Introduction to Fuzzy Logic, Classical Sets and Fuzzy Sets

7

Learning Objectives

- Definition of classical sets and fuzzy sets.
- The various operations and properties of classical and fuzzy sets.
- How functional mapping of crisp set can be carried out.
- Solved problems performing the operations and properties of fuzzy sets.

7.1 Introduction to Fuzzy Logic

In general, the entire real world is complex, and the complexity arises from uncertainty in the form of ambiguity. One should closely look into the real-world complex problems to find an accurate solution, amidst the existing uncertainties, using certain methodologies. Henceforth, the growth of fuzzy logic approach, to handle ambiguity and uncertainty existing in the complex problems. In general, fuzzy logic is a form of multi-valued logic to deal with reasoning that is approximate rather than precise. This is in contradiction with "crisp logic" that deals with precise values. Also, binary sets have binary or Boolean logic (either 0 or 1), which finds solution to a particular set of problems. Fuzzy logic variables may have a truth value that ranges between 0 and 1 and is not constrained to the two truth values of classic propositional logic. Also, as linguistic variables are used in fuzzy logic, these degrees have to be managed by specific functions.

As the complexity of a system increases, it becomes more difficult and eventually impossible to make a precise statement about its behavior, eventually arriving at a point of complexity where the fuzzy logic method born in humans is the only way to get at the problem.

(Originally identified and set forth by Lotfi A. Zadeh, Ph.D., University of California, Berkeley)

Fuzzy logic, introduced in the year 1965 by Lotfi A. Zadeh, is a mathematical tool for dealing with uncertainty. Dr. Zadeh states that the Principle of complexity and imprecision are correlated: "The closer one looks at a real world problem, the fuzzier becomes its solution." Fuzzy logic offers soft computing paradigm the important concept of computing with words. It provides a technique to deal with imprecision and information granularity. The fuzzy theory provides a mechanism for representing linguistic constructs such as "high," "low," "medium," "tall," "many." In general, fuzzy logic provides an inference structure that enables appropriate human reasoning capabilities. On the contrary, the traditional binary set theory describes crisp events, that is, events that either do or do not occur. It uses probability theory to explain if an event will occur, measuring the chance with which a given event is expected to occur. The theory of fuzzy logic is based upon the notion of relative graded membership and so are the functions of cognitive processes. The utility

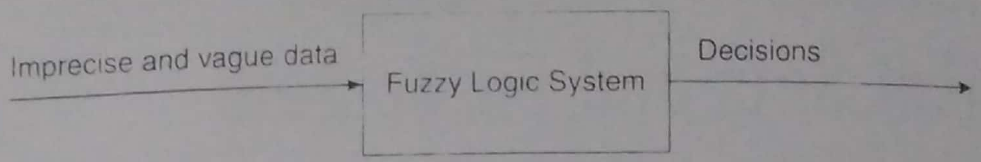

Figure 7-1 A fuzzy logic system accepting imprecise data and providing a decision.

of fuzzy sets lies in their ability to model uncertain or ambiguous data and to provide suitable decisions as i Figure 7-1.

Though fuzzy logic has been applied to many fields, from control theory to artificial intelligence, it sti remains controversial among most statisticians, who prefer Bayesian logic, and some control engineers, wh prefer traditional two-valued logic. In fuzzy systems, values are indicated by a number (called a truth valu ranging from 0 to 1, where 0.0 represents absolute falseness and 1.0 represents absolute truth. While th range evokes the idea of probability, fuzzy logic and fuzzy sets operate quite differently from probability.

Fuzzy sets that represent fuzzy logic provide means to model the uncertainty associated with vagueness imprecision and lack of information regarding a problem or a plant or a system, etc. Consider the meanin of a "short person". For an individual X, a short person may be one whose height is below 4'25". For othe individual Y, a short person may be one whose height is below or equal to 3'90". The word "short" is called linguistic descriptor. The term "short" provides the same meaning to individuals X and Y, but it can be see that they both do not provide a unique definition. The term "short" would be conveyed effectively only whe a computer compares the given height value with the pre-assigned value of "short". This variable "short" i called as *linguistic variable* which represents the imprecision existing in the system.

The basis of the theory lies in making the membership function lie over a range of real numbers from 0.0 to 1.0. The fuzzy set is characterized by (0.0,0,1.0). Real world is vague and assigning rigid values to linguistic variables means that some of the meaning and semantic value is invariably lost. The uncertainty is found to arise from ignorance, from chance and randomness, due to lack of knowledge, from vagueness (unclear), like the fuzziness existing in our natural language. Dr. Zadeh proposed the *set membership* idea to make suitable decisions when uncertainty occurs. Consider the "short" example discussed previously. If we take "short" as a height equal to or less than 4 feet, then 3'90" would easily become the member of the set "short" and 4'25" will not be a member of the set "short." The membership value is "1" if it belongs to the set and "0" if it is not a member of the set. Thus membership in a set is found to be binary, that is, either the element is a member of a set or not. It can be indicated as

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

where $\chi_A(x)$ is the membership of element x in the set A and A is the entire set on the universe.

If it is said that the height is 5'6" (or 168 cm), one might think a bit before deciding whether to consider it as short or not short (i.e., tall). Moreover, one might reckon it as short for a man but tall for a woman. Let's make the statement "John is short", and give it a truth value of 0.70. If 0.70 represented a probability value, it would be read as "There is a 70% chance that John is short," meaning that it is still believed that John is either short or not short, and there exists 70% chance of knowing which group he belongs to. But fuzzy terminology actually translates to "John's degree of membership in the set of short people is 0.70," by which it is meant that if all the (fuzzy set of) short people are considered and lined up, John is positioned 70% of the way to the shortest. In conversation, it is generally said that John is "kind of" short and recognize that there is no definite demarcation between short and tall. This could be stated mathematically as μSHORT(Russell) = 0.70, where μ is the membership function.

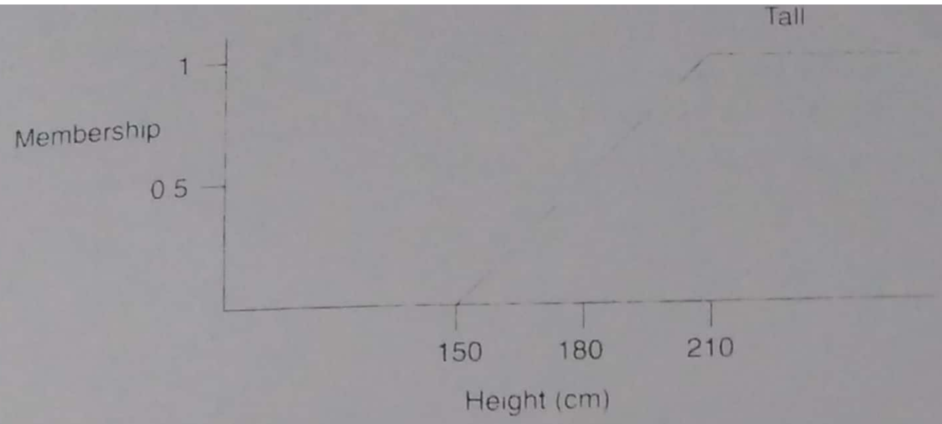

Figure 7-2 Graph showing membership functions for fuzzy set "tall."

Fuzzy logic operates on the concept of membership. For example, the statement "Elizabeth is old" can be translated as Elizabeth is a member of the set of old people and can be written symbolically as μ(OLD), where μ is the membership function that can return a value between 0.0 and 0.1 depending on the degree of membership. In Figure 7-2, the objective term "tall" has been assigned fuzzy values. At 150 cm and below, a person does not belong to the fuzzy class while for above 180, the person certainly belongs to category "tall." However, between 150 and 180 cm, the degree of membership for the class "tall" can be assigned from the curve varying linearly between 0 and 1. The fuzzy concept "tallness" can be extended into "short," "medium" and "tall" as shown in Figure 7-3. This is different from the probability approach that gives the degree of probability of an occurrence of an event (Elizabeth being old, in this instance).

The membership was extended to possess various "degrees of membership" on the real continuous interval [0,1]. Zadeh formed *fuzzy sets* as the sets on the universe X which can accommodate "degrees of membership." The concept of a fuzzy set contrasts with the classical concept of a bivalent set (crisp set) whose boundary is required to be precise, that is, a crisp set is a collection of things for which it is known irrespective of whether any given thing is inside it or not. Zadeh generalized the idea of a crisp set by extending a valuation set {1,0} (definitely in/definitely out) to the interval of real values (degrees of membership) between 1 and 0, denoted as [0,1]. We can say that the degree of membership of any particular element of a fuzzy set expresses the degree of compatibility of the element with a concept represented by fuzzy set. It means that a fuzzy set A contains an object x to degree $a(x)$, that is, $a(x) = \text{Degree}(x \in A)$, and the map $a: X \rightarrow$ {Membership Degrees} is called a *set function* or a *membership function*. The fuzzy set A can be expressed as $A = \{(x, a(x))\}, x \in X$; it imposes an elastic constrain of the possible values of elements $x \in X$, called the *possibility distribution*. Fuzzy sets tend to

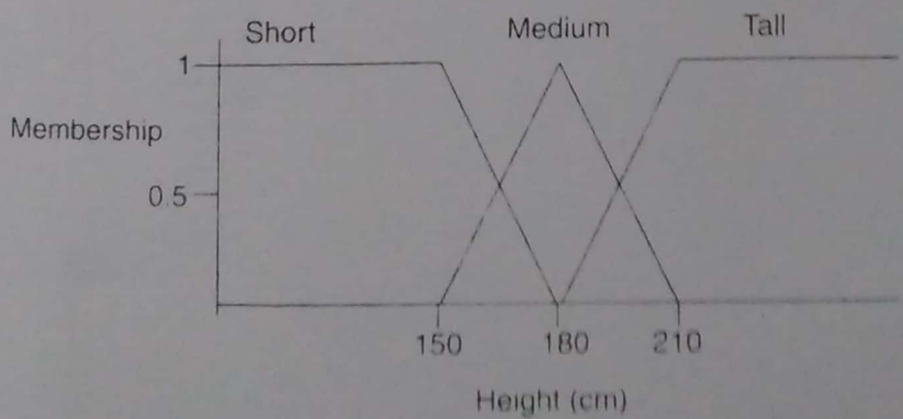

Figure 7-3 Graph showing membership functions for fuzzy sets "short," "medium" and "tall."

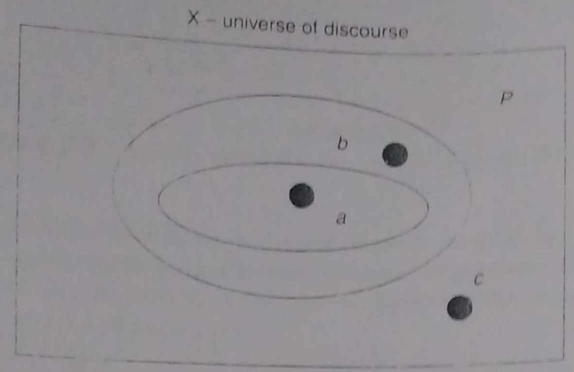

Figure 7-4 Boundary region of a fuzzy set

capture vagueness exclusively via membership functions that are mappings from a given universe of discourse X to a unit interval containing membership values. It is important to note that membership can take values between 0 and 1.

Fuzziness describes the ambiguity of an event and randomness describes the uncertainty in the occurrence of an event. It can be generally seen in classical sets that there is no uncertainty, hence they have crisp boundaries, but in the case of a fuzzy set, since uncertainty occurs, the boundaries may be ambiguously specified.

From Figure 7-4 it can be noted that "a" is clearly a member of fuzzy set P, "c" is clearly not a member of fuzzy set P and the membership of "b" is found to be vague. Hence "a" can take membership value 1, "c" can take membership value 0 and "b" can take membership value between 0 and 1 [0 to 1], say 0.4, 0.7, etc. This is said to be a partial membership of fuzzy set P.

The membership function for a set maps each element of the set to a membership value between 0 and 1 and uniquely describes that set. The values 0 and 1 describe "not belonging to" and "belonging to" a conventional set, respectively; values in between represent "fuzziness." Determining the membership function is subjective to varying degrees depending on the situation. It depends on an individual's perception of the data in question and does not depend on randomness. This concept is important and distinguishes fuzzy set theory from probability theory.

Fuzzy logic also consists of fuzzy inference engine or fuzzy rule-base to perform approximate reasoning somewhat similar to (but much more primitive than) that of the human brain. Computing with words seems to be a slightly futuristic phrase today since only certain aspects of natural language can be represented by the calculus of fuzzy sets; still fuzzy logic remains one of the most practical ways to mimic human expertise in a realistic manner. The fuzzy approach uses a premise that humans don't represent classes of objects (e.g. "*class of bald men*" or the "*class of numbers which are much greater than 50*") as fully disjoint sets but rather as sets in which there may be grades of membership intermediate between full membership and non-membership. Thus, a fuzzy set works as a concept that makes it possible to *treat fuzziness in a quantitative manner.*

Fuzzy sets form the building blocks for fuzzy *IF–THEN* rules which have the general form "*IF X is A THEN Y is B,*" where A and B are fuzzy sets. The term "fuzzy systems" refers mostly to systems that are governed by fuzzy IF–THEN rules. The IF part of an implication is called the *antecedent* whereas the THEN part is called a *consequent.* A fuzzy system is a set of fuzzy rules that converts inputs to outputs. The basic configuration of a pure fuzzy system is shown in Figure 7-5. The fuzzy inference engine (algorithm) combines fuzzy *IF–THEN* rules into a mapping from fuzzy sets in the input space X to fuzzy sets in the output space

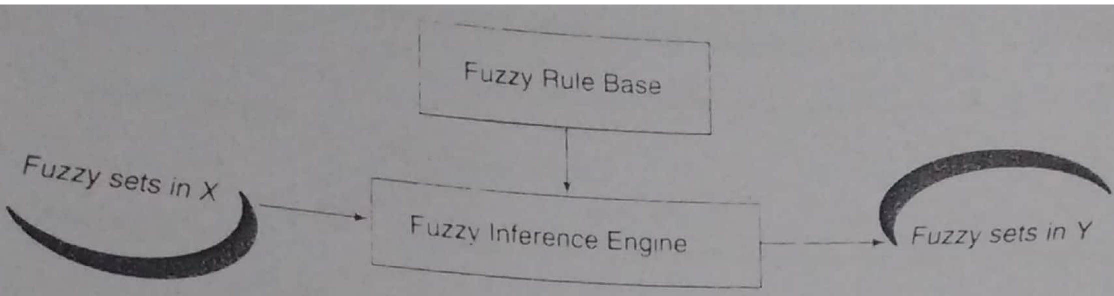

Figure 7-5 Configuration of a pure fuzzy system.

Y based on fuzzy logic principles. From a knowledge representation viewpoint, a fuzzy *IF–THEN* rule is a scheme for capturing knowledge that involves imprecision. The main feature of reasoning using these rules is its *partial matching* capability, which enables an inference to be made from a fuzzy rule even when the rule's condition is only partially satisfied.

Fuzzy systems, on one hand, are rule-based systems that are constructed from a collection of linguistic rules; on the other hand, fuzzy systems are nonlinear mappings of inputs (stimuli) to outputs (responses), that is, certain types of fuzzy systems can be written as compact nonlinear formulas. The inputs and outputs can be numbers or vectors of numbers. These rule-based systems can in theory model any system with arbitrary accuracy, that is, they work as *universal approximators*.

The Achilles' heel of a fuzzy system is its rules; smart rules give smart systems and other rules give less smart or even dumb systems. The *number of rules* increases exponentially with the dimension of the input space (number of system variables). This rule explosion is called the *curse of dimensionality* and is a general problem for mathematical models. For the last 5 years several approaches based on decomposition, (cluster) merging and fusing have been proposed to overcome this problem.

Hence, fuzzy models are not replacements for probability models. The fuzzy models are sometimes found to work better and sometimes they do not. But mostly fuzzy logic has evidently proved that it provides better solutions for complex problems.

7.2 Classical Sets (Crisp Sets)

Basically, a set is defined as a collection of objects, which share certain characteristics. A classical set is a collection of distinct objects. For example, the user may define a classical set of negative integers, a set of persons with height less than 6 feet, and a set of students with passing grades. Each individual entity in a set is called a member or an element of the set. The classical set is defined in such a way that the universe of discourse is splitted into two groups: members and nonmembers. Consider an object x in a crisp set A. This object x is either a member or a nonmember of the given set A. In case of crisp sets, no partial membership exists. A crisp set is defined by its characteristic function.

Let universe of discourse be U. The collection of elements in the universe is called whole set. The total number of elements in universe U is called cardinal number denoted by n_U. Collections of elements within a universe are called sets, and collections of elements within a set are called subsets.

We know that for a crisp set A in universe U:

1. An object x is a member of given set A ($x \in A$), i.e., x belongs to A.
2. An object x is not a member of given set A ($x \notin A$), i.e., x does not belong to A.

There are several ways for defining a set. A set may be defined using one of the following:

1. The list of all the members of a set may be given. Example

$$A = \{2, 4, 6, 8, 10\}$$

2. The properties of the set elements may be specified. Example

$$A = \{x \mid x \text{ is prime number} < 20\}$$

3. The formula for the definition of a set may be mentioned. Example

$$A = \left\{ x_i = \frac{x_i + 1}{5}, i = 1 \text{ to } 10, \quad \text{where } x_i = 1 \right\}$$

4. The set may be defined on the basis of the results of a logical operation. Example

$$A = \{x \mid x \text{ is an element belonging to } P \text{ AND } Q\}$$

5. There exists a membership function, which may also be used to define a set. The membership is denoted by the letter μ and the membership function for a set A is given by (for all values of x)

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

The set with no elements is defined as an empty set or null set. It is denoted by symbol ϕ. The occurrence of an impossible event is denoted by a null set, and the occurrence of a certain event indicates a whole set. The set which consists of all possible subsets of a given set A is called a power set and is denoted as

$$P(A) = \{x \mid x \subseteq A\}$$

For crisp sets A and B containing some elements in universe X, the notations used are given below:

$x \in A \Rightarrow x$ belongs to A
$x \notin A \Rightarrow x$ does not belong to A
$x \in X \Rightarrow x$ belongs to universe X

For classical sets A and B on X, we also have some notations:

$A \subset B \Rightarrow A$ is completely contained in B (i.e., if $x \in A$, then $x \in B$)
$A \subseteq B \Rightarrow A$ is contained in or is equivalent to B
$A = B \Rightarrow A \subseteq B$ and $B \subseteq A$

7.2.1 Operations on Classical Sets

Classical sets can be manipulated through numerous operations such as union, intersection, complement and difference. All these operations are defined and explained in the following sections.

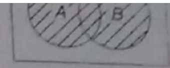

Figure 7-6 Union of two sets.

7.2.1.1 *Union*

The union between two sets gives all those elements in the universe that belong to either set A or set B or both sets A and B. The union operation can be termed as a logical OR operation. The union of two sets A and B is given as

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

The union of sets A and B is illustrated by the Venn diagram shown in Figure 7-6.

7.2.1.2 *Intersection*

The intersection between two sets represents all those elements in the universe that simultaneously belong to both the sets. The intersection operation can be termed as a logical AND operation. The intersection of sets A and B is given by

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

The intersection of sets A and B is represented by the Venn diagram shown in Figure 7-7.

7.2.1.3 *Complement*

The complement of set A is defined as the collection of all elements in universe X that do not reside in set A, i.e., the entities that do not belong to A. It is denoted by $\bar{A}$ and is defined as

$$\bar{A} = \{x \mid x \notin A, x \in X\}$$

where X is the universal set and A is a given set formed from universe X. The complement operation of set A is shown in Figure 7-8.

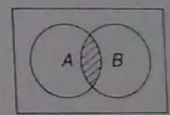

Figure 7-7 Intersection of two sets.

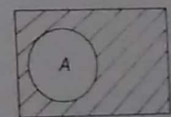

Figure 7-8 Complement of set A.

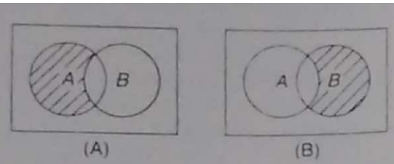

Figure 7-9 (A) Difference $A|B$ or $(A - B)$; (B) difference $B|A$ or $(B - A)$.

7.2.1.4 Difference (Subtraction)

The difference of set A with respect to set B is the collection of all elements in the universe that belong to A but do not belong to B, i.e., the difference set consists of all elements that belong to A but do not belong to B. It is denoted by $A|B$ or $A - B$ and is given by

$$A|B \text{ or } (A - B) = \{x | x \in A \text{ and } x \notin B\} = A - (A \cap B)$$

The vice versa of it also can be performed

$$B|A \text{ or } (B - A) = B - (B \cap A) = \{x | x \in B \text{ and } x \notin A\}$$

The above operations are shown in Figures 7-9(A) and (B).

7.2.2 Properties of Classical Sets

The important properties that define classical sets and show their similarity to fuzzy sets are as follows:

1. Commutativity

$$A \cup B = B \cup A; \quad A \cap B = B \cap A$$

2. Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C; \quad A \cap (B \cap C) = (A \cap B) \cap C$$

3. Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4. Idempotency

$$A \cup A = A; \quad A \cap A = A$$

5. Transitivity

$$\text{If } A \subseteq B \subseteq C, \text{ then } A \subseteq C$$

6. Identity

$$A \cup \phi = A, \quad A \cap \phi = \phi$$
$$A \cup X = X, \quad A \cap X = X$$

negation)

8. Law of excluded middle

$$\bar{\bar{A}} = A$$

9. Law of contradiction

$$A \cup \bar{A} = X$$

10. DeMorgan's law

$$A \cap \bar{A} = \phi$$

$$\overline{|A \cap B|} = \bar{A} \cup \bar{B}; \quad \overline{|A \cup B|} = \bar{A} \cap \bar{B}$$

From the properties mentioned above, we can observe the duality existing by replacing ϕ, $\cup$, $\cap$ X, $\cap$, $\cup$, respectively. It is important to know the law of excluded middle and the law of contradiction.

7.2.3 Function Mapping of Classical Sets

Mapping is a rule of correspondence between set-theoretic forms and function theoretic forms. A classic is represented by its characteristic function $\chi(x)$, where x is the element in the universe.

Now consider X and Y as two different universes of discourse. If an element x contained in X corresp to an element y contained in Y, it is called mapping from X to Y, i.e., $f: X \rightarrow Y$. On the basis of this map the characteristic function is defined as

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

where χ_A is the membership in set A for element x in the universe. The membership concept rep mapping from an element x in universe X to one of the two elements in universe Y (either to eler or 1). There exists a function-theoretic set called value set $V(A)$ for any set A defined on universe X, ba the mapping of characteristic function. The whole set is assigned a membership value 1, and the nu assigned a membership value 0.

Let A and B be two sets on universe X. The function-theoretic forms of operations performed b these two sets are given as follows:

1. Union $(A \cup B)$

$$\chi_{A \cup B}(x) = \chi_A(x) \vee \chi_B(x) = \max\{\chi_A(x), \chi_B(x)\}$$

Here $\vee$ is the maximum operator.

2. Intersection $(A \cap B)$

$$\chi_{A \cap B}(x) = \chi_A(x) \wedge \chi_B(x) = \min\{\chi_A(x), \chi_B(x)\}$$

Here $\wedge$ is the minimum operator.

3. Complement $(\bar{A})$

$$\chi_{\bar{A}}(x) = 1 - \chi_A(x)$$

7.3 Fuzzy Sets

Fuzzy sets may be viewed as an extension and generalization of the basic concepts of crisp sets. An important property of fuzzy set is that it allows partial membership. A fuzzy set is a set having degrees of membership between 1 and 0. The membership in a fuzzy set need not be complete, i.e., member of one fuzzy set can also be member of other fuzzy sets in the same universe. Fuzzy sets can be analogous to the thinking of intelligent people. If a person has to be classified as friend or enemy, intelligent people will not resort to absolute classification as friend or enemy. Rather, they will classify the person somewhere between two extremes of friendship and enmity. Similarly, vagueness is introduced in fuzzy set by eliminating the sharp boundaries that divide members from nonmembers in the group. There is a gradual transition between full membership and nonmembership, not abrupt transition.

A fuzzy set $\underset{\sim}{A}$ in the universe of discourse U can be defined as a set of ordered pairs and it is given by

$$\underset{\sim}{A} = \{(x, \mu_{\underset{\sim}{A}}(x)) \mid x \in U\}$$

where $\mu_{\underset{\sim}{A}}(x)$ is the degree of membership of x in $\underset{\sim}{A}$ and it indicates the degree that x belongs to $\underset{\sim}{A}$. The degree of membership $\mu_{\underset{\sim}{A}}(x)$ assumes values in the range from 0 to 1, i.e., the membership is set to unit interval [0, 1] or $\mu_{\underset{\sim}{A}}(x) \in [0, 1]$.

There are other ways of representation of fuzzy sets; all representations allow partial membership to be expressed. When the universe of discourse U is discrete and finite, fuzzy set $\underset{\sim}{A}$ is given as follows:

$$\underset{\sim}{A} = \left\{\frac{\mu_{\underset{\sim}{A}}(x_1)}{x_1} + \frac{\mu_{\underset{\sim}{A}}(x_2)}{x_2} + \frac{\mu_{\underset{\sim}{A}}(x_3)}{x_3} + \cdots\right\} = \left\{\sum_{i=1}^{n} \frac{\mu_{\underset{\sim}{A}}(x_i)}{x_i}\right\}$$

where "n" is a finite value. When the universe of discourse U is continuous and infinite, fuzzy set $\underset{\sim}{A}$ is given by

$$\underset{\sim}{A} = \left\{\int \frac{\mu_{\underset{\sim}{A}}(x)}{x}\right\}$$

In the above two representations of fuzzy sets for discrete and continuous universe, the horizontal bar is not a quotient but a delimiter. The numerator in each representation is the membership value in set $\underset{\sim}{A}$ that is associated with the element of the universe present in the denominator. For discrete and finite universe of discourse U, the summation symbol in the representation of fuzzy set $\underset{\sim}{A}$ does not denote algebraic summation but indicates the collection of each element. Thus the summation sign ("+") used is not the algebraic "add" but rather it is a discrete function-theoretic union. Also, for continuous and infinite universe of discourse U, the integral sign in the representation of fuzzy set $\underset{\sim}{A}$ is not an algebraic integral but is a continuous function-theoretic union for continuous variables.

A fuzzy set is universal fuzzy set if and only if the value of the membership function is 1 for all the members under consideration. Any fuzzy set $\underset{\sim}{A}$ defined on a universe U is a subset of that universe. Two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are said to be equal fuzzy sets if $\mu_{\underset{\sim}{A}}(x) = \mu_{\underset{\sim}{B}}(x)$ for all $x \in U$. A fuzzy set $\underset{\sim}{A}$ is said to be empty fuzzy set if and only if the value of the membership function is 0 for all possible members considered. The universal fuzzy set can also be called whole fuzzy set.

Also, for all $x \in U$

$$\mu_{\phi}(x) = 0, \quad \mu_{U}(x) = 1$$

7.3.1 Fuzzy Set Operations

The generalization of operations on classical sets to operations on fuzzy sets is not unique. The fuzzy set operations being discussed in this section are termed standard fuzzy set operations. These are the operations widely used in engineering applications. Let A and B be fuzzy sets in the universe of discourse U. For a given element x on the universe, the following function theoretic operations of union, intersection and complement are defined for fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ on U.

7.3.1.1 Union

The union of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$, denoted by $\underset{\sim}{A} \cup \underset{\sim}{B}$, is defined as

$$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = \max[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)] = \mu_{\underset{\sim}{A}}(x) \vee \mu_{\underset{\sim}{B}}(x) \quad \text{for all } x \in U$$

where $\vee$ indicates max operation. The Venn diagram for union operation of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is shown in Figure 7-10.

7.3.1.2 Intersection

The intersection of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$, denoted by $\underset{\sim}{A} \cap \underset{\sim}{B}$, is defined by

$$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)] = \mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(x) \quad \text{for all } x \in U$$

where $\wedge$ indicates min operator. The Venn diagram for intersection operation of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is shown in Figure 7-11.

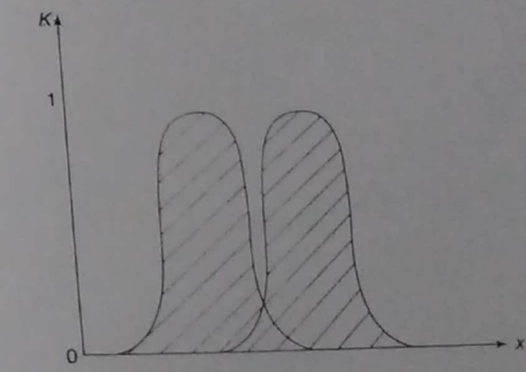

Figure 7-10 Union of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

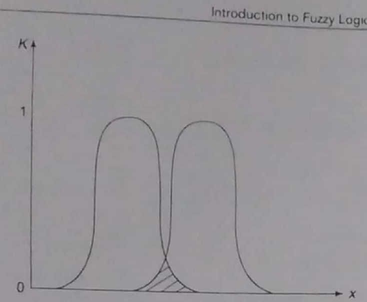

Figure 7-11 Intersection of fuzzy sets A and B.

7.3.1.3 Complement

When $\mu_A(x) \in [0, 1]$, the complement of A, denoted as $\bar{A}$ is defined by

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x) \quad \text{for all } x \in U$$

The Venn diagram for complement operation of fuzzy set A is shown in Figure 7-12.

7.3.1.4 More Operations on Fuzzy Sets

1. *Algebraic sum:* The algebraic sum $(A + B)$ of fuzzy sets, fuzzy sets A and B is defined as

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

2. *Algebraic product:* The algebraic product $(A \cdot B)$ of two fuzzy sets A and B is defined as

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$$

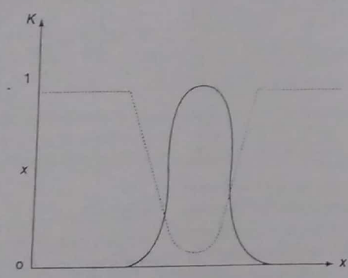

Figure 7-12 Complement of fuzzy set A.

3. *Bounded sum:* The bounded sum $(A \oplus B)$ of two fuzzy sets A and B is defined as

$$\mu_{A \oplus B}(x) = \min[1, \mu_A(x) + \mu_B(x)]$$

4. *Bounded difference:* The bounded difference $(A \odot B)$ of two fuzzy sets A and B is defined as

$$\mu_{A \odot B}(x) = \max[0, \mu_A(x) - \mu_B(x)]$$

7.3.2 Properties of Fuzzy Sets

Fuzzy sets follow the same properties as crisp sets except for the law of excluded middle and law of contradiction. That is, for fuzzy set A

$$A \cup \bar{A} \neq U; \quad A \cap \bar{A} \neq \phi$$

Frequently used properties of fuzzy sets are given as follows:

1. Commutativity

$$A \cup B = B \cup A; \quad A \cap B = B \cap A$$

2. Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

3. Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4. Idempotency

$$A \cup A = A; \quad A \cap A = A$$

5. Identity

$$A \cup \phi = A \quad \text{and} \quad A \cup U = U \text{(universal set)}$$
$$A \cap \phi = \phi \quad \text{and} \quad A \cap U = A$$

6. Involution (double negation)

$$\bar{\bar{A}} = A$$

7. Transitivity

$$\text{If } A \subseteq B \subseteq C, \text{ then } A \subseteq C$$

8. De Morgan's law

$$\overline{A \cup B} = \bar{A} \cap \bar{B}; \overline{A \cap B} = \bar{A} \cup \bar{B}$$

7.4 Summary

In this chapter, we have discussed the basic definitions, properties and operations on classical sets and fuzzy sets. Fuzzy sets are the tools that convert the concept of fuzzy logic into algorithms. Since fuzzy sets allow partial membership, they provide computer with such algorithms that extend binary logic and enable it to take human-like decisions. In other words, fuzzy sets can be thought of as a media through which the human thinking is transferred to a computer. One difference between fuzzy sets and classical sets is that the former do not follow the law of excluded middle and law of contradiction. Hence, if we want to choose fuzzy intersection and union operations which satisfy these laws, then the operations will not satisfy distributivity and idempotency. Except the difference of set membership being an infinite valued quantity instead of a binary valued quantity, fuzzy sets are treated in the same mathematical form as classical sets.

7.5 Solved Problems

1. Find the power set and cardinality of the given set $X = \{2, 4, 6\}$. Also find cardinality of power set.

Solution: Since set X contains three elements, so its cardinal number is

$$n_X = 3$$

The power set of X is given by

$$P(X) = \{\phi, \{2\}, \{4\}, \{6\}, \{2, 4\}, \{4, 6\}, \{2, 6\}, \{2, 4, 6\}\}$$

The cardinality of power set $P(X)$, denoted by $n_{P(X)}$, is found as

$$n_{P(X)} = 2^{n_X} = 2^3 = 8$$

2. Consider two given fuzzy sets

$$\tilde{A} = \left\{\frac{1}{2} + \frac{0.3}{4} + \frac{0.5}{6} + \frac{0.2}{8}\right\}$$

$$\tilde{B} = \left\{\frac{0.5}{2} + \frac{0.4}{4} + \frac{0.1}{6} + \frac{1}{8}\right\}$$

Perform union, intersection, difference and complement over fuzzy sets $\tilde{A}$ and $\tilde{B}$.

Solution: For the given fuzzy sets we have the following

(a) Union

$$\tilde{A} \cup \tilde{B} = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = \left\{\frac{1}{2} + \frac{0.4}{4} + \frac{0.5}{6} + \frac{1}{8}\right\}$$

(b) Intersection

$$\tilde{A} \cap \tilde{B} = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} = \left\{\frac{0.5}{2} + \frac{0.3}{4} + \frac{0.1}{6} + \frac{0.2}{8}\right\}$$

(c) Complement

$$\bar{\tilde{A}} = 1 - \mu_{\tilde{A}}(x) = \left\{\frac{0}{2} + \frac{0.7}{4} + \frac{0.5}{6} + \frac{0.8}{8}\right\}$$

$$\bar{\tilde{B}} = 1 - \mu_{\tilde{B}}(x) = \left\{\frac{0.5}{2} + \frac{0.6}{4} + \frac{0.9}{6} + \frac{0}{8}\right\}$$

(d) Difference

$$\tilde{A}|\tilde{B} = \tilde{A} \cap \bar{\tilde{B}} = \left\{\frac{0.5}{2} + \frac{0.3}{4} + \frac{0.5}{6} + \frac{0}{8}\right\}$$

$$\tilde{B}|\tilde{A} = \tilde{B} \cap \bar{\tilde{A}} = \left\{\frac{0}{2} + \frac{0.4}{4} + \frac{0.1}{6} + \frac{0.8}{8}\right\}$$

3. Given the two fuzzy sets

$$\tilde{B}_1 = \left\{\frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0}\right\}$$

$$\tilde{B}_2 = \left\{\frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0}\right\}$$

find the following:

(a) $\tilde{B}_1 \cup \tilde{B}_2$; (b) $\tilde{B}_1 \cap \tilde{B}_2$; (c) $\bar{\tilde{B}}_1$; (d) $\bar{\tilde{B}}_2$; (e) $\tilde{B}_1|\tilde{B}_2$; (f) $\overline{\tilde{B}_1 \cup \tilde{B}_2}$; (g) $\overline{\tilde{B}_1 \cap \tilde{B}_2}$; (h) $\tilde{B}_1 \cap \bar{\tilde{B}}_1$; (i) $\tilde{B}_1 \cup \bar{\tilde{B}}_1$; (j) $\tilde{B}_2 \cap \bar{\tilde{B}}_2$; (k) $\tilde{B}_2 \cup \bar{\tilde{B}}_2$

Solution: For the given fuzzy sets, we have the following

(a) $$\tilde{B}_1 \cup \tilde{B}_2 = \left\{\frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0}\right\}$$

(b) $$\tilde{B}_1 \cap \tilde{B}_2 = \left\{\frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0}\right\}$$

(c) $$\bar{\tilde{B}}_1 = \left\{\frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0}\right\}$$

(d) $$\bar{\tilde{B}}_2 = \left\{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0}\right\}$$

(e) $$\tilde{B}_1|\tilde{B}_2 = \tilde{B}_1 \cap \bar{\tilde{B}}_2 = \left\{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0}\right\}$$

(f) $$\overline{\tilde{B}_1 \cup \tilde{B}_2} = \left\{\frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0}\right\}$$

(g) $$\overline{\tilde{B}_1 \cap \tilde{B}_2} = \left\{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0}\right\}$$

(h) $$\tilde{B}_1 \cap \bar{\tilde{B}}_1 = \left\{\frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0}\right\}$$

(i) $$\tilde{B}_1 \cup \bar{\tilde{B}}_1 = \left\{\frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0}\right\}$$

(j) $$\tilde{B}_2 \cap \bar{\tilde{B}}_2 = \left\{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0}\right\}$$

(k) $$\tilde{B}_2 \cup \bar{\tilde{B}}_2 = \left\{\frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0}\right\}$$

4. It is necessary to compare two sensors based upon their detection levels and gain settings. The table of gain settings and sensor detection levels with a standard item being monitored providing typical membership values to represent the detection levels for each sensor is given in Table 1.

Table 1

Gain setting	Detection level of sensor 1	Detection level of sensor 2
0	0	0
10	0.2	0.35
20	0.35	0.25
30	0.65	0.8
40	0.85	0.95
50	1	1

Now given the universe of discourse $X = \{0, 10, 20, 30, 40, 50\}$ and the membership functions for the two sensors in discrete form as

$$\tilde{D}_1 = \left\{\frac{0}{0} + \frac{0.2}{10} + \frac{0.35}{20} + \frac{0.65}{30} + \frac{0.85}{40} + \frac{1}{50}\right\}$$

$$\tilde{D}_2 = \left\{\frac{0}{0} + \frac{0.35}{10} + \frac{0.25}{20} + \frac{0.8}{30} + \frac{0.95}{40} + \frac{1}{50}\right\}$$

find the following membership functions:

(a) $\mu_{\tilde{D}_1 \cup \tilde{D}_2}(x)$; (b) $\mu_{\tilde{D}_1 \cap \tilde{D}_2}(x)$; (c) $\mu_{\bar{\tilde{D}}_1}(x)$; (d) $\mu_{\bar{\tilde{D}}_2}(x)$; (e) $\mu_{\tilde{D}_1 \cup \bar{\tilde{D}}_1}(x)$; (f) $\mu_{\tilde{D}_1 \cap \bar{\tilde{D}}_1}(x)$; (g) $\mu_{\tilde{D}_2 \cup \bar{\tilde{D}}_2}$; (h) $\mu_{\tilde{D}_2 \cap \bar{\tilde{D}}_2}(x)$; (i) $\mu_{\tilde{D}_1|\tilde{D}_2}(x)$; (j) $\mu_{\tilde{D}_2|\tilde{D}_1}(x)$

Solution: For the given fuzzy sets we have

(a) $\mu_{\tilde{D}_1 \cup \tilde{D}_2}(x)$

$$= \max\{\mu_{\tilde{D}_1}(x), \mu_{\tilde{D}_2}(x)\} = \left\{\frac{0}{0} + \frac{0.35}{10} + \frac{0.35}{20} + \frac{0.8}{30} + \frac{0.95}{40} + \frac{1}{50}\right\}$$

(b) $\mu_{\tilde{D}_1 \cap \tilde{D}_2}(x)$

$$= \min\{\mu_{\tilde{D}_1}(x), \mu_{\tilde{D}_2}(x)\} = \left\{\frac{0}{0} + \frac{0.2}{10} + \frac{0.25}{20} + \frac{0.65}{30} + \frac{0.85}{40} + \frac{1}{50}\right\}$$

(c) $\mu_{\bar{\tilde{D}}_1}(x)$

$$= 1 - \mu_{\tilde{D}_1}(x) = \left\{\frac{1}{0} + \frac{0.8}{10} + \frac{0.65}{20} + \frac{0.35}{30} + \frac{0.15}{40} + \frac{0}{50}\right\}$$

(d) $\mu_{\overline{\underset{\sim}{D}_2}}(x)$

$$= 1 - \mu_{\underset{\sim}{D}_2}(x)$$

$$= \left\{\frac{1}{0} + \frac{0.65}{10} + \frac{0.75}{20} + \frac{0.2}{30} + \frac{0.05}{40} + \frac{0}{50}\right\}$$

(e) $\mu_{\underset{\sim}{D}_1 \cup \overline{\underset{\sim}{D}_1}}(x)$

$$= \max\{\mu_{\underset{\sim}{D}_1}(x), \mu_{\overline{\underset{\sim}{D}_1}}(x)\}$$

$$= \left\{\frac{1}{0} + \frac{0.8}{10} + \frac{0.65}{20} + \frac{0.65}{30} + \frac{0.85}{40} + \frac{1}{50}\right\}$$

(f) $\mu_{\underset{\sim}{D}_1 \cap \overline{\underset{\sim}{D}_1}}(x)$

$$= \min\{\mu_{\underset{\sim}{D}_1}(x), \mu_{\overline{\underset{\sim}{D}_1}}(x)\}$$

$$= \left\{\frac{0}{0} + \frac{0.2}{10} + \frac{0.35}{20} + \frac{0.35}{30} + \frac{0.15}{40} + \frac{0}{50}\right\}$$

(g) $\mu_{\underset{\sim}{D}_2 \cup \overline{\underset{\sim}{D}_2}}(x)$

$$= \max\{\mu_{\underset{\sim}{D}_2}(x), \mu_{\overline{\underset{\sim}{D}_2}}(x)\}$$

$$= \left\{\frac{1}{0} + \frac{0.65}{10} + \frac{0.75}{20} + \frac{0.8}{30} + \frac{0.95}{40} + \frac{1}{50}\right\}$$

(h) $\mu_{\underset{\sim}{D}_2 \cap \overline{\underset{\sim}{D}_2}}(x)$

$$= \min\{\mu_{\underset{\sim}{D}_2}(x), \mu_{\overline{\underset{\sim}{D}_2}}(x)\}$$

$$= \left\{\frac{0}{0} + \frac{0.35}{10} + \frac{0.25}{20} + \frac{0.2}{30} + \frac{0.05}{40} + \frac{0}{50}\right\}$$

(i) $\mu_{\underset{\sim}{D}_1 | \underset{\sim}{D}_2}(x)$

$$= \mu_{\underset{\sim}{D}_1 \cap \overline{\underset{\sim}{D}_2}}(x) = \min\{\mu_{\underset{\sim}{D}_1}(x), \mu_{\overline{\underset{\sim}{D}_2}}(x)\}$$

$$= \left\{\frac{0}{0} + \frac{0.2}{10} + \frac{0.35}{20} + \frac{0.2}{30} + \frac{0.05}{40} + \frac{0}{50}\right\}$$

(j) $\mu_{\underset{\sim}{D}_2 | \underset{\sim}{D}_1}(x)$

$$= \mu_{\underset{\sim}{D}_2 \cap \overline{\underset{\sim}{D}_1}}(x) = \min\{\mu_{\underset{\sim}{D}_2}(x), \mu_{\overline{\underset{\sim}{D}_1}}(x)\}$$

$$= \left\{\frac{0}{0} + \frac{0.35}{10} + \frac{0.25}{20} + \frac{0.35}{30} + \frac{0.15}{40} + \frac{0}{50}\right\}$$

5. Design a computer software to perform image processing to locate objects within a scene. The two fuzzy sets representing a plane and a train image are:

$$\underset{\sim}{\text{Plane}} = \left\{\frac{0.2}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.3}{\text{boat}} + \frac{0.8}{\text{plane}} + \frac{0.1}{\text{house}}\right\}$$

$$\underset{\sim}{\text{Train}} = \left\{\frac{1}{\text{train}} + \frac{0.2}{\text{bike}} + \frac{0.4}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.2}{\text{house}}\right\}$$

Find the following:

(a) $\underset{\sim}{\text{Plane}} \cup \underset{\sim}{\text{Train}}$; (b) $\underset{\sim}{\text{Plane}} \cap \underset{\sim}{\text{Train}}$;
(c) $\overline{\underset{\sim}{\text{Plane}}}$; (d) $\overline{\underset{\sim}{\text{Train}}}$;
(e) $\underset{\sim}{\text{Plane}} | \underset{\sim}{\text{Train}}$; (f) $\overline{\underset{\sim}{\text{Plane}} \cup \underset{\sim}{\text{Train}}}$;
(g) $\overline{\underset{\sim}{\text{Plane}} \cap \underset{\sim}{\text{Train}}}$; (h) $\underset{\sim}{\text{Plane}} \cup \overline{\underset{\sim}{\text{Plane}}}$;
(i) $\underset{\sim}{\text{Plane}} \cap \overline{\underset{\sim}{\text{Plane}}}$; (j) $\underset{\sim}{\text{Train}} \cup \overline{\underset{\sim}{\text{Train}}}$;
(k) $\underset{\sim}{\text{Train}} \cup \overline{\underset{\sim}{\text{Train}}}$

Solution: For the given fuzzy sets we have the following:

(a) $\underset{\sim}{\text{Plane}} \cup \underset{\sim}{\text{Train}}$

$$= \max\{\mu_{\text{Plane}}(x), \mu_{\text{Train}}(x)\}$$

$$= \left\{\frac{1.0}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.4}{\text{boat}} + \frac{0.8}{\text{plane}} + \frac{0.2}{\text{house}}\right\}$$

(b) $\underset{\sim}{\text{Plane}} \cap \underset{\sim}{\text{Train}}$

$$= \min\{\mu_{\text{Plane}}(x), \mu_{\text{Train}}(x)\}$$

$$= \left\{\frac{0.2}{\text{train}} + \frac{0.2}{\text{bike}} + \frac{0.3}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.1}{\text{house}}\right\}$$

(c) $\overline{\underset{\sim}{\text{Plane}}} = 1 - \mu_{\text{Plane}}(x)$

$$= \left\{\frac{0.8}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.7}{\text{boat}} + \frac{0.2}{\text{plane}} + \frac{0.9}{\text{house}}\right\}$$

(d) $\overline{\underset{\sim}{\text{Train}}} = 1 - \mu_{\text{Train}}(x)$

$$= \left\{\frac{0}{\text{train}} + \frac{0.8}{\text{bike}} + \frac{0.6}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.8}{\text{house}}\right\}$$

(e) $\underset{\sim}{\text{Plane}} | \underset{\sim}{\text{Train}}$

$$= \underset{\sim}{\text{Plane}} \cap \overline{\underset{\sim}{\text{Train}}}$$

$$= \min\{\mu_{\text{Plane}}(x), \mu_{\overline{\text{Train}}}(x)\}$$

$$= \left\{\frac{0}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.3}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.1}{\text{house}}\right\}$$

(f) $\overline{\underset{\sim}{\text{Plane}} \cup \underset{\sim}{\text{Train}}}$

$$= 1 - \max\{\mu_{\text{Plane}}(x), \mu_{\text{Train}}(x)\}$$

$$= \left\{\frac{0}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.6}{\text{boat}} + \frac{0.2}{\text{plane}} + \frac{0.8}{\text{house}}\right\}$$

(g) $\overline{\underset{\sim}{\text{Plane}} \cap \underset{\sim}{\text{Train}}}$

$$= 1 - \min\{\mu_{\text{Plane}}(x), \mu_{\text{Train}}(x)\}$$

$$= \left\{\frac{0.8}{\text{train}} + \frac{0.8}{\text{bike}} + \frac{0.7}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.9}{\text{house}}\right\}$$

(h) $\underset{\sim}{\text{Plane}} \cup \overline{\underset{\sim}{\text{Plane}}}$

$$= \max\{\mu_{\text{Plane}}(x), \mu_{\overline{\text{Plane}}}(x)\}$$

$$= \left\{\frac{0.8}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.7}{\text{boat}} + \frac{0.8}{\text{plane}} + \frac{0.9}{\text{house}}\right\}$$

(i) $\underset{\sim}{\text{Plane}} \cap \overline{\underset{\sim}{\text{Plane}}}$

$$= \min\{\mu_{\text{Plane}}(x), \mu_{\overline{\text{Plane}}}(x)\}$$

$$= \left\{\frac{0.2}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.3}{\text{boat}} + \frac{0.2}{\text{plane}} + \frac{0.1}{\text{house}}\right\}$$

(j) $\underset{\sim}{\text{Train}} \cup \overline{\underset{\sim}{\text{Train}}}$

$$= \max\{\mu_{\text{Train}}(x), \mu_{\overline{\text{Train}}}(x)\}$$

$$= \left\{\frac{1.0}{\text{train}} + \frac{0.8}{\text{bike}} + \frac{0.6}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.8}{\text{house}}\right\}$$

(k) $\underset{\sim}{\text{Train}} \cap \overline{\underset{\sim}{\text{Train}}}$

$$= \min\{\mu_{\text{Train}}(x), \mu_{\overline{\text{Train}}}(x)\}$$

$$= \left\{\frac{0}{\text{train}} + \frac{0.2}{\text{bike}} + \frac{0.4}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.2}{\text{house}}\right\}$$

6. For aircraft simulator data the determination of certain changes in its operating conditions is made on the basis of hard break points in the mach region. We define two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ representing the condition of "near" a mach number of 0.65 and "in the region" of a mach number of 0.65, respectively, as follows

$\underset{\sim}{A}$ = near mach 0.65

$$= \left\{\frac{0}{0.64} + \frac{0.75}{0.645} + \frac{1}{0.65} + \frac{0.5}{0.655} + \frac{0}{0.66}\right\}$$

$\underset{\sim}{B}$ = in the region of mach 0.65

$$= \left\{\frac{0}{0.64} + \frac{0.25}{0.645} + \frac{0.75}{0.65} + \frac{1}{0.655} + \frac{0.5}{0.66}\right\}$$

For these two sets find the following:

(a) $\underset{\sim}{A} \cup \underset{\sim}{B}$; (b) $\underset{\sim}{A} \cap \underset{\sim}{B}$; (c) $\overline{\underset{\sim}{A}}$;
(d) $\overline{\underset{\sim}{B}}$; (e) $\overline{\underset{\sim}{A} \cup \underset{\sim}{B}}$; (f) $\overline{\underset{\sim}{A} \cap \underset{\sim}{B}}$

Solution: For the two given fuzzy sets we have the following:

(a) $\underset{\sim}{A} \cup \underset{\sim}{B}$

$$= \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$$

$$= \left\{\frac{0}{0.64} + \frac{0.75}{0.645} + \frac{1}{0.65} + \frac{1}{0.655} + \frac{0.5}{0.66}\right\}$$

(b) $\underset{\sim}{A} \cap \underset{\sim}{B}$

$$= \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$$

$$= \left\{\frac{0}{0.64} + \frac{0.25}{0.645} + \frac{0.75}{0.65} + \frac{0.5}{0.655} + \frac{0}{0.66}\right\}$$

(c) $\overline{\underset{\sim}{A}} = 1 - \mu_{\underset{\sim}{A}}(x)$

$$= \left\{\frac{1}{0.64} + \frac{0.25}{0.645} + \frac{0}{0.65} + \frac{0.5}{0.655} + \frac{1}{0.66}\right\}$$

(d) $\overline{\underset{\sim}{B}} = 1 - \mu_{\underset{\sim}{B}}(x)$

$$= \left\{\frac{1}{0.64} + \frac{0.75}{0.645} + \frac{0.25}{0.65} + \frac{0}{0.655} + \frac{0.5}{0.66}\right\}$$

(e) $\overline{\underset{\sim}{A} \cup \underset{\sim}{B}}$

$$= 1 - \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$$

$$= \left\{\frac{1}{0.64} + \frac{0.25}{0.645} + \frac{0}{0.65} + \frac{0}{0.655} + \frac{0.5}{0.66}\right\}$$

(f) $\overline{\underset{\sim}{A} \cap \underset{\sim}{B}}$

$$= 1 - \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$$

$$= \left\{\frac{1}{0.64} + \frac{0.75}{0.645} + \frac{0.25}{0.65} + \frac{0.5}{0.655} + \frac{1}{0.66}\right\}$$

7. For the two given fuzzy sets

$$\underset{\sim}{A} = \left\{\frac{0.1}{0} + \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{1}{4}\right\}$$

$$\underset{\sim}{B} = \left\{\frac{1}{0} + \frac{0.5}{1} + \frac{0.7}{2} + \frac{0.3}{3} + \frac{0}{4}\right\}$$

find the following:

(a) $\underset{\sim}{A} \cup \underset{\sim}{B}$; (b) $\underset{\sim}{A} \cap \underset{\sim}{B}$; (c) $\overline{\underset{\sim}{A}}$;
(d) $\overline{\underset{\sim}{B}}$; (e) $\underset{\sim}{A} \cup \overline{\underset{\sim}{A}}$; (f) $\underset{\sim}{A} \cap \overline{\underset{\sim}{A}}$;
(g) $\underset{\sim}{B} \cup \overline{\underset{\sim}{B}}$; (h) $\underset{\sim}{B} \cap \overline{\underset{\sim}{B}}$; (i) $\underset{\sim}{A} \cap \overline{\underset{\sim}{B}}$;
(j) $\underset{\sim}{A} \cup \overline{\underset{\sim}{B}}$; (k) $\underset{\sim}{B} \cap \overline{\underset{\sim}{A}}$; (l) $\underset{\sim}{B} \cup \overline{\underset{\sim}{A}}$;
(m) $\overline{\underset{\sim}{A} \cup \underset{\sim}{B}}$; (n) $\overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}}$

Solution: For the given sets we have:

(a) $\underset{\sim}{A} \cup \underset{\sim}{B} = \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{1}{0} + \frac{0.5}{1} + \frac{0.7}{2} + \frac{0.6}{3} + \frac{1}{4}\right\}$$

(b) $\underset{\sim}{A} \cap \underset{\sim}{B} = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{0.1}{0} + \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.3}{3} + \frac{0}{4}\right\}$$

(c) $\overline{\underset{\sim}{A}} = 1 - \mu_{\underset{\sim}{A}}(x)$

$$= \left\{\frac{0.9}{0} + \frac{0.8}{1} + \frac{0.6}{2} + \frac{0.4}{3} + \frac{0}{4}\right\}$$

(d) $\overline{\underset{\sim}{B}} = 1 - \mu_{\underset{\sim}{B}}(x)$

$$= \left\{\frac{0}{0} + \frac{0.5}{1} + \frac{0.3}{2} + \frac{0.7}{3} + \frac{1}{4}\right\}$$

(e) $\underset{\sim}{A} \cup \overline{\underset{\sim}{A}} = \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\overline{\underset{\sim}{A}}}(x)\}$

$$= \left\{\frac{0.9}{0} + \frac{0.8}{1} + \frac{0.6}{2} + \frac{0.6}{3} + \frac{1}{4}\right\}$$

(f) $\underset{\sim}{A} \cap \overline{\underset{\sim}{A}} = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\overline{\underset{\sim}{A}}}(x)\}$

$$= \left\{\frac{0.1}{0} + \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.4}{3} + \frac{0}{4}\right\}$$

(g) $\underset{\sim}{B} \cup \overline{\underset{\sim}{B}} = \max\{\mu_{\underset{\sim}{B}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{1}{0} + \frac{0.5}{1} + \frac{0.7}{2} + \frac{0.7}{3} + \frac{1}{4}\right\}$$

(h) $\underset{\sim}{B} \cap \overline{\underset{\sim}{B}} = \min\{\mu_{\underset{\sim}{B}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{0}{0} + \frac{0.5}{1} + \frac{0.3}{2} + \frac{0.3}{3} + \frac{0}{4}\right\}$$

(i) $\underset{\sim}{A} \cap \overline{\underset{\sim}{B}} = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{0}{0} + \frac{0.2}{1} + \frac{0.3}{2} + \frac{0.6}{3} + \frac{1}{4}\right\}$$

(j) $\underset{\sim}{A} \cup \overline{\underset{\sim}{B}} = \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{0.1}{0} + \frac{0.5}{1} + \frac{0.4}{2} + \frac{0.7}{3} + \frac{1}{4}\right\}$$

(k) $\underset{\sim}{B} \cap \overline{\underset{\sim}{A}} = \min\{\mu_{\underset{\sim}{B}}(x), \mu_{\overline{\underset{\sim}{A}}}(x)\}$

$$= \left\{\frac{0.9}{0} + \frac{0.5}{1} + \frac{0.6}{2} + \frac{0.3}{3} + \frac{0}{4}\right\}$$

(l) $\underset{\sim}{B} \cup \overline{\underset{\sim}{A}} = \max\{\mu_{\underset{\sim}{B}}(x), \mu_{\overline{\underset{\sim}{A}}}(x)\}$

$$= \left\{\frac{1}{0} + \frac{0.8}{1} + \frac{0.7}{2} + \frac{0.4}{3} + \frac{0}{4}\right\}$$

(m) $\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = 1 - \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{0}{0} + \frac{0.5}{1} + \frac{0.3}{2} + \frac{0.4}{3} + \frac{0}{4}\right\}$$

(n) $\overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}} = \min\{\mu_{\overline{\underset{\sim}{A}}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{0}{0} + \frac{0.5}{1} + \frac{0.3}{2} + \frac{0.4}{3} + \frac{0}{4}\right\}$$

8. Let U be the universe of military aircraft of interest' as defined below:

$$U = \{a10,\ b52,\ c130,\ f2,\ f9\}$$

Let $\underset{\sim}{A}$ be the fuzzy set of bomber class aircraft:

$$\underset{\sim}{A} = \left\{\frac{0.3}{a10} + \frac{0.4}{b52} + \frac{0.2}{c130} + \frac{0.1}{f2} + \frac{1}{f9}\right\}$$

Let $\underset{\sim}{B}$ be the fuzzy set of fighter class aircraft:

$$\underset{\sim}{B} = \left\{\frac{0.1}{a10} + \frac{0.2}{b52} + \frac{0.8}{c130} + \frac{0.7}{f2} + \frac{0}{f9}\right\}$$

Find the following:

(a) $\underset{\sim}{A} \cup \underset{\sim}{B}$; (b) $\underset{\sim}{A} \cap \underset{\sim}{B}$; (c) $\overline{\underset{\sim}{A}}$; (d) $\overline{\underset{\sim}{B}}$;
(e) $\underset{\sim}{A} | \underset{\sim}{B}$; (f) $\underset{\sim}{B} | \underset{\sim}{A}$; (g) $\overline{\underset{\sim}{A} \cup \underset{\sim}{B}}$;
(h) $\overline{\underset{\sim}{A} \cap \underset{\sim}{B}}$; (i) $\overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}}$; (j) $\overline{\underset{\sim}{B}} \cup \underset{\sim}{A}$

Solution: We have

(a) $\underset{\sim}{A} \cup \underset{\sim}{B} = \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{0.3}{a10} + \frac{0.4}{b52} + \frac{0.8}{c130} + \frac{0.7}{f2} + \frac{1}{f9}\right\}$$

(b) $\underset{\sim}{A} \cap \underset{\sim}{B} = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{0.1}{a10} + \frac{0.2}{b52} + \frac{0.2}{c130} + \frac{0.1}{f2} + \frac{0}{f9}\right\}$$

(c) $\overline{\underset{\sim}{A}} = 1 - \mu_{\underset{\sim}{A}}(x)$

$$= \left\{\frac{0.7}{a10} + \frac{0.6}{b52} + \frac{0.8}{c130} + \frac{0.9}{f2} + \frac{0}{f9}\right\}$$

(d) $\overline{\underset{\sim}{B}} = 1 - \mu_{\underset{\sim}{B}}(x)$

$$= \left\{\frac{0.9}{a10} + \frac{0.8}{b52} + \frac{0.2}{c130} + \frac{0.3}{f2} + \frac{1}{f9}\right\}$$

(e) $\underset{\sim}{A} | \underset{\sim}{B} = \underset{\sim}{A} \cap \overline{\underset{\sim}{B}} = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{0.3}{a10} + \frac{0.4}{b52} + \frac{0.2}{c130} + \frac{0.1}{f2} + \frac{1}{f9}\right\}$$

(f) $\underset{\sim}{B} | \underset{\sim}{A} = \underset{\sim}{B} \cap \overline{\underset{\sim}{A}} = \min\{\mu_{\underset{\sim}{B}}(x), \mu_{\overline{\underset{\sim}{A}}}(x)\}$

$$= \left\{\frac{0.1}{a10} + \frac{0.2}{b52} + \frac{0.8}{c130} + \frac{0.7}{f2} + \frac{0}{f9}\right\}$$

(g) $\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = 1 - \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{0.7}{a10} + \frac{0.6}{b52} + \frac{0.2}{c130} + \frac{0.3}{f2} + \frac{0}{f9}\right\}$$

(h) $\overline{\underset{\sim}{A} \cap \underset{\sim}{B}} = 1 - \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{0.9}{a10} + \frac{0.8}{b52} + \frac{0.8}{c130} + \frac{0.9}{f2} + \frac{1}{f9}\right\}$$

(i) $\overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}} = \max\{\mu_{\overline{\underset{\sim}{A}}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{0.9}{a10} + \frac{0.8}{b52} + \frac{0.8}{c130} + \frac{0.9}{f2} + \frac{1}{f9}\right\}$$

(j) $\overline{\underset{\sim}{B}} \cup \underset{\sim}{A} = \max\{\mu_{\overline{\underset{\sim}{B}}}(x), \mu_{\underset{\sim}{A}}(x)\}$

$$= \left\{\frac{0.9}{a10} + \frac{0.8}{b52} + \frac{0.2}{c130} + \frac{0.3}{f2} + \frac{1}{f9}\right\}$$

9. Consider two fuzzy sets

$$\underset{\sim}{A} = \left\{\frac{0.2}{1} + \frac{0.3}{2} + \frac{0.4}{3} + \frac{0.5}{4}\right\}$$

$$\underset{\sim}{B} = \left\{\frac{0.1}{1} + \frac{0.2}{2} + \frac{0.2}{3} + \frac{1}{4}\right\}$$

Find the algebraic sum, algebraic product, bounded sum and bounded difference of the given fuzzy sets.

Solution: We have

(a) Algebraic sum

$$\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(x) = [\mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x)] - [\mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x)]$$

$$= \left\{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.5}{4}\right\} - \left\{\frac{0.02}{1} + \frac{0.06}{2} + \frac{0.08}{3} + \frac{0.5}{4}\right\}$$

$$= \left\{\frac{0.28}{1} + \frac{0.44}{2} + \frac{0.52}{3} + \frac{0}{4}\right\}$$

(b) Algebraic product

$$\mu_{\underset{\sim}{A} \cdot \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x)$$

$$= \left\{\frac{0.02}{1} + \frac{0.06}{2} + \frac{0.08}{3} + \frac{0.5}{4}\right\}$$

(c) Bounded sum

$$\mu_{\underset{\sim}{A} \oplus \underset{\sim}{B}}(x)$$

$$= \min[1, \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x)]$$

$$= \min\left\{1, \left\{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.5}{4}\right\}\right\}$$

$$= \left\{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.5}{4}\right\}$$

(d) Bounded difference

$$\mu_{\underset{\sim}{A} \odot \underset{\sim}{B}}(x)$$

$$= \max[0, \mu_{\underset{\sim}{A}}(x) - \mu_{\underset{\sim}{B}}(x)]$$

$$= \max\left\{0, \left\{\frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3} + \frac{0.5}{4}\right\}\right\}$$

$$= \left\{\frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3} + \frac{0.5}{4}\right\}$$

10. The discretized membership functions for a transistor and a resistor are given below:

$$\mu_T = \left\{\frac{0}{0} + \frac{0.2}{1} + \frac{0.7}{2} + \frac{0.8}{3} + \frac{0.9}{4} + \frac{1}{5}\right\}$$

$$\mu_R = \left\{\frac{0}{0} + \frac{0.1}{1} + \frac{0.3}{2} + \frac{0.2}{3} + \frac{0.4}{4} + \frac{0.5}{5}\right\}$$

Find the following: (a) Algebraic sum; (b) algebraic product; (c) bounded sum; (d) bounded difference.

Solution: We have

(a) Algebraic sum

$$\mu_{T+R}(x) = [\mu_T(x)+\mu_R(x)] - [\mu_T(x)\cdot\mu_R(x)]$$
$$= \left\{\frac{0}{0} + \frac{0.3}{1} + \frac{1.0}{2} + \frac{1.0}{3} + \frac{1.3}{4} + \frac{1.5}{5}\right\} - \left\{\frac{0}{0} + \frac{0.02}{1} + \frac{0.21}{2} + \frac{0.16}{3} + \frac{0.36}{4} + \frac{0.5}{5}\right\}$$
$$= \left\{\frac{0}{0} + \frac{0.28}{1} + \frac{0.79}{2} + \frac{0.84}{3} + \frac{0.94}{4} + \frac{1}{5}\right\}$$

(b) Algebraic product

$$\mu_{TR}(x) = \mu_T(x)\cdot\mu_R(x)$$
$$= \left\{\frac{0}{0} + \frac{0.02}{1} + \frac{0.21}{2} + \frac{0.16}{3} + \frac{0.36}{4} + \frac{0.5}{5}\right\}$$

(c) Bounded sum

$$\mu_{T\oplus R}(x) = \min\{1, \mu_T(x)+\mu_R(x)\}$$
$$= \min\left\{1, \left\{\frac{0}{0} + \frac{0.3}{1} + \frac{1.0}{2} + \frac{1.0}{3} + \frac{1.3}{4} + \frac{1.5}{5}\right\}\right\}$$
$$= \left\{\frac{0}{0} + \frac{0.3}{1} + \frac{1.0}{2} + \frac{1.0}{3} + \frac{1.0}{4} + \frac{1.0}{5}\right\}$$

(d) Bounded difference

$$\mu_{T\odot R}(x) = \max\{0, \mu_T(x)-\mu_R(x)\}$$
$$= \max\left\{0, \left\{\frac{0}{0} + \frac{0.1}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.5}{4} + \frac{0.5}{5}\right\}\right\}$$
$$= \left\{\frac{0}{0} + \frac{0.1}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.5}{4} + \frac{0.5}{5}\right\}$$

7.6 Review Questions

1. Define classical sets and fuzzy sets.
2. State the importance of fuzzy sets.
3. What are the methods of representation of a classical set?
4. Discuss the operations of crisp sets.
5. List the properties of classical sets.
6. What is meant by characteristic function?
7. Write the function theoretic form representation of crisp set operations.
8. Justify the following statement: "Partial membership is allowed in fuzzy sets."
9. Discuss in detail the operations and properties of fuzzy sets.
10. Represent the fuzzy sets operations using Venn diagram.
11. What is the cardinality of a fuzzy set? Whether a power set can be formed for a fuzzy set?
12. Apart from basic operations, state few other operations involved in fuzzy sets.