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The Fréchet and limiting subdifferentials of integral functionals on the spaces $L_1(\Omega, E)$

Nguyen Huy Chieu

Department of Mathematics, Vinh University, Vinh, Nghe An, Viet Nam

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Dedicated to Professor B.S. Mordukhovich on the occasion of his 60th birthday

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ABSTRACT

A new approach to computing the Fréchet subdifferential and the limiting subdifferential of integral functionals is proposed. Thanks to this way, we obtain formulae for computing the Fréchet and limiting subdifferentials of the integral functional $F(u) = \int_\Omega f(\omega, u(\omega))\, d\mu(\omega)$, $u \in L_1(\Omega, E)$. Here $(\Omega, \mathcal{A}, \mu)$ is a measured space with an atomless σ-finite complete positive measure, E is a separable Banach space, and $f : \Omega \times E \to \bar{\mathbb{R}}$. Under some assumptions, it turns out that these subdifferentials coincide with the Fenchel subdifferential of F.

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1. Introduction

Consider the integral functional of the form

$$F(u) = \int_\Omega f\big(\omega, u(\omega)\big)\, d\mu(\omega), \quad u \in L_p(\Omega, E), \tag{1.1}$$

where $(\Omega, \mathcal{A}, \mu)$ is a measured space with an atomless σ-finite complete positive measure, E is a separable Banach space, and $f : \Omega \times E \to \bar{\mathbb{R}}$. Recall [5, p. 223] that

$$L_p(\Omega, E) := \left\{ u : \Omega \to E \text{ is measurable} \;\middle|\; \int_\Omega \big\|u(\omega)\big\|^p\, d\mu < +\infty \right\}$$

with the norm $\|u\| := (\int_\Omega \|u(\omega)\|^p\, d\mu)^{1/p}$ $(1 \leqslant p < \infty)$ and

$$L_\infty(\Omega, E) := \big\{ u : \Omega \to E \text{ is measurable} \;\big|\; \exists \alpha > 0 \text{ such that } \big\|u(\omega)\big\| < \alpha \ \mu\text{-a.e.} \big\}$$

with the norm $\|u\| := \inf\{\alpha > 0 \mid \|u(\omega)\| < \alpha \ \mu\text{-a.e.}\}$.

The problem of computing or estimating generalized subdifferentials of integral functionals has been studied intensively in the literature; see [3,4,7] and the references therein. Results in this direction (for instance, Lemma 6.18 in [7]) can be used for variational problems involving integral functionals (see [7, Chapter 6]).

E-mail address: nghuychieu@vinhuni.edu.vn.

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The purpose of this paper is to compute the Fréchet subdifferential and the limiting subdifferential (also called the Mordukhovich subdifferential) of the function F defined by (1.1) at a given point $x \in L_1(\Omega, E)$.

To compute the Fréchet subdifferential $\hat{\partial} F(x)$ and the limiting subdifferential $\partial F(x)$, we propose a new approach which is based on a very interesting result of Giner [6] about local/global minimizers of integral functionals. Thanks to this way, we obtain formulae for computing $\hat{\partial} F(x)$ and $\partial F(x)$. Under some assumptions, it turns out that these subdifferentials coincide with the Fenchel subdifferential of F at x.

The rest of the paper is divided into two sections. Section 2 contains some definitions and results which are needed in the sequel. Main results will be presented in Section 3.

2. Preliminaries

For a set-valued mapping $G : X \rightrightarrows X^*$ between a Banach space X and its topological dual X^*, the notation

$$\operatorname*{Lim\,sup}_{u \to x} G(x) := \left\{x^* \in X^* \;\middle|\; \exists \text{ sequence } u_k \to x \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in G(u_k) \text{ for all } k = 1, 2, \ldots\right\}$$

stands for the *sequential Painlevé–Kuratowski upper limit* with respect to the norm topology of X and the weak* topology w^* of X^*. The symbols $u \xrightarrow{\varphi} x$ for a function $\varphi : X \to \bar{\mathbb{R}} := [-\infty, +\infty]$ and $u \xrightarrow{\Omega} x$ for a set $\Omega \subset X$ mean, respectively, $u \to x$ with $\varphi(u) \to \varphi(x)$ and $u \to x$ with $u \in \Omega$.

Let us recall some notions related to generalized differentiation taken from [7]. Suppose that $\varphi : X \to \bar{\mathbb{R}}$ is finite at x and $\varepsilon \geqslant 0$. The *ε-Fréchet subdifferential* of φ at x is defined by setting

$$\hat{\partial}_\varepsilon \varphi(x) := \left\{x^* \in X^* \;\middle|\; \liminf_{u \to x} \frac{\varphi(u) - \varphi(x) - \langle x^*, u - x\rangle}{\|u - x\|} \geqslant -\varepsilon\right\}. \tag{2.1}$$

If $|\varphi(x)| = \infty$ then $\hat{\partial}_\varepsilon \varphi(x) := \emptyset$. When $\varepsilon = 0$ the set $\hat{\partial}_0 \varphi(x)$, denoted by $\hat{\partial}\varphi(x)$, is called the *Fréchet subdifferential* of φ at x. The *limiting subdifferential* (or the *Mordukhovich subdifferential*) of φ at x is defined by setting

$$\partial \varphi(x) := \operatorname*{Lim\,sup}_{\substack{u \xrightarrow{\varphi} x \\ \varepsilon \downarrow 0}} \hat{\partial}_\varepsilon \varphi(u). \tag{2.2}$$

The limiting subdifferential reduces to the classical Fréchet derivative for strictly differentiable functions. We have $\hat{\partial}\varphi(x) \subset \partial\varphi(x)$ for any $\varphi : X \to \bar{\mathbb{R}}$ finite at x; see [7].

The *Fenchel subdifferential* of φ at $x \in X$ with $\varphi(x) \in \mathbb{R}$ is the set

$$\partial^{Fen} \varphi(x) := \left\{x^* \in X^* \;\middle|\; \varphi(u) - \varphi(x) \geqslant \langle x^*, u - x\rangle \ \forall u \in X\right\}.$$

If φ is convex, then $\partial\varphi(x) = \hat{\partial}\varphi(x) = \partial^{Fen}\varphi(x)$.

In the sequel, unless otherwise stated, $(\Omega, \mathcal{A}, \mu)$ is a measured space with an atomless positive σ-finite complete measure, E is a separable Banach space with its σ-Borel algebra $\mathcal{B}(E)$, and $f : \Omega \times E \to \bar{\mathbb{R}}$ is $\mathcal{A} \otimes \mathcal{B}(E)$-measurable.

Let $\mathcal{F}$ be a subset of the space $L_0(\Omega, \bar{\mathbb{R}})$ of measurable functions defined on Ω and with values in $\bar{\mathbb{R}}$. The *essential infimum function* of $\mathcal{F}$, denoted by $\operatorname{ess\,inf}_{v \in \mathcal{F}} v$, is a measurable function from Ω to $\bar{\mathbb{R}}$ satisfying the following conditions:

(i) for each $u \in \mathcal{F}$, $\operatorname{ess\,inf}_{v \in \mathcal{F}} v \leqslant u$ μ-a.e.;
(ii) if $\tilde{u} : \Omega \to \bar{\mathbb{R}}$ is a measurable function such that for each $u \in \mathcal{F}$, $\tilde{u} \leqslant u$ μ-a.e., then $\tilde{u} \leqslant \operatorname{ess\,inf}_{v \in \mathcal{F}} v$ μ-a.e.

The proof of the existence and uniqueness of the essential infimum function of $\mathcal{F}$ can be found in [8, pp. 43–44] for the case where μ is finite. The result immediately implies the existence and uniqueness of the essential infimum function of $\mathcal{F}$ when μ is σ-finite. The reader is referred to [2,5,8] for more details.

It is worthy observing that if there exists $v_0 \in \mathcal{F}$ such that for each $v \in \mathcal{F}$, $v \geqslant v_0$ μ-a.e., then $\operatorname{ess\,inf}_{v \in \mathcal{F}} v = v_0$.

Applying [2, Theorem 3.8] to the case where $X = L_1(\Omega, E)$ and $M : \Omega \rightrightarrows E$ defined by $M(\omega) = E$ for all $\omega \in \Omega$, one has

$$\operatorname*{ess\,inf}_{u \in L_1(\Omega, E)} f(u)(\omega) = \inf_{e \in E} f(\omega, e). \tag{2.3}$$

For each $u \in L_1(\Omega, E)$, we put

$$\begin{aligned} I_f(u) &:= \int_\Omega^* f\bigl(\omega, u(\omega)\bigr)\, d\mu \\ &= \inf\left\{\int_\Omega v(\omega)\, d\mu \;\middle|\; v \in L_1(\Omega, \mathbb{R}),\ v(\omega) \geqslant f\bigl(\omega, u(\omega)\bigr)\ \mu\text{-a.e.}\right\}. \end{aligned} \tag{2.4}$$

It is easy to see that if the function $\omega \mapsto f(\omega, u(\omega))$ belongs to $L_1(\Omega, \mathbb{R})$ then $I_f(u) = F(u)$, where $F(u)$ is given by (1.1). The following statement is a special case of the theorem established by E. Giner.

Theorem 2.1. *(See [6].) Suppose that $f(x) \in L_1(\Omega, \mathbb{R})$, where $f(x)(\omega) := f(\omega, x(\omega))$ for all $\omega \in \Omega$, and x is a local minimizer of I_f on $L_1(\Omega, E)$. Then, for each $u \in L_1(\Omega, E)$,*

$$f\big(\omega, x(\omega)\big) \leqslant f\big(\omega, u(\omega)\big) \quad \mu\text{-a.e.}$$

A function $u : \Omega \to E^*$ is said to be *weakly star measurable* [5, Definition 2.101(iii)] if for any $e \in E$ the function $\Omega \ni \omega \mapsto \langle u(\omega), e\rangle$ is measurable. Denote by $L^w_\infty(\Omega, E^*)$ the space of all (equivalence classes of) weakly star measurable functions $u : \Omega \to E^*$ such that the function $\Omega \ni \omega \mapsto \|u(\omega)\|$ belongs to $L_\infty(\Omega, \mathbb{R})$. The space $L^w_\infty(\Omega, E^*)$ is endowed with the norm

$$\|u\|_{L^w_\infty(\Omega,E^*)} = \operatorname*{ess\,sup}_{\omega\in\Omega} \big\|u(\omega)\big\|,$$

where $\operatorname{ess\,sup}_{\omega\in\Omega} \|u(\omega)\| := \inf\{\alpha > 0 \mid \|u(\omega)\| < \alpha \ \mu\text{-a.e.}\}$; see [5, Definition 2.111].

Recall [5, Theorem 2.112] that

$$L_1(\Omega, E)^* = L^w_\infty(\Omega, E^*),$$

i.e., we can identify each $u^* \in L_1(\Omega, E)^*$ with an $u^*(\cdot) \in L^w_\infty(\Omega, E^*)$ such that

$$\langle u^*, u\rangle = \int_\Omega \big\langle u^*(\omega), u(\omega)\big\rangle \, d\mu,$$

for all $u \in L_1(\Omega, E)$. The reader can find more information on the $L_p(\Omega, E)$-spaces and related matters in [5] and the references therein.

3. Main results

We are going to obtain some formulae for the Fréchet subdifferential and the limiting subdifferential of an integral functional of the form (1.1) with $p = 1$ and E is a separable Banach space. Let us begin with a proposition which plays a crucial role in proving the main theorem.

Proposition 3.1. *Let $I_f(\cdot) : L_1(\Omega, E) \to \bar{\mathbb{R}}$ be the function defined by (2.4), and let $x \in L_1(\Omega, E)$ satisfying $f(x) \in L_1(\Omega, \mathbb{R})$, where $f(x)(\omega) := f(\omega, x(\omega))$. Then,*

$$\begin{aligned} \hat{\partial}_\varepsilon I_f(x) &= \Big\{x^* \in L^w_\infty(\Omega, E^*) \;\Big|\; \inf_{e\in E} g_\varepsilon\big(\omega, e, x^*(\omega)\big) \geqslant 0 \ \mu\text{-a.e.}\Big\} \\ &= \big\{x^* \in L^w_\infty(\Omega, E^*) \;\big|\; I_f(u) - I_f(x) - \langle x^*, u - x\rangle + \varepsilon\|u - x\| \geqslant 0 \text{ for all } u \in L_1(\Omega, E)\big\}. \end{aligned} \tag{3.1}$$

Here $g_\varepsilon(\omega, e, e^*) := f(\omega, e) - f(\omega, x(\omega)) - \langle e^*, e - x(\omega)\rangle + \varepsilon\|e - x(\omega)\|$, $\omega \in \Omega$, $e \in E$, $e^* \in E^*$, $\varepsilon \geqslant 0$.

Proof. Let $\varepsilon \geqslant 0$, $k \in \mathbb{N}$, $x \in L_1(\Omega, E)$, and $x^* \in \hat{\partial}_\varepsilon I_f(x)$. According to (2.1), there exists $\delta_k > 0$ such that

$$I_f(u) - I_f(x) - \langle x^*, u - x\rangle + \big(\varepsilon + k^{-1}\big)\|u - x\| \geqslant 0,$$

for all $u \in \mathbb{B}(x, \delta_k) := \{v \in L_1(\Omega, E) \mid \|v - x\| \leqslant \delta_k\}$. Note that $(L_1(\Omega, E))^* = L^w_\infty(\Omega, E^*)$ and

$$\langle u^*, u\rangle = \int_\Omega \big\langle u^*(\omega), u(\omega)\big\rangle \, d\mu,$$

for all $u^* \in L^w_\infty(\Omega, E^*)$, $u \in L_1(\Omega, E)$. Hence x is a local minimizer of the function $I(\cdot)$ defined by

$$\begin{aligned} I(u) &:= I_f(u) - I_f(x) - \langle x^*, u - x\rangle + \big(\varepsilon + k^{-1}\big)\|u - x\| \\ &= \int_\Omega^* h\big(\omega, u(\omega)\big)\, d\mu \quad \big(u \in L_1(\Omega, E)\big), \end{aligned}$$

where

$$h(\omega, e) := f(\omega, e) - f\big(\omega, x(\omega)\big) - \big\langle x^*(\omega), e - x(\omega)\big\rangle + \big(\varepsilon + k^{-1}\big)\big\|e - x(\omega)\big\|, \quad (\omega, e) \in \Omega \times E.$$

Since f is $\mathcal{A}\otimes\mathcal{B}(E)$-measurable, h is also $\mathcal{A}\otimes\mathcal{B}(E)$-measurable. For each $u\in L_1(\Omega,E)$, by Theorem 2.1,

$$h\big(\omega,x(\omega)\big)\leqslant h\big(\omega,u(\omega)\big)\quad \mu\text{-a.e.}$$

Note that for any $u\in L_1(\Omega,E)$, by the $\mathcal{A}\otimes\mathcal{B}(E)$-measurability of h, the function $\Omega\ni\omega\mapsto h(u)(\omega):=h(\omega,u(\omega))$ is $\mathcal{A}$-measurable. Thus,

$$\operatorname*{ess\,inf}_{u\in L_1(\Omega,E)} h(u)(\omega)=h\big(\omega,x(\omega)\big)\quad \mu\text{-a.e.}$$

By (2.3),

$$\operatorname*{ess\,inf}_{u\in L_1(\Omega,E)} h(u)(\omega)=\inf_{e\in E}h(\omega,e)\quad \mu\text{-a.e.}$$

Hence

$$\inf_{e\in E}h(\omega,e)=h\big(\omega,x(\omega)\big)\quad \mu\text{-a.e.},$$

which implies that for $k\in\mathbb{N}$, we can find $\Omega_k\in\mathcal{A}$ with $\mu(\Omega_k)=0$ such that

$$f(\omega,e)-f\big(\omega,x(\omega)\big)-\big\langle x^*(\omega),e-x(\omega)\big\rangle+\big(\varepsilon+k^{-1}\big)\big\|e-x(\omega)\big\|\geqslant 0,$$

for all $\omega\in\Omega\setminus\Omega_k$, $e\in E$. Put $\Omega_{x^*}:=\bigcup_{k\in\mathbb{N}}\Omega_k$. Then $\mu(\Omega_{x^*})=0$ and

$$f(\omega,e)-f\big(\omega,x(\omega)\big)-\big\langle x^*(\omega),e-x(\omega)\big\rangle+\big(\varepsilon+k^{-1}\big)\big\|e-x(\omega)\big\|\geqslant 0,$$

for all $\omega\in\Omega\setminus\Omega_{x^*}$, $e\in E$, $k\in\mathbb{N}$. Passing to the limit as $k\to\infty$, we get

$$f(\omega,e)-f\big(\omega,x(\omega)\big)-\big\langle x^*(\omega),e-x(\omega)\big\rangle+\varepsilon\big\|e-x(\omega)\big\|\geqslant 0,$$

for all $\omega\in\Omega\setminus\Omega_{x^*}$, $e\in E$. Hence $\inf_{e\in E}g_\varepsilon(\omega,e,x^*(\omega))\geqslant 0$ μ-a.e., and thus,

$$\hat\partial_\varepsilon I_f(x)\subset\Big\{x^*\in L^w_\infty(\Omega,E^*)\;\Big|\;\inf_{e\in E}g_\varepsilon\big(\omega,e,x^*(\omega)\big)\geqslant 0\ \mu\text{-a.e.}\Big\}.$$

We next want to show that the set on the right-hand side of this inclusion is contained in the last set of (3.1). Let $x^*\in\{x^*\in L^w_\infty(\Omega,E^*)\mid \inf_{e\in E}g_\varepsilon(\omega,e,x^*(\omega))\geqslant 0\ \mu\text{-a.e.}\}$. Then, there exists $\Omega_{x^*}\in\mathcal{A}$ with $\mu(\Omega_{x^*})=0$ such that

$$f(\omega,e)-f\big(\omega,x(\omega)\big)-\big\langle x^*(\omega),e-x(\omega)\big\rangle+\varepsilon\big\|e-x(\omega)\big\|\geqslant 0,$$

for all $\omega\in\Omega\setminus\Omega_{x^*}$, $e\in E$. Thus, for any $u\in L_1(\Omega,E)$,

$$f\big(\omega,u(\omega)\big)-f\big(\omega,x(\omega)\big)-\big\langle x^*(\omega),u(\omega)-x(\omega)\big\rangle+\varepsilon\big\|u(\omega)-x(\omega)\big\|\geqslant 0\quad \mu\text{-a.e.}$$

Since $f(x)\in L_1(\Omega,\mathbb{R})$, it holds

$$I_f(u)-I_f(x)-\langle x^*,u-x\rangle+\varepsilon\|u-x\|\geqslant 0, \tag{3.2}$$

for all $u\in L_1(\Omega,E)$. Hence x^* belongs to the last set of (3.1). It is clear that (3.2) implies $x^*\in\hat\partial_\varepsilon I_f(x)$. Summarizing all the above, we can conclude that (3.1) holds. □

We are now ready to present the main results.

Theorem 3.2. *Let $f:\Omega\times E\to\bar{\mathbb{R}}$ be $\mathcal{A}\otimes\mathcal{B}(E)$-measurable. Suppose that $f(u)\in L_1(\Omega,\mathbb{R})$ for all $u\in L_1(\Omega,E)$, where $f(u)(\omega):=f(\omega,u(\omega))$. Put*

$$F(u):=\int_\Omega f\big(\omega,u(\omega)\big)\,d\mu\quad \big(u\in L_1(\Omega,E)\big). \tag{3.3}$$

Then,

$$\begin{aligned}\partial F(x)&=\hat\partial F(x)=\partial^{Fen}F(x)\\&=\Big\{x^*\in L^w_\infty(\Omega,E^*)\;\Big|\;\inf_{e\in E}g_0\big(\omega,e,x^*(\omega)\big)\geqslant 0\ \mu\text{-a.e.}\Big\},\end{aligned} \tag{3.4}$$

with $g_0(\omega,e,e^*):=f(\omega,e)-f(\omega,x(\omega))-\langle e^*,e-x(\omega)\rangle$, $\omega\in\Omega$, $e\in E$, $e^*\in E^*$, and $x\in L_1(\Omega,E)$.

Proof. Since $f(u) \in L_1(\Omega, \mathbb{R})$ for any $u \in L_1(\Omega, E)$, $F(u) = I_f(u)$ for all $u \in L_1(\Omega, E)$. According to Proposition 3.1,

$$\hat{\partial} F(x) = \partial^{Fen} F(x) = \left\{ x^* \in L^w_\infty(\Omega, E^*) \;\middle|\; \inf_{e \in E} g_0\big(\omega, e, x^*(\omega)\big) \geqslant 0 \ \mu\text{-a.e.} \right\}.$$

By (2.2), $\hat{\partial} F(x) \subset \partial F(x)$. We are going to show that $\partial F(x) \subset \partial^{Fen} F(x)$. Take any $x^* \in \partial F(x)$. Then, there exist sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{F} x$ and $x_k^* \xrightarrow{w^*} x^*$ such that $x_k^* \in \hat{\partial}_{\varepsilon_k} F(x_k)$ for all $k \in \mathbb{N}$. By Proposition 3.1,

$$F(u) - F(x_k) - \langle x_k^*, u - x_k \rangle + \varepsilon_k \|u - x_k\| \geqslant 0 \quad \forall u \in L_1(\Omega, E).$$

Taking limits as $k \to \infty$, we get

$$F(u) - F(x) - \langle x^*, u - x \rangle \geqslant 0 \quad \forall u \in L_1(\Omega, E).$$

Thus, $x^* \in \partial^{Fen} F(x)$ and (3.4) is valid. $\square$

Remark 3.3. (1) Under the assumptions of Theorem 3.2, it follows from (3.1) that

$$\begin{aligned} \partial F(x) = \hat{\partial} F(x) &\subset \left\{ x^* \in L^w_\infty(\Omega, E^*) \;\middle|\; x^*(\omega) \in \hat{\partial} f(\omega, \cdot)\big(x(\omega)\big) \ \mu\text{-a.e.} \right\} \\ &\subset \left\{ x^* \in L^w_\infty(\Omega, E^*) \;\middle|\; x^*(\omega) \in \partial f(\omega, \cdot)\big(x(\omega)\big) \ \mu\text{-a.e.} \right\}, \end{aligned}$$

for all $x \in L_1(\Omega, E)$.

(2) If $E = \mathbb{R}^n$ then $L^w_\infty(\Omega, E^*) = L_\infty(\Omega, \mathbb{R}^n)$, and so (3.4) becomes

$$\begin{aligned} \partial F(x) &= \hat{\partial} F(x) = \partial^{Fen} F(x) \\ &= \left\{ x^* \in L_\infty\big(\Omega, \mathbb{R}^n\big) \;\middle|\; \inf_{e \in \mathbb{R}^n} g_0\big(\omega, e, x^*(\omega)\big) \geqslant 0 \ \mu\text{-a.e.} \right\}. \end{aligned}$$

(3) If the function $F : L_1(\Omega, E) \to \mathbb{R}$ defined by (3.3) is convex then, as mentioned in Section 2,

$$\partial F(x) = \hat{\partial} F(x) = \partial^{Fen} F(x). \tag{3.5}$$

Formula (3.4) shows that (3.5) is valid without assuming the convexity of F.

We now examine an example which shows, among other things, that the integral functional F under our consideration is nonconvex in general.

Example 3.4. Let E be any nontrivial separable Banach space (for instance, we can take $E = \ell_p$ or $E = L_p[0,1]$, $1 \leqslant p < \infty$). Let $\Omega = [0,1]$, $\mathcal{A}$ the σ-algebra of the Lebesgue measurable subsets of $[0,1]$, μ the Lebesgue measure on $\mathbb{R}$, and $f : [0,1] \times E \to \mathbb{R}$ defined by $f(t, e) = |\sin(\|e\|)|$ for all $(t, e) \in [0,1] \times E$. Consider the integral functional

$$F(u) = \int_0^1 f\big(t, u(t)\big)\, dt \quad \big(u \in L_1\big([0,1], E\big)\big).$$

The following hold:

(a) F is nonconvex;
(b) $\partial F(0) = \hat{\partial} F(0) = \partial^{Fen} F(0) = \{0\}$;
(c) $\{x^* \in L^w_\infty([0,1], E^*) \mid x^*(t) \in \hat{\partial} f(t, \cdot)(0) \text{ a.e.}\} = \mathbb{B}_{L^w_\infty([0,1],E^*)}$;
(d) $\{x^* \in L^w_\infty([0,1], E^*) \mid x^*(t) \in \partial f(t, \cdot)(0) \text{ a.e.}\} = \mathbb{B}_{L^w_\infty([0,1],E^*)}$,

where $\mathbb{B}_{L^w_\infty([0,1],E^*)} := \{x^* \in L^w_\infty([0,1], E^*) \mid \|x^*\|_{L^w_\infty([0,1],E^*)} \leqslant 1\}$.

To verify (a), choose $u_1(t) := 0$ and $u_2(t) := \pi e_0$ for all $t \in [0,1]$ with $e_0 \in E$ satisfying $\|e_0\| = 1$. Then $u_1, u_2 \in L_1([0,1], E)$, $F(u_1) = F(u_2) = 0$ and $F(\frac{1}{2}u_1 + \frac{1}{2}u_2) = 1$. So, we get

$$F\left(\frac{1}{2}u_1 + \frac{1}{2}u_2\right) > \frac{1}{2}F(u_1) + \frac{1}{2}F(u_2),$$

which establishes (a).

In order to obtain (b), fix an $x^* \in \partial F(0)$. By Theorem 3.2, we can find $\Omega_{x^*} \in \mathcal{A}$ with $\mu(\Omega_{x^*}) = 0$ such that

$$\big|\sin\big(\|e\|\big)\big| \geqslant \big\langle x^*(t), e \big\rangle$$

for any $e \in E$, $t \in [0,1] \setminus \Omega_{x^*}$. It follows that

$$\left\langle x^*(t), \frac{\pi}{\|e\|}e \right\rangle \leqslant |\sin \pi| = 0 \quad \text{and} \quad \left\langle x^*(t), -\frac{\pi}{\|e\|}e \right\rangle \leqslant |\sin \pi| = 0,$$

for all $t \in [0,1] \setminus \Omega_{x^*}$, $e \in E \setminus \{0\}$. Hence $x^*(t) = 0$ a.e. We have shown that $\partial F(0) \subset \{0\}$. Since $|\sin(\|e\|)| \geqslant \langle 0, e \rangle$, by Theorem 3.2, $0 \in \partial F(0)$. Therefore, $\partial F(0) = \hat{\partial} F(0) = \partial^{Fen} F(0) = \{0\}$.

Pick any $x^* \in L^w_\infty([0,1], E^*)$. The necessary and sufficient condition for $x^*(t) \in \hat{\partial} f(t,\cdot)(0)$ a.e. $t \in [0,1]$ is that

$$\liminf_{e \to 0} \frac{|\sin(\|e\|)| - \langle x^*(t), e \rangle}{\|e\|} \geqslant 0 \quad \text{a.e. } t \in [0,1],$$

or, equivalently, $\|x^*(t)\| \leqslant 1$ a.e. $t \in [0,1]$. This means that (c) holds.

From (c) it follows that

$$\mathbb{B}_{L^w_\infty([0,1],E^*)} \subset \left\{ x^* \in L^w_\infty\big([0,1], E^*\big) \;\middle|\; x^*(t) \in \partial f(t,\cdot)(0) \text{ a.e.} \right\}.$$

Since $f(t,\cdot)$ is Lipschitz with the constant $\ell = 1$, for each $t \in [0,1]$, we have $\|x^*(t)\| \leqslant 1$ whenever $x^*(t) \in \partial f(t,\cdot)(0)$. Hence

$$\left\{ x^* \in L^w_\infty\big([0,1], E^*\big) \;\middle|\; x^*(t) \in \partial f(t,\cdot)(0) \text{ a.e.} \right\} \subset \mathbb{B}_{L^w_\infty([0,1],E^*)},$$

and the assertion (d) follows.

Corollary 3.5. *Suppose in addition to the assumptions of Theorem* 3.2 *that F is Fréchet differentiable at x and locally Lipschitz around x. Then,*

$$\lim_{k \to \infty} \big\| x^*_k - F'(x) \big\| = 0 \quad \textit{whenever } x^*_k \in \partial F(x_k) \textit{ with } x_k \to x \textit{ as } k \to \infty.$$

Consequently, if F is Fréchet differentiable and locally Lipschitz, then F is continuously differentiable.

Proof. This proof is based on the scheme given in [1, Proposition 4.7, p. 86]. Let $\gamma > 0$. Then there exists $\rho > 0$ such that

$$F(x+u) - F(x) - \big\langle F'(x), u \big\rangle \leqslant \gamma \|u\| \quad \forall u \in \rho \mathbb{B}_{L_1(\Omega,E)}, \tag{3.6}$$

where $\mathbb{B}_{L_1(\Omega,E)} := \{u \in L_1(\Omega,E) \mid \|u\| \leqslant 1\}$. Since $x^*_k \in \partial F(x_k)$, by Theorem 3.2 we have

$$\big\langle x^*_k, x + u - x_k \big\rangle \leqslant F(x+u) - F(x_k) \quad \forall u \in L_1(\Omega, E). \tag{3.7}$$

As $\sup_{\|u\|=\rho} \langle x^*_k - F'(x), \rho^{-1} u \rangle = \|x^*_k - F'(x)\| > 2^{-1} \|x^*_k - F'(x)\|$ whenever $\|x^*_k - F'(x)\| \neq 0$, we can find $u_k \in L_1(\Omega, E)$ such that $\|u_k\| = \rho$ and $\langle x^*_k - F'(x), u_k \rangle \geqslant 2^{-1} \rho \|x^*_k - F'(x)\|$ $(k = 1, 2, \ldots)$. Substituting u_k into (3.6) and (3.7) yields

$$-\gamma \|u_k\| \leqslant F(x) - F(x+u_k) + \big\langle F'(x), u_k \big\rangle$$

and

$$0 \leqslant F(x+u_k) - F(x_k) - \big\langle x^*_k, x + u_k - x_k \big\rangle.$$

Hence

$$\begin{aligned} -\gamma\rho &\leqslant F(x) - F(x_k) + \big\langle F'(x) - x^*_k, u_k \big\rangle + \big\langle x^*_k, x_k - x \big\rangle \\ &\leqslant F(x) - F(x_k) - 2^{-1}\rho \big\| x^*_k - F'(x) \big\| + \big\langle x^*_k, x_k - x \big\rangle. \end{aligned}$$

Together with the fact that F is locally Lipschitz around x (thus $\{x^*_k\}$ is bounded), this gives $\limsup_{k\to\infty} \|x^*_k - F'(x)\| \leqslant 2\gamma$. Since $\gamma > 0$ is arbitrary, $\limsup_{k\to\infty} \|x^*_k - F'(x)\| = 0$. The proof is completed. □

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