APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

FIFTH SEMESTER B.TECH DEGREE MODEL EXAM, NOVEMBER 2017

Department: Computer Science and Engineering

CS309: Graph Theory and Combinatorics

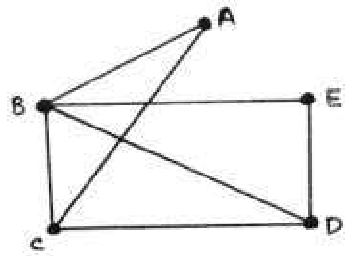
Time: 3 hrs | Max. Marks: 100

PART A

(Answer all questions. Each carries 3 marks) **Total: (12)**

1. a. Differentiate walk, path and circuit. (1 ½)
   b. Using the graph classify each sequence as a walk, a path or a circuit (1 ½)

   1. E→C→D→E
   2. A→C→D→E→B→A
   3. B→D→E→B→C
   4. A→B→C→D→B→A



2. Prove that in a simple graph with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges. (3)

3. a. Draw the graph for the following (1)

   1. Graph that is Euler and Hamiltonian
   2. Graph that is Euler but not Hamiltonian

   b. Define with example

   1. Euler and Unicursal graph
   2. Hamiltonian Circuit (2)

4. Prove that in a complete graph with n vertices there are $(n-1)/2$ edge disjoint hamiltonian circuits ,if n is an odd number $n>=3$. (3)

PART B

(Answer any 2 full questions.) **Total: (18)**

5. a. Write note on Konigsberg Bridge Problem (2)

   b. A graph has exactly 10 vertices, 4 vertices of degree 3, 4 vertices of degree 2 and 2 isolated vertices. How many edges the graph have? (2) c.Determine whether the following graph are isomorphic or not? (3)

d.Prove that maximum numbers of edges in a simple graph with n vertices is n(n-1)/2. (2)

6. a.Draw all simple graphs with 4 vertices (3)

b.Show that the maximum degree of any vertex in a simple graph with n vertices is n-1 (1 ½)

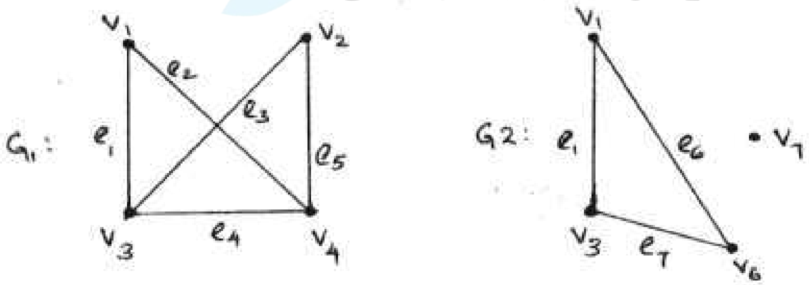
c. Explain complete digraph with example (1 ½)

d. Prove that in a connected graph G with exactly 2k odd vertices, there exists k-edge disjoint subgraph such that they together contains all edges of G and that each is a Unicursal graph (3)

7. a. A connected graph G is a Euler graph if and only if it can be decomposed into circuits (3)

b. Consider the graph G1 and G2 (3)

find G1 U G2, G1 G2, G1 G2



c. Explain the travelling salesman problem using the concept of Hamiltonian circuits (3)

PART C

(Answer all questions. Each carries 3 marks) **Total: (12)**

8. Prove that all trees will have either one or two centers (3)

9. a.What is eccentricity of a node ? How it is used in finding the center of a graph., explain with examples. (1 ½)

b. Show that Hamiltonian path is a spanning tree (1 ½)

10. Show that every circuit has an even number of edges in common with any cut set. (3)

11. Show that a connected graph with n vertices and e edges has e - n + 2 regions (3)

PART D

(Answer any 2 full questions.) **Total: (18)**

12. a.Define spanning tree. Show that a connected graph of n vertices and e edges has n-1 tree branches and e - n + 1 chords. (5)

    b.Show that a tree in which its diameter is not equal to twice the radius (2)

    c.Define rank and nullity of a graph G (2)

13. a.Draw all trees of n labeled vertices for n=3 and n=4 (4)

    b.Define binary tree and sketch all binary trees with 6 pendant vertices (2)

    c.Show that a connected planar graph with n vertices and e edges has e - n + 2 regions. (3)

14. a. Define edge connectivity and vertex connectivity (3)

    b. Show that a vertex v in a connected graph G is a cut vertex iff there exist two vertices x and y in G such that every path between x and y passes through v (3)

PART E

(Answer any 4 full questions.) **Total: (40)**

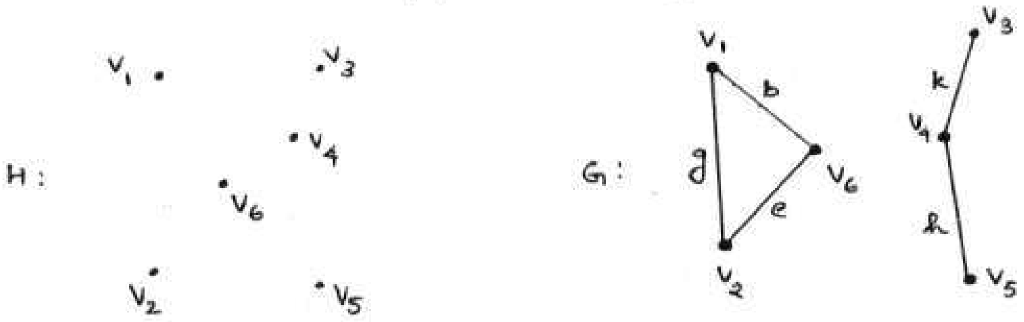
15. With flow chart explain algorithm to find the spanning tree of a given graph (10)

16. Write an algorithm to find the connectedness and components of a graph and analyse the complexity of the algorithm (10)

17. a.Write the properties of incidence matrix. (5)

    b.Write the incidence matrix for the labelled graph H and G shown below.

    Put the incidence matrix of graph G in the block diagonal form. (5)



18. Discuss about the different computer representation of a graph (10)

19. a Explain circuit matrix and its properties. (5)

    b. Let B and A be the circuit matrix and incidence matrix whose columns are arranged using the same order of edges. Show that every row of B is orthogonal to every row A. (5)

Answer Key

Department: Computer Science and Engineering

CS309: Graph Theory and Combinatorics

Part A

1.a

Walk:A finite sequence of alternating vertices and edges... A walk can be open: first and last vertex are not equal. or closed...

Path: Trail with each vertrex visited only once (except perhaps the first and last)

Circuit:A circuit can be a closed walk allowing repetitions of vertices but not edges; however, it can also be a simple cycle, so explicit definition is recommended when it is used.

1.b

Qn No	Walk	Path	Circuit
a	x	x	x
b	YES	YES	YES
c	YES	NO	NO
d	YES	NO	NO

2

1) show that a simple graph G with n vertices and k components can have atmost $(n-k)(n-k+1)/2$ edges

Let the no.of vertices in each of it components of a graph as : $n_1, n_2 \ldots n_k$. Thus we have $n_1 + n_2 + \ldots n_k = n$. $n_i \geq 1$

max no.of edges in the i^{th} component of G is $n_i(n_i-1)/2$ (theorem 3)

$\therefore$ max no.of edges in G

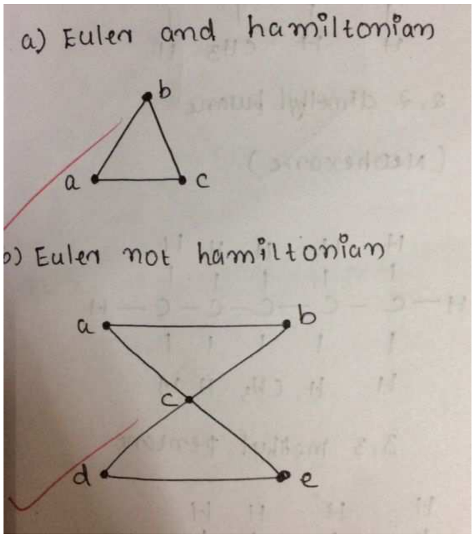
$$\sum_{i=1}^{k} \frac{1}{2} n_i(n_i-1) = \frac{1}{2}\sum_{i=1}^{k} n_i(n_i-1)$$

$$= \frac{1}{2}\sum_{i=1}^{k}(n_i)^2 - \sum_{i=1}^{k}\frac{n_i}{2}$$

$$= \Sigma \frac{1}{2}\left[n^2 - (k-1)(2n-k)\right] - n/2$$

$$= \frac{1}{2}(n-k)(n-k+1)$$

3 a



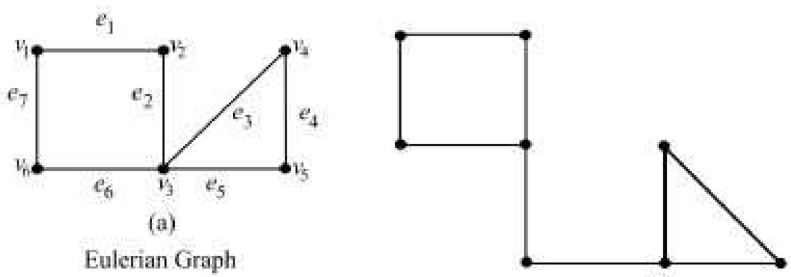
3.b.1

Euler Graphs

A closed walk in a graph G containing all the edges of G is called an *Euler line* in G. A graph containing an Euler line is called an *Euler graph*.

We know that a walk is always connected. Since the Euler line (which is a walk) contains all the edges of the graph, an Euler graph is connected except for any isolated vertices the graph may contain. As isolated vertices do not contribute anything to the understanding of an Euler graph, it is assumed now onwards that Euler graphs do not have any isolated vertices and are thus connected.

Example Consider the graph shown in Figure 3.1. Clearly, $v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_4\ v_5\ e_5\ v_3\ v_6\ e_7\ v_1$ in (a) is an Euler line, whereas the graph shown in (b) is non-Eulerian.

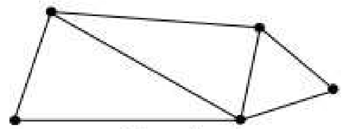


(a)
Eulerian Graph

(b)
Non-Eulerian Graph

Unicursal Graphs

An open walk that includes (or traces) all edges of a graph without retracing any edge is called a unicursal line or open Euler line. A connected graph that has a unicursal line is called a unicursal graph. Figure 3.6 shows a unicursal graph.



Unicursal graph

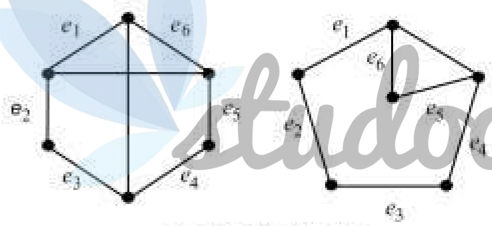
Fig. 3.6

Clearly by adding an edge between the initial and final vertices of a unicursal line, we get an Euler line.

3.b.2

Hamiltonian Graphs

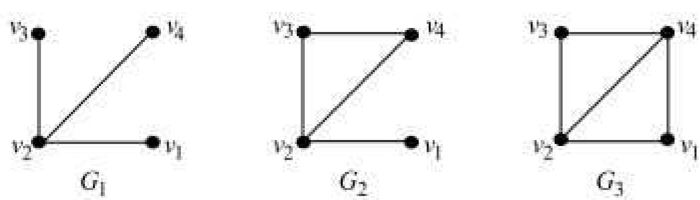
A cycle passing through all the vertices of a graph is called a *Hamiltonian cycle*. A graph containing a Hamiltonian cycle is called a *Hamiltonian graph*. A path passing through all the vertices of a graph is called a *Hamiltonian path* and a graph containing a Hamiltonian path is said to be *traceable*. Examples of Hamiltonian graphs are given in Figure 3.9.



Hamiltonian Graphs

Fig. 3.9

If the last edge of a Hamiltonian cycle is dropped, we get a Hamiltonian path. However, a non-Hamiltonian graph can have a Hamiltonian path, that is, Hamiltonian paths cannot always be used to form Hamiltonian cycles. For example, in Figure 3.10, G_1 has no Hamiltonian path, and so no Hamiltonian cycle; G_2 has the Hamiltonian path $v_1v_2v_3v_4$, but has no Hamiltonian cycle, while G_3 has the Hamiltonian cycle $v_1v_2v_3v_4v_1$.



4

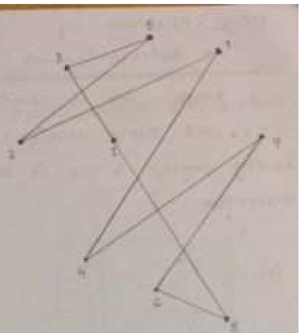
In a complete graph with n vertices there are $(n-1)/2$ edge disjoint hamatonion ckt if n is a odd number $n \geq 3$.

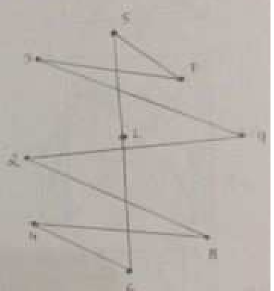
Proof

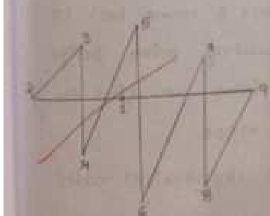
A hamatonion ckt in a complete graph with n vertices n edges. A complete graph with n vertices $n(n-1)/2$ many edges therefore we are looking for edge disjoint hamatonion ckt then the max no.of edge disjoint hamatonion ckt in a complete graph is $1/n \times n(n-1)/2$

eg:-

Let $n = 9$







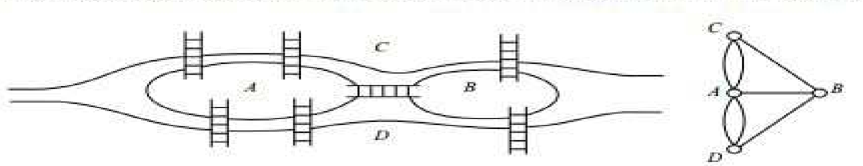
Part B

5.a

Konigsberg Bridge Problem

Two islands A and B formed by the Pregal river (now Pregolya) in Konigsberg (then the capital of east Prussia, but now renamed Kaliningrad and in west Soviet Russia) were connected to each other and to the banks C and D with seven bridges. The problem is to start at any of the four land areas, A, B, C, or D, walk over each of the seven bridges exactly once and return to the starting point.

Euler modeled the problem representing the four land areas by four vertices, and the seven bridges by seven edges joining these vertices. This is illustrated in Figure 3.3.



Konigsberg Bridges　　Underlying Graph

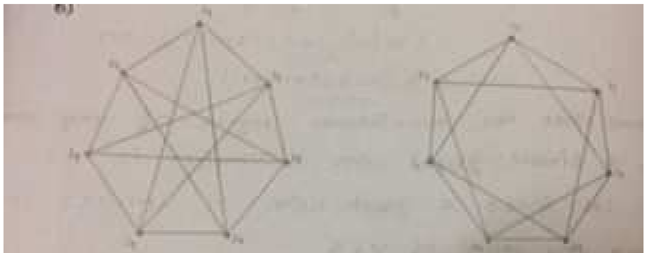
Fig. 3.3

We see from the graph G of the Konigsberg bridges that not all its vertices are of even degree. Thus G is not an Euler graph, and implies that there is no closed walk in G containing all the edges of G. Hence it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

5.b 10 edges

Sum of degree= 2 x edges

5.c



→ vertices = 7 = 7

→ edges = 14 = 14

→ degree is same

→ then it's isomorphic graph

5.d

The maximum no. of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

Proof by Induction:

If $n=1$, 0 edges are required

By subs. we get $\frac{1(1-1)}{2} = 0$.

Assume that complt graph with n vertices as $n(n-1)/2$ edges.

When we add the $(n+1)^{th}$ vertex,

we need to connect it with the n original vertices.

We get no. of edges $= \frac{n(n-1)}{2} + n$

$$= \frac{n(n-1)}{2} + n$$

$$= \frac{n(n-1)+2n}{2}$$

$$= \frac{n^2-n+2n}{2}$$

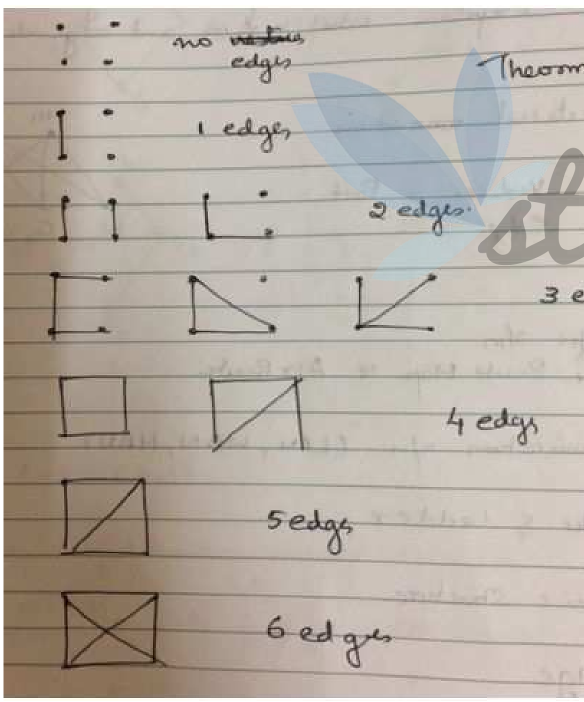
$$= \frac{n^2+n}{2}$$

$$= \frac{n(n+1)}{2}$$

So, for $(n+1)^{th}$ vertex,

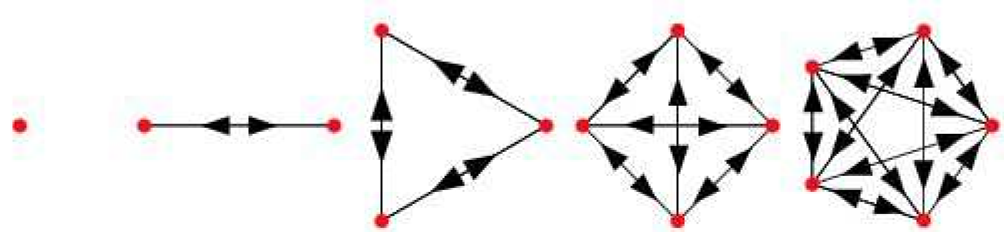
$$\text{maxi}^{c} \text{ no: of edges} = \frac{(n+1)((n+1)-1)}{2}$$

6.a



6.b

6.c.Complete digraphs are digraphs in which every pair of nodes is connected by a bidirectional edge



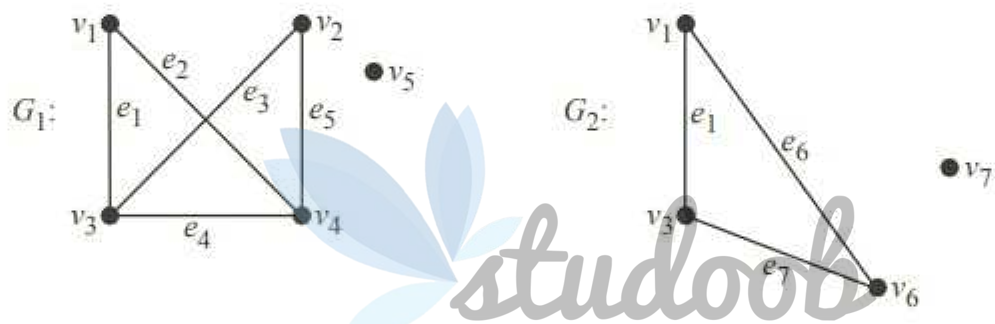
6.d

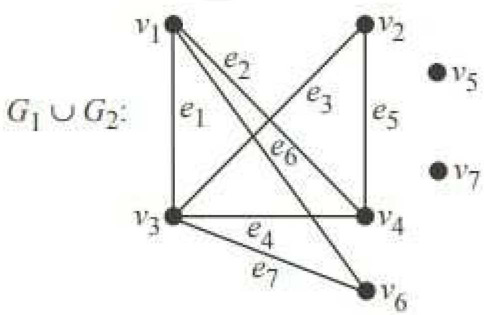
Proof: Let the odd vertices of the given graph G be named $v_1, v_2, \ldots, v_k$; $w_1, w_2, \ldots, w_k$ in any arbitrary order. Add k edges to G between the vertex pairs $(v_1, w_1), (v_2, w_2), \ldots, (v_k, w_k)$ to form a new graph G'.

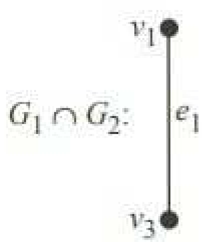
Since every vertex of G' is of even degree, G' consists of an Euler line ρ. Now if we remove from ρ the k edges we just added (no two of these edges are incident on the same vertex), ρ will be split into k walks, each of which is a unicursal line: The first removal will leave a single unicursal line; the second removal will split that into two unicursal lines; and each successive removal will split a unicursal line into two unicursal lines, until there are k of them. Thus the theorem. ■

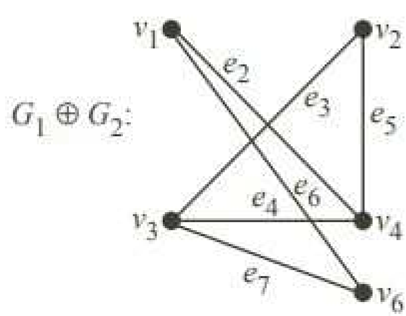
7.a

7. b









7.c

The problem of finding an optimal Hamilton Circuit in a complete weighted graph is often called the Traveling Salesman Problem (TSP). explanation (3)

Part C

8

Every tree has either one or two centers.

Proof The maximum distance, $\max d(v, v_i)$ from a given vertex v to any other vertex occurs only when v_i is a pendant vertex. With this observation, let T be a tree having more than two vertices. Tree T has two or more pendant vertices. Deleting all the pendant vertices from T, the resulting graph T' is again a tree. The removal of all pendant vertices from T uniformly reduces the eccentricities of the remaining vertices (vertices in T') by one. Therefore the centers of T are also the centers of T'. From T' we remove all pendant vertices and get another tree T''. Continuing this process, we either get a vertex, which is a center of T, or an edge whose end vertices are the two centers of T.

9.a The **eccentricity** of a graph vertex in a connected graph is the maximum graph distance between and any other vertex of . For a disconnected graph, all vertices are defined to have infinite **eccentricity**. The maximum **eccentricity** is the graph diameter.

The **center** (or Jordan **center) of a graph** is the set of all vertices of minimum eccentricity

9.b**spanning tree** is a **Hamiltonian path**. Since it has degree ≤ 2 it cannot branch and since it is **spanning** only two vertices can have degree < 2.

Part D

12.a. A **spanning tree** is a subset of Graph G, which has all the vertices covered with minimum possible number of edges. Hence, a **spanning tree** does not have cycles and it cannot be disconnected.. By this**definition**, we can draw a conclusion that every connected and undirected Graph G has at least one**spanning tree**.

Proof theorm 3.12 (3)

12 b. proof explain with example

12 c.the **nullity** of the **graph** is the **nullity** of its oriented incidence matrix, given by the formula m − n + c, where n and c are as above and m is the number of edges in the **graph**. The **nullity** is equal to the first Betti number of the **graph**. The sum of the **rank** and the **nullity** is the number of edges.

13for n= 4

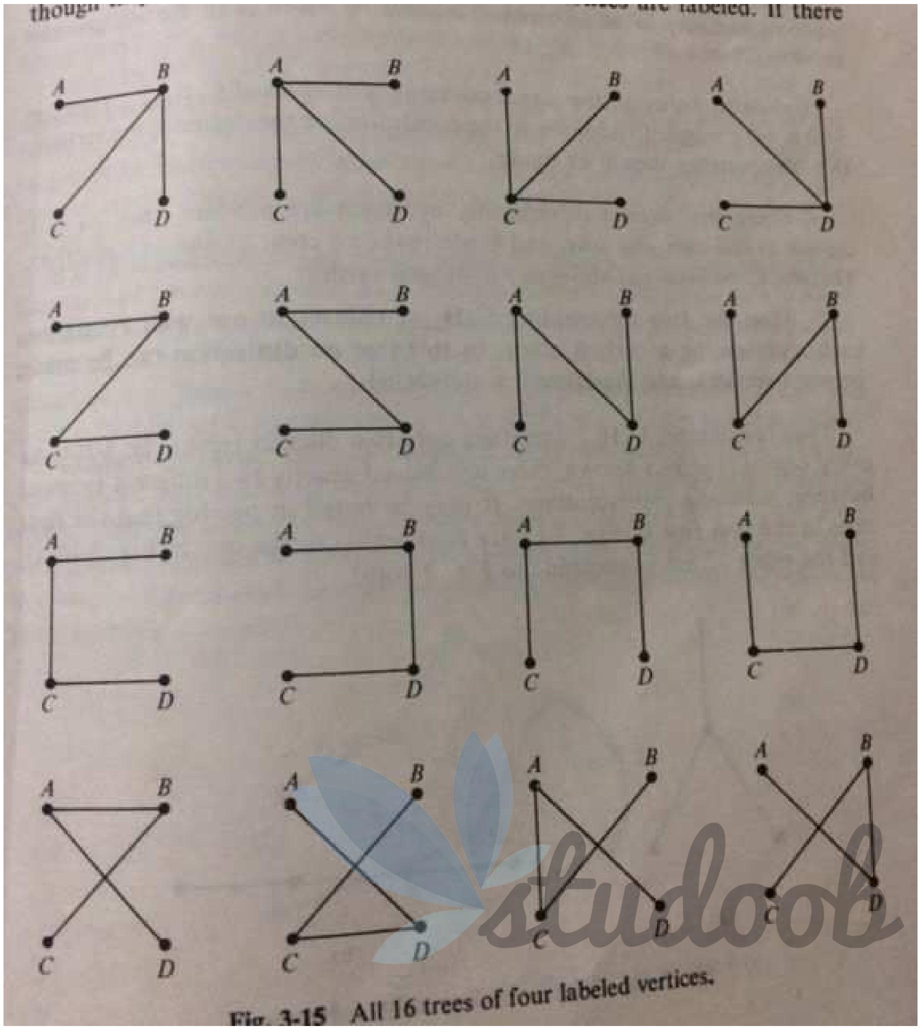
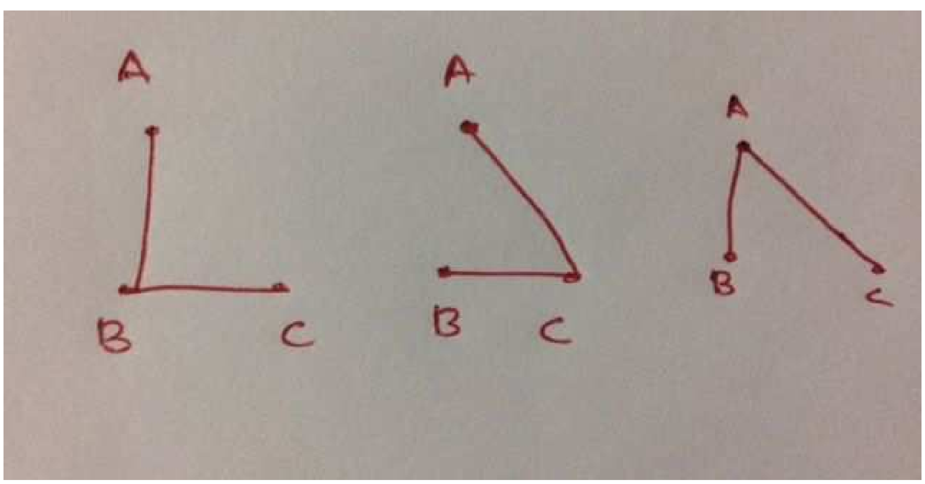
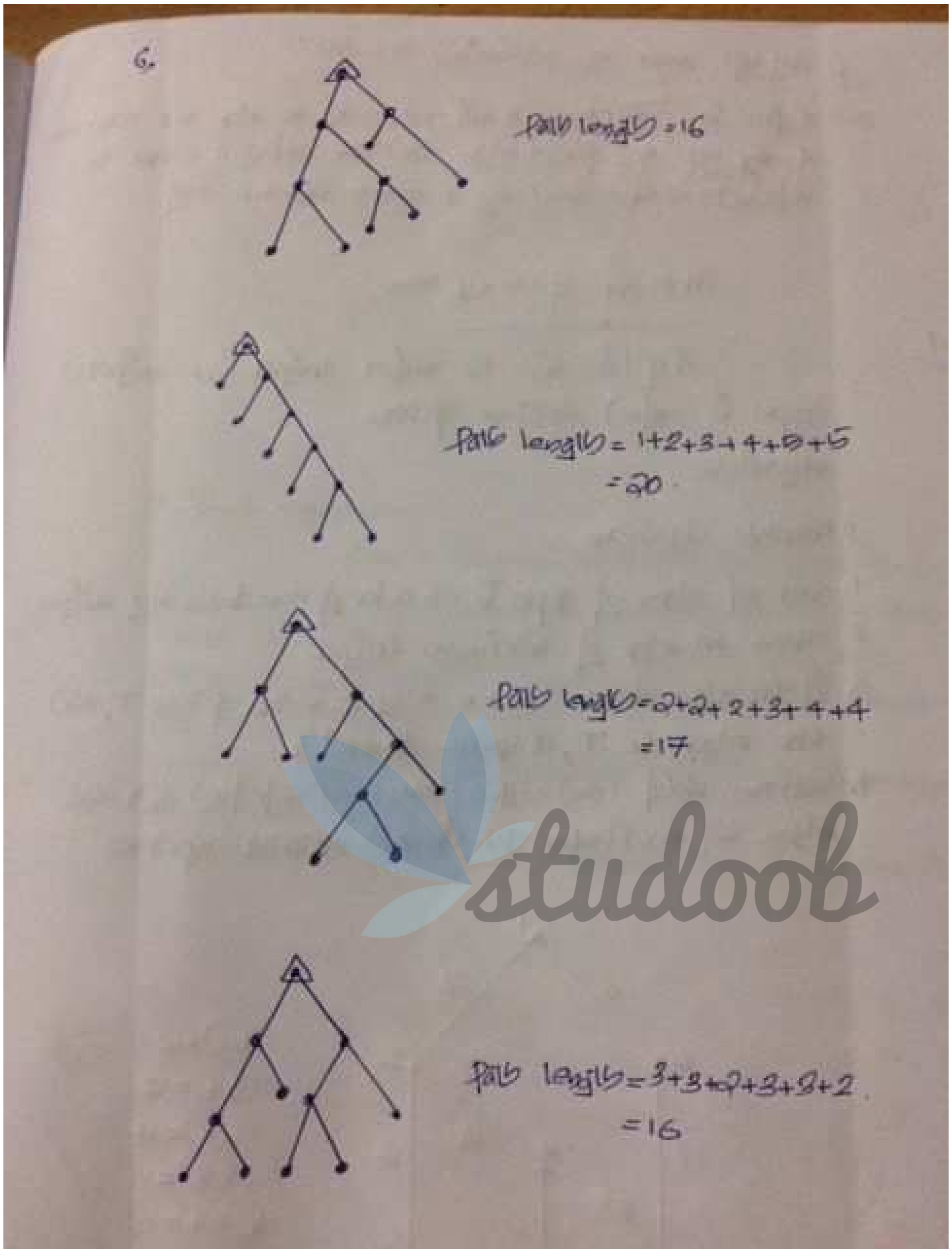


Fig. 3-15 All 16 trees of four labeled vertices.

n=3



13 b



14 a

Edge Connectivity. The minimum number of **edges** whose deletion from a graph disconnects , also called the line **connectivity**. The **edge connectivity** of a disconnected graph is 0, while that of a connected graph with a graph bridge is 1.

The vertex connectivity $\kappa(G)$ of a graph G is the minimum number of nodes whose deletion disconnects it. Vertex connectivity is sometimes called "point connectivity" or simply "connectivity."

Part E

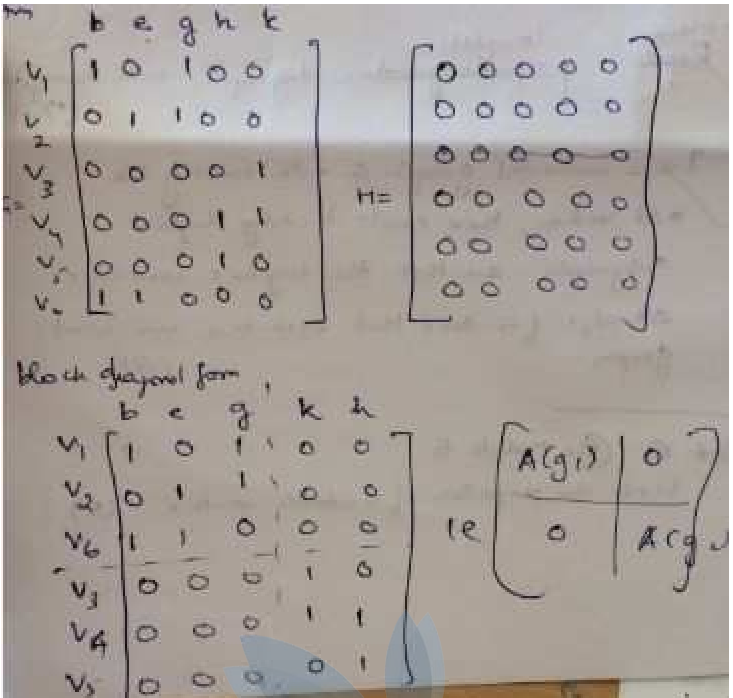
15.algm and flowchart (5+ 5)

16 algm and flowchart (5 + 5)

17.a.Incidence matrix definition

Example and its properties

b.



18. Computer representation of graph

5 forms

1. Adjacency matrix
2. Incidence matrix
3. Edge listing
4. Two linear arrays
5. Successor Listing

Explanation of each representations 5 X 2= 10

19.a.circuit matrix definition

example and properties

19.b

Proof: Consider a vertex v and a circuit Γ in the graph G. Either v is in Γ or it is not. If v is not in Γ, there is no edge in the circuit Γ that is incident on v. On the other hand, if v is in Γ, the number of those edges in the circuit Γ that are incident on v is exactly two.

With this remark in mind, consider the ith row in A and the jth row in B. Since the edges are arranged in the same order, the nonzero entries in the corresponding positions occur only if the particular edge is incident on the ith vertex and is also in the jth circuit.

If the ith vertex is not in the jth circuit, there is no such nonzero entry, and the dot product of the two rows is zero. If the ith vertex is in the jth circuit, there will be exactly two 1's in the sum of the products of individual entries. Since $1 + 1 = 0 \pmod 2$, the dot product of the two arbitrary rows—one from A and the other from B—is zero. Hence the theorem. ■