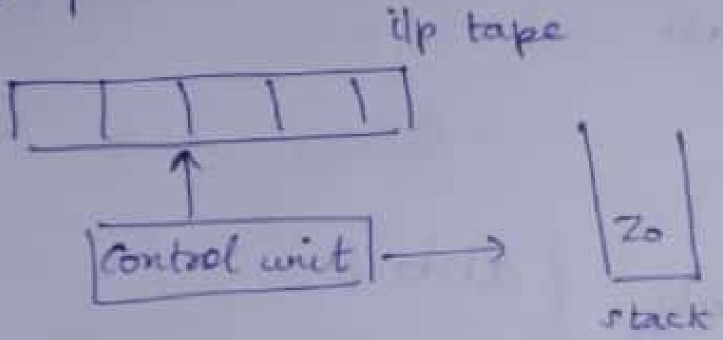
# TOC - Mod 4

## Push Down Automata

- is an NFA with stack.
- CFGs are recognised by PDA.
- components




- 7 tuples
  $(Q, \Sigma, \delta, q_0, F, Z_0, \gamma)$

  Q - set of states

  $\Sigma$ - set of i/p symbols

  $\delta$ - transition function $Q \times (\Sigma \cup \epsilon) \times \gamma \rightarrow Q \times \gamma^*$

  $Z_0$ - Initial symbol on stack

  $q_0$ - Initial state

  F - set of final states

  $\gamma$ - elements of stack

## Instantaneous Description (ID)

- Current configuration of PDA at any given time.
- 3 tuple $(q, w, \alpha)$
- Move

$(q_0, abcba, z_0) \vdash (q_0, bcba, az_0)$

## Acceptance

By empty stack

$\{ w \mid (q_0, w, z_0) \vdash^* (p, \varepsilon, \varepsilon)$

p - not final state

By final state

$\{ w \mid (q_0, w, z_0) \vdash^* (p, \varepsilon, \alpha) \quad \alpha \in \Gamma^*$

p - final state

## Deterministic & Non-Deterministic PDA

PDA is deterministic if

$\delta(q, a, z)$ has only 1 element.

If $\delta(q, \varepsilon, z)$ is not empty then

$\delta(q, a, z)$ should be empty.

i.e. If there is a $\varepsilon$-transition then, there should not be any transition from a, when top of stack is $z_0$.

i.e. $\delta(q_0, \varepsilon, z_0) = q_1, z_0$

$\delta(q_0, a, z_0) = q_0, az_0$ } no such

$\delta(q_0, b, z_0) = q_0, bz_0$ } transition

Qn] Design a PDA to accept $L = \{a^n b^n \mid n \geq 1\}$

$a^2b^2$     present state i/p TOS

push → $\delta(q_0, a, z_0) = q_0, az_0$

$\delta(q_0, a, a) = q_0, aa$

Pop-state change & $\varepsilon$

pop → $\delta(q_0, b, a) = q_1, \cancel{b}\varepsilon$

$\delta(q_1, b, a) = q_1, \varepsilon$

$\delta(q_1, \varepsilon, z_0) = q_f, z_0$

or $\delta(q_1, \varepsilon, z_0) = q_1, \varepsilon$

Move

$(q_0, aabb, z_0) \vdash (q_0, abb, az_0) \vdash$

$(q_0, bb, aaz_0)$ ~~⊢ (q0, b, ba)~~

$\vdash (q_1, b, az_0) \vdash (q_1, \varepsilon, z_0)$

$\vdash (q_f, \varepsilon, z_0)$

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**Push**

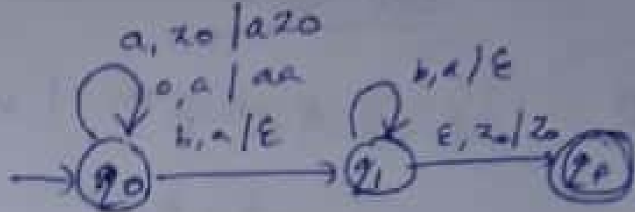
$\delta(q_0, a, a) = q_0, aa$

**Pop**

$\delta(q_0, b, a) = q_1, \varepsilon$

**Read**

$\delta(q_0, b, a) = q_0, a$ → no state change needed




$L = a^n \; b \; c^n$

$a^n$ ↓ push, $b$ ↓ read, $c^n$ ↓ pop

$n = 2$

$a^2 b c^2$

$$\left.\begin{aligned} &\delta(q_0, a, z_0) = q_0, az_0 \\ &\delta(q_0, a, a) = q_0, aa \end{aligned}\right\} \text{PUSH}$$

$\delta(q_0, \underline{b}, a) = q_1, a$ ⇒ b is only READ, a is TOS

$$\left.\begin{aligned} &\delta(q_1, c, a) = q_2, \varepsilon \\ &\delta(q_2, c, a) = q_2, \varepsilon \end{aligned}\right\} \text{POP}$$

$\delta(q_2, \varepsilon, z_0) = q_f, z_0$

Dec 2017

13. $L = \{a^n b^{2n} \mid n > 0\}$  $n = 3$.

$L = a^3 b^6$


$$\left.\begin{array}{l} \delta(q_0, a, Z_0) = q_0, aZ_0 \\ \delta(q_0, a, a) = q_0, aa \\ \cancel{\delta(q_0, a, a) = q_0, aa} \end{array}\right\} \text{PUSH}$$

$\delta(q_0, b, a) = q_0, a \longrightarrow$ READ b

$\delta(q_0, b, a) = q_1, \varepsilon \longrightarrow$ POP a

$\delta(q_1, b, a) = q_1, a \longrightarrow$ Read b

$\delta(q_1, b, a) = q_1, \varepsilon \longrightarrow$ Pop a

$\cancel{\delta(q_1, b, a) = q_1, a} \longrightarrow$ ~~Read b~~

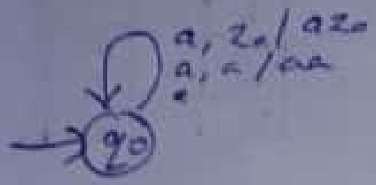
$\cancel{\delta(q_1, b, a) = q_1, \varepsilon} \longrightarrow$ ~~Pop last a~~

$\delta(q_1, \varepsilon, Z_0) = q_f, Z_0$

Move

$(q_0, aaabbbbbb, Z_0) \vdash (q_0, aabbbbbb, aZ_0) \vdash$
$(q_0, abbbbbb, aaZ_0) \vdash (q_0, bbbbbb, aaaZ_0) \vdash$
$(q_0, \overset{\text{read}}{bbbbb}, aaaZ_0) \vdash (q_1, bbbb, aaZ_0) \vdash$
$\vdash (q_1, bbb, aaZ_0) \vdash (q_1, bb, aZ_0) \vdash (q_1, b, aZ_0)$

$\vdash (q_1, \cancel{z_0}\ \varepsilon, z_0) \vdash (q_f, \varepsilon, z_0)$.




(aa bb)(c ...)

April 2018

empty stack as final condn.

Q(i) $L = wcw^R \mid w$ in $(a+b)^*$

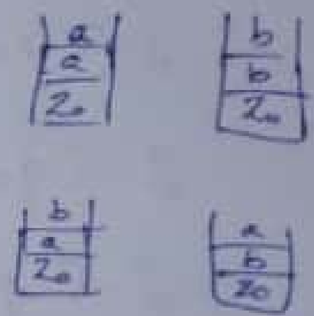
-7-

push | pop

aca
bcb
abcab

$\delta(q_0, a, z_0) = q_0, az_0$

$\delta(q_0, b, z_0) = q_0, bz_0$



$\delta(q_0, a, a) = q_0, aa$

$\delta(q_0, b, a) = q_0, \cancel{ab}\ ba$

$\delta(q_0, b, b) = q_0, bb$

$\delta(q_0, a, b) = q_0, ab$

$\delta(q_0, c, a) = q_1, a$ → Read ×

$\delta(q_1, c, b) = q_1, b$ → Read ×

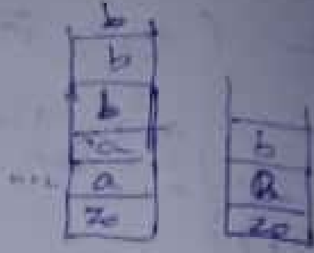
$\delta(q_1, a, a) = q_2, \varepsilon$.

$\delta(q_2, b, b) = q_2, \varepsilon$

$\delta(q_2, \cancel{z_0}\ \varepsilon, z_0) = q_f, \cancel{z_0}\ \varepsilon$

$\cancel{\delta(q_1, a, \varepsilon) = q_f \times \varepsilon}$

Qn] $a^n b^m c^{n+m}$, $n, m > 0$



$\delta(q_0, a, z_0) = q_0, az_0$

~~$\delta(q_0, b, z_0) = q_0, bz_0$~~

$\delta(q_0, a, a) = q_0, aa$

$\delta(q_0, b, a) = q_0, ba$

$\delta(q_0, b, b) = q_0, bb$

$\delta(q_0, c, b) = q_1, \varepsilon$

$\delta(q_1, c, b) = q_1, \varepsilon$

$\delta(q_1, c, a) = q_1, \varepsilon$

$\delta(q_1, \varepsilon, z_0) = q_f, z_0$

Qn] $N_a(\omega) = N_b(\omega)$

Note: No need of state change

$\frac{a}{\text{PUSH}}$ $\frac{b}{\text{POP}}$ $\left|\frac{a}{z_0}\right|$ $\frac{a}{\text{POP}}$ $\frac{b}{\text{PUSH}}$ $\left|\frac{b}{z_0}\right|$

$\delta(q_0, a, z_0) = q_0, az_0$

$\delta(q_0, b, z_0) = q_0, bz_0$

$\delta(q_0, a, a) = q_0, aa$

$\delta(q_0, b, a) = q_1, \varepsilon$

$\delta(q_0, a, b) = q_1, \varepsilon$

$\delta(q_0, b, b) = q_0, bb$

$\delta(q_1, \varepsilon, z_0) = q_f, z_0$

12.

(ii) ID from initial ID (start state, abcba, initial stack symbol) $\vdash^*$ to final ID (state, $\varepsilon$, $\varepsilon$)

$(q_0, \underset{push}{\underline{ab}}c\underset{pop}{\underline{ba}}, z_0) \vdash^* (q_0, bcba, az_0) \vdash$
$(q_0, cba, baz_0) \vdash (q_0, ba, baz_0) \vdash$
$(q_1, a, az_0) \vdash (q_1, \varepsilon, z_0) \vdash$
$(q_f, \varepsilon, \varepsilon)$

abab

$WW^R$

$\delta(q_0, a, z_0) = q_0, az_0$

$\delta(q_0, b, z_0) = q_0, bz_0$

$\delta(q_0, b, a) = q_0, ba$

$\delta(q_0, a, b) = q_0, ab$

$\delta(q_0, a, a) = (q_0, aa)\ (q_1, \varepsilon)$

$\delta(q_0, b, b) = (q_0, bb)\ (q_1, \varepsilon)$

$\delta(q_1, a, a) = q_1, \varepsilon$

$\delta(q_1, b, b) = q_1, \varepsilon$

$\delta(q_1, \varepsilon, z_0) = q_f, z_0$

$\cancel{\delta(q_0, a, a) = q_1, \varepsilon}$

$\cancel{\delta(q_0, b, b) = q_1, \varepsilon}$

# CFG to PDA

1. Convert grammar to GNF.
2. Let $q_0$ be start state & $z_0$ be initial symbol on stack, w/o consuming i/p, push $S$ onto stack & state change to $q_1$

$$\delta(q_0, \epsilon, z_0) = q_1, Sz_0$$

3. For each production of the form

$$A \rightarrow a\alpha$$

Introduce transition

$$\delta(q_1, a, A) = q_1, \alpha$$

4. Finally in state $q_1$, without consuming input, change state to $q_f$.

$$\delta(q_1, \epsilon, z_0) = q_f, z_0$$

Qn] $S \rightarrow aABC$

$A \rightarrow aB \mid a$

$B \rightarrow bA \mid b$

$C \rightarrow a$

Design PDA:

It is in GNF.

$$\delta(q_0, \epsilon, z_0) = q_1, Sz_0$$

$S \rightarrow aABC$ $\quad \delta(q_1, a, S) = q_1, ABC$

$A \rightarrow aB$ $\quad \delta(q_1, a, A) = q_1, B$

$A \rightarrow a$ $\quad \delta(q_1, a, A) = q_1, \epsilon$

$B \rightarrow bA$ $\quad \delta(q_1, b, B) = q_1, A$

$B \rightarrow b$ $\quad\quad\quad\quad$ $\delta(q_1, b, B) = q_1, \varepsilon$

$C \rightarrow a$ $\quad\quad\quad\quad$ $\delta(q_1, a, C) = q_1, \varepsilon$

$\delta(q_1, \varepsilon, z_0) = q_f, z_0$

Derive any string.

Move

$S \rightarrow aABC$

$\rightarrow aaBC$

$\rightarrow aabC$

$S \rightarrow aaba$

$(q_0, aaba, z_0) \vdash (q_1, aaba, Sz_0)$

$\vdash (q_1, aba, ABCz_0) \vdash (q_1, ba, BCz_0)$

$\vdash (q_1, a, Cz_0) \vdash (q_1, \varepsilon, z_0) \vdash$

$(q_f, \varepsilon, z_0)$

$B \rightarrow A$ is not in GNF

$B \rightarrow aBB \mid a$

Closure (X)

if cycle → infinite

| | |
|---|---|
| $a^n b^m$ $\quad$ $n > m$ | $0^n 1^m 0^m 1^n$ |

$(q_1, \varepsilon, a) = q_1, \varepsilon$

~~$(q_f, \varepsilon, a)$~~ → final state

$(q_1, b, z_0)$

$a^m b^n$

$a^n b^m$ $m > n$

$\delta(q_1, b, z_0)$

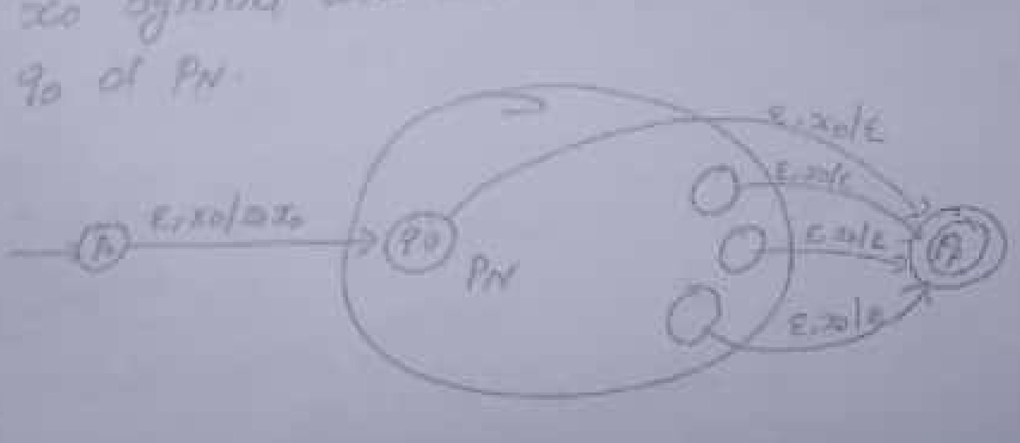
$z a$

## Equivalance of Acceptance by empty stack & final state

If there exist a PDA $P_N$ that accepts a language by empty stack, then there exist a PDA $P_F$ that accepts a language by final state.

Here $X_0$ is the symbol marked on the bottom of the stack for $P_N$. $P_N$ goes on processing the input if it sees any other symbol on the stack except $X_0$.

If $P_N$ sees $X_0$, then it finishes processing the string. So construct $P_F$ with new starting state $p_0$ and new final state $p_f$. $p_0$, starting state of $P_F$ is used to push $Z_0$ to make $X_0$ at $Z_0$ $X_0$ symbol on stack, when it reads the bottom of the stack, $X_0$ symbol and enters the initial state $q_0$ of $P_N$.



$P_0$ is used to push the symbol $z_0$ to the stack when it sees $x_0$ on the top of the stack and enters state $q_0$. $q_0$ - initial state of $P_N$. So $P_F$ stimulates $P_N$ until empties its stack. $P_F$ detects that $P_N$ emptied its stack when $P_F$ sees $x_0$ & $P_F$ go to final state $P_F$.

PDA $P_F$
$= (Q \cup \{P_0, P_F\}, \Sigma, \Gamma \cup \{x_0\}, \delta_F, P_0 x_0, P_F)$

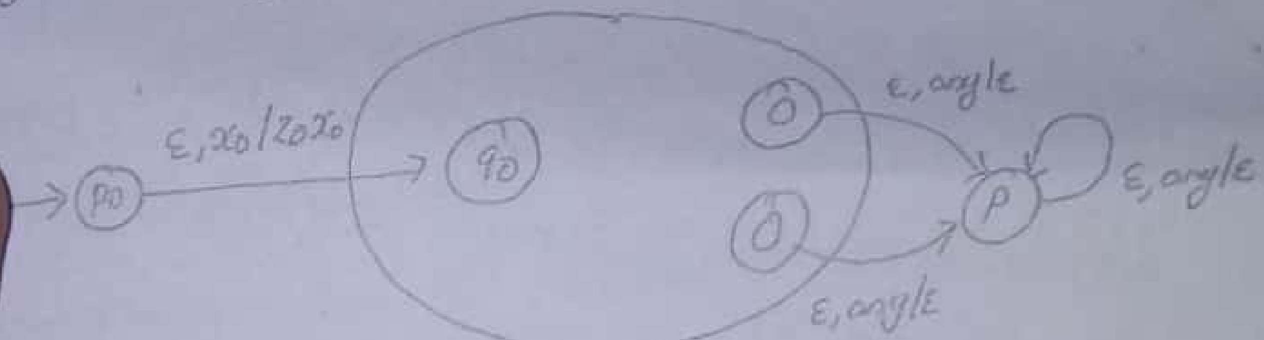
$\delta_F (P_0, \epsilon, x_0) = (P_0, z_0 x_0)$

$\delta_F (q, \epsilon, x_0) = (P_F, \epsilon)$

We can conclude that $\omega$ is in $L(P_F)$ iff $\omega$ is in $N(P_N)$. The ID of $P_F$ after stimulating $P_N$ is

$(P_0, \omega, x_0) \vdash (P_0, \omega, z_0 x_0) \vdash (q, \epsilon, x_0) \vdash (P_F, \epsilon, \epsilon)$

Thus PDA $P_F$ accepts $\omega$ by final state if there exist a PDA $P_F$ that accept language by reaching final state then there exist a PDA $P_N$ that accept a language by empty stack.




$q_0$ - initial state of $P_F$

Here initial state $p_0$ of $P_N$ push the stack symbol $z_0$ which is the start symbol of $P_F$ on to the stack & enters state $q_0$, initial state of $P_F$

Then $P_F$ after consuming its input ω, it enters any one of the final states.

For each accepting states of $P_F$, add a transition to the new final state p on ε with any symbol on stack & delete stack symbol.

Thus whenever $P_F$ enters the final state after consuming ω, $P_N$ will empty its stack after consuming string ω

$P_N = \{ Q \cup \{p_0, p\}, \Sigma, \Gamma \cup \{x_0\}, \delta_N, p_0, x_0 \}$

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$\delta_N(p_0, \varepsilon, x_0) = (q_0, z_0 x_0)$

$\delta_N(q, a, y)$ contain every pair of $\delta_F(q, a, y)$ since $P_N$ simulates $P_F$

For all accepting states of $q$ in $F$ & stack symbol $y$ in $\Gamma$ $\delta_N(q, \varepsilon, y)$ contain $(p, \varepsilon)$.
For all stack symbol $y$ in $\Gamma$ $\delta_N(p, \varepsilon, y) = (p, \varepsilon)$

We conclude that $w$ is in $N(P_N)$ IFF $w$ is in $L(P_F)$

ID of $P_N$

$(p_0, w, x_0) \underset{P_N}{\vdash} (q_0, w, z_0 x_0) \underset{P_N}{\overset{*}{\vdash}} (q, \varepsilon, \alpha x_0)$

$\underset{P_N}{\overset{*}{\vdash}} (p, \varepsilon, \varepsilon)$

Thus PDA $P_N$ accepts $w$ by emptying its stack.

# Equivalence of PDA & CFG

Let $L$ is a CFL then there exists a PDA $M$ such that $L = N(M)$

Let $G = (V, T, P, S)$ be a CFG in GNF generating $L$.

Let $M = (\{q\}, T, V, \delta, q, S, \phi)$

where $\delta(q, a, A)$ contains $(q, \gamma)$ whenever $A \to a\gamma$ is in $P$.

PDA stimulates leftmost derivation of $G$.

$S \xrightarrow{*} x\alpha$ by leftmost derivation

iff $(q, x, S) \overset{*}{\underset{M}{\vdash}} (q, \epsilon, \alpha)$

Suppose that $(q, x, S) \overset{i}{\vdash} (q, \epsilon, \alpha)$ and show by induction on $i$ that $S \xrightarrow{*} x\alpha$. The basis $i=0$ is trivial since $x = \epsilon$ & $\alpha = S$. For induction, suppose $i \geq 1$ & let $x = ya$

$(q, ya, S) \overset{i-1}{\vdash} (q, a, \beta) \vdash (q, \epsilon, \alpha)$

If we remove 'a' from end of i/p string in first $i$ ID's of the sequence, we discover that $(q, y, S) \overset{i-1}{\vdash} (q, \epsilon, \beta)$

By inductive hypothesis $S \overset{*}{\Rightarrow} y\beta$

The move $(q, a, \beta) \vdash (q, \varepsilon, \alpha)$ implies that $\beta = A\gamma'$ for some $A$ in $V$, $A \to a\eta$ is a production of $G$ and $\alpha = \eta\gamma'$ where

$$S \overset{*}{\Rightarrow} y\beta \Rightarrow ya\eta\gamma' = x\alpha$$

Suppose that $S \overset{i}{\Rightarrow} x\alpha$ by LMD

We show by induction on $i$ that $(q, x, S) \overset{*}{\vdash} (q, \varepsilon, \alpha)$ The basis $i=0$ is again trivial. Let $i \geq 1$ & Suppose

$$S \overset{i-1}{\Rightarrow} yA\gamma' \Rightarrow ya\eta\gamma'$$

where $x = ya$ & $\alpha = \eta\gamma'$

By inductive hypothesis

$(q, y, S) \overset{*}{\vdash} (q, \varepsilon, A\gamma')$ and thus $(q, ya, S) \overset{*}{\vdash} (q, a, A\gamma')$ since $A \to a\eta$ is a production. It follows that $\delta(q, a, A)$ contains $(q, \eta)$ thus

$$(q, x, S) \overset{*}{\vdash} (q, a, A\gamma') \vdash (q, \varepsilon, \alpha)$$

with $\alpha = \varepsilon$ so $S \overset{*}{\Rightarrow} x$ iff $(q, x, S) \overset{*}{\vdash} (q, \varepsilon$

& $x$ is in $L(G)$ iff $x$ is in $N(M)$)

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