Graph theory - solutions to problem set 4

Exercises

1. In this exercise we show that the sufficient conditions for Hamiltonicity that we saw in the lecture are "tight" in some sense.

   (a) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has $\binom{n-1}{2} + 1$ edges.
   Solution: Consider the complete graph on $n-1$ vertices K_{n-1}. Add a new vertex v and connect it to a vertex $V(K_{n-1})$. This graph has $\binom{n-1}{2} + 1$ edges and it is non-Hamiltonian: every cycle uses 2 edges at each vertex, but v has only one adjacent edge.

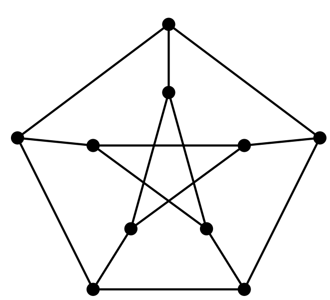
   (b) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has minimum degree $\lceil \frac{n}{2} \rceil - 1$.
   Solution: Let G_1 be a complete graph on $\lceil \frac{n}{2} \rceil$ vertices and G_2 be a complete graph on $\lfloor \frac{n}{2} \rfloor$ vertices which is disjoint from G_1. Fix a vertex $v \in V(G_1)$ and connect it to all the vertices of G_2. Let G be the resulting graph: it has minimum degree $\lceil \frac{n}{2} \rceil - 1$ and it is non-Hamiltonian, since every cycle passing through all the vertices of G has to pass through v at least twice.

   (c) For every $k, n \geq 2$, find a graph G on *at least* n vertices such that $\delta(G) = k$ but G contains no cycle longer than $k+1$. **Solution:** Let $a = \lceil n/k \rceil$, and take a disjoint copies of K_{k+1}. Then this graph has $a(k+1) > n$ vertices, each of them has degree k, but there is no cycle longer than $k+1$. We can actually find a connected such graph if join all of these cliques at one vertex, creating a star of cliques.

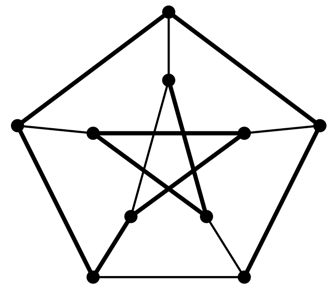
2. Check that the proof of Dirac's Theorem also proves the following statement (called Ore's theorem): If for all non-adjacent vertices u, v in an n-vertex graph G we have $d(u) + d(v) \geq n$, then G has a Hamilton cycle.

   Solution: There were two places in the proof of Dirac's Theorem where we used the condition that $\delta(G) \geq \frac{n}{2}$: To show that G is connected, and to show that there is an edge in P that is both type-1 and type-2. The proof of the first statement is very similar in this case: if u and v were in different components, then the component of u would contain at least $d(u) + 1$ vertices, and the component of v would have at least $d(v) + 1$ vertices, which would give more than n vertices in total. The second statement follows because there are $d(v_1)$ type-1 edges and $d(v_k)$ type-2 edges. But then if v_1 and v_k are not adjacent, then P has $n - 1 < d(v_1) + d(v_k)$ edges by assumption, so some edge is both type-1 and type-2 and we can continue the argument. Otherwise, we get a cycle $v_1 \dots v_k v_1$ that we can use as C for the rest of the proof.

3. The graph below is called the Petersen graph. Does it have a Hamilton path? And a Hamilton cycle?



   Solution: Here is a Hamilton path:



Let us show that there is no Hamilton cycle in the Petersen graph P. One can check that the girth of P is 5 (i.e. P has no 3-cycle or 4-cycle).

Assume there is a Hamilton cycle C in P. Since C must go through each vertex, C is actually C_{10} (i.e. the Petersen graph contains C_{10}). Then there are five more edges in P. If each of the latter edges connects two opposite vertices on C, then there is a 4-cycle. Therefore, some edge e joins vertices at distance 4 in C (why cannot it be 2 or 3?). Let e be incident to vertices A and B, and D be the opposite vertex to A in C. The vertex D must be connected to one of the neighbours of A in C (Why?), let us call it F. Then $ABDFA$ is a 4-cycle. This is a contradiction.

Problems

4. Use Ore's theorem to give a short proof of the fact that any n-vertex graph G with more than $\binom{n-1}{2}+1$ has a Hamilton cycle.

Solution: Set $|V(G)| = n$. If the graph is complete then it has a Hamilton cycle. Consider two non-adjacent vertices u, v. Remove them to get a graph $G-u-v$ with $n-2$ vertices and $|E(G)|-d(u)-d(v)$ edges. It has at most $\binom{n-2}{2}$ edges, so we have

$$\binom{n-2}{2} \geq |E(G)| - d(u) - d(v) > \binom{n-1}{2} + 1 - d(u) - d(v)$$

$$\implies \quad d(u) + d(v) > \binom{n-1}{2} - \binom{n-2}{2} + 1 = n - 1.$$

Thus the condition of Exercise 2 holds, so G has a Hamilton cycle.

5. Let G be a connected graph on n vertices with minimum degree δ. Show that

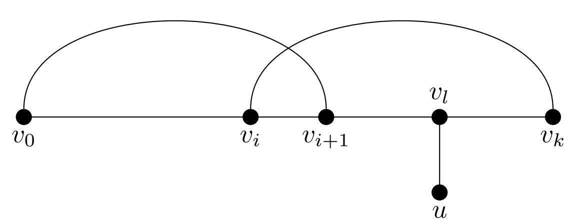
   (a) if $\delta \leq \frac{n-1}{2}$ then G contains a path of length 2δ, and

   (b) if $\delta \geq \frac{n-1}{2}$ then G contains a Hamiltonian path.

Solution: We show that G contains a path of length at least $\min\{2\delta, n-1\}$.Take a longest path $P = v_0 v_1 \cdots v_k$. If $k = n-1$, then we are done, so we can assume that there is a vertex not on P. Then by connectedness there must be a vertex u that is not on P but adjacent to a vertex of P, let's say v_l.

Observe that v_l is not v_0 or v_k, since then we could extend P to a longer path. Also, v_0 and v_k are not adjacent, otherwise $uv_l \cdots v_k v_0 \cdots v_{l-1}$ would be a longer path. And the neighbourhoods $N(v_0)$ and $N(v_k)$ are contained in $\{v_1, \ldots, v_k\}$, again since otherwise we could the extend P.

One more observation, which is a bit trickier: we cannot have anything like (not showing all the vertices on P)



since this would also give a longer path:

$$v_{l+1} \cdots v_k v_i \cdots v_0 v_{i+1} \cdots v_l u.$$

Note that something similar works when $i > l$. Thus we cannot have v_i adjacent to v_k and at the same time v_{i+1} adjacent to v_0.

The set $\{v_1, \dots, v_k\}$ contains the two sets $N(v_0)$ and $\{v_{i+1} : v_i \in N(v_k)\}$, both of which have size at least δ. Our last observation implies that these two sets are disjoint, which tells us that $k \geq 2\delta$.

6. Prove that the only acyclic tournament (with no directed cycle) is the transitive tournament.

   Solution. We have seen that every tournament has a directed Hamilton path $v_1 \dots v_n$. If the tournament is acyclic, then the edge $v_i v_j$ $(i < j)$ is directed from v_i to v_j. Indeed, otherwise $v_j v_i$ and the subpath of the Hamilton path from v_i to v_j would form a cycle. But then there is an ordering of the vertices of this tournament where every edge is directed from the smaller vertex towards the larger one, so it is isomorphic to the transitive tournament.

7. Prove that if a tournament contains a directed cycle (i.e., it is not the transitive tournament) then it contains a directed triangle (3-cycle), as well.

   Solution. Take a shortest directed cycle in the tournament $C = v_1 \dots v_k$. If $k > 3$ then C has a "diagonal": v_1 and v_3 are connected by an edge in some direction. If $v_1 \to v_3$ then $v_1 v_3 v_4 \dots v_k$ is a directed cycle of length $k-1$. If $v_3 \to v_1$ then $v_1 v_2 v_3$ is a directed cycle of length 3. Either way, there is a shorter cycle, contradicting our assumption.

8. Suppose each edge of the complete graph K_n has either a red or a blue color. Prove that this colored graph has a Hamilton path that is the union of a red path and a blue path. (We allow the case when one of the paths has length 0, i.e., the Hamilton path uses only one color.)

   Solution. Take a longest such path $P = v_k \dots v_1 w u_1 \dots u_l$, where $v_k \dots v_1 w$ is a red path and $w u_1 \dots u_l$ is a blue path. If P is not Hamiltonian, then there is a vertex x not contained in it. Look at the edge wx. If it is red, then the path $v_k \dots v_1 w x u_1 \dots u_l$ satisfies the required property, and it is longer (no matter if $x u_1$ is red or blue). If wx is blue, then $v_k \dots v_1 x w u_1 \dots u_l$ is a longer such path. This contradiction establishes that P is a Hamilton path.

   Alternatively, one can prove this statement by induction on n, using the same idea of looking at wx.