Classical Relations and Fuzzy Relations

8

Learning Objectives

- Definition of classical relations and fuzzy relations.
- Formulation of Cartesian product of a relation.
- Operations and properties of classical relations and fuzzy relations.
- Composition of relations – max-min and max-product composition.
- Description on classical and fuzzy equivalence and tolerance relations.
- A short note on noninteractive fuzzy sets.

8.1 Introduction

Relationships between objects are the basic concepts involved in decision making and other dynamic system applications. The relations are also associated with graph theory, which has a great impact on designs and data manipulations. Relations represent mappings between sets and connectives in logic. A classical binary relation represents the presence or absence of a connection or interaction or association between the elements of two sets. Fuzzy binary relations are a generalization of crisp binary relations, and they allow various degrees of relationship (association) between elements. In other words, fuzzy relations impart degrees of strength to such connections and associations. In a fuzzy binary relation, the degree of association is represented by membership grades in the same way as the degree of set membership is represented in a fuzzy set. This chapter discusses the basic concepts and operations on fuzzy relations, and the composition between relations is studied via the max-min and max-product compositions. The properties and the cardinality of fuzzy relations are also discussed. Other topics discussed include the tolerance and equivalence relations on both crisp and fuzzy relations.

8.2 Cartesian Product of Relation

An ordered r-tuple is an ordered sequence of r-elements expressed in the form $(a_1, a_2, a_3, \ldots, a_r)$. An unordered r-tuple is a collection of r-elements without any restrictions in order. For $r = 2$, the r-tuple is called an ordered pair. For crisp sets $A_1, A_2, \ldots, A_r$, the set of all r-tuples $(a_1, a_2, a_3, \ldots, a_r)$, where $a_1 \in A_1, a_2 \in A_2, \ldots, a_r \in A_r$, is called the Cartesian product of $A_1, A_2, \ldots, A_r$ and is denoted by $A_1 \times A_2 \times \cdots \times A_r$. The Cartesian product of two or more sets is not the same as the arithmetic product of two or more sets. If all the a_r's are identical and equal to A, then the Cartesian product $A_1 \times A_2 \times \cdots \times A_r$ is denoted as A^r.

8.3 Classical Relation

An r-ary relation over $A_1, A_2, \ldots, A_r$ is a subset of the Cartesian product $A_1 \times A_2 \times \cdots \times A_r$. When $r = 2$, the relation is a subset of the Cartesian product $A_1 \times A_2$. This is called a binary relation from A_1 to A_2. When three, four or five sets are involved in the subset of full Cartesian product then the relations are called ternary, quaternary and quinary, respectively. Generally, the discussions are centered on binary relations.

Consider two universes X and Y: their Cartesian product $X \times Y$ is given by

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

Here the Cartesian product forms an ordered pair of every $x \in X$ with every $y \in Y$. Every element in X is completely related to every element in Y. The characteristic function, denoted by χ, gives the strength of the relationship between ordered pair of elements in each universe. If it takes unity as its value, then complete relationship is found; if the value is zero, then there is no relationship, i.e.,

$$\chi_{X \times Y}(x, y) = \begin{cases} 1, & (x, y) \in X \times Y \\ 0, & (x, y) \notin X \times Y \end{cases}$$

When the universes or sets are finite, then the relation is represented by a matrix called relation matrix. An r-dimensional relation matrix represents an r-ary relation. Thus, binary relations are represented by two-dimensional matrices.

Consider the elements defined in the universes X and Y as follows:

$$X = \{2, 4, 6\}; \quad Y = \{p, q, r\}$$

The Cartesian product of these two sets leads to

$$X \times Y = \{(p, 2), (p, 4), (p, 6), (q, 2), (q, 4), (q, 6), (r, 2), (r, 4), (r, 6)\}$$

From this set one may select a subset such that

$$R = \{(p, 2), (q, 4), (r, 4), (r, 6)\}$$

Subset R can be represented using a coordinate diagram as shown in Figure 8-1.

The relation could equivalently be represented using a matrix as follows:

R	p	q	r
2	1	0	0
4	0	1	1
6	0	0	1

The relation between sets X and Y may also be expressed by mapping representations as shown in Figure 8-2.

A binary relation in which each element from first set X is not mapped to more than one element in second set Y is called a function and is expressed as

$$R : X \rightarrow Y$$

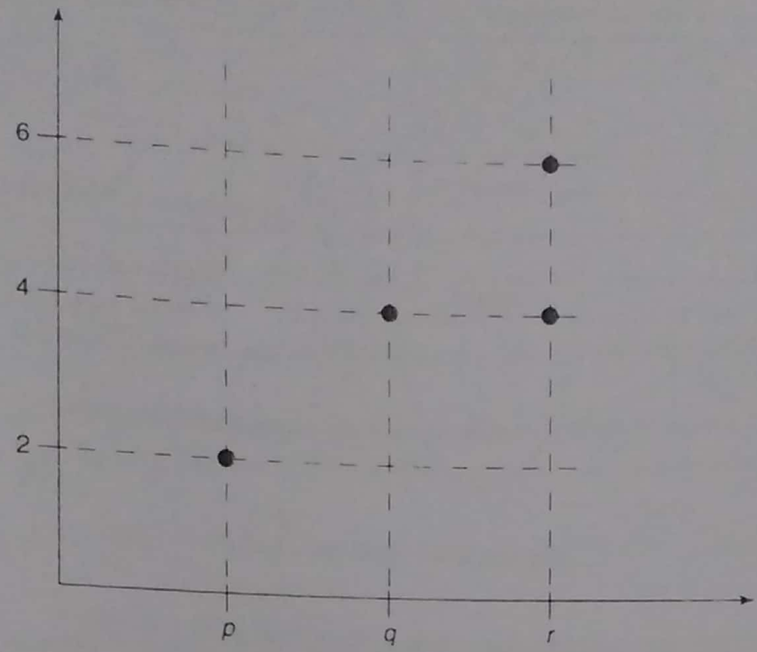


Figure 8-1 Coordinate diagram of a relation.

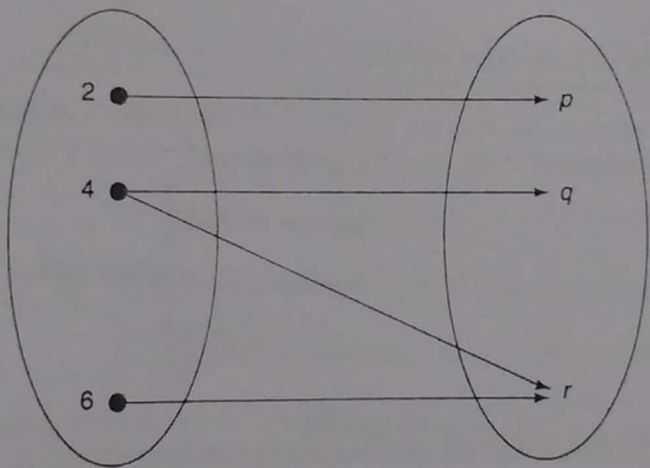


Figure 8-2 Mapping representation of a relation.

Figures 8-3 (A) and (B) show the illustration of $R: X \to Y$. Figure 8-3 shows mapping of an unconstrained relation. A more general crisp relation, R, exists when matches between elements in two universes are constrained. The characteristic function is used to assign values of relationship in the mapping of the Cartesian space $X \times Y$ to the binary values (0, 1) and is given by

$$\chi_R(x, y) = \begin{cases} 1, & (x, y) \in R \\ 0, & (x, y) \notin R \end{cases}$$

The constrained Cartesian product for sets when $r = 2$ (i.e., $A \times A = A^2$) is called identity relation, and the unconstrained Cartesian product for sets when $r = 2$ is called universal relation. Consider set $A = \{2, 4, 6\}$.

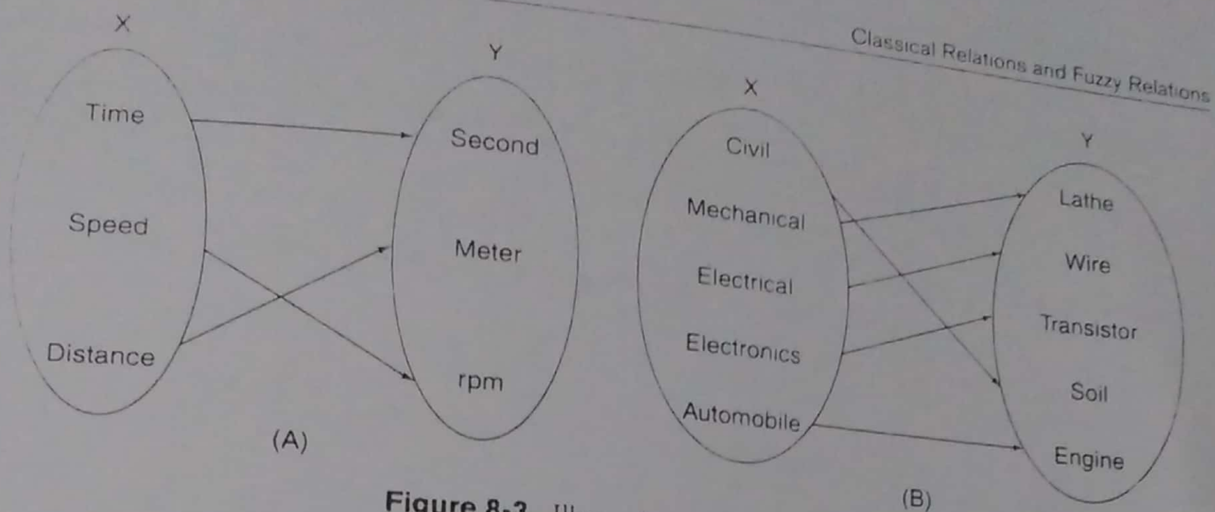


Figure 8-3 Illustrations of $R : X \to Y$.

Then universal relation (U_A) and identity relation (I_A) are given as follows:

$$U_A = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(2,6),(4,6),(6,6)\}$$
$$I_A = \{(2,2),(4,4),(6,6)\}$$

8.3.1 Cardinality of Classical Relation

Consider n elements of universe X being related to m elements of universe Y. When the cardinality of $X = n_X$ and the cardinality of $Y = n_Y$, then the cardinality of relation R between the two universes is

$$n_{X\times Y} = n_X \times n_Y$$

The cardinality of the power set $P(X \times Y)$ describing the relation is given by

$$n_{P(X\times Y)} = 2^{(n_X n_Y)}$$

8.3.2 Operations on Classical Relations

Let R and S be two separate relations on the Cartesian universe $X \times Y$. The null relation and the complete relation are defined by the relation matrices ϕ_R and E_R. An example of a 3×3 form of the ϕ_R and E_R matrices is given below:

$$\phi_R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Function-theoretic operations for the two crisp relations (R, S) are defined as follows:

1. Union

$$R \cup S \to \chi_{R\cup S}(x,y) : \chi_{R\cup S}(x,y) = \max[\chi_R(x,y), \chi_S(x,y)]$$

2. Intersection

$$R \cap S \to \chi_{R\cap S}(x,y) : \chi_{R\cap S}(x,y) = \min[\chi_R(x,y), \chi_S(x,y)]$$

3. Complement

$$\bar{R} \to \chi_{\bar{R}}(x, y) : \chi_{\bar{R}}(x, y) = 1 - \chi_R(x, y)$$

4. Containment

$$R \subset S \to \chi_R(x, y) \quad \chi_R(x, y) \le \chi_S(x, y)$$

5. Identity

$$\phi \to \phi_R \quad \text{and} \quad X \to E_R$$

8.3.3 Properties of Crisp Relations

The properties of classical set operations such as commutativity, associativity, distributivity, involution and idempotency also hold good for classical relations. Similarly De Morgan's law and excluded middle laws hold good for crisp relations as they do for crisp sets. The null relation ϕ_R is analogous to null set ϕ and complete relation E_R is analogous to whole set X.

8.3.4 Composition of Classical Relations

The operation executed on two compatible binary relations to get a single binary relation is called composition.

Let R be a relation that maps elements from universe X to universe Y and S be a relation that maps elements from universe Y to universe Z. The two binary relations R and S are compatible if

$$R \subseteq X \times Y \quad \text{and} \quad S \subseteq Y \times Z$$

In other words, the second set in R must be the same as the first set in S. On the basis of this explanation, a relation T can be formed that relates the same elements of universe X contained in R with the same elements of universe Z contained in S. This type of relation can be obtained by performing the composition operation over the two given relations. The composition between the two relations is denoted by $R \circ S$. Consider the universal sets given by

$$X = \{a_1, a_2, a_3\}; \quad Y = \{b_1, b_2, b_3\}; \quad Z = \{c_1, c_2, c_3\}$$

Let the relations R and S be formed as

$$R = X \times Y = \{(a_1, b_1), (a_1, b_2), (a_2, b_2), (a_3, b_3)\}$$
$$S = Y \times Z = \{(b_1, c_1), (b_2, c_3), (b_3, c_2)\}$$

Relations R and S are illustrated in Figure 8-4. From Figure 8-4, it can be inferred that

$$T = R \circ S = \{(a_1, c_1), (a_2, c_3), (a_3, c_2), (a_1, c_3)\}$$

The representation of relations R and S in matrix form is given as

$$R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}; \quad S = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

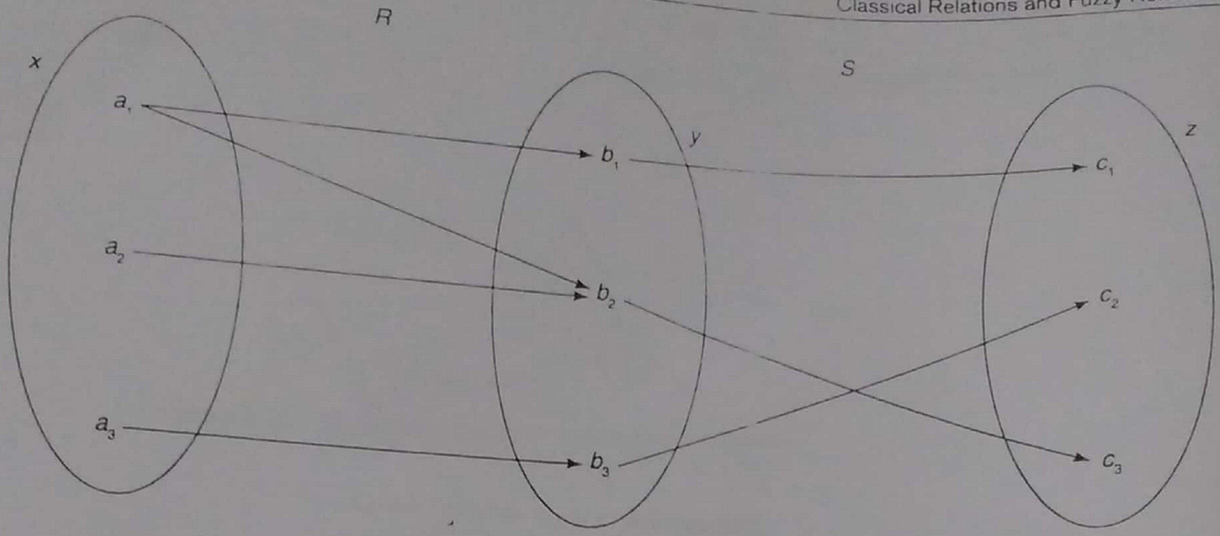


Figure 8-4 Illustration of relations R and S.

Composition $T = R \circ S$ is represented in matrix form as

$$T = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

This matrix also leads to

$$T = R \circ S = \{(a_1, c_1), (a_2, c_3), (a_3, c_2), (a_1, c_3)\}$$

as expected. The composition operations are of two types:

1. Max-min composition
2. Max-product composition.

The max-min composition is defined by the function theoretic expression as

$$T = R \circ S$$
$$\chi_T(x, z) = \bigvee_{y \in Y} [\chi_R(x, y) \wedge \chi_S(y, z)]$$

The max-product composition is defined by the function theoretic expression as

$$T = R \circ S$$
$$\chi_T(x, z) = \bigvee_{y \in Y} [\chi_R(x, y) \cdot \chi_S(y, z)]$$

The max-product composition is sometimes also referred to as max-dot composition. Some properties of the composition operation are described in Table 8-1.

Table 8-1 Few properties of composition operation

Associative	$(R \circ S) \circ M = R \circ (S \circ M)$
Commutative	$R \circ S \neq S \circ R$
Inverse	$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

8.4 Fuzzy Relations

Fuzzy relations relate elements of one universe (say X) to those of another universe (say Y) through the Cartesian product of the two universes. These can also be referred to as fuzzy sets defined on universal sets, which are Cartesian products. A fuzzy relation is based on the concept that everything is related to some extent or unrelated.

A fuzzy relation is a fuzzy set defined on the Cartesian product of classical sets $\{X_1, X_2, \ldots, X_n\}$ where tuples $(x_1, x_2, \ldots, x_n)$ may have varying degrees of membership $\mu_R(x_1, x_2, \ldots, x_n)$ within the relation. That is,

$$R(X_1, X_2, \ldots, X_n) = \int_{X_1 \times X_2 \times \cdots \times X_n} \mu_R(x_1, x_2, \ldots, x_n) | (x_1, x_2, \ldots, x_n), \quad x_i \in X_i$$

A fuzzy relation between two sets X and Y is called binary fuzzy relation and is denoted by $R(X, Y)$. A binary relation $R(X, Y)$ is referred to as bipartite graph when $X \neq Y$. The binary relation on a single set X is called directed graph or digraph. This relation occurs when $X = Y$ and is denoted as $R(X, X)$ or $R(X^2)$.

Let

$$\underset{\sim}{X} = \{x_1, x_2, \ldots, x_n\} \quad \text{and} \quad \underset{\sim}{Y} = \{y_1, y_2, \ldots, y_m\}$$

Fuzzy relation $\underset{\sim}{R}(\underset{\sim}{X}, \underset{\sim}{Y})$ can be expressed by an $n \times m$ matrix as follows:

$$\underset{\sim}{R}(\underset{\sim}{X}, \underset{\sim}{Y}) = \begin{bmatrix} \mu_R(x_1, y_1) & \mu_R(x_1, y_2) & \cdot & \cdot & \mu_R(x_1, y_m) \\ \mu_R(x_2, y_1) & \mu_R(x_2, y_2) & \cdot & \cdot & \mu_R(x_2, y_m) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_R(x_n, y_1) & \mu_R(x_n, y_2) & \cdot & \cdot & \mu_R(x_n, y_m) \end{bmatrix}$$

The matrix representing a fuzzy relation is called fuzzy matrix. A fuzzy relation $\underset{\sim}{R}$ is a mapping from Cartesian space $X \times Y$ to the interval [0, 1] where the mapping strength is expressed by the membership function of the relation for ordered pairs from the two universes $[\mu_{\underset{\sim}{R}}(x, y)]$.

A fuzzy graph is a graphical representation of a binary fuzzy relation. Each element in $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ corresponds to a node in the fuzzy graph. The connection links are established between the nodes by the elements of $\underset{\sim}{X} \times \underset{\sim}{Y}$ with nonzero membership grades in $\underset{\sim}{R}(\underset{\sim}{X}, \underset{\sim}{Y})$. The links may also be present in the form of arcs. These links are labeled with the membership values as $\mu_{\underset{\sim}{R}}(x_i, y_j)$. When $X \neq Y$, the link connecting the two nodes is an undirected binary graph called bipartite graph. Here, each of the sets X and Y can be represented by a set of nodes such that the nodes corresponding to one set are clearly differentiated from the nodes representing the other set. When $X = Y$, a node is connected to itself and directed links are used; in such a case, the fuzzy graph is called directed graph. Here, only one set of nodes corresponding to set X is used.

The domain of a binary fuzzy relation $R(X, Y)$ is the fuzzy set, *dom* $R(X, Y)$, having the membership function as

$$\mu_{\text{domain } R}(x) = \max_{y \in Y} \mu_R(x, y) \quad \forall x \in X$$

The range of a binary fuzzy relation $R(X, Y)$ is the fuzzy set, *ran* $R(X, Y)$, having the membership function as

$$\mu_{\text{range } R}(y) = \max_{x \in X} \mu_R(x, y) \quad \forall y \in Y$$

Consider a universe $X = \{x_1, x_2, x_3, x_4\}$ and the binary fuzzy relation on X as

$$\underset{\sim}{R}(X, X) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0.2 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0.7 & 0.8 \\ 0.1 & 0 & 0.4 & 0 \\ 0 & 0.6 & 0 & 1 \end{bmatrix} \end{array}$$

The bipartite graph and simple fuzzy graph of $\underset{\sim}{R}(X, X)$ is shown in Figures 8-5(A) and (B), respectively.

Let

$$\underset{\sim}{X} = \{x_1, x_2, x_3, x_4\} \quad \text{and} \quad \underset{\sim}{Y} = \{y_1, y_2, y_3, y_4\}$$

Let $\underset{\sim}{R}$ be a relation from $\underset{\sim}{X}$ to $\underset{\sim}{Y}$ given by

$$\underset{\sim}{R} = \frac{0.2}{(x_1, y_3)} + \frac{0.4}{(x_1, y_2)} + \frac{0.1}{(x_2, y_2)} + \frac{0.6}{(x_2, y_3)} + \frac{1.0}{(x_3, y_3)} + \frac{0.5}{(x_3, y_1)}$$

The corresponding fuzzy matrix for relation $\underset{\sim}{R}$ is

$$\underset{\sim}{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 0.2 \\ 0 & 0.1 & 0.6 \\ 0.5 & 0 & 1.0 \end{bmatrix} \end{array}$$

The graph of the above relation $\underset{\sim}{R} = \underset{\sim}{X} \times \underset{\sim}{Y}$ is shown in Figure 8-6.

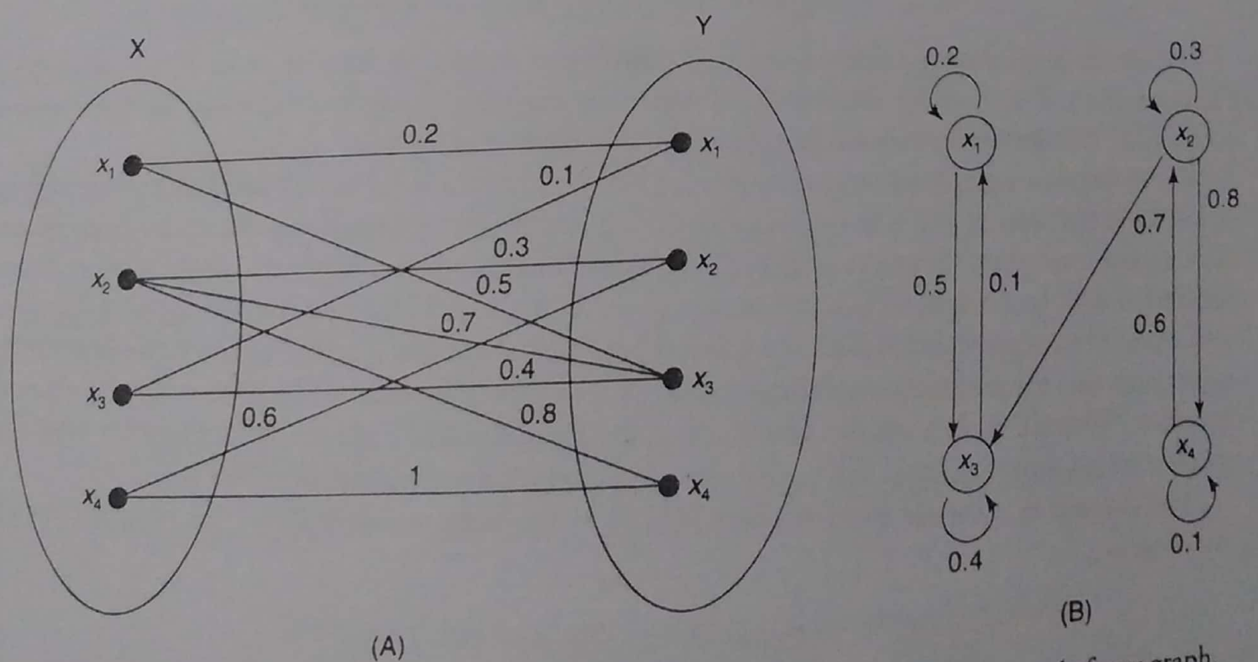


Figure 8-5 Graphical representation of fuzzy relations: (A) Bipartite graph; (B) simple fuzzy graph.

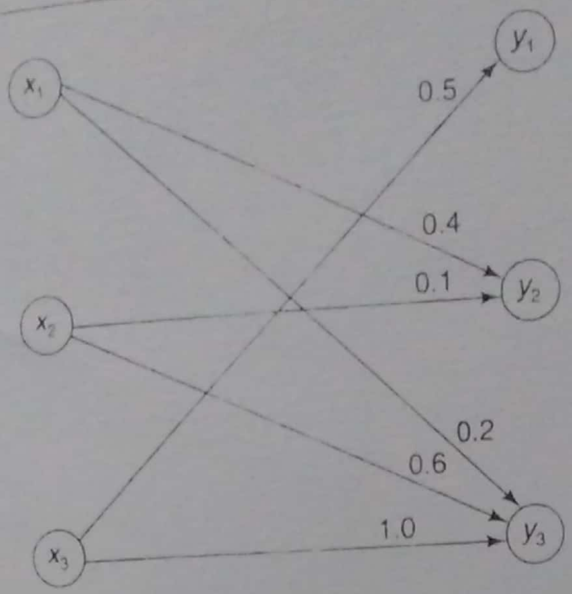


Figure 8-6 Graph of fuzzy relation.

8.4.1 Cardinality of Fuzzy Relations

The cardinality of fuzzy sets on any universe is infinity; hence the cardinality of a fuzzy relation between two or more universes is also infinity. This is mainly a result of the occurrence of partial membership in fuzzy sets and fuzzy relations.

8.4.2 Operations on Fuzzy Relations

The basic operations on fuzzy sets also apply on fuzzy relations. Let $\underset{\sim}{R}$ and $\underset{\sim}{S}$ be fuzzy relations on the Cartesian space $X \times Y$. The operations that can be performed on these fuzzy relations are described below:

1. Union

$$\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x, y) = \max\left[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)\right]$$

2. Intersection

$$\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x, y) = \min\left[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)\right]$$

3. Complement

$$\mu_{\overline{\underset{\sim}{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y)$$

4. Containment

$$\underset{\sim}{R} \subset \underset{\sim}{S} \Rightarrow \mu_{\underset{\sim}{R}}(x, y) \leq \mu_{\underset{\sim}{S}}(x, y)$$

5. Inverse: The inverse of a fuzzy relation R on $X \times Y$ is denoted by R^{-1}. It is a relation on $Y \times X$ defined by $R^{-1}(y, x) = R(x, y)$ for all pairs $(y, x) \in Y \times X$.
6. Projection: For a fuzzy relation $R(X, Y)$, let $[R \downarrow Y]$ denote the projection of R onto Y. Then $[R \downarrow Y]$ is a fuzzy relation in Y whose membership function is defined by

$$\mu_{[R \downarrow Y]}(x, y) = \max_{x} \mu_{\underset{\sim}{R}}(x, y)$$

The projection concept can be extended to an n-ary relation $R(x_1, x_2, \ldots, x_n)$.

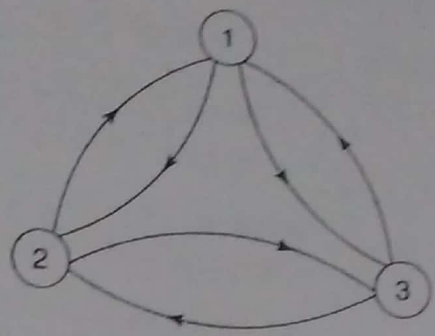


Figure 8-8 Three-vertex node – symmetry property.

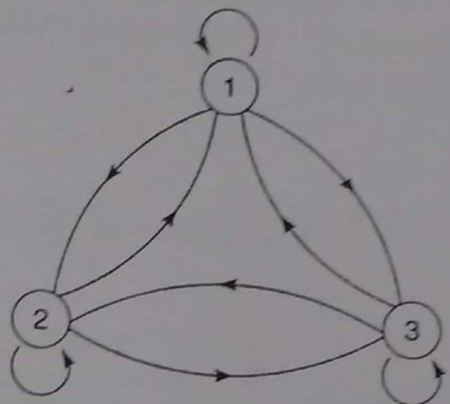


Figure 8-9 Three-vertex graph – transitive property.

Figure 8-9 represents a transitive relation. Here an arrow points from node 1 to node 2 and another arro extends from node 2 to node 3. There is also an arrow from node 1 to node 3.

8.5.1 Classical Equivalence Relation

Let relation R on universe X be a relation from X to X. Relation R is an equivalence relation if the followi three properties are satisfied:

1. Reflexivity
2. Symmetry
3. Transitivity

The function theoretic forms of representation of these properties are as follows:

1. Reflexivity

$$\chi_R(x_i, x_i) = 1 \text{ or } (x_i, x_i) \in R$$

2. Symmetry

$$\chi_R(x_i, x_j) = \chi_R(x_j, x_i)$$
$$\text{i.e., } (x_i, x_j) \in R \Rightarrow (x_j, x_i) \in R$$

3. Transitivity

$$\chi_R(x_i, x_j) \text{ and } \chi_R(x_j, x_k) = 1, \text{ so } \chi_R(x_i, x_k) = 1$$
$$\text{i.e., } (x_i, x_j) \in R(x_j, x_k) \in R, \text{ so } (x_i, x_k) \in R$$

The best example of an equivalence relation is the relation of similarity among triangles.

8.5.2 Classical Tolerance Relation

A tolerance relation R_1 on universe X is one where the only the properties of reflexivity and symmetry are satisfied. The tolerance relation can also be called proximity relation. An equivalence relation can be formed from tolerance relation R_1 by $(n-1)$ compositions within itself, where n is the cardinality of the set that defines R_1, here it is X, i.e.

$$\underbrace{R_1^{n-1}}_{\substack{\text{Tolerance}\\ \text{relation}}} = R_1 \circ R_1 \circ \cdots \circ R_1 = \underbrace{R}_{\substack{\text{Equivalence}\\ \text{relation}}}$$

8.5.3 Fuzzy Equivalence Relation

Let $\underset{\sim}{R}$ be a fuzzy relation on universe X, which maps elements from X to X. Relation $\underset{\sim}{R}$ will be a fuzzy equivalence relation if all the three properties – reflexive, symmetry and transitivity – are satisfied. The membership function theoretic forms for these properties are represented as follows:

1. Reflexivity

$$\mu_{\underset{\sim}{R}}(x_i, x_i) = 1 \quad \forall x \in X$$

If this is not the case for few $x \in X$, then $R(X, X)$ is said to be irreflexive.

2. Symmetry

$$\mu_{\underset{\sim}{R}}(x_i, x_j) = \mu_{\underset{\sim}{R}}(x_j, x_i) \text{ for all } x_i, x_j \in X$$

If this is not satisfied for few $x_i, x_j \in X$, then $R(X, X)$ is called asymmetric.

3. Transitivity

$$\mu_{\underset{\sim}{R}}(x_i, x_j) = \lambda_1 \quad \text{and} \quad \mu_{\underset{\sim}{R}}(x_j, x_k) = \lambda_2$$
$$\Rightarrow \mu_{\underset{\sim}{R}}(x_i, x_k) = \lambda$$

where

$$\lambda = \min[\lambda_1, \lambda_2]$$
$$\text{i.e., } \mu_{\underset{\sim}{R}}(x_i, x_j) \geq \max_{x_j \in X} \min[\mu_{\underset{\sim}{R}}(x_i, x_j), \mu_{\underset{\sim}{R}}(x_j, x_k)] \quad \forall (x_i, x_k) \in X^2$$

This can also be called max-min transitive. If this is not satisfied for some members of X, then $R(X, X)$ is nontransitive. If the given transitivity inequality is not satisfied for all the members $(x_i, x_k) \in X^2$, then the relation is called as antitransitive.

The max-product transitive can also be defined. It is given by

$$\mu_{\underset{\sim}{R}}(x_i, x_k) \geq \max_{x_j \in X} \cdot [\mu_{\underset{\sim}{R}}(x_i, x_j) \bullet \mu_{\underset{\sim}{R}}(x_j, x_k)] \quad \forall (x_i, x_k) \in X^2$$

The equivalence relation discussed can also be called similarity relation.

8.5.4 Fuzzy Tolerance Relation

A binary fuzzy relation that possesses the properties of reflexivity and symmetry is called fuzzy tolerance relation or resemblance relation. The equivalence relations are a special case of the tolerance relation. The fuzzy tolerance relation can be reformed into fuzzy equivalence relation in the same way as a crisp tolerance relation is reformed into crisp equivalence relation, i.e.,

$$\underbrace{\underset{\sim}{R}_1^{n-1}}_{\substack{\text{Fuzzy}\\ \text{tolerance}\\ \text{relation}}} = \underset{\sim}{R}_1 \circ \underset{\sim}{R}_1 \circ \cdots \circ \underset{\sim}{R}_1 = \underbrace{\underset{\sim}{R}}_{\substack{\text{Fuzzy}\\ \text{equivalence}\\ \text{relation}}}$$

where "n" is the cardinality of the set that defines R_1.

8.6 Noninteractive Fuzzy Sets

The independent events in probability theory are analogous to noninteractive fuzzy sets in fuzzy theory. A noninteractive fuzzy set is defined as follows. We are defining fuzzy set $\underset{\sim}{A}$ on the Cartesian space $X = X_1 \times X_2$. Set $\underset{\sim}{A}$ is separable into two noninteractive fuzzy sets called orthogonal projections, if and only if

$$\underset{\sim}{A} = \text{OPr}_{X_1}(\underset{\sim}{A}) \times \text{OPr}_{X_2}(\underset{\sim}{A})$$

where

$$\mu_{\text{OPr}_{X_1}(\underset{\sim}{A})}(x_1) = \max_{x_2 \in X_2} \mu_{\underset{\sim}{A}}(x_1, x_2) \quad \forall x_1 \in X_1$$

$$\mu_{\text{OPr}_{X_2}(\underset{\sim}{A})}(x_2) = \max_{x_1 \in X_1} \mu_{\underset{\sim}{A}}(x_1, x_2) \quad \forall x_2 \in X_2$$

The equations represent membership functions for the orthographic projections of $\underset{\sim}{A}$ on universes X_1 and X_2, respectively.

8.7 Summary

This chapter discussed the properties and operations of crisp and fuzzy relations. The relation concept is most powerful, and is used for nonlinear simulation, classification and control. The description on composition of relations gives a view of extending fuzziness into functions. Tolerance and equivalence relations are helpful for solving similar classification problems. The noninteractivity between fuzzy sets is analogous to the assumption of independence in probability modeling.

8.8 Solved Problems

1. The elements in two sets A and B are given as

$$A = \{2, 4\} \quad \text{and} \quad B = \{a, b, c\}$$

Find the various Cartesian products of these two sets.

Solution: The various Cartesian products of these two given sets are

$$A \times B = \{(2, a), (2, b), (2, c), (4, a), (4, b), (4, c)\}$$

$$B \times A = \{(a, 2), (a, 4), (b, 2), (b, 4), (c, 2), (c, 4)\}$$

$$A \times A = A^2 = \{(2, 2), (2, 4), (4, 2), (4, 4)\}$$

$$B \times B = B^2 = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$$

2. Consider the following two fuzzy sets:

$$\underset{\sim}{A} = \left\{ \frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{1}{x_3} \right\}$$

$$\text{and } \underset{\sim}{B} = \left\{ \frac{0.4}{y_1} + \frac{0.9}{y_2} \right\}$$

Perform the Cartesian product over these given fuzzy sets.

Solution: The fuzzy Cartesian product performed over fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ results in fuzzy relation $\underset{\sim}{R}$ given by $\underset{\sim}{R} = \underset{\sim}{A} \times \underset{\sim}{B}$. Hence

$$\underset{\sim}{R} = \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.7 \\ 0.4 & 0.9 \end{bmatrix}$$

The calculation for $\underset{\sim}{R}$ is as follows:

$$\begin{aligned}
\mu_{\underset{\sim}{R}}(x_1, y_1) &= \min[\mu_{\underset{\sim}{A}}(x_1), \mu_{\underset{\sim}{B}}(y_1)] \\
&= \min(0.3, 0.4) = 0.3 \\
\mu_{\underset{\sim}{R}}(x_1, y_2) &= \min[\mu_{\underset{\sim}{A}}(x_1), \mu_{\underset{\sim}{B}}(y_2)] \\
&= \min(0.3, 0.9) = 0.3 \\
\mu_{\underset{\sim}{R}}(x_2, y_1) &= \min[\mu_{\underset{\sim}{A}}(x_2), \mu_{\underset{\sim}{B}}(y_2)] \\
&= \min(0.7, 0.4) = 0.4 \\
\mu_{\underset{\sim}{R}}(x_2, y_2) &= \min[\mu_{\underset{\sim}{A}}(x_2), \mu_{\underset{\sim}{B}}(y_2)] \\
&= \min(0.7, 0.9) = 0.7 \\
\mu_{\underset{\sim}{R}}(x_3, y_1) &= \min[\mu_{\underset{\sim}{A}}(x_3), \mu_{\underset{\sim}{B}}(y_1)] \\
&= \min(1, 0.4) = 0.4 \\
\mu_{\underset{\sim}{R}}(x_3, y_2) &= \min[\mu_{\underset{\sim}{A}}(x_3), \mu_{\underset{\sim}{B}}(y_2)] \\
&= \min(1, 0.9) = 0.9
\end{aligned}$$

Thus, the Cartesian product between fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are obtained.

3. Two fuzzy relations are given by

$$\underset{\sim}{R} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.6 & 0.3 \\ 0.2 & 0.9 \end{bmatrix} \end{array}$$

$$\text{and } \underset{\sim}{S} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 1 & 0.5 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{bmatrix} \end{array}$$

Obtain fuzzy relation $\underset{\sim}{T}$ as a composition between the fuzzy relations.

Solution: The composition between two given fuzzy relations is performed in two ways as

(a) Max-min composition

(b) Max-product composition

(a) Max-min composition

$$\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.6 & 0.5 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{bmatrix} \end{array}$$

The calculations for obtaining $\underset{\sim}{T}$ are as follows:

$$\begin{aligned}
\mu_{\underset{\sim}{T}}(x_1, z_1) &= \max\{\min[\mu_{\underset{\sim}{R}}(x_1, y_1), \mu_{\underset{\sim}{S}}(y_1, z_1)], \\
&\qquad \min[\mu_{\underset{\sim}{R}}(x_1, y_2), \mu_{\underset{\sim}{S}}(y_2, z_1)]\} \\
&= \max[\min(0.6, 1), \min(0.3, 0.8)] \\
&= \max(0.6, 0.3) = 0.6 \\
\mu_{\underset{\sim}{T}}(x_1, z_2) &= \max[\min(0.6, 0.5), \min(0.3, 0.4)] \\
&= \max(0.5, 0.3) = 0.5 \\
\mu_{\underset{\sim}{T}}(x_1, z_3) &= \max[\min(0.6, 0.3), \min(0.3, 0.7)] \\
&= \max(0.3, 0.3) = 0.3 \\
\mu_{\underset{\sim}{T}}(x_2, z_1) &= \max[\min(0.2, 1), \min(0.9, 0.8)] \\
&= \max(0.2, 0.8) = 0.8 \\
\mu_{\underset{\sim}{T}}(x_2, z_2) &= \max[\min(0.2, 0.5), \min(0.9, 0.4)] \\
&= \max(0.2, 0.4) = 0.4 \\
\mu_{\underset{\sim}{T}}(x_2, z_3) &= \max[\min(0.2, 0.3), \min(0.9, 0.7)] \\
&= \max(0.2, 0.7) = 0.7
\end{aligned}$$

(b) Max-product composition

$$\underset{\sim}{T} = \underset{\sim}{R} \bullet \underset{\sim}{S}$$

Calculations for $\underset{\sim}{T}$ are as follows:

$$\begin{aligned}
\mu_{\underset{\sim}{T}}(x_1, z_1) &= \max\{[\mu_{\underset{\sim}{R}}(x_1, y_1) \bullet \mu_{\underset{\sim}{S}}(y_1, z_1)], \\
&\qquad [\mu_{\underset{\sim}{R}}(x_1, y_2) \bullet \mu_{\underset{\sim}{S}}(y_2, z_1)]\} \\
&= \max(0.6, 0.24) = 0.6
\end{aligned}$$

$$\mu_{\underset{\sim}{T}}(x_1, z_2) = \max[(0.6 \times 0.5), (0.3 \times 0.4)]$$
$$= \max(0.3, 0.12) = 0.3$$
$$\mu_{\underset{\sim}{T}}(x_1, z_3) = \max[(0.6 \times 0.3), (0.3 \times 0.7)]$$
$$= \max(0.18, 0.21) = 0.21$$
$$\mu_{\underset{\sim}{T}}(x_2, z_1) = \max[(0.2 \times 1), (0.9 \times 0.8)]$$
$$= \max(0.2, 0.72) = 0.72$$
$$\mu_{\underset{\sim}{T}}(x_2, z_2) = \max[(0.2 \times 0.5), (0.9 \times 0.4)]$$
$$= \max(0.1, 0.36) = 0.36$$
$$\mu_{\underset{\sim}{T}}(x_2, z_3) = \max[(0.2 \times 0.3), (0.9 \times 0.7)]$$
$$= \max(0.06, 0.63) = 0.63$$

The fuzzy relation $\underset{\sim}{T}$ by max-product composition is given as

$$\underset{\sim}{T} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.6 & 0.3 & 0.21 \\ 0.72 & 0.36 & 0.63 \end{bmatrix} \end{array}$$

4. For a speed control of DC motor, the membership functions of series resistance, armature current and speed are given as follows:

$$\underset{\sim}{R_{se}} = \left\{ \frac{0.4}{30} + \frac{0.6}{60} + \frac{1.0}{100} + \frac{0.1}{120} \right\}$$

$$\underset{\sim}{I_a} = \left\{ \frac{0.2}{20} + \frac{0.3}{40} + \frac{0.6}{60} + \frac{0.8}{80} + \frac{1.0}{100} + \frac{0.2}{120} \right\}$$

$$\underset{\sim}{N} = \left\{ \frac{0.35}{500} + \frac{0.67}{1000} + \frac{0.97}{1500} + \frac{0.25}{1800} \right\}$$

Compute relation $\underset{\sim}{T}$ for relating series resistance to motor speed, i.e., $\underset{\sim}{R_{se}}$ to $\underset{\sim}{N}$. Perform max-min composition only.

Solution: For relating series resistance to motor-speed, i.e., $\underset{\sim}{R_{se}}$ to $\underset{\sim}{N}$, we have to perform the following operations – two fuzzy cross-products and one fuzzy composition (max-min):

$$\underset{\sim}{R} = \underset{\sim}{R_{se}} \times \underset{\sim}{I_a}$$
$$\underset{\sim}{S} = \underset{\sim}{I_a} \times \underset{\sim}{N}$$
$$\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S}$$

Relation $\underset{\sim}{R}$ is obtained as the Cartesian product of $\underset{\sim}{R_{se}}$ and $\underset{\sim}{I_a}$, i.e.,

$$\underset{\sim}{R} = \underset{\sim}{R_{se}} \times \underset{\sim}{I_a}$$

$$= \begin{array}{c c} & \begin{array}{cccccc} 20 & 40 & 60 & 80 & 100 & 120 \end{array} \\ \begin{array}{c} 30 \\ 60 \\ 100 \\ 120 \end{array} & \begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.4 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.6 & 0.6 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.6 & 0.8 & 1.0 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix} \end{array}$$

Relation $\underset{\sim}{S}$ is obtained as the Cartesian product of $\underset{\sim}{I_a}$ and $\underset{\sim}{N}$, i.e.,

$$\underset{\sim}{S} = \underset{\sim}{I_a} \times \underset{\sim}{N} = \begin{array}{c c} & \begin{array}{cccc} 500 & 1000 & 1500 & 1800 \end{array} \\ \begin{array}{c} 20 \\ 40 \\ 60 \\ 80 \\ 100 \\ 120 \end{array} & \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 & 0.25 \\ 0.35 & 0.6 & 0.6 & 0.25 \\ 0.35 & 0.67 & 0.8 & 0.25 \\ 0.35 & 0.67 & 0.97 & 0.25 \\ 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

Relation $\underset{\sim}{T}$ is obtained as the composition between relations $\underset{\sim}{R}$ and $\underset{\sim}{S}$, i.e.,

$$\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S} = \begin{array}{c c} & \begin{array}{cccc} 500 & 1000 & 1500 & 1800 \end{array} \\ \begin{array}{c} 30 \\ 60 \\ 100 \\ 120 \end{array} & \begin{bmatrix} 0.35 & 0.4 & 0.4 & 0.25 \\ 0.35 & 0.6 & 0.6 & 0.25 \\ 0.35 & 0.67 & 0.97 & 0.25 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix} \end{array}$$

5. Consider two fuzzy sets given by

$$\underset{\sim}{A} = \left\{ \frac{1}{\text{low}} + \frac{0.2}{\text{medium}} + \frac{0.5}{\text{high}} \right\}$$

$$\underset{\sim}{B} = \left\{ \frac{0.9}{\text{positive}} + \frac{0.4}{\text{zero}} + \frac{0.9}{\text{negative}} \right\}$$

(a) Find the fuzzy relation for the Cartesian product of $\underset{\sim}{A}$ and $\underset{\sim}{B}$, i.e., $\underset{\sim}{R} = \underset{\sim}{A} \times \underset{\sim}{B}$.

(b) Introduce a fuzzy set $\underset{\sim}{C}$ given by

$$\underset{\sim}{C} = \left\{ \frac{0.1}{\text{low}} + \frac{0.2}{\text{medium}} + \frac{0.7}{\text{high}} \right\}$$

Find the relation between $\underset{\sim}{C}$ and $\underset{\sim}{B}$ using Cartesian product, i.e., find $\underset{\sim}{S} = \underset{\sim}{C} \times \underset{\sim}{B}$.

(c) Find $\underset{\sim}{C} \circ \underset{\sim}{R}$ using max-min composition.

(d) Find $\underset{\sim}{C} \circ \underset{\sim}{S}$ using max-min composition.

Solution:

(a) The Cartesian product between $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is obtained as

$$\underset{\sim}{R} = \underset{\sim}{A} \times \underset{\sim}{B} = \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)]$$

$$\begin{array}{rl} & \quad\text{positive} \quad \text{zero} \quad \text{negative} \\ \begin{array}{r}\text{low}\\ =\text{medium}\\ \text{high}\end{array} & \begin{bmatrix} 0.9 & 0.4 & 0.9 \\ 0.2 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.5 \end{bmatrix} \end{array}$$

(b) The new fuzzy set is

$$\underset{\sim}{C} = \left\{\frac{0.1}{\text{low}} + \frac{0.2}{\text{medium}} + \frac{0.7}{\text{high}}\right\}$$

The Cartesian product between $\underset{\sim}{C}$ and $\underset{\sim}{B}$ is obtained as

$$\underset{\sim}{S} = \underset{\sim}{C} \times \underset{\sim}{B} = \min[\mu_{\underset{\sim}{C}}(x), \mu_{\underset{\sim}{B}}(y)]$$

$$\begin{array}{rl} & \quad\text{positive} \quad \text{zero} \quad \text{negative} \\ \begin{array}{r}\text{low}\\ =\text{medium}\\ \text{high}\end{array} & \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.2 \\ 0.7 & 0.4 & 0.7 \end{bmatrix} \end{array}$$

(c)

$$\underset{\sim}{C} \circ \underset{\sim}{R} = [0.1 \quad 0.2 \quad 0.7]_{1\times 3} \begin{bmatrix} 0.9 & 0.4 & 0.9 \\ 0.2 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.5 \end{bmatrix}_{3\times 3}$$

$$= [0.5 \quad 0.4 \quad 0.5]$$

For instance,

$$\mu_{\underset{\sim}{C} \circ \underset{\sim}{R}}(x_1, y_1) = \max[\min(0.1, 0.9), \min(0.2, 0.2), \min(0.7, 0.5)]$$
$$= \max(0.1, 0.2, 0.5) = 0.5$$

(d)

$$\underset{\sim}{C} \circ \underset{\sim}{S} = [0.1 \quad 0.2 \quad 0.7]_{1\times 3} \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.2 \\ 0.7 & 0.4 & 0.7 \end{bmatrix}_{3\times 3}$$

$$= [0.7 \quad 0.4 \quad 0.7]$$

Hence max-min composition was used to find the relations.

6. Consider a universe of aircraft speed near the speed of sound as $X = \{0.72, 0.725, 0.75, 0.775, 0.78\}$ and a fuzzy set on this universe for the speed "near mach 0.75" $= \underset{\sim}{M}$ where

$$\underset{\sim}{M} = \left\{\frac{0}{0.72} + \frac{0.8}{0.725} + \frac{1}{0.75} + \frac{0.8}{0.775} + \frac{0}{0.78}\right\}$$

Define a universe of altitudes as $Y = \{21, 22, 23, 24, 25, 26, 27\}$ in k-feet and a fuzzy set on this universe for the altitude fuzzy set "approximately 24,000 feet" $= \underset{\sim}{N}$ where

$$\underset{\sim}{N} = \left\{\frac{0}{21k} + \frac{0.2}{22k} + \frac{0.7}{23k} + \frac{1}{24k} + \frac{0.7}{25k} + \frac{0.2}{26k} + \frac{0}{27k}\right\}$$

(a) Construct a relation $\underset{\sim}{R} = \underset{\sim}{M} \times \underset{\sim}{N}$

(b) For another aircraft speed, say $\underset{\sim}{M}_1$, in the region of mach 0.75 where

$$\underset{\sim}{M}_1 = \left\{\frac{0}{0.72} + \frac{0.8}{0.725} + \frac{1}{0.75} + \frac{0.6}{0.775} + \frac{0}{0.78}\right\}$$

find relation $\underset{\sim}{S} = \underset{\sim}{M}_1 \circ \underset{\sim}{R}$ using max-min composition.

Solution: The two given fuzzy sets are

$$\underset{\sim}{M} = \left\{\frac{0}{0.72} + \frac{0.8}{0.725} + \frac{1}{0.75} + \frac{0.8}{0.775} + \frac{0}{0.78}\right\}$$

$$\underset{\sim}{N} = \left\{\frac{0}{21k} + \frac{0.2}{22k} + \frac{0.7}{23k} + \frac{1}{24k} + \frac{0.7}{25k} + \frac{0.2}{26k} + \frac{0}{27k}\right\}$$

(a) Relation $\underset{\sim}{R} = \underset{\sim}{M} \times \underset{\sim}{N}$ is obtained by using Cartesian product

$$\underset{\sim}{R} = \min[\mu_{\underset{\sim}{M}}(x), \mu_{\underset{\sim}{N}}(y)]$$

$$= \begin{array}{c} \\ 0.72 \\ 0.725 \\ 0.75 \\ 0.775 \\ 0.78 \end{array} \begin{array}{c} \begin{array}{ccccccc} 21k & 22k & 23k & 24k & 25k & 26k & 27k \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.7 & 0.8 & 0.7 & 0.2 & 0 \\ 0 & 0.2 & 0.7 & 1 & 0.7 & 0.2 & 0 \\ 0 & 0.2 & 0.7 & 0.8 & 0.7 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(b) Relation $\underset{\sim}{S} = \underset{\sim}{M}_1 \circ \underset{\sim}{R}$ is found by using max-min composition

$$\underset{\sim}{S} = \max\{\min[\mu_{\underset{\sim}{M}}(x), \mu_{\underset{\sim}{R}}(x, y)]\}$$

$$= [0 \quad 0.8 \quad 1 \quad 0.6 \quad 0]_{1\times 5}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.7 & 0.8 & 0.7 & 0.2 & 0 \\ 0 & 0.2 & 0.7 & 1 & 0.7 & 0.2 & 0 \\ 0 & 0.2 & 0.7 & 0.8 & 0.7 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 7}$$

$$\underset{\sim}{S} = [0 \quad 0.2 \quad 0.7 \quad 1 \quad 0.7 \quad 0.2 \quad 0]_{1\times 7}$$

7. Consider two relations

$$\underset{\sim}{R} = \begin{array}{c} \\ 9 \\ 18 \\ 27 \\ 36 \end{array} \begin{array}{c} \begin{array}{ccccc} -100 & -50 & 0 & 50 & 100 \end{array} \\ \begin{bmatrix} 0.2 & 0.5 & 0.7 & 1 & 0.9 \\ 0.3 & 0.5 & 0.7 & 1 & 0.8 \\ 0.4 & 0.6 & 0.8 & 0.9 & 0.4 \\ 0.9 & 1 & 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}.$$

and

$$\underset{\sim}{S} = \begin{array}{c} \\ -100 \\ -50 \\ 0 \\ 50 \\ 100 \end{array} \begin{array}{c} \begin{array}{ccccc} 2 & 4 & 8 & 16 & 20 \end{array} \\ \begin{bmatrix} 1 & 0.8 & 0.6 & 0.3 & 0.1 \\ 0.7 & 1 & 0.7 & 0.5 & 0.4 \\ 0.5 & 0.6 & 1 & 0.8 & 0.8 \\ 0.3 & 0.4 & 0.6 & 1 & 0.9 \\ 0.9 & 0.3 & 0.5 & 0.7 & 1 \end{bmatrix} \end{array}$$

If $\underset{\sim}{R}$ is a relationship between frequency and temperature and $\underset{\sim}{S}$ represents a relation between temperature and reliability index of a circuit, obtain the relation between frequency and reliability index using (a) max-min composition and (b) max-product composition.

Solution:

(a) Max-min composition is performed as follows.

$$\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S} = \max\{\min[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)]\}$$

$$= \begin{array}{c} \\ 9 \\ 18 \\ 27 \\ 36 \end{array} \begin{array}{c} \begin{array}{ccccc} 2 & 4 & 8 & 16 & 20 \end{array} \\ \begin{bmatrix} 0.9 & 0.6 & 0.7 & 1 & 0.9 \\ 0.8 & 0.6 & 0.7 & 1 & 0.9 \\ 0.6 & 0.6 & 0.8 & 0.9 & 0.9 \\ 0.9 & 1 & 0.8 & 0.8 & 0.8 \end{bmatrix} \end{array}$$

(b) Max-product composition is performed as follows.

$$\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S} = \max\{\min[\mu_{\underset{\sim}{R}}(x, y) \times \mu_{\underset{\sim}{S}}(x, y)]\}$$

$$= \begin{array}{c} \\ 9 \\ 18 \\ 27 \\ 36 \end{array} \begin{array}{c} \begin{array}{ccccc} 2 & 4 & 8 & 16 & 20 \end{array} \\ \begin{bmatrix} 0.81 & 0.5 & 0.7 & 1.0 & 0.9 \\ 0.72 & 0.5 & 0.7 & 1.0 & 0.9 \\ 0.4 & 0.6 & 0.8 & 0.9 & 0.81 \\ 0.9 & 1.0 & 0.8 & 0.64 & 0.64 \end{bmatrix} \end{array}$$

Thus the relation between frequency and reliability index has been found using composition techniques.

8. Three fuzzy sets are given as follows:

$$\underset{\sim}{P} = \left\{ \frac{0.1}{2} + \frac{0.3}{4} + \frac{0.7}{6} + \frac{0.4}{8} + \frac{0.2}{10} \right\}$$

$$\underset{\sim}{Q} = \left\{ \frac{0.1}{0.1} + \frac{0.3}{0.2} + \frac{0.3}{0.3} + \frac{0.4}{0.4} + \frac{0.5}{0.5} + \frac{0.2}{0.6} \right\}$$

$$\underset{\sim}{T} = \left\{ \frac{0.1}{0} + \frac{0.7}{0.5} + \frac{0.3}{1} \right\}$$

The following operations are performed over the fuzzy sets:

(a) $\underset{\sim}{R} = \underset{\sim}{P} \times \underset{\sim}{Q} = \min[\mu_{\underset{\sim}{P}}(x), \mu_{\underset{\sim}{Q}}(y)]$

$$= \begin{array}{c} \\ 2 \\ 4 \\ 6 \\ 8 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccccc} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 \end{array} \\ \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 \\ 0.1 & 0.3 & 0.3 & 0.4 & 0.5 & 0.2 \\ 0.1 & 0.3 & 0.3 & 0.4 & 0.4 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

(b) $\underset{\sim}{S} = \underset{\sim}{Q} \times \underset{\sim}{T} = \min[\mu_{\underset{\sim}{Q}}(x), \mu_{\underset{\sim}{T}}(y)]$

$$= \begin{array}{c} \\ 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 0.5 & 1 \end{array} \\ \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.3 \\ 0.1 & 0.4 & 0.3 \\ 0.1 & 0.5 & 0.3 \\ 0.1 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

(c) $\underset{\sim}{M} = \underset{\sim}{R} \circ \underset{\sim}{S} = \max\{\min[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)]\}$

$$= \begin{array}{c} \\ 2 \\ 4 \\ 6 \\ 8 \\ 10 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 0.5 & 1 \end{array} \\ \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.3 \\ 0.1 & 0.5 & 0.3 \\ 0.1 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

(d) $\underset{\sim}{M} = \underset{\sim}{R} \circ \underset{\sim}{S} = \max[\mu_{\underset{\sim}{R}}(x, y) \times \mu_{\underset{\sim}{S}}(x, y)]$

$$= \begin{array}{c} \\ 2 \\ 4 \\ 6 \\ 8 \\ 10 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 0.5 & 1 \end{array} \\ \begin{bmatrix} 0.01 & 0.05 & 0.03 \\ 0.03 & 0.05 & 0.09 \\ 0.05 & 0.25 & 0.15 \\ 0.04 & 0.20 & 0.12 \\ 0.02 & 0.0 & 0.06 \end{bmatrix} \end{array}$$

Thus the operations were performed over the given fuzzy sets.

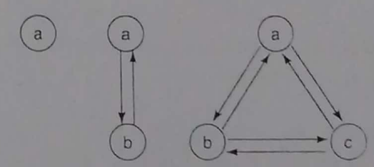
9. Which of the following are equivalence relations?

No.	Set	Relation on the set
(i)	People	is the brother of
(ii)	People	has the same parents as
(iii)	Points on a map	is connected by a road to
(iv)	Lines in plane geometry	is perpendicular to
(v)	Positive integers	for some integer k, equals 10^k times

Draw graphs of the equivalence relations.

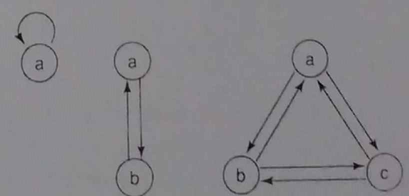
Solution:

(a) The set is people. The relation of the set "is the brother of." The relation (figure below) is not equivalence relation because people considered cannot be brothers to themselves. So, reflexive property is not satisfied. But symmetry and transitive properties are satisfied.



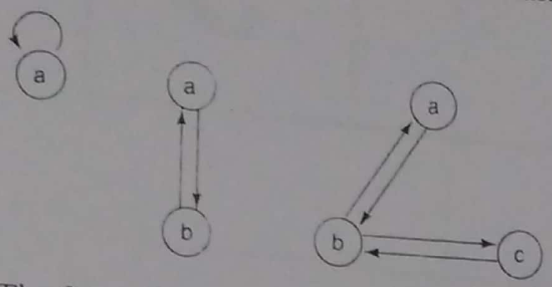
The figure illustrates that the relation is not an equivalence relation.

(b) The set is people. The relation is "has the same parents as." In this case (figure below), all the three properties are satisfied, hence it is an equivalence relation.



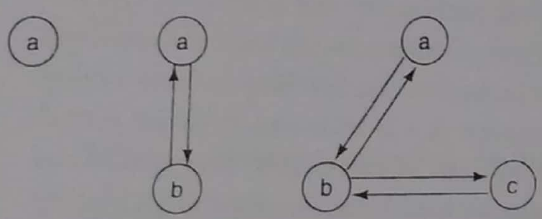
Thus the relation is an equivalence relation.

(c) The set is "points on a map." The relation is "is connected by a road to." This relation (figure on next page) is not an equivalence relation because the transitive property is not satisfied. The road may connect 1st point and 2nd point; 2nd point and 3rd point; but it may not connect 1st and 3rd points. Thus, transitive property is not satisfied.



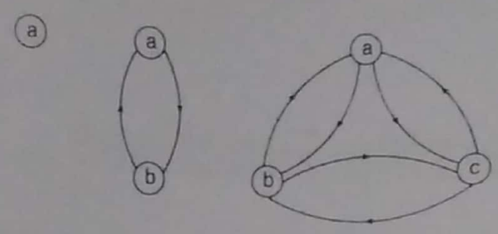
The figure illustrates that the relation is not an equivalence relation.

(d) The set is "lines in plane geometry." The relation "is perpendicular to." The relation (figure below) defined here is not an equivalence relation because both reflexive and transitive properties are not satisfied. A line cannot be perpendicular to itself, hence reflexivity is not satisfied. Also transitivity property is not satisfied because 1st line and 2nd line may be perpendicular to each other, 2nd line and 3rd line may also be perpendicular to each other, but 1st line and 3rd line will not be perpendicular to each other. However, symmetry property is satisfied.



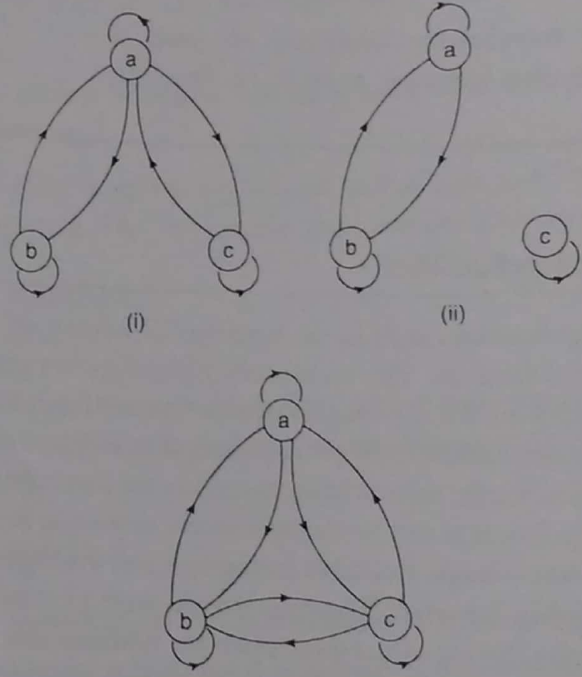
The figure illustrates that the relation is not an equivalence relation.

(e) The set is "positive integers". The relation is "for some integer k, equals 10^k times." In this case (figure below), reflexivity is not satisfied because a positive integer, for some integer k, equals 10^k times is not possible. Symmetry and transitivity properties are satisfied. Thus, the relation is not an equivalence relation.



The figure illustrates that the relation is not an equivalence relation.

10. The following figure shows three relations on the universe $X = \{a, b, c\}$. Are these relations equivalence relations?



(i) (ii) (iii)

Solution:

(a) The relation in (i) is not equivalence relation because transitive property is not satisfied.

(b) The relation in (ii) is not equivalence relation because transitive property is not satisfied.

(c) The relation in (iii) is equivalence relation because reflexive, symmetry and transitive properties are satisfied.

8.9 Review Questions

1. Define classical relations and fuzzy relations.
2. State the Cartesian product of a relation.
3. How are the relations represented in various forms?
4. What is one-one mapping of a relation?