The Myhill-Nerode Theorem

Priti Shankar
priti@csa.iisc.ernet.in

Department of Computer Science and Automation
Indian Institute of Science

Outline

1. Equivalence Relations
2. Right Invariance
3. Equivalence Relations Induced by DFA's
4. The Myhill Nerode theorem
5. Applications of the Myhill Nerode Theorem

Definition

A binary relation R on a set S is a subset of $S \times S$. An equivalence relation on a set satisfies

Reflexivity: For all x in S, xRx

- Symmetry: For $x, y \in S$ $xRy \iff yRx$
- Transitivity: For $x, y, z \in S$ xRy and $yRz \implies xRz$
- Every equivalence relation on S partitions S into equivalence classes. The number of equivalence classes is called the index of the relation.

Definition

A binary relation R on a set S is a subset of $S \times S$. An equivalence relation on a set satisfies

- Reflexivity: For all x in S, xRx
- Symmetry: For $x, y \in S$ $xRy \iff yRx$
- Transitivity: For $x, y, z \in S$ xRy and $yRz \implies xRz$
- Every equivalence relation on S partitions S into equivalence classes. The number of equivalence classes is called the index of the relation.

Definition

A binary relation R on a set S is a subset of $S \times S$. An equivalence relation on a set satisfies

Reflexivity: For all x in S, xRx

Symmetry: For $x, y \in S$ $xRy \iff yRx$

Transitivity: For $x, y, z \in S$ xRy and $yRz \implies xRz$

- Every equivalence relation on S partitions S into equivalence classes. The number of equivalence classes is called the index of the relation.

Definition

A binary relation R on a set S is a subset of $S \times S$. An equivalence relation on a set satisfies

Reflexivity: For all x in S, xRx

Symmetry: For $x, y \in S$ $xRy \iff yRx$

Transitivity: For $x, y, z \in S$ xRy and $yRz \implies xRz$

Every equivalence relation on S partitions S into equivalence classes. The number of equivalence classes is called the index of the relation.

Example 1

Let $S = \Sigma^+$ where $\Sigma = \{a, b\}$.

- Define R as xRy whenever x and y both end in the same symbol of Σ.
- How many equivalence classes does R partition S into?

Example 1

Let $S = \Sigma^+$ where $\Sigma = \{a, b\}$.

Define R as xRy whenever x and y both end in the same symbol of Σ.

- How many equivalence classes does R partition S into?

Example 1

Let $S = \Sigma^+$ where $\Sigma = \{a, b\}$.

Define R as xRy whenever x and y both end in the same symbol of Σ.

How many equivalence classes does R partition S into?

Right invariance

An equivalence relation on Σ^* is said to be right invariant with respect to concatenation if $\forall x, y \in \Sigma^*$ and a in Σ xRy implies that $xaRya$.

Example:

- Let $S = \Sigma^*$ where $\Sigma = \{a, b\}$ and R be defined as follows:
- xRy if x and y have the same number of $a's$.
- How many equivalence classes does R partition S into?
- Is R right invariant?

Right invariance

An equivalence relation on Σ^* is said to be right invariant with respect to concatenation if $\forall x, y \in \Sigma^*$ and a in Σ xRy implies that $xaRya$.

Example:

Let $S = \Sigma^*$ where $\Sigma = \{a, b\}$ and R be defined as follows:

- xRy if x and y have the same number of $a's$.
- How many equivalence classes does R partition S into?
- Is R right invariant?

Right invariance

An equivalence relation on Σ^* is said to be right invariant with respect to concatenation if $\forall x, y \in \Sigma^*$ and a in Σ xRy implies that $xaRya$.

Example:

Let $S = \Sigma^*$ where $\Sigma = \{a, b\}$ and R be defined as follows:

xRy if x and y have the same number of $a's$.

- How many equivalence classes does R partition S into?
- Is R right invariant?

Right invariance

An equivalence relation on Σ^* is said to be right invariant with respect to concatenation if $\forall x, y \in \Sigma^*$ and a in Σ xRy implies that $xaRya$.

Example:

Let $S = \Sigma^*$ where $\Sigma = \{a, b\}$ and R be defined as follows:

xRy if x and y have the same number of $a's$.

How many equivalence classes does R partition S into?

- Is R right invariant?

Right invariance

An equivalence relation on Σ^* is said to be right invariant with respect to concatenation if $\forall x, y \in \Sigma^*$ and a in Σ xRy implies that $xaRya$.

Equivalence relations induced by DFA's

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA.

- Define a relation R_M as follows:
- For $x, y \in \Sigma^*$ $xR_My \iff \delta(q_0, x) = \delta(q_0, y)$
- Is this an equivalence relation?
- If so how many equivalence classes does it have? That is, what is its index?

Equivalence relations induced by DFA's

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA.

Define a relation R_M as follows:

- For $x, y \in \Sigma^*$ $xR_My \iff \delta(q_0, x) = \delta(q_0, y)$
- Is this an equivalence relation?
- If so how many equivalence classes does it have? That is, what is its index?

Equivalence relations induced by DFA's

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA.

Define a relation R_M as follows:

For $x, y \in \Sigma^*$ $xR_My \iff \delta(q_0, x) = \delta(q_0, y)$

- Is this an equivalence relation?
- If so how many equivalence classes does it have? That is, what is its index?

Equivalence relations induced by DFA's

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA.

Define a relation R_M as follows:

For $x, y \in \Sigma^*$ $xR_My \iff \delta(q_0, x) = \delta(q_0, y)$

Is this an equivalence relation?

- If so how many equivalence classes does it have? That is, what is its index?

Equivalence relations induced by DFA's

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA.

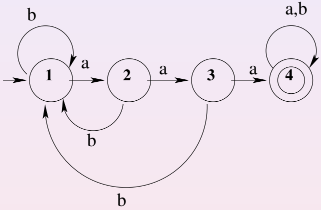
Define a relation R_M as follows:

For $x, y \in \Sigma^*$ $xR_My \iff \delta(q_0, x) = \delta(q_0, y)$

Is this an equivalence relation?

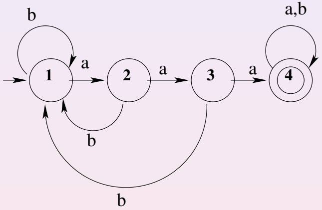
If so how many equivalence classes does it have?
That is, what is its index?

Example



- C_1: All strings not containing more than 2 consecutive $a's$ and which end in a b.
- C_2: All strings not containing more than 2 consecutive $a's$ and which end in a
- C_3: All strings not containing more than 2 consecutive $a's$ and which end in aa
- C_4: All strings containing at least three consecutive $a's$

Example

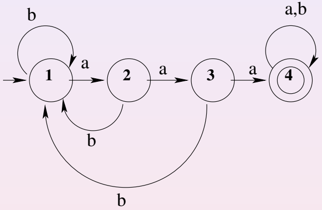


C_1: All strings not containing more than 2 consecutive $a's$ and which end in a b.

C_2: All strings not containing more than 2 consecutive $a's$ and which end in a

- C_3: All strings not containing more than 2 consecutive $a's$ and which end in aa
- C_4: All strings containing at least three consecutive $a's$

Example



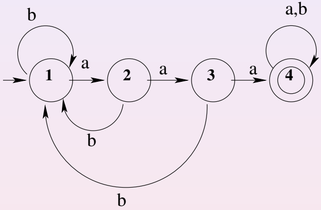
C_1: All strings not containing more than 2 consecutive $a's$ and which end in a b.

C_2: All strings not containing more than 2 consecutive $a's$ and which end in a

C_3: All strings not containing more than 2 consecutive $a's$ and which end in aa

- C_4: All strings containing at least three consecutive $a's$

Example



C_1: All strings not containing more than 2 consecutive $a's$ and which end in a b.

C_2: All strings not containing more than 2 consecutive $a's$ and which end in a

C_3: All strings not containing more than 2 consecutive $a's$ and which end in aa

C_4: All strings containing at least three consecutive $a's$

Refinements of equivalence relations

An equivalence relation R_1 is a refinement of R_2 if $R_1 \subseteq R_2$ i.e. $(x, y) \in R_1 \implies (x, y) \in R_2$

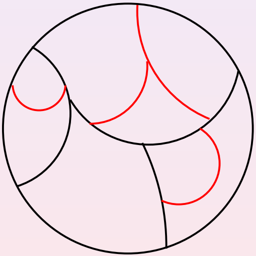
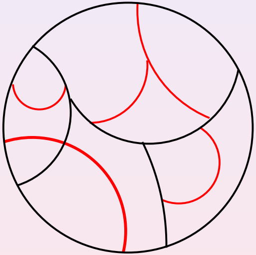


Figure: R_1 equivalence class boundaries–red lines and black lines, R_2 equivalence class boundaries– black lines

Example



Here R_1 is not a refinement of R_2 as $(x, y) \in R_1$ does not imply $(x, y) \in R_2$

Example of a refinement of an equivalence relation

Let $S = \{0, 1\}^*$

- For $x, y \in S$ xR_1y if x and y have the same parity.
- xR_2y if x and y have the same parity and the same length.
- What is the index of R_1?
- What is the index of R_2?
- Is one of them a refinement of the other?

Example of a refinement of an equivalence relation

Let $S = \{0,1\}^*$

For $x, y \in S$ xR_1y if x and y have the same parity.

- xR_2y if x and y have the same parity and the same length.
- What is the index of R_1?
- What is the index of R_2?
- Is one of them a refinement of the other?

Example of a refinement of an equivalence relation

Let $S = \{0, 1\}^*$

For $x, y \in S$ xR_1y if x and y have the same parity.

xR_2y if x and y have the same parity and the same length.

- What is the index of R_1?
- What is the index of R_2?
- Is one of them a refinement of the other?

Example of a refinement of an equivalence relation

Let $S = \{0, 1\}^*$

For $x, y \in S$ xR_1y if x and y have the same parity.

xR_2y if x and y have the same parity and the same length.

What is the index of R_1?

- What is the index of R_2?
- Is one of them a refinement of the other?

Example of a refinement of an equivalence relation

Let $S = \{0, 1\}^*$

For $x, y \in S$ xR_1y if x and y have the same parity.

xR_2y if x and y have the same parity and the same length.

What is the index of R_1?

What is the index of R_2?

- Is one of them a refinement of the other?

Example of a refinement of an equivalence relation

Let $S = \{0, 1\}^*$

For $x, y \in S$ xR_1y if x and y have the same parity.

xR_2y if x and y have the same parity and the same length.

What is the index of R_1?

What is the index of R_2?

Is one of them a refinement of the other?

The Myhill Nerode theorem

Let $L \subseteq \Sigma*$. The following statements are equivalent:

1. L is recognized by a DFA
2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.
3. Define an equivalence relation R_L as follows. For $x, y \in \Sigma^*$ $(x, y) \in R_L \iff \forall z \in \Sigma^* xz \in L$ whenever $yz \in L$. Then R_L has finite index.

To prove this, we need to prove that
$1 \implies 2 \implies 3 \implies 1$

The Myhill Nerode theorem

Let $L \subseteq \Sigma*$. The following statements are equivalent:

1. L is recognized by a DFA
2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.
3. Define an equivalence relation R_L as follows. For $x, y \in \Sigma^*$ $(x, y) \in R_L \iff \forall z \in \Sigma^* xz \in L$ whenever $yz \in L$. Then R_L has finite index.

To prove this, we need to prove that
$1 \implies 2 \implies 3 \implies 1$

The Myhill Nerode theorem

Let $L \subseteq \Sigma*$. The following statements are equivalent:

1. L is recognized by a DFA
2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.
3. Define an equivalence relation R_L as follows. For $x, y \in \Sigma^*$ $(x, y) \in R_L \iff \forall z \in \Sigma^* xz \in L$ whenever $yz \in L$. Then R_L has finite index.

To prove this, we need to prove that
$1 \implies 2 \implies 3 \implies 1$

To prove 1 $\implies$ 2

1. L is recognized by a DFA
2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA that recognizes L.
- The relation R_M defined earlier as follows: for $x, y \in \Sigma^*$ $xR_My \iff \delta(q_0, x) = \delta(q_0, y)$ is an equivalence relation which is of finite index and is also right invariant as M is a DFA.
- L is just the union of the equivalence classes corresponding to the final states of M

To prove 1 $\implies$ 2

1. L is recognized by a DFA
2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA that recognizes L.
- The relation R_M defined earlier as follows: for $x, y \in \Sigma^*$ $xR_My \iff \delta(q_0, x) = \delta(q_0, y)$ is an equivalence relation which is of finite index and is also right invariant as M is a DFA.
- L is just the union of the equivalence classes corresponding to the final states of M

To prove 1 $\implies$ 2

1. L is recognized by a DFA
2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.

Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA that recognizes L.

- The relation R_M defined earlier as follows: for $x, y \in \Sigma^*$ $xR_My \iff \delta(q_0, x) = \delta(q_0, y)$ is an equivalence relation which is of finite index and is also right invariant as M is a DFA.
- L is just the union of the equivalence classes corresponding to the final states of M

To prove 1 $\implies$ 2

1. L is recognized by a DFA
2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.

Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA that recognizes L. The relation R_M defined earlier as follows: for $x, y \in \Sigma^*$ $xR_My \iff \delta(q_0, x) = \delta(q_0, y)$ is an equivalence relation which is of finite index and is also right invariant as M is a DFA.

- L is just the union of the equivalence classes corresponding to the final states of M

To prove 1 $\implies$ 2

1. L is recognized by a DFA
2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.

Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA that recognizes L.

The relation R_M defined earlier as follows: for $x, y \in \Sigma^*$ $xR_My \iff \delta(q_0, x) = \delta(q_0, y)$ is an equivalence relation which is of finite index and is also right invariant as M is a DFA.

L is just the union of the equivalence classes corresponding to the final states of M

We next prove 2 $\implies$ 3

2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.
3. Define an equivalence relation R_L as follows. For $x, y \in \Sigma^*$ $(x, y) \in R_L \iff \forall z \in \Sigma^* xz \in L$ whenever $yz \in L$. Then R_L has finite index.

- Let E be the equivalence relation in (2). We show that E is a refinement of R_L. Hence since E has finite index, so does R_L.
- We show that if $(x, y) \in E$ then $(x, y) \in R_L$, i.e. $E \subseteq R_L$ and hence E is a refinement of R_L.
- $(x, y) \in E \implies \forall z \in \Sigma^*, (xz, yz) \in E$ by repeated use of right invariance. Hence xz is in an equivalence class in L iff yz is, thereby implying that $(x, z) \in R_L$. Thus R_L has finite index.

We next prove $2 \implies 3$

2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.
3. Define an equivalence relation R_L as follows. For $x, y \in \Sigma^*$ $(x, y) \in R_L \iff \forall z \in \Sigma^* xz \in L$ whenever $yz \in L$. Then R_L has finite index.

- Let E be the equivalence relation in (2). We show that E is a refinement of R_L. Hence since E has finite index, so does R_L.
- We show that if $(x, y) \in E$ then $(x, y) \in R_L$, i.e. $E \subseteq R_L$ and hence E is a refinement of R_L.
- $(x, y) \in E \implies \forall z \in \Sigma^*, (xz, yz) \in E$ by repeated use of right invariance. Hence xz is in an equivalence class in L iff yz is, thereby implying that $(x, z) \in R_L$. Thus R_L has finite index.

We next prove 2 $\implies$ 3

2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.
3. Define an equivalence relation R_L as follows. For $x, y \in \Sigma^*$ $(x, y) \in R_L \iff \forall z \in \Sigma^* xz \in L$ whenever $yz \in L$. Then R_L has finite index.

   Let E be the equivalence relation in (2). We show that E is a refinement of R_L. Hence since E has finite index, so does R_L.

- We show that if $(x, y) \in E$ then $(x, y) \in R_L$, i.e. $E \subseteq R_L$ and hence E is a refinement of R_L.
- $(x, y) \in E \implies \forall z \in \Sigma^*, (xz, yz) \in E$ by repeated use of right invariance. Hence xz is in an equivalence class in L iff yz is, thereby implying that $(x, z) \in R_L$. Thus R_L has finite index.

We next prove $2 \implies 3$

2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.
3. Define an equivalence relation R_L as follows. For $x, y \in \Sigma^* \ (x, y) \in R_L \iff \forall z \in \Sigma^* xz \in L$ whenever $yz \in L$. Then R_L has finite index.

Let E be the equivalence relation in (2). We show that E is a refinement of R_L. Hence since E has finite index, so does R_L.

We show that if $(x, y) \in E$ then $(x, y) \in R_L$, i.e. $E \subseteq R_L$ and hence E is a refinement of R_L.

- $(x, y) \in E \implies \forall z \in \Sigma^*, (xz, yz) \in E$ by repeated use of right invariance. Hence xz is in an equivalence class in L iff yz is, thereby implying that $(x, z) \in R_L$. Thus R_L has finite index.

We next prove 2 $\implies$ 3

2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.
3. Define an equivalence relation R_L as follows. For $x, y \in \Sigma^*$ $(x, y) \in R_L \iff \forall z \in \Sigma^* xz \in L$ whenever $yz \in L$. Then R_L has finite index.

   Let E be the equivalence relation in (2). We show that E is a refinement of R_L. Hence since E has finite index, so does R_L.
   We show that if $(x, y) \in E$ then $(x, y) \in R_L$, i.e. $E \subseteq R_L$ and hence E is a refinement of R_L.
   $(x, y) \in E \implies \forall z \in \Sigma^*, (xz, yz) \in E$ by repeated use of right invariance. Hence xz is in an equivalence class in L iff yz is, thereby implying that $(x, z) \in R_L$.
   Thus R_L has finite index.

We now prove that 3 $\implies$ 1

3. Define an equivalence relation R_L as follows. For $x, y \in \Sigma^*$ $(x, y) \in R_L \iff \forall z \in \Sigma^*, xz \in L$ whenever $yz \in L$. Then R_L has finite index.
1. L is recognized by a DFA

We now prove that 3 $\implies$ 1

3. Define an equivalence relation R_L as follows. For $x, y \in \Sigma^*$ $(x, y) \in R_L \iff \forall z \in \Sigma^*, xz \in L$ whenever $yz \in L$. Then R_L has finite index.
1. L is recognized by a DFA

1. Construct a DFA $M_L = (Q_L, \Sigma, \delta_L, q_{0L}, F_L)$.
2. We first show that R_L is right invariant. To see this assume that $(x, y) \in R_L$. Assume R_L is not right invariant. Then there exists $a \in \Sigma$ such that $(xa, ya) \notin R_L$. Let z be a "distinguishing" string for the pair (xa, ya). Then az is a distinguishing string for (x, y) contradicting the assumption that $(x, y) \in R_L$. Therefore R_L is right invariant.
3. Let $[x]$ denote the equivalence class containing x. The state set $Q_L = \{[x] : x \in \Sigma^*\}$. The transition function δ_L is defined as $\delta_L([x], a) = [xa]$. The definition is consistent because had we chosen some other representative $[y]$ of the equivalence class, $[xa] = [ya]$ by right invariance of R_L so the transition is to the same state of M_L

1. Construct a DFA $M_L = (Q_L, \Sigma, \delta_L, q_{0L}, F_L)$.
2. We first show that R_L is right invariant. To see this assume that $(x, y) \in R_L$. Assume R_L is not right invariant. Then there exists $a \in \Sigma$ such that $(xa, ya) \notin R_L$. Let z be a "distinguishing" string for the pair (xa, ya). Then az is a distinguishing string for (x, y) contradicting the assumption that $(x, y) \in R_L$. Therefore R_L is right invariant.
3. Let $[x]$ denote the equivalence class containing x. The state set $Q_L = \{[x] : x \in \Sigma^*\}$. The transition function δ_L is defined as $\delta_L([x], a) = [xa]$. The definition is consistent because had we chosen some other representative $[y]$ of the equivalence class, $[xa] = [ya]$ by right invariance of R_L so the transition is to the same state of M_L

1. Construct a DFA $M_L = (Q_L, \Sigma, \delta_L, q_{0L}, F_L)$.
2. We first show that R_L is right invariant. To see this assume that $(x, y) \in R_L$. Assume R_L is not right invariant. Then there exists $a \in \Sigma$ such that $(xa, ya) \notin R_L$. Let z be a "distinguishing" string for the pair (xa, ya). Then az is a distinguishing string for (x, y) contradicting the assumption that $(x, y) \in R_L$. Therefore R_L is right invariant.
3. Let $[x]$ denote the equivalence class containing x. The state set $Q_L = \{[x] : x \in \Sigma^*\}$. The transition function δ_L is defined as $\delta_L([x], a) = [xa]$. The definition is consistent because had we chosen some other representative $[y]$ of the equivalence class, $[xa] = [ya]$ by right invariance of R_L so the transition is to the same state of M_L

1. Construct a DFA $M_L = (Q_L, \Sigma, \delta_L, q_{0L}, F_L)$.
2. We first show that R_L is right invariant. To see this assume that $(x, y) \in R_L$. Assume R_L is not right invariant. Then there exists $a \in \Sigma$ such that $(xa, ya) \notin R_L$. Let z be a "distinguishing" string for the pair (xa, ya). Then az is a distinguishing string for (x, y) contradicting the assumption that $(x, y) \in R_L$. Therefore R_L is right invariant.
3. Let $[x]$ denote the equivalence class containing x. The state set $Q_L = \{[x] : x \in \Sigma^*\}$. The transition function δ_L is defined as $\delta_L([x], a) = [xa]$. The definition is consistent because had we chosen some other representative $[y]$ of the equivalence class, $[xa] = [ya]$ by right invariance of R_L so the transition is to the same state of M_L

We need to prove that $x \in L \iff \delta_L(q_{0L}, x) \in F_L$ i.e. x is accepted by M_L

- This is easy as $\delta_L(q_{0L}, x) = [x]$ and x is accepted by M_L iff $[x] \in F_L \iff x \in L$.

We need to prove that $x \in L \iff \delta_L(q_{0L}, x) \in F_L$ i.e. x is accepted by M_L

This is easy as $\delta_L(q_{0L}, x) = [x]$ and x is accepted by M_L iff $[x] \in F_L \iff x \in L$.

The MN theorem can be used to show that a particular language is regular without actually constructing the automaton or to show conclusively that a language is not regular.

- Example. Is the following language regular $L_1 = \{xy : |x| = |y|, x, y \in \Sigma^*\}$?
- Example. What about the language $L_2 = \{xy : |x| = |y|$ and y ends with a $1\}$?
- Example. What about the language $L_3 = \{xy : |x| = |y|$ and y contains a $1\}$?

The MN theorem can be used to show that a particular language is regular without actually constructing the automaton or to show conclusively that a language is not regular.

Example. Is the following language regular $L_1 = \{xy : |x| = |y|, x, y \in \Sigma^*\}$?

- Example. What about the language $L_2 = \{xy : |x| = |y|$ and y ends with a $1\}$?
- Example. What about the language $L_3 = \{xy : |x| = |y|$ and y contains a $1\}$?

The MN theorem can be used to show that a particular language is regular without actually constructing the automaton or to show conclusively that a language is not regular.

Example. Is the following language regular $L_1 = \{xy : |x| = |y|, x, y \in \Sigma^*\}$?

Example. What about the language $L_2 = \{xy : |x| = |y| \text{ and } y \text{ ends with a } 1\}$?

- Example. What about the language $L_3 = \{xy : |x| = |y| \text{ and } y \text{ contains a } 1\}$?

The MN theorem can be used to show that a particular language is regular without actually constructing the automaton or to show conclusively that a language is not regular.

Example. Is the following language regular $L_1 = \{xy : |x| = |y|, x, y \in \Sigma^*\}$?

Example. What about the language $L_2 = \{xy : |x| = |y| \text{ and } y \text{ ends with a } 1\}$?

Example. What about the language $L_3 = \{xy : |x| = |y| \text{ and } y \text{ contains a } 1\}$?

For the language L_1 there are two equivalence classes of R_{L_1}. The first C_1 contains all strings of even length and the second C_2 all strings of odd length. All strings z of odd length when concatenated with any string in C_1 take us to C_2, and all strings z of even length when concatenated with any string in C_1 take us back to C_1 which is nothing but L_1. Similarly all strings z of even length when concatenated with any string of C_2 take us back to C_2 and all strings of odd length to C_1

For L_2 we have the additional constraint that y ends with a 1. Class C_2 remains the same as that for L_1. Class C_1 is refined into classes C_1' which contains all strings of even length that end in a 1 and C_1'' which contains all strings of even length which end in a 0. Thus L_1 and L_2 are both regular.

For L_3 we have to distinguish for example, between the even length strings in the sequence 01, 0001, 000001,... and in general 0^i1 for i odd, as 00 distinguishes the first string from all the others after it in the sequence (concatenation of 00 to 01 gives a string not in L_3 but concatenation of 00 to all the others gives a string in L_3), 0000 distinguishes the second from all the others after it in the sequence, and in general $(00)^{2i}$ distinguishes the i^{th} string from all the others after it in the sequence. As there are an infinite number of strings that can be distinguished from one another, the relation R_L has infinite index and therefore L_3 is not regular.