Module 5

Matrix Representation of Graphs

MATRIX REPRESENTATION

Ques 1) List all the methods to represent the graphs in form of matrix.

Ans: Matrix Representation of Graphs

A diagrammatic representation of a graph has limited usefulness. Furthermore, such a representation is only possible when the number of nodes and edges is reasonably small. If nodes and edges are large then method of matrix representation is used. Such a method of representation has several advantages. It is easy to store and manipulate matrices and hence the graphs represented by them in a computer. Well-known operations of matrix algebra can be used to calculate paths, cycles, and other characteristics of a graph.

Methods of Representation of Graphs

1) Adjacency Matrix,
2) Incidence Matrix
3) Circuit Matrix, and
4) Path Matrix.

Ques 2) What adjacency matrix? Explain using suitable diagram and example.

Ans: Adjacency Matrix

The adjacency matrix of a graph G with n vertices is an $n \times n$ matrix AG such that each entry a_{ij} is the number of edges connecting v_i and v_j. Thus, $a_{ij} = 0$ if there is no edge from v_i to v_j.

The adjacency matrix of a graph G with n vertices and parallel edges/self-loops is an $n \times n$ matrix.

$$A(G) = [a_{ij}]$$

given by

$a_{ij} = N$, where N is the number of edges between i^{th} and j^{th} vertices, and

$a_{ij} = 0$, if there is no edge between them.

Let G be a graph with n vertices $V_1, V_2, \ldots V_n$. The adjacency matrix of G with respect to this particular listing of n vertices is the $n \times n$ matrix $A(g) = a_{ij}$, where a_{ij} is the number of edges joining the vertex v_i to v_j. If G has no loops then all the entries of the main diagonal will be 0 and if G has no parallel edges then the entries of A(G) are either 0 or 1. If the graph has no self-loops and no parallel edges, the degree of a vertex equals the number of ones in the corresponding row or column of A(G).

The adjacency matrix of a graph is a matrix with rows and column labeled by the vertices and such that its entry in row i, column j, $i \neq j$, is the number of edges incident on i and j. **For example**, the following is the adjacency matrix of the graph of **figure 5.1**:

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

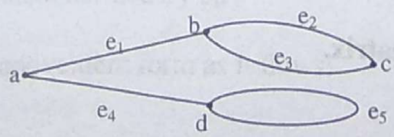


Figure 5.1

One of the uses of the adjacency matrix A of a simple graph G is to compute the number of paths between two vertices, namely entry (i, j) of A^n is the number of paths of length n from i to j.

Ques 3) Find the adjacency matrix M of the graph D with four vertices d_1, d_2, d_3 and d_4 shown in figure 5.2:

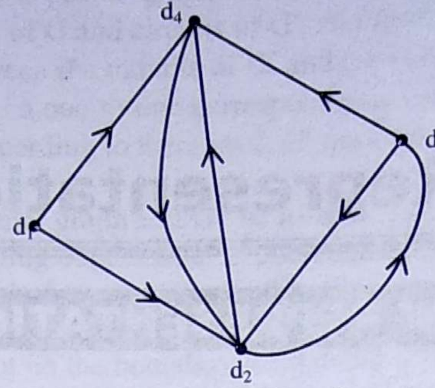


Figure 5.2: Digraph

Ans: The adjacency matrix M of diagraph D.

$$M = \begin{array}{c} \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & d_3 & d_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Ques 4) Discuss about the incidence matrix. Write the theorem used. Also give an example.

Ans: Incidence Matrix

The incidence matrix of a graph G is a matrix with rows labeled by vertices, and columns labeled by edges, so that entry for row v column e is 1 if e is incident on v, and 0 otherwise.

Suppose that G has n vertices listed as $v_1, v_2, \ldots v_n$ and t edges listed as $e_1, e_2, \ldots e_t$. The incidence matrix of G is the $n \times t$ matrix $M(G) = [m_{ij}]$, where m_{ij} is the number of times that the vertex v_i is incident with the edge e_j, i.e.,

$m_{ij} = 0$, if v_i is not an end of e_j

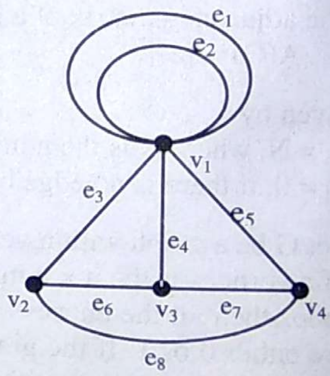
$m_{ij} = 1$, if v_i is an end of the non-loop e_j.

$m_{ij} = 2$, if v_i is an end of the loop e_j.

Sum of the elements in the i^{th} row of M(G) gives the degree of the vertex v_i.

Theorem: Let M(G) is an incidence matrix of a connected graph with m vertices; then the rank of M(G) is m – 1.

For example, calculate the incidence matrix of the graph shown in the **figure aside**:



Incidence Matrix of graph shown aside is given below:

$$M(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Ques 5) Give the concept of circuit matrix.

Ans: Circuit Matrix

Let G be a graph with n vertices and e edges. Suppose that the number of circuits in G is c, then we define a circuit matrix $C(G) = [c_{ij}]$ of the order $c \times e$ as follows:

Rows of the matrix will correspond to different circuits and the columns will correspond to edges.

The element $c_{ij} = 1$ if j^{th} edge belongs to i^{th} circuit.
$= 0$, otherwise

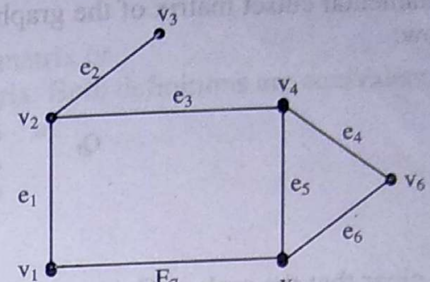


Figure 5.3

For example, The given graph has three circuits:

Circuit $c_1 = (e_1, e_3, e_4, e_6, e_7)$

Circuit $c_2 = (e_1, e_3, e_5, e_7)$

Circuit $c_3 = (e_4, e_5, e_6)$

Here the corresponding circuit matrix will be as given below:

$$\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Theorem: If B is a circuit matrix of a connected graph of e edges and n vertices, then prove that rank of $B = e - n + 1$.

Ques 6) Discuss about the cut matrix.

Ans: Cut Matrix

Consider a cut (V_a, V_b) in a connected directed graph G with n vertices and m edges. (V_a, V_b) consists of all those edges connecting vertices in V_a to V_b. This cut may be assigned an orientation from V_a to V_b or from V_b to V_a. Suppose the orientation of (V_a, V_b) is from V_a to V_a. Then the orientation of an edge (v_i, v_j) is said to agree with the cut orientation if $v_i, \in V_a$, and $V_j \in V_b$.

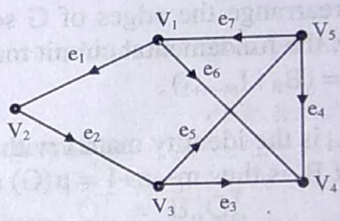


Figure 5.4: A Directed Graph

The cut matrix $Q_c = [q_{ij}]$ of G has m columns, one for each edge, and has one row for each cut. The element is defined as follows:

$$q_{ij} = \begin{cases} 1, \text{if the } j^{th} \text{ edge is in the } i^{th} \text{ cut and its orientation agrees with the cut orientation} \\ -1, \text{if the } j^{th} \text{ edge is in the } i^{th} \text{ cut and its orientation does not agree with the cut orientation} \\ 0, \text{if the } j^{th} \text{ edge is not in the } i^{th} \text{ cut} \end{cases}$$

Each row of Q_c is called the cut vector. The edges incident on a vertex forms a cut. Thus it follows that the matrix A_c is a submatrix of Q_c. Next to identify another important submatrix of Q_c.

Each branch of a spanning tree T of connected graph G defines a fundamental cutset. The submatrix of Q_c corresponding to the n – 1 fundamental cutsets defined by T is called the fundamental cutset matrix Q_f of G with respect to T.

Let $b_1, b_2, ..., b_{n-1}$ denote the branches of T. Let us assume that the orientation of a fundamental cutset is chosen so as to agree with that of the defining branch. Suppose arranging the rows and the columns of Q_f so that the i^{th} column corresponds to the fundamental cutset defined by b_i.

Then the matrix Q_f can be displayed in a convenient form as follows:

$$Q_f = [U \mid Q_{fc}]$$

Where U is the unit matrix of order n – 1, and its columns correspond to the branches of T. As an example, the fundamental cutset matrix of the graph in **figure 5.4** with respect to the spanning tree T = (e_1, e_2, e_5, e_6) is given below:

$$Q_f = \begin{array}{c} \\ e_1 \\ e_2 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_5 & e_6 & e_3 & e_4 & e_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

It is clear that the rank of Q_f is n – 1. Hence,

rank (Q_c) ≥ n – 1

Ques 7) What is fundamental circuit matrix and its rank? Give suitable example.

Ans: Fundamental Circuit Matrix

If the graph G is connected and contains at least one circuit, then it has a co-spanning tree T* and the corresponding fundamental circuits. By choosing the corresponding rows of the circuit matrix B, we get an (m – n + 1) × m matrix B_f, called the **fundamental circuit matrix**.

Similarly, a connected digraph G with at least one circuit has a fundamental circuit matrix: the direction of a fundamental circuit is the same as the direction of the corresponding link in T*.

When we rearrange the edges of G so that the links of T* come last and sort the fundamental circuits in the same order, the fundamental circuit matrix takes the form

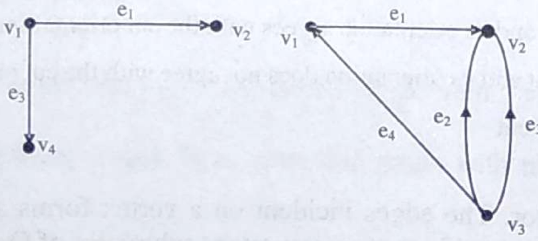
$B_f = (B_{ft} \mid I_{m-n+1})$,

where I_{m-n+1} is the identity matrix with m – n+1 rows.

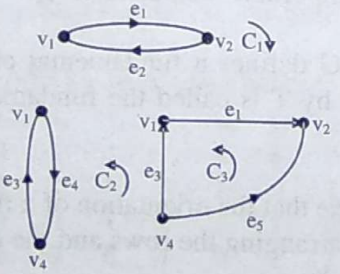
The rank of B_f is thus m– n+1 = μ(G) and the rank of B is ≥ m – n + 1.

For example, in the figure below we left out vertex v3 so we get a connected digraph and we chose the spanning tree.

Fundamental Circuit Matrix



The fundamental circuits are



and

$$B_f = \begin{array}{c} \begin{array}{ccccc} e_1 & e_3 & e_2 & e_4 & e_5 \end{array} \\ \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \end{array}$$

Rank of a Matrix

The rank of a matrix is defined as:

1) The maximum number of linearly independent column vectors in the matrix or
2) The maximum number of linearly independent row vectors in the matrix. Both definitions are equivalent.

For an $m \times n$ matrix,

1) If m is less than n, then the maximum rank of the matrix is m.
2) If m is greater than n, then the maximum rank of the matrix is n.

The rank of a matrix would be zero only if the matrix had no non-zero elements. If a matrix had even one non-zero element, its minimum rank would be one.

Ques 8) Verify the statement "If the digraph G contains at least one circuit, then the rank of its circuit matrix B is μ(G). Furthermore, if G is connected, then the circuit matrix B can be expressed as $B = B_2B_f$, where the matrix B_2 consists of 0's and ±1's, and the cut matrix Q can be expressed as $Q = Q_1Q_f$, where the matrix Q_1 consists of 0's and ±1's".

Ans: First we consider the case when G is connected. We choose a spanning tree T of G and rearrange the m edges of G so that the branches of T come first and the links of T* come last. We sort the fundamental cut sets in the same order as the branches and links. Then

$Q_f = (I_{n-1} \mid Q_{fc})$ and $B_f = (B_{ft} \mid I_{m-n+1})$

The blocks of B can be constructed in a similar way:

$B = (B_1 \mid B_2)$

Since Q_f is a submatrix of Q and B_f is a submatrix of B, it follows from theorem that

$$O = B_f Q_f^T = (B_{ft} \mid I_{m-n+1})\,(I_{n-1} \mid Q_{fc})^T = (B_{ft} \mid I_{m-n+1})\begin{pmatrix} I_{n-1} \\ \hline Q_{fc}^T \end{pmatrix} = B_{ft}I_{n-1} + I_{m-n+1}\,Q_{fc}^T = B_{ft}Q_{fc}^T$$

Hence, $B_{ft} = -Q_{fc}^T$

Furthermore, since Q_f is a submatrix of Q, we can use the same theorem to get

$$O = BQ_f^T = (B_1 \mid B_2)\,(I_{n-1} \mid Q_{fc})^T = (B_1 \mid B_2)\begin{pmatrix} I_{n-1} \\ \hline Q_{fc}^T \end{pmatrix} = B_1I_{n-1} + B_2\,Q_{fc}^T = B_1 - B_2B_{ft}.$$

Hence, $B = (B_2B_{ft} \mid B_2) = B_2\,(B_{ft} \mid I_{m-n+1}) = B_2B_f$, as claimed.

In the same way, Q can be expressed as $Q = Q_1Q_f$, as claimed, which is clear anyway since the rank of Q is $n-1$ and its elements are 0's and ±1's.

Every row of B is a linear combination of the rows corresponding to the fundamental circuits and the rank of B is at most equal to the rank of $B_f = m - n + 1$. On the other hand, as we pointed out earlier, the rank of B is $\geq m - n + 1$. Thus, $\text{rank}(B) = m - n + 1$ ($= \mu(G)$) for a connected digraph.

In the case of a disconnected digraph G (which contains at least one circuit), it is divided into components ($k \geq 2$ components) and the circuit matrix B is divided into blocks corresponding to the components, in which case

$$\text{rank}(B) = \sum_{i=1}^{k}(m_i - n_i + 1) = m - n + k = \mu(G)$$

Note: The proof also gives the formula, $B_{ft} = -Q_{fc}^T$, which connects the fundamental cut matrix and the fundamental circuit matrix.

Ques 9) Discuss the cut set matrix using suitable example.

Ans: Cut Set Matrix

Let G be a graph with m edges and q cutsets. The cut-set matrix $C = [c_{ij}]_{q \times m}$ of G is a (0, 1) matrix with

$$c_{ij} = \begin{cases} 1, \text{if } i^{th} \text{ cutset contains } j^{th} \text{ edge} \\ 0, \text{otherwise.} \end{cases}$$

For example, consider the graphs shown in **figure 5.5**,

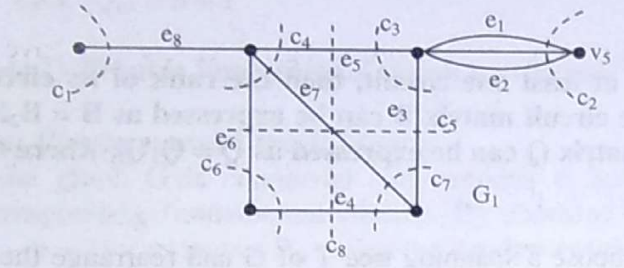


Figure 5.5(a)

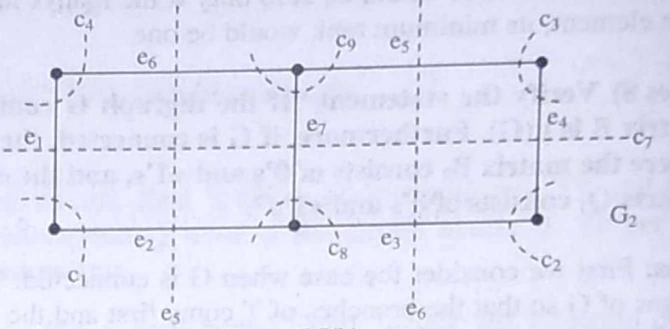


Figure 5.5(b)

In the graph G_1, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$.

The cut-sets are $c_1 = \{e_8\}$, $c_2 = \{e_1, e_2\}$, $c_3 = \{e_3, e_5\}$, $c_4 = \{e_5, e_6, e_7\}$, $c_5 = \{e_3, e_6, e_7\}$, $c_6 = \{e_4, e_6\}$, $c_7 = \{e_3, e_4, e_7\}$ and $c_8 = \{e_4, e_5, e_7\}$. The cut-sets for the graph G_2 are $c_1 = \{e_1, e_2\}$, $c_2 = \{e_3, e_4\}$, $c_3 = \{e_4, e_5\}$, $c_4 = \{e_1, e_6\}$, $c_5 = \{e_2, e_6\}$, $c_6 = \{e_3, e_5\}$, $c_7 = \{e_1, e_4, c_7\}$, $c_8 = \{e_2, e_3, e_7\}$ and $c_9 = \{e_5, e_6, e_7\}$. Thus the cut-set matrices are given by

$$C(G_1) = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}, \text{ and}$$

$$C(G_2) = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

We have the following observations about the cut-set matrix C(G) of a graph G:

1) The permutation of rows or columns in a cut-set matrix corresponds simply to renaming of the cut-sets and edges respectively.
2) Each row in C(G) is a cut-set vector.
3) A column with all zeros corresponds to an edge forming a self-loop.
4) Parallel edges form identical columns in the cut-set matrix.

5) In a non-separable graph, since every set of edges incident on a vertex is a cut-set, therefore every row of incidence matrix A(G) is included as a row in the cut-set matrix C(G). That is, for a non-separable graph G, C(G) contains A(G). For a separable graph, the incidence matrix of each block is contained in the cut-set matrix. **For example,** in the graph G_1 of **figure 5.5**, the incidence matrix of the block $\{e_3, e_4, e_5, e_6, e_7\}$ is the 4×5 submatrix of C, left after deleting rows c_1, c_2, c_5, c_8 and columns e_1, e_2, e_8.

6) It follows from observation 5, that rank $C(G) \geq$ rank $A(G)$. Therefore, for a connected graph with n vertices, rank $C(G) \geq n-1$.

Ques 10) Prove that "If G is a connected graph, then the rank of a cut-set matrix C(G) is equal to the rank of incidence matrix A(G), which equals the rank of graph G".

Ans: Let A(G), B(G) and C(G) be the incidence, cycle and cut-set matrix of the connected graph G. Then we have

rank $C(G) \geq n-1$ …(1)

Since the number of edges common to a cut-set and a cycle is always even, every row in C is orthogonal to every row in B, provided the edges in both B and C are arranged in the same order

Thus,

$BC^T = CB^T \equiv 0 \pmod 2$ …(2)

Now, applying Sylvester's theorem to **equation (2)**, we have

rank B+ rank $C \leq m$.

For a connected graph, we have rank $B = m - n+1$.

Therefore, rank $C \leq m-$ rank $B = m-(m-n+1) = n-1$.

So,

rank $C \leq n-1$ …(3)

It follows from equation (1) and (3) that rank $C = n-1$.

Ques 11) Write a short note on path matrix. Also give suitable example.

Ans: Path Matrix

Let G be a graph with n vertices and e edges. v_i and v_j are any two vertices of the graph G.

Let the k different paths between the two vertices v_i and v_j be denoted by $W_1, W_2 \ldots\ldots W_k$, $k \geq 1$. Then the path matrix between v_i and v_j is given by:

$W(v_i, v_j) = [w_{ij}]$ of order k by e.

where $w_{ij} = 1$ if i^{th} path contains j^{th} edge

$= 0$, otherwise.

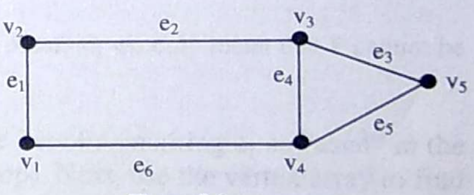


Figure 5.6

For example, find the path graph between v_2 and v_5 for the following graph:

Paths between v_2 and v_5 are $W_1 = \{e_2, e_3\}$, $W_2 = \{e_2, e_4, e_5\}$, $W_3 = \{e_1, e_6, e_5\}$ and $W_4 = \{e_1, e_6, e_4, e_3\}$.

Thus, the path matrix of G between v_2 and v_5 is given below:

$$W(v_2, v_5) = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$