\mathbf{W}_2^2]) + \frac{dq}{2(1-q^2)}\mathrm{Tr}[\mathbf{W}_1\mathbf{W}_2]} \mathrm{d}\mathbf{W}_1 \mathrm{d}\mathbf{W}_2. \tag{73}$$

The last term in eq. (72) is independent of q, and the second term is clearly smooth around $q = 0$, and is independent of n, d up to a multiplicative factor. Recalling that $n/d^2 \to \tau$, it is clear that to prove Lemma 4.2 it is sufficient to show:

$$\limsup_{d\to\infty} \sup_{|q|\le\varepsilon} \left|\frac{\partial^3}{\partial q^3}\left[\frac{1}{d^2}\log \mathcal{Z}_d(q)\right]\right| \le C(\varepsilon, \kappa), \tag{74}$$

We define

$$H_d(q, \mathbf{W}_1, \mathbf{W}_2) \coloneqq -\frac{d(\mathrm{Tr}[\mathbf{W}_1^2] + \mathrm{Tr}[\mathbf{W}_2^2])}{4(1-q^2)} + \frac{dq}{2(1-q^2)}\mathrm{Tr}[\mathbf{W}_1\mathbf{W}_2], \tag{75}$$

so that

$$\mathcal{Z}_d(q) = \int \mathbb{1}\{\|\mathbf{W}_1\|_{\mathrm{op}}, \|\mathbf{W}_2\|_{\mathrm{op}} \le \kappa\} e^{H_d(q,\mathbf{W}_1,\mathbf{W}_2)} \mathrm{d}\mathbf{W}_1 \mathrm{d}\mathbf{W}_2.$$

And finally, recall that we defined in eq. (39):

$$\langle \cdot \rangle_{q,\kappa} \coloneqq \frac{\int (\cdot) \mathbb{1}\{\|\mathbf{W}_1\|_{\mathrm{op}}, \|\mathbf{W}_2\|_{\mathrm{op}} \le \kappa\} e^{H_d(q,\mathbf{W}_1,\mathbf{W}_2)} \mathrm{d}\mathbf{W}_1 \mathrm{d}\mathbf{W}_2}{\int \mathbb{1}\{\|\mathbf{W}_1\|_{\mathrm{op}}, \|\mathbf{W}_2\|_{\mathrm{op}} \le \kappa\} e^{H_d(q,\mathbf{W}_1,\mathbf{W}_2)} \mathrm{d}\mathbf{W}_1 \mathrm{d}\mathbf{W}_2}.$$

To lighten notations, we use in what follows the notation $\langle \cdot \rangle$ rather than $\langle \cdot \rangle_{q,\kappa}$. With these notations, we obtain successively:

$$\begin{aligned}
\frac{\partial}{\partial q} \log \mathcal{Z}_d(q) &= \langle \partial_q H_d \rangle, \\
\frac{\partial^2}{\partial q^2} \log \mathcal{Z}_d(q) &= \langle \partial_q^2 H_d \rangle + \langle (\partial_q H_d)^2 \rangle - \langle \partial_q H_d \rangle^2, \\
\frac{\partial^3}{\partial q^3} \log \mathcal{Z}_d(q) &= \underbrace{\langle \partial_q^3 H_d \rangle}_{=:I_1(q)} + \underbrace{3\left[\langle (\partial_q^2 H_d)(\partial_q H_d) \rangle - \langle \partial_q^2 H_d \rangle \langle \partial_q H_d \rangle\right]}_{=:I_2(q)} \\
&\quad + \underbrace{\langle (\partial_q H_d)^3 \rangle - 3\langle (\partial_q H_d)^2 \rangle \langle \partial_q H_d \rangle + 2\langle \partial_q H_d \rangle^3}_{=:I_3(q)}.
\end{aligned} \tag{76}$$

We will control all terms $\{I_a\}_{a=1}^4$ in eq. (76). We achieve this by noting that for any $p \ge 0$, the derivative $\partial_q^p H_d$ can be written as $d\mathrm{Tr}[P_q(\mathbf{W}_1, \mathbf{W}_2)]$, where P is a polynomial of degree 2 in non-commutative random variables which is independent of d, and whose coefficients are bounded and smooth functions of q around $q = 0$. This has the following two consequences.

- (i) Since $\mathrm{Tr}[\mathbf{W}_1^2] \le d\kappa^2$ and $\mathrm{Tr}[\mathbf{W}_1\mathbf{W}_2] \le d\kappa^2$ on the support of $\langle \cdot \rangle$, it is clear that for any $\varepsilon \in (0,1)$, any $p, k \ge 0$, and $d \ge 1$:

$$\sup_{|q| \le \varepsilon} |\partial_q^p H_d|^k \le C_1(k, p, \varepsilon, \kappa) \cdot d^{2k}, \quad \langle \cdot \rangle - \text{almost surely}. \tag{77}$$

- (ii) As a consequence of Lemma 3.5, we further have for any $\varepsilon \in (0,1)$, any $p \ge 0$, and $d \ge 1$:

$$\sup_{|q| \le \varepsilon} \left\| \partial_q^p H_d - \langle \partial_q^p H_d \rangle \right\|_{\psi_2} \le C_2(k, p, \varepsilon, \kappa) \cdot d, \tag{78}$$

  and in particular:

$$\sup_{|q| \le \varepsilon} \left[\langle (\partial_q^p H_d)^2 \rangle - \langle \partial_q^p H_d \rangle^2\right] \le C_3(k, p, \varepsilon, \kappa) \cdot d^2. \tag{79}$$

Eq. (77) implies that

$$\sup_{|q| \le \varepsilon} |I_1(q)| \le C_1(\varepsilon, \kappa) \cdot d^2. \tag{80}$$

We further have

$$I_2(q) = 3\langle(\partial_q^2 H_d - \langle\partial_q^2 H_d\rangle)(\partial_q H_d - \langle\partial_q H_d\rangle)\rangle.$$

Using the Cauchy-Schwarz inequality and (79), this yields

$$\sup_{|q|\leq\varepsilon} |I_2(q)| \leq C_2(\varepsilon, \kappa) \cdot d^2. \tag{81}$$

FIXME: How to control I_3? It is the third cumulant of $\partial_q H_d$ (which is also the third central moment), but my bound only shows that it is $\mathcal{O}(d^3)$, which is not enough... Maybe some symmetry can help me? I am not sure...

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