深蓝学院
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计算机视觉课程

——从运动到结构

主讲人 隋博士

课程内容

✓ 针孔相机模型

- ✓ 针孔相机模型
- ✓ 径向畸变

✓ 2D-2D:对极几何

- ✓ 对极约束
- ✓ 本质/单应矩阵
- ✓ 直接线性变换法

✓ 3D-2D: PnP问题

- ✓ 三角量测
- ✓ 直接线性变换法
- ✓ 非线性优化

✓ 捆绑调整Bundle Adjustment

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  - ✓ 直接线性变换法
- ✓ **3D-2D: PnP问题**
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  - ✓ 非线性优化
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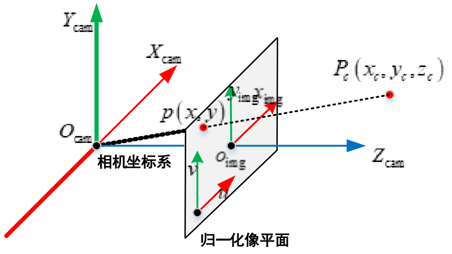
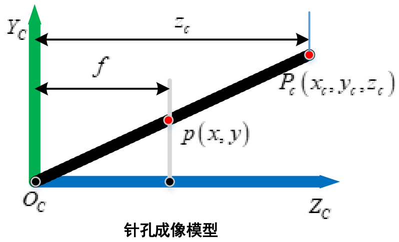
针孔相机模型-内参数矩阵

相机坐标系到图像坐标系



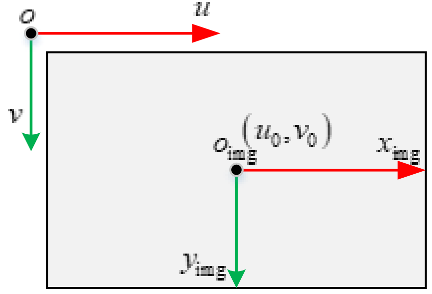


$$x = f\frac{x_c}{z_c}$$

$$y = f\frac{y_c}{z_c}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \frac{1}{z_c}\begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix}$$

图像坐标系到像素坐标系



$$u = \underline{\alpha f}\frac{x_c}{z_c} + u_0$$

$$v = \underline{\beta f}\frac{y_c}{z_c} + v_0$$

毫米到像素

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \frac{1}{z_c}\boxed{\begin{pmatrix} f_\alpha & 0 & u_0 \\ 0 & f_\beta & v_0 \\ 0 & 0 & 1 \end{pmatrix}}\begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} = \frac{1}{z_c}\boldsymbol{K}\begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix}$$

内参数矩阵

通常情况下 $f = f_\alpha = f_\beta$

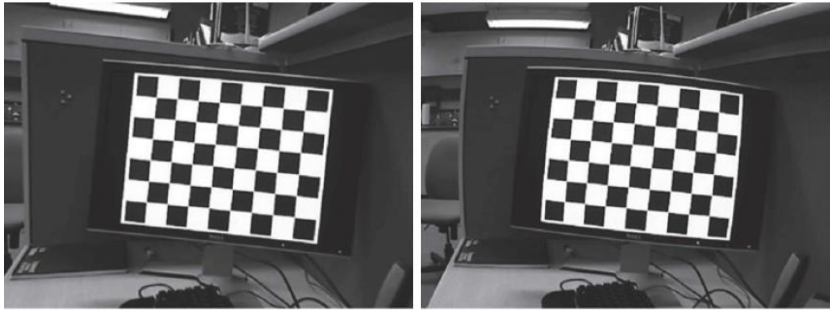
针孔相机模型–径向畸变

成因：透镜不能完全满足针孔模型假设

$$\begin{bmatrix}\tilde{x}_c\\ \tilde{y}_c\end{bmatrix}=\frac{1}{z_c}\left(1+k_1r_c^2+k_2r_c^4\right)\begin{bmatrix}x_c\\ y_c\end{bmatrix},r_c^2=x_c^2+y_c^2$$

$$\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix}=\begin{bmatrix}f\tilde{x}_c\\ f\tilde{y}_c\end{bmatrix}+\begin{bmatrix}u_0\\ v_0\end{bmatrix}=\left(1+k_1r_c^2+k_2r_c^4\right)\begin{bmatrix}f\dfrac{x_c}{z_c}\\ f\dfrac{y_c}{z_c}\end{bmatrix}+\begin{bmatrix}u_0\\ v_0\end{bmatrix}$$

$$=\left(1+k_1r_c^2+k_2r_c^4\right)\begin{bmatrix}u-u_0\\ v-v_0\end{bmatrix}+\begin{bmatrix}u_0\\ v_0\end{bmatrix}$$



由图像中心往外畸变程度越来越大

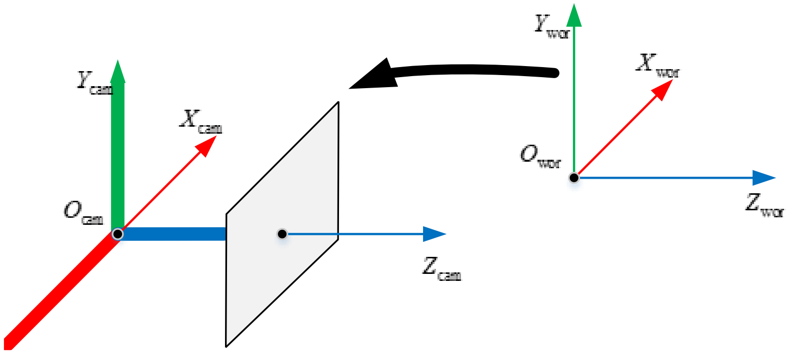
$$\begin{bmatrix}\left(u-u_0\right)r_c^2 & \left(u-u_0\right)r_c^4\\ \left(v-v_0\right)r_c^2 & \left(v-v_0\right)r_c^4\end{bmatrix}\begin{bmatrix}k_1\\ k_2\end{bmatrix}=\begin{bmatrix}\tilde{u}-u\\ \tilde{v}-v\end{bmatrix}$$

$\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix}$ 是观察到（带畸变）的像素点坐标

$\begin{bmatrix}u\\ v\end{bmatrix}$ 是理理想（无畸变）的像素点坐标

针孔相机模型–外参数矩阵

世界坐标系到相机坐标系



$$X_c = RX_w + T$$

$$\begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{T} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix}$$

透视矩阵

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \frac{1}{z_c} \boldsymbol{K} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{T} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix} = \frac{1}{z_c} \boldsymbol{K} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{T} \end{bmatrix} \begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix}$$

$$\boldsymbol{P} = \boldsymbol{K} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{T} \end{bmatrix}$$

6个外参数，5个内参数 $f_\alpha = f_\beta$， k_1, k_2, u_0, v_0

相机标定： **确定内参数**

姿态估计： **确定外参数**

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2D-2D:对极几何-对极约束

对极约束

$$\mathbf{x}_2^T \boldsymbol{F} \mathbf{x}_1 = 0 \qquad \hat{\mathbf{x}}_2^T \boldsymbol{E} \hat{\mathbf{x}}_1 = 0$$

其中 $\boldsymbol{E} = \boldsymbol{K}_2^{-T} \boldsymbol{F} \boldsymbol{K}_1 \quad \hat{\mathbf{x}}_1 = \boldsymbol{K}_1^{-1} \mathbf{x}_1 \quad \hat{\mathbf{x}}_2 = \boldsymbol{K}_2^{-1} \mathbf{x}_2$

公式推导

$$\boldsymbol{P}_1 = \boldsymbol{K}_1 [\boldsymbol{I}, \quad \mathbf{0}] \qquad \boldsymbol{P}_2 = \boldsymbol{K}_2 [\boldsymbol{R}, \quad \boldsymbol{t}]$$

$$d_1 \mathbf{x}_1 = \boldsymbol{K}_1 \boldsymbol{X} \Rightarrow d_1 \boldsymbol{K}_1^{-1} \mathbf{x}_1 = \boldsymbol{X} = d_1 \hat{\mathbf{x}}_1$$

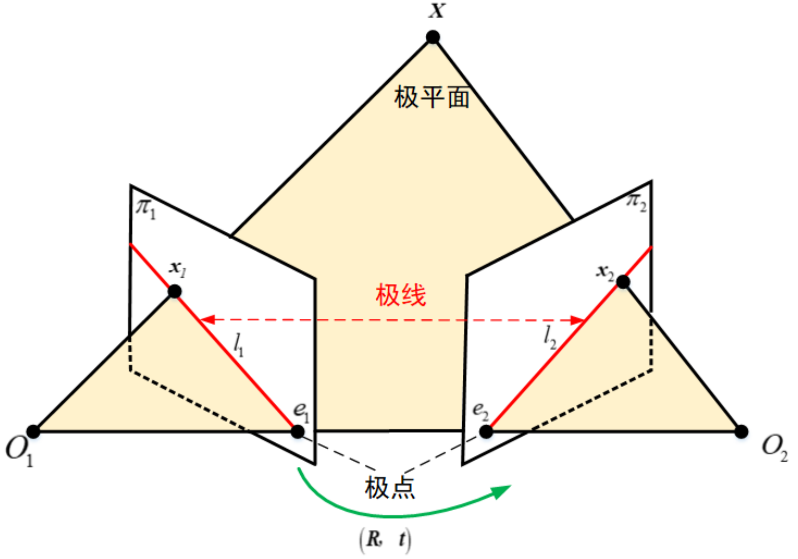
$$d_2 \mathbf{x}_2 = \boldsymbol{K}_2 (\boldsymbol{R}\boldsymbol{X} + \boldsymbol{t}) \Rightarrow d_2 \boldsymbol{K}_2^{-1} \mathbf{x}_2 = \boldsymbol{R}\boldsymbol{X} + \boldsymbol{t} = d_1 \boldsymbol{R} \hat{\mathbf{x}}_1 + \boldsymbol{t}$$

$$\Rightarrow d_2 [\boldsymbol{t}]_\times \hat{\mathbf{x}}_2 = d_1 [\boldsymbol{t}]_\times \boldsymbol{R} \hat{\mathbf{x}}_1 + [\boldsymbol{t}]_\times \boldsymbol{t}$$

$$\Rightarrow d_2 \hat{\mathbf{x}}_2^T [\boldsymbol{t}]_\times \hat{\mathbf{x}}_2 = d_1 \hat{\mathbf{x}}_2^T [\boldsymbol{t}]_\times \boldsymbol{R} \hat{\mathbf{x}}_1 = 0$$

$$\hat{\mathbf{x}}_2^T [\boldsymbol{t}]_\times \boldsymbol{R} \hat{\mathbf{x}}_1 = \hat{\mathbf{x}}_2^T \boldsymbol{E} \hat{\mathbf{x}}_1 = 0 \qquad \mathbf{x}_2^T \boldsymbol{K}_2^{-T} [\boldsymbol{t}]_\times \boldsymbol{R} \boldsymbol{K}_1^{-1} \mathbf{x}_1 = \mathbf{x}_2^T \boldsymbol{F} \mathbf{x}_1 = 0$$

$$\boldsymbol{E} = [\boldsymbol{t}]_\times \boldsymbol{R}$$



2D-2D:对极几何-基础矩阵F

基础矩阵性质

- ✓ 3x3的矩阵，秩为2
- ✓ 具有7个自由度
- ✓ 奇异值为 $[\sigma_1, \ \sigma_2, \ 0]^T$
- ✓ 极线约束 $l_1 = \boldsymbol{F}^T \boldsymbol{x}_2, \ l_2 = \boldsymbol{F}\boldsymbol{x}_1$

$$\boldsymbol{x}_2^T \boldsymbol{F} \boldsymbol{x}_1 = 0$$

基础矩阵求解方法

- ➢ 直接线性变换法
    - 8点法
    - 最小二乘法
- ➢ 基于RANSAC的鲁棒方法

2D-2D:对极几何-基础矩阵 F

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直接线性变换法

$$x_2^T F x_1 = \mathbf{0}, \quad x_1 = [u_1, \quad v_1, \quad 1]^T, \quad x_2 = [u_2, \quad v_2, \quad 1]^T$$

$$\begin{pmatrix} u_1 & v_1 & 1 \end{pmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{pmatrix} u_2 \\ v_2 \\ 1 \end{pmatrix} = 0$$

令 $f = [F_{11}, \quad F_{12}, \quad F_{13}, \quad F_{21}, \quad F_{22}, \quad F_{23}, \quad F_{31}, \quad F_{32}, \quad F_{33}]^T$

对每一对匹配点

$$[u_1 u_1, \quad u_1 v_2, \quad u_1, \quad v_2 u_1, \quad v_1 v_2, \quad v_1, \quad u_2, \quad v_2, \quad 1] \, f = \mathbf{0}$$

对n个对匹配点

$$A = \begin{pmatrix} u_1^{(1)} u_1^{(1)}, & u_1^{(1)} v_2^{(1)}, & u_1^{(1)}, & v_1^{(1)} u_2^{(1)}, & v_1^{(1)} v_2^{(1)}, & v_1^{(1)}, & u_2^{(1)}, & v_2^{(1)}, & 1 \\ u_1^{(2)} u_1^{(2)}, & u_1^{(2)} v_2^{(2)}, & u_1^{(2)}, & v_1^{(2)} u_2^{(2)}, & v_1^{(2)} v_2^{(2)}, & v_1^{(2)}, & u_2^{(2)}, & v_2^{(2)}, & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^{(n)} u_1^{(n)}, & u_1^{(n)} v_2^{(n)}, & u_1^{(n)}, & v_1^{(n)} u_2^{(n)}, & v_1^{(n)} v_2^{(n)}, & v_1^{(n)}, & u_2^{(n)}, & v_2^{(n)}, & 1 \end{pmatrix}$$

$$Af = \mathbf{0}$$

至少需要8对匹配点 $A = UDV^T$

2D-2D:对极几何-基础矩阵 F

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代数方法

$$\min \left\| \boldsymbol{A}\boldsymbol{f} \right\|^2 = \min \ \boldsymbol{f}^T \boldsymbol{A}^T \boldsymbol{A}\boldsymbol{f}$$

最小特征值对应的特征向量

几何方法

$$\min \sum_j \left(\left\| \boldsymbol{x}_1^{(j)} - \boldsymbol{F}\boldsymbol{x}_2^{(j)} \right\|^2 + \left\| \boldsymbol{x}_2^{(j)} - \boldsymbol{F}\boldsymbol{x}_1^{(j)} \right\|^2 \right)$$

采用非线性优化的方法求解, ceres

几何方法的求解效果要优于代数方法

2D-2D:对极几何-基础矩阵F

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奇异值约束

$$\boldsymbol{x}_2^T \boldsymbol{F} \boldsymbol{x}_1 = \boldsymbol{0}, \quad \boldsymbol{x}_1 = [u_1, \quad v_1, \quad 1]^{\mathrm{T}}, \quad \boldsymbol{x}_2 = [u_2, \quad v_2, \quad 1]^{\mathrm{T}}$$

$$\hat{\boldsymbol{F}} = \begin{bmatrix} \hat{F}_{11} & \hat{F}_{12} & \hat{F}_{13} \\ \hat{F}_{21} & \hat{F}_{22} & \hat{F}_{23} \\ \hat{F}_{31} & \hat{F}_{32} & \hat{F}_{33} \end{bmatrix}$$

基础矩阵只有两个非零的奇异值

$$\left\| \boldsymbol{F} - \hat{\boldsymbol{F}} \right\|, \quad \text{wrt. svd}(\boldsymbol{F}) = [\sigma 1, \quad \sigma 2, \quad 0]$$

$$\hat{\boldsymbol{F}} = \boldsymbol{USV}^T$$

$$\text{with } \boldsymbol{S} = \text{diag}(\sigma_1, \sigma_2, \sigma_3)$$

重新构造基础矩阵

$$\boldsymbol{F} = \boldsymbol{U}\text{diag}(\sigma_1, \quad \sigma_2, \quad 0)\boldsymbol{V}^T$$

2D-2D:对极几何-RANSAC

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RANSAC-随机一致性采样

采用尽可能少的点估计模型参数，并扩大得到模型参数的影响范围

N –样本点个数 K –求解模型需要最少的点的个数

1. 随机采样 K 个点
2. 对该 K 个点拟合模型
3. 计算其它点到拟合模型的距离
   小于一定阈值，当作内点，统计内点个数
4. 重复1–3 m次，选择内点数最多的模型
5. 利用所有的内点重新估计模型

采样次数的计算

p 表示内点的概率

p^K 表示 K个点都是内点概率

$z = 1 - \left(1 - p^K\right)^m$ 表示 m 采样中只少有一次都是内点的概率

$$m = \frac{\log(1-z)}{\log(1-p^K)}$$

2D-2D:对极几何-基础矩阵F

基于RANSAC的鲁邦方法-针对匹配对中存在外点的情况

算法流程

1. 随机采样8对匹配点 $\left(x_1^{(n)},\ x_2^{(n)}\right)$
2. 8点法求解基础矩阵 $\hat{F}$
3. 奇异值约束获取基础矩阵 F
4. 计算误差，并统计内点个数
5. 重复上述过程，选择内点数最多的结果
6. 对所有内点执行2, 3，重新计算 F

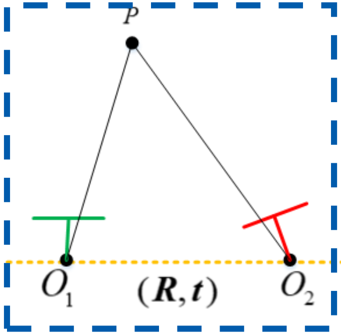
内点判断标准 $E\left(\boldsymbol{x}_1,\ \boldsymbol{x}_2,\ \boldsymbol{F}\right)<\tau$

$$E\left(\boldsymbol{x}_1,\ \boldsymbol{x}_2,\ \boldsymbol{F}\right)=d\left(\boldsymbol{x}_1,\boldsymbol{F}\boldsymbol{x}_2\right)^2+d\left(\boldsymbol{x}_2,\boldsymbol{F}\boldsymbol{x}_1\right)^2$$

2D-2D:对极几何-本征矩阵 E

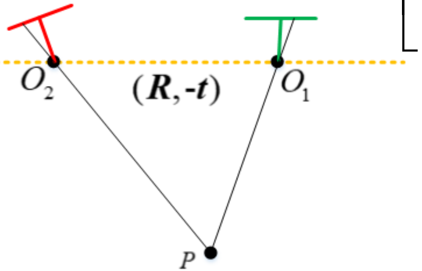
本征矩阵性质

- ✓ 3x3的矩阵，秩为2
- ✓ 具有5个自由度
- ✓ 奇异值为 $[\sigma, \ \sigma, \ 0]^T$



O_2 的世界坐标 $-\boldsymbol{R}^T\boldsymbol{t}$

$$\begin{bmatrix} \boldsymbol{r}_1^T & \boldsymbol{r}_2^T & \boldsymbol{r}_3^T & -\boldsymbol{R}^T\boldsymbol{t} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \boldsymbol{r}_3^T$$

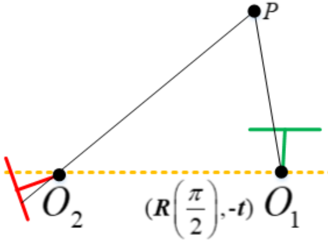


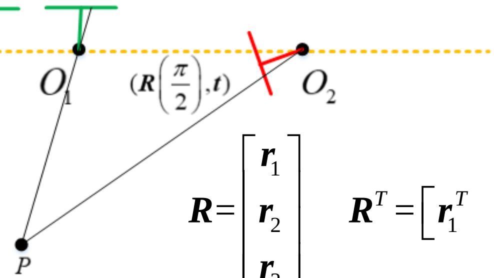
本质矩阵与相机姿态

$$\boldsymbol{E} = \boldsymbol{U}\Sigma\boldsymbol{V}^T, \ \Sigma = \mathrm{diag}(\sigma, \ \sigma, \ 0)$$

$$\boldsymbol{t}_1 = \boldsymbol{U}\boldsymbol{R}_Z\left(\frac{\pi}{2}\right)\boldsymbol{U}^T \qquad \boldsymbol{R}_1 = \boldsymbol{U}\boldsymbol{R}_Z^T\left(\frac{\pi}{2}\right)\boldsymbol{V}^T$$

$$\boldsymbol{t}_2 = \boldsymbol{U}\boldsymbol{R}_Z\left(-\frac{\pi}{2}\right)\boldsymbol{U}^T \qquad \boldsymbol{R}_2 = \boldsymbol{U}\boldsymbol{R}_Z^T\left(\frac{\pi}{2}\right)\boldsymbol{V}^T$$





$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{r}_3 \end{bmatrix} \qquad \boldsymbol{R}^T = \begin{bmatrix} \boldsymbol{r}_1^T & \boldsymbol{r}_2^T & \boldsymbol{r}_3^T \end{bmatrix}$$

2D-2D:对极几何-单应矩阵H

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空间中特征点位于一平面上

$$n^T X + d = 0 \quad \Rightarrow \quad -\frac{n^T X}{d} = 1$$

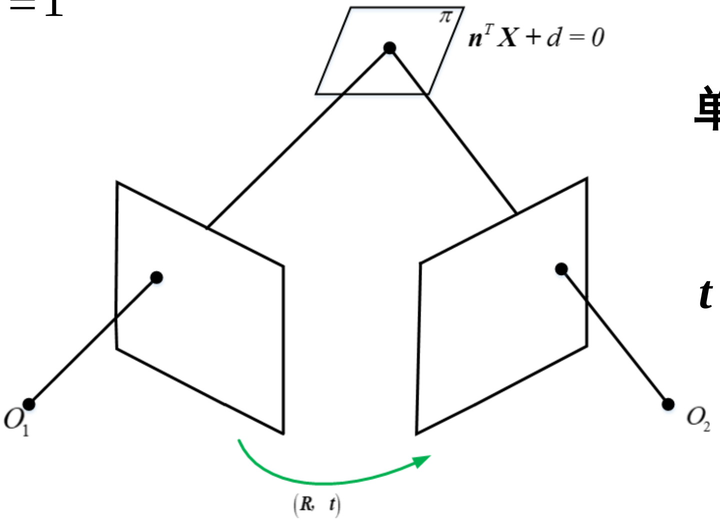
$$x_2 = K_2(RX + t)$$

$$= K_2\left(RX + t \cdot \left(-\frac{n^T X}{d}\right)\right)$$

$$= K_2\left(R - \frac{tn^T}{d}\right)X$$

$$= K_2\left(R - \frac{tn^T}{d}\right)K_1^{-1}x_1$$

$$\boxed{x_2 = Hx_1, \quad H = K_2\left(R - \frac{tn^T}{d}\right)K_1^{-1}}$$



单应矩阵是满秩的

$$x_1 = H^{-1}x_2$$

$t = 0$ 时，对应纯旋转

$$H = K_2 R K_1^{-1}$$

2D-2D:对极几何-单应矩阵H

直接线性变换法

$$\begin{pmatrix} u_2 \\ v_2 \\ 1 \end{pmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{H_{11}u_1 + H_{12}v_1 + H_{13}}{H_{31}u_1 + H_{32}v_1 + H_{33}}$$

$$v_2 = \frac{H_{21}u_1 + H_{22}v_1 + H_{23}}{H_{31}u_1 + H_{32}v_1 + H_{33}}$$

8个自由度，每对点有两个约束

$$H_{11}u_1 + H_{12}v_1 + H_{13} - H_{31}u_1u_2 - H_{32}u_2v_1 - H_{33}u_2 = 0$$

$$H_{21}u_1 + H_{22}v_1 + H_{23} - H_{31}u_1v_2 - H_{32}v_1v_2 - H_{33}v_2 = 0$$

令 $H_{33}=1$ 总共需要4对特征点

$$\boldsymbol{A} = \begin{pmatrix} u_1^{(1)}, & v_1^{(1)}, & 1, & 0, & 0, & 0, & -u_1^{(1)}u_2^{(1)}, & -u_2^{(1)}v_1^{(1)} \\ 0, & 0, & 0, & u_1^{(1)}, & v_1^{(1)}, & 1, & -u_1^{(1)}v_2^{(1)}, & -v_1^{(1)}v_2^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^{(4)}, & v_1^{(4)}, & 1, & 0, & 0, & 0, & -u_1^{(4)}u_2^{(4)}, & -u_2^{(4)}v_1^{(4)} \\ 0, & 0, & 0, & u_1^{(4)}, & v_1^{(4)}, & 1, & -u_1^{(4)}v_2^{(4)}, & -v_1^{(4)}v_2^{(4)} \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \end{pmatrix} = \begin{pmatrix} u_2^{(1)} \\ v_2^{(1)} \\ u_2^{(2)} \\ v_2^{(2)} \\ u_2^{(3)} \\ v_2^{(3)} \\ u_2^{(4)} \\ v_2^{(4)} \end{pmatrix}$$

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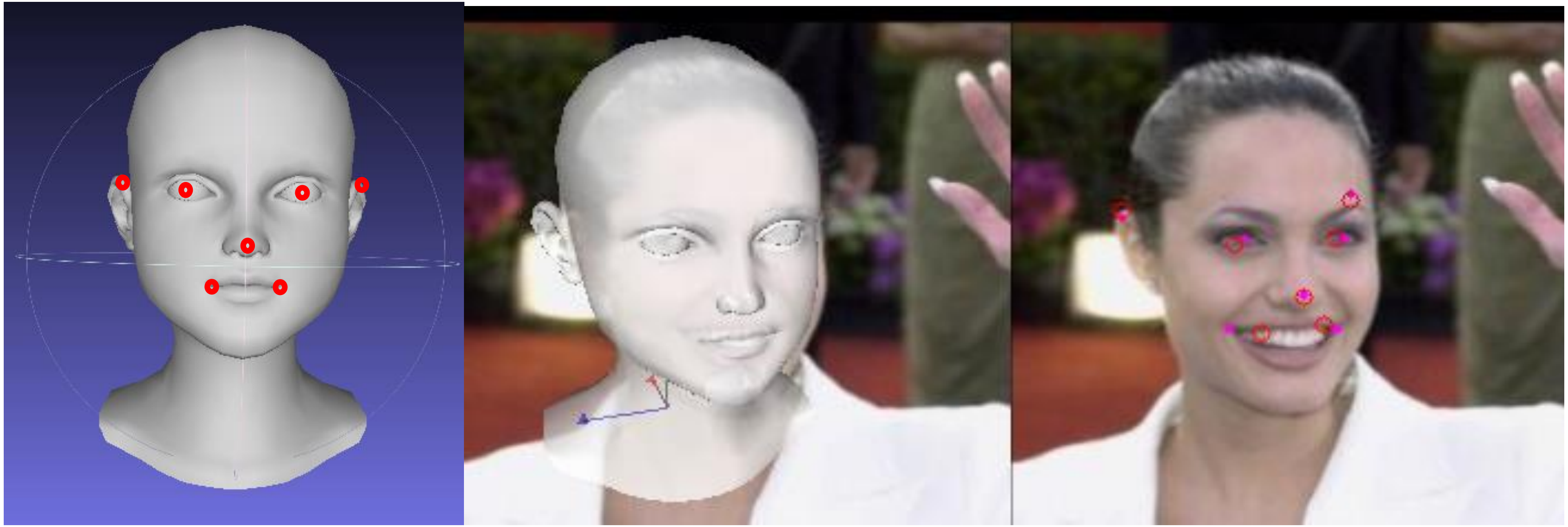
3D-2D:PnP问题-PnP

已知三维点和对应二维点求解相机内外参数



3D-2D: PnP问题 -三角量测

已知**相机参数**和**匹配点**恢复三维点的深度

原理：三维点到所有射线距离最小

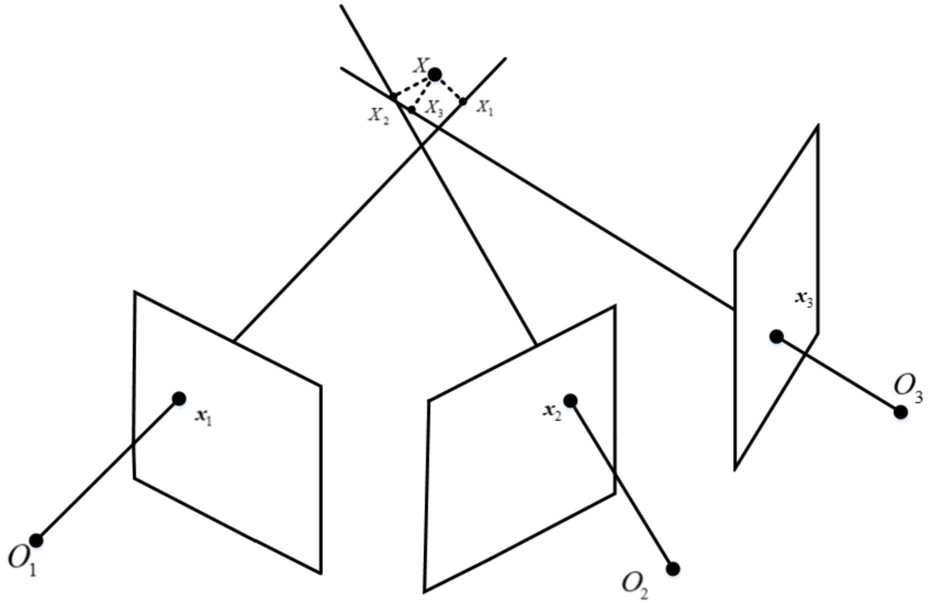
相机投影矩阵：$P_i = K_i[R_i, t_i]$

$x_1 = R_i x_2 + t_i \Leftrightarrow x_2 = R_i^T(x_2 - t_i)$

相机在世界坐标系中的位置：$c_i^w = R_i^T(0 - t_i) = -R_i^T t_i$

特征点对应世界坐标系射线：$v_i^w = -R_i^T K_i^{-1} x_i$

$$\min_{X,\ \lambda_1,\ \lambda_i,\ldots} \left(c_i^w + \lambda_i v_i^w - X\right)^2$$



3D-2D: PnP问题 -三角量测

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已知**相机参数**和**匹配点**恢复三维点的深度

1. 固定 X 优化 $\lambda_1, \dots \lambda_i, \dots$

$$\min_{\lambda_1,\dots\lambda_i,\dots} \left(\boldsymbol{c}_i^w + \lambda_i \boldsymbol{v}_i^w - \boldsymbol{X} \right)^2$$

对 λ_i 求导数并令为 0

$$\boldsymbol{v}_i^w \cdot \left(\boldsymbol{c}_i^w + \lambda_i \boldsymbol{v}_i^w - \boldsymbol{X} \right) = 0$$

$$\lambda_i^* = \boldsymbol{v}_i^w \cdot \left(\boldsymbol{X} - \boldsymbol{c}_i^w \right)$$

2. 将 $\lambda_1^*, \dots \lambda_i^*, \dots$ 代入原式中

$$\min_{\boldsymbol{X}} E(\boldsymbol{X}) = \min_{\boldsymbol{X}} \sum_i \left(\boldsymbol{c}_i^w + \lambda_i^* \boldsymbol{v}_i^w - \boldsymbol{X} \right)^2$$

$$= \min_{\boldsymbol{X}} \sum_i \left(\left(\boldsymbol{v}_i^w \left(\boldsymbol{v}_i^w \right)^T - \boldsymbol{I} \right) \left(\boldsymbol{X} - \boldsymbol{c}_i^w \right) \right)^2$$

$$\frac{\partial E(\boldsymbol{X})}{\partial \boldsymbol{X}} = \sum_i \left(\boldsymbol{v}_i^w \left(\boldsymbol{v}_i^w \right)^T - \boldsymbol{I} \right)^T \left(\boldsymbol{v}_i^w \left(\boldsymbol{v}_i^w \right)^T - \boldsymbol{I} \right) \left(\boldsymbol{X} - \boldsymbol{c}_i^w \right)$$

$$= \sum_i \left(\boldsymbol{I} - \boldsymbol{v}_i^w \left(\boldsymbol{v}_i^w \right)^T \right) \left(\boldsymbol{X} - \boldsymbol{c}_i^w \right) = 0$$

$$\boldsymbol{X} = \left(\sum_i \left(\boldsymbol{I} - \boldsymbol{v}_i^w \left(\boldsymbol{v}_i^w \right)^T \right) \right)^{-1} \left(\sum_i \left(\boldsymbol{I} - \boldsymbol{v}_i^w \left(\boldsymbol{v}_i^w \right)^T \right) \boldsymbol{c}_i^w \right)$$

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附录–三角量测公式推导

$$E(\boldsymbol{X})=\sum_i\left(\boldsymbol{c}_i^w+\lambda_i^*\boldsymbol{v}_i^w-\boldsymbol{X}\right)^2$$

$$=\sum_i\left(\boldsymbol{c}_i^w+\left(\boldsymbol{v}_i^w\right)^T\left(\boldsymbol{X}-\boldsymbol{c}_i^w\right)\boldsymbol{v}_i^w-\boldsymbol{X}\right)^2$$

$$=\sum_i\left(\boldsymbol{c}_i^w+\left(\boldsymbol{v}_i^w\right)^T\boldsymbol{X}\boldsymbol{v}_i^w-\left(\boldsymbol{v}_i^w\right)^T\boldsymbol{c}_i^w\boldsymbol{v}_i^w-\boldsymbol{X}\right)^2$$

$$=\sum_i\left(\boldsymbol{c}_i^w+\boldsymbol{v}_i^w\left(\boldsymbol{v}_i^w\right)^T\boldsymbol{X}-\boldsymbol{v}_i^w\left(\boldsymbol{v}_i^w\right)^T\boldsymbol{c}_i^w-\boldsymbol{X}\right)^2$$

$$=\sum_i\left(\left(\boldsymbol{v}_i^w\left(\boldsymbol{v}_i^w\right)^T-\boldsymbol{I}\right)\boldsymbol{X}+\boldsymbol{c}_i^w-\boldsymbol{v}_i^w\left(\boldsymbol{v}_i^w\right)^T\boldsymbol{c}_i^w\right)^2$$

$$=\sum_i\left(\left(\boldsymbol{v}_i^w\left(\boldsymbol{v}_i^w\right)^T-\boldsymbol{I}\right)\left(\boldsymbol{X}-\boldsymbol{c}_i^w\right)\right)^2$$

$$\frac{\partial E(\boldsymbol{X})}{\partial \boldsymbol{X}}=\sum_i\left(\boldsymbol{v}_i^w\left(\boldsymbol{v}_i^w\right)^T-\boldsymbol{I}\right)^T\left(\boldsymbol{v}_i^w\left(\boldsymbol{v}_i^w\right)^T-\boldsymbol{I}\right)\left(\boldsymbol{X}-\boldsymbol{c}_i^w\right)$$

$$=\sum_i\left(\boldsymbol{I}-\boldsymbol{v}_i^w\left(\boldsymbol{v}_i^w\right)^T\right)\left(\boldsymbol{X}-\boldsymbol{c}_i^w\right)=0$$

$$\sum_i\left(\boldsymbol{I}-\boldsymbol{v}_i^w\left(\boldsymbol{v}_i^w\right)^T\right)\boldsymbol{X}=\sum_i\left(\boldsymbol{I}-\boldsymbol{v}_i^w\left(\boldsymbol{v}_i^w\right)^T\right)\boldsymbol{c}_i^w$$

$$\boldsymbol{X}=\left(\sum_i\left(\boldsymbol{I}-\boldsymbol{v}_i^w\left(\boldsymbol{v}_i^w\right)^T\right)\right)^{-1}\left(\sum_i\left(\boldsymbol{I}-\boldsymbol{v}_i^w\left(\boldsymbol{v}_i^w\right)^T\right)\boldsymbol{c}_i^w\right)$$

3D-2D:PnP问题–PnP

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已知三维点和对应二维点求解相机内外参数

$$s\begin{pmatrix}u\\v\\1\end{pmatrix}=K[R,t]\begin{pmatrix}X\\Y\\Z\\1\end{pmatrix}=\begin{bmatrix}T_{11}, & T_{12}, & T_{13}, & T_{14}\\T_{21}, & T_{22}, & T_{23}, & T_{24}\\T_{31}, & T_{32}, & T_{33}, & T_{34}\end{bmatrix}\begin{pmatrix}X\\Y\\Z\\1\end{pmatrix}=\begin{pmatrix}\boldsymbol{r}_1^T\\\boldsymbol{r}_2^T\\\boldsymbol{r}_3^T\end{pmatrix}\boldsymbol{X}$$

$$u=\frac{T_{11}X+T_{12}Y+T_{13}Z+T_{14}}{T_{31}X+T_{32}Y+T_{33}Z+T_{34}}=\frac{\boldsymbol{X}^T\boldsymbol{r}_1}{\boldsymbol{X}^T\boldsymbol{r}_3}$$

$$v=\frac{T_{21}X+T_{22}Y+T_{23}Z+T_{24}}{T_{31}X+T_{32}Y+T_{33}Z+T_{34}}=\frac{\boldsymbol{X}^T\boldsymbol{r}_2}{\boldsymbol{X}^T\boldsymbol{r}_3}$$

$$\boldsymbol{X}^T\boldsymbol{r}_1-\boldsymbol{X}^T\boldsymbol{r}_3u=0$$

$$\boldsymbol{X}^T\boldsymbol{r}_2-\boldsymbol{X}^T\boldsymbol{r}_3v=0$$

$$\begin{pmatrix}\boldsymbol{X}_1^T & 0 & -u\boldsymbol{X}_1^T\\0 & \boldsymbol{X}_1^T & -v\boldsymbol{X}_1^T\\\vdots & \vdots & \vdots\\\boldsymbol{X}_N^T & 0 & -u\boldsymbol{X}_N^T\\0 & \boldsymbol{X}_N^T & -v\boldsymbol{X}_N^T\end{pmatrix}\begin{pmatrix}\boldsymbol{r}_1\\\boldsymbol{r}_2\\\boldsymbol{r}_3\end{pmatrix}=\boldsymbol{0}$$

共需要至少6对3D-2D对应点

$T=[KR,Kt]$ **矩阵QR分解获取K,R,t**

课程内容

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- ✓ 针孔相机模型
  - ✓ 针孔相机模型
  - ✓ 径向畸变
- ✓ 2D-2D:对极几何
  - ✓ 对极约束
  - ✓ 本质/单应矩阵
  - ✓ 直接线性变换法
- ✓ 3D-2D: PnP问题
  - ✓ 三角量测
  - ✓ 直接线性变换法
  - ✓ 非线性优化
- ✓ **捆绑调整Bundle Adjustment**

捆绑调整Bundle Adjustment

同时对三维点位置和相机参数进行非线性优化

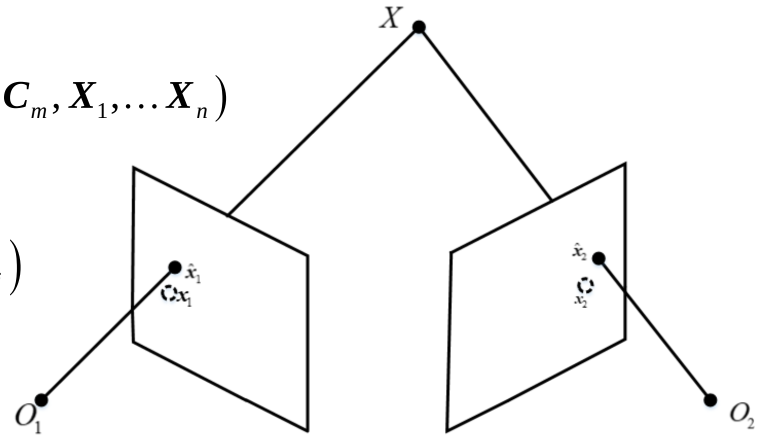
观测点

三维点坐标 $\boldsymbol{X}_i=(X_i,Y_i,Z_i)^T$

$$g(\boldsymbol{\theta})=\frac{1}{2}\sum_{i}^{n}\sum_{j}^{m}\chi_{ij}\left\|\boldsymbol{u}_{ij}-\hat{\boldsymbol{u}}_{ij}(\boldsymbol{C}_j,\boldsymbol{X}_i)\right\|^2=\frac{1}{2}\sum_{i}^{n}\sum_{j}^{m}\chi_{ij}e_{ij},\quad \boldsymbol{\theta}=(\boldsymbol{C}_1,\ldots,\ \boldsymbol{C}_m,\boldsymbol{X}_1,\ldots\boldsymbol{X}_n)$$

投影点　相机参数 $\boldsymbol{C}_j=(f_j,k_{1j},k_{2j},\boldsymbol{R}j,\boldsymbol{t}_j)$

$\chi_{ij}=1$　表示第 i 个点在第 j 个相机中可见



捆绑调整Bundle Adjustment

非线性优化

$$\min_{\theta} g(\boldsymbol{\theta}) = \min_{\boldsymbol{\theta}} \left\| \mathbf{x} - f(\boldsymbol{\theta}) \right\|^2$$

$$g(\boldsymbol{\theta}) = g(\boldsymbol{\theta}_t) + \frac{\partial g}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}_t)^T (\boldsymbol{\theta} - \boldsymbol{\theta}_t) + \frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\theta}_t)^T \frac{\partial^2 g}{\partial \boldsymbol{\theta}^2}(\boldsymbol{\theta}_t)(\boldsymbol{\theta} - \boldsymbol{\theta}_t)$$

$$\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \frac{\partial g}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}_t) + \frac{\partial^2 g}{\partial \boldsymbol{\theta}^2}(\boldsymbol{\theta}_t)(\boldsymbol{\theta} - \boldsymbol{\theta}_t) = 0$$

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \left(\frac{\partial^2 g}{\partial \boldsymbol{\theta}^2}(\boldsymbol{\theta}_t) \right)^{-1} \frac{\partial g}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}_t) \qquad \delta(\boldsymbol{\theta}) = (\boldsymbol{\theta}_{t+1} - \boldsymbol{\theta}_t)$$

$$\delta\boldsymbol{\theta} = \left(\frac{\partial^2 g}{\partial \boldsymbol{\theta}^2}(\boldsymbol{\theta}_t) \right)^{-1} \frac{\partial g}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}_t)$$

1.选取初始点$\theta_t\,(t=0)$，终止控制条件ε

2.计算$\delta\boldsymbol{\theta} = \left(\frac{\partial^2 g}{\partial \boldsymbol{\theta}^2}(\boldsymbol{\theta}_t) \right)^{-1} \frac{\partial g}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}_t)$，更新$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t + (-\delta\boldsymbol{\theta})$

3.判断 $\left\| \mathbf{x} - f(\boldsymbol{\theta}) \right\|^2 < \varepsilon$ 时终止，否则重复上述第2步

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Levenberg-Marquardt

$$\boxed{\left(\frac{\partial^2 g}{\partial \boldsymbol{\theta}^2}(\boldsymbol{\theta}_t)\right) = J^T(\boldsymbol{\theta})J(\boldsymbol{\theta}) + \lambda_t \boldsymbol{I}}$$

$$\boxed{\begin{aligned}\frac{\partial g}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}_t) &= -\frac{\partial f}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}_t)^T\left(\mathbf{x} - f(\boldsymbol{\theta}_t)\right) \\ &= -J^T(\boldsymbol{\theta}_t)\left(\mathbf{x} - f(\boldsymbol{\theta}_t)\right)\end{aligned}}$$

$$\delta(\boldsymbol{\theta}) = -\left(\boxed{\frac{\partial^2 g}{\partial \boldsymbol{\theta}^2}(\boldsymbol{\theta}_t)}\right)^{-1} \boxed{\frac{\partial g}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}_t)}$$

$$\left(\frac{\partial^2 g}{\partial \boldsymbol{\theta}^2}(\boldsymbol{\theta}_t)\right)\delta(\boldsymbol{\theta}) = J^T(\boldsymbol{\theta}_t)\left(\mathbf{x} - f(\boldsymbol{\theta}_t)\right)$$

正规方程 $\left(J^T(\boldsymbol{\theta})J(\boldsymbol{\theta}) + \lambda_t \boldsymbol{I}\right)\delta(\boldsymbol{\theta}) = (\boldsymbol{\theta}_{t+1} - \boldsymbol{\theta}_t) = J^T(\boldsymbol{\theta}_t)\left(\mathbf{x} - f(\boldsymbol{\theta}_t)\right)$

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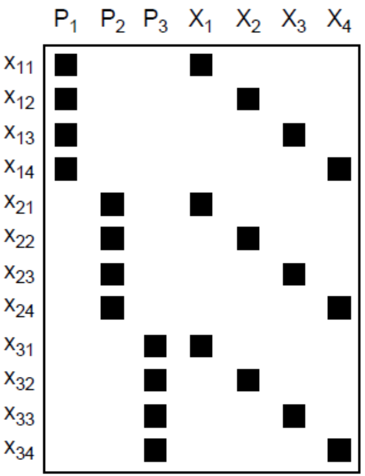
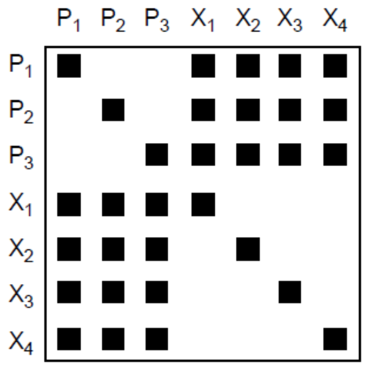
Levenberg-Marquardt

对称、稀疏 $\boxed{\left(J^T(\theta)J(\theta)+\lambda_t \mathbf{I}\right)}\delta(\theta)=\left(\theta_{t+1}-\theta_t\right)=J^T(\theta_t)\left(\mathbf{x}-f(\theta_t)\right)$



(a)

$J^T(\theta)$



(b)

$J^T(\theta)J(\theta)$

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Levenberg-Marquardt

$$\boxed{\left(J^T(\boldsymbol{\theta})J(\boldsymbol{\theta})+\lambda_t \boldsymbol{I}\right)}\boxed{\delta(\boldsymbol{\theta})}=\boxed{J^T(\boldsymbol{\theta}_t)\left(\mathbf{x}-f(\boldsymbol{\theta}_t)\right)}$$

$$\boxed{\begin{bmatrix} \boldsymbol{J}_{CC} & \boldsymbol{J}_{CX} \\ \boldsymbol{J}_{CX} & \boldsymbol{J}_{XX} \end{bmatrix}}\boxed{\begin{bmatrix} \delta_C \\ \delta_X \end{bmatrix}}=\boxed{\begin{bmatrix} b_C \\ b_X \end{bmatrix}}$$

左乘 $\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{J}_{CX}\boldsymbol{J}_{XX}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}$

$$\downarrow$$

$$\begin{bmatrix} \boldsymbol{J}_{CC}-\boldsymbol{J}_{CX}\boldsymbol{J}_{XX}^{-1}\boldsymbol{J}_{CX} & \boldsymbol{0} \\ \boldsymbol{J}_{CX} & \boldsymbol{J}_{XX} \end{bmatrix}\begin{bmatrix} \delta_C \\ \delta_X \end{bmatrix}=\begin{bmatrix} b_C-\boldsymbol{J}_{CX}\boldsymbol{J}_{XX}^{-1}b_X \\ b_X \end{bmatrix}$$

$$\downarrow$$

$$\left(\boldsymbol{J}_{CC}-\boldsymbol{J}_{CX}\boldsymbol{J}_{XX}^{-1}\boldsymbol{J}_{CX}\right)\boldsymbol{\delta}_C=\boldsymbol{b}_C-\boldsymbol{J}_{CX}\boldsymbol{J}_{XX}^{-1}\boldsymbol{b}_X$$

$$\boldsymbol{J}_{XX}\boldsymbol{\delta}_X=\boldsymbol{b}_X-\boldsymbol{J}_{CX}\boldsymbol{\delta}_C$$

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Levenberg-Marquardt 计算流程

1. $t=0$ 时, 选取初始点 $\boldsymbol{\theta}_0$, 终止控制常数 ε, 令 $e^0=\|\mathbf{x}-f(\boldsymbol{\theta}_0)\|^2$, $\lambda_0=10^{-3}$

2. 计算 $J^T(\boldsymbol{\theta}_t)$

3. 构造增量正规方程 $\left(J^T(\boldsymbol{\theta})J(\boldsymbol{\theta})+\lambda_t\boldsymbol{I}\right)\delta(\boldsymbol{\theta})=(\boldsymbol{\theta}_{t+1}-\boldsymbol{\theta}_t)=J^T(\boldsymbol{\theta}_t)\left(\mathbf{x}-f(\boldsymbol{\theta}_t)\right)$

4. 通过求解增量正规方程，得到 $\delta(\boldsymbol{\theta})$

如果 $\|\mathbf{x}-f(\boldsymbol{\theta}_t+\delta\boldsymbol{\theta})\|^2<e^k$，令 $\boldsymbol{\theta}_{t+1}=\boldsymbol{\theta}_t+\delta(\boldsymbol{\theta})$,

如果 $\|\delta(\boldsymbol{\theta})\|<\varepsilon$, 终止迭代;

否则, 令 $\lambda_{t+1}=0.1\lambda_t$, $t=t+1$, 执行第2步

否则 $\|\mathbf{x}-f(\boldsymbol{\theta}_t+\delta\boldsymbol{\theta})\|^2\geq e^k$，令 $\lambda_{t+1}=10\lambda_t$, 执行第3步

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雅阁比矩阵

$$e_{ij}=\frac{1}{2}\left\|\boldsymbol{e}_{ij}\right\|^2=\frac{1}{2}\left\|\boldsymbol{u}_{ij}-\hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j,\boldsymbol{X}_i\right)\right\|^2=\frac{1}{2}\left(\left(u_{ij}-\hat{u}_{ij}\left(\boldsymbol{C}_j,\boldsymbol{X}_i\right)\right)^2+\left(v_{ij}-\hat{v}_{ij}\left(\boldsymbol{C}_j,\boldsymbol{X}_i\right)\right)^2\right)$$

$$\xi_{ij}=\begin{bmatrix}\boldsymbol{C}_j\\ \boldsymbol{X}_i\end{bmatrix},\quad \boldsymbol{C}_j=\left(f_j,k_1,k_2,\boldsymbol{R}_j,\boldsymbol{t}_j\right)$$

$$\frac{\partial e_{ij}}{\xi_{ij}^T}=\boldsymbol{e}_{ij}^T\frac{\partial \boldsymbol{e}_{ij}}{\xi_{ij}^T}=-\boldsymbol{e}_{ij}^T\frac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j,\boldsymbol{X}_i\right)}{\xi_{ij}^T}$$

$$\frac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j,\boldsymbol{X}_i\right)}{\partial \xi_{ij}^T}=\begin{bmatrix}\dfrac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j,\boldsymbol{X}_i\right)}{\partial \boldsymbol{C}_j^T} & \dfrac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j,\boldsymbol{X}_i\right)}{\boldsymbol{X}_i^T}\end{bmatrix}$$

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雅阁比矩阵-投影矩阵

世界坐标系到相机坐标系

$$\boldsymbol{x}_i^c = \begin{bmatrix} x_i^c \\ y_i^c \\ z_i^c \end{bmatrix} = \boldsymbol{R}\boldsymbol{X}_i + \boldsymbol{t}$$

径向畸变

$$\begin{bmatrix} \tilde{x}_i^c \\ \tilde{y}_i^c \end{bmatrix} = \left(1 + k_{j1} r_c^2 + k_{j2} r_c^4\right) \begin{bmatrix} x_i^c \\ y_i^c \end{bmatrix},$$

$$r_c^2 = \left(x_i^c\right)^2 + \left(y_i^c\right)^2$$

相机坐标系到图像坐标系

$$\hat{\boldsymbol{u}}_{ij} = \begin{bmatrix} \hat{u}_{ij} \\ \hat{v}_{ij} \end{bmatrix} = \begin{bmatrix} f_j \dfrac{\tilde{x}_i^c}{z_i^c} + c_x \\ f_j \dfrac{\tilde{y}_i^c}{z_i^c} + c_y \end{bmatrix}$$

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雅阁比矩阵- $\frac{\partial \hat{\mathbf{u}}_{ij}\left(\boldsymbol{C}_j, \boldsymbol{X}_i\right)}{\partial \boldsymbol{C}_j^T}$

焦距：

$$\frac{\partial \hat{\mathbf{u}}_{ij}\left(\boldsymbol{C}_j, \boldsymbol{X}_i\right)}{\partial f_j} = \begin{bmatrix} \frac{\tilde{x}_i^c}{z_i^c} \\ \frac{\tilde{y}_i^c}{z_i^c} \end{bmatrix}$$

径向畸变系数：

$$\frac{\partial \hat{\mathbf{u}}_{ij}\left(\boldsymbol{C}_j, \boldsymbol{X}_i\right)}{\partial k_{j1}} = f_j \frac{1}{z_i^c} \begin{bmatrix} \frac{\partial \tilde{x}_i^c}{\partial k_{j1}} \\ \frac{\partial \tilde{y}_i^c}{\partial k_{j1}} \end{bmatrix} = f_j \frac{1}{z_i^c} r_c^2 \begin{bmatrix} x_i^c \\ y_i^c \end{bmatrix}$$

$$\frac{\partial \hat{\mathbf{u}}_{ij}\left(\boldsymbol{C}_j, \boldsymbol{X}_i\right)}{\partial k_{j2}} = f_j \frac{1}{z_i^c} \begin{bmatrix} \frac{\partial \tilde{x}_i^c}{\partial k_{j2}} \\ \frac{\partial \tilde{y}_i^c}{\partial k_{j2}} \end{bmatrix} = f_j \frac{1}{z_i^c} r_c^4 \begin{bmatrix} x_i^c \\ y_i^c \end{bmatrix}$$

$$\boldsymbol{x}_i^c = \begin{bmatrix} x_i^c \\ y_i^c \\ z_i^c \end{bmatrix} = \boldsymbol{R}\boldsymbol{X}_i + \boldsymbol{t}$$

$$\begin{bmatrix} \tilde{x}_i^c \\ \tilde{y}_i^c \end{bmatrix} = \left(1 + k_{j1} r_c^2 + k_{j2} r_c^4\right) \begin{bmatrix} x_i^c \\ y_i^c \end{bmatrix},$$

$$r_c^2 = \left(x_i^c\right)^2 + \left(y_i^c\right)^2$$

$$\hat{\mathbf{u}}_{ij} = \begin{bmatrix} \hat{u}_{ij} \\ \hat{v}_{ij} \end{bmatrix} = \begin{bmatrix} f_j \frac{\tilde{x}_i^c}{z_i^c} + c_x \\ f_j \frac{\tilde{y}_i^c}{z_i^c} + c_y \end{bmatrix}$$

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雅阁比矩阵- $\dfrac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j, \boldsymbol{X}_i\right)}{\partial \boldsymbol{C}_j^T}$

$$\frac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j, \boldsymbol{X}_i\right)}{\partial \left(\boldsymbol{x}_i^c\right)^T} = \begin{bmatrix} \dfrac{\partial \hat{u}_{ij}}{\partial x_i^c}, & \dfrac{\partial \hat{u}_{ij}}{\partial y_i^c}, & \dfrac{\partial \hat{u}_{ij}}{\partial z_i^c} \\ \dfrac{\partial \hat{v}_{ij}}{\partial x_i^c}, & \dfrac{\partial \hat{v}_{ij}}{\partial z_i^c}, & \dfrac{\partial \hat{v}_{ij}}{\partial z_i^c} \end{bmatrix}$$

$$= \left(1 + k_1 r_c^2 + k_2 r_c^4\right) \begin{bmatrix} f_j \dfrac{1}{z_i^c}\left(1 + \dfrac{\left(2k_{j1} + 4k_{j2} r_c^2\right)\left(x_i^c\right)^2}{1 + k_1 r_c^2 + k_2 r_c^4}\right), & f_j \dfrac{1}{z_i^c}\left(\dfrac{\left(2k_{j1} + 4k_{j2} r_c^2\right) x_i^c y_i^c}{1 + k_1 r_c^2 + k_2 r_c^4}\right), & -f_j \dfrac{x_i^c}{\left(z_i^c\right)^2} \\ f_j \dfrac{1}{z_i^c}\left(\dfrac{\left(2k_{j1} + 4k_{j2} r_c^2\right) x_i^c y_i^c}{1 + k_1 r_c^2 + k_2 r_c^4}\right), & f_j \dfrac{1}{z_i^c}\left(1 + \dfrac{\left(2k_{j1} + 4k_{j2} r_c^2\right) y_i^c}{1 + k_1 r_c^2 + k_2 r_c^4}\right), & -f_j \dfrac{y_i^c}{\left(z_i^c\right)^2} \end{bmatrix}$$

平移向量：

$$\frac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j, \boldsymbol{X}_i\right)}{\partial \boldsymbol{t}_j^T} = \frac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j, \boldsymbol{X}_i\right)}{\partial \left(\boldsymbol{x}_i^c\right)^T} \frac{\partial \left(\boldsymbol{x}_i^c\right)}{\partial \boldsymbol{t}^T}$$

$$\frac{\partial \left(\boldsymbol{x}_i^c\right)^T}{\partial \boldsymbol{t}^T} = \boldsymbol{I}$$

$$\boldsymbol{x}_i^c = \begin{bmatrix} x_i^c \\ y_i^c \\ z_i^c \end{bmatrix} = \boldsymbol{R}\boldsymbol{X}_i + \boldsymbol{t}$$

$$\begin{bmatrix} \tilde{x}_i^c \\ \tilde{y}_i^c \end{bmatrix} = \left(1 + k_{j1} r_c^2 + k_{j2} r_c^4\right) \begin{bmatrix} x_i^c \\ y_i^c \end{bmatrix},$$

$$r_c^2 = \left(x_i^c\right)^2 + \left(y_i^c\right)^2$$

$$\hat{\boldsymbol{u}}_{ij} = \begin{bmatrix} \hat{u}_{ij} \\ \hat{v}_{ij} \end{bmatrix} = \begin{bmatrix} f_j \dfrac{\tilde{x}_i^c}{z_i^c} + c_x \\ f_j \dfrac{\tilde{y}_i^c}{z_i^c} + c_y \end{bmatrix}$$

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雅阁比矩阵- $\dfrac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j, \boldsymbol{X}_i\right)}{\partial \boldsymbol{C}_j^T}$

$$\frac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j, \boldsymbol{X}_i\right)}{\partial \left(\boldsymbol{x}_i^c\right)^T} = \begin{bmatrix} \dfrac{\partial \hat{u}_{ij}}{\partial x_i^c}, & \dfrac{\partial \hat{u}_{ij}}{\partial y_i^c}, & \dfrac{\partial \hat{u}_{ij}}{\partial z_i^c} \\ \dfrac{\partial \hat{v}_{ij}}{\partial x_i^c}, & \dfrac{\partial \hat{v}_{ij}}{\partial z_i^c}, & \dfrac{\partial \hat{v}_{ij}}{\partial z_i^c} \end{bmatrix}$$

$$= \left(1 + k_1 r_c^2 + k_2 r_c^4\right) \begin{bmatrix} f_j \dfrac{1}{z_i^c}\left(1 + \dfrac{\left(2k_{j1} + 4k_{j2} r_c^2\right)\left(x_i^c\right)^2}{1 + k_1 r_c^2 + k_2 r_c^4}\right), & f_j \dfrac{1}{z_i^c}\left(\dfrac{\left(2k_{j1} + 4k_{j2} r_c^2\right) x_i^c y_i^c}{1 + k_1 r_c^2 + k_2 r_c^4}\right), & -f_j \dfrac{x_i^c}{\left(z_i^c\right)^2} \\ f_j \dfrac{1}{z_i^c}\left(\dfrac{\left(2k_{j1} + 4k_{j2} r_c^2\right) x_i^c y_i^c}{1 + k_1 r_c^2 + k_2 r_c^4}\right), & f_j \dfrac{1}{z_i^c}\left(1 + \dfrac{\left(2k_{j1} + 4k_{j2} r_c^2\right) y_i^c}{1 + k_1 r_c^2 + k_2 r_c^4}\right), & -f_j \dfrac{y_i^c}{\left(z_i^c\right)^2} \end{bmatrix}$$

旋转矩阵：

$$\boldsymbol{R}_j$$

参数化?

$$\boldsymbol{x}_i^c = \begin{bmatrix} x_i^c \\ y_i^c \\ z_i^c \end{bmatrix} = \boldsymbol{R}\boldsymbol{X}_i + \boldsymbol{t}$$

$$\begin{bmatrix} \tilde{x}_i^c \\ \tilde{y}_i^c \end{bmatrix} = \left(1 + k_{j1} r_c^2 + k_{j2} r_c^4\right) \begin{bmatrix} x_i^c \\ y_i^c \end{bmatrix},$$

$$r_c^2 = \left(x_i^c\right)^2 + \left(y_i^c\right)^2$$

$$\hat{\boldsymbol{u}}_{ij} = \begin{bmatrix} \hat{u}_{ij} \\ \hat{v}_{ij} \end{bmatrix} = \begin{bmatrix} f_j \dfrac{\tilde{x}_i^c}{z_i^c} + c_x \\ f_j \dfrac{\tilde{y}_i^c}{z_i^c} + c_y \end{bmatrix}$$

捆绑调整Bundle Adjustment

旋转矩阵参数化　*Rodrigues' formula*

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\boldsymbol{R} = \cos\|\boldsymbol{w}\| \boldsymbol{I}_{3x3} + \frac{[\boldsymbol{w}]_\times}{\|\boldsymbol{w}\|}\sin(\|\boldsymbol{w}\|) + \frac{[\boldsymbol{w}]_\times^2}{\|\boldsymbol{w}\|^2}(1-\cos\|\boldsymbol{w}\|)$$

$\Leftrightarrow$

$$\boldsymbol{w} = [w_1 \quad w_2 \quad w_3]$$

$$\|\boldsymbol{w}\| = \cos^{-1}\left(\frac{trace(\boldsymbol{R})-1}{2}\right), \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{1}{2\sin(\|\boldsymbol{w}\|)}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

微小变化　$\delta \boldsymbol{R} \approx \boldsymbol{I}_{3x3} + [\boldsymbol{w}]_\times$

$$\frac{\partial \delta \boldsymbol{R}\boldsymbol{x}}{\partial \boldsymbol{x}^T} \approx \boldsymbol{I}_{3x3} + [\boldsymbol{x}]_\times$$

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雅阁比矩阵- $\frac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j,\boldsymbol{X}_i\right)}{\partial \boldsymbol{C}_j^T}$

$$\frac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j,\boldsymbol{X}_i\right)}{\partial \left(\boldsymbol{x}_i^c\right)^T}=\begin{bmatrix}\frac{\partial \hat{u}_{ij}}{\partial x_i^c}, & \frac{\partial \hat{u}_{ij}}{\partial y_i^c}, & \frac{\partial \hat{u}_{ij}}{\partial z_i^c}\\ \frac{\partial \hat{v}_{ij}}{\partial x_i^c}, & \frac{\partial \hat{v}_{ij}}{\partial z_i^c}, & \frac{\partial \hat{v}_{ij}}{\partial z_i^c}\end{bmatrix}$$

$$=\left(1+k_1 r_c^2+k_2 r_c^4\right)\begin{bmatrix} f_j\frac{1}{z_i^c}\left(1+\frac{\left(2k_{j1}+4k_{j2}r_c^2\right)\left(x_i^c\right)^2}{1+k_1 r_c^2+k_2 r_c^4}\right), & f_j\frac{1}{z_i^c}\left(\frac{\left(2k_{j1}+4k_{j2}r_c^2\right)x_i^c y_i^c}{1+k_1 r_c^2+k_2 r_c^4}\right), & -f_j\frac{x_i^c}{\left(z_i^c\right)^2}\\ f_j\frac{1}{z_i^c}\left(\frac{\left(2k_{j1}+4k_{j2}r_c^2\right)x_i^c y_i^c}{1+k_1 r_c^2+k_2 r_c^4}\right), & f_j\frac{1}{z_i^c}\left(1+\frac{\left(2k_{j1}+4k_{j2}r_c^2\right)y_i^c}{1+k_1 r_c^2+k_2 r_c^4}\right), & -f_j\frac{y_i^c}{\left(z_i^c\right)^2}\end{bmatrix}$$

旋转矩阵： $\boldsymbol{R}_j\leftrightarrow \boldsymbol{w}_j=\left(w_{j1},w_{j2},w_{j3}\right)^T \quad \boldsymbol{C}_j\leftrightarrow\left(f_j,k_{j1},k_{j2},\boldsymbol{w}_j,\boldsymbol{t}_j\right)^T$

$$\frac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j,\boldsymbol{X}_i\right)}{\partial \boldsymbol{w}_j^T}=\frac{\partial \hat{\boldsymbol{u}}_{ij}\left(\boldsymbol{C}_j,\boldsymbol{X}_i\right)}{\partial \left(\boldsymbol{x}_i^c\right)^T}\frac{\partial \left(\boldsymbol{x}_i^c\right)^T}{\partial \boldsymbol{w}_j^T}$$

$$\frac{\partial \left(\boldsymbol{x}_i^c\right)}{\partial \boldsymbol{w}_j^T}=\boldsymbol{I}+\left[\boldsymbol{X}_i\right]_\times$$

$$\boldsymbol{x}_i^c=\begin{bmatrix}x_i^c\\ y_i^c\\ z_i^c\end{bmatrix}=\boldsymbol{R}\boldsymbol{X}_i+\boldsymbol{t}$$

$$\begin{bmatrix}\tilde{x}_i^c\\ \tilde{y}_i^c\end{bmatrix}=\left(1+k_{j1}r_c^2+k_{j2}r_c^4\right)\begin{bmatrix}x_i^c\\ y_i^c\end{bmatrix},$$

$$r_c^2=\left(x_i^c\right)^2+\left(y_i^c\right)^2$$

$$\hat{\boldsymbol{u}}_{ij}=\begin{bmatrix}\hat{u}_{ij}\\ \hat{v}_{ij}\end{bmatrix}=\begin{bmatrix}f_j\frac{\tilde{x}_i^c}{z_i^c}+c_x\\ f_j\frac{\tilde{y}_i^c}{z_i^c}+c_y\end{bmatrix}$$

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雅阁比矩阵- $\frac{\partial \hat{\mathbf{u}}_{ij}\left(\mathbf{C}_j, \mathbf{X}_i\right)}{\partial \mathbf{X}_i^T}$

$$\frac{\partial \hat{\mathbf{u}}_{ij}\left(\mathbf{C}_j, \mathbf{X}_i\right)}{\partial \left(\mathbf{x}_i^c\right)^T} = \begin{bmatrix} \frac{\partial \hat{u}_{ij}}{\partial x_i^c}, & \frac{\partial \hat{u}_{ij}}{\partial y_i^c}, & \frac{\partial \hat{u}_{ij}}{\partial z_i^c} \\ \frac{\partial \hat{v}_{ij}}{\partial x_i^c}, & \frac{\partial \hat{v}_{ij}}{\partial z_i^c}, & \frac{\partial \hat{v}_{ij}}{\partial z_i^c} \end{bmatrix}$$

$$= \left(1 + k_1 r_c^2 + k_2 r_c^4\right) \begin{bmatrix} f_j \frac{1}{z_i^c}\left(1 + \frac{\left(2k_{j1} + 4k_{j2} r_c^2\right)\left(x_i^c\right)^2}{1 + k_1 r_c^2 + k_2 r_c^4}\right), & f_j \frac{1}{z_i^c}\left(\frac{\left(2k_{j1} + 4k_{j2} r_c^2\right) x_i^c y_i^c}{1 + k_1 r_c^2 + k_2 r_c^4}\right), & -f_j \frac{x_i^c}{\left(z_i^c\right)^2} \\ f_j \frac{1}{z_i^c}\left(\frac{\left(2k_{j1} + 4k_{j2} r_c^2\right) x_i^c y_i^c}{1 + k_1 r_c^2 + k_2 r_c^4}\right), & f_j \frac{1}{z_i^c}\left(1 + \frac{\left(2k_{j1} + 4k_{j2} r_c^2\right) y_i^c}{1 + k_1 r_c^2 + k_2 r_c^4}\right), & -f_j \frac{y_i^c}{\left(z_i^c\right)^2} \end{bmatrix}$$

对三维点：

$$\frac{\partial \hat{\mathbf{u}}_{ij}\left(\mathbf{C}_j, \mathbf{X}_i\right)}{\partial \mathbf{X}_j^T} = \frac{\partial \hat{\mathbf{u}}_{ij}\left(\mathbf{C}_j, \mathbf{X}_i\right)}{\partial \mathbf{X}_j^T} \frac{\partial \left(\mathbf{x}_i^c\right)}{\partial \mathbf{X}_j^T}$$

$$\frac{\partial \left(\mathbf{x}_i^c\right)}{\partial \mathbf{X}_i^T} = \mathbf{R}_j$$

$$\mathbf{x}_i^c = \begin{bmatrix} x_i^c \\ y_i^c \\ z_i^c \end{bmatrix} = \mathbf{R}\mathbf{X}_i + \mathbf{t}$$

$$\begin{bmatrix} \tilde{x}_i^c \\ \tilde{y}_i^c \end{bmatrix} = \left(1 + k_{j1} r_c^2 + k_{j2} r_c^4\right) \begin{bmatrix} x_i^c \\ y_i^c \end{bmatrix},$$

$$r_c^2 = \left(x_i^c\right)^2 + \left(y_i^c\right)^2$$

$$\hat{\mathbf{u}}_{ij} = \begin{bmatrix} \hat{u}_{ij} \\ \hat{v}_{ij} \end{bmatrix} = \begin{bmatrix} f_j \frac{\tilde{x}_i^c}{z_i^c} + c_x \\ f_j \frac{\tilde{y}_i^c}{z_i^c} + c_y \end{bmatrix}$$

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```
struct SnavelyReprojectionError {
    SnavelyReprojectionError(double observed_x, double observed_y)
            : observed_x(observed_x), observed_y(observed_y) {}

    template <typename T>
    bool operator()(const T* const camera,
                    const T* const point,
                    T* residuals) const {
        // camera[0,1,2] are the angle-axis rotation.
        T p[3];
        AngleAxisRotatePoint(camera, point, p);

        // camera[3,4,5] are the translation.
        p[0] += camera[3];
        p[1] += camera[4];
        p[2] += camera[5];

        // Compute the center of distortion.
        const T xp =  p[0] / p[2];
        const T yp =  p[1] / p[2];

        // Apply second and fourth order radial distortion.
        const T& l1 = camera[7];
        const T& l2 = camera[8];
        const T r2 = xp*xp + yp*yp;
        const T distortion = 1.0 + r2  * (l1 + l2  * r2);

        // Compute final projected point position.
        const T& focal = camera[6];
        const T predicted_x = focal * distortion * xp;
        const T predicted_y = focal * distortion * yp;

        // The error is the difference between the predicted and observed position.
        residuals[0] = predicted_x - observed_x;
        residuals[1] = predicted_y - observed_y;

        return true;
    }
```

旋转平移

径向畸变

投影过程

$$\mathbf{x}_i^c = \begin{bmatrix} x_i^c \\ y_i^c \\ z_i^c \end{bmatrix} = \boldsymbol{R}\boldsymbol{X}_i + \boldsymbol{t}$$

$$\begin{bmatrix} \tilde{x}_i^c \\ \tilde{y}_i^c \end{bmatrix} = \left(1 + k_{j1} r_c^2 + k_{j2} r_c^4\right) \begin{bmatrix} x_i^c \\ y_i^c \end{bmatrix},$$

$$r_c^2 = \left(x_i^c\right)^2 + \left(y_i^c\right)^2$$

$$\hat{\boldsymbol{u}}_{ij} = \begin{bmatrix} \hat{u}_{ij} \\ \hat{v}_{ij} \end{bmatrix} = \begin{bmatrix} f_j \dfrac{\tilde{x}_i^c}{z_i^c} + c_x \\ f_j \dfrac{\tilde{y}_i^c}{z_i^c} + c_y \end{bmatrix}$$

捆绑调整Bundle Adjustment

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```
// 构建bundle adjustment 问题
ceres::examples::BALProblem bal_problem(n_cameras
                                        ,n_points
                                        ,observations.size()
                                        ,point_index
                                        ,camera_index
                                        ,observations
                                        ,params);


// 获取观测到的图像点
const double* observations_ptr = bal_problem.observations();
// 对于每一个观察点，构造一个残差方程 |predicted_x - observed_x|^2 + |predicted_y - observed_y|^2
ceres::Problem problem;
for (int i = 0; i < bal_problem.num_observations(); ++i) {
    // 每一个残差模块输入一个相机和一个三维点，输出一个二维的残差，实际上残差反映了冲投影误差
    ceres::CostFunction* cost_function =
            ceres::examples::SnavelyReprojectionError::Create(observations_ptr[2 * i + 0], observations_ptr[2 * i + 1]);
    problem.AddResidualBlock(cost_function,
                             NULL /* squared loss */,
                             bal_problem.mutable_camera_for_observation(i),
                             bal_problem.mutable_point_for_observation(i));
}
```

每一个观察点和一个相机
构建一个 residual block

$$g(\boldsymbol{\theta})=\frac{1}{2}\sum_{i}^{n}\sum_{j}^{m}\chi_{ij}\boxed{\left\|\mathbf{u}_{ij}-\hat{\mathbf{u}}_{ij}\left(\boldsymbol{C}_j,\boldsymbol{X}_i\right)\right\|^2}=\frac{1}{2}\sum_{i}^{n}\sum_{j}^{m}\chi_{ij}e_{ij},$$

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三维建模课程地址

感谢各位聆听！

Thanks for Listening