ASSIGNMENT

GRAPH THEORY AND COMBINATORICS

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Question no 1

Let B and A be, respectively, the circuit matrix and the incidence matrix of a self-loop-free graph. Then prove that $A \cdot B^T = 0 \pmod 2$

Answer

Consider a vertex v and a circuit Γ in the graph G. Either v is in Γ or it is not. If v is not in Γ, there is no edge in the circuit Γ that is incident on v. On the other hand, if v is in Γ, the number of those edges in the circuit Γ that are incident on v is exactly two.

With this remark in mind, consider the ith row in A and the jth row in B. Since the edges are arranged in the same order, the nonzero entries in the corresponding positions occur only if the particular edge is incident on the ith vertex and is also in the jth circuit.

If the ith vertex is not in the jth circuit, there is no such nonzero entry, and the dot product of the two rows is zero. If the ith vertex is in the jth circuit, there will be exactly two 1s in the sum of the products of individual entries. Since $1+1=0 \pmod 2$, the dot product of the two arbitrary rows - one from A and one from B - is zero.

Hence proved.

Question no 2

Show that for a simple disconnected graph of k components, n vertices and e edges the ranks of matrices A, B and C are $n-k$, $e-n+k$ and $n-k$ respectively where A is the incidence matrix, B is the circuit matrix and C is the cut-set matrix.

Answer

This question can be solved in three parts

Part - I

To prove in a simple disconnected graph of k components, n vertices and e edges, rank of incidence matrix A is $n-k$

Proof:

Rank of the incidence matrix : Each row in an incidence matrix $A(G)$ may be regarded as a vector over $GF(2)$ in the vector space of graph G.

Let the vector in the first row be called A_1, in the second row A_2 and so on. Thus

$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$$

Since there are exactly two 1's in every column of A, the sum of all these vectors is 0 (this being a modulo 2 sum of the corresponding entries). Thus vectors $A_1, A_2, \dots A_n$ are not linearly independent. Therefore the rank of A is less than n; that is, rank $A \le n-1$

Now consider the sum of any m of these n vectors ($m \leq n-1$). If the graph is connected, $A(G)$ cannot be partitioned, as in the equation, such that $A(g_1)$ is with m rows and $A(g_2)$ with $n-m$ rows. In other words, no m by m submatrix of $A(G)$ can be found, for $m \leq n-1$, such that the modulo 2 sum of those m rows is equal to zero.

Since there are only two constants 0 and 1 in this field, the additions of all vectors taken m at a time for $m = 1, 2, \ldots n-1$ exhausts all possible linear combinations of $n-1$ row vectors.

Thus we have just shown that no linear combination of m row vectors of A ($m \leq n-1$) can be equal to zero. Therefore, the rank of $A(G)$ must be at least $n-1$.

Since the rank of $A(G)$ is no more than $n-1$ and is no less than $n-1$, it must be exactly equal to $n-1$.

This statement can be extended to prove that the rank of $A(G)$ is $n-k$, if G is a disconnected graph with n vertices and k components. This is the reason why the number $n-k$ has been called the rank of a graph with k components.

Part II

To prove that in a simple disconnected graph G with n vertices, e edges and k components, rank of circuit matrix B is $e-n+k$

Proof:

Let B be the circuit matrix of the disconnected graph G with n vertices, e edges and k components.

Let the k components be $G_1, G_2, \ldots G_k$ with $n_1, n_2 \ldots n_k$ vertices and $e_1, e_2, \ldots e_k$ edges respectively

Then $n_1+n_2+\cdots+n_k = n$ and $e_1+e_2+\cdots+e_k = e$

Let $B_1, B_2, \ldots B_k$ be the circuit matrices of $G_1, G_2, \ldots, G_k$

Then
$$B(G) = \begin{bmatrix} B_1(G_1) & 0 & 0 & \cdots & 0 \\ 0 & B_2(G_2) & 0 & \cdots & 0 \\ 0 & 0 & B_3(G_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & B_k(G_k) \end{bmatrix}$$

We know rank of $B_i = e_i - n_i + 1$ for $1 \le i \le k$

Therefore, rank of B = rank of $B_1 + \cdots +$ rank of B_k

$$= (e_1-n_1+1)+\cdots+(e_k-n_k+1)$$

$$= (e_1+e_2+\cdots+e_k) - (n_1+n_2+\cdots+n_k)+k$$

$$= e - n+k$$

$\therefore$ rank of $B = e-n+k$

Part III

To prove that in a disconnected graph with n vertices and e edges and with k components, the rank of cut-set matrix C is n-k

Proof:

We know that rank of incidence matrix is n-k for a disconnected graph.

Let $B(G)$, $A(G)$, $C(G)$ be the circuit, incidence and cut-set matrix of the connected graph G, then we have

$$\text{Rank of } C(G) \geq n-1 \quad -①$$

Since the number of edges common to an edge cutset and a circuit is always even.

Every row in C is orthogonal to every row in B. Provided the edges in both. B and C are arranged in the same order.

Thus $BC^T = CB^T \cong 0 \pmod 2$

Rank of B + Rank of C $\leq e$

For a connected graph, Rank of B = $e-n+1$

$\therefore$ Rank of C $\leq e$ − Rank of B

Rank of C $\leq e-(e-n+1)$

Rank of C $\leq n-1 \quad -②$

$\therefore$ From ① and ② Rank of $C(G) = n-1$ = Rank of $A(G)$

$\therefore$ Rank of C = Rank of A

$\therefore$ For a disconnected graph, rank of C = rank of incidence matrix = $n-k$

$\therefore$ Rank of C = $n-k$ for a disconnected graph

Hence proved.

Part I, II and III is proved.