CH20B012

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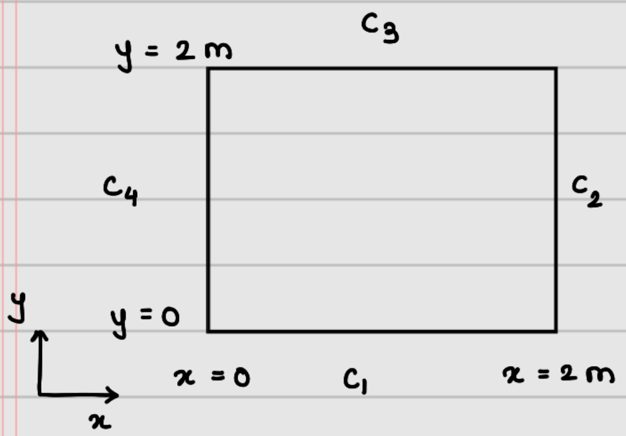
AM5450 Assignment 3

Q1)



$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + 2\frac{\partial^4 u}{\partial y^4} + \frac{q}{D} = 0$$

$$D = \frac{Et^3}{12(1-\nu^2)}$$

$q = 50 \ N/m^2$, $t = 1 \ mm$

For a simply supported plate, the boundary conditions are given as,

At C_1, $u(x,0) = 0$ and $\frac{\partial^2 u}{\partial y^2}(x,0) = 0$

At C_3, $u(x,2) = 0$ and $\frac{\partial^2 u}{\partial y^2}(x,2) = 0$

At C_2, $u(2,y) = 0$ and $\frac{\partial^2 u}{\partial x^2}(2,y) = 0$

At C_4, $u(0,y) = 0$ and $\frac{\partial^2 u}{\partial x^2}(0,y) = 0$

$\downarrow$ EBCs $\qquad\qquad$ NBCs

① Modified Galerkin method -

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} + \frac{q}{D} = 0$$

Multiplying by the weight function w_i and integrating over the domain,

$$\iint \left(\frac{\partial^4 u}{\partial x^4} + \frac{2\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} + \frac{q}{D} \right) w_i \, dx \, dy = 0$$

Consider the term $\iint \frac{\partial^4 u}{\partial x^4} w_i \, dx \, dy$

Using Green's theorem,

$$\iint \frac{\partial^4 u}{\partial x^4} w_i \, dx\, dy = -\iint \frac{\partial^3 u}{\partial x^3} \frac{\partial w_i}{\partial x} dx\, dy + \int \frac{\partial^3 u}{\partial x^3} w_i \, n_x \, ds$$

Applying Green's theorem to the first term again,

$$\therefore \iint \frac{\partial^4 u}{\partial x^4} w_i \, dx\, dy = \iint \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 w_i}{\partial x^2} dx\, dy + \int \left(\frac{\partial^3 u}{\partial x^3} w_i - \frac{\partial^2 u}{\partial x^2} \frac{\partial w_i}{\partial x} \right) n_x \, ds$$

For the second (line integral) term, we get only C_2 and C_4 as they also have unit normal in x direction

$$\therefore \int_{C_1, C_4} \left(\frac{\partial^3 u}{\partial x^3} w_i - \frac{\partial^2 u}{\partial x^2} \frac{\partial w_i}{\partial x} \right) ds = 0$$

From the NBC, we know that $\frac{\partial^2 u}{\partial x^2} = 0$ on C_1 and C_4

and based on the EBC, w_i should also be 0 on C_1, C_4

Hence the entire line integral term goes to 0.

$$\therefore \iint \frac{\partial^4 u}{\partial x^4} w_i \, dx\, dy = \iint \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 w_i}{\partial x^2} dx\, dy$$

Similarly, for the term $\iint \frac{\partial^4 u}{\partial y^4} w_i \, dx\, dy$ we can use

Green's theorem twice to get

$$\iint \frac{\partial^4 u}{\partial y^4} w_i \, dx\, dy = \iint \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 w_i}{\partial y^2} dx\, dy + \int \left(\frac{\partial^3 u}{\partial y^3} w_i - \frac{\partial^2 u}{\partial y^2} \frac{\partial w_i}{\partial y} \right) n_y \, ds$$

Again for the line integral term, only C_1 and C_3 will be used as they are normal to n_y.

From the NBC, we know that $\frac{\partial^2 u}{\partial y^2} = 0$ on C_1 and C_3

and from the EBC, w_i should be zero on C_1 and C_3

Hence, we get

$$\iint \frac{\partial^4 u}{\partial y^4} w_i \, dx\, dy = \iint \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 w_i}{\partial y^2} \, dx\, dy$$

For the mixed derivative term,

$$\iint \frac{\partial^4 u}{\partial x^2 \partial y^2} w_i \, dx\, dy = -\iint \frac{\partial^3 u}{\partial x^2 \partial y} \frac{\partial w_i}{\partial y} \, dx\, dy$$

$$+ \int \frac{\partial^3 u}{\partial^2 x\, \partial y} w_i \, n_y \, ds = \iint \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 w_i}{\partial x \partial y} \, dx\, dy +$$

$$\int \left(\frac{\partial^3 u}{\partial^2 x\, \partial y} w_i \, n_x - \frac{\partial^2 u}{\partial x \partial y} \frac{\partial w_i}{\partial x} n_y \right) \cdot ds$$

Again, using the BCs, the line integral term goes to 0

$$\therefore \iint \frac{\partial^4 u}{\partial x^2 \partial y^2} w_i \, dx\, dy = \iint \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 w_i}{\partial x \partial y} \, dx\, dy$$

For the remaining term,

$\iint \frac{q}{D} w_i \, dx\, dy$, we do not use the Green's theorem

and use it as it is.

Combining all the terms, we get the weak form as

$$\iint \left\{ \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 w_i}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 w_i}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 w_i}{\partial y^2} + \frac{q}{D} w_i \right\} dx\, dy = 0$$

② Variational approach -

$$\frac{\partial^4 u}{\partial x^4} + \frac{2 \partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} + \frac{q}{D} - 0$$

$$\therefore \iint \left(\frac{\partial^4 u}{\partial x^4} + \frac{2 \partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} + \frac{q}{D} \right) \delta u \, dx\, dy = 0$$

Again using Green's theorem and the appropriate BCs, we get a form similar to MGM which is given as

$$\iint \left\{ \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 \delta u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 \delta u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 \delta u}{\partial y^2} + \frac{q}{D}\delta u \right\} dx\,dy = 0$$

we know that $\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x} = -\frac{1}{2}\delta\left(\frac{\partial u}{\partial x}\right)^2$

$\therefore$ similarly we can get,

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 \delta u}{\partial x^2} = -\frac{1}{2}\delta\left(\frac{\partial^2 u}{\partial x^2}\right)^2$$

$$\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 \delta u}{\partial y^2} = -\frac{1}{2}\delta\left(\frac{\partial^2 u}{\partial y^2}\right)^2$$

$$\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 \delta u}{\partial x \partial y} = -\frac{1}{2}\delta\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2$$

Substituting the above, we get,

$$\iint \left\{ -\frac{\delta}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - \delta\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 - \frac{\delta}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)^2 + \frac{q}{D}\delta u \right\} dx\,dy = 0$$

$\therefore$ The weak form can be written as $\delta[I(u)] = 0$ where,

$$I(u) = \iint \left\{ \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)^2 - \frac{q}{D}u \right\}$$

Now, to solve using this the weak form, the trial solution must satisfy the EBCs.

Consider $u(x) = xy(x-2)(y-2) \times f(x,y)$

This trial solution satisfies all the EBCs given above

$[\, u(0,y) = u(2,y) = u(x,0) = u(x,2) = 0 \,]$

As the governing equation is of order 4, let $f(x,y)$ be a general 2^{nd} order function so that u is a 4^{th} order polynomial. $\therefore$ A suitable trial function is,

$\therefore\ u(x) = xy(x^2-4)(y^2-4)(c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 y^2 + c_6 xy)$

Q1)

MATLAB Code using Modified Galerkin Method:

```
clc; clear all; close all; format compact; format shortg;
syms x y c1 c2 c3 c4 c5 c6;

%Constants
E = 80*10^9; %Young's modulus
nu = 0.3; %Poisson Ratio
%E and nu for aluminium are taken from online data as they were not specified in
the problem
q = 50; %Load
t = 10^-3; %Thickness
D = E*t^3/(12*(1-nu^2)); %Bending Stiffness

%Dimensions
x0 = 0; xL = 2; y0 = 0; yL = 2;

%Trial solution (Assumed in a way that EBCs are satisfied)
u = x*y*(x^2-4)*(y^2-4)*(c1 + c2*x + c3*y + c4*x^2 + c5*y^2 + c6*x*y);
eqns = sym(zeros(6, 1));

for i = 1:6
    %Weights for MGM
    weight = diff(u, eval(['c' num2str(i)]), 1);

    %Weighted residual equation for MGM
    term1 = int(int(diff(u,x,2) * diff(weight,x,2), x,x0,xL), y,y0,yL);
    term2 = 2*int(int(diff(diff(u,x,1),y,1) * diff(diff(weight,x,1),y,1),
x,x0,xL), y,y0,yL);
    term3 = int(int(diff(u, y, 2) * diff(weight, y, 2), y, 0, 2), x, 0, 2);
    term4 = int(int(weight*(q/D), x,x0,xL), y,y0,yL);
    eqns(i) = term1 + term2 + term3 + term4;
end

%Solving the system of equations
sol = solve(eqns, [c1 c2 c3 c4 c5 c6]);
u = subs(u, [c1 c2 c3 c4 c5 c6], [sol.c1 sol.c2 sol.c3 sol.c4 sol.c5 sol.c6]);
disp(vpa(u,4)); %Displaying the solution

%Plotting the solution
figure;
fsurf(u, [0 2 0 2]);
xlabel('x');
ylabel('y');
zlabel('u(x,y)');
title('u(x,y) vs x,y in 3D');
colormap(jet); colorbar;
grid on;


figure;
fcontour(u, [0 2 0 2], 'LineWidth', 2);
xlabel('x');
ylabel('y');
title('Contour plot for u(x,y) vs x,y');
colormap(jet); colorbar;
grid on;
```

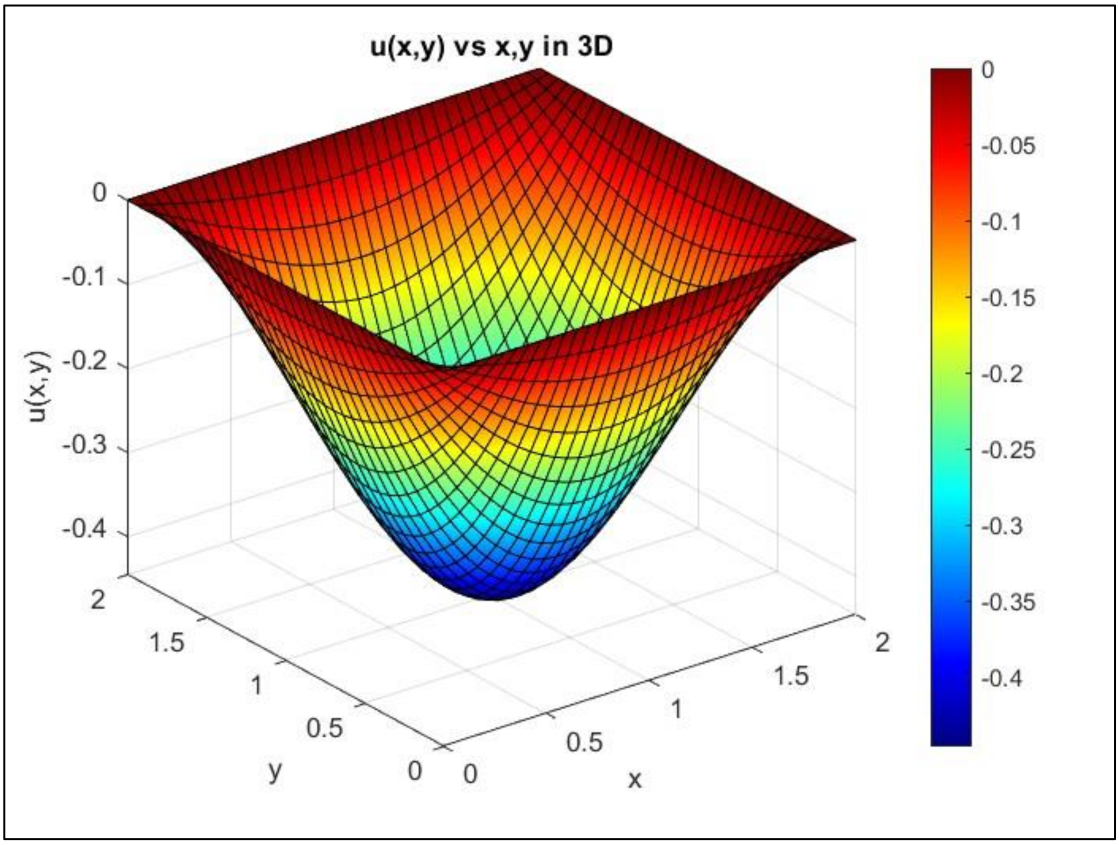


Fig 1: Surface plot of the displacement of the plate

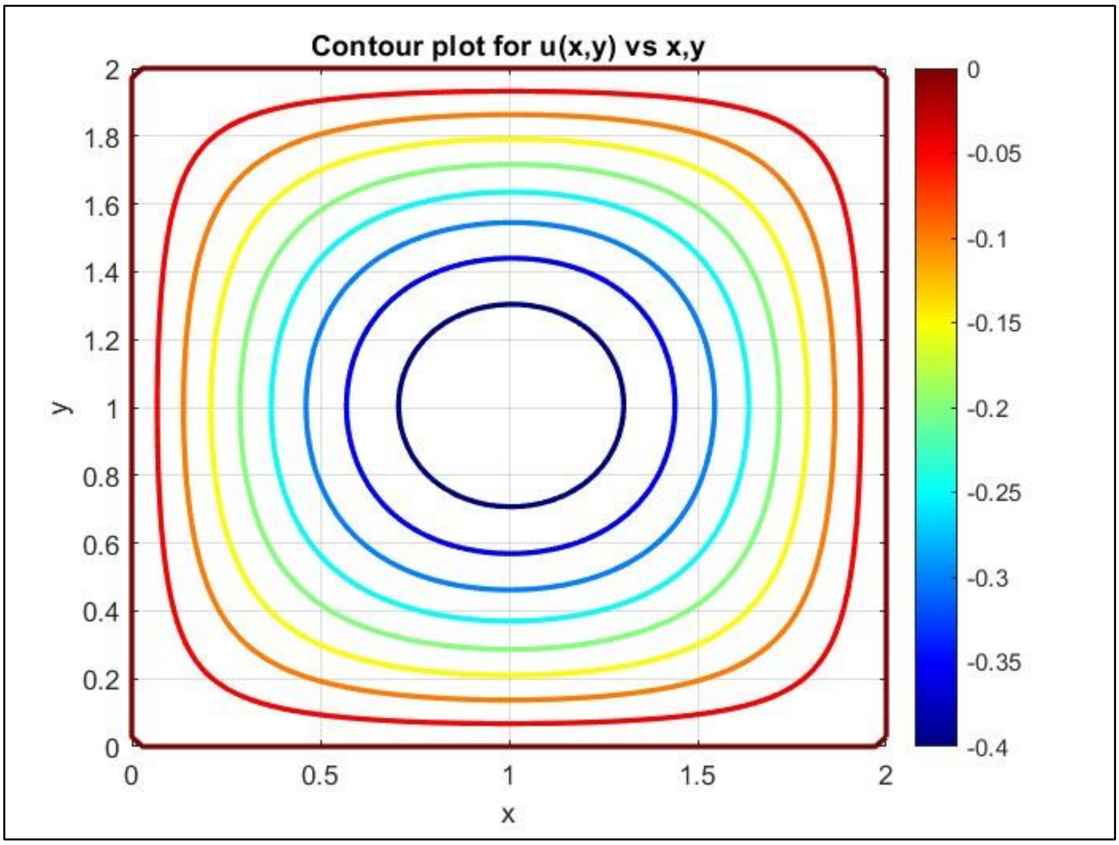


Fig 2: Contour plot of the displacement of the plate

Q2)

a) $$\frac{d^2u}{dx^2} + f(x) = 0$$

$$\int_{x_0}^{x_L} \left[\frac{d^2u}{dx^2} + f(x)\right] \delta u \, dx = 0$$

$$\therefore \int_{x_0}^{x_L} \frac{d^2u}{dx^2} \delta u \, dx + \int_{x_0}^{x_L} f(x)\, \delta u \, dx = 0$$

$$\therefore -\int_{x_0}^{x_L} \frac{du}{dx} \frac{d\delta u}{dx} dx + \left(\delta u \frac{du}{dx}\right)_{x_0}^{x_L} + \int_{x_0}^{x_L} f(x)\, \delta u \, dx = 0$$

$$\therefore -\int_{x_0}^{x_L} \frac{1}{2} \delta \left(\frac{du}{dx}\right)^2 dx + \delta u(x_L) \frac{du}{dx}(x_L) - \delta u(x_0) \frac{du}{dx}(x_0)$$

$$+ \int_{x_0}^{x_L} f(x)\, \delta u \, dx = 0$$

To further simplify this, we need f(x) and the BCs.
But in general, we can write

$$\delta \left\{ \int_{x_0}^{x_L} -\frac{1}{2} \left(\frac{du}{dx}\right)^2 dx + u \frac{du}{dx}(x_L) - u \frac{du}{dx}(x_0) + \int_{x_0}^{x_L} f(x)\, u \, dx \right\} = 0$$

$$\therefore \delta [I(u)] = 0$$

$$I(u) = \int_{x_0}^{x_L} -\frac{1}{2} \left(\frac{du}{dx}\right)^2 dx + u \frac{du}{dx}(x_L) - u \frac{du}{dx}(x_0) + \int_{x_0}^{x_L} f(x)\, u \, dx$$

b) $$\frac{d^3u}{dx^3} + 2u = x^2$$

$$\therefore \int_{x_0}^{x_L} \left[\frac{d^3u}{dx^3} + 2u - x^2\right] \delta u \, dx = 0$$

$$\therefore \int_{x_0}^{x_L} \frac{d^3u}{dx^2} \delta u + \int_{x_0}^{x_L} 2u\, \delta u \, dx - \int_{x_0}^{x_L} x^2 \delta u \, dx = 0$$

$$\therefore -\int_{x_0}^{x_L} \frac{d^2u}{dx^2} \frac{d\delta u}{dx} + \left(\delta u \frac{d^2u}{dx^2}\right)_{x_0}^{x_L} - \int_{x_0}^{x_L} \delta u^2 \, dx -$$

$$\int_{x_0}^{x_L} x^2 \delta u \, dx = 0$$

For the first term, again integrating by parts will increase the order of derivative of δu. So we should not use this. But there is no way to simplify the term $\frac{d^2u}{dx^2} \frac{d\delta u}{dx}$ to seperate out δ.

$$\therefore \; - \int_{x_0}^{x_L} \frac{d^2u}{dx^2} \frac{d\delta u}{dx} \, dx + \delta \left\{ u \frac{d^2u}{dx^2}(x_L) - u \frac{d^2u}{dx^2}(x_0) - \int_{x_0}^{x_L} u^2 dx + \int_{x_0}^{x_L} x^2 u \, dx \right\} = 0$$

As term 1 cannot be further simplified, the given eqn. does not have an equivalent function form of the type $\delta[I(u)] = 0$