SESSION #5 16.05.2018

§2. Finite dimensional algebras

- Quotient ring: $R = \mathbb{K}[x_1, \dots, x_n]$, $I \subseteq R$ an ideal.
  The quotient R/I consists of all the cosets

$$[f] = f + I = \{ f + h : h \in I \}$$ equivalence classes modulo I.

$$[f] = [g] \iff f - g \in I.$$

- Example from highschool

$$\mathbb{Z}/p\mathbb{Z} = \{ 0 + p\mathbb{Z}, 1 + p\mathbb{Z}, 2 + p\mathbb{Z}, \dots, (p-1) + p\mathbb{Z} \}$$

  → This is arithmetic modulo p.

- In R/I we do arithmetic modulo I, and for these computations we use Gröbner bases.

- Let G be a G.B for I. Recall from Chp 1 that for $f \in \mathbb{K}[x_1, \dots, x_n]$ when we divide by G, we obtain

$$f = h_1 g_1 + \cdots + h_t g_t + \underbrace{\overline{f}^G}$$ This remainder is unique.

and $\overline{f}^G$ is a linear combination of the monomials $x^\alpha \notin \langle LT(I) \rangle$

→ Also, $f \in I \iff \overline{f}^G = 0$ i.e remainder is zero.

- From $f = h_1 g_1 + \cdots + h_t g_t + \overline{f}^G$ we see that this remainder can be chosen as a representative for $[f]$. Therefore

$$\text{remainders} \longleftrightarrow \text{cosets}$$
$$\overline{f}^G \longleftrightarrow [f]$$

- $\overline{f}^G$ is a standard representative of its coset $[f] \in R/I$.

We can
- add cosets $[f] + [g] = [f+g]$
- multiply cosets $[f] \cdot [g] = [f \cdot g]$
- multiply cosets times scalars
$$c \cdot [f] = [c \cdot f] \qquad c \in \mathbb{K}$$

- Since we can add elements an multiply times scalars, R/I is a vector space over $\mathbb{K}$,
  → But R/I is also a ring so we call it a $\mathbb{K}$-algebra.

<u>Q</u>: What is the vector space structure of $A := R/I$ and how does it relate to $\mathbb{V}(I)$?
- $\dim A$?
- A basis for A?

- We will look at these for the case that $\mathbb{V}(I)$ is a finite set of points.

<u>Obsv 1</u>: The remainders in R/I are linear combinations of the monomials $\mathbf{x}^\alpha \notin \langle LT(I) \rangle$.

→ This set is linearly independent in A for otherwise $LT(I)$ would divide some $\mathbf{x}^\alpha \notin LT(I)$.

- Therefore the monomials $B = \{ \mathbf{x}^\alpha : \mathbf{x}^\alpha \notin \langle LT(I) \rangle \}$ form a basis for A.

↳ Basis monomials
Standard monomials.

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Example 1: $x>y>z$ LEX

$$\begin{cases} x^2+y+z-1=0 \\ x+y^2+z-1=0 \\ x+y+z^2-1=0 \end{cases} \xrightarrow{\text{Compute G.B.}} \begin{array}{l} g_1 = z^6-4z^4+4z^3-z^2 \\ g_2 = 2yz^2+z^4-z^2 \\ g_3 = y^2-y-z^2+z \end{array} \quad g_4 = x+y+z^2-1$$

$\Rightarrow$ $\langle LT(I)\rangle = \langle z^6, 2yz^2, y^2, x\rangle$

$\Rightarrow$ $B = \{ x^\alpha : x^\alpha \notin \langle LT(I)\rangle \} = \{1, y, yz, z, z^2, z^3, z^4, z^5\}$

$\Rightarrow$ $\dim A = 8$

Important: A is a finite dimensional vector space.

Finiteness Theorem: Let $K \subseteq \mathbb{C}$ be a field and let $I \subseteq \mathbb{K}[x_1, \dots, x_n]$ be an ideal. Then the following conditions are equivalent:

(a) The algebra A is a finite dimensional vector space over $\mathbb{K}$.

(b) The variety $\mathbb{V}(I) \subseteq \mathbb{C}^n$ is a finite set

(c) If G is a Gröbner basis for I, then for each i, $1 \leq i \leq n$ there is an $m_i \geq 0$ such that $x_i^{m_i} = LT(g)$ for some $g \in G$.

* An ideal satisfying these conditions is called a zero-dimensional ideal.

Example 1: (continued) Check that the conditions hold in this case.

Example 2: (Last time) $\begin{cases} x^2+y^2+z^2-1=0 \\ xyz-1=0 \end{cases} \xrightarrow{\text{G.B}} \begin{cases} y^4z^2+y^2z^4-y^2z^2+1 \\ x+y^3z+yz^3-yz \end{cases}$

$$\langle LT(I)\rangle = \langle y^4z^2, x\rangle$$

Consequence: I is a zero dimensional ideal $\Leftrightarrow$ there is a nonzero polynomial in $I \cap \mathbb{K}[x_i]$ for each $i=1, \dots, n$.

To compute it use a LEX order with x_i last.

Example 1: Do this to get

$I \cap \mathbb{K}[x] = \langle x^6-4x^4+4x^3-x^2 \rangle$ $\qquad$ $I \cap K[z] = z^6-4z^4+4z^3-z^2$

$I \cap \mathbb{K}[y] = \langle y^6-4y^4+4y^3-y^2 \rangle$ $\qquad$ $\rightarrow$ symmetry of the eqns!

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Def: Let $I \subseteq \mathbb{K}[X_1, \dots, X_n]$ be an ideal. The radical of I is the set

$$\sqrt{I} = \{ g \in \mathbb{K}[X_1, \dots, X_n] : g^m \in I \text{ for some } m \}$$

- An ideal I is said to be radical if $\sqrt{I} = I$.

Example: $\langle x^2, y \rangle$, $x^2 \in I \Rightarrow x \in \sqrt{I}$ in fact $\sqrt{I} = \langle x, y \rangle$.

- "The radical is the square-free part of I"

Important fact: $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$

$\rightarrow I \subseteq \sqrt{I} \Rightarrow \mathbb{V}(\sqrt{I}) = \mathbb{V}(I)$