CH20B012

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AM5450 Assignment 2

Q1) $\frac{d^2u}{dx^2} - 6x = 0$; $0 \leq x \leq 3$; $u(0) = 0$ and $u(3) = 27$

A) Galerkin method -

Strong form has d^2u/dx^2

i) First order trial function $u(x) = a_0 + a_1 x$

we cannot use this as we use the strong form for the Galerkin method. So, $u(x)$ must be of atleast order 2.

ii) Second order trial function $u(x) = a_0 + a_1 x + a_2 x^2$

$u(0) = 0 \Rightarrow a_0 = 0$

$u(3) = 27 \Rightarrow 3a_1 + 9a_2 = 27$, $a_1 = 9 - 3a_2$

Trial function should satisfy EBCs

$\therefore u(x) = (9 - 3a_2)x + a_2 x^2$

$\therefore \frac{d^2u}{dx^2} = 2a_2$

So we do not consider boundary residual

$$R = \frac{d^2u}{dx^2} - 6x = 2a_2 - 6x$$

$$\int_\Omega w_i R \, d\Omega = 0$$

For the Galerkin Method, as we have only one unknown a_2, $w_i = w = \frac{\partial u}{\partial a_2} = -3x + x^2$

$$\therefore \int_0^3 (-3x + x^2)(2a_2 - 6x) \, dx = 0$$

$$\therefore 2\int_0^3 (a_2 - 3x)(-3x + x^2) \, dx = 0$$

$$\therefore \int_0^3 (-3a_2 x + a_2 x^2 + 9x^2 - 3x^3) \, dx = 0$$

$$\therefore \left(-\frac{3a_2 x^2}{2} + \frac{a_2 x^3}{3} + \frac{9x^3}{3} - \frac{3x^4}{4} \right)_{x=0}^{x=3} = 0$$

$$\therefore -\frac{3a_2}{2} + a_2 + 9 - \frac{27}{4} = 0$$

$$\therefore -18a_2 + 12a_2 + 108 - 81 = 0$$

$$\therefore 6a_2 = 27$$

$$\therefore a_2 = \frac{27}{6} = 4.5$$

$$a_1 = 9 - 3a_2 = -4.5$$

$$\therefore \boxed{u(x) = 4.5\,(x^2 - x)}$$

iii) Third order trial function $u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$u(0) = 0 \Rightarrow a_0 = 0$

$u(3) = 27 \Rightarrow a_1 = 9 - 9a_3 - 3a_2$ } EBCs

$$\therefore u(x) = (9 - 9a_3 - 3a_2)x + a_2 x^2 + a_3 x^3 = 0$$

$$\therefore \frac{d^2u}{dx^2} = 2a_2 + 6a_3 x$$

$$R = \frac{d^2u}{dx^2} - 6x = 2a_2 + 6x(a_3 - 1)$$

$$\int_\Omega w_i R \, d\Omega = 0$$

For Galerkin method, $w_i = \frac{\partial u}{\partial a_i}$ and as we have two

unknowns a_2 and a_3,

$$w_1 = \frac{\partial u}{\partial a_2} = -3x + x^2$$

$$w_2 = \frac{\partial u}{\partial a_3} = -9x + x^3$$

$\therefore$ we have two equations

$$\int_0^3 (-3x + x^2)\,[2a_2 + 6x(a_3 - 1)]\, dx = 0 \quad \text{—— } (1)$$

$$\int_0^3 (-9x + x^3)[2a_2 + 6x(a_3-1)]\,dx = 0 \quad \text{—— ②}$$

From ①, $2\int_0^3 [-3a_2 x - 9x^2(a_3-1) + a_2 x^2 + 3x^3(a_3-1)]\,dx = 0$

$$\therefore \left[-\frac{3a_2 x^2}{2} - 3x^3(a_3-1) + \frac{a_2 x^3}{3} + \frac{3x^4}{4}(a_3-1)\right]_{x=0}^{x=3} = 0$$

$$\therefore -\frac{3a_2}{2} - 9(a_3-1) + a_2 + \frac{27}{4}(a_3-1) = 0$$

$$\therefore -6a_2 - 36(a_3-1) + 4a_2 + 27(a_3-1) = 0$$

$$\therefore 2a_2 + 9a_3 = 9 \quad \text{—— ③}$$

From ②, $2\int_0^3 [-9a_2 x - 27x^2(a_3-1) + a_2 x^3 + 3x^4(a_3-1)]\,dx = 0$

$$\therefore \left[-\frac{9a_2 x^2}{2} - 9x^3(a_3-1) + \frac{a_2 x^4}{4} + \frac{3x^5}{5}(a_3-1)\right]_{x=0}^{x=3} = 0$$

$$\therefore -\frac{9a_2}{2} - 27(a_3-1) + \frac{9a_2}{4} + \frac{81}{5}(a_3-1) = 0$$

$$\therefore -\frac{9a_2}{4} - \frac{54}{5}(a_3-1) = 0$$

$$\therefore 45a_2 + 216a_3 = 216 \quad \text{—— ④}$$

Solving ③ and ④, we get $a_2 = 0$, $a_3 = 1$

$\therefore a_1 = 9 - 9a_3 - 3a_2 = 0$

$$\therefore \boxed{u(x) = x^3}$$

B) Least Squares method

i) First order trial function $u(x) = a_0 + a_1 x$

we cannot use this as we use the strong form for the Least square method. So, $u(x)$ must be of atleast order 2.

ii) Second order trial function $u(x) = a_0 + a_1 x + a_2 x^2$

$u(0) = 0 \Rightarrow a_0 = 0$
$u(3) = 27 \Rightarrow 3a_1 + 9a_2 = 27, \quad a_1 = 9 - 3a_2$ } EBCs

$\therefore u(x) = (9 - 3a_2)x + a_2 x^2$

$\therefore \dfrac{d^2u}{dx^2} = 2a_2$

$R = \dfrac{d^2u}{dx^2} - 6x = 2a_2 - 6x$

$\displaystyle\int_\Omega w_i R \, d\Omega = 0$

For the least squares method, $w_i = \dfrac{\partial R}{\partial a_i}$. As we have

only one unknown a_2, $w = \dfrac{\partial R}{\partial a_2} = 2$

$\therefore \displaystyle\int_0^3 2(2a_2 - 6x)\,dx = 0$

$\therefore \left[2a_2 x - 3x^2\right]_{x=0}^{x=3} = 0$

$\therefore 2a_2 - 9 = 0$

$\therefore a_2 = \dfrac{9}{2} = 4.5, \quad a_1 = 9 - 3a_2 = -4.5$

$\therefore \boxed{u(x) = 4.5\,(x^2 - x)}$

iii) Third order trial function $u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$u(0) = 0 \Rightarrow a_0 = 0$
$u(3) = 27 \Rightarrow a_1 = 9 - 9a_3 - 3a_2$ } EBCs

$\therefore u(x) = (9 - 9a_3 - 3a_2)x + a_2 x^2 + a_3 x^3 = 0$

$\therefore \dfrac{d^2u}{dx^2} = 2a_2 + 6a_3 x$

$R = \dfrac{d^2u}{dx^2} - 6x = 2a_2 + 6x(a_3 - 1)$

$$\int_{\Omega} w_i R \, d\Omega = 0$$

For the Least squares method, $w_i = \frac{\partial R}{\partial a_i}$.

As we have 2 unknowns a_2 and a_3.

$w_1 = \frac{\partial R}{\partial a_2} = 2$ and $w_2 = \frac{\partial R}{\partial a_3} = 6x$

$\therefore$ We get two equations as follows -

$$\int_0^3 2 [2a_2 + 6x (a_3 - 1)] \, dx = 0 \quad \text{—— (1)}$$

$$\int_0^3 6x [2a_2 + 6x (a_3 - 1) \, dx] = 0 \quad \text{—— (2)}$$

From (1), $4 \int_0^3 [a_2 + 3x (a_3 - 1)] \, dx = 0$

$$\therefore \left[a_2 x + \frac{3x^2}{2} (a_3 - 1) \right]_{x=0}^{x=3} = 0$$

$$\therefore \quad a_2 + \frac{9}{2} (a_3 - 1) = 0$$

$$\therefore \quad 2a_2 + 9a_3 = 9 \quad \text{—— (3)}$$

From (2), $12 \int_0^3 [a_2 x + 3x^2 (a_3 - 1)] \, dx = 0$

$$\therefore \left[a_2 \frac{x^2}{2} + x^3 (a_3 - 1) \right]_{x=0}^{x=3} = 0$$

$$\therefore \quad \frac{a_2}{2} + 3 (a_3 - 1) = 0$$

$$\therefore \quad a_2 + 6a_3 = 6 \quad \text{—— (4)}$$

Solving equations (3) and (4),

$a_2 = 0$, $a_3 = 1$

$\therefore a_1 = 9 - 9a_3 - 3a_2 = 0$

$\therefore \boxed{u(x) = x^3}$

$\therefore$ We can see that the solutions for Galerkin and Least squares method are same for the same trial function

c) Analytical solution -

$$\frac{d^2u}{dx^2} - 6x = 0 \; ; \; 0 \leq x \leq 3 \; ; \; u(0) = 0 \text{ and } u(3) = 27$$

$$\frac{d^2u}{dx^2} = 6x$$

Integrating both sides, $\int \frac{d^2u}{dx^2}\, dx = \int 6x\, dx$

$$\therefore \frac{du}{dx} = 3x^2 + c_1$$

Integrating again, $\int \frac{du}{dx}\, dx = \int (3x^2 + c_1)\, dx$

$$\therefore u = x^3 + c_1 x + c_2$$

Now, $u(0) = 0 \Rightarrow c_2 = 0$

$\therefore u = x^3 + c_1 x$

$u(3) = 27 \Rightarrow 27 + 3c_1 = 27, \; c_1 = 0$

$\therefore$ The exact solution is $\boxed{u(x) = x^3}$

$\therefore$ The solution from LSM and GM considering an order 3 trial function is the same as the exact, analytical solution

For 1^{st} order polynomial in both methods, let

$u(x) = a_0 + a_1 x \; ; \; u(0) = 0$ and $u(3) = 27$

As EBCs must be satisfied, we get $\boxed{u(x) = 9x}$

Residuals for each order of polynomial is calculated as

$$\frac{d^2u}{dx^2} - 6x = e(x)$$

$$\therefore e_1(x) = \frac{d^2}{dx^2}(9x) - 6x = -6x \quad \text{— Order 1}$$

$$e_2(x) = \frac{d^2}{dx^2}[4.5(x^2 - x)] - 6x = 9 - 6x \quad \text{— Order 2}$$

$$e_3(x) = \frac{d^2}{dx^2}(x^3) - 6x = 0 \quad \text{— Order 3}$$

Q2)

MATLAB Code:

```
clc; clear all; close all; format compact; format shortg;

x0 = 0; xL = 3;
x_vec = x0:0.1:xL;

%Solutions using LSM and GM are the same for same order of trial function
%(When both essential BCs are satisfied by the trial function)

%First order trial function
u1 = 9*x_vec;
u1_prime = 9*ones(numel(x_vec),1);
r1 = -6*x_vec; %Residual

%Second order trial function
u2 = 4.5*(x_vec.^2 - x_vec);
u2_prime = 4.5*(2*x_vec - 1);
r2 = 9 - 6*x_vec;

%Third order trial function
r3 = zeros(numel(x_vec),1);
u3 = x_vec.^3;
u3_prime = 3*x_vec.^2;

%Analytical Solution
u = x_vec.^3;
u_prime = 3*x_vec.^2;

%Plotting
%u vs x
figure;
hold on
plot(x_vec,u1,'-b')
plot(x_vec,u2,'-r')
plot(x_vec,u3,'-g')
plot(x_vec,u,'xk')
legend("Order 1","Order 2","Order 3","Exact")
xlabel("x")
ylabel("u")
title("u vs x")

%u' vs x
figure;
hold on
plot(x_vec,u1_prime,'-b')
plot(x_vec,u2_prime,'-r')
plot(x_vec,u3_prime,'-g')
plot(x_vec,u_prime,'xk')
legend("Order 1","Order 2","Order 3","Exact")
xlabel("x")
ylabel("u'")
title("u' vs x")


%e(x) vs x
figure;
```

```
hold on
plot(x_vec,r1,'-r')
plot(x_vec,r2,'-b')
plot(x_vec,r3,'-g')
legend("Order 1","Order 2","Order 3")
xlabel("x")
ylabel("Residual")
title("e(x) vs x")
```

u(x) vs x:

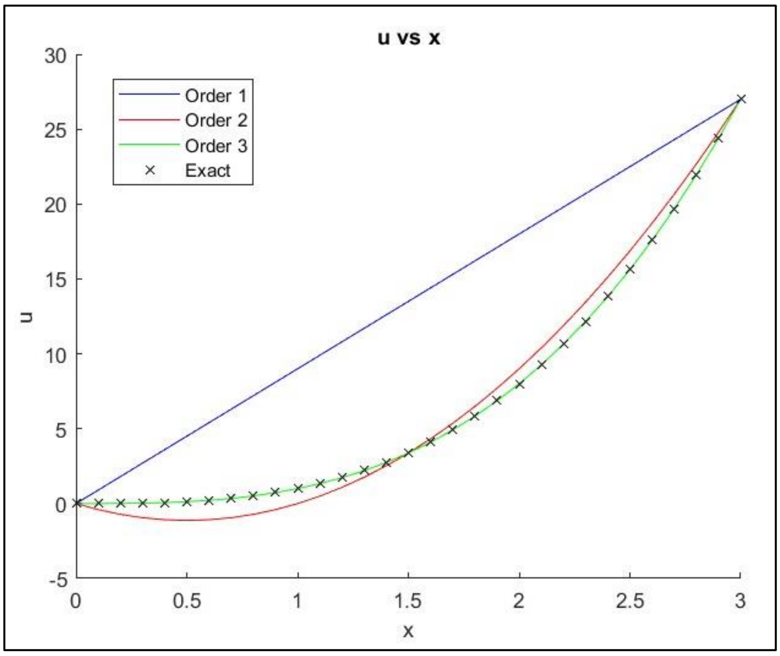

Fig 1: u(x) vs x

u’(x) vs x:

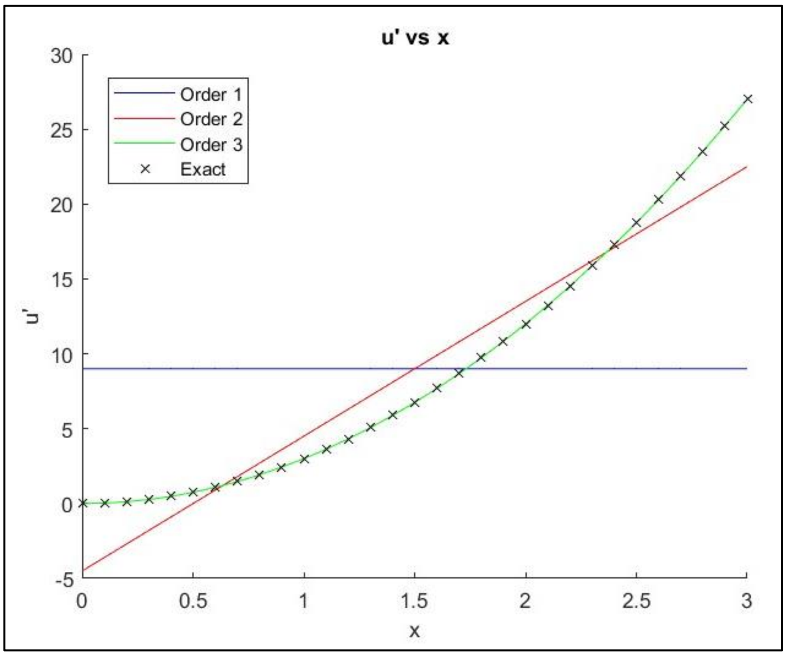

Fig 2: u’(x) vs x

e(x) vs x:

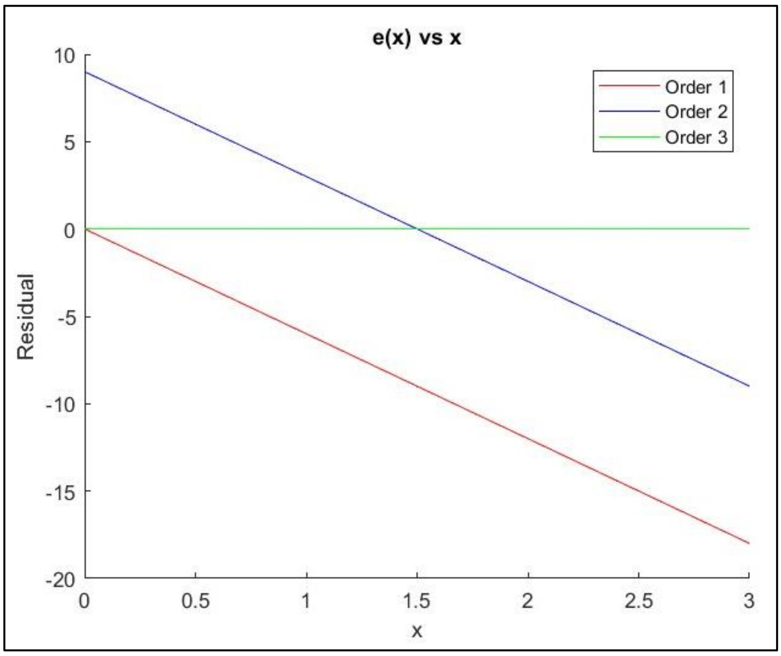

Fig 3: e(x) vs x

Q2)

a) Highest order polynomial for which the mean of the residual is below 1e-6 is **N = 14** for both the Galerkin and the Least Squares Method. MATLAB code to calculate the order of polynomial and plot the required graphs is given below.

MATLAB Code:

```
clc; clear all; close all; format compact; format shortg;

%Parameters
syms x
k = 5;
ubar = 10;
x0 = 0; xL = 1;

%True Solution
x_vec = x0:0.005:xL;
u_true = 2*sin(5*(x0:0.05:xL))/cos(5);

%Galerkin Method
figure(1); hold on;
figure(2); hold on;

Rt1_mean = 1; %Mean residual
N1 = 1;
while (Rt1_mean > 1e-6)
    N1 = N1+1;
    [u1,Rt1] = galerkin(N1, x0, xL, k, ubar);

    figure(1);
    plot(x_vec, subs(u1, x_vec), '-', 'DisplayName', num2str(N1), 'LineWidth',
1.3);

    figure(2);
```

```
    plot(x_vec, subs(Rt1, x_vec), '-', 'DisplayName', num2str(N1), 'LineWidth',
1.3);
    Rt1_mean = mean(abs(subs(Rt1,x_vec)));
    fprintf('Error for Galerkin Method = %.5e with an order %d polynomial\n',
Rt1_mean,N1);
end
fprintf('\n');

figure(1);
plot(x0:0.05:xL, u_true, 'xk', 'DisplayName', 'Exact');

figure(1); legend("show"); xlabel('x'); ylabel('u(x)'); title('u(x) vs x using
Galerkin Method')
figure(2); legend("show"); xlabel('x'); ylabel('Residual'); title('e(x) vs x using
Galerkin Method')

%Least Squares Method
figure(3); hold on;
figure(4); hold on;

Rt2_mean = 1; %Mean Residual
N2 = 1;
while (Rt2_mean > 1e-6)
    N2 = N2+1;
    [u2,Rt2] = least_squares(N2, x0, xL, k, ubar);

    figure(3);
    plot(x_vec, subs(u2, x_vec), '-', 'DisplayName', num2str(N2), 'LineWidth',
1.3);

    figure(4);
    plot(x_vec, subs(Rt2, x_vec), '-', 'DisplayName', num2str(N2), 'LineWidth',
1.3);
    Rt2_mean = mean(abs(subs(Rt2,x_vec)));
    fprintf('Error for Least Squares Method = %.5e with an order %d polynomial\n',
Rt2_mean,N2);
end

figure(3);
plot(x0:0.05:xL, u_true, 'xk', 'DisplayName', 'Exact');

figure(3); legend("show"), xlabel('x'); ylabel('u(x)'); title('u(x) vs x using
Least Squares Method')
figure(4); legend("show"); xlabel('x'); ylabel('Residual'); title('e(x) vs x using
Least Squares Method')

%Function Definitions
%Galerkin Method
function [u,Rt] = galerkin(N,x0,xL,k,ubar)
    syms x
    a = sym('a', [1,N+1]);
    b = zeros(1,N+1);

    %Trial Function
    u = poly2sym(flip(a));
    dudx = diff(u,x,1);
    du2dx2 = diff(u,x,2);

    %Essential Boundary Condition
```

```
    BC1 = subs(u,x0) == 0;
    b(1) = solve(BC1,a(1));

    %Domain Residual
    Rd = du2dx2 + k^2*subs(u,a(1),b(1));
    %Boundary Residual
    Rb = subs(dudx - ubar,x,xL); %Natural BC

    %Weighted residual equations
    eq = sym(zeros(N,1));
    for i = 2:N+1
        w = diff(u,a(i),1);
        wb = subs(diff(u,a(i),1),x,xL);
        eq(i-1) = int(w*Rd,x,x0,xL) + wb*Rb == 0;
    end

    sol = solve(eq,a(2:end));
    for i = 2:N+1
        b(i) = sol.(['a',num2str(i)]);
    end

    u = subs(u,a,b);
    %Total Residual
    Rt = subs(Rd,a,b) + subs(Rb,a,b);
end

%Least Squares Method
function [u,Rt] = least_squares(N,x0,xL,k,ubar)
    syms x
    a = sym('a', [1,N+1]);
    b = zeros(1,N+1);

    %Trial Function
    u = poly2sym(flip(a));
    dudx = diff(u,x,1);
    du2dx2 = diff(u,x,2);

    %Essential BC
    BC1 = subs(u,x0) == 0;
    b(1) = solve(BC1,a(1));

    %Domain Residual
    Rd = du2dx2 + k^2*subs(u,a(1),b(1));
    %Boundary Residual
    Rb = subs(dudx - ubar,x,xL); %Natural BC

    %Weighted residual equations
    eq = sym(zeros(N,1));
    for i = 2:N+1
        w = diff(Rd,a(i),1);
        wb = diff(Rb,a(i),1);
        eq(i-1) = int(w*Rd,x,x0,xL) + wb*Rb == 0;
    end

    sol = solve(eq,a(2:end));
    for i = 2:N+1
        b(i) = sol.(['a',num2str(i)]);
    end
```

```
    u = subs(u,a,b);
    %Total Residual
    Rt = subs(Rd,a,b) + subs(Rb,a,b);
end
```

Output:

Command Window

```
Error for Galerkin Method = 9.59309e+00 with an order 2 polynomial
Error for Galerkin Method = 8.32347e+01 with an order 3 polynomial
Error for Galerkin Method = 1.19323e+01 with an order 4 polynomial
Error for Galerkin Method = 6.68136e+00 with an order 5 polynomial
Error for Galerkin Method = 3.29204e+00 with an order 6 polynomial
Error for Galerkin Method = 4.86375e-01 with an order 7 polynomial
Error for Galerkin Method = 1.20086e-01 with an order 8 polynomial
Error for Galerkin Method = 1.45950e-02 with an order 9 polynomial
Error for Galerkin Method = 2.80809e-03 with an order 10 polynomial
Error for Galerkin Method = 2.69851e-04 with an order 11 polynomial
Error for Galerkin Method = 4.22224e-05 with an order 12 polynomial
Error for Galerkin Method = 3.35761e-06 with an order 13 polynomial
Error for Galerkin Method = 4.44696e-07 with an order 14 polynomial

Error for Least Squares Method = 8.53272e+00 with an order 2 polynomial
Error for Least Squares Method = 9.03825e+00 with an order 3 polynomial
Error for Least Squares Method = 9.04440e+00 with an order 4 polynomial
Error for Least Squares Method = 5.45767e+00 with an order 5 polynomial
Error for Least Squares Method = 2.53489e+00 with an order 6 polynomial
Error for Least Squares Method = 4.58243e-01 with an order 7 polynomial
Error for Least Squares Method = 1.13903e-01 with an order 8 polynomial
Error for Least Squares Method = 1.38828e-02 with an order 9 polynomial
Error for Least Squares Method = 2.63186e-03 with an order 10 polynomial
Error for Least Squares Method = 2.52799e-04 with an order 11 polynomial
Error for Least Squares Method = 3.89340e-05 with an order 12 polynomial
Error for Least Squares Method = 3.08778e-06 with an order 13 polynomial
Error for Least Squares Method = 4.02083e-07 with an order 14 polynomial
fx >>
```

Galerkin Method:

u(x) vs x:

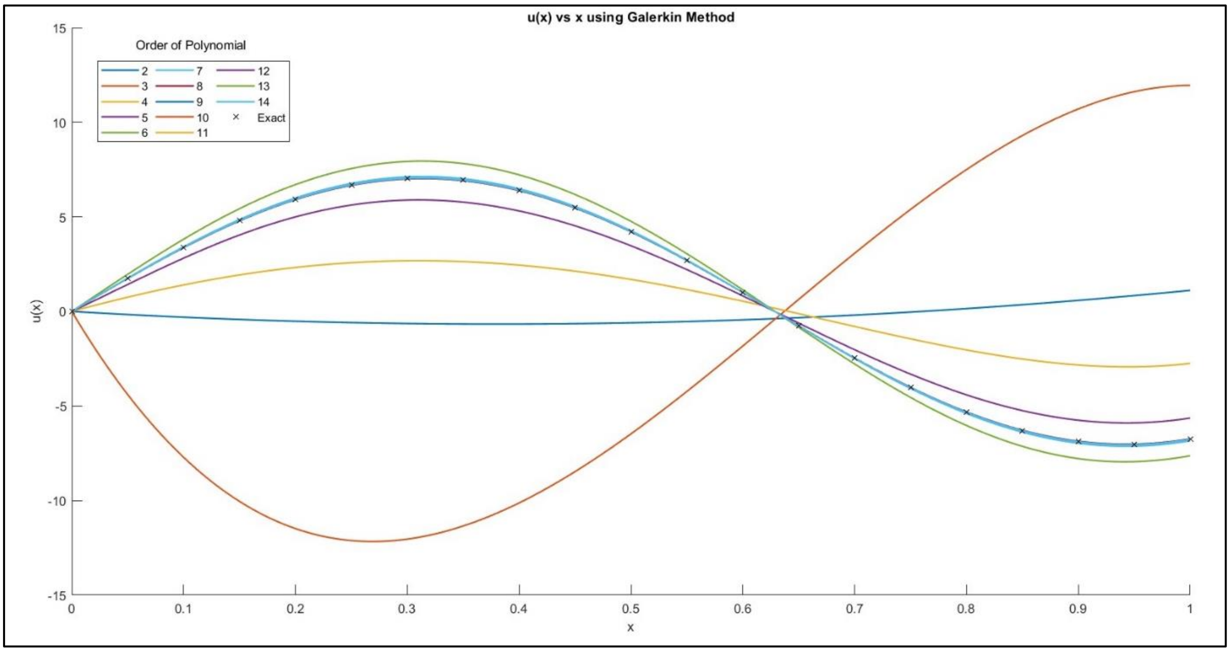

Fig 4: u(x) vs x for different order polynomials using Galerkin Method

e(x) vs x:

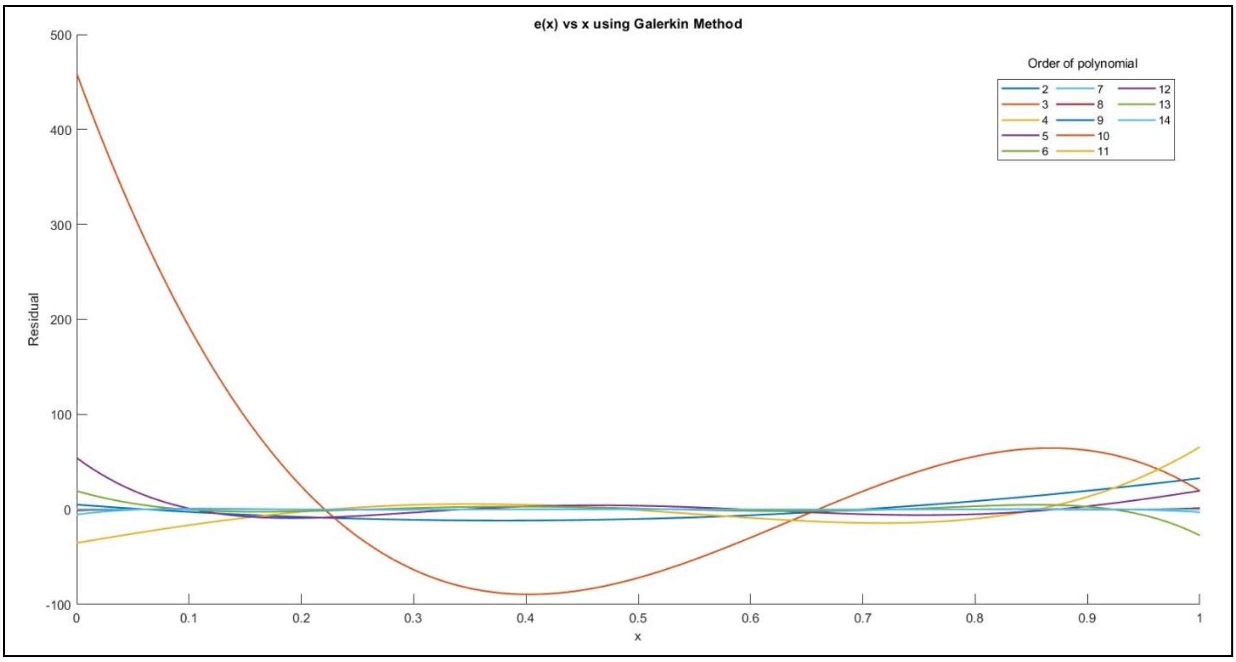

Fig 5: e(x) vs x for different order polynomials using Galerkin Method

Least Squares Method:

u(x) vs x:

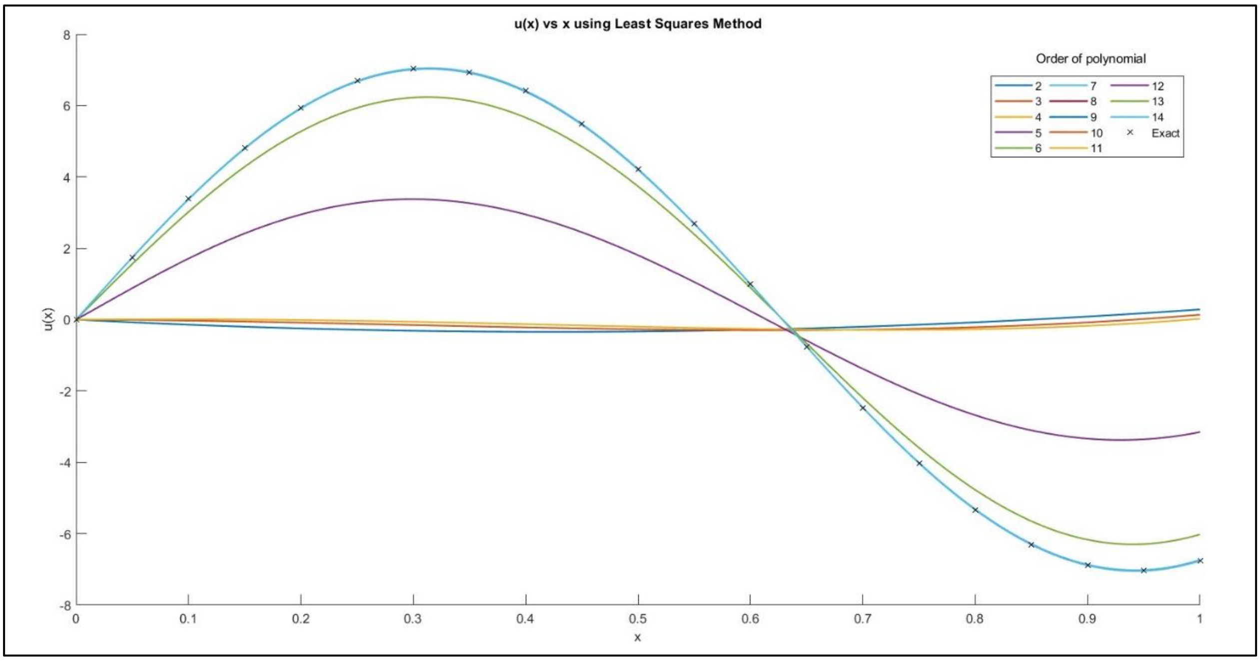

Fig 6: u(x) vs x for different order polynomials using Least Squares Method

e(x) vs x:

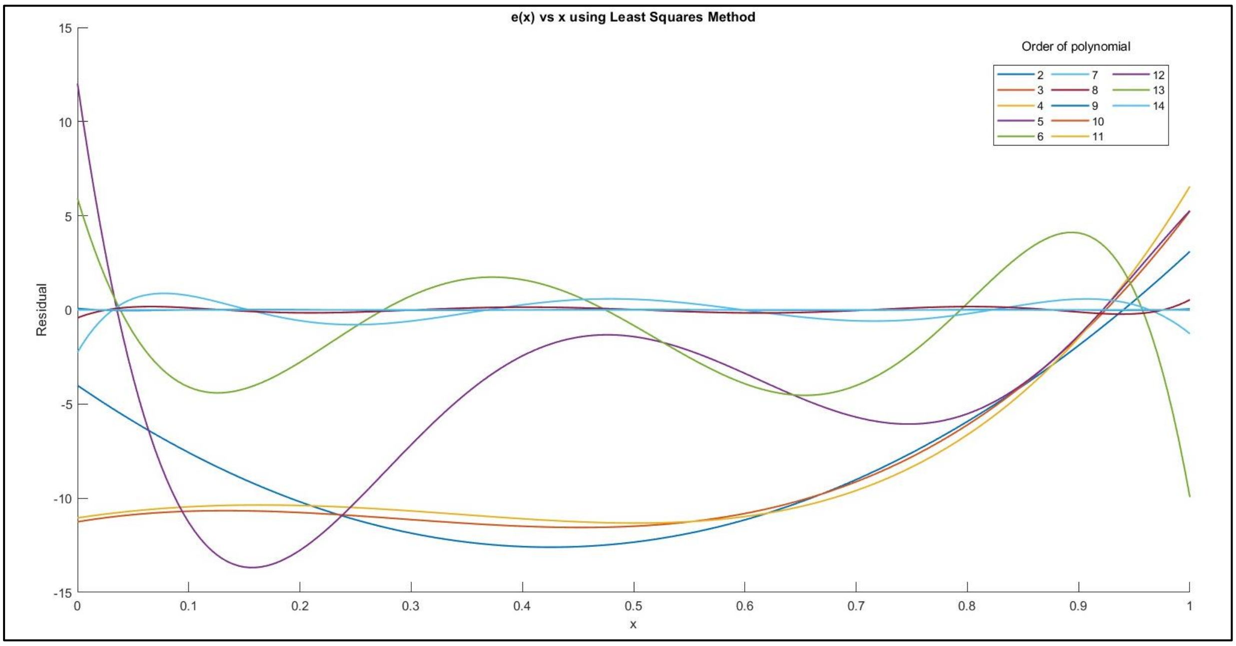

Fig 7: e(x) vs x for different order polynomials using Least Squares Method