Approximately convex sets

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Abstract

We study two classes of sets whose associated distance functions satisfy properties akin to approximate convexity. Since the class of approximately convex functions is known to enjoy nice properties, one may expect analogous properties for this class of sets. We present characterizations and delineate links with the concept of approximately convex functions through epigraphs and sublevel sets.

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1 Introduction

The abundance of constructions in nonsmooth analysis enables to attack various problems by using adapted techniques. However, this abundance of variants is often considered as an inconvenience. Therefore, it is of interest to show that in some classes of sets or functions these variants coincide. This has been done in [32, Thm 3.6], [43], [49] for some favorable classes of functions, for the family of approximately convex functions and for the class of $\alpha(\cdot)$-paraconvex functions respectively. In this note we study the notion of approximate convexity for sets studied in [11, section 5] under the name of property (ω) and called submoothness in [2]. We also introduce a notion of intrinsically convex set. Both notions are given in terms of the distance function to the set. In a companion paper ([47]) we study pointwise variants, namely the concepts of approximate starshapedness and intrinsic approximate starshapedness for sets. Since these notions are more general than the ones studied here, we refer the reader to that paper for what concerns regularity properties, i.e. properties ensuring that various concepts of tangent or normal cones to a set coincide at a given point. Such properties are important as they show some unified character of nonsmooth analysis. On the other hand, the class of sets we study here is more general than the class of weakly convex sets considered in [51], [11], [2]. Directional versions of the concepts we study can be introduced, but for the sake of brevity we do not consider them here. Our main goal is to give characterizations of approximate convexity for sets, thus establishing a parallel with the study conducted in [38] for functions.

We adopt a versatile approach which allows one to deal with a large spectrum of notions of normal cones and subdifferentials. This enables one to combine the advantages of these various notions or to use the notion which is the best adapted to a specific problem.

Finally, we endeavor to relate the concept of approximate convexity of sets to the notion of approximate convexity of functions introduced in [32] and studied in a number of papers ([2], [15], [38]...). It is as follows.

Definition 1 *([32]) A function $f : X \to \mathbb{R}\cup\{+\infty\}$ on a normed vector space X is said to be approximately convex around $\overline{x} \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in B(\overline{x}, \delta)$ and any $t \in [0, 1]$ one has*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t)\left\|x - y\right\|.$$

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In ([32, Prop. 3.1]) it is shown that this class of functions is stable under finite sums, under finite suprema and under composition with a continuous affine function; moreover, this class contains the family of functions which are strictly differentiable at $\overline{x}$. Such functions are locally Lipschitzian on the interior of their domains by [32, Prop. 3.2].

These nice properties incite to guess that the class of approximately convex sets also enjoys pleasant properties.

2 Preliminaries

In the sequel, X is a Banach space with topological dual space X^*. The open ball with center $\overline{x} \in X$ and radius $\rho > 0$ is denoted by $B(\overline{x}, \rho)$, while $\overline{B}_X$ (resp. $\overline{B}_{X^*}$) stands for the closed unit ball of X (resp. X^*) and S_X stands for the unit sphere. Given a subset E of X, the *distance function* d_E associated with E is given by $d_E(x) := \inf_{e \in E} d(x, e)$ and the *indicator function* ι_E of E is the function defined by $\iota_E(x) = 0$ if $x \in E$, $\iota_E(x) = \infty$ if $x \in X \backslash E$. We write $x \xrightarrow{E} a$ for $x \to a$ and $x \in E$.

Since our study is of geometrical nature, we have to introduce some geometrical concepts. The *tangent cone* to a subset E of X at some $\overline{x} \in \mathrm{cl}(E)$ is the set $T(E, \overline{x})$ of vectors $v \in X$ such that there exist sequences $(t_n) \to 0_+$, $(x_n) \xrightarrow{E} \overline{x}$ (i.e. $(x_n) \to \overline{x}$ and $x_n \in E$ for each $n \in \mathbb{N}$) for which $(t_n^{-1}(x_n - x)) \to v$. The *normal cone* $N(E, \overline{x})$ to E at $\overline{x}$ is the polar cone of $T(E, \overline{x})$. Both play a crucial role in nonlinear analysis and optimization.

The *firm normal cone* (or Fréchet normal cone) to E at $\overline{x}$ is given by

$$x^* \in N^-(E, \overline{x}) \Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 : \langle x^*, x - \overline{x} \rangle \leq \varepsilon \left\| x - \overline{x} \right\| \quad \forall x \in E \cap B(\overline{x}, \delta).$$

The *Clarke normal cone* to E at $\overline{x}$ is defined as the polar cone to the *Clarke tangent cone* $T^{\uparrow}(E, \overline{x})$, where

$$T^{\uparrow}(E, \overline{x}) := \{v \in X : \forall (t_n) \to 0_+, \forall (x_n) \to \overline{x}, \; \exists (v_n) \to v, \; x_n + t_n v_n \in E \; \forall n \in \mathbb{N}\}.$$

To any notion $N^?$ of normal cone one can associate a notion of subdifferential $\partial^?$ by setting

$$\partial^? f(\overline{x}) := \{x^* \in X^* : (x^*, -1) \in N^?(E_f, e_f)\},$$

where $E_f := \{(x, r) \in X \times \mathbb{R} : r \geq f(x)\}$ is the epigraph of f and $e_f := (\overline{x}, f(\overline{x}))$. The notion of subdifferential we adopt here is as versatile as possible: given a Banach space X and a subset $\mathcal{F}(X)$ of the set $\mathcal{S}(X)$ of lower semicontinuous (l.s.c.) functions $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$) a *subdifferential* on $\mathcal{F}(X)$ will be just a correspondence $\partial : \mathcal{F}(X) \times X \rightrightarrows X^*$ which assigns a subset $\partial f(x)$ of the dual space X^* of X to any f in $\mathcal{F}(X)$ and any $x \in X$ at which f is finite; we assume it satisfies the following natural property:

(M) $0 \in \partial f(\overline{x})$ when $\overline{x}$ is a minimizer of a Lipschitzian function f.

Conversely, with any subdifferential $\partial^?$ is associated a notion of *normal cone* obtained by setting for a subset E of X and $e \in E$

$$N^?(E, e) := \mathbb{R}_+ \partial^? \iota_E(e),$$

where ι_E is the indicator function of E. In the cases $\partial^? = \partial$, $\partial^? = \partial^{\uparrow}$ and $\partial^? = \partial^-$ we get the *normal cones* $N(E, \overline{x})$, $N^{\uparrow}(E, \overline{x})$ and $N^-(E, \overline{x})$ to E at $\overline{x} \in E$ in the senses of Bouligand, Clarke and Fréchet respectively, as defined above. We refer to [29] and [34] for the definitions of the approximate subdifferential ∂^A and the moderate subdifferential $\partial^\diamond$ respectively. We say that X is a $\partial^?$*-subdifferentiability space* if for any Lipschitz function f on X the domain of $\partial^? f$ is dense in X; this notion is close to the one introduced in [27]. We say that a subdifferential $\partial^?$ is *Lipschitz-valuable on* X, if for any Lipschitz function f on X and any $a, b \in X$ there exists c in the segment joining a to b and $c^* \in \partial^? f(c)$ such that

$$f(b) - f(a) \leq \langle c^*, b - a \rangle.$$

Some of the subdifferentials of current use are related to generalized concepts of directional derivatives (but not all). The *Clarke-Rockafellar* derivative or circa-derivative of a function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ finite at $\overline{x}$ is given by the following formulas in which $E_f := \{(x,r) \in X \times \mathbb{R} : r \geq f(x)\}$, $e_f := (\overline{x}, f(\overline{x}))$

$$\begin{aligned} f^{\uparrow}(\overline{x}, v) &:= \inf_{r>0} \limsup_{\substack{(t,y)\to(0_+,\overline{x}) \\ f(y)\to f(\overline{x})}} \inf_{w\in B(v,r)} \frac{1}{t}(f(y+tw) - f(y)) \\ &= \inf\{r \in \mathbb{R} : (v,r) \in T^{\uparrow}(E_f, e_f)\}. \end{aligned}$$

The (lower) *directional derivative* (or contingent derivative or lower epiderivative or lower Hadamard derivative) of f at $\overline{x}$ is given by

$$\begin{aligned} f'(\overline{x}, v) &:= \liminf_{(t,w)\to(0_+,v)} \frac{1}{t}(f(\overline{x}+tw) - f(\overline{x})) \\ &= \inf\{r \in \mathbb{R} : (v,r) \in T(E_f, e_f)\}. \end{aligned}$$

In particular, one has

$$\begin{aligned} \partial f(\overline{x}) &= \{x^* \in X^* : x^* \leq f'(\overline{x}, \cdot)\}, \\ \partial^{\uparrow} f(\overline{x}) &= \{x^* \in X^* : x^* \leq f^{\uparrow}(\overline{x}, \cdot)\}. \end{aligned}$$

We will need the following results of independent interest. The first one is obtained by an easy argument taken from the proof of [45, Prop. 1].

Lemma 2 *Let E be a nonempty subset of a Banach space X and let $x \in E$, $v \in X$. Then, one has*

$$d_E^{\uparrow}(x,v) = \limsup_{t\to 0_+,\ e \xrightarrow{E} x} \frac{1}{t} d_E(e + tv).$$

Proof. By definition of $d_E^{\uparrow}$ there exist some sequences $(t_n) \to 0_+$, (e_n) in E such that

$$d_E^{\uparrow}(x,v) = \lim_n t_n^{-1} \left[d_E(x_n + t_n v) - d_E(x_n) \right].$$

We can find $(x_n) \to x$ in X such that $\|e_n - x_n\| \leq d_E(x_n) + t_n^2$. Now $d_E(x_n + t_n v) \leq d_E(e_n + t_n v) + \|e_n - x_n\|$. Thus

$$d_E^{\uparrow}(x,v) \leq \lim_n t_n^{-1} \left[d_E(e_n + t_n v) + \|e_n - x_n\| - (\|e_n - x_n\| - t_n^2) \right] = \lim_n t_n^{-1} d_E(e_n + t_n v).$$

The reverse inequality $d_E^{\uparrow}(x,v) = \limsup_{t\to 0_+,\ e \xrightarrow{E} x} \frac{1}{t} d_E(e+tv)$ being always valid, equality is proved. $\square$

The second result we need is close to [29, Lemma 5], [45, Lemma 1], [40, Lemma 3.6] and [2, Lemma 3.7] but it contains a crucial additional information. Recall that the norm of X is said to satisfy the *Kadec-Klee property* if for every $x \in X$, a sequence (x_n) of X converges to x whenever it weakly converges to x and $(\|x_n\|) \to \|x\|$.

Lemma 3 *Suppose that E is a closed nonempty subset of an Asplund space X and that $w^* \in \partial^- d_E(w)$ with $w \in X \backslash E$. Then $\|w^*\| = 1$ and there exist sequences (x_n), (x_n^*) of E and X^* respectively such that $x_n^* \in \partial^- d_E(x_n)$ for each $n \in \mathbb{N}$ and*

$$(||x_n - w||) \to d_E(w), \quad (\langle x_n^*, w - x_n \rangle) \to d_E(w), \quad (||x_n^* - w^*||) \to 0.$$

If moreover X is reflexive and if its norm has the Kadec-Klee property then a subsequence of (x_n) converges to some best approximation x of w in E and one has $\langle w^*, w - x \rangle = ||x - w|| = d_E(w)$.

Proof. The fact that $\|w^*\| = 1$ for each $w^* \in \partial^- d_E(w)$ is well-known (see [4, Prop. 1.4], for instance). By [40, Lemma 3.6] or [2, Lemma 3.7], given a sequence $(\varepsilon_n) \to 0_+$, one can find sequences (x_n), (x_n^*) of E and X^* respectively such that $x_n^* \in \partial^- d_E(x_n)$ for each $n \in \mathbb{N}$ and $(||x_n - w\|) \to d_E(w)$, $(||x_n^* - w^*\|) \to 0$. It remains to apply [40, Lemma 3.6] which asserts that for any sequence (x_n) of E satisfying $(||x_n - w\|) \to d_E(w)$ one has $(\langle w^*, w - x_n \rangle) \to d_E(w)$. Since $(||x_n^* - w^*\|) \to 0$ and since $(w - x_n)$ is bounded, one gets $(\langle x_n^*, w - x_n \rangle) \to d_E(w)$.

The last assertion is taken from [6, Lemma 6]. □

Given a subdifferential $\partial^?$ one can associate to it a corresponding *limiting subdifferential* $\overline{\partial^?}$ by setting for a l.s.c. function f and a point x of it domain

$$\overline{\partial^?} f(x) := w^* - \limsup_{(u,f(u) \to (x,f(x))} \partial^? f(u).$$

Similarly, to any notion of normal cone one can associate a corresponding *limiting normal cone* by setting

$$\overline{N^?}(E, x) := w^* - \limsup_{u \xrightarrow{E} x} \partial^? N(E, u).$$

Here, the w^*-limsup of a family $(F_t)_{t \in T}$ of subsets of X^* parametrized by some topological space T with a base point 0, is the set w^*-$\limsup_t F_t$ of weak* limits of bounded families $(x_t^*)_{t \in S}$ as $t \to 0$, where S is some subset of T containing 0 in its closure $\mathrm{cl}(S)$ and $x_t^* \in F_t$ for each $t \in S$. For the limiting firm normal cone $\overline{N^-}(E, a) := \limsup_{x \xrightarrow{E} a} N^-(E, x)$ we have the following result we will use later on. Recall that an *Asplund space* is a space all of which separable subspaces have a separable dual.

Corollary 4 *Let E be a closed nonempty subset of an Asplund space X, let $\overline{x} \in E$ and $\overline{x}^* \in \overline{\partial^-} d_E(\overline{x})$. Then $\|\overline{x}^*\| \leq 1$ and there exist sequences (x_n), (x_n^*) of E and X^* respectively such that $(x_n) \to \overline{x}$, $(x_n^*) \to \overline{x}^*$ weak* and $x_n^* \in \partial^- d_E(x_n)$ for each $n \in \mathbb{N}$.*

Proof. The inequality $\|\overline{x}^*\| \leq 1$ stems from the fact that $\overline{x}^*$ is a weak* limit of a sequence (w_n^*) of X^* such that $w_n^* \in \partial^- d_E(w_n)$ for each $n \in \mathbb{N}$ where $(w_n) \to \overline{x}$. Let $N := \{n \in \mathbb{N} : w_n \in E\}$. If N is infinite, it suffices to consider the subsequence $(x_n)_{n \in N}$. When N is finite, using the preceding lemma, for each $k \in K := \mathbb{N} \backslash N$ we pick $x_k \in E$ and $x_k^* \in \partial^- d_E(x_k)$ such that $\|x_k - w_k\| \leq 2d_E(w_k)$, $\|x_k^* - w_k^*\| \leq 1/k$. Then $(x_k) \to \overline{x}$, $(x_k^*) \to \overline{x}^*$ weak*. □

3 Approximate convexity of sets

We observe that using the notion of approximate convexity for the indicator function ι_E of a subset E of X would lead to convexity of E and not to a relaxed form of convexity. Therefore, we rather use the distance function d_E. In the sequel $\overline{x}$ is a point of E.

Definition 5 *A subset E of X is said to be approximately convex around $\overline{x}$ if its associated distance function d_E is approximately convex around $\overline{x}$.*

Example. The set $E := \{(r, s) \in \mathbb{R}^2 : s \geq |r| - r^2\}$ is approximately convex at each of its points (for an appropriate norm) but is nonconvex. This example is a special instance of Proposition 14 below.
Example. If $\overline{x}$ is an isolated point of E, then E is approximately convex around $\overline{x}$.

It is not obvious to decide whether the preceding definition depends on the choice of the norm in an equivalent class; on the contrary, the variant presented in the next section will not depend on the choice of the norm inducing the topology. In order to look for characterizations, we need the following notion.

Definition 6 *A multimapping $M : X \rightrightarrows X^*$ is said to be approximately monotone around $\overline{x}$ on E if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for any $x_1, x_2 \in E \cap B(\overline{x}, \delta)$, $x_1^* \in M(x_1)$, $x_2^* \in M(x_2)$ one has*

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\| . \tag{1}$$

For $E = X$ one simply says that M is approximately monotone around $\overline{x}$.

The following result is an easy consequence of [15] when $\partial^? = \partial^\uparrow$. However, since we use here an arbitrary subdifferential contained in the Clarke subdifferential, we have to use [38] with the distance function to get the implication (d)⇒(a).

Theorem 7 *Let $\partial^?$ be a subdifferential on the family $\mathcal{L}(X)$ of Lipschitz functions on X such that $\partial^? f \subset \partial^\uparrow f$ for any $f \in \mathcal{L}(X)$ and let $\overline{x}$ be an element of a subset E of X. Then, among the following assertions, one has the implications (a)⇒(b)⇒(c)⇒(d). If moreover $\partial^?$ is Lipschitz-valuable on X, in particular if $\partial^? := \partial^\uparrow$, ∂^A, $\partial^\diamond$, all these assertions are equivalent.*

(a) E is approximately convex around $\overline{x}$ in the sense that d_E is approximately convex around $\overline{x}$;

(b) for any $\varepsilon > 0$ there exists $\rho > 0$ such that for any $x \in B(\overline{x}, \rho)$ and any $v \in B(0, \rho)$ one has

$$d_E^\uparrow(x, v) \leq d_E(x + v) - d_E(x) + \varepsilon \|v\| ; \tag{2}$$

(c) for any $\varepsilon > 0$ there exists $\rho > 0$ such that for any $x \in B(\overline{x}, \rho)$, any $x^* \in \partial^? d_E(x)$ and any $(u, t) \in S_X \times (0, \rho)$ one has

$$\langle x^*, u \rangle \leq \frac{d_E(x + tu) - d_E(x)}{t} + \varepsilon; \tag{3}$$

(c') for any $\varepsilon > 0$ there exists $\rho > 0$ such that for any $x \in B(\overline{x}, \rho)$, any $x^* \in \partial^? d_E(x)$ and any $v \in \rho \overline{B}_X$ one has

$$\langle x^*, v \rangle \leq d_E(x + v) - d_E(x) + \varepsilon \|v\| ; \tag{4}$$

(d) $\partial^? d_E$ is approximately monotone around $\overline{x}$;

Corollary 8 *If E is approximately convex around $\overline{x}$ then, for any subdifferential $\partial^?$ such that $\partial^- \subset \partial^? \subset \partial^\uparrow$ one has $\partial^- d_E(\overline{x}) = \partial^? d_E(\overline{x}) = \partial^\uparrow d_E(\overline{x})$.*

Proof. Let $\overline{x}^* \in \partial^\uparrow d_E(\overline{x})$ and let $\varepsilon > 0$ be given. By (c) with $\partial^? = \partial^\uparrow$, we can find $\rho > 0$ such that for each $v \in \rho B_X$, setting $x := \overline{x}$, $t := \|v\|$, $u := t^{-1} v$ in (4), we have

$$\langle \overline{x}^*, v \rangle \leq d_E(\overline{x} + v) - d_E(\overline{x}) + \varepsilon \|v\| .$$

That shows that $\overline{x}^* \in \partial^- d_E(\overline{x})$. □

For a related assertion about normal cones, we refer to [47]

4 Intrinsic approximate convexity

The terminology of the definition we adopt now is justified by the fact that the notion we introduce is obtained by restricting the requirement on the distance function to the subset E. Thus this notion is more general than the preceding notion. It is an open problem to decide whether the two notions coincide.

Definition 9 *A subset E of X is said to be intrinsically approximately convex around $\overline{x} \in E$ if for any $\varepsilon > 0$ there exists $\rho > 0$ such that for any $x_1, x_2 \in E \cap B(\overline{x}, \rho)$, $t \in [0, 1]$, one has*

$$d_E((1 - t)x_1 + tx_2) \leq \varepsilon t(1 - t) \|x_1 - x_2\| . \tag{5}$$

It is intrinsically approximately convex if it is intrinsically approximately convex around each of its points.

Let us note that this definition does not depend on the choice of the norm among the ones inducing the same topology. It is sufficiently strong to eliminate pathological subsets.
Example. Let $X := \mathbb{R}$ and let $E := \{0\} \cup \{x_n\}$, where (x_n) is a decreasing sequence of $(0, +\infty)$ with limit 0. Then E is not approximately convex around 0 since for $w \in [x_{n+1}, x_n]$ one has $d_E(w) = \min(x_n - w, w - x_{n+1})$.
Example. If E is paraconvex around $\overline{x}$ (i.e. locally weakly convex in the sense of [51]), then it is intrinsically approximately convex since for any given $c > 0$, $\varepsilon > 0$, one has $ct(1-t)\left\|x_1 - x_2\right\|^2 \leq \varepsilon t(1-t)\left\|x_1 - x_2\right\|$ when $\left\|x_i - \overline{x}\right\| \leq c^{-1}\varepsilon/2$ for $i = 1, 2$ (see [51, Prop. 3.4]).

Characterizations can be given as follows; they are not as complete as in the preceding theorem. However, we will supplement them in some special cases later on. The equivalence (c)⇔(c') of our first statement is nothing but a reformulation. However, it shows a link with the study made in [11, Prop. 4.2, 4.4]. When one of the assertions (b)-(d) holds, we say that E is *$\partial^?$-intrinsically approximately convex around $\overline{x}$*.

Theorem 10 *Let E be a nonempty closed subset of X and let $\partial^?$ be a subdifferential such that $\partial^? f \subset \partial^\uparrow f$ for any Lipschitz function f on X. Then the following assertions (b), (c), (c'), (d) are equivalent and are implied by assertions (a) and (e): (a)⇒(b)⇔(c)⇔(c')⇔(d)⇐(e). When X is a $\partial^?$-subdifferentiability space one has (e)⇒(a).*

(a) E is intrinsically approximately convex around $\overline{x}$;

(b) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, x' \in E \cap B(\overline{x}, \delta)$, one has

$$d_E^\uparrow(x, x' - x) \leq \varepsilon \left\|x - x'\right\|; \tag{6}$$

(c) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, x' \in E \cap B(\overline{x}, \delta)$, $x^* \in \partial^? d_E(x)$, one has

$$\langle x^*, x' - x\rangle \leq \varepsilon \left\|x - x'\right\|; \tag{7}$$

(c') there exists a function $\alpha : E \times E \to \mathbb{R}_+ \cup \{+\infty\}$ such that $\alpha(x, x') \to 0$ as $x, x' \xrightarrow{E} \overline{x}$ and

$$\langle x^*, x' - x\rangle \leq \alpha(x, x') \left\|x - x'\right\| \quad \textit{for any } (x, x') \in E \times E,\ x^* \in \partial^? d_E(x); \tag{8}$$

(d) $\partial^? d_E(\cdot)$ is approximately monotone around $\overline{x}$ on E : for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_1, x_2 \in E \cap B(\overline{x}, \delta)$, $x_1^* \in \partial^? d_E(x_1)$, $x_2^* \in \partial^? d_E(x_2)$ one has

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq -\varepsilon \left\|x_1 - x_2\right\|; \tag{9}$$

(e) for any $\varepsilon > 0$ there exists $\rho > 0$ such that for any $w \in B(\overline{x}, \rho)$, $x \in E \cap B(\overline{x}, \rho)$, $w^* \in \partial^? d_E(w)$ one has

$$d_E(w) + \langle w^*, x - w\rangle \leq \varepsilon \left\|x - w\right\|. \tag{10}$$

Proof. (a)⇒(b) Given $\varepsilon > 0$ let $\rho > 0$ be as in Definition 9 and let $x, x' \in E \cap B(\overline{x}, \rho)$. By Lemma 2 , we have

$$d_E^\uparrow(x, x' - x) \leq \limsup_{t \to 0_+,\ e \xrightarrow{E} x} \frac{1}{t} d_E(e + t(x' - x)).$$

Now, since $d_E(e + t(x' - x)) \leq d_E(e + t(x' - e)) + t\left\|e - x\right\| \leq \varepsilon t(1-t)\left\|x' - e\right\| + t\left\|e - x\right\|$, we get

$$d_E^\uparrow(x, x' - x) \leq \varepsilon \left\|x' - x\right\|.$$

(b)⇒(c) is a consequence of the inclusion $\partial^? d_E(x) \subset \partial^\uparrow d_E(x)$ and of the definition of $\partial^\uparrow d_E(x)$
(c)⇒(c') It suffices to set for $(x, x') \in E \times E$, $\alpha(x, x') := 0$ if $x = x'$ and for $x \neq x'$,

$$\alpha(x, x') := \sup\{\langle x^*, \frac{x' - x}{\left\|x - x'\right\|}\rangle :\ x^* \in \partial^? d_E(x)\}.$$

Then (b) ensures that $\alpha(x, x') \to 0$ as $x, x' \to \overline{x}$.

(c')⇒(c) is obvious.

(c)⇒(d) Given $\varepsilon > 0$ let $\delta > 0$ be as in assertion (c) and let $x_1, x_2 \in E \cap B(\overline{x}, \delta)$, $x_1^* \in \partial^? d_E(x_1)$, $x_2^* \in \partial^? d_E(x_2)$. Taking $x = x_1$, $x^* = x_1^*$, $x' = x_2$ and adding inequality (7) to the corresponding one obtained by choosing $x = x_2$, $x^* = x_2^*$, $x' = x_1$ we get relation (9) with ε changed into 2ε.

(d)⇒(c) is obtained in taking $x_1 = x$, $x_1^* = x^*$, $x_2 = x'$, $x_2^* = 0$ in assertion (d), using the fact that x_2 is a minimizer of d_E, so that $0 \in \partial^? d_E(x_2)$.

(e)⇒(c) is obvious (change x into x' in (10) and take $w = x \in E$).

(e)⇒(a) when X is a $\partial^?$-subdifferentiability space. Given $\varepsilon > 0$, let $\rho > 0$ be as in assertion (e) and let $x_1, x_2 \in E \cap B(\overline{x}, \rho)$, $t \in [0, 1]$, $w := (1-t)x_1 + tx_2$. Since X is a $\partial^?$-subdifferentiability space, there exist sequences $(w_n) \to w$, (w_n^*) such that $w_n^* \in \partial^? d_E(w_n)$ for each $n \in \mathbb{N}$. Then, as $\partial^? d_E(w_n) \subset \partial^{\uparrow} d_E(w_n) \subset \overline{B}_{X^*}$, we have $(\langle w_n^*, w - w_n\rangle) \to 0$. Since by convexity $w \in B(\overline{x}, \rho)$, we have $w_n \in B(\overline{x}, \rho)$ for n large enough, hence

$$\begin{aligned}(1-t)d_E(w_n) + (1-t)\langle w_n^*, x_1 - w_n\rangle &\leq (1-t)\varepsilon \|x_1 - w_n\|, \\ t d_E(w_n) + t\langle w_n^*, x_2 - w_n\rangle &\leq t\varepsilon \|x_2 - w_n\|.\end{aligned}$$

Adding the corresponding sides of these relations we get

$$d_E(w_n) + \langle w_n^*, w - w_n\rangle \leq (1-t)\varepsilon \|x_1 - w_n\| + t\varepsilon \|x_2 - w_n\|,$$

and, passing to the limit,

$$d_E(w) \leq (1-t)\varepsilon \|x_1 - w\| + t\varepsilon \|x_2 - w\| = 2\varepsilon t(1-t) \|x_1 - x_2\|.$$

□

Now let us give some specializations to some specific subdifferentials and normal cones. We start with the firm normal cone.

Corollary 11 *Suppose that E is a closed subset of an Asplund space X and let ∂ be the Fréchet subdifferential ∂^-. Then all the assertions of Theorem 10 are equivalent to the following assertions:*

(f) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, x' \in E \cap B(\overline{x}, \delta)$, $x^* \in N^-(E, x)$ one has

$$\langle x^*, x' - x\rangle \leq \varepsilon \|x^*\| \|x - x'\|; \tag{11}$$

(g) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_1, x_2 \in E \cap B(\overline{x}, \delta)$, $x_1^* \in N^-(E, x_1)$ $x_2^* \in N^-(E, x_2)$ one has

$$\langle x_1^* - x_2^*, x_2 - x_1\rangle \leq \varepsilon \max(\|x_1^*\|, \|x_2^*\|) \|x_1 - x_2\|. \tag{12}$$

Proof. The relation $\partial^- d_E(x) = N^-(E, x) \cap \overline{B}_{X^*}$ for each $x \in E$, yields equivalence of assertions (c) and (f) by an homogeneity argument.

(f)⇒(g) by summation, changing ε into $\varepsilon/2$. The reverse implication is obtained by taking $x_1 := x$, $x_2 := x'$, $x_1^* = x^*$, $x_2^* = 0$.

(c)⇒(e) Given $\varepsilon > 0$, let $\delta > 0$ be as in (c). Let $\rho := \delta/6$ and let $w \in B(\overline{x}, \rho)$, $x \in E \cap B(\overline{x}, \rho)$, $w^* \in \partial^- d_E(w)$. By Lemma 3 we can find sequences (x_n) in E and (x_n^*) in X^* such that $x_n^* \in \partial^- d_E(x_n)$ for each $n \in \mathbb{N}$ and

$$(\|x_n - w\|) \to d_E(w), \qquad (\langle x_n^*, w - x_n\rangle) \to d_E(w), \qquad (\|x_n^* - w^*\|) \to 0. \tag{13}$$

Relation (10) being trivial if $w = x$, without loss of generality, we may suppose $w \neq x$ and $\|x_n - w\| \leq 2\|x - w\| \leq 4\rho$ for each $n \in \mathbb{N}$; then $\|x_n - x\| \leq \|x_n - w\| + \|w - x\| \leq 3\|x - w\|$ and $x_n \in B(\overline{x}, 6\rho) \subset B(\overline{x}, \delta)$.

Now, for n large enough, relation (13) implies the inequality of the first line below, while assertion (c) ensures the passage from the second line to the third one:

$$\begin{aligned} d_E(w) + \langle w^*, x - w\rangle &\leq (\langle x_n^*, w - x_n\rangle + \varepsilon \|x - w\|) + (\langle x_n^*, x - w\rangle + ||x_n^* - w^*\| \, \|x - w\|) \\ &\leq \langle x_n^*, x - x_n\rangle + \varepsilon \|x - w\| + \varepsilon \|x - w\| \\ &\leq \varepsilon \|x - x_n\| + 2\varepsilon \|x - w\| \leq 5\varepsilon \|x - w\| \, . \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, assertion (e) holds.

Finally, since X is Asplund, it is a ∂^--subdifferentiability space, so that (e) ensures that E is intrinsically approximately convex by Theorem 10. $\square$

The case of the limiting subdifferential can be easily derived from the preceding corollary.

Corollary 12 *Suppose that E is a closed subset of an Asplund space X and let ∂ be the limiting Fréchet subdifferential $\overline{\partial^-}$. Then all the assertions of Theorem 10 are equivalent.*

Proof. Using Corollary 4, the assertions (c), (d), (e) of Theorem 10 with $\overline{\partial^-}$ follow from the corresponding assertions with ∂^- by a passage to the weak* limit for a bounded net; the reverse implications are obvious. $\square$

Now let us turn to the Clarke subdifferential. It would be interesting to know whether one can get rid of the assumption of the last assertion that X is an Asplund space.

Corollary 13 *For a closed subset E of a Banach space X and $\partial^? = \partial^\uparrow$, among the assertions of Theorem 10 the following implications hold: (e)$\Rightarrow$(a)$\Rightarrow$(b)$\Leftrightarrow$(c)$\Leftrightarrow$(c')$\Leftrightarrow$(d). If moreover X is an Asplund space, all these assertions are equivalent.*

Proof. The implications follow from the choice $\partial^? = \partial^\uparrow$ in Theorem 10 since any Banach space is a $\partial^\uparrow$-subdifferentiability space. When X is an Asplund space the equivalences can be deduced from the preceding corollary: since $\partial^\uparrow d_E(x) = \overline{\text{co}}^*(\overline{\partial^-} d_E(x))$ one has the equivalence (e^-)$\Leftrightarrow$($\overline{e^-}$)$\Leftrightarrow$($\text{e}^\uparrow$) (where ($\text{e}^\uparrow$), ($\overline{e^-}$), (e^-) are (e) for $\partial^? = \partial^\uparrow$, $\overline{\partial^-}$, ∂^- respectively), hence ($\text{c}^\uparrow$)$\Rightarrow$(c^-)$\Rightarrow$(e^-)$\Rightarrow$($\text{e}^\uparrow$)$\Rightarrow$(a). $\square$

5 Approximately convex sets and functions

Some links between geometrical properties and analytical properties are contained in the next statements. Unless otherwise specified, we endow the product space $X := W \times \mathbb{R}$ of a n.v.s. W with $\mathbb{R}$ with a product norm, i.e. a norm such that the projections and the insertions $w \mapsto (w, 0)$ and $r \mapsto (0, r)$ are nonexpansive. Then, for each $(w, r) \in W \times \mathbb{R}$ one has

$$\max (\|w\| \, , |r|) \leq \|(w, r)\| \leq \|w\| + |r| \, .$$

Proposition 14 *Let W be a normed vector space and let $f : W \to \mathbb{R}\cup\{\infty\}$ be a l.s.c. function which is approximately convex around $\overline{w} \in W$. Then, for any $\overline{r} \geq f(\overline{w})$ the epigraph E of f is intrinsically approximately convex around $\overline{x} := (\overline{w}, \overline{r})$.*

Proof. Given $\varepsilon > 0$, let $\rho > 0$ be such that

$$f((1-t)w_1 + tw_2) \leq (1-t)f(w_1) + tf(w_2) + \varepsilon t(1-t) \|w_1 - w_2\|$$

for any $w_1, w_2 \in B(\overline{w}, \rho)$, $t \in [0, 1]$. Let $x_i := (w_i, r_i)$ $(i = 1, 2)$ be elements of the epigraph E of f in $B(\overline{x}, \rho)$ and let $t \in [0, 1]$, $w := (1-t)w_1 + tw_2$, $r := (1-t)r_1 + tr_2$, $x := (w, r)$. Then, as $w_1, w_2 \in B(\overline{w}, \rho)$

and $(w, f(w)) \in E$ one has $d_E(x) = 0$ if $f(w) \leq r$ and $d_E(x) \leq \|(w,r) - (w, f(w))\| \leq f(w) - r$ if $f(w) > r$, so that

$$d_E(x) \leq \max(0, f(w) - r) \leq \max(0, (1-t)(f(w_1) - r_1) + t(f(w_2) - r_2) + \varepsilon t(1-t) \|w_1 - w_2\|)$$
$$\leq \varepsilon t(1-t) \|w_1 - w_2\| \leq \varepsilon t(1-t) \|x_1 - x_2\| .$$

Thus E is intrinsically approximately convex around $\overline{x}$. □

Let us give a kind of converse to the preceding proposition.

Theorem 15 *Let W be a Banach space and let $f : W \to \mathbb{R}$ be a function which is locally Lipschitzian around $\overline{w} \in W$ and such that the epigraph E of f is an intrinsically approximately convex subset of $X := W \times \mathbb{R}$ around $\overline{x} := (\overline{w}, f(\overline{w}))$. Then f is an approximately convex function around $\overline{w}$.*

Proof in the case W is an Asplund space. In view of the characterization of approximate convexity of a function given in [38] it suffices to prove that $\partial^- f$ is approximately monotone around $\overline{w}$. Let ℓ be the Lipschitz rate of f on some ball $B(\overline{x}, \rho_0)$. Given $\varepsilon > 0$ there exists some $\rho \in (0, \rho_0)$ such that for any $x_1, x_2 \in B(\overline{x}, \rho)$ and any $x_1^* \in N^-(E, x_1) \cap (\ell+1)\overline{B}_{X^*}$, $x_2^* \in N^-(E, x_2) \cap (\ell+1)\overline{B}_{X^*}$ one has

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\| .$$

Then, for $w_1, w_2 \in B(\overline{w}, \rho/(\ell+1))$, $w_i^* \in \partial^- f(w_i)$ for $i = 1, 2$, setting $x_i := (w_i, f(w_i))$, $x_i^* := (w_i^*, -1)$ one has $x_i \in B(\overline{x}, \rho)$ and $x_i^* \in N^-(E, x_i) \cap (\ell+1)\overline{B}_{X^*}$, hence

$$\langle w_1^* - w_2^*, w_1 - w_2 \rangle \geq -\varepsilon \|x_1 - x_2\| ,$$
$$\langle w_2^* - w_1^*, w_2 - w_1 \rangle \geq -\varepsilon \|x_2 - x_1\| .$$

Adding the corresponding sides of these inequalities we get

$$\langle w_1^* - w_2^*, w_1 - w_2 \rangle \geq -\varepsilon \|x_1 - x_2\| \geq -\varepsilon(\ell+1) \|w_1 - w_2\| .$$

Since ε is arbitrarily small, we get that f is approximately convex around $\overline{w}$. □

The proof in the general case relies on the following lemma extracted from the proof of [45, Prop. 10 c]; it is close to previous results of that kind due to F.H. Clarke [9] and to A.D. Ioffe ([29], [30, Prop. 2.1] for bornological subdifferentials) in the case $x = (\overline{w}, f(\overline{w}))$.

Lemma 16 *Let $f : W \to \mathbb{R}$ be a function which is Lipschitzian with rate c on a ball $B(\overline{w}, \rho)$ of W. Then, for $\sigma \in (0, \rho)$ small enough and for any $w \in B(\overline{w}, \sigma)$ and any $w^* \in \partial^\uparrow f(w)$ one has $(w^*, -1) \in \partial^\uparrow d_E(x)$, where E is the epigraph of f and $x := (w, f(w))$, $X := W \times \mathbb{R}$ being endowed with the norm given by $\|(w, r)\| = c \|w\| + |r|$.*

Proof. By [24], [28] one can find $\sigma \in (0, \rho)$ such that

$$d_E(w, r) = (f(w) - r)_+$$

for $(w, r) \in B(\overline{x}, \sigma)$ with $\overline{x} := (\overline{w}, f(\overline{w}))$, X being endowed with the norm described in the statement; here for $t \in \mathbb{R}$, t_+ stands for $\max(t, 0)$. Let $w \in B(\overline{w}, \sigma)$ and $w^* \in \partial^\uparrow f(w)$; we have to prove that for any $(v, s) \in X$ we have

$$\langle (w^*, -1), (v, s) \rangle \leq d_E^\uparrow((w, r), (v, s)).$$

Since $w^* \in \partial^\uparrow f(w)$ there exist sequences $(\varepsilon_n) \to 0_+$, $(w_n) \to w$, $(t_n) \to 0_+$ such that

$$t_n^{-1} (f(w_n + t_n v) - f(w_n)) > \langle w^*, v \rangle - \varepsilon_n$$

for each n. Setting $r_n := f(w_n)$ and observing that, for n large enough,

$$\begin{aligned} t_n^{-1} d_E(w_n + t_n v, r_n + t_n s) &= t_n^{-1} \left(f(w_n + t_n v) - r_n - t_n s \right)_+ \\ &\geq t_n^{-1} \left(f(w_n + t_n v) - r_n - t_n s \right) \\ &\geq \langle w^*, v \rangle - \varepsilon_n - s, \end{aligned}$$

using Lemma 2 we get the expected inequality:

$$d_E^{\uparrow}((w,r),(v,s)) \geq \limsup_n t_n^{-1} d_E(w_n + t_n v, r_n + t_n s) \geq \langle w^*, v \rangle - s.$$

Proof of the theorem in the general case. Since intrinsic approximate convexity is preserved when using an equivalent norm, we may use the norm described in the lemma and take $\sigma > 0$ as there. We use the implication (a)⇒(c) of Corollary 13: for any $\varepsilon > 0$ there exists $\delta \in (0, \sigma/2)$ such that for any $x, x' \in E \cap B(\overline{x}, \delta)$, $x^* \in \partial^{\uparrow} d_E(x)$, one has

$$\langle x^*, x' - x \rangle \leq \varepsilon \left\| x - x' \right\|. \tag{14}$$

Now, by the preceding lemma, for every $w \in B(\overline{w}, \delta)$ and $w^* \in \partial^{\uparrow} f(w)$ we have $(w^*, -1) \in \partial^{\uparrow} d_E(x)$ with $x := (w, f(w))$. Let $u \in \delta \overline{B}_X$ be such that $w' := w + u \in B(\overline{w}, \delta)$. Then we have that $x := (w, f(w))$, $x' := (w', f(w')) \in E \cap B(\overline{x}, \sigma)$, $x^* := (w^*, -1) \in \partial^{\uparrow} d_E(x)$, hence, by inequality (14),

$$\langle w^*, u \rangle - (f(w') - f(w)) \leq \varepsilon \left(\left\| w' - w \right\| + \left| f(w') - f(w) \right| \right) \leq \varepsilon (c+1) \left\| u \right\|.$$

Thus for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $w \in B(\overline{w}, \delta)$, any $w^* \in \partial^{\uparrow} f(w)$ and any $u \in \overline{B}(0, \delta)$ with $w + u \in B(\overline{w}, \delta)$ one has $f(w+u) - f(w) \geq \langle w^*, u \rangle - \varepsilon (c+1) \left\| u \right\|$, so that f is approximately convex around $\overline{w}$ by [15, Thm 2] or [38]. □

Let us complete the preceding results with the following one.

Proposition 17 *Let $f : W \to \mathbb{R}$ be a function which is Lipschitz with rate c on some ball $B(\overline{w}, \rho)$. Suppose $X := W \times \mathbb{R}$ is endowed with the norm given by $\left\| (w,r) \right\| = c \left\| w \right\| + |r|$. If f is approximately convex around $\overline{w}$, then, for any $\overline{r} \geq f(\overline{w})$, the epigraph E of f is approximately convex around $\overline{x} := (\overline{w}, \overline{r})$.*

Proof. Let us endow X with the norm described in the statement. By [24], [28] we can find $\rho' \in (0, \rho)$ such that

$$d_E(w,r) = (f(w) - r)_+$$

for $(w,r) \in B(\overline{x}, \rho')$ with $\overline{x} := (\overline{w}, f(\overline{w}))$. Given $\varepsilon > 0$, let $\delta \in (0, \rho')$ be such that

$$f((1-t)w_1 + t w_2) \leq (1-t) f(w_1) + t f(w_2) + \varepsilon t (1-t) \left\| w_1 - w_2 \right\|$$

for any $w_1, w_2 \in B(\overline{w}, \delta)$, $t \in [0,1]$. Let $x_i := (w_i, r_i) \in B(\overline{x}, \delta)$ for $i = 1, 2$ and let $w := (1-t) w_1 + t w_2$, $r := (1-t) r_1 + t r_2$, $x := (w, r)$. Then we have

$$f(w) - r \leq (1-t)(f(w_1) - r_1)_+ + t (f(w_2) - r_2)_+ + \varepsilon t (1-t) \left\| w_1 - w_2 \right\|$$

hence, assuming without loss of generality that $c \geq 1$, so that $\left\| w_1 - w_2 \right\| \leq \left\| x_1 - x_2 \right\|$,

$$d_E(x) \leq (1-t) d_E(x_1) + t d_E(x_2) + \varepsilon t (1-t) \left\| x_1 - x_2 \right\|.$$

□

The preceding results enable us to give a partial answer to the question of the relationships between intrinsic approximate convexity and approximate convexity. We restrict our attention to sets satisfying the *cone property* (the so-called *epi-Lipschitzian sets*). Recall that E satisfies the cone property around $\overline{x}$ if there exist $r, \rho > 0$ and $u \in S_X$ such that for each $x \in E \cap B(\overline{x}, \rho)$, $v \in B(u, r)$, $t \in (0, r)$ one has $x + tv \in E$. Thus, our argument is close to the one in [2, Thm 4.14], even if intrinsic approximate convexity is not considered there.

Corollary 18 *Suppose E satisfies the cone property around $\overline{x}$. Then E is intrinsically approximately convex around $\overline{x}$ if, and only if, it is approximately convex around $\overline{x}$ for some compatible norm on X.*

Proof. It suffices to prove the only if condition. Since E satisfies the cone property around $\overline{x}$ there exist $\rho, \sigma > 0$, some hyperplane W of X and some $u \in S_X$ such that $X = W \oplus \mathbb{R}u$ and a Lipschitzian function $f : B(0, \rho) \cap W \to \mathbb{R}$ with $E \cap B(\overline{x}, \sigma) = \{\overline{x} + w + ru : w \in B(0, \rho), r \geq f(w)\} \cap B(\overline{x}, \sigma)$. Thus, identifying X with $W \times \mathbb{R}u$, locally E is the epigraph of a Lipschitzian function, and by Theorem 15, since E is intrinsically approximately convex around $\overline{x}$, f is approximately convex around 0. Then, by Proposition 17, we can endow the product $W \times \mathbb{R}u$ with a norm for which E is approximately convex around $\overline{x}$. $\square$

Finally, let us turn to sublevel sets.

Proposition 19 *Let X be a Banach space with a norm which is Fréchet differentiable off 0 and let $f : X \to \mathbb{R}$ be a continuous function. Suppose f is approximately convex around $\overline{x} \in S := \{x \in X : f(x) \leq 0\}$ and there exist $c > 0$, $r > 0$ such that $\|x^*\| \geq c$ for each $x \in (X \backslash S) \cap B(\overline{x}, r)$ and each $x^* \in \partial^- f(x)$. Then S is intrinsically approximately convex around $\overline{x}$.*

Proof. Without loss of generality we may suppose f takes the value $+\infty$ on $X \backslash U$, where $U := B(\overline{x}, r)$. Then, by [53], [12], [44, Thm 9.1], [40, Thm 3.2] and several other contributions we have $f_+(x) \geq c d_S(x)$ for $x \in U$, where $f_+ := \max(f, 0)$. Let $\varepsilon > 0$ be given. Using [38, Thm 7] we can find $\delta \in (0, r)$ such that

$$\langle x^*, x' - x\rangle \leq f(x') - f(x) + \varepsilon \|x' - x\|$$

for any $x, x' \in B(\overline{x}, \delta)$, $x^* \in \partial^- f(x)$. Given $x, x' \in S \cap B(\overline{x}, \delta)$, $x^* \in \partial^- d_S(x)$, using [40, Cor. 4.1] we can find sequences (λ_n), (x_n), (x_n^*) in $[0,1]$, X, X^* respectively such that $(x_n) \to x$, $(\lambda_n x_n^*) \to x^*$ and $x_n^* \in \partial^- f(x_n)$ for each $n \in \mathbb{N}$. Let us first suppose $x^* \neq 0$. Then $f(x) = 0$ because f is continuous at x and x cannot belong to the interior of S. For each n so large that $x_n \in B(\overline{x}, \delta)$, we have

$$\langle x_n^*, x' - x_n\rangle \leq f(x') - f(x_n) + \varepsilon \|x' - x_n\|.$$

Thus

$$\langle x^*, x' - x\rangle = \lim_n \langle \lambda_n x_n^*, x' - x_n\rangle \leq \limsup_n \lambda_n \left(f(x') - f(x_n) + \varepsilon \|x' - x_n\|\right) \leq \varepsilon \|x' - x\|$$

since $f(x') \leq 0$ and $(f(x_n)) \to 0$. When $x^* = 0$, the inequality $\langle x^*, x' - x\rangle \leq \varepsilon \|x' - x\|$ is obvious. Thus, assertion (c) of Theorem 10 is satisfied for $\partial := \partial^-$ and $E := S$ so that S is intrinsically approximately convex around $\overline{x}$. $\square$

6 Approximately convex sets and projections

The following result shows that approximate convexity of a distance function is equivalent to its continuous differentiability in the context of uniformly smooth spaces.

First, we need the following lemma which gives the firm regularity of $-d_E$ on uniformly smooth spaces. The result could be deduced from [23, Thm 5.6] or from the fact that an approximately convex function is firmly regular and from the study of marginal functions made in [38]. However, for the reader's convenience, we present a direct proof inspired by [5] where the Gâteaux regularity of $-d_E$ has been established.

Lemma 20 *Let X be Fréchet uniformly smooth and E be a closed subset of X. Then $-d_E(.)$ is firmly (Clarke) regular at any $w \in X \setminus E$ in the sense that $\partial^{\uparrow}(-d_E) = \partial^-(-d_E)$.*

Proof. Let us denote by j the reduced duality mapping, i.e. the derivative of the function $\|\cdot\|$ on $X\backslash\{0\}$. Let $w \in X \setminus E$. By ([52, Thm 3.7.4]) and the uniform smoothness of X, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{1}{t}\left(||x+tu|| - ||x||\right) - \langle j(x), u\rangle\right| < \varepsilon \qquad \forall x, u \in S_X,\ |t| < \delta. \tag{15}$$

Let $u \in S_X$. There exists $(w_n) \to w, (t_n) \to 0_+$, such that

$$(-d_E)^{\uparrow}(w,u) = \lim_{n\to\infty} \frac{1}{t_n}\left(-d_E(w_n + t_n u) + d_E(w_n)\right).$$

For each n, we can find $x_n \in E$ such that

$$-d_E(w_n + t_n u) \leqslant -||w_n + t_n u - x_n|| + t_n^2.$$

Therefore,

$$\frac{1}{t_n}\left(-d_E(w_n + t_n u) + d_E(w_n)\right) \leqslant \frac{1}{t_n}\left(-||w_n + t_n u - x_n|| + ||w_n - x_n||\right) + t_n.$$

Setting $r_n := ||w_n - x_n||$, $u_n := r_n^{-1}(w_n - x_n)$, and observing that $\left(t_n r_n^{-1}\right) \to 0$ as $\left(r_n^{-1}\right) \to 1/d_E(w)$, using (15), we have, for n large enough and $t \in (-\delta d_E(w), \delta d_E(w))$,

$$\begin{aligned}
\frac{1}{t_n}\left(-||w_n + t_n u - x_n|| + ||w_n - x_n||\right) &= \frac{1}{t_n r_n^{-1}}\left(-||u_n + t_n r_n^{-1} u|| + \|u_n\|\right)\\
&\leqslant -\langle j(u_n), u\rangle + \varepsilon\\
&\leqslant \frac{1}{t r_n^{-1}}\left(-||u_n + t r_n^{-1} u|| + ||u_n||\right) + 2\varepsilon\\
&\leqslant t^{-1}\left(-||w_n + tu - x_n|| + ||w_n - x_n||\right) + 2\varepsilon\\
&\leqslant t^{-1}\left(-d_E(w_n + tu) + ||w_n - x_n||\right) + 2\varepsilon.
\end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain for any $u \in S_X$, $t \in (-\delta d_E(w), \delta d_E(w))$

$$(-d_E)^{\uparrow}(w,u) \leqslant \frac{1}{t}\left(-d_E(w + tu) + d_E(w)\right) + 2\varepsilon.$$

This inequality proves the firm regularity of $-d_E(.)$ ■

The following result is reminiscent of [10, Thm 4.1] which takes place in a Hilbert space. However, here U is not a uniform entourage of E; it may be small (or large) and far from E.

Theorem 21 *Suppose that the norm of X is Fréchet differentiable on $X\backslash\{0\}$. Let E be a closed subset of X and let U be an open subset of X. Consider the following assertions*

(a) Each $w \in U$ has a unique metric projection $P_E(w)$ in E and the mapping $w \mapsto P_E(w)$ is continuous on $U \setminus E$.

(b) $d_E(\cdot)$ is continuously differentiable on $U \setminus E$.

(c) $d_E(\cdot)$ is approximately convex on $U \setminus E$.

Then, one has (a) $\Rightarrow$ (b) $\Rightarrow$ (c). If X is uniformly Fréchet smooth, then (a)$\Rightarrow$(b)$\Longleftrightarrow$(c).
If, in addition, X is strictly convex and the norm of X has the Kadec-Klee property, then (a)$\Longleftrightarrow$(b)$\Longleftrightarrow$(c).

Proof. (a)$\Rightarrow$(b) For any $w \in U\backslash E, v \in X$, we have

$$\begin{aligned}
d_E^{\uparrow}(w,v) &= \limsup_{y\to w, t\downarrow 0} \frac{1}{t}\left(d_E(y+tv) - d_E(y)\right)\\
&\leqslant \limsup_{y\to w, t\downarrow 0} \frac{1}{t}\left(||y + tv - P_E(y)|| - ||y - P_E(y)||\right)\\
&\leqslant \limsup_{y\to w, t\downarrow 0} \sup_{\theta\in[0,1]} \langle j(y - P_E(y) + \theta t v), v\rangle = \langle j(w - P_E(w)), v\rangle.
\end{aligned}$$

Thus $d_E^{\uparrow}(w,\cdot) = \langle j(w - P_E(w)), \cdot\rangle$ and $\partial^{\uparrow} d_E(w) = j(w - P_E(w))$. Since the norm is differentiable and convex, the duality mapping $j(.)$ is continuous. Thus $d_E(.)$ is continuously differentiable on $U \setminus E$.

(b)⇒(c) is obvious.

(c)⇒(b) when X is uniformly Fréchet smooth. Assume that $d_E(\cdot)$ is approximately convex on $U \setminus E$. Then $d_E(\cdot)$ is firmly regular at all $w \in U \setminus E$. Moreover, by the preceding lemma, $-d_E(\cdot)$ is firmly regular at all $w \in X \setminus E$. Thus $\partial^- d_E(w) \neq \emptyset$ and $\partial^-(-d_E)(w) \neq \emptyset$ at any $w \in U \setminus E$. Therefore, $d_E(\cdot)$ is Fréchet differentiable on $U \setminus E$.

Let us prove that $d'_E(\cdot)$ is continuous on $U \setminus E$. Let $w \in U \setminus E$ and $0 < \varepsilon < 1$ be given. There exists $\delta > 0$ such that for all $v \in B(0, \delta)$ we have

$$d_E(w + v) - d_E(w) - \langle d'_E(w), v\rangle \leqslant \varepsilon ||v||,$$

On the other hand, by the approximate convexity of $d_E(\cdot)$, there exists $\rho \in (0, \delta)$ such that

$$\langle d'_E(x), v\rangle \leqslant d_E(x + v) - d_E(x) + \varepsilon ||v||$$

for all $x \in B(w, \rho)$, $B(0, \rho)$. Thus, for any $x \in B(w, \varepsilon\rho)$, $v \in B(0, \rho)$, we have

$$\langle d'_E(x) - d'_E(w), v\rangle \leqslant 2||x - w|| + 2\varepsilon ||v|| \leqslant 4\varepsilon\rho$$

Hence $||d'_E(x) - d'_E(w)|| \leqslant 4\varepsilon$ for $x \in B(w, \varepsilon\rho)$ and $d'_E(\cdot)$ is continuous at w.

(b)⇒(a) when X is strictly convex, uniformly smooth and its norm has the Kadec-Klee property. We follow the argument of [6, Lemma 6]. The uniform smoothness of X ensures that X is reflexive by the Milman-Pettis theorem ([3], [21, Thm 9.12]) and by Lemma 3, for any $w \in U \setminus E$, there exists $x \in E$ such that

$$\langle d'_E(w), w - x\rangle = ||w - x|| = d_E(w). \tag{16}$$

Since $\|d'_E(w)\| \leq 1$, we have $\|d'_E(w)\| = 1$ and $j(w - x) = ||w - x|| d'_E(w) = (\frac{1}{2} d_E^2)'(w)$. Since the space is strictly convex, j is injective, so that x is the unique point of E satisfying (16). In order to prove that $P_E(\cdot)$ is continuous, let us consider a sequence $(w_n) \to w$. Let $x_n := P_E(w_n)$ and let z be a weak limit point of (x_n). Since d'_E is continuous and the norm is weakly lower semicontinuous, passing to the limit in the equality

$$\langle d'_E(w_n), w_n - x_n\rangle = d_E(w_n) = ||w_n - x_n||$$

we get, since $\|d'_E(w)\| \leq 1$,

$$||w - z|| \geq \langle d'_E(w), w - z\rangle = d_E(w) = \lim_n ||w_n - x_n|| \geq ||w - z||.$$

By the Kadec-Klee property we obtain that $(x_n) \to z$, so that $z \in E$ and $z = P_E(w)$. ■

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