ON LICQ AND THE UNIQUENESS OF LAGRANGE MULTIPLIERS

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ABSTRACT. Kyparisis proved in 1985 that a strict version of the Mangasarian-Fromovitz constraint qualification (MFCQ) is equivalent to the uniqueness of Lagrange multipliers. However, the definition of this strict version of MFCQ requires the existence of a Lagrange multiplier and is *not* a constraint qualification (CQ) itself. In this note we show that LICQ is the weakest CQ which ensures (existence and) uniqueness of Lagrange multipliers. We also recall the relations between other CQs and properties of the set of Lagrange multipliers.

1. INTRODUCTION

In this paper we study the optimization problem

$$\left.\begin{aligned} \text{Minimize} \quad & f(x), \\ \text{such that} \quad & g(x) \le 0 \\ \text{and} \quad & h(x) = 0. \end{aligned}\right\} \tag{P}$$

Here, $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^k$ are given functions which are continuously differentiable. Strongly related to this problem is the Lagrangian

$$\mathcal{L}(x, \lambda, \mu) = f(x) + (\lambda,\, g(x))_{\mathbb{R}^m} + (\mu,\, h(x))_{\mathbb{R}^k} \tag{1}$$

with $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^k$. Here, $(\cdot,\, \cdot)_{\mathbb{R}^l}$ refers to the standard inner product in $\mathbb{R}^l$. In fact, if $\bar{x}$ is a local optimum of (**P**) and if an additional assumption is satisfied for g and h, then there exist multipliers $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^m \times \mathbb{R}^k$ such that the Karush-Kuhn-Tucker-system (KKT-system)

$$\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu}) = 0 \tag{2a}$$

$$\bar{\lambda} \ge 0 \tag{2b}$$

$$(\bar{\lambda},\, g(\bar{x}))_{\mathbb{R}^m} = 0 \tag{2c}$$

is satisfied. This first order necessary optimality condition generalizes Fermat's theorem which states $\nabla f(\bar{x}) = 0$ for unconstrained minimizers $\bar{x}$ of f. In Section 2 we recall why (2) cannot be a necessary optimality condition without further assumptions on g and h. These conditions which render (2) an optimality system are called constraint qualifications (CQs). Three important CQs are recalled in Section 3, see also [6].

Depending on the CQ which is satisfied, one has additional information about the set of multipliers $(\bar{\lambda}, \bar{\mu})$ satisfying (2). We will review the known relationships between constraint qualifications and properties of the set of multipliers satisfying (2) in Section 4. Moreover, we show that the linear independence CQ (LICQ) is the weakest CQ which implies uniqueness of the Lagrange multipliers, see Theorem 2.

Finally, we give some comments on the strict Mangasarian-Fromovitz condition in Section 5. In particular, we highlight that this condition is not a CQ.

Key words and phrases. Lagrange multiplier, constraint qualification.

2. Necessary optimality conditions

In this section we want to review well-known facts of the optimality system (2). We denote the feasible set of the problem (**P**) by

$$K = \{x \in \mathbb{R}^n : g(x) \le 0 \text{ and } h(x) = 0\}.$$

An immediate consequence of the local optimality of $\bar{x}$ for (**P**) is

$$(\nabla f(\bar{x}),\, d)_{\mathbb{R}^n} \ge 0 \quad \text{for all } d \in \mathcal{T}_K(\bar{x}), \tag{3}$$

where

$$\mathcal{T}_K(\bar{x}) = \{d \in \mathbb{R}^n : \text{there exist } \{x_n\} \subset \mathbb{R}^n \text{ and } \{t_n\} \subset \mathbb{R}, \\ \text{such that } x_n \to \bar{x}, t_n \searrow 0 \text{ and } (x_n - \bar{x})/t_n \to d\}$$

is the tangent cone of K at $\bar{x}$. Using the notion of the polar cone

$$\mathcal{T}_K(\bar{x})^\circ = \{u \in \mathbb{R}^n : (u,\, d)_{\mathbb{R}^n} \le 0 \text{ for all } d \in \mathcal{T}_K(\bar{x})\},$$

condition (3) is equivalent to

$$-\nabla f(\bar{x}) \in \mathcal{T}_K(\bar{x})^\circ. \tag{4}$$

On the other hand, using Farkas' Lemma, we find that the KKT-system (2) is equivalent to

$$-\nabla f(\bar{x}) \in \mathcal{T}_K^{\text{lin}}(\bar{x})^\circ. \tag{5}$$

Here,

$$\mathcal{T}_K^{\text{lin}}(\bar{x}) = \{d \in \mathbb{R}^n : \nabla h(\bar{x})\, d = 0, \nabla g_i(\bar{x})\, d \le 0 \text{ for all } i \in \mathcal{A}(\bar{x})\}$$

is the linearized tangent cone and $\mathcal{A}(\bar{x}) = \{i \in \mathbb{N} : 1 \le i \le m \text{ and } g_i(\bar{x}) = 0\}$ is the set of active indices. Note that $\mathcal{T}_K^{\text{lin}}(\bar{x})$ depends actually on the representation of K via g and h and not on K itself.

It is easy to show that $\mathcal{T}_K(\bar{x}) \subset \mathcal{T}_K^{\text{lin}}(\bar{x})$ and hence $\mathcal{T}_K^{\text{lin}}(\bar{x})^\circ \subset \mathcal{T}_K(\bar{x})^\circ$. Moreover, its not hard to construct examples where $\mathcal{T}_K^{\text{lin}}(\bar{x})$ is significantly larger than $\mathcal{T}_K(\bar{x})$. This implies that the condition (5) is stronger than (4) in the general case. Therefore, it cannot be a necessary optimality condition without any further assumptions.

3. Constraint qualifications

Constraint qualifications (CQs) are assumptions on the constraints g and h which ensure that condition (5) is a necessary optimality condition for the problem (**P**). The weakest CQ is Guignard's constraint qualification

$$\mathcal{T}_K(\bar{x})^\circ = \mathcal{T}_K^{\text{lin}}(\bar{x})^\circ, \tag{GCQ}$$

see [3]. It immediately implies that (4) is equivalent to (5) and hence, the condition (5) is a necessary optimality condition. Note that Theorem 1 (ii) indeed shows that GCQ is the *weakest possible* CQ.

A stronger CQ is the constraint qualification of Mangasarian and Fromovitz

$$\begin{aligned} &\text{the set } \{\nabla h_i(\bar{x})\}_{i=1}^k \text{ is linearly independent, and} \\ &\text{there exists } d \in \mathbb{R}^n, \text{such that } (\nabla g_i(\bar{x}),\, d)_{\mathbb{R}^n} < 0 \text{ for } i \in \mathcal{A}(\bar{x}) \\ &\qquad\qquad \text{and } (\nabla h_i(\bar{x}),\, d)_{\mathbb{R}^n} = 0 \text{ for } i = 1, \dots, k, \end{aligned} \tag{MFCQ}$$

see [5, (3.4)–(3.6)].

The strongest CQ is the linear independence constraint qualification

$$\text{the set } \{\nabla h_i(\bar{x})\}_{i=1}^k \cup \{\nabla g_i(\bar{x})\}_{i \in \mathcal{A}(\bar{x})} \text{ is linearly independent.} \tag{LICQ}$$

It is easy to see that (LICQ) implies (MFCQ).

We remark that many other CQs (e.g. Abadie CQ, constant rank CQ) can be found in the literature. However, we restrict ourselves to the above ones since they imply certain properties of the set of Lagrange multipliers, as explained in the next section.

4. Connections betweens CQs and properties of the set of Lagrange multipliers

Let us remark that all constraint qualifications above are independent of the objective function f. Hence, if a CQ would imply a certain property for the multipliers satisfying (2), this property would hold for all objective functions (such that $\bar{x}$ is a local minima). We fix an arbitrary feasible point $\bar{x} \in K$ and consider all objective functions f such that $\bar{x}$ is a local minimum. We define

$$\mathcal{F} = \{f \in C^1(\mathbb{R}^n; \mathbb{R}) : \bar{x} \text{ is a local minimizer of } (\mathbf{P})\}.$$

Moreover, for $f \in \mathcal{F}$ we define the set of Lagrange multipliers

$$\Lambda(f) = \{(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^m \times \mathbb{R}^k : (2) \text{ is satisfied}\}.$$

The following connections between CQs and the set of feasible multipliers $\Lambda(f)$ are well-known.

Theorem 1.

(i) *The set $\Lambda(f)$ is closed and convex for all $f \in \mathcal{F}$.*
(ii) *The set $\Lambda(f)$ is non-empty for all $f \in \mathcal{F}$ if and only if (GCQ) is satisfied.*
(iii) *Let (MFCQ) be satisfied. Then, the set $\Lambda(f)$ is compact for all $f \in \mathcal{F}$.*
(iv) *If there exists $f \in \mathcal{F}$, such that $\Lambda(f)$ is compact, then (MFCQ) is satisfied.*
(v) *Let (LICQ) be satisfied. Then, the set $\Lambda(f)$ is a singleton for all $f \in \mathcal{F}$.*

Assertions (i) and (v) are easy to prove. In [2, Section 4] the proof of (ii) can be found and [1] shows (iii) and (iv).

Some comments on Theorem 1 are in order. Note that each of (GCQ), (MFCQ) and (LICQ) implies the corresponding property for all objectives $f \in \mathcal{F}$. For (GCQ) also the converse is true, i.e., if $\Lambda(f)$ is non-empty for all $f \in \mathcal{F}$, (GCQ) holds. The relation of the compactness of $\Lambda(f)$ and (MFCQ) is a little bit different. Namely, the compactness of $\Lambda(f)$ for some *arbitrary* $f \in \mathcal{F}$ implies (MFCQ), giving in turn the compactness of $\Lambda(f)$ for *all* $f \in \mathcal{F}$.

As announced in the introduction, we show a converse of Theorem 1 (v).

Theorem 2. *Let $\bar{x}$ be a feasible point of* (**P**)*. The set of Lagrange multipliers $\Lambda(f)$ is a singleton for all $f \in \mathcal{F}$ if and only if (LICQ) is satisfied.*

Proof. If (LICQ) is satisfied, Theorem 1 (v) implies the uniqueness of Lagrange multipliers.

To prove the contrary, we assume that $\Lambda(f)$ is a singleton for all $f \in \mathcal{F}$. We set

$$f = -\sum_{i \in \mathcal{A}(\bar{x})} g_i.$$

Since $f(\bar{x}) = 0$ and $f(x) \geq 0$ for all feasible $x \in \mathbb{R}^n$, $\bar{x}$ is a local minimizer of f, i.e., $f \in \mathcal{F}$. It can be seen easily, that

$$\bar{\lambda}_i = \begin{cases} 1 & \text{if } i \in \mathcal{A}(\bar{x}) \\ 0 & \text{else} \end{cases} \quad \text{and} \quad \bar{\mu} = 0$$

are Lagrange multipliers. Let us take numbers $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^k$ such that

$$\sum_{i \in \mathcal{A}(\bar{x})} a_i \nabla g_i(\bar{x}) + \sum_{i=1}^{k} b_i \nabla h_i(\bar{x}) = 0 \quad \text{and} \quad a_i = 0 \text{ for all } i \notin \mathcal{A}(\bar{x}).$$

We define $C = 1 + \max_{i \in \mathcal{A}(\bar{x})}\{|a_i|\}$. It can be seen easily that the multipliers

$$\lambda = \bar{\lambda} + \frac{a}{C} \quad \text{and} \quad \mu = \bar{\mu} + \frac{b}{C}$$

satisfy (2). Since the Lagrange multipliers are unique by assumption, this shows $a = 0$ and $b = 0$, giving in turn (LICQ). ☐

5. Uniqueness of multipliers and the strict Mangasarian-Fromovitz condition

In this section, we recall the *strict Mangasarian-Fromovitz condition* (SMFC) and its relation to the uniqueness of Lagrange multipliers. Let $\bar{x}$ be a feasible point of (**P**), such that Lagrange multipliers $(\bar{\lambda}, \bar{\mu}) \in \Lambda(f)$ exist. We say that SMFC is satisfied at $\bar{x}$ with multipliers $\bar{\lambda}$ if

$$\begin{aligned}
&\text{the set } \{\nabla h_i(\bar{x})\}_{i=1}^k \cup \{\nabla g_i(\bar{x})\}_{i \in \mathcal{A}^+(\bar{x},\bar{\lambda})} \text{ is linearly independent, and} \\
&\text{there exists } d \in \mathbb{R}^n, \text{such that } (\nabla g_i(\bar{x}),\, d)_{\mathbb{R}^n} < 0 \text{ for } i \in \mathcal{A}^0(\bar{x},\bar{\lambda}), \\
&\qquad\qquad\qquad\qquad\qquad (\nabla g_i(\bar{x}),\, d)_{\mathbb{R}^n} = 0 \text{ for } i \in \mathcal{A}^+(\bar{x},\bar{\lambda}) \\
&\qquad\qquad\qquad\qquad \text{and } (\nabla h_i(\bar{x}),\, d)_{\mathbb{R}^n} = 0 \text{ for } i = 1, \ldots, k,
\end{aligned} \tag{SMFC($\bar{\lambda}$)}$$

hold, where $\mathcal{A}^+(\bar{x},\bar{\lambda}) = \{i \in \mathcal{A}(\bar{x}) : \bar{\lambda}_i > 0\}$ denotes the strictly active inequality constraints and $\mathcal{A}^0(\bar{x},\bar{\lambda}) = \{i \in \mathcal{A}(\bar{x}) : \bar{\lambda}_i = 0\} = \mathcal{A}(\bar{x}) \setminus \mathcal{A}^+(\bar{x},\bar{\lambda})$ denotes the weakly active inequality constraints. Note that this condition depends on the multiplier $\bar{\lambda}$. Since (SMFC($\bar{\lambda}$)) relies on the existence of Lagrange multipliers and since it depends (indirectly) on the objective f, we refrain from calling this a constraint qualification as it is sometimes used in the literature.

The relation between (SMFC($\bar{\lambda}$)) and the uniqueness of Lagrange multipliers is given in the following theorem.

Theorem 3 ([4, Proposition 1.1])**.** *Let $\bar{x} \in \mathbb{R}^n$ be given such that Lagrange multipliers $(\bar{\lambda}, \bar{\mu}) \in \Lambda(f)$ exist. The strict Mangasarian-Fromovitz condition* (SMFC($\bar{\lambda}$)) *is equivalent to $\Lambda(f)$ being a singleton, i.e., to the uniqueness of Lagrange multipliers.*

Let us highlight the difference of this theorem with Theorem 2. Theorem 3 shows that, if we already know multipliers $(\bar{\lambda}, \bar{\mu})$ such that (SMFC($\bar{\lambda}$)) is satisfied, these multipliers are unique. As already said, we have to assume the existence of multipliers a-priori since (SMFC($\bar{\lambda}$)) is not a CQ. On the other hand, (LICQ) is a CQ and ensures the existence *and* uniqueness of Lagrange multipliers.

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