SESSION 2

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Goal: Introduce Gröbner bases and talk about the geometry of affine varieties.

§3. Gröbner Bases

The division algorithm gives us a way to write a polynomial as

$$f = a_1 f_1 + \cdots + a_s f_s + r$$

But it could be the case that $r \neq 0$ and yet $f \in \langle f_1, \ldots, f_s \rangle$.

We would like a division algorithm that can clearly tell us when

$$f \in \langle f_1, \ldots, f_s \rangle \text{ and when it doesn't.}$$

- Fix a term order $>$ in $\mathbb{K}[X_1, \ldots, X_n]$.

  $I \subseteq \mathbb{K}[X_1, \ldots, X_n]$ an ideal.

  $\Rightarrow I = \langle f_1, \ldots, f_s \rangle$ for some $f_i \in \mathbb{K}[X_1, \ldots, X_n]$.

The ideal $LT_>(I)$ is defined by

$$LT_>(I) := \langle LT_>(f) : f \in I \rangle$$

↳ "the leading term ideal of I"

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- Using Hilbert's Basis Thm,

$$LT_>(I) = \langle m_1, \dots, m_k \rangle$$

i.e. $LT_>(I)$ is finitely generated. In fact the generators of $LT_>(I)$ are monomials.

Def: A Gröbner basis $G_>$ of a polynomial ideal $I \subseteq \mathbb{K}[X_1, \dots, X_n]$ with respect to the term order $>$ is a finite set of polynomials $g_1, \dots, g_t \in I$ such that

$$\langle LT_>(g_1), \dots, LT_>(g_t) \rangle = LT_>(I).$$

Warning: It is possible to have $\langle f_1, \dots, f_s \rangle = I$ BUT $\langle LT_>(f_1), \dots, LT_>(f_s) \rangle \subsetneq I$. So, Gröbner bases in general have more terms.

Examples: (1) $J = \langle x + z, y - z \rangle$. The generators of J form a G.B. of J w.r.t. LEX.

(2) $\mathbb{Q}[x, y, z]$ $\quad x > y > z$, we use LEX order.

$p = x^2 + \frac{1}{2} y^2 z - 1$ $\qquad LT_>(p) = x^2$

$f_1 = x^2 + z^2 - 1$ $\qquad LT_>(f_1) = x^2$

$f_2 = x^2 + y^2 + (z-1)^2 - 4$ $\qquad LT_>(f_2) = x^2$

$LT_>(\langle f_1, f_2 \rangle) = \langle x^2, y^2 \rangle$

$p = 1 \cdot f_1 + 0 \cdot f_2 + \underbrace{\frac{1}{2} y^2 z - z - z^2}$ The remainder is not zero but is in I

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Proposition: Let $G = \{g_1, g_2, \dots, g_t\}$ be a Gröbner Basis for $I \subseteq \mathbb{K}[X_1, \dots, X_n]$ and let $f \in \mathbb{K}[X_1, \dots, X_n]$. Then there exists a unique $r \in \mathbb{K}[X_1, \dots, X_n]$ such that

(1) No term of r is divisible by any of $LT_>(g_1), \dots, LT_>(g_t)$.

(2) There is $g \in I$ s.t. $f = g + r$

In particular, r is the remainder of division of f by G no matter how the elements of G are listed when using the division algorithm.

→ The remainder r is called the normal form of f w.r.t. G.

§4. Affine Varieties

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- The set $\mathbb{K}^n = \{(a_1, \dots, a_n) ; a_i \in \mathbb{K}^n\}$ is called the affine n-dimensional space over $\mathbb{K}$
- The geometric objects studied in algebraic geometry are subsets of affine space defined by one or more polynomial equations.

Def: • The set of all simultaneous solutions $(a_1, \dots, a_n) \in \mathbb{K}^n$ of a system of equations

$$\begin{cases} f_1(X_1, \dots, X_n) = 0 \\ f_2(X_1, \dots, X_n) = 0 \\ \vdots \\ f_s(X_1, \dots, X_n) = 0 \end{cases}$$

is known as the affine variety defined by $f_1, \dots, f_s$ and is denoted by

$$\mathbb{V}(f_1, \dots, f_s).$$

- A subset $V \subset \mathbb{K}^n$ is said to be an affine variety if $V = \mathbb{V}(f_1, \dots, f_s)$ for some collection of $f_i \in \mathbb{K}[X_1, \dots, X_n]$

Examples of Affine varieties

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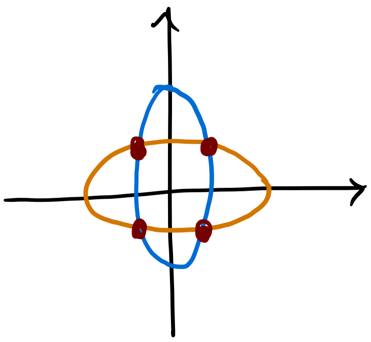
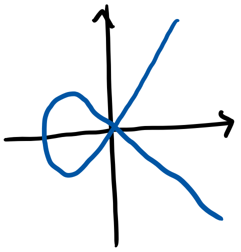
POINTS:



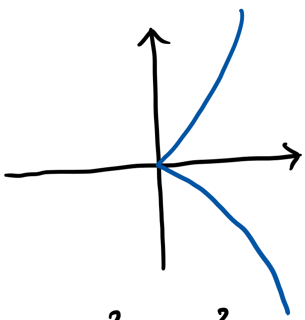
$$\begin{cases} 4x^2 + y^2 - 4 = 0 \\ x^2 + 4y^2 - 4 = 0 \end{cases}$$

→ Intersection of two curves.

CURVES:

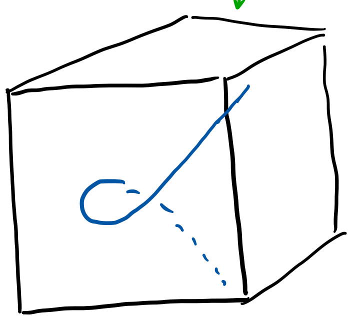


$y^2 - x^2(x+1) = 0$



$y^2 - x^3 = 0$

Intersection of two surfaces.



$$xz - y^2 = 0$$
$$x - zy = 0.$$

- If there is more than one equation in the system, the resulting variety can be considered as the intersection of other varieties.

⚠ There exist subsets of $\mathbb{K}^n$ that are not affine varieties. Example: $\mathbb{K}^n - \{0\}$.

POINTS in $\mathbb{R}^n$:

$$\underbrace{\begin{cases} x_1^2 - x_1 = 0 \\ \vdots \\ x_n^2 - x_n = 0 \end{cases}}$$

$x_i(x_i - 1) = 0$

$\Rightarrow x_i = 0, 1$

$\Rightarrow \mathbb{V}(\quad) = \{\text{all } 0/1 \text{ vectors in } \mathbb{R}^n\}$

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- An affine variety $\mathbb{V} \subseteq \mathbb{K}^n$ can be described by many different systems of equations.

- If $p = p_1 f_1 + \cdots + p_s f_s$ and $(a_1, \ldots, a_n) \in \mathbb{V}(f_1, \ldots, f_s)$, then $p(a_1, \ldots, a_n) = 0$ because each f_i evaluated at $(a_1, \ldots, a_n)$ is zero.

- (Equal ideals have equal varieties)
  If $\langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$ in $\mathbb{K}[X_1, \ldots, X_n]$ then $\mathbb{V}(f_1, \ldots, f_s) = \mathbb{V}(g_1, \ldots, g_t)$
  i.e. the variety only depends on the ideal.