Total Pages:3

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College of Engineering Thalassery

APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

FIFTH SEMESTER B.TECH DEGREE MODEL EXAMINATION, NOVEMBER 2017
Course Code: CS309
Course Name: GRAPH THEORY AND COMBINATORICS (CS)

Max. Marks: 100 **Duration: 3 Hours**

PART A

Answer all questions, each carries 3 marks.

1. Is it possible to construct a graph with 12 vertices such that 2 of the vertices have degree 3 and the remaining vertices have degree 4? Justify 2. (3)
2. Define isomorphism with an example (3)
3. Define Euler and Hamiltonian graphs. Give examples of a Euler graph which is not a Hamiltonian and vice versa. (3)
4. State the Dirac's theorem for hamiltonicity and plot the graph. (3)

PART B

Answer any two full questions, each carries 9 marks.

5. Explain any three applications of graph theory. (9)
6. a) Prove that a simple graph with *n* vertices and *k* components can have at most *(n-k)(n-k+1)/2* edges. (4)

   b) Prove that a given connected graph *G* is an Euler graph if and only if all vertices of *G* are of even degree. (5)
7. "An Euler graph *G* is arbitrarily traceable from vertex *v* in *G* if and only if every circuit in *G* contains *v*". Prove it. (9)

PART C

Answer all questions, each carries 3 marks.

8. How the distance between any two vertices can be measured? Explain using suitable example. (3)

9. Plot a maximum level and minimum level binary trees with 11 vertices. (3)

10. Define a planar graph. Show that K5 is not a planar graph. (3)

11. If $G(V, E)$ is a connected graph, then v is a cut vertex if there exist vertices $u, w \in V - \{v\}$ such that every $u - w$ path in G passes through v. (3)

PART D

Answer any two full questions, each carries 9 marks.

12. Let T be a graph with n vertices. Then prove that following statements are equivalent.

    a) T is a tree.

    b) T contains no cycles and has $(n-1)$ edges.

    c) T is connected and has $(n-1)$ edges. (9)

13.a) Define binary tree. Then prove that number of pendant vertices in a binary tree is $(n+1)/2$. (4)

   b) Explain the term vertex connectivity k and edge connectivity λ. Prove that for any graph G, $k <= \lambda$. (5)

14. Prove that a connected planar graph with v vertices, e edges and r regions, then $v-e+r=2$. (9)

PART E

Answer any four full questions, each carries 10 marks.

15. Describe adjacency matrix. Explain using suitable diagram. (10)

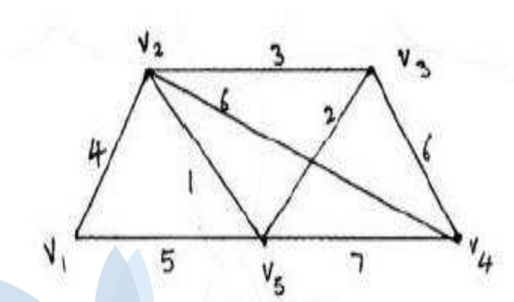
16. Describe incidence matrix. Then prove that rank of incidence matrix is $n-1$. (10)

17. Discuss about the cut matrix. (10)

18. Describe the Dijikstra's algorithm. Write the steps of the algorithm. Also give a suitable example to explain the algorithm. (10)

19 . Explain Floyd Warshall algorithm with suitable example (10)

20. Using Prim's algorithm, find a minimal spanning tree for the following weighted graph.



(10)

APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

FIFTH SEMESTER B.TECH DEGREE MODEL EXAMINATION, NOVEMBER 2017

CS 309 Graph Theory and Combinatorics

– Answer key

PART A (Each question carries 3 marks)

(1) Yes, (1/2, explanation 2.5 mark)

Suppose it is possible to construct a graph with 12 vertices out of which 2 of them are having degree and remaining vertices are having degree 4.

Hence by fundamental theorem,

$$\sum_{i=1}^{12} d(v_i) = 2e$$ where e is the number of edges.

According to given conditions

$$(2\times3) + (10\times4) = 2e$$

$$\Rightarrow 6+40 = 2e$$

$$2e = 46$$

$$e = 23$$

It is possible to construct a graph with 23 edges and 12 vertices which satisfy given conditions.

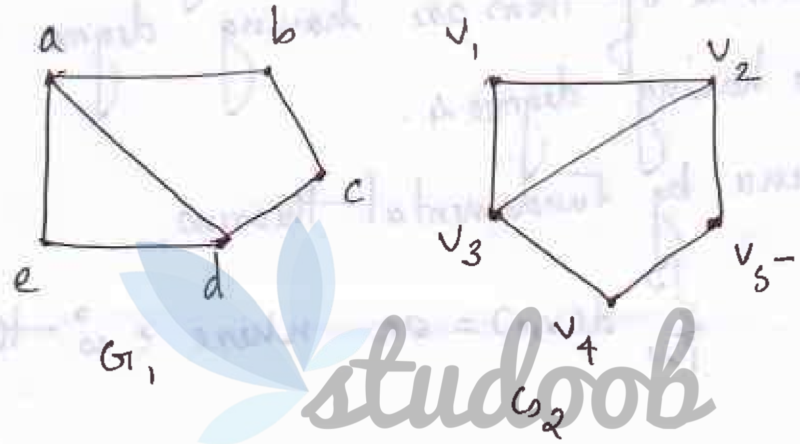
(2) Isomorphism :- Some graphs have the same structure i.e. same vertices and edges but differ only in the way of representation.

Conditions for isomorphism are,

1. The same number of vertices
2. The same number of edges
3. An equal number of vertices with a given degree.

Eg: Two graphs G_1 and G_2 are called isomorphic graphs if there is a one to one correspondence b/w vertices and b/w their edges.

Eg:



Here G_1 & G_2 have equal number of vertices and edges.

The pairs of vertices in decreasing order of degree are as follows:

$d(a) \leftrightarrow d(v_2)$, $d(d) \leftrightarrow d(v_3)$, $d(b) \leftrightarrow d(v_1)$, $d(e) \leftrightarrow d(v_4)$, $d(c) \leftrightarrow d(v_5)$

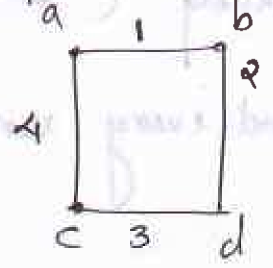
Since both the graphs contain vertices having same degree, hence they are isomorphic.

③ Euler graph (Definition 1 mark)

If some closed walk in a graph which contains all the edges of the graph then walk is called an Euler line.

and graph is called Euler graph.

eg:



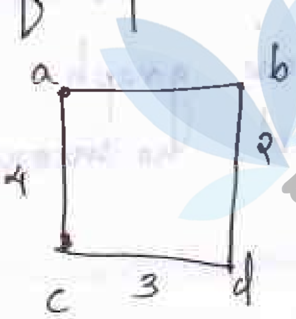
closed walk a1b2d3c4a contains all the edge of the graph. This is an Euler line, then the graph is called Euler graph.

Hamiltonian Graph (Definition 1 mark)

A graph is Hamiltonian, if it has a hamiltonian circuit.

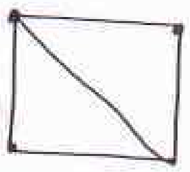
A hamiltonian circuit is a graph G is a circuit that contains each vertex of G once (Except for the starting and ending vertex), which

eg:



~~Closed walk~~ circuit a1b2d3c4a contains each vertex of G once (except for starting and ending vertex). This a Hamiltonian graph.

Eg: Hamiltonian but not Euler (1/2 mark)



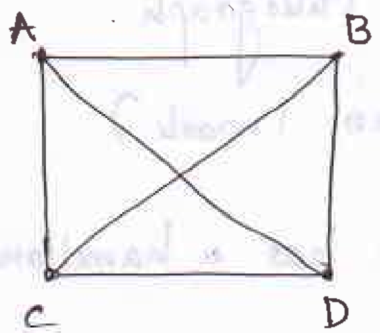
Eg: Euler but not hamiltonian (1/2 mark)

④ Dirac's theorem for hamiltonicity. (1 mark)

A graph with $n \geq 3$ vertices and every vertex having degree $\geq n/2$ is hamiltonian.

Example with explanation (2 mark)

eg:



This graph vertices = 4.

i.e. $n \geq 3$

And all the vertex having degree 3. i.e. $d(v_i) \geq n/2$. i.e. $d(v_i) \geq 2$

So this is a hamiltonian graph.

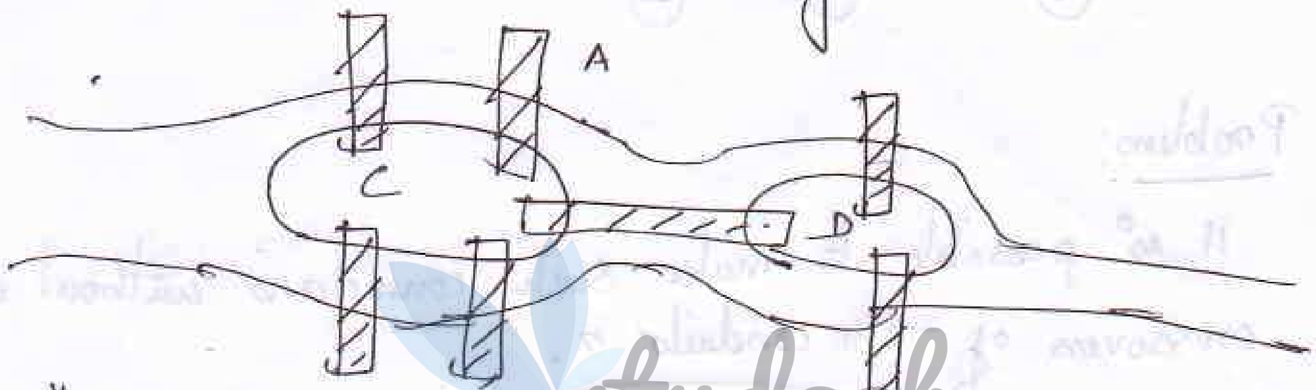
(This condition is sufficient, no means necessary).

PART B (each question carries 9 marks) ③

⑤ Explain any 3 examples (each carries 3 marks).

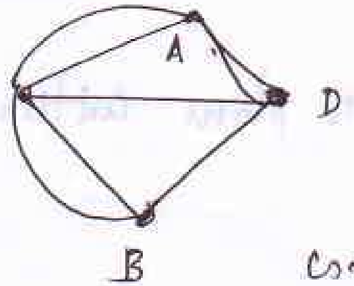
(a) Konigsberg bridge Problem.

Two islands C and D formed by the Pregel River in Konigsberg where connected to each other and to the banks A & B with seven bridges.



The Problem was to start at any of the four land areas of the city - A, B, C, or D walk over each of the seven bridges exactly once, and return to the starting Point.

- Euler represented this solution by means of a graph



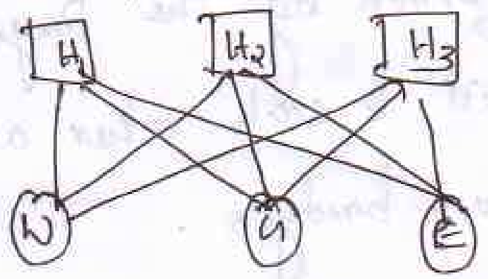
Vertex - land Area.
edge - bridges

Graph of konigsberg bridge problem.

— No Solution for this problem.

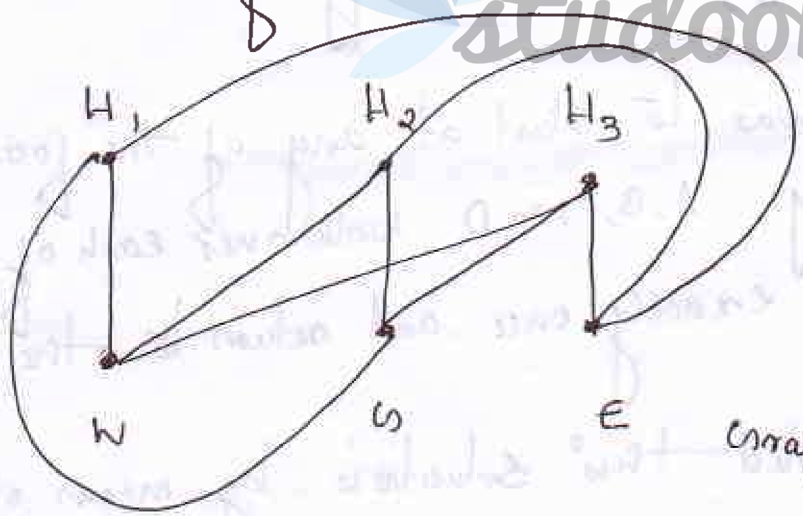
(b) Utilities Problem.

— There are 3 houses H_1, H_2, H_3 each to be connected to each of the three utilities water(W), gas(G) and electricity (E) by means of conduits.



Problem:

It is possible to make such connections without any crossovers of the conduits ?



Conduits — Edges
houses & utility — vertices

Graph of three utilities Problem.

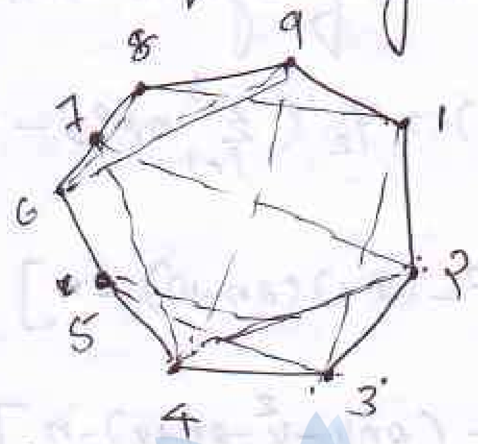
— Cannot be drawn in the plane without edges crossing ones.

(c) Seating Problem.

— Nine members of a new club meet each day for lunch at a round table. They decide to sit such that every member has different neighbours at each lunch. How many days can this arrangement last?

→ Vertices – Nine members.

– An edge joining two vertices represents the relationship of sitting next to each other.



2 seating arrangements – 1 2 3 4 5 6 7 8 9 1 (solid lines) and 1 3 5 2 7 4 9 6 8 1 (dashed lines).

– It can be shown by graph-theoretic considerations that there are only two more arrangements possible.

– 1 5 7 3 9 2 8 4 6 1

– 1 7 9 5 8 3 6 2 4 1

6 (4 marks)

a) Let the number of vertices in each of the k components of a graph be,

$n_1, n_2 \ldots n_k$. Thus we have

$n_1 + n_2 + \ldots + n_k = n$

$n_i \geq 1$.

– The proof of the theorem depends on an algebraic inequality

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k) \quad \text{--- (2)}$$

- Now the max. number of edges in the ith component of G is $\frac{1}{2} n_i (n_i - 1)$.

∴ The max. number of edges in G is

$$\frac{1}{2} \sum_{i=1}^{k} (n_i - 1)(n_i) = \frac{1}{2} \left(\sum_{i=1}^{k} n_i^2 \right) - n/2.$$

$$= \frac{1}{2} \left[(n^2 - (k-1)(2n-k)) - n \right] \quad \text{from (2)}$$

$$= \frac{1}{2} \left[n^2 - (2nk - k^2 - 2n + k) - n \right]$$

$$= \frac{1}{2} \left[n^2 - 2nk + k^2 + 2n - k - n \right]$$

$$= \frac{1}{2} \left[(n-k)^2 + (n-k) \right]$$

$$= \frac{1}{2} (n-k)(n-k+1)$$

(6 marks)

(6)(b) ~~The~~ 5 marks)

- Suppose that G is an Eulergraph, so it contains an Euler line
- In tracing this walk we observe that every walk meets a vertex v it goes through two "new" edges incident on v with one we entered "v" and with the other exited.
- This is true not only of all intermediate vertices of

the walk but also of the terminal vertex, because we "exited" and "entered" the same vertex at the beginning and end of the walk respectively. Thus if G is an Euler graph, the degree of every vertex is even.

To prove the sufficiency of the conditions, assume that all vertices of G are of even degree,

- Then construct a walk starting at an arbitrary vertex v and going through the edges of such that no edge is traced more than once.
- Since every vertex is of even degree, we can exit from every vertex we enter, tracing cannot stop at any vertex but v. And since v is also even degree, we shall eventually reach v when tracing comes to an end.
- If this closed walk h includes all the edges of G, G is an Euler graph.
- If not, remove from G all the edges in h and obtain a subgraph h' of G.
- Then again construct a new walk from h' and then this walk in h' can be combined with h to form a new walk, which starts and ends at vertex v and has more edges than h.
- This process can be repeated until obtain a closed walk that traverses all the edges of G. Thus G is an Euler graph.

(7) (4 marks)

Necessity: Let the Euler graph G be arbitrarily traceable from a vertex v. Assume there is a circuit C not passing through v.

Let H = G - E(C). Then every vertex of H has an even degree and the component of H containing v is Eulerian. This component of H can be traversed as an Euler line z, starting and ending with v and contain all those edges of G which are incident at v. Clearly, this v-v walk cannot be extended to contain the edges of C also contradicting that G contains v. Thus every circuit in G contains v.

Sufficiency: Let every circuit of the Euler graph G pass through the vertex v of G. We show that G is arbitrarily traceable from v.

Assume, on the contrary, that G is not arbitrarily traceable from v. Then there is v-v closed walk w of G containing all the edges of G incident with v and yet not containing all the edges of G. Let one such edge be incident at a vertex u on W. So every vertex of H = G - E(W) is of even degree and v is an isolated vertex of H and u is not. The component of H containing u is therefore Euler subgraph of G not passing through v. This is contradicting the assumption.

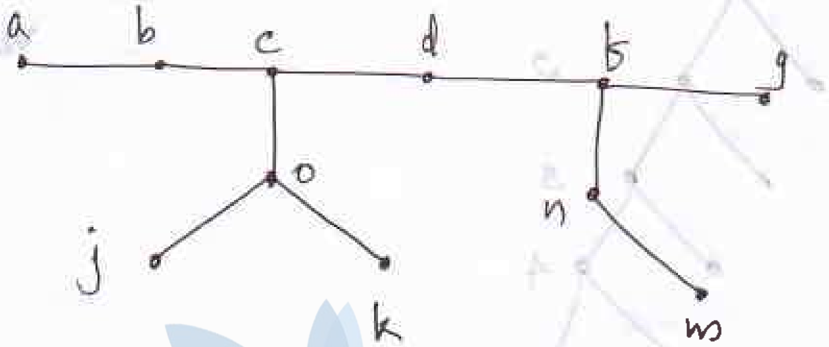
PART C (Each questions carries 3 marks)

(both definitions and example carries 1.5 marks)

(8) Distance b/w two vertices

In a connected graph G, the distance $d(v_i, v_j)$ b/w two its vertices v_i and v_j is the length of the shortest path (ie. the number of edges in the shortest path) b/w them.

eg:



In this figure, distance b/w the vertices b and o is 2. ie. $d(b, o) = 2$. ie. length of the shortest path b/w b and o. ie. Since, there are two edges (b, o) and (c, o) between the vertices b and o. Similarly $d(a, k) = 4$, $d(b, m) = 5$, $d(j, f) = 4$, $d(k, g) = 6$, $d(a, m) = 6$ and so on.

(9)

Equation for maximum level

$$\max l_{max} = \frac{n-1}{2} \qquad n = 11$$

$$= \frac{11-1}{2}$$

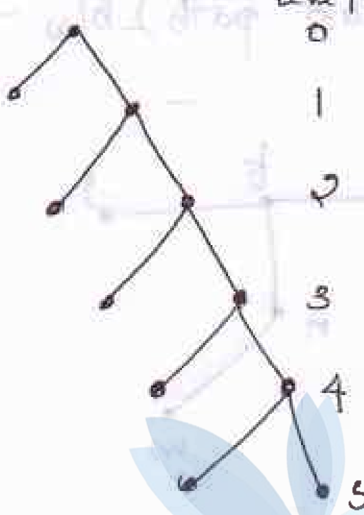
$$= 10/2 = 5$$

$$\text{Min } l_{max} = \lceil \log_2(n+1) - 1 \rceil$$

$$= \lceil (\log_2 12) - 1 \rceil$$

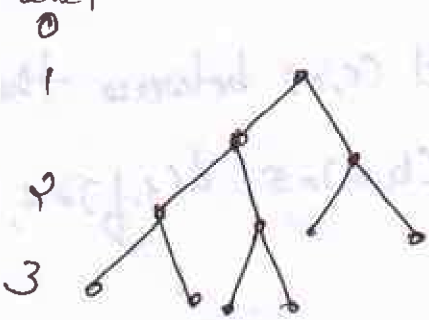
$$= 3$$

Maximum level binary tree with 11 vertices



— Plot max. 2 node vertex at each level.

Minimum level binary tree with 11 vertices



— Plot maximum node vertex at each level

(10) Planar graph (Definition 1 mark)

A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect.

7

Proof (2 marks)

k_5 is not planar.

Assume that k_5 is planar.

So using the equations for the planar graph.

$e \leq 3v-6$ e - edges

v - vertices

In k_5 $v = 5$

$e = {}^5C_2 = \frac{5!}{(5-2)!\,2!} = \frac{5\cdot 4}{1\cdot 2} = 10$

$\therefore e \leq 3v-6$

$\Rightarrow 10 \leq 3\times 5 - 6$

$10 \leq 9$ ⇒ This is contradiction, our assumption is wrong.

$\therefore$ k_5 is not planar

(11) Proof.

Let $G(V,E)$ be a connected graph and let v be the cut vertex of G. Then $G-v$ is disconnected. Let $G_1, G_2 \ldots G_k$ be the components of $G-v$. Let $U = V(G_1)$ and $W = \bigcup_{i=2}^{k} V(G_i)$. Also let $u \in U$ and $w \in W$ and to be definite, let $w \in V(G_i)$ $i \neq 1$. If there is a $u-w$ path P in G not passing through v then P connects u and w in G−v also. Therefore $G_1 \cup G_1$ is a single component in G−v, contradicting our assumption. Thus every u-w path in G passes through v.

Conversely, let there be vertices $u, w \in V - \{v\}$, such that every u-w path in G passes through v. Then there is no u-w path in G−v. Therefore u and w belong to different components of G−v. Thus G−v is disconnected and v is a cut vertex of G.

PART D (each questions carries 9 marks)

(12) Prove all the statements are equivalents.
(a) ⇒ (b)

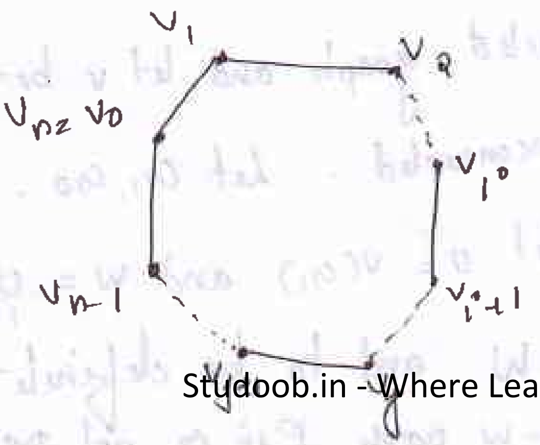
Assume T is a tree.

To prove T is acyclic and has (n−1) edges.

— To prove T is acyclic

If Assume that T contains atleast one cycle.

∴ T is a tree, T contains a simple cycle.



consider the vertices v_i and v_j. Then there exist.

- $P_1: v_i v_{i+1} \ldots v_{j+1} v_j$.

$P_2: v_i \ldots v_2 v_1 \ldots v_{j+1} v_j$.

P_1 and P_2 are distinct simple path from v_i to v_j. This contradicts the fact that T is a tree.

- our assumption is wrong.

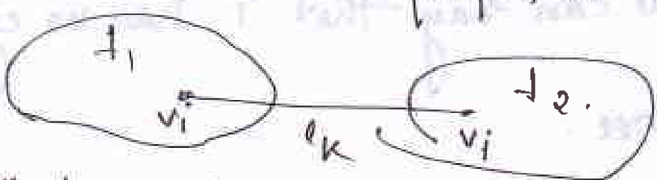
i.e. T has no cycle.

- Now want to prove that T has (n-1) edges.

- Proved by induction on the number of vertices.

- True for n=1, 2, 3 etc. Assume that the theorem holds for all trees with fewer than n vertices.

- Now consider a tree T with n vertices. In T let e_k be an edge with end vertices v_i and v_j. Deletion of e_k from T will disconnect the graph, as shown in figure.



t_1 & t_2 have fewer than n vertices each, and therefore, by the induction hypothesis, each contains one less edge than the number of vertices in H. Thus $T - e_k$ consists of $n-2$ edges.

Hence T has exactly n-1 edges.

b ⟹ c

Assume T is acyclic and (n-1) edges

To prove T is connected and (n-1) edges

- already assume that T has (n-1) edges.
- we want to prove that T is connected.
- Assume that T is disconnected

let $T_1, T_2 \cdots T_n$ be the components.

= 1/5 T_i has n_i vertices, no. of edges $(n_i - 1)$

$\therefore$ Total no. of edges $= (n_1 - 1) + (n_2 - 1) + \cdots (n_k - 1)$

$= (n_1 + \cdots n_k) - k$

$= n - k < n - 1$

our assumption is wrong.

$\therefore$ T is connected.

(c) ⟺ (a)

Assume T is connected & (n-1) edges.

From that we can say that T has no cycle.

$\therefore$ T is tree.

(13) A) Binary tree (definition 1 mark).

Binary tree is defined as a tree in which there is exactly one vertex of degree two. and each of the remaining vertices is of degree one or three.

Proof (3 mark)

Let p be the number of pendant vertices in binary tree T.
Then $n \to$ no. of vertices
$n-p-1$ is the number of vertices of degree three.

$\therefore$ The number of edges in T equals.

$$\frac{1}{2}[p+3(n-p-1)+2] = n-1;$$

$$\frac{1}{2}[p+3n-3p-3+2] = n-1$$

$$p+3n-3p-1 = (n-1)2$$

$$\cancel{p}+3n-2p-1 = 2n-2$$

$$-2p-1+2 = 2n-3n$$

$$-2p+1 = -n$$

$$-2p = -n-1$$

$$2p = n+1$$

$$p = \frac{n+1}{2}$$

(13) B) Vertex connectivity, Edge connectivity (Definition 2 marks).

Vertex connectivity

Vertex connectivity $k(G)$ of a graph G is the minimum number of nodes whose deletion disconnects it.

Edge connectivity (λ)

Let 'G' be a Connected graph. The minimum number of edges whose removal makes 'G' disconnected is called edge connectivity of G.

Proof (3 marks)

We want to prove that vertex connectivity of any graph G can never exceed the edge connectivity of G.

Let α denote the edge connectivity of G. Therefore, there exist a cut set S in G with α edges. Let S partition the vertices of G into subsets V_1 and V_2. By removing at most α vertices from V_1 (or V_2) on which the edges in S are incident, we can effect the removal of S from G.

(14) Proof

V- vertices, e- edges, r regions, then $v-e+r=2$

The proof is by induction on no. of edges.

If $e=0$

Then possible graph

G : $\cdot$ $\quad e-e+1=1-0+1=2$.

$v_1=1$ $\quad r=1$

$e=0$

If $e = 1$

Then possible graph is

G_1 : $v_1 \longrightarrow v_2$ $\qquad$ G_2 .

$v = 2$ $\qquad$ $v = 1$

$e = 1$ $\qquad$ $e = 1$

$r = 1$ $\qquad$ $r = 2$

$v - e + r = 1 - 1 + 2 = 2$.

Assume the result is true for every connected planar graph with no. of edges $\le k$.

→ Let $G = (V, E)$ be a connected planar graph with v vertices, r regions and $e = k+1$ edges.

Let $\{a, b\}$ be an edge of G.

Let $H = G - \{a, b\}$

Case 1 : H is connected

– H is connected planar graph with k edges by induction assumption result is true for H.

$v - k + r - 1$

– Then H has v vertices and k edges and $r-1$ regions

So $v - k + r - 1 = 2$

$v - k - 1 + r = 2$

$v - (k+1) + r = 2$

$v - e + r = 2$.

– The result is true for G.

Case 2: H is Disconnected

Suppose H has 2 components H_1 and H_2.

Then H_1 and H_2 are connected planar graph.

Let v_1, e_1, r_1 and v_2, e_2, r_2 be the no. of vertices, edges, and regions of H_1 & H_2 respectively.

- Since $e_1 \le k$, $e_2 \le k$. The result is true for H_1 & H_2.

$$v_1 - e_1 + r_1 = 2 \quad \text{---(1)}$$

$$v_2 - e_2 + r_2 = 2 \quad \text{---(2)}$$

(1) + (2) $\quad v_1 + v_2 - e_1 + e_2 + r_1 + r_2 = 4$

$$v - (k) + r + 1 = 4$$

$$v - k + r = 3$$

$$v - k - 1 + r = 3 - 1$$

$$v - (k+1) + r = 2$$

$$v - e + r = 2$$

Result is true for G.

PART E (each questions carries 10 marks)

(15) Adjacency Matrix (6 marks).

The adjacency matrix of a graph G with n vertices is an $n \times n$ matrix $A(G)$ such that each entry a_{ij} is the number of edges connecting v_i and v_j. Thus $a_{ij} = 0$, if there is no edge from v_i to v_j.

The adjacency matrix of a graph G with n vertices and parallel edges / self-loops is an $n \times n$ matrix

$$A(G) = [a_{ij}]$$

given by,

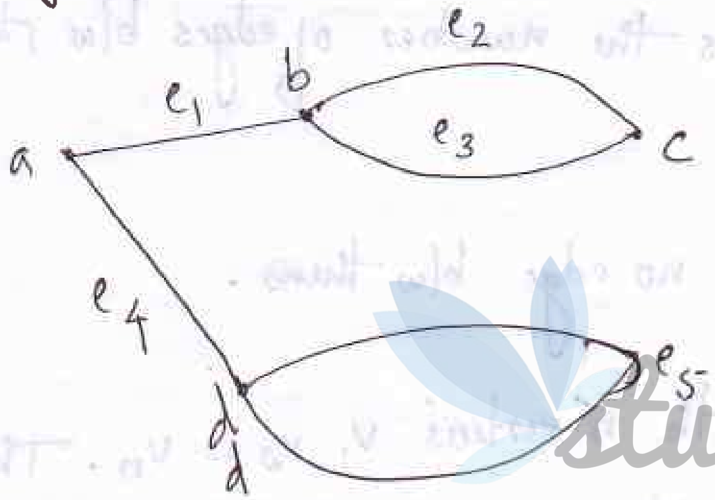
$a_{ij} = N$, where N is the number of edges b/w i^{th} and j^{th} vertices and

$a_{ij} = 0$, if there is no edge b/w them.

Let G be a graph with n vertices $v_1, v_2 \dots v_n$. The adjacency matrix of G with respect to this particular listing of n vertices is the $n \times n$ matrix $A(G) = a_{ij}$, where a_{ij} is the number of edges joining the vertex v_i to v_j. If G has no loops then all the entries of the main diagonal will be 0 and if G has no parallel edges then the entries of $A(G)$ are either 0 or 1. If the graph has no self-loops and no parallel edges, the degree of a vertex equals the number of ones in the corresponding row or column of A(G).

The adjacency matrix of a graph is a matrix with rows and column labeled by the vertices and such that its entry in row i, column j, $i \neq j$, is the number of edges incident on i and j.

eg: (4 marks).



Graph

$$
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & & & & \\
b & & & & \\
c & & & & \\
d & & & &
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

Adjacency matrix.

16 Incidence Matrix. (6 marks)

The incidence matrix of a graph G is a matrix with rows labeled by vertices, and columns labeled by edges, so that entry for row v column e is 1 if e is incident on v, and 0 otherwise.

Suppose that G has n vertices listed as $v_1, v_2, \ldots v_n$ and t edges listed as $e_1, e_2 \ldots e_t$. Then incidence matrix of G is the $n \times t$ matrix $M(G) = [m_{ij}]$, where m_{ij} is the number of times that the vertex v_i is incident with the edge e_j. ie.

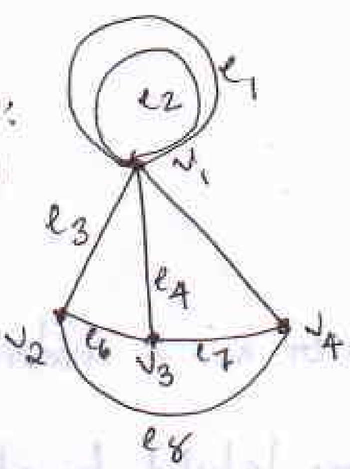
$m_{ij} = 0$, if v_i is not an end of e_j

$m_{ij} = 1$, if v_i is an end of the ~~loop~~ non-loop e_j.

$m_{ij} = 2$, if v_i is an end of the loop e_j.

Sum of the elements in the i^{th} row of $M(G)$ gives the degree of the vertex v_i.

eg:



$$M(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

incidence matrix.

Proof (4 mark).

Prove that Rank of $M(G) = n-1$.

Suppose x is a vector in the left null space of $Q := Q(G)$ i.e. $x'Q = 0$. Then $x_i - x_j = 0$ whenever $i \sim j$. It follows that $x_i = x_j$ whenever there is an ij-path. Since G is connected, x must have all components equal. Thus, the left null space of Q is at most one-dimensional and therefore the rank of Q is at least $n-1$. The rows of Q are linearly dependent and therefore rank $Q \leq n-1$.

Hence $Q = n-1$.

(17) Cut Matrix.

Consider a cut $m(v_a, v_b)$ is a connected directed graphs with n vertices and m edges. (v_a, v_b) consists of all those edges connecting vertices in V_a to V_b. This cut may be assigned an orientation from V_a to V_b or from V_b to V_a. Suppose the orientation of (V_a, V_b) is from V_a to V_a. Then the orientation of an edge (v_i, v_j) is said to agree with the cut orientation if $v_i \in V_a$ and $v_j \in V_b$.

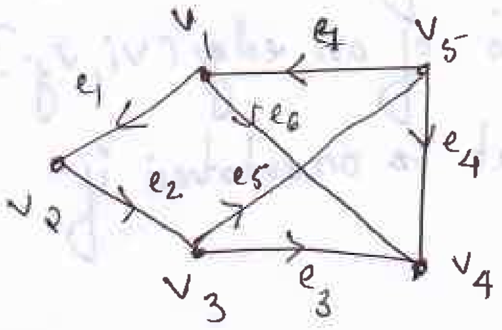
The cut matrix $Q_c = [q_{ij}]$ of G has m columns, one for each edge. and has one row for each cut. The element is defined as follows:

$$q_{ij} = \begin{cases} 1, & \text{if the } j^{th} \text{ edge is in the } i^{th} \text{ cut and its orientation agrees with the cut orientation} \\ -1, & \text{if the } j^{th} \text{ edge is in the } i^{th} \text{ cut and its orientation does not agree with the cut orientation} \\ 0, & \text{if the } j^{th} \text{ edge is not in the } i^{th} \text{ cut.} \end{cases}$$

* Each row of Q_c is called the cut ~~vertex~~ vector. The edges incident on a vertex forms a cut. Thus it follows that the matrix A_c is a submatrix of Q_c.
* Each branch of a spanning tree T of connected graph G defines a fundamental cutset. The submatrix of Q_c corresponding to the n-1 fundamental cutsets defined by T is

called the fundamental cutset matrix Q_b of G with respect to T.

eg: fundamental cutset matrix,



Directed graph.

$$Q_b = \begin{array}{c} \\ e_1 \\ e_2 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_5 & e_6 & e_3 & e_4 & e_7 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \end{array}$$

* rank of Q_b is $n-1$. Then rank of $(Q_c) \geq n-1$.

(18) **Dijkstra's Algorithm** (2 marks)

Dijkstra's algorithm solves the problem of finding the shortest path from a point in a graph (the source) to a destination. It turns out that one can find the shortest paths from a given source to all points in a graph in the same time, hence this problem is sometimes called the single-source shortest paths problem. This algorithm can be used for directed as well as un-directed graphs.

Steps of Dijkstra's Algorithm (4 marks)

Following are the steps of Dijkstra's algorithm:

Step 1) Temporarily assign C(A) = 0 and C(x) = infinity for all other x.

C(A) means the cost of A

C(x) means the current cost of getting to node x.

Step 2) Find the node x with the smallest temporary value of C(x).

If there are no temporary nodes or if C(x) = infinity, then stop.

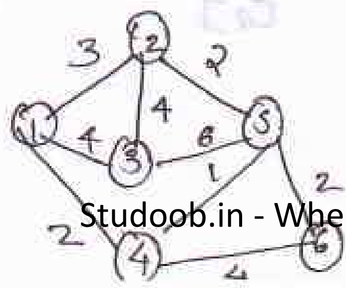
Node x is now labeled as permanent. Node x is now labeled as the current node C(x) and parent of x will not change again.

Step 3): For each temporary node labeled vertex y adjacent to x, make the following comparison:

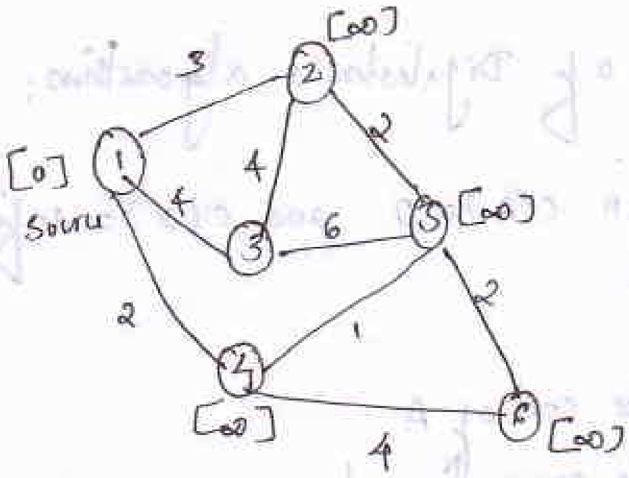
If $C(x) + W_{xy} < C(y)$, then C(y) is changed to $C(x) + W_{xy}$.

Step 4): Return to step 2.

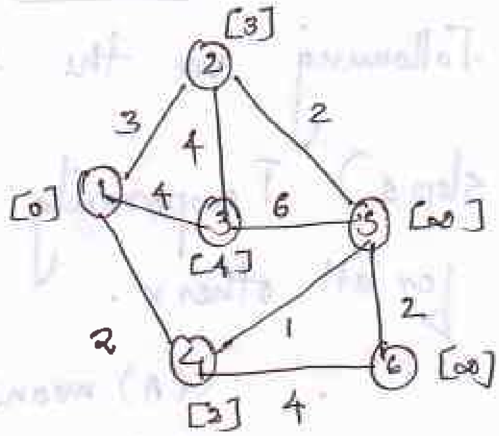
eg: (4 marks)



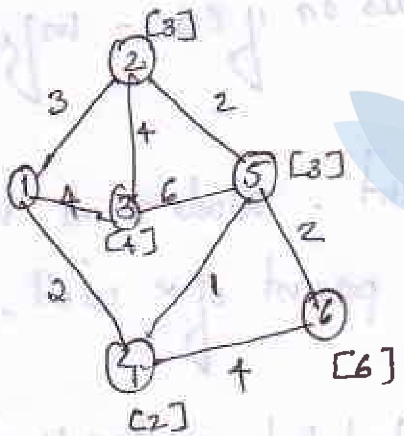
Step 1.



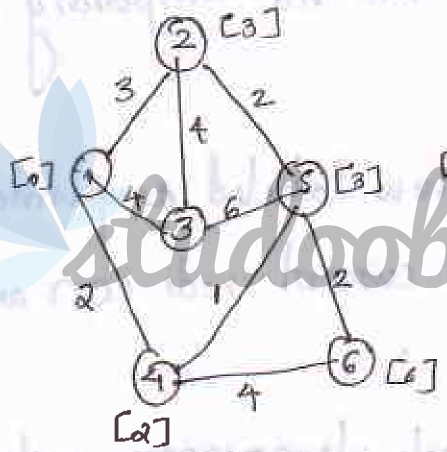
Step 2



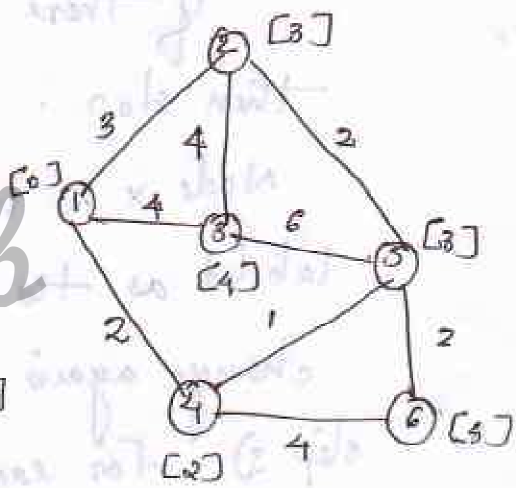
Step 3



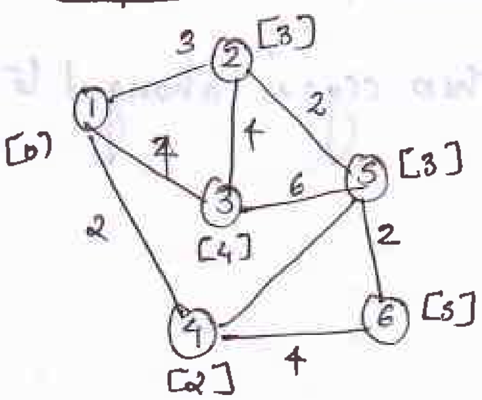
Step 4



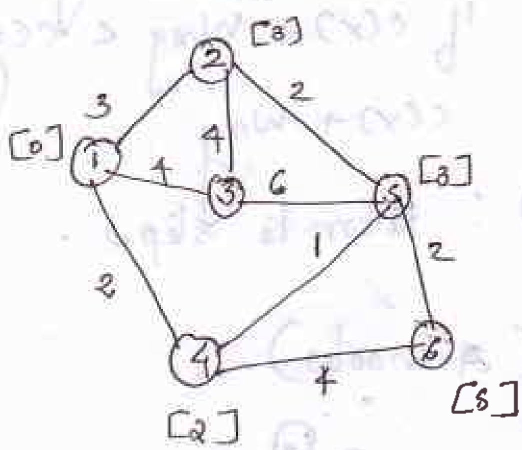
Step 5



Step 6



Step 7



(both examples & description carries 5 marks).

(19) Floyd-Warshall Algorithm is a used to find the length of the shortest path b/w any pair of vertices in G.

Description of the Algorithm

Starting with the n by n matrix $D = [d_{ij}]$ of direct distance, n different matrices $D_1, D_2, \ldots, D_n$ are constructed sequentially. Matrix D_k, $1 \le k \le n$, may be thought of as the matrix whose (i, j) th entry gives the length of the shortest path directed path among all directed paths from i to j with vertices $1, 2 \ldots k$ allowed as the intermediate vertices. Matrix $D_k = [d_{ij}^{(k)}]$ is constructed from D_{k-1} according to the following rule:

$$d_{ij}^{(k)} = \min [d_{ij}^{(k-1)}, (d_{ik}^{(k-1)} + d_{kj}^{(k-1)}],$$

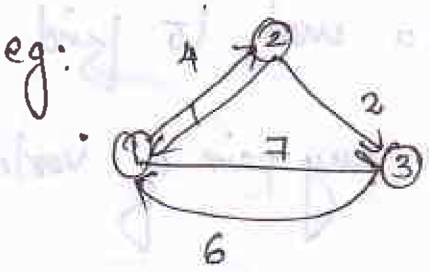
$$k = 1, 2 \ldots, n$$

$$d_{ij}^{(0)} = d_{ij}$$

ie, in Iteration 1, vertex 1 is inserted in the path from vertex i to j if $d_{ij} > d_{i1} + d_{1j}$. In Iteration 2, vertex 2 is inserted, and so on.

(can write any examples).

eg:



$$D^0 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 4 & 7 \\ 2 & 1 & 0 & 2 \\ 3 & 6 & \infty & 0 \end{array}$$

$$D^{(1)} = \begin{bmatrix} 0 & 4 & 7 \\ 1 & 0 & 2 \\ 6 & 10 & 0 \end{bmatrix}$$

Consider vertex 1

$D(3,2) = D(3,1) + D(1,2)$

$$D^{(2)} = \begin{bmatrix} 0 & 4 & 6 \\ 1 & 0 & 2 \\ 6 & 10 & 0 \end{bmatrix}$$

Consider vertex 2

$D(1,3) = D(1,2) + D(2,3)$

$$D^{(3)} = \begin{bmatrix} 0 & 4 & 6 \\ 1 & 0 & 2 \\ 6 & 10 & 0 \end{bmatrix}$$

Consider vertex 3

Nothing changes.

(20) Find Explain Prim's Algorithm (5 marks)

A Prim's Algorithm is used to find a minimum spanning of a connected weighted graph.

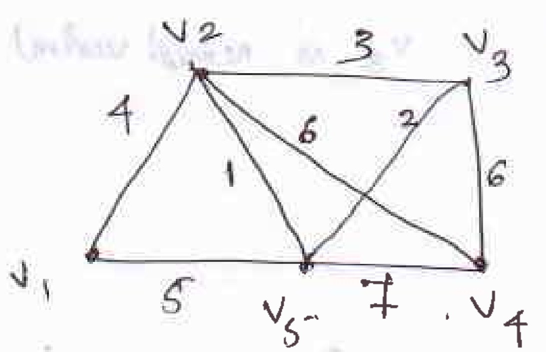
Steps are.

1) Draw n isolated vertices and label them $v_1, v_2 \ldots v_n$

→ Tabulate the given weights of the edges of G in an n by n table.

- start from vertex v_1 and connect ~~this~~ to its nearest neighbours say v_k.
- Now consider v_1 and v_k as one subgraph, and connect this subgraph to its closest neighbour (i.e. to a vertex other than v_1 and v_k that has the smallest entry among all entries in row 1 and k).
- let this new vertex be v_i.
- Next regard the tree with vertices v_1, v_k & v_i as one subgraph
- continue the process until all n vertices have been connected by (n-1) edges.

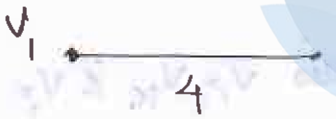
~~Find~~ Min spanning tree (5 marks)



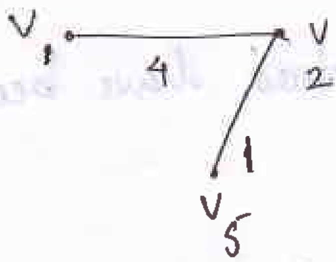
Adj matrix

$$
\begin{array}{c|c}
 & \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \\
\begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} &
\begin{bmatrix}
- & 4 & \infty & \infty & 5 \\
4 & - & 3 & 6 & 1 \\
\infty & 3 & - & 6 & 2 \\
\infty & 6 & 6 & - & 7 \\
5 & 1 & 2 & 7 & -
\end{bmatrix}
\end{array}
$$

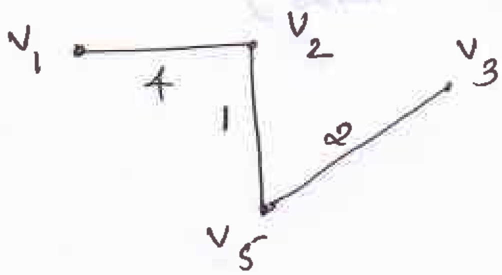
First start with v_1



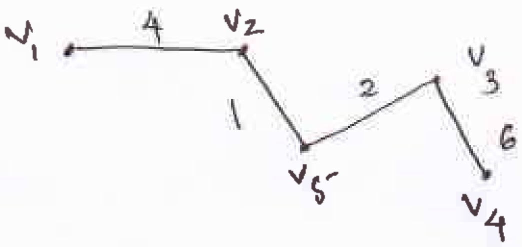
(v_2 is neighbour vertex)



(v_5 is next nearest vertex)



(v_3 is nearest vertex)



→ Minimum spanning tree.

Weight = 13//