where $u_k = f(x^k) - f^*$, or $u_k = \|x^k - x^*\|^2$, or $u_k = \|\nabla f(x^k)\|$. The estimate $u_k \le u_0 q^k$ follows from (2). In proving Newton's method (Sec. 1.5), we obtained for $u_k = \|\nabla f(x^k)\|$:

$$u_{k+1} \le c u_k^2 , \quad c > 0 , \tag{3}$$

yielding $u_k \le c^{-1}(cu_0)^{2^k}$ and if $cu_0 < 1$ then $u_k \to 0$.

In other problems, relation (1) is more complex and the analysis is not quite so trivial.

We start with linear inequalities of the form

$$u_{k+1} \le q_k u_k + \alpha_k , \quad q_k \ge 0 , \tag{4}$$

implying

$$u_k \le q_{k-1} q_{k-2} \cdots q_0 u_0 + q_{k-1} \cdots q_1 \alpha_0 + \cdots + q_{k-1}\alpha_{k-2} + \alpha_{k-1} . \tag{5}$$

Now we consider some special cases.

LEMMA 1. Let

$$u_{k+1} \le q u_k + \alpha , \quad 0 \le q < 1 , \ \alpha > 0 . \tag{6}$$

Then

$$u_k \le \alpha/(1-q) + (u_0 - \alpha/(1-q)) q^k . \tag{7}$$

PROOF. Setting $v_k = u_k - \alpha/(1-q)$, we obtain from (6) that $v_{k+1} \le v_k q$, and therefore (7). □

Thus, u_k converges geometrically into the region $u \le \alpha/(1-q)$ with ratio q.

LEMMA 2. Let $u_k \ge 0$ and let

$$u_{k+1} \le (1 + \alpha_k) u_k + \beta_k , \quad \alpha_k \ge 0 , \ \beta_k \ge 0 ,$$
$$\sum_{k=0}^{\infty} \alpha_k < \infty , \quad \sum_{k=0}^{\infty} \beta_k < \infty . \tag{8}$$

Then $u_k \to u \ge 0$.

The proof is the same as that of the more general Lemma 9 below. □

LEMMA 3. Let

$$u_{k+1} \le q_k u_k + \alpha_k , \quad 0 \le q_k < 1 , \ \alpha_k \ge 0 ,$$
$$\sum_{k=0}^{\infty} (1 - q_k) = \infty , \quad \alpha_k/(1 - q_k) \to 0 . \tag{9}$$

Then $\lim_{k\to\infty} u_k \le 0$. In particular, if $u_k > 0$, then $u_k \to 0$. □

COROLLARY. If in (9) $q_k \equiv q < 1$, $\alpha_k \to 0$, $u_k \ge 0$, then $u_k \to 0$. □

Under the conditions of Lemma 3, one can also estimate the rate of convergence for a number of cases.

LEMMA 4 (Chung). Let $u_k \ge 0$ and

$$u_{k+1} \le \left(1 - \frac{c}{k}\right) u_k + \frac{d}{k^{p+1}} , \quad d > 0 , \ p > 0 , \ c > 0 . \tag{10}$$

Then

$$u_k \le d(c-p)^{-1} k^{-p} + o(k^{-p}) \quad \text{for } c > p , \tag{11}$$
$$u_k = O(k^{-c} \log k) \quad \text{for } p = c , \tag{12}$$
$$u_k = O(k^{-c}) \quad \text{for } p > c . \tag{13}$$

PROOF. For any relation between c and p we have that Lemma 3 is applicable since

$$1 - q_k = c/k , \quad \sum_{k=0}^{\infty} (1 - q_k) = \infty , \quad \alpha_k (1-q_k)^{-1} = d c^{-1} k^{-p} \to 0 ,$$

and hence $u_k \to 0$. Let $c > p$. Also, let $v_k = k^p u_k - d(c-p)^{-1}$. Then

$$\begin{aligned}
v_{k+1} &= (k+1)^p u_{k+1} - \frac{d}{c-p} \le k^p \left(1 + \frac{1}{k}\right)^p \left(\left(1 - \frac{c}{k}\right) u_k + \frac{d}{k^{p+1}}\right) - \frac{d}{c-p} \\
&= k^p u_k \left(1 - \frac{c-p}{k} + o\left(\frac{1}{k}\right)\right) + \frac{d}{k}\left(1 + \frac{p}{k} + o\left(\frac{1}{k}\right)\right) - \frac{d}{c-p} \\
&= \left(v_k + \frac{d}{c-p}\right)\left(1 - \frac{c-p}{k} + o\left(\frac{1}{k}\right)\right) + \frac{d}{k}\left(1 + \frac{p}{k} + o\left(\frac{1}{k}\right)\right) - \frac{d}{c-p} \\
&= v_k \left(1 - \frac{c-p}{k} + o\left(\frac{1}{k}\right)\right) + \frac{dp}{k^2} + o\left(\frac{1}{k^2}\right) .
\end{aligned}$$